L^2-constraint minimizers for the planar nonlinear Schrödinger-Newton system and the nonlinear Schrödinger-Poisson system with harmonic potential

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Abstract

In this paper, we study the existence of L^2-constraint minimizers for the planar nonlinear Schrödinger-Newton system and the planar nonlinear Schrödinger-Poisson system with a harmonic trapping potential. Especially, we are interested in the correspondence between minimizers and ground state solutions. For the planar nonlinear Schrödinger-Newton system, we are able to completely give this correspondence. We also investigate the asymptotic behavior of minimizers when the harmonic trapping potential vanishes.

Key words: planar Schrödinger-Newton system, planar Schrödinger-Poisson system, L^2-constraint minimization problem, ground state solution, asymptotic behavior.

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1 Introduction

In this paper, we consider the following planar elliptic system:

$$\begin{cases} -\Delta u + \omega u + \kappa |x|^2 u \pm e\phi u = |u|^{p-1}u \\ -\Delta \phi = \frac{e}{2}|u|^2 \end{cases} \quad \text{in } \mathbb{R}^2, \qquad (P_\pm)$$

where $e > 0$, $\kappa \in (0,1]$, $p > 1$ and $\omega \in \mathbb{R}$. Although both (P_-) and (P_+) are sometimes denominated by the same name, to make the difference clearer, we call (P_-) the *nonlinear Schrödinger-Newton system* (*NLSN*) and

(P_+) the *nonlinear Schrödinger-Poisson system* (*NLSP*) respectively. We are interested in the existence of L^2-constraint minimizers associated with $(P_\pm)$, which plays an important role in the study of the stability of standing waves. In this case, the constant ω appears as a Lagrange multiplier.

It is known that the Schrödinger-Newton system and the Schrödinger-Poisson system appear in various fields of physics, such as black holes in gravitation, quantum mechanics, plasma physics and semi-conductor theory; see e.g. [21, 24, 30, 32, 33, 37]. The constant e represents the strength of the interaction and $(P_\pm)$ reduces to the nonlinear Schrödinger equation if $e = 0$.

In the last two decades, special attention has been paid to systems NLSN and NLSP in $\mathbb{R}^3$. We refer to [3, 5, 6, 9, 22, 29, 36, 37, 39, 41, 42] and references therein. A first study using numerical analysis on the 2D Schrödinger-Newton system was made in [40]. After a pioneer work [13], the planar NLSN and NLSP has been widely studied in [1, 2, 8, 10, 11, 12, 19, 20, 28, 38]. Especially, NLSN and NLSP with the harmonic potential $\kappa|x|^2$ or with general unbounded potentials were considered in [29, 42] for 3D case and in [1, 2, 20, 38] for 2D case respectively. It is important to mention that solutions of 2D NLSN and NLSP can be obtained as a reduction of those of the 3D problem, which is referred as *adiabatic approximation*; see [7, 34]. When carrying out this process, the presence of the harmonic potential plays a fundamental role. On one hand, from this point of view, it is not unnatural to consider $(P_\pm)$ which includes $\kappa|x|^2$. This strong trapping potential makes easier the existence of L^2-constraint minimizers of $(P_\pm)$. On the other hand, from a scaling point of view, fibering maps associated with L^2-invariant scaling becomes more complex to use. Therefore, the presence of the harmonic potential brings new difficulties when one wants to determine qualitative properties of minimizers. We also refer to [31] for the solvability of Cauchy problem of the corresponding time-evolution NLSP in 2D.

As most relevant to this paper, the existence of solutions with prescribed L^2-norm (*normalized solutions*) of $(P_\pm)$ without the harmonic potential $\kappa|x|^2$ was addressed in [12]. Especially, in the case $\kappa = 0$ and $1 < p < 3$ (L^2-subcritical), it was shown that (P_-) has a L^2-constraint minimizer. Furthermore, regarding (P_+) with $\kappa = 0$, the existence of two normalized solutions was obtained. (See also [38] for the case $\kappa = 1$.) However, qualitative properties of minimizers are not mentioned in [12, 38], and in particular the relationship between minimizers and ground state solutions is unknown for $(P_\pm)$. The correspondence between minimizers and ground state solutions is expected to be useful to carry out further investigations on the stability of standing waves.

To state our main results, let us introduce the variational formulation of

($P_\pm$). In the 2D case, one can solve the Poisson equation as follows:

$$\phi(x) = \frac{e}{2}(-\Delta)^{-1}|u(x)|^2 = -\frac{e}{4\pi}\int_{\mathbb{R}^2} \log|x-y||u(y)|^2\,dy =: eS(u)(x).$$

Then ($P_\pm$) can be rewritten as the following nonlocal elliptic equation:

$$-\Delta u + \omega u + \kappa|x|^2 u \pm e^2 S(u)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^2. \tag{1.1}$$

