Indag. Mathem., N.S., 6 (4), 439–451 December 18, 1995

On one-dimensional separated rigid spaces

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Communicated by Prof. T.A. Springer at the meeting of October 31, 1994

1. INTRODUCTION

This note answers a question which arose in [JP], Remark (2.5.11). The question is the following:

Let X be a connected one-dimensional and separated rigid space over some complete, non-archimedean valued field k. Does X have an admissible covering by affinoid subsets $\{X_i\}$, such that each X_i meets only finitely many X_j?

In the terminology of [JP], such a covering is called *locally finite* and an X having such a covering is called *paracompact*. We note that if X is connected and has a locally finite admissible covering by affinoids, then this covering is at most countable.

The analogue of the question over the field of complex numbers is the theorem of Radon which states that a connected Riemann surface is a countable union of open subsets isomorphic to the open unit disk in $\mathbf{C}$. We will prove that the question above has an affirmative answer and we will show the following stronger statement:

There exists an admissible formal scheme $\mathcal{X}$ over the valuation ring R of k, (i.e. $\mathcal{X}$ is flat over R and locally topologically of finite type) which is separated, such that $\mathcal{X}$ has a locally finite covering by affine subsets and such that its 'generic fibre' $\mathcal{X} \otimes k$ coincides with X.

Special cases of the statement above have already been proved in [FP], p. 34, [L], Corollary 2.1.1 and [R], 3.2.3(1). In Section 3, after proving a Mittag–Leffler decomposition theorem, we show that X is a quasi-Stein space if no irreducible component of X is complete. Counterexamples for higher dimensional rigid spaces complete this note.

2. ONE-DIMENSIONAL SPACES OF COUNTABLE TYPE

A rigid space X is called *of countable type* if X has a countable (or finite) admissible covering by affinoids. The main step in proving structure theorems for connected, separated one-dimensional spaces X is to show that such an X is of countable type. This is done in the present section.

Theorem 2.1. *If X is a connected, separated one-dimensional rigid space then X is of countable type.*

Proof. The proof has many ingredients. In particular we will need the notion of *relatively completeness.*

Definition 2.2. Let V be an affinoid space of dimension 1, let U be an affinoid open subset of V. We call U *relatively complete in* V if there is a reduction $r : V \to \bar{V}$ such that $U = r^{-1}(\bar{U})$ for some open subset $\bar{U}$ of $\bar{V}$ and no connected component of $\bar{V} \backslash \bar{U}$ is complete

This property doesn't depend on the chosen reduction $r : V \to \bar{V}$ such that U is the preimage of some open subset of $\bar{V}$.

For example, if U is relatively compact in V, then U is relatively complete in V. But the converse is false.

Let X be a one-dimensional separated connected rigid space over k. We will say that an affinoid open subset U of X is *relatively complete in* X if U is relatively complete in every affinoid $V \subseteq X$ containing U.

The proof of (2.1) starts by a reduction step.

Reduction to the case of an algebraically closed ground field

Let K denote the completion of a separable algebraic closure of k. It is well known that K is algebraically closed. In other words, K is also the completion of the algebraic closure of k. Consider a connected, separated one-dimensional rigid space X over k. We want to show that X is of countable type by comparing X with $X \hat{\otimes}_k K$. By assumption every connected component of the space $X \hat{\otimes}_k K$ is of countable type. The next lemma asserts implies that $X \hat{\otimes}_k K$ is also of countable type and so $X \hat{\otimes}_k K$ has a countable admissible affinoid covering $\{Z_n\}$. Let $\{X_i\}_{i \in I}$ denote an admissible affinoid covering of X. Then every Z_n is contained in finitely many $X_i \hat{\otimes}_k K$. This implies the existence of a subset $J \subset I$ which is at most countable and satisfies $\{X_j \hat{\otimes}_k K\}_{j \in J}$ is an admissible affinoid covering of $X \hat{\otimes}_k K$. It is easily seen that $\{X_j\}_{j\in J}$ is a countable admissible covering of X.

Lemma 2.3. *The number of connected components of $X \hat{\otimes}_k K$ is finite.*

Proof. Let l be a finite normal extension of k, contained in K, such that $X(l) \neq \emptyset$. Let l^s denote the separable closure of k in l. The Galois group $\mathrm{Gal}(l^s/k)$ acts transitively on the connected components of $X \otimes_k l^s$. Since $X \otimes_k l \to X \otimes_k l^s$ is a homeomorphism, $X \otimes_k l$ has finitely many connected components and each one of them has a rational point over l. This reduces the proof of (2.3) to the case $X(k) \neq \emptyset$.

Let $p \in X(k)$. The family of the open connected affinoid subsets U with $p \in U$ is an admissible covering of X. It suffices to show that for such a U the affinoid space $V := U \hat{\otimes}_k K$ is connected.

