Let k be a field and C/k a proper, smooth, geometrically connected curve and $f : C \to \mathbb{P}^1_k$ a separable morphism; we denote by br (resp. ram) the branch (resp. ramification) divisor.

Let R be a complete (or henselian) DVR with residue field k.

Proposition 0.1. *We assume that the cardinality of k is at least as big as $\deg f$. Let $\mathbb{P}^1_R$ be a smooth lifting for $\mathbb{P}^1_k$; fix* Br $\in \mathbb{P}^1_R$ *a lifting divisor for* br*; then there is an R-curve C_R/R and a morphism $F : C_R \to \mathbb{P}^1_R$ which lifts f and whose branch locus is* Br*. In fact we will see along the proof that it is aswell possible to fix in a natural sense a lifting for* ram.

The proof is an adaptation to the case of wild ramification of Fulton's paper Ann math 1969. Note that the interest of Fulton's paper is the lifting of tame Galois action; this is clearly not always possible if we admit wild ramification!

In fact Proposition 4.1 in Fulton generalizes to the wild context.

Lemma 0.2. *Let A be a discrete valuation ring with maximal ideal $m = (t)$, resdidue field k, and quotient field K. Let L be a finite separable field extension of K of degree n, and B be the integral closure of A in L. Assume the cardinality of k is at least n.*

Let $m_1, ..., m_s$ be the maximal ideals of B, $k_i = B/m_i$, $f_i = [k_i : k]$, and $mB = \prod m_i^{e(i)}$. Then there is a $z \in B$ and there are monic polynomials $F_i \in A[Z]$ of degree f_i, $i = 1, ..., s$ such that

(i) $B = A[z]$

(ii) $m_i = (F_i(z))$

(iii) Each residue $\bar{F}_i \in k[Z]$ is the minimal polynomial of the image of z in k_i over k and $\bar{F}_1, ..., \bar{F}_s$ are distinct.

Proof. We follow Fulton's proof for ii) and iii) ... In order to be complete we repeat the proof. There are $z_i \in k_i$ such that $k_i = k(z_i)$, and such that the minimal polynomials $\bar{F}_i$ of z_i are distinct for $i = 1, ..., s$. This uses the assumption $\operatorname{card}(k) \geq n$.

Choose $z_0 \in B$ which has image $z_i \in k_i$ for $i = 1, ..., s$, and choose $F_i \in A[X]$ monic of degree f_i such that F_i reduces to $\bar{F}_i$ mod m. Let v_i be the m_i-adic valuation. Choose $h \in B$ such that $v_i(h) = 1$ if $v_i(F_i(z_0)) > 1$ and $v_i(h) > 1$ if $v_i(F_i(z_0)) = 1$. Let $z = z_0 + h$ then (ii) and (iii) are done.

Now we show that z is such that $B = A[z]$. Note that $L = K[z]$. We denote by $\hat{A}$ (resp. $\hat{K}$) the completion for the t adic valuation. Let (Ω, v) be an algebraic closure of $\hat{K}$ where v is a prolongation of the t-adic valuation of K. Let $\hat{L}_i$ the completion of L for the m_i-adic valuation; then $\hat{L}_i$ endowed with the m_i adic valuation v_i embeds isometrically in (Ω, v) by sending z to a convenient root $\hat{z}_i$ of $F(X)$ in Ω; namely we write $F(X) = \prod_{1 \leq i \leq s} \varphi_i(X)$ the irreducible decomposition over $\hat{A}$ (here $\varphi_i(Z) \in \hat{A}[X]$, is monic and irreducible); as $n = \sum e(i) f_i$ (see Fulton) a look at $F(X)$ mod t shows that $\deg \varphi_i(X) \geq e(i) f_i$ and $\hat{z}_i$ is a root of $\varphi_i(X)$. From ii) we know that $F_i(\hat{z}_i)$ is a uniformizing parameter of v_i in $\hat{L}_i$ and as $\deg F_i(X) = f_i$; it follows from ([Serre], Corps locaux, lemme 4 p. 66) that the valuation ring of $\hat{L}_i$ is $\hat{A}[z_i]$. The equality $B = A[z]$, then follows from the faithfullflatness of $\hat{A}$ over A or a comparaison of $v_i(F'(z))$ with the local different $v_i(\varphi'(z_i))$. ☐

Proof for the Proposition .

A) Analytic lifting of local morphisms.

