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A STRUCTURE THEOREM FOR SIMPLE TRANSCENDENTAL EXTENSIONS OF VALUED FIELDS

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ABSTRACT. The fundamental inequality for a finite algebraic extension of a valued field relates the degree of the extension to the ramification indices and residue degrees, and of primary importance is the question of when this inequality becomes equality. An analogous question for simple transcendental extensions is treated here as an application of a fundamental structure theorem for such extensions.

Let $K_0 \subset K = K_0(x)$ be fields with x transcendental (abbreviated tr.) over K_0; let v_0 be a valuation of K_0 and v be an extension of v_0 to K; and let $V_0 \subset V$, $k_0 \subset k$, and $G_0 \subset G$ be the respective valuation rings, residue fields, and value groups. There are two possibilities for the residue field extension k/k_0: either (i) k/k_0 is algebraic (possibly of infinite degree) or (ii) k/k_0 is finitely generated of deg of transcendence 1 (cf. [**9**, p. 203, §1.3]). We shall be interested here in extensions for which (ii) holds, the *residually tr. extensions* (also called the residually nonalgebraic extensions). Henceforth we assume throughout the paper that $(K_0, v_0) \subset (K = K_0(x), v)$ is a (simple tr.) residually tr. extension.

For such extensions there exists $t \in K$ such that $v(t) = 0$ and t^* is tr. over k_0, where * denotes image under the canonical homomorphism $V \to V/m_v = k$; such a t will be called a *residually tr. element of* K (or, more precisely, of the extension $(K_0, v_0) \subset (K, v)$). For any $s \in K \backslash K_0$, we define *deg* $s = [K : K_0(s)]$. By a *residually tr. element of* K *of minimal deg* we mean a residually tr. element t of K such that $\deg t \leq \deg s$ for every residually tr. element s of K.

Now let t be a residually tr. element of K of minimal deg, and consider the extensions $(K_0, v_0) \subset (K_0(t), v_t) \subset (K, v)$, where $v_t = v|K_0(t)$. The assertion that t is residually tr. is equivalent to the assertion that v_t is the *inf extension of* v_0 *w.r.t.* t, i.e. to the assertion that for all $b_0, \ldots, b_n \in K_0$, $v_t(b_0 + b_1 t + \cdots + b_n t^n) = \inf\{v_0(b_i) | i = 0, \ldots, n\}$. The residue field for such a v_t is $k_0(t^*)$ and the value group remains G_0; cf. [**1**, p. 161, Proposition 2]. As for the further (finite algebraic) extension $(K_0(t), v_t) \subset (K, v)$, we have

0.1 THEOREM. *Let $(K_0, v_0) \subset (K_0(x), v)$ be a residually tr. extension, let t be a residually tr. element of $K_0(x)$ of minimal deg, and let $v_t = v|K_0(t)$. Then v is the unique extension of v_t to $K_0(x)$, up to dependence.*

(Recall that two nontrivial valuations of K are called dependent if they have a common valuation overring $< K$ (where $<$ indicates proper inclusion). In the rank-1 case, dependence coincides with equivalence.)

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Theorem 0.1 is a corollary to

0.2 THEOREM. *Assume the hypothesis of* 0.1. *Then*

$$[K_0(x) : K_0(t)] = [K_0(x)^\frown : K_0(t)^\frown],$$

where $^\frown$ *denotes completion.*

Consider now the following three integers ≥ 1, which depend only on the extension $(K_0, v_0) \subset (K, v)$:

E (the extension degree) $= \min\{[K : K_0(t)] | t$ is a residually tr. element of $K\}$;

R (the residue degree) $= [k_0' : k_0]$, where k_0' is the algebraic closure of k_0 in k;

I (the index) $= [G : G_0]$.

It is immediate that, for any residually tr. element t of $K_0(x)$ of minimal deg, E and I are equal, respectively, to the deg and index of the finite algebraic extension $(K_0(t), v_t) \subset (K_0(x), v)$; moreover, it follows from [**11**, p. 17, Theorem 3.3] that $R =$ the residue degree of this extension. Thus, we see that the degree, index, and residue degree of the extension v/v_t are independent of the choice of t, subject to the stipulation that t should be chosen residually tr. of minimal degree. Another number that may be associated to a finite algebraic extension of valued fields is the *defect*, which is defined by: defect = (the local degree)/(index)(residue degree). In terms of the extension v/v_t, where t is residually tr. of minimal deg, this means

$$\operatorname{def}(v/v_t) = [K_0(x)^\frown : K_0(t)^\frown]/IR = E/IR,$$

the latter equality by 0.2. Thus, we see that $\operatorname{def}(v/v_t)$ is also independent of the choice of t.

In the rank-1 case the defect of a finite algebraic extension is a classical concept and is known to have the properties needed to prove

0.3 THEOREM. *Let* $(K_0, v_0) \subset (K_0(x), v)$ *be a residually tr. extension, and assume* $\operatorname{rk} v_0$ $(= \operatorname{rk} v) = 1$. *Then* (i) $E = IR$ *if* v_0 *is discrete or* $\operatorname{char} k_0 = 0$; *and* (ii) $E = IRp^i$ *for some integer* $i \geq 0$ *if* $\operatorname{char} k_0 = p > 0$.