Any connected component of V is already defined over a finite Galois extension l of k, contained in K. Indeed, such fields are dense in K. The Galois group $\mathrm{Gal}(l/k)$ acts transitively on the connected components of $U \otimes_k l$. Since $U(k) \neq \emptyset$, it follows that $U \otimes_k l$ and also V is connected. □

The case of an algebraically closed base field *k*

We start again with a reduction step. The field k is supposed to be algebraically closed and X is a connected, separated rigid space over k of dimension one. Let $\{X_i\}_{i\in I}$ denote the set of irreducible components of X. Let $n : X' \to X$ denote the normalization of X. Then X' is the disjoint union of the normalizations X_i' of the X_i. Let S denote the set of singular points of X. This set is discrete in the sense that the intersection of S with any affinoid subset of X is finite. The preimage S' of S under n is also discrete. Suppose that we have shown that each X_i' has a countable admissible covering by affinoids. Then $S' \cap X_i'$ is at most countable. It follows that X_i meets at most countably many of the X_j. Since X is connected it follows that I is at most countable. X_i as image of the space X_i' of countable type is also of countable type. Therefore we have reduced the general case to the case where X is connected and non-singular over the algebraically closed base field k. We may suppose moreover that X is not quasi-compact.

Let x be a point of X and consider the family $\mathcal{F}$ of all connected affinoid subspaces of X containing x. This family is an admissible affinoid covering of X (see [L], Proposition 1.2). The family $\mathcal{F}$ is in general not countable and we will replace $\mathcal{F}$ by a smaller subfamily $\mathcal{F}_1$.

Consider an U in $\mathcal{F}$ and a reduction $r : U \to \bar{U}$. Let $\hat{\bar{U}}$ denote the complete curve over $\bar{k}$, the residue field of k, containing $\bar{U}$ such that $\hat{\bar{U}}\backslash\bar{U}$ consists of a finite number of regular points. (This set does not depend on the chosen reduction.) *We will call this set $H(U)$.*

One can easily show that $U \subset V \subset W$; $U, V, W \in \mathcal{F}$, U relatively complete in V and V relatively complete in W implies that U is relatively complete in W. If U is not relatively complete in X then: there is a $U \subset V \in \mathcal{F}$, a reduction $r : V \to T$, an open $\bar{U} \subset T$ with preimage U and a connected component C of $T \backslash \bar{U}$ which is complete. Define U' to be the preimage under r of $\bar{U} \cup C$. Then clearly the cardinal of $H(U')$ is less than the one of $H(U)$. After a finite number of steps of this kind one finds a connected affinoid U'' containing U which is relatively complete in X. Let $\mathcal{F}_1$ denote the set of elements of $\mathcal{F}$ which are relatively complete in X. Then $\mathcal{F}_1$ is an admissible affinoid covering of X.

We will use in the sequel that every connected, non-singular one-dimensional affinoid space U has a unique stable reduction $\bar{U}^{st}$ and that every semi-stable reduction of U is obtained from $\bar{U}^{st}$ by a sequence of 'blowing-ups' of points. The statements above follow easily from the existence and uniqueness of a stable reduction of any complete non-singular curve over the algebraically closed field k (see [BL1] or [P2]).

Consider finitely many elements $U_1, \ldots, U_n, V$ of $\mathcal{F}_1$ such that all $U_i \subset V$. There is a unique reduction $r : V \to T$ having the following properties:

1. T is semi-stable.
2. For every i there exists an open subset Z_i of T such that $U_i = r^{-1} Z_i$.
3. r is minimal in the sense that: if L is an irreducible component of T, isomorphic to the projective line over $\bar{k}$, meeting the other components of T in less than three points, then there exists an i such that $L \cap Z_i \neq \emptyset, L$.

This reduction r will be called *the stable reduction of* $(U_1, \ldots, U_n, V)$. Let $r : V \to T$ be the stable reduction of (U, V) and let Z be the open subset of T with $r^{-1} Z = U$. Then the map $\hat{Z} \to \hat{T}$ is injective. Let Z^{cl} denote the Zariski-closure of Z in T. Then one can identify $Z^{cl} \backslash Z$ with a subset of $H(U)$. Let a 'new' irreducible component L of T be given (i.e. L is not a component of Z^{cl}). Then either $L \cap Z^{cl} = \emptyset$ or $L \cap Z^{cl} \neq \emptyset$ and $L \cap Z^{cl}$ is a finite subset B of $Z^{cl} \backslash Z \subset H(U)$.

In the latter case one calls B a *V-bridge.* Further B is called *a stable V-bridge* if the component L of T with $B = L \cap Z^{cl}$ satisfies one of the following properties:

1. The arithmetic genus of L is > 0.
2. $\#(\hat{L} \backslash L) + \#\{\text{the intersection of } L \text{ with the other components of } T\} \geq 3$.

Using that U is relatively complete in V one sees that the following holds: If B does not satisfy one of the conditions above then L must be A^1_k and L meets the other components of T only in B and B consists of one point.

If B is a stable V-bridge then B is also a stable W-bridge for every $W \in \mathcal{F}_1$ with $V \subset W$. Indeed, the conditions (1) or (2) imply that the component 'L' in the stable reduction for (U, V, W) is not contracted in the stable reduction for (U, W). Further 'L' has still B as intersection with Z^{cl}.