Let $K := \operatorname{Fr} R$; we denote by $r : \mathbb{P}^1_K \to \mathbb{P}^1_k$; the analytic reduction induced by the model $\mathbb{P}^1_R$. We denote by $\mathcal{O}^{an}$ the sheaf of analytic functions on $\mathbb{P}^1_K$. Let $x \in \mathbb{P}^1_k$ and U_x an affine open containing x; let A (resp. B) be the ring of regular functions over U_x (resp. $f^{-1}(U_x)$). After a suitable localisation one can assume that $B = A[z]$ (this is the lemma above). Let $p(X) \in A[X]$ the irreducible polynomial of z. Let $T \in k(C)$ with poles outside U_x; then we can write $\mathcal{A} := \mathcal{O}^{an}(r^{-1}(U_x)) = K < T, 1/Q(T) >$ (affinoid algebra). Let $P(X) \in \mathcal{A}[X]$ a monic polynomial which lifts $p(X)$. Let $\Omega \supset K(T)$ an algebraically closed field which is complete relatively to the T-Gauss valuation (we mean $v(\sum a_i T^i) = \inf_i(v(a_i))$, and let $Z \in \Omega$ a root of $P(X)$ which lifts z. Let $\mathcal{B}$ the affinoid algebra $\mathcal{A}[Z]$; the inclusion $\mathcal{A} \subset \mathcal{B}$ provides us with an analytic lifting of the morphism $f_{|f^{-1}(U_x)}$.

B) Unicity (up to isomorphism) of the lifting in the étale case.

C) Conclusion. Let $\mathcal{U}$ be a covering of $\mathbb{P}^1_k$ with affine opens U_x such that $f_{|f^{-1}(U_x)}$ is monogenous; moreover one can assume that it is ramified in at most one fiber. We can produce using A) an analytic presheaf of $\mathcal{O}^{an}_{\mathbb{P}^1_K}$-algebras which by B) is a coherent $\mathcal{O}^{an}_{\mathbb{P}^1_K}$-algebra and so induces a finite analytic cover $\operatorname{Spec}\mathcal{B} \to {\mathbb{P}^1_K}^{an}$ which is a lifting of f. We conclude with rigid GaGa.

D) Ramification conditions (see also K. Kato, Vanishing cycles, ramification of valuations,.... Duke Math. J. Vol. 55). The branch locus of the morphism is locally given by the discriminants of the polynomials $P(X)$ choosen in A). Say for example that the morphism f is totally ramified at $t = 0$; then $p(X)$ is an Eisenstein polynomial whose z is a root; say $p(X) = X^n + a_{n-1}(t)X^{n-1} + ... + a_0(t)$ where $t|a_i(t)$ and say $a_0(t) = t$. Then we can view $t = \varphi(z) \in k[[z]]$. Let $P(X) := X^n + A_{n-1}(T)X^{n-1} + ... + A_0(T)$ a lifting with coefficients in $R[T]$ and choose $A_0(T) = T$, then if Z is like in A), Z is now a parameter for the open disc over the disc $|T(x)| < 10$ and $T = \Phi(Z) \in R[[Z]]$. Then the ramification locus in the disc $|Z(?)| < 1$ is given by the roots of $\Phi'(Z)$; in general the ramification locus is supported by many points (say $Z \in \{z_1, ..., z_r\}$ with ramification indices $e_1, ..., e_r$. In the inequal characteristic case one get the condition $\sum_{1\leq i\leq r}(e_i - 1) = v_z(p'(z))$ i.e. the contribution of the differents in the formal fiber above $z = 0, t = 0$ is the same as the different at the special fiber at $t = 0$. It is a difficult question to "classify" the possible p-adic geometry for the ramification (or branch) locus. Nevertheless it is easy to impose the lifting morphism to be totally ramified above one point (i.e. we impose for example $e_1 = n$). If the ramification at the special fiber is tame then this point will be the only ramification point in the formal fiber above $z = 0, t = 0$; but if there is wild ramification there will be necessarily other points ($r > 1$) and in fact more than one branch point!!! In order to get this we take for $P(X)$ an Eisenstein polynomial at $T = 0$; i.e $T|A_i(T)$ and $A_0(T) = T$. It is interesting in this case to have a look at the formal power series $t = \varphi(z)$ and $T = \Phi(Z)$. Assume $p|n$ and $t = \varphi(z) = z^n(1 + c_m z^m + ...)$ where $(p, m) = 1$ and $c_m \neq 0$; then $v_z(\varphi'(z)) = n - 1 + m$ and say $T = \Phi(Z) = Z^n(1 + C_1 Z + ... + C_m Z^m + ...$ where $v(C_i) > 0$ for $i < m$ and $v(C_m) = 0$. Then $\Phi'(Z) = nZ^{n-1} + (n + 1)C_1 Z^n + ... + (n + m)C_m Z^{n-1+m} + ...$ which following Weierstrass preparation factorises in $Z^{n-1}Q(Z)(1+...)$ where $Z^{n-1}Q(Z)$ is a distinguished polynomial whose zeroes have special tree like geometry....

E) Separability condition. The same statement would work if one replace $\mathbb{P}^1_R$ by a smooth R curve. The fact that the morphism is to $\mathbb{P}^1_k$ is usefull in order to eliminate the separability condition. In the proposition assume that k is also perfect then one can decompose the morphism f in a separable one and a radiciel one between $\mathbb{P}^1_k$ and $\mathbb{P}^1_k$ and this last morphism is easily lifted.

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