We define the function spaces $\mathcal{X}_0$ and $\mathcal{X}$ by

$$\mathcal{X}_0 := \left\{ u \in H^1(\mathbb{R}^2, \mathbb{C}) \mid \int_{\mathbb{R}^2} \log(1+|x|)|u|^2\,dx < +\infty \right\},$$
$$\mathcal{X} := \left\{ u \in H^1(\mathbb{R}^2, \mathbb{C}) \mid \int_{\mathbb{R}^2} |x|^2|u|^2\,dx < +\infty \right\}.$$

It is clear that $\mathcal{X} \subset \mathcal{X}_0$. Let us also introduce the functionals $V_0 : \mathcal{X}_0 \to \mathbb{R}$ and $V : \mathcal{X} \to \mathbb{R}$ defined by

$$V_0(u) := \int_{\mathbb{R}^2} \log(1+|x|)|u|^2\,dx \quad \text{and} \quad V(u) := \int_{\mathbb{R}^2} |x|^2|u|^2\,dx.$$

The energy functional associated with ($P_\pm$) is given by

$$E_{\kappa,\pm}(u) := E_{0,\pm}(u) + \frac{\kappa}{2}V(u), \quad E_{0,\pm}(u) := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^2} |u|^{p+1}\,dx \pm e^2 A(u),$$

$$A(u) = \frac{1}{4}\int_{\mathbb{R}^2} S(u)|u|^2\,dx = -\frac{1}{16\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y||u(x)|^2|u(y)|^2\,dx\,dy.$$

We can see that $E_{0,\pm}$ is well-defined in $\mathcal{X}_0$, while $E_{\kappa,\pm}$ is well-defined only on $\mathcal{X}$. It is important to point out that $E_{0,\pm}$ is translation invariant, although a natural norm $\|u\|_{\mathcal{X}_0}^2 := \|u\|_{H^1}^2 + V_0(u)$ on $\mathcal{X}_0$ is not invariant under translation.

Let us consider the following minimization problem:

$$c_{\kappa,\pm}(\mu) := \inf_{u \in \mathcal{B}_\mu \cap \mathcal{X}} E_{\kappa,\pm}(u),$$

where $\mu > 0$ and $\mathcal{B}_\mu := \left\{ u \in H^1(\mathbb{R}^2, \mathbb{C}) \mid \|u\|_{L^2(\mathbb{R}^2)}^2 = \mu \right\}$. For simplicity, except when one wants to emphasize the dependence of κ, we denote $E_{\kappa,\pm}(u)$ and $c_{\kappa,\pm}(\mu)$ by $E_\pm(u)$ and $c_\pm(\mu)$ respectively. We also define the action functional corresponding to (1.1) by

$$I_\pm(u) := E_\pm(u) + \frac{\omega}{2}\int_{\mathbb{R}^2} |u|^2\,dx.$$

A solution w of (1.1) is said to be a *ground state solution*, if w satisfies

$$I_{\pm}(w) = \inf\{I_{\pm}(u) \mid u \in \mathcal{X} \text{ and } u \text{ is a nontrivial solution of (1.1)}\}.$$

First we consider the nonlocal elliptic problem:

$$-\Delta u + \omega u + \kappa |x|^2 u - e^2 S(u) u = |u|^{p-1} u \quad \text{in } \mathbb{R}^2, \tag{1.2}$$

which corresponds to the nonlinear Schrödinger-Newton system. Our main results are summarized as follows.

Theorem 1.1 (Existence of minimizers and positivity of Lagrange multiplier)**.**

(i) *Suppose that* $1 < p < 3$ *and* $\kappa \in (0, 1]$. *Then for any* $\mu > 0$, $c_-(\mu)$ *admits a minimizer* $u_- \in \mathcal{B}_\mu \cap \mathcal{X}$.

(ii) *Suppose that* $2 \le p < 3$ *and* $\kappa \in (0, 1]$. *Then there exists* $\mu_- > 0$ *such that the Lagrange multiplier* ω_- *associated with* $c_-(\mu)$ *is positive.*

Theorem 1.2 (Correspondence of minimizers and ground state solutions)**.** *Suppose that* $2 \le p < 3$, $\kappa \in (0, 1]$ *and assume that* $\mu > \mu_-$.

(i) *Any minimizer* $u_- \in \mathcal{B}_\mu \cap \mathcal{X}$ *for* $c_-(\mu)$ *is a ground state solution of* (1.2) *with* $\omega = \omega_-$.

(ii) *Let us denote by* $\Omega_-(\mu)$ *the set of Lagrange multipliers associated with minimizers for* $c_-(\mu)$, *that is,*

$$\Omega_-(\mu) := \Big\{\omega_- > 0 \mid \omega_- \textit{ is a Lagrange multiplier corresponding to } c_-(\mu)\Big\}.$$

Any ground state solution w_μ *of* (1.2) *with* $\omega = \omega_- \in \Omega_-(\mu)$ *is a minimizer for* $c_-(\mu)$.