One calls $B \subset H(U)$ a *stable bridge* if B is a stable V-bridge for some $V \in \mathcal{F}_1$.

A subset B of $H(U)$ is called an *unstable bridge* if B is a W-bridge for some $W \in \mathcal{F}_1$ and such that for every $V \in \mathcal{F}_1$ with $U \subset W \subset V$, B remains a V-bridge but is not a stable V-bridge. In other words, for $V \supset W$, $V \in \mathcal{F}_1$, the irreducible component L of T with $T \cap Z^{cl} = B$ satisfies:

(a) $L \cong A^1_{\bar{k}}$.

(b) L meets the other components of T only in B and B consists of one point.

The main combinatorial statement about 'bridges' is the following:

Lemma 2.4.

1. *Let $B \subset H(U)$ be a stable bridge. There exists a unique minimal $V \in \mathcal{F}_1$, such that $U \subset V$ and B is a stable V-bridge.*

2. *Let $B \subset H(U)$ be an unstable bridge. There exists a sequence of elements V_n (finite or infinite) in $\mathcal{F}_1$ such that:*

(a) *$U \subset V_n \subset V_{n+1}$ for every n.*

(b) *B is a V_n-bridge and the stable reduction of (U, V_n) is the union of Z with an affine line (i.e. isomorphic to $A^1_{\bar{k}}$) over the residue field $\bar{k}$ of k.*

(c) *Every $V \in \mathcal{F}_1$ containing U, such that B is a V-bridge and such that the stable reduction of the pair (U, V) consists of Z and an affine line over the residue field of k, is contained in some V_n.*

Proof. 1. Consider $S := \{V \in \mathcal{F}_1 \mid U \subset V \text{ and } B \text{ is a } V\text{-stable bridge}\}$.

1(a) We will show first that any element $V \in S$ contains a minimal element of S. Consider the stable reduction $r : V \to T$ for (U, V) with open subset $Z \subset T$ such that $r^{-1}Z = U$. Let L denote the irreducible component of T with $L \cap Z^{cl} = B$. Suppose that there is a new irreducible component $M \neq L$ of T which is not complete. Let T' denote the connected component of $T \backslash M$ which contains Z and let $V' = r^{-1}T'$. Then V' is connected and relatively complete in V. It follows that $V' \in S$ and $V' \neq V$.

After finitely many steps of this type we may suppose that every new irreducible component $M \neq L$ of T is complete. It follows that $T \backslash Z$ has precisely one connected component and that L is the only non-complete irreducible component of $T \backslash Z$. We claim now that V is a minimal element of S.

Let $W \subset V$ and $W \in S$. After 'blowing-up' some points in the stable reduction $r : V \to T$ for (U, V) one obtains the stable reduction $r_1 : V \to T_1$ for (U, W, V). Let $O \supset Z$ denote the open subset with $r_1^{-1}O = W$. Since W is relatively complete in V, $T_1 \backslash O$ is a disjoint union of non-complete connected components. The reasoning above implies that L is the only new irreducible component of T_1 which is not complete. Since B is W-stable, O must contain L. It follows that $T_1 \backslash O = \emptyset$ and so $W = V$. This proves 1(a).

1(b) We will show that S has a unique minimal element.

Let V_1, V_2 be minimal elements of S. Consider $V \supset V_1, V_2$ with $V \in S$ and the stable reduction $r : V \to T$ for (U, V_1, V_2, V). Let Z, O_1, O_2 denote the open subsets of T with $r^{-1}Z = U$, $r^{-1}O_1 = V_1$, $r^{-1}O_2 = V_2$. Let L denote the irreducible component of T with $L \cap Z^{cl} = B$. According to 1(a) one knows that O_1 satisfies: $O_1 \backslash Z$ has only one connected component and has only one non-complete irreducible component which is a Zariski-open subset of L. This determines O_1 as open subset of T. Hence $O_1 = O_2$ and $V_1 = V_2$.

2. Let $B = \{p\}$ be an unstable bridge.

Put $S := \{V \in \mathcal{F}_1 \mid U \subset V$ and B is the only V-bridge$\}$. According to the definition of 'B unstable', one has that the stable reduction $r : V \to T$ of (U, V) has the form $Z \cup L$ with $L \cong \boldsymbol{A}^1_{\bar{k}}$, $r^{-1}Z = U$ and $L \cap Z^{cl} = \{p\}$.

Consider two elements $V_1, V_2 \in S$. Choose a $V \in \mathcal{F}_1$ containing both V_1 and V_2. Let $r : V \to T$ denote the stable reduction for (U, V_1, V_2, V), let Z, O_1, O_2 denote the open subsets of T with $r^{-1}Z = U$, $r^{-1}O_1 = V_1$, $r^{-1}O_2 = V_2$. Let L denote the irreducible component of T with $L \cap Z^{cl} = B = \{p\}$. Since B is not a stable V-bridge one has the following possibilities:

(a) $L = \boldsymbol{A}^1_{\bar{k}}$ and L meets the other components of T only at p. This implies easily that $V_1 = V_2$.