The discrete case of (i) is due to Mathieu [**5**, p. 88, Satz 4.1], and (i) was conjectured in [**10**] and proved there for $I = 1$ (the second author was unaware of Mathieu's thesis at the time). It should be noted that the proof given here of the general theorem is more direct than the proofs of these special cases. Moreover, an example is given in [**9**, p. 218, §7.2], in residue char 0, of a rank-2 discrete v_0 for which $E = 2$ and $IR = 1$, so the rank-1 hypothesis is needed in 0.3.

To get a feeling for the equality $E = IR$, note, for example, that the $I = R = 1$ case of 0.3(i) yields: Assume v_0 is rank 1 and either discrete or of residue char 0. Then v is the inf extension of v_0 w.r.t. some generator of K/K_0 if (and only if) $G_0 = G$ and k_0 is algebraically closed in k.

Some preliminary technical results are proved in §1; these are then applied in §2 to derive the above theorems. The final part of §2 is devoted to proving that in the rank-1 case E/IR equals the defect of the extension v/v_t for arbitrary residually tr. elements t of K, and not just for those of minimal degree.

Finally, §3 contains an existence theorem: Given a nontrivially-valued field (K_0, v_0), a totally ordered group extension $G_0 \subset G$ of finite index, and a finite algebraic field extension $k_0 \subset k_0'$, there exists a residually tr. extension v of v_0 to

$K_0(x)$ such that the value group of v is G, the algebraic closure of k_0 in the residue field of v is k_0', and $E = IR$.

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1. Preliminaries. We fix throughout §1 a field L and a valuation w of L having value group H.

1.1. Let A be a subset of L. We shall say elements $s_0, \ldots, s_n$ of L satisfy the bounded jump condition over A if the following holds:

(BJ/A) There exists $\gamma \geq 0$ in H such that for all $a_0, \ldots, a_n$ in A,

$$w(a_0 s_0 + \cdots + a_n s_n) \leq \inf\{w(a_i s_i) | i = 0, \ldots, n\} + \gamma.$$

We shall say that an element t in L satisfies the inf condition over A if the following holds:

$(\inf/A)$ For every integer $m \geq 0$ and all $a_0, \ldots, a_m \in A$,

$$w(a_0 + a_1 t + \cdots + a_m t^m) = \inf\{w(a_i) | i = 0, \ldots, m\}.$$

1.2 PROPOSITION. *Let L_0 be a subfield of L; let $s_0 = 1, s_1, \ldots, s_n$ be elements of L; and let t be an element of L which satisfies* $(\inf/(L_0 s_0 + \cdots + L_0 s_n))$. *If $s_0, \ldots, s_n$ satisfy (BJ/L_0), then they also satisfy $(BJ/L_0(t))$.*

PROOF. Let $\gamma \in H$ be given by (BJ/L_0); and let $a = a_0 s_0 + \cdots + a_n s_n$, $a_j \in L_0(t)$. It suffices to prove

$$w(a) \leq \inf\{w(a_j s_j) | j = 0, \ldots, n\} + \gamma. \tag{1.2.1}$$

We can write $a_j = (a_{0j} + a_{1j} t + \cdots + a_{mj} t^m)/d$, where $a_{ij} \in L_0$ and $d \in L_0[t]$. Since $w(d) =$ inf of values of the coefficients of d (by $(\inf/(L_0 s_0 + \cdots + L_0 s_n))$, using $s_0 = 1$), by dividing the numerators and denominator of the a_j by a d-coefficient of least value, we may assume $w(d) = 0$.

We have

$$\begin{aligned} da &= (a_{00} s_0 + a_{01} s_1 + \cdots + a_{0n} s_n) + (a_{10} s_0 + a_{11} s_1 + \cdots + a_{1n} s_n) t \\ &\quad + \cdots + (a_{m0} s_0 + a_{m1} s_1 + \cdots + a_{mn} s_n) t^m \\ &= b_0 + b_1 t + \cdots + b_m t^m, \end{aligned}$$

where $b_i = (a_{i0} s_0 + \cdots + a_{in} s_n)$. Then $w(a) = w(da) = \inf\{w(b_i) | i = 0, \ldots, m\}$, the second $=$ by $(\inf/(L_0 s_0 + \cdots + L_0 s_n))$. By the choice of γ, for all $i = 0, \ldots, m$,

$$w(b_i) \leq \inf\{w(a_{ij} s_j) | j = 0, \ldots, n\} + \gamma.$$

Therefore,

$$w(a) \leq \inf\{w(a_{ij} s_j) | j = 0, \ldots, n,\ i = 0, \ldots, m\} + \gamma. \tag{1.2.2}$$

But $w(a_j) = w(da_j) = \inf\{w(a_{ij}) | i = 0, \ldots, m\}$ (by $(\inf/(L_0 s_0 + \cdots + L_0 s_n))$, using $s_0 = 1$); so $w(a_j s_j) = \inf\{w(a_{ij} s_j) | i = 0, \ldots, m\}$. Therefore,

(1.2.3)

$$\inf\{w(a_j s_j) | j = 0, \ldots, n\} + \gamma = \inf\{w(a_{ij} s_j) | i = 0, \ldots, m,\ j = 0, \ldots, n\} + \gamma.$$

Putting together (1.2.3) and (1.2.2), we have (1.2.1).