It is worth mentioning that Theorem 1.1 and Theorem 1.2 hold even for $\kappa = 0$. The correspondence between minimizers and ground state solutions, provided for example by Theorem 1.2, are of great interest; see [15, 16, 17, 18, 23]. However, up to authors' knowledge, there is no such result for the planar nonlinear Schrödinger-Newton system. Therefore, we emphasize that Theorem 1.2 is new even for the case $\kappa = 0$. As we will see later, in the process of the proof of Theorem 1.2, the positivity of the Lagrange multiplier ω_- established in Theorem 1.1 (ii) plays an essential role. Moreover, the positivity of ω_- can be shown by combining the Nehari identity, the Pohozaev identity and the fact $c_-(\mu) < 0$ for $\mu > \mu_-$. For the 3D Shrödinger-Poisson system, as done in [5, 9, 15, 39], the negativity of the minimum of the energy can be obtained by scaling and continuity arguments. In the 2D case, since the nonlocal term A has a bad scaling property, the negativity of $c_-(\mu)$ cannot be derived by such simple arguments. Instead, we construct a suitable test function with appropriate parameters.

Next we make more precise the asymptotic behavior of minimizers as $\kappa \to 0$.

Theorem 1.3 (Asymptotic behavior of minimizers as $\kappa \to 0$)**.**
Suppose that $2 \le p < 3$, $\kappa \in (0,1]$ *and assume that* $\mu > \mu_-$. *Let* $u_{\kappa,-} \in \mathcal{B}_\mu \cap \mathcal{X}$ *be a minimizer for* $c_{\kappa,-}(\mu)$. *Then there exist* $\kappa_j \to 0$, $\{y_j\} \subset \mathbb{R}^2$ *and* $u_0 \in \mathcal{B}_\mu \cap \mathcal{X}_0$ *such that*

$$u_{\kappa_j}(\cdot - y_j) \to u_0 \ \textit{in } \mathcal{X}_0.$$

Moreover u_0 *is a minimizer for* $c_{0,-}(\mu)$.

Note that, since $\{u_{\kappa_j}\}_{j\in\mathbb{N}}$ can be shown to be bounded in $\mathcal{X}_0$, it converges weakly in $\mathcal{X}_0$. Moreover, standard arguments provide the convergence of the minimum energy $c_{\kappa,-}(\mu)$ to $c_{0,-}(\mu)$ as $\kappa \to 0$, as well as the strong convergence $u_{\kappa_j} \to u_0$ in H^1. However, it is not trivial to obtain the strong convergence of $\{u_{\kappa_j}\}_{j\in\mathbb{N}}$ in the stronger topology $\mathcal{X}_0$. It requires a mutual estimate between the nonlocal term A and the functional V_0. It should be mentioned that we are able to obtain a similar result for 3D problem, leading to the strong convergence but only in H^1. In other words, the strong convergence in $\mathcal{X}_0$ is a 2D-specific result.

Next we introduce our main result for the nonlocal elliptic problem:

$$-\Delta u + \omega u + \kappa|x|^2 u + e^2 S(u)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^2, \tag{1.3}$$

which corresponds to the nonlinear Schrödinger-Poisson system. In this case, we obtain the following result.

Theorem 1.4.

(i) *Suppose that* $1 < p < 3$ *and* $\kappa \in (0,1]$. *Then for any* $\mu > 0$, $c_+(\mu)$ *admits a minimizer* $u_+ \in \mathcal{B}_\mu \cap \mathcal{X}$.

(ii) *Suppose that* $1 < p < 3$ *and let* $\mu > 0$ *be given. Then there exists* $\kappa_+ = \kappa_+(e,\mu) \in (0,1)$ *such that for* $0 < \kappa < \kappa_+$,

$$\frac{e^2\mu^2}{8\pi}\log\kappa \le c_+(\mu) \le \frac{e^2\mu^2}{64\pi}\log\kappa.$$

Theorem 1.4 (ii) means that we cannot expect a similar result as in Theorem 1.3 for minimizers of (1.3). Moreover since we cannot determine the sign of the Lagrange multiplier, it is not possible to prove a similar result as in Theorem 1.2. See Remark 4.9 below for more details.

We also mention that the restriction of $\kappa \le 1$ in Theorems 1.1 and 1.2 is not essential, while so it is in Theorem 1.4.

This paper is organized as follows. In Section 2, we introduce basic properties of the function space $\mathcal{X}_0$ and the nonlocal term A. Although most of them are already known, we need to investigate the dependence of estimates on $\mathcal{X}$ with respect to κ precisely. In Section 3, we study the nonlinear Schrödinger-Newton system. First, the existence of L^2-constraint minimizers and their properties are established. Next, we consider the link between minimizers and ground state solutions of (1.2). The asymptotic behavior of minimizers as $\kappa \to 0$ will be studied in the last subsection. Finally, we consider the nonlinear Schrödinger-Poisson system and prove Theorem 1.4 in Section 4.

Hereafter in this paper, unless otherwise specified, we write $\|u\|_{L^p(\mathbb{R}^2)} = \|u\|_p$. We also denote by $B(y, R)$ the 2-dimensional ball of radius R centered at a point $y \in \mathbb{R}^2$.