(b) $L = \boldsymbol{P}^1_{\bar{k}}$ and L meets the other components of T in p and q. Then $O_1 \cap O_2 \supset L \backslash \{q\}$. If also $O_1 \cap O_2 \supset L$, then L can be contracted in the reduction for (U, V_1, V_2, V). So we may assume that $q \notin O_1$. It follows that $O_1 = Z \cup (L \backslash \{q\})$ and that $O_1 \subset O_2$. Hence $V_1 \subset V_2$. We have shown that the set S is totally ordered by inclusion.

Fix a $V_1 \in S$ and consider a $V \in S$ with $V_1 \subset V$ and $V \neq V_1$. The stable reduction $r : V \to T$ for (U, V_1, V) with open subsets $Z \subset O_1$ satisfying $r^{-1}Z = U$, $r^{-1}O_1 = V_1$ has according to the reasoning above two new irreducible components L, M. Hence $L \cong \boldsymbol{P}^1_{\bar{k}}$ and L meets the other components of T in the points p and q. Further $M \cong \boldsymbol{A}^1_{\bar{k}}$ and meets the other components in q. It follows that $T \backslash Z$ is the union of two affine lines over $\bar{k}$ meeting (normally) at one point. It is well known that the preimage of this set under r is isomorphic to a ring domain $\{z \in k \mid 1 \leq |z| \leq \rho\}$ for a unique $\rho > 1$, $\rho \in |k^*|$. We associate to every $V \in S$, containing V_1 as a proper subset, this real number $\rho = \rho(V)$. One easily shows that for $V, V' \in S$, properly containing V_1, one has:

$\rho(V) = \rho(V')$ if and only if $V = V'$.

$\rho(V) \leq \rho(V')$ if and only if $V \subset V'$.

This implies that S has a countable or finite cofinal subset $\{V_n \mid n = 1, \ldots\}$ and ends the proof of (2.4). $\square$

Notations 2.5. Let B be a stable bridge for U, then U_B is the unique element of (2.4.1). If B is an unstable bridge for U then $U_B(n)$ is the chosen cofinal sequence in (2.4.2).

For $U \subset V$; $U, V \in \mathcal{F}_1$ and $U \neq V$, one considers the stable reduction

$r : V \to T$ for (U, V). We will write $d(U, V)$ denote the number of new irreducible components of T. If $U = V$, then $d(U, V)$ is supposed to be 0.

Lemma 2.6. *Given $U \subset V$; $U, V \in \mathcal{F}_1$ and $U \neq V$. There are two possibilities:*
1. *There exists a stable bridge B of U such that*

$$U_B \cup V \in \mathcal{F}_1 \qquad \text{and} \qquad d(U_B, U_B \cup V) < d(U, V).$$

2. *There is an unstable bridge B of U and an integer n such that*

$$d(U_B(n), U_B(n) \cup V) < d(U, V) \qquad \text{and} \qquad U_B(n) \cup V \in \mathcal{F}_1.$$

Proof. Let $r : V \to T$ and $Z \subset T$ have the usual meaning for the stable reduction of (U, V). Let L be a new irreducible component of T such that $C := L \cap Z^{cl}$ is not empty. If C is contained in a stable bridge B of U then one can verify that (1) holds. If C happens to be an unstable bridge then one can see that (2) holds. □

Corollary 2.7. *If X is a connected, separated non-singular one-dimensional rigid space over the algebraically closed field k, then X is of countable type.*

Proof. Fix a $U \in \mathcal{F}_1$. For every $V \in \mathcal{F}_1$ we suppose that the at most countable sets of new elements V_B and $V_B(n)$ of $\mathcal{F}_1$ is chosen. Let $\mathcal{F}_2$ denote the smallest subset of $\mathcal{F}_1$ such that $U \in \mathcal{F}_2$ and such that for any $V \in \mathcal{F}_2$ all V_B and $V_B(n)$ belong to $\mathcal{F}_2$. Clearly $\mathcal{F}_2$ is countable. Using (2.6) one sees that every $V \in \mathcal{F}_1$ is contained in a finite union of elements of $\mathcal{F}_2$. Thus $\mathcal{F}_2$ is a countable admissible affinoid covering of X. □

This ends the proof of (2.1). □

3. FORMAL SCHEMES AND QUASI-STEIN SPACES

Let X denote again a connected separated one-dimensional rigid space. As we have shown X is of countable type. In case X is quasi-compact (i.e. a finite union of affinoids) then it is known that (see [FM], Theorem 2, p. 176) every irreducible component of X is either affinoid or the analytification of a projective curve over k. In both cases there exists a separated formal scheme $\mathcal{X}$ over the valuation ring R of k, which is flat over R and has as 'generic fibre' $\mathcal{X} \otimes k$ a rigid space which is isomorphic to X. In the sequel we will suppose (without loss of generality) that no irreducible component of X is quasi-compact.

