On the Local Skolem Property

By

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Recently much attention has focused on Skolem type problems, where criteria are given for the solvability of algebraic diophantine equations in the ring of algebraic integers over a local field, respectively a global field. Over an *algebraically closed valued field* Robinson, [Rob], has shown there is a decision procedure, and this implies the solvability of local Skolem problems over such fields is decidable. In 1982 Cantor and Roquette, [C–R], proved a Local–Global Principle for the class of those diophantine equations over ˜ which belong to a rationally parametrizable variety. Following this Rumely proved the Local-Global Principle in general for algebraic diophantine equations. Together with Robinson's result this provided a decision procedure for global Skolem problems and gave a positive solution to Hilbert's 10^{th} problem over ˜. Rumely's Local–Global Principle has motivated much of the subsequent work on Skolem type problems. Moret-Bailly has given a scheme theoretic interpretation of this result with applications in [M-B]. In [R–P–G], Roquette, Pop and Green have given a more direct proof of the Local–Global Principle, taking rationality conditions as well as archimedean primes into account. In the local case Polzin, [Po], has studied valued fields satisfying a Local–Skolem property using rigid analytic methods and has given equivalent forms of this property for varieties defined over a rank 1 valued field. Polzin's paper may be thought of as the precursor to the present paper.

Let (K, v) denote a valued field and $(\tilde{K}, \tilde{v})$ a valued algebraic closure of it. The valuation ring of v, respectively $\tilde{v}$, is denoted by $\mathcal{O}_v$, respectively $\mathcal{O}_{\tilde{v}}$, and $\tilde{\mathcal{O}}_v$ denotes the integral closure of $\mathcal{O}_v$ in $\tilde{K}$.

The Local–Skolem property is concerned with systems of algebraic diophantine equations which define a geometrically integral variety over K, and asserts the existence of a solution in $\tilde{\mathcal{O}}_v$ provided there is a solution in $\mathcal{O}_{\tilde{v}}$.

Our aim in this paper is to study the class of valued fields satisfying the property above for any such algebraic diophantine system. The valued fields in this class will be said to have the Local–Skolem property. It follows directly from the definition that the henselian valued fields have the Local–Skolem property. However this is not a necessary condition and it is known from the work of Rumely, [Ru1], Roquette, [R2], and Moret-Bailly, [M-B], that the global fields equipped with non-archimedean valuations have the Local–Skolem property. In the third section of this paper we will give general conditions on the value group and residue field which imply this property for the valued field and include the class above. In the definition, if we require only that each geometrically integral variety of dimension 1 satisfies the Local–Skolem property, then this turns out to be no weakening but an equivalent form of the definition. Using a Bertini type argument it follows that the property for 1 dimensional varieties implies the same for all higher dimensions.

In the first section we show that each valued field satisfying the Local–Skolem propert is dense in its henselisation. We also give two equivalent criteria for the Local–Skolem property. The first asserts the existence of elements with the uniqueness property for constant reductions of function fields in one variable over the valued field. This condition, which appeared in the work of Polzin, [Po], suggests that valued fields having the Local–Skolem property be called *"geometric henselian"*. It is also connected with the existence of a finite morphism for proper integral normal curves defined over the valuation ring $\mathcal{O}_v$ of the valued field, to $^1_{\mathcal{O}_v}$. The second criterion is a weak form of the Jacobi–Rumely existence theorem, see [Ru1]. This criterion asserts the existence of finitely many points of the variety, integral over a henselisation of $\mathcal{O}_v$, whose sum is linearly equivalent to a divisor rational over the valued field.

In the second section we study permanance properties of valued fields having the Local–Skolem property. For a henselian valued field each coarsening and residue field is henselian; further a finite extension of a henselian field is henselian — we show the same properties are true for fields having the Local Skolem property. The first result follows from the observations concerning density in the first section while the second is established by making descent of the variety defined over the extended field.

In the last section we use the criteria given in sections 1 and 2 together with the non-archimedean uniformisation of abelian varieties to give valuation side conditions, i.e., conditions on the value group and residue field, which are equivalent to the Local–Skolem property. We show that the class of valued fields (K, v) having residue field algebraic over a finite field and value group of rational rank 1, satisfies the Local–Skolem property. Observe that this class contains the discretely valued fields with residue field algebraic over a finite field, mentioned above. Moreover these are the most general valuation side conditions which are equivalent to the Local–Skolem property. Indeed if Γ is a totally ordered abelian group and κ an arbitrary field with *either the rational rank of* Γ *greater than 1, or* κ *not algebraic over a finite field,* then there exist valued fields (K, v) with value group Γ and residue field κ, not satisfying the Local–Skolem property.

1. Characterizations of the Local–Skolem property

We first fix the notations we will be using throughout the paper:
Let K be a field. For a valuation v defined on K we denote by $\mathcal{O}_v$, Kv and $v(K)$ the valuation ring, residue field and value group respectively. Let $\tilde{K}$ denote a fixed algebraic closure and $\tilde{\mathcal{O}}_v$ the integral closure of $\mathcal{O}_v$ in $\tilde{K}$. For a fixed prolongation $\tilde{v}$ of v to $\tilde{K}$ let $(K^{\mathrm{h}}, v^{\mathrm{h}})$ be the henselisation of (K, v) which prolongs to $(\tilde{K}, \tilde{v})$.

Let X be a geometrically integral variety defined over K. Equivalently its function field $K(X)$ is a regular field extension of K. For any overfield $K \subseteq L$ we denote by $X(L)$ the set of L-rational points of X, precisely $X(L) := \mathrm{Hom}_{\mathrm{Spec}\,K}\big(\mathrm{Spec}\,L, X\big)$. Suppose $S \subset L$ and $\boldsymbol{x} = (x_1, \ldots, x_n)$ is a tuple of rational functions on X defined over K. We say that a point $P \in X(L)$ is S-rational with respect to $\boldsymbol{x}$ if $x_i(P) \in S$, $(1 \leq i \leq n)$. We denote the set of all S-rational points with respect to $\boldsymbol{x}$ by $X_{\boldsymbol{x}}(S)$.

With the notations above X is said to satisfy the *Local–Skolem property* at v if for each tuple $\boldsymbol{x} = (x_1, \ldots, x_n)$ of rational functions on X defined over K,

$$X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}}) \neq \emptyset \Rightarrow X_{\boldsymbol{x}}(\tilde{\mathcal{O}}_v) \neq \emptyset.$$

Remarks. With the above notations:
i) Suppose $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ is non-empty. Then $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ is Zariski dense in X. Furthermore there exists a K-rational function x_0 on X such that for $\boldsymbol{y} = (x_0, x_1, \ldots, x_n)$, $X_{\boldsymbol{y}}(\mathcal{O}_{\tilde{v}}) \subset X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ is non-empty and consists only of smooth points.

We sketch the proof that $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ is Zariski dense: Take $P \in X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ and let w' be a valuation on $\tilde{K}(X)$ over $\tilde{K}$ which dominates the local ring of P, so that $\boldsymbol{x}w' = \boldsymbol{x}P$, i.e., $x_i w' = x_i(P)$ for $1 \leq i \leq n$. We next refine w' by the valuation $\tilde{v}$ and denote the resulting valuation by w. Let $\tilde{w}$ be a prolongation of w to the algebraic closure, say Ω, of $\tilde{K}(X)$. Then in this situation we have:

a) $(\Omega, \tilde{w})$ is a valued field extension of $(\tilde{K}, \tilde{v})$ and hence by Robinson's theorem, [Rob], they are elementary equivalent.

b) The set of all Ω-rational points $Q \in X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{w}})$ is Zariski dense as it contains the generic point of X.

Hence by b), for every non-zero K-rational function f on X there exists a Ω-rational point $Q \in X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{w}})$ such that $f(Q) \neq 0$. By a) it follows that there exist such $Q \in X(\tilde{K}) \cap X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{w}}) \subset X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$. This proves that $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ is Zariski dense in X.

ii) If $X \longrightarrow Y$ is a birational morphism of geometrically integral varieties defined over K, then Y has the *Local–Skolem property* at v if and only if X does.

Proof of ii). It suffices to show that a geometrically integral variety X has the Local–Skolem property at v if and only if any affine open U of it does.