2 Preliminaries

In this section, we collect some basic properties of the function space $\mathcal{X}_0$ and the nonlocal term $A(u)$. First we begin with the embedding theorem of $\mathcal{X}_0$. Since the weight function $\log(1 + |x|)$ is unbounded at infinity, the following compact embedding lemma holds. (See e.g. [4, 35] for the proof.)

Lemma 2.1 (Compact embedding)**.** *$\mathcal{X}_0 \hookrightarrow L^q(\mathbb{R}^2)$ is compact for all $q \in [2, \infty)$.*

Next we prepare a scaling property of the nonlocal term $A(u)$.

Lemma 2.2 (Scaling property)**.** *Let $u \in \mathcal{X}_0$, $\lambda > 0$, $(a, b) \in \mathbb{R}^2$ and define $u_\lambda(x) = \lambda^a u(\lambda^b x)$. Then it follows that*

$$S(u_\lambda)(x) = \lambda^{2a-2b}\Big(\frac{b\log\lambda}{4\pi}\|u\|_2^2 + S(u)(\lambda^b x)\Big),$$
$$A(u_\lambda) = \lambda^{4a-4b}\Big(\frac{b\log\lambda}{16\pi}\|u\|_2^4 + A(u)\Big).$$

Proof. Let us introduce the change of variable $z = \lambda^b y$. Then we can compute as follows.

$$\begin{aligned}
S(u_\lambda)(x) &= -\frac{\lambda^{2a}}{4\pi}\int_{\mathbb{R}^2} \log|x-y||u(\lambda^b y)|^2\,dy = -\frac{\lambda^{2a-2b}}{4\pi}\int_{\mathbb{R}^2}\log|x-\lambda^{-b}z||u(z)|^2\,dz \\
&= -\frac{\lambda^{2a-2b}}{4\pi}\int_{\mathbb{R}^2}\log|\lambda^{-b}(\lambda^b x - z)||u(z)|^2\,dz \\
&= -\frac{\lambda^{2a-2b}}{4\pi}\int_{\mathbb{R}^2}\big\{-b\log\lambda + \log|\lambda^b x - z|\big\}|u(z)|^2\,dz \\
&= \lambda^{2a-2b}\Big(\frac{b\log\lambda}{4\pi}\|u\|_2^2 + S(u)(\lambda^b x)\Big).
\end{aligned}$$

This also yields that

$$\begin{aligned}
A(u_\lambda) = \frac{1}{4}\int_{\mathbb{R}^2} S(u_\lambda)|u_\lambda|^2\,dx &= \frac{\lambda^{4a-2b}}{4}\int_{\mathbb{R}^2}\left\{\frac{b\log\lambda}{4\pi}\|u\|_2^2|u(\lambda^b x)|^2 + S(u)(\lambda^b x)|u(\lambda^b x)|^2\right\}dx \\
&= \frac{\lambda^{4a-4b}}{4}\int_{\mathbb{R}^2}\left\{\frac{b\log\lambda}{4\pi}\|u\|_2^2|u(y)|^2 + S(u)(y)|u(y)|^2\right\}dy \\
&= \lambda^{4a-4b}\Big(\frac{b\log\lambda}{16\pi}\|u\|_2^4 + A(u)\Big),
\end{aligned}$$

from which we conclude. $\square$

Now we state several estimates for nonlocal terms.

Lemma 2.3 (Estimates for nonlocal terms)**.**

(i) *For any $u \in \mathcal{X}_0$, it follows that*

$$-A_1(u) \le A(u) \le A_2(u),$$

where

$$\begin{aligned}
A_1(u) &:= \frac{1}{16\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(1+|x-y|)|u(x)|^2|u(y)|^2\,dx\,dy, \\
A_2(u) &:= \frac{1}{16\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log\left(1+\frac{1}{|x-y|}\right)|u(x)|^2|u(y)|^2\,dx\,dy.
\end{aligned}$$

(ii) *For any $u \in \mathcal{X}_0$, it holds that*

$$\begin{aligned}
&0 \le A_1(u) \le \frac{1}{8\pi}\|u\|_2^2 V_0(u), \\
&0 \le A_2(u) \le C\|u\|_{\frac{8}{3}}^4 \le C\|\nabla u\|_2\|u\|_2^3
\end{aligned}$$

for some $C > 0$.