Theorem 3.1. *Let X be one-dimensional, separated, connected. Then:*

1. *X is paracompact.*
2. *There exists a separated formal scheme $\mathcal{X}$ over R which is flat over R such that its generic fibre is X.*

Proof. Assuming that no irreducible component of X is the analytification of a projective curve over k and using again [FM], corollary of Theorem 2, one finds that X has an admissible affinoid covering $\{X_n\}_{n \in N}$ such that $X_n \subset X_{n+1}$ for all n. Using the next lemma, one finds a sequence of formal schemes $\mathcal{Z}_n$ such that:

1. $\mathcal{Z}_n$ is separated, flat over R.
2. The generic fibre of $\mathcal{Z}_n$ is X_n.
3. $\mathcal{Z}_n$ is an open subscheme of $\mathcal{Z}_{n+1}$.

Let $\mathcal{X}$ denote the formal scheme obtained as the union of the $\mathcal{Z}_n$. Then $\mathcal{X}$ has the properties stated in (3.1.2). Further, the topological space $\mathcal{X}$ has a locally finite covering by affine open subsets $\mathcal{A}_n$. The corresponding affinoid subsets $A_n := \mathcal{A}_n \otimes k$ of X form an admissible affinoid covering of locally finite type. This proves (3.1.1). □

Lemma 3.2. *Let $A \subset B$ denote two (pure) one-dimensional affinoids and let a separated formal scheme $\mathcal{A}$, flat over R, be given with generic fibre A. Then there exists a separated formal scheme $\mathcal{B}$, flat over R, with generic fibre B such that $\mathcal{A}$ is an open formal subscheme of $\mathcal{B}$.*

Proof. Since B is affinoid there exists a formal scheme $\mathcal{B}''$, flat over R, such that $\mathcal{B}'' \otimes k = B$. According to [M], 4.3.8 or [BL2], Section 5, there are formal admissible blowing-ups $\pi : \mathcal{A}' \to \mathcal{A}$, $\rho : \mathcal{B}' \to \mathcal{B}''$ and an open immersion $i : \mathcal{A}' \to \mathcal{B}'$ such that $i \otimes k : A \to B$ is the open immersion of A into B. The support of the blowing-up π is a finite set F and so proper over R. Since π itself is proper it follows that $\pi^{-1}F$ is proper and that $i(\pi^{-1}F)$ is closed in $\mathcal{B}'$. One can therefore glue $\mathcal{A}$ and $\mathcal{B}' \backslash i(\pi^{-1}F)$ over $\mathcal{A} \backslash \pi^{-1}F$. The resulting formal scheme $\mathcal{B}$ satisfies the conditions of the lemma. □

Corollary 3.3. *If one assumes moreover that the field k is algebraically closed and that X is non-singular, then there exists a separated admissible formal scheme $\mathcal{X}$, with generic fibre X, such that the special fibre of $\mathcal{X}$ has only ordinary double points as singularities. We will call such a formal scheme semi-stable.*

Proof. In the sequel we will use the following: If X is affinoid and if $\mathcal{X}$ is an admissible formal scheme over R with generic fibre X then a certain blow-up of $\mathcal{X}$ is semi-stable. This follows from the stable reduction theorem for algebraic curves over k. One can now apply (3.2) and the method of (3.1) with formal schemes which are semi-stable. □

In what follows, we want to show that X is a quasi-Stein space if no irreducible component of X is complete.

Recall that a rigid analytic space X is a *quasi-Stein space* in the sense of Kiehl [K] if there exists an admissible affinoid covering $\{X_n\}_{n \in N}$ of X such that $X_n \subseteq X_{n+1}$ and the restriction map $\mathcal{O}_X(X_{n+1}) \to \mathcal{O}_X(X_n)$ is dense for all n. For such a space, one has $H^q(X, \mathcal{F}) = 0$ for all $q \geq 1$ and for every coherent sheaf $\mathcal{F}$ ([K], Satz 2.4). The following proposition is previously known for spaces of finite genus ([L], 2.1.1).

Theorem 3.4. *Let X be a one-dimensional separated and connected rigid space without complete irreducible components. Then X is a quasi-Stein space.*

The following lemma is a kind of Mittag–Leffler decomposition theorem. The same result over a discrete valuation field has been proved by Raynaud [R], 3.5.2.

Lemma 3.5. *Assume the base field k to be algebraically closed. Let C be a projective smooth curve over k, let U be an affinoid open subset of X, analytification of C.*
(a) *The space $X\setminus U$ has only finitely many connected components $D_1, \ldots, D_m$.*
(b) *Fix a point $p_i \in D_i$ for all $1 \leq i \leq m$. Then the restriction map*

$$\mathcal{O}_C(C\setminus\{p_1, \ldots, p_m\}) \to \mathcal{O}_X(U)$$

is dense.

Proof. (a) There is a reduction $r_0 : X \to T_0$ such that the canonical reduction $\bar{U}^c$ of U is a dense open subset of T_0. Then the connected components of $X\setminus U$ correspond to the points of $T_0\setminus\bar{U}^c$.

(b) Let $r : X \to T$ be a semi-stable reduction such that $U = r^{-1}(\bar{U})$ for some open subset $\bar{U}$ of T and $\bar{p}_i := r(p_i)$ is smooth for all i. Then D_i is just the pre-image of the connected component of $T\setminus\bar{U}$ containing $\bar{p}_i$. Let Γ denote the Zariski closure of $\bar{U}$ in T. We will make induction on the number d of irreducible components of $T\setminus\Gamma$.