Suppose X has the Local–Skolem property at v and let $\boldsymbol{x} = (x_1, ..., x_n)$, $x_i \in K(U) = K(X)$, be given with $U_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}}) \neq \phi$. Let $u \in \mathcal{O}_X(U)$ and $D(u) \subset U$ be the principal open associated to u. Since $U_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ is Zariski dense, it follows that

$$U_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}}) \cap D(u) \neq \phi.$$

Hence there exists $P_v \in U_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ such that $u(P_v) \neq 0$. Choosing $c \in K$ such that $\tilde{v}(cu^{-1}(P_v)) \geq 0$ it follows that for $\boldsymbol{u} = (cu^{-1}, \boldsymbol{x})$

$$P_v \in U_{\boldsymbol{u}}(\mathcal{O}_{\tilde{v}}) \subset X_{\boldsymbol{u}}(\mathcal{O}_{\tilde{v}}).$$

By hypothesis there exists $P \in X_{\boldsymbol{u}}(\tilde{\mathcal{O}}_v)$, and as $u(P) \neq 0$ we have $P \in D(u) \subset U$. This implies $P \in U_{\boldsymbol{u}}(\tilde{\mathcal{O}}_v) \subset U_{\boldsymbol{x}}(\tilde{\mathcal{O}}_v)$, showing that U has the Local–Skolem property at v.

The converse follows from the observation that by Zariski density, if $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}}) \neq \phi$ then $U_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}}) \neq \phi$.

iii) Using ii) above it follows that for geometrically integral varieties X and Y defined over K, if $X \longrightarrow Y$ is a dominant morphism defined over K then X has the *Local–Skolem property* at v implies the same holds for Y. Note that as the morphism is dominant the image contains an affine open of Y, and so using ii) it is sufficient to consider the case when the morphism is surjective. Here the result follows directly from the definitions.

Lemma 1.1. *Let (K, v) be a valued field and X a geometrically integral variety defined over K which has the Local–Skolem property at v. Then (K, v) is dense in its henselisation $(K^{\mathrm{h}}, v^{\mathrm{h}})$.*

Proof. Let $X \longrightarrow {}^1_K$, be a dominant morphism. By iii) above it follows that ${}^1_K$ has the *Local–Skolem property* at v. Applying ii) it follows that ${}^1_K = \operatorname{Spec} K[t]$ has the Local–Skolem property at v.

Let v^1 be a coarsening of v and $v^0 = v/v^1$ its quotient defined on $K^0 = Kv^1$. The corresponding valuation ideals are denoted by $\mathcal{M}_{v^1}$ and $\mathcal{M}_{v^0}$. To prove the lemma it suffices to show that (K^0, v^0) is henselian. For this it is sufficient that every separable polynomial of the form $P^0(T) = T^n - T^{n-1} + f^0(T)$ with $f^0(T) \in \mathcal{M}_{v^0}[T]$, and $\deg f^0 \leq n-2$ has a root in K^0. Let $P(T) = T^n - T^{n-1} + f(T)$ be a lifting in $K[T]$ with $\deg f = \deg f^0$.

Now we set $\boldsymbol{x} = (t, t^{-1}, P(t)/\pi)$ with $\pi \in \mathcal{M}_{v^1}$ and observe that as P has a unique root which is an $\mathcal{O}_{\tilde{v}}$-unit, ${}^1_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}}) \neq \phi$. By the Local–Skolem property we deduce ${}^1_{\boldsymbol{x}}(\tilde{\mathcal{O}}_v) \neq \phi$, i.e. there exists $a \in \tilde{K}$ such that $a, a^{-1} \in \tilde{\mathcal{O}}_v$, so a is a unit of $\tilde{\mathcal{O}}_v$, and $P(a) \in \pi\tilde{\mathcal{O}}_v$. We remark the same is true for any conjugate σa of a, $\sigma \in G_K$. Now

$$P(\sigma a) \in \pi\tilde{\mathcal{O}}_v$$

for each $\sigma \in G_K$ and consequently $(P(\sigma a))\tilde{v}^1 = P^0(\sigma a\tilde{v}^1) = 0$ ($\tilde{v}^1$ is the coarsening of $\tilde{v}$ defined by v^1). Hence $\sigma a\tilde{v}^1$ are roots of P^0 and units in $\mathcal{O}_{\tilde{v}^0} = \mathcal{O}_{\tilde{v}}/\mathcal{M}_{\tilde{v}^1}$. As P^0 has a unique root in K^0 which is a unit it follows that $a\tilde{v}^1 \in K^0$ as P^0 is separable over K^0.

Definition. The valued field (K, v) is said to satisfy the *Local–Skolem property* at v if each geometrically integral variety X over K satisfies the Local–Skolem property at v.

Using a Bertini type argument one shows that K has the Local–Skolem property at v iff every geometrically integral curve X/K satisfies the Local–Skolem property at v (for a proof see [Ru1], or [Po], lemma 3.). Further by the remark ii) above it follows that a geometrically integral curve X/K satisfies the Local–Skolem property iff some open of it does, respectively iff the normalisation of a scheme theoretic completion does. Therefore for the rest of the paper we shall concentrate on the case of curves.

Let X be a geometrically integral curve defined over K with function field $K(X) = F$. A constant reduction of $F|K$ relative to v is defined to be a valuation v of F prolonging v such that the residue field $F\mathrm{v}$ is a function field in one variable over Kv. Let V be a set of constant reductions of $F|K$. A non-constant function $f \in F$ is said to be an element with the uniqueness property for V, if V is the full set of prolongations (denoted V_f) of the Gauß valuation v_f on $K(f)$ induced by v, to F. If K is henselian, then for each set V as above there always exist elements with uniqueness property, [G–M–P 2] theorem 3.1.

The main result of this section gives equivalent criteria for the Local-Skolem property at v.

Theorem 1.2. *Let (K, v) be a valued field and $\tilde{v}$ a prolongation to $\tilde{K}$. Let X be a geometrically integral curve defined over K and $F|K$ its function field. Then the following assertions are equivalent:*

i) X satisfies the Local-Skolem property with respect to v.

ii) For each finite set of constant reductions V of $F|K$ prolonging v there exist elements with the uniqueness property.

iii) (K, v) is dense in its henselisation $(K^{\mathrm{h}}, v^{\mathrm{h}})$ and for every $\boldsymbol{x} = (x_1, \ldots, x_n)$, $x_i \in F$, if $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}}) \neq \emptyset$ then there exists a positive K-rational divisor D and $P_1, \ldots, P_d \in X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ such that $D \sim P_1 + P_2 + \cdots + P_d$ over $\tilde{K}$.

Before proving this theorem we recall certain facts concerning constant reductions and curves defined over the valuation ring $\mathcal{O}_{\tilde{v}}$:

1) For a finite set V of constant reductions of $F|K$ let V^{h} be the set of all constant reductions v^{h} of $FK^{\mathrm{h}}|K^{\mathrm{h}}$ which prolong both v and v^{h} to FK^{h}. We define $\tilde{V}$ correspondingly on $F\tilde{K}$ and remark that $\tilde{V}|_{FK^{\mathrm{h}}} = V^{\mathrm{h}}$ and $V^{\mathrm{h}}|_F = V$.

We show that for each $\mathrm{v} \in V$ *there is a unique* $\mathrm{v}^{\mathrm{h}} \in V^{\mathrm{h}}$ *prolonging it.*

Proof. We first remark that it is sufficient to prove the assertion for some set of constant reductions V' containing V. Choose $x \in F$ so that $V \subseteq V_x =: V'$. We shall prove the assertion for V'. Note that $\mathrm{v}^{\mathrm{h}} \in {V'}^{\mathrm{h}}$ if and only if it restricts to some $\mathrm{v} \in V'$ on F and to v^{h} on K^{h}. Equivalently v^{h} restricts to v_x^{h} on $K^{\mathrm{h}}(x)$. Hence we only have to prove there is a one to one correspondence between the prolongations of v_x to F and those of v_x^{h} to FK^{h}. The prolongations of v_x from $K(x)$ to F, are determined by the decomposition of $F \otimes_{K(x)} (K(x))^{\mathrm{h}}$ and the prolongations of v_x^{h} from $K^{\mathrm{h}}(x)$ to FK^{h} by the decomposition of $FK^{\mathrm{h}} \otimes_{K^{\mathrm{h}}(x)} (K(x))^{\mathrm{h}}$. However, as $K^{\mathrm{h}} \subset (K(x))^{\mathrm{h}}$, one has

$$F \otimes_{K(x)} (K(x))^{\mathrm{h}} \cong FK^{\mathrm{h}} \otimes_{K^{\mathrm{h}}(x)} (K(x))^{\mathrm{h}}$$

which completes the proof of the claim.