(iii) *Let* $C_\kappa := -\frac{\log \kappa}{2} + \kappa^{\frac{1}{2}} - 1$ *for* $\kappa \in (0,1]$*. For any* $u \in \mathcal{X}$*, it follows that*

$$0 \le A_1(u) \le \frac{1}{8\pi}\|u\|_2^2 \left(C_\kappa \|u\|_2^2 + \kappa^{\frac{1}{2}} \|u\|_2 V(u)^{\frac{1}{2}} \right).$$

(iv) A_1 *is weakly lower semi-continuous on* $\mathcal{X}_0$ *and* A_2 *is continuous on* $L^{\frac{8}{3}}(\mathbb{R}^2)$*. Moreover, the functional* A *is of the class* C^1 *on* $\mathcal{X}_0$*.*

Proof. (i) First we observe that for any $(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$\begin{aligned} \log|x-y| &= \log(1+|x-y|) - \log(1+|x-y|) + \log|x-y| \\ &= \log(1+|x-y|) - \log\left(1 + \frac{1}{|x-y|}\right). \end{aligned}$$

This yields that

$$\begin{aligned} A(u) = &-\frac{1}{16\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|)|u(x)|^2|u(y)|^2\,dx\,dy \\ &+ \frac{1}{16\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right)|u(x)|^2|u(y)|^2\,dx\,dy \end{aligned}$$

and hence $A(u) = -A_1(u) + A_2(u)$. Since $\log(1+|x-y|) \ge 0$ and $\log\left(1+\frac{1}{|x-y|}\right) \ge 0$ on $\mathbb{R}^2 \times \mathbb{R}^2$, we obtain (i).

(ii) By the triangular inequality, it follows that

$$\log(1+|x-y|) \le \log(1+|x|+|y|) \le \log(1+|x|) + \log(1+|y|).$$

As a consequence, one has

$$\begin{aligned} &A_1(u) = \frac{1}{16\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|)|u(x)|^2|u(y)|^2\,dx\,dy \\ &\le \frac{1}{16\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x|)|u(x)|^2|u(y)|^2\,dx\,dy + \frac{1}{16\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|y|)|u(x)|^2|u(y)|^2\,dx\,dy \\ &= \frac{1}{8\pi}\|u\|_2^2 \int_{\mathbb{R}^2} \log(1+|x|)|u(x)|^2\,dx = \frac{1}{8\pi}\|u\|_2^2 V_0(u). \end{aligned}$$

Next by the inequality $\log\left(1+\frac{1}{s}\right) \le \frac{1}{s}$ for $s \ge 0$ and the Hardy-Littlewood Inequality [25], we find that

$$\begin{aligned} A_2(u) &= \frac{1}{16\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right)|u(x)|^2|u(y)|^2\,dx\,dy \\ &\le \frac{1}{16\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|u(x)|^2|u(y)|^2}{|x-y|}\,dx\,dy \\ &\le C\left\||u|^2\right\|_{\frac{4}{3}}^2 = C\|u\|_{\frac{8}{3}}^4. \end{aligned}$$

Then applying the interpolation inequality and the Gagliardo-Nirenberg inequality, we get the last estimate.

(iii) An elementary calculation shows that the function $f(s) := \kappa^{\frac{1}{2}} s - \log(1+s)$ takes its global minimum at $s = \kappa^{-\frac{1}{2}} - 1$ and $f(\kappa^{-\frac{1}{2}} - 1) = \frac{\log \kappa}{2} - \kappa^{\frac{1}{2}} + 1 = -C_\kappa < 0$. Then by the Cauchy-Schwarz inequality, one deduces that

$$\begin{aligned} V_0(u) &= \int_{\mathbb{R}^2} \log(1+|x|)|u|^2 \, dx \le \int_{\mathbb{R}^2} (C_\kappa + \kappa^{\frac{1}{2}}|x|)|u|^2 \, dx \\ &= C_\kappa \|u\|_2^2 + \kappa^{\frac{1}{2}} \int_{\mathbb{R}^2} |x||u|^2 \, dx \le C_\kappa \|u\|_2^2 + \kappa^{\frac{1}{2}} \|u\|_2 V(u)^{\frac{1}{2}}. \end{aligned}$$

Thus from (ii), we conclude.

(iv) We refer to [13, Lemma 2.2] for the proof. □

By Lemma 2.3 (iv), it follows that the functionals $E_\pm$ and $I_\pm$ are of the class C^1 on $\mathcal{X}$. Next we prepare the following compactness result.

Lemma 2.4. *Let $\{u_n\} \subset \mathcal{X}_0$ be a sequence satisfying*

$$\|u_n\|_2^2 = \mu \quad \text{and} \quad A_1(u_n) \le M \quad \text{for all } n \in \mathbb{N} \text{ and some } M > 0. \tag{2.1}$$

Then there exist a sequence $n_j \to \infty$, $\{y_j\} \subset \mathbb{R}^2$ and $u_0 \in L^2(\mathbb{R}^2, \mathbb{C})$ such that

$$u_{n_j}(\cdot - y_j) \to u_0 \text{ in } L^2(\mathbb{R}^2).$$

Especially it holds that $\|u_0\|_2^2 = \mu$.

Proof. Although the same result has been established in [12, Lemma 2.5 and Lemma 2.6], we give the proof for the sake of completeness.