First assume that $T = \Gamma$ (i.e. $d = 0$). Then $T = \bar{U} \cup \{\bar{p}_1, \ldots, \bar{p}_m\}$. Since $\bar{p}_i$ is smooth, there exists an affine open subset W_i of T containing $\bar{p}_i$ and $h_i \in \mathcal{O}(r^{-1}(W_i))$ with $\{p_i\}$ as zero set. Using exactly the same construction as in [FM], § 1.3, Proposition 1, one gets a regular function $F \in \mathcal{O}_C(C\setminus\{p_1, \ldots p_m\})$ such that

$$U = \{x \in X \mid F \in \mathcal{O}_{X,x} \text{ and } |F(x)| \leq 1\}.$$

This implies obviously that $\mathcal{O}_C(C\setminus\{p_1, \ldots, p_m\}) \to \mathcal{O}_X(U)$ has dense image.

Suppose now that $T \neq \Gamma$ and that the proposition is true for all couples (U_0, C_0) such that the associated number d_0 is less than d. It suffices to consider the case $T\setminus\Gamma$ connected (i.e. $m = 1$). Actually, put $U_i = X\setminus D_i$. Then $U = \bigcap_i U_i$ and the canonical map

$$\mathcal{O}(U_1) \hat{\otimes}_k \cdots \hat{\otimes}_k \mathcal{O}(U_m) \to \mathcal{O}(U)$$

is surjective. So if $\mathcal{O}(C\setminus\{p_i\}) \to \mathcal{O}(U_i)$ is dense for all i, then

$$\mathcal{O}(C\setminus\{p_1, \ldots, p_m\}) \to \mathcal{O}(U)$$

is dense. Note that the number of irreducible components of $T\setminus r(U_i)$ is $\leq d$.

We suppose now that $T\setminus\Gamma$ is connected. Let Z be the irreducible component of T containing $\bar{p} = \bar{p}_1$, Z is not contained in Γ. Put $S =$ Zariski closure of $T\setminus Z$ in T and $V = r^{-1}(T\setminus Z) \supseteq U$. There is a smooth projective curve E over k and a reduction $s: E \to \bar{E} = S$ such that $V = s^{-1}(T \backslash Z)$ (see [P], Theorem 1.1). Fix points $q_1, \ldots, q_n \in E \backslash V$ in the connected components of $E \backslash U$. Since the number of irreducible components of $S \backslash \Gamma$ is less than d, $\mathcal{O}(E \backslash \{q_1, \ldots, q_n\}) \to \mathcal{O}(U)$ is dense. But $V \subset E \backslash \{q_1, \ldots, q_n\}$, so $\mathcal{O}(V) \to \mathcal{O}(U)$ is dense. On the other hand, put $W = r^{-1}(T \backslash \{\bar{p}\})$, then by the first case ($T = \Gamma$), we have that $\mathcal{O}(C \backslash \{p\}) \to \mathcal{O}(W)$ is dense. Finally, we are reduced to prove that $\mathcal{O}(W) \to \mathcal{O}(V)$ is dense.

Let $\bar{g}$ be a rational function over T with a unique pole $\bar{p}$ and which vanishes on $Z \cap S$ (a such function exists according to Riemann–Roch). Then $\bar{g}$ lifts to a holomorphic function $g \in \mathcal{O}(W)$, and there is a $\rho_0 \in k$ such that $|g(x)| \le |\rho_0| < 1$ for all $x \in V$. For all $\rho \in k$, let W_ρ denote the union of the connected components of $\{x \in W \mid |g(x)| \le |\rho|\}$ which meet V. If $t \in S \cap Z$, then one has an isomorphism $\phi_t : r^{-1}(t) \to \{z \in k \mid |\pi_t| < |z| < 1\}$ for some $\pi_t \in k$. The isomorphism ϕ_t can be chosen so that the 'boundary' $\phi_t^{-1}(\{|z| = 1\})$ belongs to V. Therefore, when $|\rho| < 1$ is close enough to 1, W_ρ is the union of V with the open annulis $\{|\pi_t'| \le |z| < 1\}$, $t \in S \cap Z$, $|\pi_t'| > |\pi_t|$. Using the same process as above (embedding W_ρ in some curve), one sees that $\mathcal{O}(W_\rho) \to \mathcal{O}(V)$ is dense. Since $\mathcal{O}(W) \to \mathcal{O}(W_\rho)$ is obviously dense, the lemma is proved. □

Corollary 3.6. *Lemma* (3.5) *is valid for an arbitrary base field k.*

Proof. The method of Lemma (3.8) can be applied. □

Lemma 3.7. *Let $U \subseteq V$ be one-dimensional affinoids. Then the property 'U is relatively complete in V' is preserved under normalization and base change $k \subseteq K = \widehat{k^{alg}}$.*