2) For a finite set $\tilde{V}$ of constant reductions of $F\tilde{K}$ relative to $\tilde{v}$, we denote by $\mathcal{X} = \mathcal{X}_{\tilde{V}}$ the $\mathcal{O}_{\tilde{v}}$-curve defined by $\tilde{V}$ as in [G–M–P 3] section 1, by X its generic fibre and by $\mathcal{X}_{\mathcal{M}}$ its closed fibre. By [G–M–P 2], proposition 2.5, the following assertion holds:

$(*)$ There exist finite subsets $S \subset \mathcal{X}_{\mathcal{M}}$ with the property: if D is a positive divisor of $F\tilde{K}|\tilde{K}$ such that the specialisations of $\operatorname{supp} D$ lie in the complement of S and meet all the components of $\mathcal{X}_{\mathcal{M}}$, then for n large enough there exist $f \in F\tilde{K}$ with pole divisor nD and $\tilde{V}_f = \tilde{V}$.

3) For $\tilde{V} \subset \tilde{V}'$ by [G–M–P 3], lemma 1.3, there is a commutative diagram of $\mathcal{O}_{\tilde{v}}$-curves:

$$\begin{array}{ccccc} Y & \longrightarrow & \mathcal{Y} & \longleftarrow & \mathcal{Y}_{\mathcal{M}} \\ \downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow{\scriptstyle pr} \\ X & \longrightarrow & \mathcal{X} & \longleftarrow & \mathcal{X}_{\mathcal{M}} \end{array}$$

where the columns are surjective and $\mathcal{Y}$ is the $\mathcal{O}_{\tilde{v}}$-curve defined by $\tilde{V}'$.

There exist finite subsets $S \subset \mathcal{X}_{\mathcal{M}}$ which have the properties:

- $\mathcal{X}$ satisfies $(*)$ with respect to S.
- the preimage $T = pr^{-1}(S)$ contains finite subsets S' such that $\mathcal{Y}_{\mathcal{M}}$ has the property $(*)$ with respect to S'.
- T contains a finite subset which can be prescribed in advance.

4) *Reciprocity.* Suppose x is an element with the uniqueness property for $\tilde{V}'$ and let $y \in \mathcal{O}_{\tilde{v}}[x]'$, where $'$ denotes taking the integral closure in $F\tilde{K}$. Then if $P \in X_x(\mathcal{O}_{\tilde{v}})$ one has

$$y(P)\tilde{v} = \overline{y}(\overline{Q})$$

where $\overline{y} := (y\tilde{\mathrm{v}})_{\tilde{\mathrm{v}}} \in \prod_{\tilde{\mathrm{v}}\in\tilde{V}'} F\tilde{K}\tilde{\mathrm{v}}$, the ring of functions of $\mathcal{Y}_{\mathcal{M}}$, and $\overline{Q}$ is any point in the normalisation of $\mathcal{Y}_{\mathcal{M}}$ which lies over the specialisation of P to $\mathcal{Y}_{\mathcal{M}}$.

Proof. First as $y \in \mathcal{O}_{\tilde{v}}[x]'$ and $P \in X_x(\mathcal{O}_{\tilde{v}})$ it follows $\tilde{v}(y(P)) \geq 0$. By [G–M–P 3], $\mathcal{O}_{\tilde{v}}[x]'$ determines an affine open of $\mathcal{Y}$, $\mathcal{O}_{\tilde{v}}[x]' \otimes_{\tilde{K}} \tilde{K}\tilde{v}$ the corresponding affine open of $\mathcal{Y}_{\mathcal{M}}$ and $\prod_{\tilde{\mathrm{v}}\in\tilde{V}'} \left(\tilde{K}\tilde{v}[x\tilde{\mathrm{v}}]\right)'$ that of its normalisation. Denoting the ideal associated to P in $\mathcal{O}_{\tilde{v}}[x]'$ by $[P]$ and that of any $\overline{Q}$ lying over its specialisation to $\mathcal{Y}_{\mathcal{M}}$ by $[\overline{Q}]$ in $\prod_{\tilde{\mathrm{v}}\in\tilde{V}'} \left(\tilde{K}\tilde{v}[x\tilde{\mathrm{v}}]\right)'$ we have

$$(y - y(P))\mathcal{O}_{\tilde{v}}[x]' \subset [P] \quad \text{and} \quad \left((y - y(P))\tilde{\mathrm{v}}\right)_{\tilde{\mathrm{v}}} \prod_{\tilde{\mathrm{v}}\in\tilde{V}'} \left(\tilde{K}\tilde{v}[x\tilde{\mathrm{v}}]\right)' \subset [\overline{Q}].$$

Therefore $\overline{Q}$ is a zero of $\left((y - y(P))\tilde{\mathrm{v}}\right)_{\tilde{\mathrm{v}}} = \left(y\tilde{\mathrm{v}} - y(P)\tilde{v}\right)_{\tilde{\mathrm{v}}}$, i.e., $(y\tilde{\mathrm{v}})_{\tilde{\mathrm{v}}}(\overline{Q}) = y(P)\tilde{v}$.

We return to the proof of the theorem.

Proof. i) $\Rightarrow$ ii) By [G–M–P 2], proposition 3.4, it is sufficient to prove the assertion ii) in the case where V consists of a unique constant reduction v of $F|K$.

Now we shall construct a suitable tuple $\boldsymbol{x}$ which will be used to show that the Local–Skolem property implies the existence of elements with the uniqueness property for v.

Let $x \in F$ be chosen so that $V' := V_x$ contains V and $\tilde{V}$, $\tilde{V}'$ be defined as in 1) above. Then:

- As x is V' residually transcendental, for every finite subset $S' \subset \mathcal{Y}'_{\mathcal{M}}$ given in advance there exist polynomials $z \in \mathcal{O}_v[x]$ such that S' is contained in the (scheme theoretical) specialisation of the divisor support of z. We choose S' to satisfy the condition 3) and z as above.
- Next we choose $y \in \mathcal{O}_v[x]' \subset F$ such that $y\mathrm{v} \neq 0$ and $y\mathrm{v}' = 0$ for $\mathrm{v}' \in V' \backslash V$. By [G–M–P 2] proposition 2.1 proof, $\mathcal{O}_{\tilde{v}}[x]' \otimes_{\tilde{K}} \tilde{K}\tilde{v}$ is of finite index in $\prod_{\tilde{\mathrm{v}}\in\tilde{V}'} \left(\tilde{K}\tilde{v}[x\tilde{\mathrm{v}}]\right)'$ so it is possible to choose such an element y. Using 4) above it follows that the scheme theoretical specialisation of $X_{(x,y,y^{-1})}(\mathcal{O}_{\tilde{v}})$ is an open dense subset of the union of the irreducible components of $\mathcal{Y}_{\mathcal{M}}$ corresponding to the $\tilde{\mathrm{v}} \in \tilde{V}$.

Setting $\boldsymbol{x} = (x, y, y^{-1}, z, z^{-1})$ and letting $\mathcal{D}$ denote the scheme theoretical specialisation of $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ in $\mathcal{Y}_{\mathcal{M}}$, it follows that $\mathcal{Y}_{\mathcal{M}} \backslash \mathcal{D}$ has the properties required of T in 3). In particular for every positive $\tilde{K}$-rational divisor A with $\mathrm{supp}(A) \subset X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$, whose specialisation meets all the components of $\mathcal{X}_{\mathcal{M}}$, there exist $\tilde{V}$ elements with the uniqueness property $h \in F\tilde{K}$ such that $(h)_\infty = nA$ for n big enough.