Step 1: We claim that under the assumption (2.1), for all $j \ge 1$, there exist $n_j \to \infty$, $R_j > 0$ and $\{y_j\} \subset \mathbb{R}^2$ such that

$$\int_{B(y_j, R_j)} |u_{n_j}(x)|^2 \, dx > \mu - \frac{1}{j}. \tag{2.2}$$

Suppose by contradiction that (2.2) does not hold. Then there exists $\varepsilon_0 \in (0, \mu)$ such that

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B(y,R)} |u_n(x)|^2 \, dx \le \mu - \varepsilon_0 \quad \text{for all } R > 0.$$

Let us put

$$\varepsilon_n := \sup_{y\in\mathbb{R}^2} \int_{B(y,R)} |u_n(x)|^2 \, dx$$

Up to a subsequence, we may assume that $\lim_{n\to\infty} \varepsilon_n \le \mu - \varepsilon_0$. Then we find that

$$16\pi A_1(u_n) \ge \iint_{|x-y|\ge R} \log(1+|x-y|)|u_n(x)|^2|u_n(y)|^2\,dx\,dy$$
$$\ge \log(1+R)\left\{\mu^2 - \iint_{|x-y|\le R} |u_n(x)|^2|u_n(y)|^2\,dx\,dy\right\}.$$

Moreover one has

$$\iint_{|x-y|\le R} |u_n(x)|^2|u_n(y)|^2\,dx\,dy \le \int_{\mathbb{R}^2} |u_n(y)|^2 \left(\int_{B(y,R)} |u_n(x)|^2\,dx\right) dy \le \mu\varepsilon_n,$$

from which we deduce that

$$16\pi A_1(u_n) \ge \log(1+R)\mu(\mu-\varepsilon_n) \ge \frac{1}{2}\log(1+R)\mu\varepsilon_0 \quad \text{for sufficiently large } n \in \mathbb{N}.$$

Since $R > 0$ is arbitrary, letting $R \to \infty$, this contradicts (2.1) and hence (2.2) holds.

Step 2: Now we put $v_j(x) := u_{n_j}(x + y_j)$. From (2.2), it follows that

$$\int_{B(0,R_j)} |v_j(x)|^2\,dx > \mu - \frac{1}{j}. \tag{2.3}$$

Since $\|v_j\|_2^2 = \|u_j\|_2^2 = \mu$, passing to a subsequence, we may assume that $v_j \rightharpoonup u_0$ in $L^2(\mathbb{R}^2)$ for some $u_0 \in L^2(\mathbb{R}^2, \mathbb{C})$. From (2.3) and the concentration compactness principle [26, 27], we conclude that $v_j \to u_0$ in $L^2(\mathbb{R}^2)$. $\square$

Next by Lemma 2.3 (ii), we know that $A_1(u)$ is controlled by $V_0(u)$. Conversely, we have the following result.

Lemma 2.5. *Let $u \in \mathcal{X}_0$ and assume that there exist $\delta_0 > 0$, $R_0 > 0$ and a measurable subset $A_0 \subset B(0, R_0)$ such that*

$$|u(x)|^2 \ge \delta_0 \quad \textit{for all } x \in A_0. \tag{2.4}$$

Then it follows that

$$V_0(u) \le C\left(A_1(u) + \|u\|_2^2\right)$$

for some $C > 0$.

Proof. We argue as in [13, Lemma 2.1]. For $R > 0$, we first observe that

$$1 + |x-y| \ge 1 + \frac{|y|}{2} \ge \sqrt{|1+|y|} \quad \text{for } x \in B(0,R) \text{ and } y \in \mathbb{R}^2 \setminus B(0,2R).$$

Then from (2.4), one finds that

$$\begin{aligned}16\pi A_1(u) &\geq \int_{\mathbb{R}^2\setminus B(0,2R_0)}\int_{A_0}\log(1+|x-y|)|u(x)|^2|u(y)|^2\,dx\,dy\\ &\geq \frac{\delta_0|A_0|}{2}\int_{\mathbb{R}^2\setminus B(0,2R_0)}\log(1+|y|)|u(y)|^2\,dy\\ &\geq \frac{\delta_0|A_0|}{2}\left(V_0(u)-\int_{B(0,2R_0)}\log(1+|y|)|u(y)|^2\right)\\ &\geq \frac{\delta_0|A_0|}{2}\left(V_0(u)-\log(1+2R_0)\|u\|_2^2\right),\end{aligned}$$

from which we conclude. □

Next we define a symmetric bilinear form:

$$B_1(u,v) := \operatorname{Re}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(1+|x-y|)u(x)\overline{v(y)}\,dx\,dy.$$

Then the following properties hold.

Proposition 2.6.

(i) *Let $\{u_n\}$, $\{v_n\}$, $\{w_n\}\subset\mathcal{X}_0$ be bounded sequences and assume that $u_n\rightharpoonup u$ in $\mathcal{X}_0$. Then for any $\phi\in\mathcal{X}_0$, it holds that*

$$B_1\big(v_nw_n,\phi(u_n-u)\big)\to 0 \quad \text{as } n\to\infty.$$

(ii) *Let $\{u_n\}\subset L^2(\mathbb{R}^2)$ be a sequence which satisfies $u_n\to u$ a.e. in $\mathbb{R}^2$ for some $u\in L^2(\mathbb{R}^2,\mathbb{C})\setminus\{0\}$. Moreover let $\{v_n\}\subset L^2(\mathbb{R}^2)$ be a bounded sequence such that*

$$B_1(|u_n|^2,|v_n|^2)\to 0 \quad \text{and} \quad \|v_n\|_2\to 0 \quad \text{as } n\to\infty.$$

Then it holds that $V_0(v_n)\to 0$ as $n\to\infty$.