1.3 *The valuation topology.* Recall [**1**, p. 117, §5.1] that the w-topology of the valued field (L,w) is defined by taking $\{W_\gamma | \gamma \in H\}$, where $W_\gamma = \{a \in L | w(a) > \gamma\}$, to be a fundamental system of neighborhoods of 0. (This differs from the w-topology defined in [**14**], but only in the case of the trivial valuation.) Let L_0 be a subfield of L, let $w_0 = w|L_0$, and let H_0 be the value group of L_0. We shall say that L_0 is *cofinal* in L if for every $\gamma \in H$, there exists $\gamma_0 \in H_0$ such that $\gamma_0 \geq \gamma$. This condition insures that the w_0-topology of L_0 coincides with the subspace topology of L_0 inherited from the w-topology of L; it is satisfied, for example, if H/H_0 is a torsion group. (In the applications of §2, the group G/G_0 will, in fact, be finite.)

1.4 *Completions.* Assume (L,w) is complete, and suppose L_0 is a cofinal subfield of L. Then the topological closure of L_0 in L is a completion of $(L_0, w|L_0)$ and will be denoted $\widehat{L_0}$. Recall (cf. [**1**, p. 121 or **14**, §§4 and 5]) that (i) the residue field and value group remain unaltered in passing to the completion, and (ii) if $L_0 \subset L_1 \subset L$ and L_1/L_0 is finite algebraic, then $\widehat{L_1} = \widehat{L_0}(L_1)$. It follows from (ii) that if $s_0, \ldots, s_n$ is a vector space basis of L_1/L_0, then $\widehat{L_1} = \widehat{L_0}s_0 + \cdots + \widehat{L_0}s_n$; hence $[\widehat{L_1} : \widehat{L_0}] \leq [L_1 : L_0]$.

1.5 PROPOSITION. *Assume (L,w) is complete, let L_0 be a cofinal subfield of L, and let $s_0, \ldots, s_n$ be nonzero elements of L. Then the following are equivalent:*
(i) *$s_0, \ldots, s_n$ satisfy (BJ/L_0).*
(ii) *$s_0, \ldots, s_n$ are linearly independent over $\widehat{L_0}$.*
(iii) *$s_0, \ldots, s_n$ satisfy $(BJ/\widehat{L_0})$.*

PROOF. (i)$\Rightarrow$(ii). Let $\gamma \geq 0$ in H be given by (i), and suppose there exist $\widehat{a_i} \in \widehat{L_0}$, not all zero, such that $\widehat{a_0}s_0 + \cdots + \widehat{a_n}s_n = 0$. For each $\widehat{a_i}$, we choose a corresponding $a_i \in L_0$ as follows: if $\widehat{a_i} = 0$, we let $a_i = 0$; and if $\widehat{a_i} \neq 0$, since L_0 is dense in $\widehat{L_0}$, we can choose $a_i \in L_0$ such that $w(a_i - \widehat{a_i}) > w(\widehat{a_i}) + \gamma$. Note that this forces the equality $w(a_i) = w(\widehat{a_i})$. Then

$$\begin{aligned} w(a_0 s_0 + \cdots + a_n s_n) &= w((a_0 - \widehat{a_0})s_0 + \cdots + (a_n - \widehat{a_n})s_n) \\ &\geq \inf\{w((a_i - \widehat{a_i})s_i) \,|\, i = 0, \ldots, n\} \\ &> \inf\{w(\widehat{a_i}s_i) + \gamma \,|\, i = 0, \ldots, n\} \\ &= \inf\{w(a_i s_i) \,|\, i = 0, \ldots, n\} + \gamma, \end{aligned}$$

which contradicts (BJ/L_0).

(ii)$\Rightarrow$(iii). This argument is classical; cf. [**14**, pp. 46–47 or **1**, p. 120, Proposition 4].

(iii)$\Rightarrow$(i). Trivial.

1.6 COROLLARY. *Assume (L,w) is complete, let L_0 be a cofinal subfield of L, let $s_0 = 1, s_1, \ldots, s_n$ be elements of L, and let t be an element of L which satisfies $(\inf/(L_0 s_0 + \cdots + L_0 s_n))$. If $s_0, \ldots, s_n$ are linearly independent over $\widehat{L_0}$, then they are also linearly independent over $L_0(t)\widehat{\ }$.*

PROOF. Apply 1.2 and 1.5.

1.7 *Dependent valuations* (*cf.* [**1**, *p.* 134, §7.2 *or* **14**, §II]). Recall that two nontrivial valuations w_1, w_2 of a field L are said to be *equivalent* if they have the same valuation ring and *dependent* if their valuation rings have a common valuation overring $< L$. As for the trivial valuation, we shall assume that the only valuation equivalent to it or dependent on it is itself. Equivalence and dependence are both equivalence relations on the set of valuations of L, and dependence = equivalence on the subset of rank-1 valuations. Moreover, the valuations w_1 and w_2 are dependent iff they define the same topology on L.