Now since $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ is non-empty, by hypothesis there exists $P \in X_{\boldsymbol{x}}(\tilde{\mathcal{O}}_v)$. We let $B = r\sum P'$ where P' is a system of G_K conjugates of P and r is a large enough power of the characteristic exponent of K so that B is K-rational. By the discussion above we conclude there exists a $\tilde{V}$-regular function $h \in F\tilde{K}$ with $(h)_\infty = nB$ provided n is large enough.

Using 1) and taking into account that V consists only of v it follows that $\tilde{V}$ is $G_{K^{\mathrm{h}}}$ invariant. Hence for each $\sigma \in G_{K^{\mathrm{h}}}$ one has $\tilde{V}^\sigma = \tilde{V}$ and consequently σh is a $\tilde{V}$ element with uniqueness property with $(h)_\infty = (\sigma h)_\infty$. Set $y = \left(\prod \overline{\sigma} h\right)^r$, where the $\overline{\sigma} h$ are a system of representatives for the $G_{K^{\mathrm{h}}}$ conjugates of h and r is a large enough power of the characteristic exponent of K so that $y \in FK^{\mathrm{h}}$. As the $\overline{\sigma} h$ all have the same pole divisor it follows that y is a $\tilde{V}$ element with the uniqueness property. Since it lies in FK^{h} it is an element with the uniqueness property for v^{h}. We denote the pole divisor of y by mB.

Finally as (K, v) is dense in $(K^{\mathrm{h}}, v^{\mathrm{h}})$ we can find g in the linear space $\mathcal{L}_K(mB)$ such that $(g)_\infty = mB$ and $\mathrm{v}^{\mathrm{h}}(y - g) > 0$. Then using the fundamental inequality:

$$\begin{aligned}
[FK^{\mathrm{h}} : K^{\mathrm{h}}(y)] &= [FK^{\mathrm{h}} : K^{\mathrm{h}}(g)] \\
&\geq e_{\mathrm{v}^{\mathrm{h}}} \delta_{\mathrm{v}^{\mathrm{h}}} [FK^{\mathrm{h}}\mathrm{v}^{\mathrm{h}} : K^{\mathrm{h}}(g)\mathrm{v}^{\mathrm{h}}] \quad \text{as } \mathrm{v}^{\mathrm{h}} \in V_g^{\mathrm{h}} \\
&\geq e_{\mathrm{v}^{\mathrm{h}}} \delta_{\mathrm{v}^{\mathrm{h}}} [FK^{\mathrm{h}}\mathrm{v}^{\mathrm{h}} : K^{\mathrm{h}}(y)\mathrm{v}^{\mathrm{h}}] = [FK^{\mathrm{h}} : K^{\mathrm{h}}(y)].
\end{aligned}$$

Thus $[FK^{\mathrm{h}} : K^{\mathrm{h}}(g)] = e_{\mathrm{v}^{\mathrm{h}}} \delta_{\mathrm{v}^{\mathrm{h}}} [FK^{\mathrm{h}}\mathrm{v}^{\mathrm{h}} : K^{\mathrm{h}}(g)\mathrm{v}^{\mathrm{h}}]$ and hence g is an element with the uniqueness property for v^{h}. Since g lies in F it is an element with uniqueness property for v.

ii) $\Rightarrow$ iii) We assume that every finite set of constant reductions V of $F|K$ prolonging v has elements with the uniqueness property. In the first part of the proof we show that (K, v) is dense in $(K^{\mathrm{h}}, v^{\mathrm{h}})$. Equivalently this will follow if for each coarsening v^1 of v, the quotient valuation v^0 on $K^0 := Kv^1$ is henselian.

We first make the following remark which is of more general character: Let v be a constant reduction of $F|K$ (relative to v), v^1 denote the coarsening of v which is defined by v^1 and set $F^0 = F\mathrm{v}^1$. Then $F^0|K^0$ is a function field of one variable and $\mathrm{v}^0 := \mathrm{v}/\mathrm{v}^1$ is a constant reduction of $F^0|K^0$ (relative to v^0). With these notations it follows that if h is an element with the uniqueness property for v, then $h^0 = h\mathrm{v}^1$ is an element with the uniqueness property for v^0. In particular, if $K^{0\prime}$ is the constant field of F^0 then $v^0_{h^0}$ has a unique prolongation to F^0. Therefore, if v has elements with the uniquenes property, then v^0 has a unique prolongation to $K^{0\prime}$.

We proceed as follows: Let $L^0|K^0$ be a finite galois extension and $P^0(T)$ a monic polynomial over K^0 whose root generates L^0. Let $P(T)$ be a representative for $P^0(T)$ in $K[T]$ and set $u = P(t)/\pi$ with $v^1(\pi) > 0$ and $t \in F \setminus K$. Let v be some prolongation of v_u to F. As $P(t) = \pi u$ it follows that $t\mathrm{v}^1$ is a root of $P^0(T)$ which is contained in F^0, hence in the constant field $K^{0\prime}$ of F^0. Hence L^0 is contained in $K^{0\prime}$. On the other hand, by the remarks above, v^0 has a unique prolongation to $K^{0\prime}$, and hence to L^0. From this it follows that v^0 is henselian, completing the first part of the proof.

In the second part we show that for every $\boldsymbol{x} = (x_1, \ldots, x_n)$, $x_i \in F$, if $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}}) \neq \emptyset$ then there exists a positive K-rational divisor D and $P_1, \ldots, P_d \in X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ such that $D \sim P_1 + P_2 + \cdots + P_d$ over $\tilde{K}$.

We first show there exists $f \in FK^{\mathrm{h}}$ such that $X_f(\mathcal{O}_{\tilde{v}}) \subset X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$. First for $P \in X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ we remark that all its $\mathrm{G}_{K^{\mathrm{h}}}$-conjugates P' lie in $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$. Set $A = p^e \sum P'$, where p is the characteristic exponent of K and e is sufficiently large such that A is K^{h}-rational and by the Riemann-Roch there exists $f_1 \in FK^{\mathrm{h}}$ with $(f_1)_0 = A$. Next let $u_i(T) := \mathrm{Irr}(x_i(P)|K^{\mathrm{h}})$ and set $y_i = u_i(x_i)$. Clearly $\operatorname{supp} A \subset \operatorname{supp}(y_i)_0$, hence f_1^{-1} is integral over $\tilde{K}[y_i^{-1}]$. From the irreducible equations of f_1^{-1} over the $\tilde{K}[y_i^{-1}]$, one deduces that for all $\pi \in K^{\mathrm{h}}$ with $v^{\mathrm{h}}(\pi)$ sufficiently large $f = f_1/\pi$ satisfies the claim.

Let V_f^{h} be the set of all prolongations of the Gauß valuation v_f^{h} to FK^{h} and V its restriction to F. By the remarks made in 1) before the proof of the theorem $V_f^{\mathrm{h}} = V^{\mathrm{h}}$. Let $g \in F$ be an element with the uniqueness property for V, which exists by hypothesis. Then once again by 1) it follows that g is an element with the uniqueness property for V^{h}. It follows that f is integral over $\mathcal{O}_{v_g^{\mathrm{h}}}$ and so also over $\mathcal{O}_{\tilde{v}_g}$ when viewed in $F\tilde{K}|\tilde{K}$. Hence one has

$$f^n + r_{n-1}(g) f^{n-1} + \cdots + r_0(g) = 0,$$

where $r_k(g) = p_k(g)/q_k(g)$ and $\tilde{v}_g(q_k(g)) = 0 \leq \tilde{v}_g(p_k(g))$. By the maximum principle there exists $\alpha \in \mathcal{O}_{\tilde{v}}$ such that $\tilde{v}(q_k(\alpha)) = 0 \leq \tilde{v}(p_k(\alpha))$. We show that the zeros P of $g - \alpha$ lie in $X_f(\mathcal{O}_{\tilde{v}})$: Let P be a zero of $g - \alpha$, i.e. $g(P) = \alpha$. Then

$$(f(P))^n + r_{n-1}(\alpha)(f(P))^{n-1} + \cdots + r_0(\alpha) = 0.$$

As the $r_k(\alpha) \in \mathcal{O}_{\tilde{v}}$ it follows that $f(P) \in \mathcal{O}_{\tilde{v}.}$ Hence all the zeros P_i of $g - \alpha$ lie in $X_f(\mathcal{O}_{\tilde{v}})$ and taking $D = (g)_\infty$ a K-rational divisor, the result follows.

iii) $\Rightarrow$ i) Under the hypothesis of iii) we first show that without loss of generality we can suppose that $P_i \notin \operatorname{supp} D$ (and $P_i \neq P_j$ for $i \neq j$).