For the proof, we refer to [13, Lemma 2.1 and Lemma 2.6]. Finally we introduce the following result, whose proof can be found in [13, Proposition 2.3].

Proposition 2.7. *Suppose that $1<p<3$ and $\omega>0$. Let $u\in\mathcal{X}_0$ be a nontrivial solution of*

$$-\Delta u+\omega u-e^2S(u)u=|u|^{p-1}u \quad \text{in } \mathbb{R}^2.$$

Then u decays exponentially at infinity.

3 L^2-constraint Minimizer for the nonlinear Schrödinger-Newton system

In this section, we consider the existence of a L^2-constraint minimizer of the following nonlocal elliptic problem:

$$-\Delta u + \omega u + \kappa |x|^2 u - e^2 S(u)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^2, \tag{3.1}$$

which corresponds to the nonlinear Schrödinger-Newton system. We will show that the L^2-constraint minimizer is actually a ground state solution of (3.1). We will also investigate the asymptotic behavior of the minimizer as $\kappa \to 0$.

3.1 Existence of a minimizer and its properties

In this subsection, we establish that $c_-(\mu)$ has a minimizer for all $\mu > 0$. First we begin with the following result.

Lemma 3.1. *Suppose that $1 < p < 3$, $\kappa \in (0,1]$ and let $\mu > 0$ be given. Then E_- is bounded from below on $\mathcal{B}_\mu \cap \mathcal{X}$.*

Proof. First by the Gagliardo-Nirenberg inequality and the Young inequality, one has

$$\|u\|_{p+1}^{p+1} \le C\|\nabla u\|_2^{p-1}\|u\|_2^2 \le \frac{p+1}{4}\|\nabla u\|_2^2 + C\mu^{\frac{2}{3-p}}$$

for some $C > 0$. Then from Lemma 2.3 (i) and (ii), one finds that

$$\begin{aligned} E_-(u) &\ge \frac{1}{4}\|\nabla u\|_2^2 + \frac{\kappa}{2}V(u) - C\mu^{\frac{2}{3-p}} - e^2 A_2(u) \\ &\ge \frac{1}{8}\|\nabla u\|_2^2 + \frac{\kappa}{2}V(u) - C\mu^{\frac{2}{3-p}} - Ce^4\mu^3, \end{aligned} \tag{3.2}$$

for any $u \in \mathcal{B}_\mu \cap \mathcal{X}$. Thus implies that E_- is bounded from below on $\mathcal{B}_\mu \cap \mathcal{X}$. □

The next lemma deals with the compactness of any minimizing sequences for $c_-(\mu)$.

Lemma 3.2. *Suppose that $1 < p < 3$, $\kappa \in (0,1]$ and let $\mu > 0$ be given.*

Let $\{u_j\} \subset \mathcal{X}$ be a sequence satisfying $\|u_j\|_2^2 \to \mu$ and $E_-(u_j) \to c_-(\mu)$. Then there exist a subsequence of $\{u_j\}$ which is still denoted by the same and $u_- \in \mathcal{X}$ such that $u_j \to u_-$ in $H^1(\mathbb{R}^3, \mathbb{C})$ and $E_-(u_-) = c_-(\mu)$.

Proof. First by replacing u_j by $\frac{\sqrt{\mu}}{\|u_j\|_2}u_j$, we may assume that $\{u_j\}$ is a minimizing sequence of $c_-(\mu)$. Moreover from (3.2), we find that $\{u_j\}$ is bounded in $\mathcal{X}$. Thus there exists $u_- \in \mathcal{X}$ such that $u_j \rightharpoonup u_-$ in $\mathcal{X}$ for some $u_- \in \mathcal{X}$.

Now by Lemma 2.1, it follows that

$$u_j \to u_- \quad \text{in } L^q(\mathbb{R}^2) \text{ for all } q \in [2, \infty). \tag{3.3}$$

Especially one has $\|u_-\|_2^2 = \mu$. By the weak lower semi-continuity of $\|\nabla \cdot\|_2$, we also have

$$\liminf_{j\to\infty} \|\nabla u_j\|_2^2 \geq \|\nabla u_-\|_2^2, \tag{3.4}$$

while Fatou's Lemma implies that

$$\liminf_{j\to\infty} V(u_j) \geq V(u_-). \tag{3.5}$$

Next by symmetry, we have the following estimate

$$\begin{aligned}
\big|A(u_j) - A(u_-)\big| &= \Big|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y|\big(|u_j(x)|^2|u_j(y)|^2 - |u_-(x)|^2|u_-(y)|^2\big)\,dx\,dy\Big| \\
&\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (|x|+|y|)\big||u_j(x)|^2|u_j(y)|^2 - |u_-(x)|^2|u_-(y)|^2\big|\,dx\,dy \\
&= 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |x|\big||u_j(x)|^2|u_j(y)|^2 - |u_-(x)|^2|u_-(y)|^2\big|\,dx\,dy.
\end{aligned}$$