1.8 *Local deg* (*cf.* [**14**, *p.* 49, §8, **1**, *p.* 140, *Proposition* 2]). Let $(L_0, w_0) \subset (L, w)$ be a finite algebraic extension. The integer $[(L,w)\hat{}: (L_0, w_0)\hat{}]$ is called the *local degree* of $(L,w)/(L_0,w_0)$. If $w_1, \ldots, w_n$ is a complete set of representatives for the dependency classes of the set of extensions of w_0 to L, and if $E_i\hat{} = [(L,w_i)\hat{}: (L_0,w_0)\hat{}]$, then $[L : L_0] \geq E_1\hat{} + \cdots + E_n\hat{}$ (more precisely, $[L : L_0] = E_1\hat{} q_1 + \cdots + E_n\hat{} q_n$, where $q_i = [L : L_0]_{\text{insep}}/[(L,w_i)\hat{}: (L_0,w_0)\hat{}]_{\text{insep}}$). In particular, if $[L : L_0] = E_i\hat{}$ for some i, then $n = 1$ and all the extensions of w_0 to L are dependent.

1.9 *The defect of a finite algebraic extension.* Let $(L_0, w_0) \subset (L, w)$ be a finite algebraic extension of valued fields having value groups $H_0 \subset H$ and residue fields $l_0 \subset l$. We shall call the rational number $[(L,w)\hat{} : (L_0,w_0)\hat{}]/[H : H_0][l : l_0]$ the defect of the extension $(L_0, w_0) \subset (L, w)$ (written def( )).

If $\operatorname{rk} w_0$ $(= \operatorname{rk} w) = 1$, this is a classical concept and is known to have the following properties (cf. [**12**, p. 355 and **1**, p. 148, Corollary 2]).

(i) If either w_0 is discrete or $\operatorname{char} l_0 = 0$, then $\operatorname{def}(L/L_0) = 1$; while if $\operatorname{char} l_0 = p > 0$, then $\operatorname{def}(L/L_0) = p^i$ for some $i \geq 0$.

(ii) $\operatorname{def}(L/L_0)$ is multiplicative, i.e. if $L_0 \subset L_1 \subset L$ are finite algebraic extensions of rank-1 valued fields, then $\operatorname{def}(L/L_0) = \operatorname{def}(L/L_1)\operatorname{def}(L_1/L_0)$. (This follows from the fact that the other expressions in the definition of defect are multiplicative.)

Note that, while (ii) clearly does not involve the rank-1 assumption, (i) is false for valuations of arbitrary rank. A more useful concept in the general case may be that of henselian defect, which is defined as above using the henselization in place of the completion; cf. [**8**].

2. Applications to residually tr. extensions. We now return to the notation of the introduction. Thus, $(K_0, v_0) \subset (K_0(x), v)$ is a residually tr. extension having value groups $G_0 \subset G$ and residue fields $k_0 \subset k$.

For any element $s \in K_0(x)\backslash K_0$, we have defined $\deg s$ to be $[K_0(x) : K_0(s)]$. In the proof of the next theorem we need the following alternative characterization of $\deg s$ (cf. [**15**, p. 197, Theorem]): if $s = f(x)/g(x)$, where $f(x)$ and $g(x)$ are relatively prime elements of $K_0[x]$, then $\deg s = \max\{\deg_x f(x), \deg_x g(x)\}$.

2.1 THEOREM. *Let $(K_0, v_0) \subset (K_0(x), v)$ be a residually tr. extension, and let t be a residually tr. element of $K_0(x)$ of minimal degree* $(= E)$. *Then*

$$[K_0(x) : K_0(t)] = [K_0(x)\hat{} : K_0(t)\hat{}].$$

PROOF. By definition of E, $[K_0(x) : K_0(t)] = E = \deg x$ over $K_0(t)$; so $B = \{1, x, \ldots, x^{E-1}\}$ is a vector space basis of $K_0(x)/K_0(t)$. Since x is tr. over $K_0\hat{}$ (because otherwise k/k_0 would be algebraic), B is linearly independent over $K_0\hat{}$. The lemma below shows B satisfies the hypothesis of 1.6 (by taking $L_0 = K_0$ and $L = K_0(x)\hat{}$ in 1.6), so, by 1.6, B is linearly independent over $K_0(t)\hat{}$. Since $K_0(x)\hat{} = K_0(t)\hat{} + K_0(t)\hat{}x + \cdots + K_0(t)\hat{}x^{E-1}$, we are done.

LEMMA. *Under the hypothesis of Theorem* 2.1, *t satisfies the condition* $(\inf/(K_0 + K_0x + \cdots + K_0x^{E-1}))$.

PROOF. We must show: if $b_0, \ldots, b_n$ are in $K_0 + K_0x + \cdots + K_0x^{E-1}$, then $v(b_0 + b_1t + \cdots + b_nt^n) = \inf\{v(b_i) | i = 0, \ldots, n\}$. This is immediate if all the b_i are

0, so we may assume some one of them is $\neq 0$. Let $s = b_0 + b_1 t + \cdots + b_n t^n$, and let b_j be an element of minimal value from $b_0, \ldots, b_n$. We must then show $v(s/b_j) = 0$, or equivalently, $(s/b_j)^* \neq 0$. But

$$(s/b_j)^* = (b_0/b_j)^* + (b_1/b_j)^* t^* + \cdots + (b_n/b_j)^* t^{*n},$$

and the b_i/b_j $(i = 0, \ldots, n)$ are all of degree $< E$; so, by definition of E, the $(b_i/b_j)^*$ are algebraic over k_0. Since t^* is tr. over k_0, we conclude $(s/b_j)^* \neq 0$.