Replacing D by a multiple we can assume that $\dim D \gg 0$. Let $f \in \mathcal{L}_{\tilde{K}}(D)$ with $(f) = \sum P_i - D$. One can approximate $f \in \mathcal{L}_{\tilde{K}}(D)$ arbitrarily closely $\tilde{v}$-adically by functions g with $(g)_\infty = D$. Then by the continuity of the roots (see [R–P–G]) as $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ is a $\tilde{v}$-adically open neighborhood of the P_i in $X(\tilde{K})$ it follows there exists $g \in F\tilde{K}$ with $(g)_\infty = D$ and $(g)_0 = \sum P_i'$, and the $P_i' \in X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$, $P_j' \neq P_j'$. This finishes the first part of the proof.

Suppose that the $P_i \notin \operatorname{supp} D$. Choose $f \in F\tilde{K}$ with $(f) = \sum P_i - D$ and set $h = \left(\prod \overline{\sigma} f\right)^r$, where the $\overline{\sigma} f$ form a system of $G_{K^{\mathrm{h}}}$-conjugates of f and r is a suitable power of the characteristic exponent of K such that $h \in FK^{\mathrm{h}}$. Let $L|K$ be a finite galois extension over which h is defined and set $v_L = \tilde{v}|_L$. We denote by $\left(v_L^{\overline{\tau}}\right)_{\overline{\tau}}$ the set of all prolongations of v to L, where the $\overline{\tau}$ form a system of representatives for the cosets of $\mathrm{Gal}\big(L|K\big)/Z(v_L|v)$. For each $\overline{\tau}$ let $\overline{\tau}h$ be the corresponding conjugate of h. As K is dense in K^{h} it follows that the $v_L^{\overline{\tau}}$ are independent valuations of L, and so the diagonal embedding

$$L \hookrightarrow \prod_{\overline{\tau}} \big(L, v_L^{\overline{\tau}}\big)$$

is dense. Therefore the diagonal embedding

$$\mathcal{L}_L(D) \hookrightarrow \prod_{\overline{\tau}} \big(\mathcal{L}_L(D), v_L^{\overline{\tau}}\big)$$

is dense. In particular, there exist $g \in \mathcal{L}_L(D)$ such that $g \approx \overline{\tau}h$ in the $v_L^{\overline{\tau}}$-adic topology for each $\overline{\tau}$. For $\sigma \in G_K$ let $\overline{\tau}$ be its representative in $\mathrm{Gal}\big(L|K\big)/Z(v_L|v)$. Then $\sigma h = \overline{\tau}h$ and hence $\sigma^{-1}g \approx h$ in the $\tilde{v}$-adic topology, as $g \approx \overline{\tau}h$ in the $v_L^{\overline{\tau}}$-adic topology. Consequently the zeros of $\sigma^{-1}g$ are $\tilde{v}$-close to those of h. By the choice of h, $\mathrm{supp}(h)_0 \subseteq X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$, hence the same is true for the zeros of $\sigma^{-1}g$. Finally, if P is a zero of g then $P^\sigma \in X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ for all $\sigma \in G_K$. Equivalently, $P \in X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}^\sigma})$ for all $\sigma \in G_K$, hence $P \in X_{\boldsymbol{x}}(\tilde{\mathcal{O}}_v)$ completing the proof.

2. Permanence Properties

In this section we study the behaviour of the Local–Skolem property under algebraic extensions $K'|K$. Let v' be a fixed prolongation of v on K to K'. We first remark:

2.1. *Suppose that v has a unique prolongation v' to K'. Then (K, v) has the Local-Skolem property iff (K', v') has this property.*

Proof. The direction " $\Rightarrow$ " will follow from the theorem below.

$\Leftarrow$ Let X be a geometrically integral variety defined over K and $\boldsymbol{x}$ a set of K-rational functions. Then setting $X' = X \times_K K'$ we have $K(X) \subset K'(X')$ and so X' together with $\boldsymbol{x}$ define Skolem data for (K', v'). Moreover $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}}) \neq \phi$ iff $X'_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}'}) \neq \phi$. Now take any $P' \in X'_{\boldsymbol{x}}(\tilde{\mathcal{O}}_{v'})$. As v' is the unique prolongation of v to K' it follows that $\tilde{\mathcal{O}}_v = \tilde{\mathcal{O}}_{v'}$, hence $P' \in X_{\boldsymbol{x}}(\tilde{\mathcal{O}}_v)$.

More generally we prove:

Theorem 2.2. *Let (K, v) be a valued field and $K' \supset K$ a subfield of $(\tilde{K}, \tilde{v})$ equipped with $v' := \tilde{v}|_{K'}$. Assume that (K, v) satisfies the Local–Skolem property, then (K', v') satisfies the Local–Skolem property.*

Proof. Since the data for a given variety over K satisfying the Local–Skolem property is defined over a finite extension of K, we can assume $K'|K$ is finite. We first consider the case where $K'|K$ is finite and purely inseparable.

Let φ be a power of the Frobenius map such that $K'^{\varphi} \subseteq K$. For X/K' a geometrically integral variety, X^{φ}/K'^{φ} is geometrically integral and so $Y = X^{\varphi} \times_{K'^{\varphi}} K$ is a geometrically integral K-variety. If $\boldsymbol{x}$ is a set of K'-rational functions then $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}}) \neq \phi$ (resp. $X_{\boldsymbol{x}}(\tilde{\mathcal{O}}_v) \neq \phi$) iff $Y_{\boldsymbol{x}^{\varphi}}(\mathcal{O}_{\tilde{v}}) \neq \phi$ (resp. $Y_{\boldsymbol{x}^{\varphi}}(\tilde{\mathcal{O}}_v) \neq \phi$) and so the result follows from the Local–Skolem property for K at v.

Using the above case and 2.1 ($\Leftarrow$) we can assume that K and hence K' are perfect fields.

Let X be a given geometrically integral affine curve over K' and $\boldsymbol{x}$ a finite set of rational functions which have no poles on X. We suppose that $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}'}) \neq \phi$ and show that $X_{\boldsymbol{x}}(\tilde{\mathcal{O}}_{v'}) \neq \phi$. This will be done by descending to (K, v) where the Local-Skolem property holds by assumption.

Construction of the descent data. We begin by describing the descent procedure we are going to use:

Let $M|K$ be a finite galois extension containing K' and set $G = \mathrm{Gal}(M|K)$ and $X_M = X \times_{K'} M$. To every $\tau \in G$ we associate a K-isomorphic copy X_τ of X_M via an isomorphism $X_\tau \xrightarrow{\tau} X_M$ which restricts to τ on the constants. Then G identifies with a group of (right) K-automorphisms on $Y = \times_M X_\tau$ $(\tau \in G)$. In particular, Y has descent to a variety which is defined over K, and this variety is nothing but $Z = Y/G$. The isomorphisms $X_\tau \xrightarrow{\tau} X_M$ induce K-isomomorphisms (which we also denote by τ) between the corresponding function fields $M(X)\xrightarrow{\tau}M(X_\tau)$ and the canonical projections $Y\longrightarrow X_\tau$ give rise to K-embeddings $M(X_M)\xrightarrow{\tau}M(Y)$. Every M-rational point $P\in X(M)$ of X defines a M-rational point $P_\tau\in X_\tau(M)$ by $\tau\circ P_\tau=P\circ\tau$ and a M-rational point of Y which is $Q=(P_\tau)_\tau$. For every rational function x on X one has $(\tau x)(P_\tau)=\tau\big(x(P)\big)$.

