# Lifting the curve $(X^p - X)(Y^p - Y) = 1$ and its automorphism group

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## 1 Introduction

In [O], F. Oort remarked that the curve $C : (x^p - x)(y^p - y) = 1$ has genus $g = (p-1)^2$ and automorphism group $G$ whose order is $2(p-1)p^2$ if $p > 2$ (for $p = 2$, $G$ is the automorphism group of the elliptic curve $(x^2 - x)(y^2 - y) = 1$). One has $|G| > 84(g-1)$ iff $p \geq 41$, contradicting Hurwitz bound in characteristic 0 for $p \geq 41$. In particular this is a numerical obstruction for the lifting of the pair $(C, G)$ in characteristic 0; F. Oort asked what about for $p < 41$.

In the sequel we show that the liftability is possible iff $p = 2, 3$. The proof illustrates a local-global principle ([G-M1,2], [B-M]).

## 2 Notations

Let $k$ be an algebraically closed field with characteristic $p > 0$. We denote by $\theta$ a primitive $(p-1)$-th root of unity. Let $u := x^p - x$; we consider the following tower of extensions: $k(u) \subset k(x) \subset k(x,y)$ with $(x^p - x)(y^p - y) = 1$. Let $\tau_1, \tau_2, \iota, \sigma \in G$ such that

$\tau_1(x) = x + 1, \tau_1(y) = y,$

$\tau_2(x) = x, \tau_2(y) = y + 1,$

$\iota(x) = y, \iota(y) = x,$

$\sigma(x) = \theta x, \sigma(y) = \theta^{-1} y.$

Then $G =< \tau_1, \tau_2, \iota, \sigma >$ and we have the relation

$(*) \quad \sigma^{-1}\tau_1\sigma = \tau_1^{\theta}.$

We shall denote by $H$ the subgroup $< \tau_1, \tau_2 >$.

We denote by $a$ the place in $k(u)$ corresponding to $u = \infty$ then its fiber in $k(x)$ (resp. $k(x,y)$) is $b$ corresponding to $x = \infty$ resp. $c_i, i = 0, ..., p-1$ corresponding to $(x = \infty, y = i)$.

The automorphism $\sigma$ fixes $c_0$ and induces a cycle on the $c_i, 1 \leq i \leq p-1$.

## 3 Proof

**Lemma 3.1.** *We assume that $(\mathcal{C}, G)$ is a lifting of $(C, G)$ over $(R, v)$ a complete DVR with residue field $k$ which contains $p$-th roots of unity, then necessarily $p = 2, 3$.*

*Proof.* : Denote by $\zeta$ a primitive $p$-th root of unity and define $\lambda = \zeta - 1$. Then $v(p) = (p-1)v(\lambda)$.

Let *Ram* resp. *Br* be the ramification (resp. branch) locus in the cover $(\mathcal{C}, H)$ (which is a lifting of $(C, H)$).

Then $Br = \{A, \tilde{A}\}$ as the conductor in the Artin-Schreier cover $x^p - x = u$ is 1 (see [G-M1]) and $Ram = \{(C_i, \tilde{C}_i), i = 0, ..., p-1\}$ where $C_i, \tilde{C}_i$ specialize in the same place $c_i$.

As $\sigma$ fixes $c_0$, it follows that $\sigma$ induces a permutation on $C_0, \tilde{C}_0$; in particular the group $< \tau_1, \sigma^2 >$ lies in the inertia group at $C_0$ and so is cyclic as char. $K = 0$.

From relation $*$ it follows that $\theta^2 = 1$ and so $p - 1 | 2$; i.e. $p = 2, 3$. □

**Lemma 3.2.** *Let $G_1$ be the subgroup of $G$ generated by $\tau_1, \tau_2, \iota, \sigma^s$ where $s = 1$ if $p = 2$ and $\frac{p-1}{2}$ if $p > 2$. Then the pair $(C, G_1)$ is liftable over $R = \mathbb{Z}_p[\zeta, (1-\lambda^{2p})^{1/p}]$. In particular if $p = 2, 3$ one has $G_1 = G$; so then $(C, G)$ is liftable.*

*Proof.* : We use the deformation from Artin-Schreier to Kummer (see [G-M1], [O-S-S]). Let $\mathcal{C}$ the $R$-curve which is the normalisation of the curve given by the equation

$(**)$ $\quad ((\lambda X+1)^p - 1)((\lambda Y+1)^p - 1) = \lambda^{2p}$.

We first show that $\mathcal{C}$ is a smooth $R$-curve. The generic fiber is birationnal to the $p$-cyclic cover of the projective line defined by the equation:

$(***) U^p = \frac{V^p - (1-\lambda^{2p})}{V^p - 1}$

which is totally ramified at $2p$ points and so has genus $(p-1)^2$; this shows the smoothness (see [G-M1], [O-S-S])

Let $\alpha$ such that $\alpha^p = 1 - \lambda^{2p}$.

Let $\tilde{\tau}_i, i = 1, 2$ defined by

$\tilde{\tau}_1(X) = \zeta X + 1, \tilde{\tau}_2(Y) = Y$,

$\tilde{\tau}_2(X) = X, \tilde{\tau}_2(Y) = \zeta Y + 1$,

$\tilde{\iota}(X) = Y$, $\tilde{\iota}(Y) = X$,

$\gamma(X) = \frac{\frac{\alpha - 1}{\lambda} - X}{1 + \lambda X}$ $\gamma(Y) = \frac{\frac{\alpha - 1}{\lambda} - Y}{1 + \lambda Y}$

We remark that $v(\alpha - 1) = (p+1)v(\lambda)$, then $\gamma$ is an $R$-automorphism of $\mathcal{C}$ and $\gamma^2 = id$.

Moreover $\gamma(X) = -X \mod \pi$ ($\pi$ is a uniformising element in $R$) and $\gamma(Y) = -Y \mod \pi$. Let $p > 2$, then $< \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\iota}, \gamma >$ is an automorphism group of $\mathcal{C}$ which is isomorphic to $G'$.

If $p = 2$ the same results holds with $< \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\iota} >$.

□

**Remark 3.3.** *In general metacyclic actions induce serious local obstructions; note that in [G-M2] Prop. 2.2.2. we consider $G = \langle \sigma, \tau \rangle$, with $\mathrm{o}(\sigma) = p$, $\mathrm{o}(\tau) = p-1$ and $\tau\sigma\tau^{-1} = \sigma^{h^{-1}}$ ($h$ is a primitive $(p-1)$-th root of 1 modulo $p$) and prove that if $p > 2$, there is a $G$-cover $k[\![z]\!]/k[\![z]\!]^G$, which can be lifted to a $G$-cover $R[\![Z]\!]/R[\![Z]\!]^G$, for $R = \mathbb{Z}_p[\zeta]$ and $\zeta$ a primitive $p$-th root of* 1.

# References


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