2.2 COROLLARY. *Assume the hypothesis of* 2.1, *and let* $v_t = v|K_0(t)$. *Then* v *is the unique extension, up to dependence, of* v_t *to* $K_0(x)$, *i.e. any two extensions of* v_t *to* $K_0(x)$ *are dependent.*

PROOF. Apply 1.8 and 2.1.

2.3 REMARK. It should be emphasized that in the rank-1 case dependence in 2.2 is the same as equivalence. Moreover, if v_0 is complete rank 1, the uniqueness property of 2.2 generalizes to 1-dim function fields (cf. [**7**, p. 197, Theorem 3]); and if v_0 is not complete, this result remains true if the function field satisfies an additional property which is trivial to verify in the case of a simple tr. extension (cf. [**13**]).

2.4 *The defect of a residually tr. extension.* Let $(K_0, v_0) \subset (K_0(x), v)$ be a residually tr. extension having extension $\deg E$, index I, and residue $\deg R$, as defined in the introduction. (Recall that this means: $E = \deg t$, where t is any residually tr. element of $K_0(x)$ of minimal deg; $I = [G : G_0]$; and $R = [k_0' : k_0]$, where k_0' is the algebraic closure of k_0 in k.)

We shall now define the defect of $K_0(x)/K_0$ to be the rational number E/IR: $\operatorname{def}(K_0(x)/K_0) = E/IR$.

2.4.1 COROLLARY. *If* t *is any residually tr. element of* $K_0(x)$ *of minimal degree, then* $\operatorname{def}(K_0(x)/K_0) = \operatorname{def}(K_0(x)/K_0(t))$.

PROOF. By definition (cf. 1.9), $\operatorname{def}(K_0(x)/K_0(t)) = [K_0(x)\hat{} : K_0(t)\hat{}]/I_t R_t$, where I_t is the index and R_t is the residue degree of $K_0(x)/K_0(t)$. By 2.1,

$$[K_0(x)\hat{} : K_0(t)\hat{}] = [K_0(x) : K_0(t)] = E.$$

As for I_t and R_t, since t is residually tr., the value group of $K_0(t)$ is G_0 and the residue field is $k_0(t^*)$ (cf. [**1**, p. 161, Proposition 2]); so $I_t = [G : G_0] = I$, and $R_t = [k : k_0(t^*)]$, which, by [**11**, p. 17, Theorem 3.3], $= [k_0' : k_0] = R$. Q.E.D.

Note that we have actually proved

2.4.2 COROLLARY. *If* t *is any residually tr. element of* $K_0(x)$ *of minimal degree, then the extension degree, index, residue degree, and defect of the* (*simple tr.*) *extension* $K_0(x)/K_0$ *are, respectively, equal to the degree, index, residue degree, and defect of the* (*simple algebraic*) *extension* $K_0(x)/K_0(t)$ (*and therefore these latter quantities are independent of the choice of* t).

By applying the remarks of 1.9 to the equality $E = IR \operatorname{def}(K_0(x)/K_0(t))$ given by 2.4.1, we have

2.4.3 COROLLARY. *Assume* $\operatorname{rk} v_0$ $(= \operatorname{rk} v) = 1$. *If* v_0 *is discrete or* $\operatorname{char} k_0 = 0$, *then* $E = IR$; *while if* $\operatorname{char} k_0 = p > 0$, *then* $E = IRp^i$ *for some integer* $i \geq 0$.

As noted in the introduction, 2.4.3 is false without the rank-1 hypothesis; moreover, special cases of the corollary appear in [**5**, **9**, and **10**], and it affirms some conjectures of [**10**]. (It was also conjectured in [**10**], and proved in the case $I = 1$, that if v_0 is henselian of arbitrary rank and char $k_0 = 0$, then $E = IR$; this conjecture has now been proved and will appear in [**8**].)

The next theorem asserts that in the rank-1 case 2.4.1 holds for arbitrary residually tr. elements t of $K_0(x)$, and not just for those of minimal deg. The rank-1 hypothesis cannot be omitted, as we shall see in Example 2.6.

2.5 THEOREM. *Let $(K_0, v_0) \subset (K_0(x), v)$ be a residually tr. extension, and assume* $\operatorname{rk} v_0 = 1$. *If t is any residually tr. element of $K_0(x)$, then* $\operatorname{def}(K_0(x)/K_0) = \operatorname{def}(K_0(x)/K_0(t))$.

The proof requires two lemmas, the first of these being a special case of the theorem.

2.5.1 LEMMA. *The theorem is true if there exists a generator of $K_0(x)/K_0$ which is residually tr.; in fact, then both defects are* 1.