Returning to the Skolem problem for X with data $\boldsymbol{x}$, let $P\in X_{\boldsymbol{x}}(\mathcal{O}_{\tilde v})$ be fixed and $M|K$ be a finite galois extension containing K' and such that P is M-rational. Let $(v_k)_k$ with $v_1=\tilde v|_M$ be the set of all prolongations of v to M and set $\boldsymbol{x}=(x_i)_i$. For each i let $a_i\in M$ be chosen such that $a_i\approx 1$ in the v_1-adic topology on M and $a_i\approx x_i(P)$ in the v_k-adic topology otherwise. (Such a_i always exist as the set of all prolongations of v to M are independent.) Set $x'_i=x_ia_i^{-1}$ and define $\boldsymbol{x}'=(x'_i)_i$. Hence in the v_1-adic topology $\boldsymbol{x}'(P)\approx\boldsymbol{x}(P)$ and $\boldsymbol{x}'(P)\approx 1$ otherwise. Notice that as $a_i\approx 1$ in the $\tilde v$ adic topology on $\tilde K$ we have $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde v})=X_{\boldsymbol{x}'}(\mathcal{O}_{\tilde v})$.

We now consider the following Local–Skolem data for $Z=Y/G$. First set $\boldsymbol{y}=(\tau x'_i)_{i,\tau}$ let $z_{i,l}$ be the elementary symmetric polynomial of degree l in $\tau x'_i$ ($\tau\in G$) We define $\boldsymbol{z}=(z_{i,l})_{i,l}$ and observe the $\tau x'_i$ are integral over $[\boldsymbol{z}]$.

Claim: $Z_{\boldsymbol{z}}(\mathcal{O}_{\tilde v})$ is non-empty.

Let $Q=(P_\tau)_\tau\in Y(M)$ be the M-rational point of Y associated to P. As $Y(M)\cong Z(M)$, it follows that Q is an M-rational point of Z. We show that Q lies in $Y_{\boldsymbol{y}}(\mathcal{O}_{\tilde v})$. Indeed:

$$(\tau x'_i)(Q)=(\tau x'_i)(P_\tau)=\tau\big(x'_i(P)\big)=\tau\big(a_i^{-1}x_i(P)\big),$$

hence $\tilde v\big((\tau x'_i)(Q)\big)=v_k(a_i^{-1}x_i(P))\geq 0$ with $v_k=v_1^\tau$. Thus $(\tau x'_i)(Q)\in\mathcal{O}_{\tilde v}$. As $z_{i,l}(Q)$ is an elementary symmetric polynomial in $(\tau x'_i)(Q)$ it follows that $Q\in Z_{\boldsymbol{z}}(\mathcal{O}_{\tilde v})$, hence the claim.

Conclusion. As (K,v) satisfies the Local–Skolem property, $Z_{\boldsymbol{z}}(\tilde{\mathcal{O}}_v)$ is non-empty. Let $Q^0\in Z_{\boldsymbol{z}}(\tilde{\mathcal{O}}_v)$. As the $\tau x'_i$ are integral over $[\boldsymbol{z}]$ for all τ, it follows that viewing Q^0 as a $\tilde K$-rational point of Y one has $Q^0\in Y_{\boldsymbol{y}}(\tilde{\mathcal{O}}_v)$. Hence setting $Q^0=(P^0_\tau)_\tau$ it follows that the $(\tau x'_i)(P^0_\tau)$ lie in $\tilde{\mathcal{O}}_v$ for all $\tau\in G$ and all i. Let P^0 be the $\tilde K$-rational point of X which is defined by $x(P^0):=(\varepsilon x)(P^0_\varepsilon)$ (all $x\in M(X)$), where $\varepsilon\in G$ is the neutral element. We show that $P^0\in X_{\boldsymbol{x}}(\tilde{\mathcal{O}}_{v'})$. Indeed, since $x'_i(P^0)=(\varepsilon x'_i)(P^0_\varepsilon)\in\tilde{\mathcal{O}}_v$ and by definition $x'_i(P^{0^\sigma})=\sigma\big(x'_i(P^0)\big)$, for $\sigma\in G_{K'}$ viewed as a $K'(X)$-automorphism of $\tilde K(X)$, it follows that $x'_i(P^{0^\sigma})$ lies in $\tilde{\mathcal{O}}_v$. In particular, P^{0^σ} lies in $X_{\boldsymbol{x}'}(\mathcal{O}_{\tilde v})$. Thus we deduce that all the conjugates of P^0 (when viewed as $\tilde K$-rational points of X) with respect to elements from $G_{K'}$ lie in $X_{\boldsymbol{x}'}(\mathcal{O}_{\tilde v})$. As $X_{\boldsymbol{x}'}(\mathcal{O}_{\tilde v})=X_{\boldsymbol{x}}(\mathcal{O}_{\tilde v})$, it follows that all the conjugates of P^0 with respect to $G_{K'}$ lie in $X_{\boldsymbol{x}}(\mathcal{O}_{\tilde v})$, hence $P^0\in X_{\boldsymbol{x}}(\tilde{\mathcal{O}}_{v'})$ completing the proof.

3. Valuation theoretic criteria on the base

A natural way of looking at valued fields is by studying their value groups and residue fields. The question arises whether special conditions on the value group and residue field can imply the Local–Skolem property for the valued field. For instance, if the value group is isomorphic to and the residue field is algebraic over a finite field, it was shown by Rumely, [Ru1], that such valued fields satisfy the condition iii) in theorem 1.2 and hence have the Local–Skolem property. This condition is a special case of his Jacobian Principle, which he used in the generalisation of the classical capacity theory to the case of curves.

In this section we give valuation side conditions which are equivalent to the Local–Skolem property. We show that the class of valued fields (K, v) having residue field algebraic over a finite field and value group of rational rank 1, satisfies the Local–Skolem property. Clearly this class contains the discretely valued fields with residue field algebraic over a finite field, mentioned above. Moreover these are the most general valuation side conditions, i.e. conditions on the value group and the residue field, which are equivalent to the Local–Skolem property. One has:

Theorem 3.1. *Let Γ be a totally ordered abelian group and κ an arbitrary field. Then the following are equivalent:*

i) Every valued field (K, v) with $vK \cong \Gamma$ and $Kv \cong \kappa$ has the Local–Skolem property.

ii) The rational rank of Γ is 1 and κ is algebraic over a finite field.

Before proving this theorem we remark that condition ii) is equivalent to saying that each valued subfield of the maximal completion of ${}_p$ or ${}_p\widetilde{((t))}$ satisfies the Local–Skolem property for each prime p.

Proof. ii) $\Rightarrow$ i) The main technical tool in the proof is the non-archimedean uniformisation of abelian varieties, see for example [vdP–R] theorem 2.1, [B–L], or [F–vdP]. From this it follows that under the hypothesis of ii) the m-adic multiplication on abelian varieties is continuous. For the convenience of the reader we sketch this proof in the next lemma.

Lemma 3.2. *Let (L, v) be an algebraically closed, complete, rank 1 valued field. We assume $rr(\Gamma) = 1$ (i.e. $vL \cong$) and that the residue field is algebraic over ${}_p$.*

Let A be an abelian variety of dimension r over L, then 0 is an accumulation point for the sequence $(ma)_{m\in}$ where $a \in A(L)$.

Proof. [Compare with [C], theorem 4.1.] Recall the following facts which give the non-archimedean uniformisation of A/L.

- There is an exact sequence of $\mathcal{O}_v$ group schemes

$$0 \longrightarrow T \longrightarrow G \longrightarrow B \longrightarrow 0,$$

where T is a torus and B is an abelian scheme.

- There is a lattice $\Lambda \subset G(L)$ of rank r and $\Lambda \cap G(\mathcal{O}_v) = \{0\}$.
- There is an exact sequence of analytic maps (uniformisation theorem)

$$0 \longrightarrow \Lambda \longrightarrow G(L) \xrightarrow{\psi} A(L) \longrightarrow 0$$

Claim 1: With the above notations $A(L)/\psi(G(\mathcal{O}_v))$ is torsion.