Proof. Let $r : V \to \bar{V}$ be a reduction such that $U = r^{-1}(\bar{U})$ for some open subset $\bar{U}$ of $\bar{V}$. Let $f : W \to V$ denote the normalization of V or the projection $V \hat{\otimes}_k K \to V$. Then $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is the normalization of U or the projection $U \hat{\otimes}_k K \to U$. Furthermore, r induces a reduction $r' : W \to \bar{W}$ and a finite integral morphism $\bar{f} : \bar{W} \to \bar{V}$ such that $r \circ f = \bar{f} \circ r'$. Since any connected component of $\bar{W} \backslash \bar{f}^{-1}(\bar{U})$ maps surjectively and finitely to a connected component of $\bar{V} \backslash \bar{U}$, U is relatively complete in V if and only if so is $f^{-1}(U)$ in W. □

Lemma 3.8. *Let U and V be as above. If U is relatively complete in V, then the restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$ is dense.*

Proof. It suffices to prove that $\mathcal{O}(V) \hat{\otimes}_k K \to \mathcal{O}(U) \hat{\otimes}_k K$ has a dense image. This follows from the fact that one can approximate the restriction of the valuation of K to every finite dimensional subspace E over k of K, by a norm for which there is an orthogonal basis on E. According to Lemma (3.7), we may now suppose that k is algebraically closed.

Let $V' \to V$ be the normalization of V. Since V' is a smooth one-dimensional affinoid space over k, there exists a smooth projective curve C over k such that V' is an affinoid open subset of X, the analytification of C ([P], Theorem 1.1). Let $U' \to U$ denote the normalization of U. Since U' is relatively complete in V' (Lemma (3.7)), each connected component D_i of $X \backslash U'$ contains a point $p_i \in X \backslash V'$ so that $\mathcal{O}_C(C \backslash \{p_i\}_i)$ is a subring of $\mathcal{O}(V')$. Therefore, according to Lemma (3.5), $\mathcal{O}(V')$ is dense in $\mathcal{O}(U')$.

By Runge's theorem, $\mathcal{O}(V)$ is dense in $\mathcal{O}(U)$ if and only if U is a Weierstrass subset of V. Since the latter property depends only on the reduced structure over V, one may assume that V is reduced.

Let A denote the closure of $\mathcal{O}(V)$ in $\mathcal{O}(U)$. Since $\dim_k(\mathcal{O}(V')/\mathcal{O}(V)) < \infty$ and $\mathcal{O}(V')$ is dense in $\mathcal{O}(U')$, one has $\dim_k(\mathcal{O}(U)/A) < \infty$. So A is an affinoid algebra over k. Let $\pi : U \to \operatorname{Spm} A$ (resp. $\phi : \operatorname{Spm} A \to V$) be the morphism corresponding to $A \subseteq \mathcal{O}(U)$ (resp. $\mathcal{O}(V) \subseteq A$), then $\phi \circ \pi = i$ the canonical open immersion of U in V. Furthermore π is surjective, so $\Im(\phi) \subseteq U$. Therefore, ϕ induces a morphism $\operatorname{Spm} A \to U$ which the inverse of π. So $A = \mathcal{O}(U)$ and $\mathcal{O}(V)$ is dense in $\mathcal{O}(U)$. □

Proof of Theorem 3.4. As we have seen in Section 2, there is a countable affinoid admissible covering $\{U_i\}_i$ of X. Enlarging if necessary U_i one can assume that U_i is relatively complete in X. Since no irreducible component of X is complete, any finite union of affinoids of X is still affinoid. So by taking finite unions of U_i's, one gets an admissible affinoid covering $\{X_n\}_{n \in N}$ of X such that $X_n \subseteq X_{n+1}$ and X_n is relatively complete in X (X_n is then relatively complete in X_{n+1}). Now the Lemma (3.8) says that $\mathcal{O}_X(X_{n+1}) \to \mathcal{O}_X(X_n)$ is dense and the proof is finished. □

Remark 3.9. Let X be as in Proposition (3.4), let $\mathcal{X}$ be an admissible formal scheme with generic fibre X. It can be seen that X is a Stein space (i.e. there exists an admissible affinoid covering $\{X_n\}_{n \in N}$ of X such that $X_n \subseteq X_{n+1}$ and X_n is relatively compact in X_{n+1} for all n) if and only if all irreducible components of the special fibre $\mathcal{X}_s$ are complete.

4. COUNTEREXAMPLES IN HIGHER DIMENSION

One considers a subset Λ of R, the valuation ring of k, such that the restriction of the residue map $R \to \bar{k}$ to Λ is injective. Let $\bar{\Lambda}$ denote the image of Λ. We suppose that Λ and $\bar{k} \backslash \bar{\Lambda}$ are infinite sets. Let π denote an element of R with $0 < |\pi| < 1$. For each $\lambda \in \Lambda$ one considers the affinoid subset X_λ of the polydisk $\{(s, t) \in k^2 \mid |s| \leq 1, |t| \leq 1\}$ given by the inequality $|s - \lambda t| \leq |\pi|$. Let X_Λ denote the union of the X_λ.