PROOF. We might as well assume x is residually tr.; then the value group G of $K_0(x)$ is G_0 and the residue field k is $k_0(x^*)$. Thus, $E = I = R = 1$ and $\operatorname{def}(K_0(x)/K_0) = E/IR = 1$. Since $G = G_0$, the index of $K_0(x)/K_0(t)$ is also 1, and therefore, by definition, $\operatorname{def}(K_0(x)/K_0(t)) = [K_0(x)\hat{}: K_0(t)\hat{}]/[k_0(x^*) : k_0(t^*)]$; so the lemma is equivalent to the assertion: $[K_0(x)\hat{}: K_0(t)\hat{}] = [k_0(x^*) : k_0(t^*)]$. By the remarks of 1.4, $K_0(x)\hat{} = K_0(t)\hat{}(x)$; so it suffices to prove: $\deg x$ over $K_0(t)\hat{} = \deg x^*$ over $k_0(t^*)$.

Write $t = g(x)/h(x)$, where $g(x), h(x)$ are relatively prime elements of $K_0[x]$. By dividing the coefficients of $g(x)$ and $h(x)$ by an element of least value from among all of these coefficients, we may assume the coefficients of $g(x)$ and $h(x)$ have value ≥ 0. Then

$$g^*(X) - t^*h^*(X) = a_1(X)b_1(X, t^*) \tag{2.5.2}$$

in $k_0[t^*, X]$, where $a_1(X) = \gcd\{g^*(X), h^*(X)\}$ in $k_0[X]$.

Note that $b_1(X, t^*)$ is the irreducible polynomial for x^* over $k_0(t^*)$ (cf. [**15**, p. 197]), and $a_1(X)$ and $b_1(X, t^*)$ are relatively prime in $k_0(t^*)[X]$. By Hensel's lemma (which requires the rank-1 hypothesis; cf. [**3**, p. 120]), the factorization (2.5.2) in $k_0(t^*)[X]$ lifts to a corresponding factorization in $K_0(t)\hat{}[X]$: $g(X) - th(X) = a(X)b(X)$, where $a^*(X) = a_1(X)$, $b^*(X) = b_1(X)$ and $\deg b(X) = \deg b_1(X)$. Then $0 = a(x)b(x)$; and since $a^*(x^*) = a_1(x^*) \neq 0$, we must have $b(x) = 0$. Therefore $\deg x$ over $K_0(t)\hat{} \leq \deg x^*$ over $k_0(t^*)$, so = holds.

2.5.3 LEMMA. *Let $(K_0, v_0) \subset (K_0(x), v)$ be a residually tr. extension, let K_1 be a finite algebraic extension of K_0, and assume v has been extended (arbitrarily) to a valuation of $K_1(x)$. If t is any residually tr. element of $K_0(x)$, then*

$$\operatorname{def}(K_1(t)/K_0(t)) = \operatorname{def}(K_1/K_0).$$

PROOF. Since t is residually tr., the index and residue degree of $K_1(t)/K_0(t)$ are, respectively, equal to the index and residue degree of K_1/K_0. Thus, the conclusion of the lemma is equivalent to $[K_1(t)\hat{}: K_0(t)\hat{}] = [\hat{K_1} : \hat{K_0}]$. (These completions may be assumed to lie inside a fixed completion of $K_1(x)$.)

Let $s_0 = 1, s_1, \ldots, s_n \in \hat{K_1}$ be a vector space basis of $\hat{K_1}/\hat{K_0}$. Since t is residually tr. over k_0, it is also residually tr. over the residue field of $K_0(s_0, \ldots, s_n)$;

and therefore t satisfies the condition $(\inf /(K_0 s_0 + \cdots + K_0 s_n))$ of 1.1. It follows that the hypotheses of 1.6 are satisfied (with $L = K_1(x)^\frown$ and $L_0 = K_0$), so by 1.6 we conclude that $s_0, \ldots, s_n$ are linearly independent over $K_0(t)^\frown$. Since $K_1(t)^\frown = K_0(t)^\frown s_0 + \cdots + K_0(t)^\frown s_n$, we are done.

PROOF OF 2.5. Let K_0^a be an algebraic closure of K_0, and extend v to $K_0^a(x)$. Since $K_0(x)/K_0$ is residually tr., there exist $a, b \in K_0^a$ such that $(x-a)/b$ is residually tr. (over k_0). (This is seen as follows: Choose t to be any residually tr. element of $K_0(x)$, and write $t = f/g$, where $f, g \in K_0[x]$. Since the value group of K_0^a is divisible, there exists $c \in K_0^a$ such that $v(f) = v(g) = v(c)$. Then $(f/c)^*/(g/c)^* = t^*$ implies $(f/c)^*$, say, is tr. over k_0. Now factor f/c in $K_0^a[x]$. After dividing the linear factors by appropriate elements of K_0^a again, one of these factors is residually tr.)