First we observe that

$$A(L)/\psi(G(\mathcal{O}_v)) \cong G(L)/(G(\mathcal{O}_v) + \Lambda) \cong \big(G(L)/G(\mathcal{O}_v)\big)/\big((G(\mathcal{O}_v) + \Lambda)/G(\mathcal{O}_v)\big).$$

We show that $G(L)/G(\mathcal{O}_v)$ is an r-dimensional vector space and that $\big(G(\mathcal{O}_v) + \Lambda\big)/G(\mathcal{O}_v)$ is a full lattice contained in it. Let $X(T)$ be the group of characters of T. Then by the canonical sequence of isomorphisms (see [vdP–R] introduction)

$$G(L)/G(\mathcal{O}_v) \cong T(L)/T(\mathcal{O}_v) \cong \mathrm{Hom}\big(X(T), L^\times/\mathcal{O}_v^\times\big) \cong {}^r,$$

and the image of Λ in ${}^r \cong G(L)/G(\mathcal{O}_v)$ is a full lattice. This implies the result.

Claim 2: The m-adic multiplication on $G(\mathcal{O}_v)$ is continuous.

Let $G(\mathcal{O}_v) \longrightarrow G(Lv)$ be the specialisation map induced by $\mathcal{O}_v \longrightarrow Lv$. Because Lv is algebraic over a finite field it follows that $G(Lv)$ is torsion, hence there exists $m > 0$ such that ma lies in the kernel of the specialisation map. As $0 \in G(\mathcal{O}_v)$ is smooth and the specialisation map is continuous, using the maximum modulus principle it follows that the group law on the kernel is given by an $\mathcal{O}_v$-formal group. Now the result follows from an elementary exercise dealing with formal groups, namely:

Lemma 3.3. *Let $F(\boldsymbol{X}, \boldsymbol{Y})$ be an n-dimensional formal group over $\mathcal{O}_v$ and $[m]$ denote the multiplication by $m \in$ on the formal group. Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{O}_v^n$, with $\inf v(x_i) = \rho > 0$. Then $\inf v\big(([p^k]\boldsymbol{x})_i\big) \geq k - a$ with $a = \max(0, -\log\rho\,/\log 2)$.*

Proof. One has $F(\boldsymbol{X}, \boldsymbol{Y})_i = X_i + Y_i$ modulo higher degree terms and so if $p = \mathrm{char}\, Lv$ one has $[p](\boldsymbol{X})_i = pX_i + h(\boldsymbol{X})$ modulo higher degree terms and $h(.)$ is a quadratic form with coefficients in $\mathcal{O}_v$. From these observations the announced result easily follows.

Using lemma 3.2 above one proves that (K, v) satisfies the Local–Skolem property as follows: First by 2.1 we can assume that K is perfect. Further by the remarks at the beginning of section 1 it is sufficient to consider geometrically integral projective curves X over K which are normal, and hence smooth in this case as K is perfect. For such a curve X let J_X denote its Jacobian. Then by the continuity of the m-adic multiplication on $J_X(\tilde{K})$ as in 3.2, and an argument of the type given in [R–P–G], section 2, or [Ru2], section 1.3, it follows that the third criterion for the

Local–Skolem property is satisfied. For the convenience of the reader we include a sketch of the proof here.

Let $P_0 \in X(\tilde{K})$ be fixed and $\phi : X_{\tilde{K}} \longrightarrow J_X$ the $\tilde{K}$-embedding with base point P_0. We also let ϕ denote the canonically induced map

$$\mathrm{Div}(X_{\tilde{K}}) \longrightarrow J_X(\tilde{K}) \quad \text{defined by} \quad \sum n_P P \;\mapsto\; \mathrm{class}(\sum n_P(P - P_0)).$$

For $\boldsymbol{x} = (x_i),\ x_i \in K(X)$, we set $U = X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ which is assumed to be non-empty. It follows by the implicit function theorem that $\phi(U) + \cdots + \phi(U)$ $(g = g_X$ times) contains a $\tilde{v}$-adically open neighborhood $\mathcal{U}$ of $J_X(\tilde{K})$. Hence for some fixed $a_0 \in \mathcal{U}$ the translate $\mathcal{U} - a_0$ is a neighborhood of 0 in $J_X(\tilde{K})$. Note that $a_0 = \phi(A)$ for some $A = \sum_{k=1}^{g} Q_k,\ Q_k \in U$. Let $D \in \mathrm{Div}(X_{\tilde{K}})$ be a positive K-rational divisor and set $d = \deg(D) > 0$. Setting $a = \phi(gD - dA)$, by 3.2 it follows there exists $m > 0$ such that $ma \in \mathcal{U} - a_0$. Equivalently $mgD - mdA \sim P_1 + \cdots + P_g - (Q_1 + \cdots + Q_g)$ for some $P_1, P_2, \ldots, P_g \in U$. Hence $mgD \sim P_1 + \cdots + P_g + (md - 1)(Q_1 + \cdots + Q_g)$, completing the proof of ii) $\Rightarrow$ i).

i) $\Rightarrow$ ii) *We show that if either $rr(\Gamma) > 1$ or κ is not algebraic over a finite field then there exist valued fields (K, v) with value group Γ and residue field κ not satisfying the Local–Skolem property. Moreover setting $p = \mathrm{char}\,\kappa$, if $p > 0$ there exist such K having either characteristic 0 or p.*

Let $(\gamma_i)_{i \in I} \subset \Gamma$ be a positive basis of the -vector space $\Gamma \otimes$ and set $\Gamma_0 = \sum \gamma_i$. Then Γ/Γ_0 is a torsion group. If $p = 0$ set $K_0 = \kappa$. If $p > 0$ we let K_0 be either κ or a minimal subfield of $\mathrm{Quot}\, W(\kappa)$ such that $K_0 v_0 = \kappa$, where $W(\kappa)$ is the Witt ring and v_0 is its canonical valuation. If $K_0 = \kappa$ we let v_0 denote the trivial valuation on K_0. We set $\kappa(\Gamma) = K_0(t_i)_{i \in I}$, t_i indeterminates, if $K_0 = \kappa$ and $\kappa(\Gamma) = K_0(t_i)_{i \in I \setminus \{i_0\}}$ for a fixed i_0 otherwise. When there is no confusion we will write K for $\kappa(\Gamma)$.

We define $v\colon K \longrightarrow \Gamma$ by

$$v(\sum_{\boldsymbol{n}} a_{\boldsymbol{n}} t^{\boldsymbol{n}}) = \begin{cases} \inf_{a_{\boldsymbol{n}} \neq 0}(v_0(a_{\boldsymbol{n}})\gamma_{i_0} + \sum\limits_{j \neq i_0} n_j \gamma_j) & \text{if } K_0 \neq \kappa \\ \inf_{a_{\boldsymbol{n}} \neq 0}(\sum\limits_{j} n_j \gamma_j) & \text{if } K_0 = \kappa \end{cases}$$

where $a_{\boldsymbol{n}} \in K_0$ and $t^{\boldsymbol{n}} = t_{i_0}^{n_0} \ldots t_{i_s}^{n_s}$. Clearly $v(K) = \Gamma_0$ and the residue field is $Kv = \kappa$. Using Zorn's lemma and noting that Γ/Γ_0 is torsion one finds totally ramified extensions K' of K so that the unique prolongation v' of v to K' has the value group $v'(K') = \Gamma$ and $K'v' = \kappa$. By 2.1) it follows that (K, v) has the Local–Skolem property iff (K', v') does. Hence to show that (K', v') does not have the Local–Skolem property it suffices to show that the same is true for (K, v). Thus we can suppose $\Gamma = \Gamma_0$, i.e. Γ is free abelian on the system $(\gamma_i)_i$.

First case: $rr(\Gamma) > 1$. We first show that Γ has rank equal to 1 as an ordered group. Indeed if $\Delta \subset \Gamma$ is a saturated isolated subgroup then considering the exact sequence $0 \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow \Gamma/\Delta = \overline{\Gamma} \longrightarrow 0$ we can choose the system $(\gamma_i)_i$ with $(\gamma_i)_i = (\gamma_\delta)_\delta \cup (\gamma_{\overline{\imath}})_{\overline{\imath}}$ with $(\gamma_\delta)_\delta \subset \Delta$ and the $(\gamma_{\overline{\imath}})_{\overline{\imath}}$ projecting onto a free set of generators of $\overline{\Gamma}$. Further if $K_0 \neq \kappa$ we prescribe $\gamma_{i_0} \in \Delta$. Setting $\kappa_1 := \kappa(\Delta)$ it follows that $\kappa_1(\overline{\Gamma}) = \kappa(\Gamma)$ and by construction κ_1 is not henselian. By lemma 1.1 this contradicts the Local–Skolem property unless the rank is 1.