Lemma 4.1. *For every affinoid subset Z of X_Λ there is a finite subset Λ' of Λ such that is contained in $X_{\Lambda'} := \bigcup_{\lambda \in \Lambda'} X_\lambda$.*

Proof. Z can be written as the union of two affinoids $Z_1 = \{(s, t) \in Z \mid |t| \leq |\pi|\}$

and $Z_2 = \{(s,t) \in Z \mid |t| \geq |\pi|\}$. The first set is contained in any X_λ. On Z_2 we consider the function $f := s/t$. For each point z of $Z_2 \backslash Z_1$ there is a λ such that $z \in X_\lambda$ and so $|f(z) - \lambda| \leq |\pi|/|t(z)| < 1$. The corresponding map $\bar{f} : \bar{Z}_2 \to A^1_{\bar{k}}$, where $\bar{Z}_2$ denotes the canonical reduction of Z_2, has the property $\bar{f}(\bar{Z}_2 \backslash \overline{(Z_1 \cap Z_2)}) \subset \bar{\Lambda}$. Since this image is also a constructible subset of $A^1_{\bar{k}}$ it is finite, say $\{\bar{\lambda}_1, \ldots, \bar{\lambda}_m\}$. It follows that Z is contained in the union of the $\{X_{\lambda_i}\}_{i=1,\ldots,m}$. □

Remark 4.2. The same proof also yields a somewhat stronger result:

Let $g : Z \to \{(s,t) \in k^2 \mid |s| \leq 1, |t| \leq 1\}$ be a morphism of affinoid spaces such that the image of g is contained in X_Λ, then the image of g is contained in $X_{\Lambda'}$ for some finite subset Λ' of Λ.

The remark implies that X_Λ and the covering by the affinoids X_λ satisfies the conditions of Proposition 2 on p. 914 of [BGR]. Hence X_Λ is an admissible open subspace of the two-dimensional polydisk and $\{X_\lambda\}_{\lambda \in \Lambda}$ is an admissible affinoid covering of X_Λ. The structure of X_Λ as rigid space is induced by its embedding in the polydisk.

Proposition 4.3.

1. *X_Λ is separated.*
2. *X_Λ has no locally finite admissible covering by affinoid subsets.*
3. *If Λ is not countable then X_Λ has no countable admissible covering by affinoid subsets.*

Proof. 1. X_Λ is a subspace of the unit polydisk and therefore separated.

2. Suppose that a locally finite admissible covering $\{A_i\}_{i \in I}$ by connected affinoids of X_Λ exists. Then every affinoid subspace Z of X_Λ meets only finitely many of the A_i. Let us take $Z = \{(s,t) \mid |s| \leq |\pi|, |t| \leq |\pi|\}$. If $A_i \cap Z = \emptyset$ then $A_i \subset X_\lambda \backslash Z$ for some λ. Indeed, one has supposed that A_i is connected and one easily sees that the connected components of $X_\Lambda \backslash Z$ are the $X_\lambda \backslash Z$. Since λ is infinite there exists a $\lambda_0 \in \Lambda$ such that for every $i \in I$ with $A_i \cap Z \neq \emptyset$ the affinoid A_i is contained in a finite union of X_λ with $\lambda \neq \lambda_0$. The restriction of the admissible covering $\{A_i\}$ to X_{λ_0} consists of Z and affinoid subsets of $X_{\lambda_0} \backslash Z$. Such an admissible covering does not exist.

3. Suppose that $\{Z_n\}$ is a countable admissible covering of X_Λ by affinoid subsets. Since each Z_n is contained in finitely many X_λ, we find a countable subset Λ' of Λ such that $X_{\Lambda'} = X_\Lambda$. For $\lambda \in \Lambda \backslash \Lambda'$ and $\lambda' \in \Lambda'$ one has that $X_\lambda \cap X_{\lambda'} = \{(s,t) \mid |s| \leq |\pi|; |t| \leq |\pi|\}$. This shows that $\Lambda = \Lambda'$ and that Λ is countable. □

Remarks 4.4. 1. The example X_Λ supposes that the residue field $\bar{k}$ is infinite. For finite residue fields $\bar{k}$ one can make an analogous construction. This uses not only linear inequalities $|s - \lambda t| \leq |\pi|$ but also inequalities $|P(s,t)| \leq |\pi|^d$

where P is a homogeneous polynomial of degree d with coefficients in the valuation ring R of k and such that its reduction is an irreducible form of degree d over the residue field $\bar{k}$. Since there are countably many homogeneous polynomials of this type one can imitate the construction of X_Λ. In particular, for any local field k there is a separated rigid open subspace X of the two-dimensional polydisk over k, such that X is of countable type but does not have a locally finite admissible covering.

2. The condition 'separated' for the result stated in the introduction is necessary. Let us give an example of a one-dimensional quasi-separated space, which is not separated. Let D denote the unit disk. Let I be any set and let $\{D_i\}_{i \in I}$ denote a family of copies of D. The space X is made by glueing each D_i to D over their subdisks $\{z \mid |z| \leq |\pi|\}$. The result X is quasi-separated and not separated. If I is not countable then X is not of countable type.

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