Let $K_1 = K_0(a, b)$, and note that $K_1(x)/K_1$ has a residually tr. generator (namely $(x-a)/b$):

$$\begin{array}{ccc} K_1(t) & — & K_1(x) \\ | & & | \\ K_0(t) & — & K_0(x) \end{array}$$

By the multiplicative property of def( ) (cf. 1.9), we have
(2.5.4)

$$\operatorname{def}(K_1(x)/K_1(t))\operatorname{def}(K_1(t)/K_0(t)) = \operatorname{def}(K_1(x)/K_0(x))\operatorname{def}(K_0(x)/K_0(t)).$$

By 2.5.1, $\operatorname{def}(K_1(x)/K_1(t)) = 1$, and, by 2.5.3, $\operatorname{def}(K_1(t)/K_0(t)) = \operatorname{def}(K_1/K_0)$. Substituting in (2.5.4), we obtain

$$\operatorname{def}(K_1/K_0) = \operatorname{def}(K_1(x)/K_0(x))\operatorname{def}(K_0(x)/K_0(t)),$$

which shows $\operatorname{def}(K_0(x)/K_0(t))$ is independent of the choice of t. In particular, $\operatorname{def}(K_0(x)/K_0(t)) =$ the defect given in 2.4.1. Q.E.D.

2.6. An example to show 2.5.1 (and a fortiori 2.5) is false without the rank-1 hypothesis. Let y and z be indeterminates over a field k_0, let $K_0 = k_0(y, z)$, and consider the places $p_{w_0} \colon K_0 \to k_0(y)$ and $p_{u_0} \colon k_0(y) \to k_0$ whose valuation rings are $W_0 = k_0(y)[z]_{(z)}$ and $U_0 = k_0[y]_{(y)}$. Let $p_{v_0} = p_{u_0} \circ p_{w_0}$ be the composite place $K_0 \to k_0$, and let V_0 be the associated discrete rank-2 valuation ring of K_0. Now extend u_0, v_0, w_0 via infs w.r.t. x to valuations u, v, w of their respective fields with x adjoined. Let * denote image under the residue map for w and ** image under the residue map for v. Since the valuation ring of v is contained in the valuation ring of w, the topologies defined by v and w on $K_0(x)$ coincide (cf. 1.7); let $^\frown$ denote completion w.r.t. this topology.

Let, say, $t = yx^2 + x$. By considering the rank-1 extension w/w_0, we conclude $[K_0(x)^\frown : K_0(t)^\frown] = [k_0(y)(x^*) : k_0(y)(t^*)] = 2$, the latter equality since $t^* = yx^{*2} + x^*$. But $t^{**} = x^{**}$, so $[k_0(x^{**}) : k_0(t^{**})] = 1$. Thus, as noted at the start of the proof of 2.5.1, $\operatorname{def}(K_0(x)/K_0(t)) = 2$ and $\operatorname{def}(K_0(x)/K_0) = 1$.

2.7 *A defect for function fields.* In [**6**, **7**] the defect has been defined for rank-1 valued function fields which residually conserve dim. Thus, let $(L_0, w_0) \subset (L, w)$ be an arbitrary finitely generated extension of rank-1 valued fields with value groups $H_0 \subset H$ and residue fields $l_0 \subset l$, and assume tr. deg of $L/L_0 =$ tr. deg of l/l_0. If $t_1, \ldots, t_n$ is any set of elements of L of value 0 such that $t_1^*, \ldots, t_n^*$ is a tr. basis of l/l_0, then $\operatorname{def}(L/L_0(t_1, \ldots, t_n))$ can be seen to be independent of the choice of $t_1, \ldots, t_n$ and may therefore be defined to be the defect of L/L_0. One can give a proof related to that of 2.5, but a stronger form of 2.5.1, one which uses a theorem of Grauert and Remmert [**4**, p. 119], is needed. The result is a consequence of the following (cf. [**6**, §1.4.2, Corollary 1, **7**, p. 190, §I-3]):

$\operatorname{def}(L/L_0(t_1,\ldots,t_n)) = \sup_S\{\dim_{L_0^\frown} S/\sum_{j\in J}\dim_{l_0} S_j^*\}$, where the sup is taken over all $S \neq 0$ contained in $L^\frown$ (or $\subset L$ if L_0 is complete) which are finite dim vector spaces over $\widehat{L_0}$, J is a set of representatives in H for (the finite group) H/H_0, and S_j^* is the l_0-vector space $\{s \in S|w(s) \geq j\}/\{s \in S|w(s) > j\}$.

3. An existence theorem.

3.1 THEOREM. *Let (K_0, v_0) be a nontrivially valued field having value group G_0 and residue field k_0, let $G_0 \subset G_1$ be an inclusion of totally ordered groups such that $[G_1 : G_0]$ is finite, and let k_1 be a finite algebraic extension of k_0. There exists $t \in K_0[x]$ of degree $[G_1 : G_0][k_1 : k_0]$ with the property that if v is any extension of v_0 to $K_0(x)$ such that $v(t) = 0$, then the value group of v contains G_1 and the residue field of v contains k_1.*

(To be precise, the value group of v contains a G_0-order-isomorphic copy of G_1 and the residue field of v contains a k_0-isomorphic copy of k_1.)

3.2 COROLLARY. *There exists a residually tr. extension v of v_0 to $K_0(x)$ such that the value group of v is G_1, the algebraic closure of k_0 in the residue field of v is k_1, and $E = IR$.*

PROOF OF 3.2. Choose t by 3.1. First extend v_0 to $K_0(t)$ by assigning value 0 to t and taking infs, and then further extend arbitrarily from $K_0(t)$ to a valuation v of $K_0(x)$. Since $v(t) = 0$, by 3.1, $G_1 \subset$ value group of v and $k_1 \subset$ residue field of v. Then $\deg_x t \geq E \geq IR \geq [G_1 : G_0][k_1 : k_0]$. But by 3.1 the first and last terms of this chain are equal. Q.E.D.