It remains to analyse the case of rank 1. Now take $\gamma_{i_1} \neq \gamma_{i_0}$ such that the rational rank $rr(\gamma_{i_1} + \gamma_{i_0}) = 2$. Next, setting $t = t_{i_1}$, consider the projective elliptic curve over K

$$E: \begin{cases} y^2 z = x(x-z)(x-zt) & \text{if } \mathrm{char}(K) \neq 2 \\ t(x^3+y^3+z^3) - 3xyz = 0 & \text{if } \mathrm{char}(K) = 2. \end{cases}$$

Then in each case by Igusa, [I], $E(K)$ is finite. See also Shioda, [S], remark 5.6. Also with respect to v, E has split multiplicative reduction and non-integral j invariant in both cases, and so is Tate curve over the completion.

Let $_K$ be the completion of $(\tilde{K}, \tilde{v})$. Then $E(\tilde{K})$ is dense in $E(_K)$ and the opens of the form $E_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ are dense in $E_{\boldsymbol{x}}(\mathcal{O}_{_K})$ for $\boldsymbol{x} \in K(E)$. Moreover the sets $E_{\boldsymbol{x}}(\mathcal{O}_{_K})$ represent a basis for the open $\mathrm{G}_{\hat{K}}$-invariant subsets of $E(_K)$, $\hat{K}$ denoting the completion of K. Further as a $_K$-analytic group, $E(_K)$ has the structure

$$0 \longrightarrow \langle q \rangle \longrightarrow \overset{\times}{_K} \xrightarrow{\pi} E(_K) \longrightarrow 0$$

where q is the Tate-period, $\tilde{v}(q) > 0$, and π is an analytic group homomorphism defined over $\hat{K}$, (see [R1], page 50; [F–vdP], page 141.).

Let $U_{_K}$ denote the kernel (units) of the group homomorphism $\tilde{v}\colon \overset{\times}{_K} \longrightarrow \Gamma\otimes$ and take $q' \in K$ such that $rr(v(q) + v(q')) = 2$. Then $U = q' U_{_K}$ is $\mathrm{G}_{\hat{K}}$-invariant and open, hence its image in $E(_K)$ is a $\mathrm{G}_{\hat{K}}$-invariant open. In particular $\pi(U)$ contains some $E_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}}) \neq \phi$. On the other hand $\tilde{v}(u) = \tilde{v}(q')$ for all $u \in U$ and therefore $\big(U...U\langle q\rangle\big)/\langle q \rangle \cong \big(\pi(U)...\pi(U)\big)$ (n-fold product) does not contain torsion elements. As $E(K)$ is torsion it follows that $\big(\pi(U)...\pi(U)\big) \cap E(K) = \phi$, and so by theorem 1.2 (iii), (K, v) does'nt satisfy the Local–Skolem property.

Second case: $rr(\Gamma) = 1$ and κ not algebraic over a finite field.

First by the remarks above we may assume Γ is discrete. If $\kappa \neq K_0$ then $\{\gamma_i\} = \{\gamma_{i_0}\}$, hence $K = K_0$ and $v = v_0$. If $\kappa = K_0$ then $K = \kappa(t)$ and v is the t-adic valuation. Now κ is not algebraic over a finite field. Take K' the maximal inert extension of K and v' the unique prolongation of v to K'. Then $v'K' = vK \cong$ and $K'v' = \tilde{\kappa}$.

By the permanence theorem it is sufficient to show that K' does not satisfy the Local–Skolem property with respect to of v'. Hence replacing K' by K we can assume that Kv is algebraically closed, i.e. $\kappa = \tilde{\kappa}$.

Let X/K be a geometrically integral smooth projective curve of genus $g > 0$, such that the K_0 trace of J_X is trivial so that by the Lang–Neron theorem $J_X(K)$ is finitely generated. We also suppose that X has good reduction over v. Then $X_{\tilde{K}}$ has good reduction over $\tilde{v}$ and $X_{\tilde{K}}\tilde{v} = Xv$. Let $\boldsymbol{r}\colon \mathrm{Div}(X_{\tilde{K}}) \longrightarrow \mathrm{Div}(Xv)$ be the usual divisor reduction map and v, resp. ṽ, the constant reductions of the function fields of X and $X_{\tilde{K}}$ determining the good reductions.

We show that for every $p \in Xv$ there exist rational functions $\boldsymbol{x} = (x_1, x_2)$ in $K(X)$ such that $\boldsymbol{r}\big(X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})\big) = \{p\}$.

Using [R3] let $P \in X(\tilde{K})$ be chosen such that $\boldsymbol{r}(P) = p$, and B be a K-rational divisor of $X_{\tilde{K}}$ such that $\mathrm{supp}(B) = \{P^\sigma : \sigma \in G_K\}$. We next choose $f \in K(X)$ which is an element with the uniqueness property for v and $(f)_0 = nB$ for some large $n \in$. The function f is also an element with the uniqueness property for ṽ, and by [R3] for each $Q \in \mathrm{Div}(X_{\tilde{K}})$, $f(Q)\tilde{v} = f\tilde{\mathrm{v}}(\boldsymbol{r}(Q))$. Setting $x_1 = f/a$, for $a \in K$ with $v(a) > 0$, it follows that

$$\boldsymbol{r}(X_{x_1}(\mathcal{O}_{\tilde{v}})) = \{\boldsymbol{r}(P^\sigma) : \sigma \in G_K\}.$$

We next choose x_2 as follows: Let C be a K-rational divisor of $X_{\tilde{K}}$ such that $\boldsymbol{r}(C)$ has support disjoint from the support of $\boldsymbol{r}(X_{x_1}(\mathcal{O}_{\tilde{v}}))$. Now, using Riemann–Roch, we choose a function $\bar{t} \in \mathcal{L}_{Kv}(n\boldsymbol{r}(C))$, for n large, such that $(\bar{t})_\infty = n\boldsymbol{r}(C)$, $\bar{t}(p) = 0$ and $\bar{t}(q) \neq 0$ for $q \in \boldsymbol{r}(X_{x_1}(\mathcal{O}_{\tilde{v}})) \setminus \{p\}$. Let $t \in \mathcal{L}_K(nC) \subset K(X)$ be a lifting of $\bar{t}$. Then by comparing the degree of t in $K(X)$ and $\bar{t}$ in $Kv(Xv)$ we see that t is an element with the uniqueness property for v. Let $x_2 = t/a$ with a as before, then $\boldsymbol{r}\big(X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})\big) = \{p\}$ as asserted.

Now since $J_X(K)$ is finitely generated, its image by the specialisation map $J_X \longrightarrow J_{Xv}$ is a finitely generated subgroup of $J_{Xv}(\kappa)$. As κ is not algebraic over a finite field $J_{Xv}(\kappa)$ has infinite rational rank. Hence there exists $p \in Xv$ such that the subgroup $\langle p \rangle$ generated by p in $J_{Xv}(\kappa)$ has trivial intersection with $\boldsymbol{r}\big(J_X(K)\big)$. Let P and $\boldsymbol{x} = (x_1, x_2)$ be chosen as in the paragraph above with $\boldsymbol{r}(P) = p$ and $\boldsymbol{r}\big(X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})\big) = \{p\}$. Then the subgroup $\langle \boldsymbol{r}(P) \rangle$ in $J_{Xv}(\kappa)$ has trivial intersection with $\boldsymbol{r}\big(J_X(K)\big)$ and moreover, for any $P_1, \ldots, P_n \in X_{\boldsymbol{x}}(\mathcal{O}_{\tilde{v}})$ and K-rational divisor D, it follows $P_1 + \cdots + P_n \not\sim D$. If this were not the case then $\boldsymbol{r}(P_1) + \cdots + \boldsymbol{r}(P_n) = n\,\boldsymbol{r}(P) \sim \boldsymbol{r}(D)$, a contradiction. By theorem 1.2 (iii) it follows the Local–Skolem property is not satisfied in (K, v). This completes the proof of the theorem.

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