The proof of 3.1 involves putting together two special cases of the theorem.

Case (i). $G_1 = G_0$. This case is disposed of by applying [**2**, p. 90, Lemma 22.4]. (*Note.* The proof of the corresponding result in [**3**, p. 206] is quite different and does not apply to the present situation. Perhaps this explains why the second author previously overlooked [**2**] in writing p. 595 of [**10**], which also contains a proof of Case (i).)

Case (ii). $k_0 = k_1$. This case is disposed of by the

LEMMA. *There exists $t \in K_0[x]$ of degree $[G_1 : G_0]$ such that if v is any extension of v_0 to $K_0(x)$ such that $v(t) = 0$, then the value group of v contains G_1 and $v(x) > 0$.*

PROOF. Since G_1/G_0 is finite, hence a direct sum of cyclics, there exist $g_1, \ldots, g_m \in G_1$, all > 0, and integers $n_1, \ldots, n_m \geq 1$ such that $G_1 = G_0 + Zg_1 + \cdots + Zg_m$; $n_1g_1, \ldots, n_mg_m$ are in G_0; and $n_1 \cdots n_m = [G_1 : G_0]$. Choose $b_1, \ldots, b_m \in K_0$ such that $n_ig_i = v_0(b_i)$ $(i = 1, \ldots, m)$. Note that $v_0(b_i) > 0$ since $g_i > 0$.

Now let $t_1 = x^{n_1}/b_1, t_2 = (t_1 - 1)^{n_2}/b_2, \ldots, t_m = (t_{m-1} - 1)^{n_m}/b_m$. By induction on m we see $t_m \in K_0[x]$ and $\deg t_m = n_1 \cdots n_m$. Note that, for $i = 2, \ldots, m$, $v(t_i) = 0$ implies $n_iv(t_{i-1} - 1) = v(b_i) > 0$, which implies $v(t_{i-1}) = 0$; so if $v(t_m) = 0$, then $v(t_{m-1}) = \cdots = v(t_1) = 0$. Thus, we can let $t = t_m$. Then $v(t) = 0$ implies $v(t_{i-1} - 1) = g_i$ $(i = 2, \ldots, m)$. Also, $v(t_1) = 0$ implies $n_1v(x) = v(b_1)$, hence $v(x) = g_1 > 0$. Q.E.D.

Now we can complete the

PROOF OF 3.1. Choose $t_1 \in K_0[x]$ by Case (i). Let $y = t_1 - 1$, and apply the lemma to $K_0(y)$ to obtain $t_2 \in K_0[y]$. We claim $t = t_2$ has the required properties. First,

$$\begin{aligned}\deg_x t &= [K_0(x) : K_0(t)] = [K_0(x) : K_0(t_1)][K_0(t_1) : K_0(t_2)] \\ &= (\deg_x t_1)(\deg_y t_2) = [k_1 : k_0][G_1 : G_0].\end{aligned}$$

Next, let v be any extension of v_0 to $K_0(x)$ such that $v(t_2) = 0$. By the lemma, we have $v(y) > 0$, which implies $v(t_1) = 0$. By Case (i) $k_1 \subset$ residue field of v, and by Case (ii) $G_1 \subset$ value group of v.

3.3 REMARKS. The existence proof of §3 differs from the proof of the corresponding existence theorem for algebraic extensions in [**2**, pp. 84–97, §22] in only a few details. In fact, it appears that if Endler's Theorem 22.6 were formulated in the generality of his Lemma 22.4, then it would include our 3.1. To carry this a bit further, the proof of 3.1 actually yields the following additional result: Let $t = t(x)$ be the polynomial of 3.1 and let a be an element of an extension field of K_0. If v is any extension of v_0 to $K_0(a)$ such that $v(t(a)) = 0$, then the value group of v contains G_1 and the residue field of v contains k_1. For example, let a be a root of the polynomial $t(x) - 1$. Then $[G_1 : G_0][k_1 : k_0] = \deg t(x) \geq [K_0(a) : K_0] \geq [G_1 : G_0][k_1 : k_0]$, so $G_1 =$ value group of $K_0(a), k_1 =$ residue field of $K_0(a)$, and $t(x) - 1$ is irreducible over K_0.

It is also possible to specify in 3.1 that $t(x)$ should have a term of the form cx with $0 \neq c \in K_0$. (This condition will insure that the algebraic extension $K_0(a)/K_0$ in the above example is separable and that $K_0(x)/K_0(t)$ is separable in 3.2.) To see this, one should first note that the definition of t shows that its leading coefficient has minimum value (< 0) among its coefficients; this forces $v(x)$ to be ≥ 0 whenever $v(t) \geq 0$. Therefore, if c is any nonzero element of K_0 of value > 0 and $t' = t - cx$, then $v(t') = 0$ iff $v(t) = 0$; so t may be replaced by t' in 3.1.

Finally, note that the extension v/v_0 constructed in 3.2 is rather special in that there is a polynomial t in $K_0[x]$ (and not just a rational function in $K_0(x)$) which is residually tr. of minimal degree.

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