Observing that

$$|u_j(x)|^2|u_j(y)|^2 - |u_-(x)|^2|u_-(y)|^2 = |u_j(x)|^2\big(|u_j(y)|^2 - |u_-(y)|^2\big) + |u_-(y)|^2\big(|u_j(x)|^2 - |u_-(x)|^2\big),$$

we deduce that

$$\begin{aligned}
\big|A(u_j) - A(u_-)\big| &\leq 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |x||u_j(x)|^2\big||u_j(y)|^2 - |u_-(y)|^2\big|\,dx\,dy \\
&\quad + 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |x||u_-(y)|^2\big||u_j(x)|^2 - |u_-(x)|^2\big|\,dx\,dy \\
&=: \mathcal{A}_j^1 + \mathcal{A}_j^2.
\end{aligned}$$

Moreover we can rewrite $\mathcal{A}_j^1$ as follows

$$\mathcal{A}_j^1 = 2\left(\int_{\mathbb{R}^2} |x||u_j(x)|^2\,dx\right)\left(\int_{\mathbb{R}^2} \big||u_j(y)|^2 - |u_-(y)|^2\big|\,dy\right).$$

By the Cauchy-Schwarz inequality, one has

$$\int_{\mathbb{R}^2} |x||u_j(x)|^2\,dx \leq \left(\int_{\mathbb{R}^2} |x|^2|u_j(x)|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2} |u_j(x)|^2\,dx\right)^{\frac{1}{2}},$$

$$\int_{\mathbb{R}^2} \big||u_j(y)|^2 - |u_-(y)|^2\big|\,dy \leq \left(\int_{\mathbb{R}^2} \big(|u_j(y)| + |u_-(y)|\big)^2\,dy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2} \big(|u_j(y)| - |u_-(y)|\big)^2\,dy\right)^{\frac{1}{2}}.$$

Since $u_j \to u_-$ in $L^2(\mathbb{R}^2)$, we deduce that

$$\mathcal{A}_j^1 \le C\|u_j - u_-\|_2 \to 0. \tag{3.6}$$

Similarly, one finds that

$$\lim_{j\to\infty} \mathcal{A}_j^2 = 0. \tag{3.7}$$

From (3.3)-(3.7), we finally obtain

$$c_-(\mu) = \liminf_{j\to\infty} E_-(u_j) \ge E_-(u_-) \ge c_-(\mu). \tag{3.8}$$

(3.8) implies that the sequence u_j converges to u_- strongly in $H^1(\mathbb{R}^2)$, which ends the proof. □

By Lemma 3.1 and Lemma 3.2, we are able to obtain the following existence result.

Proposition 3.3. *Suppose that $1 < p < 3$, $\kappa \in (0,1]$ and let $\mu > 0$ be arbitrarily given. Then $c_-(\mu)$ admits a minimizer $u_- \in \mathcal{B}_\mu \cap \mathcal{X}$.*

Next we investigate some properties of the minimizer u_-. First we prepare the Nehari identity and the Pohozaev identity associated with (3.1).

Lemma 3.4. *Let $u \in \mathcal{X}$ be a solution of (3.1). Then u satisfies the following identities:*

$$0 = N_-(u) := \|\nabla u\|_2^2 + \omega\|u\|_2^2 + \kappa V(u) - 4e^2 A(u) - \|u\|_{p+1}^{p+1}, \tag{3.9}$$

$$0 = P_-(u) := \omega\|u\|_2^2 + 2\kappa V(u) - 4e^2 A(u) + \frac{e^2}{16\pi}\|u\|_2^4 - \frac{2}{p+1}\|u\|_{p+1}^{p+1}. \tag{3.10}$$

Proof. To obtain (3.9), we just multiply (4.1) by $\overline{u}$ and integrate over $\mathbb{R}^2$. The proof of (3.10) is more delicate, since the function $x \cdot \nabla S(u)|u|^2$ may not belong to $L^1(\mathbb{R}^2)$. To overcome this difficulty, we adopt the method used in [14]. Let $\psi \in \mathcal{D}(\mathbb{R}^2)$ be such that $\psi \ge 0$, $\operatorname{supp}\psi \subset B(0,2)$ and $\psi \equiv 1$ on $B(0,1)$. For all $n \in \mathbb{N}$, we set $\psi_n(x) = \psi(\frac{x}{n})$. We then multiply (3.1) by $\psi_n x \cdot \nabla\overline{u}$, and take the real part of the resulting equation. The only non-straightforward term is the following one.

$$\begin{aligned}
&\operatorname{Re}\int_{\mathbb{R}^2} S(u)u\psi_n x \cdot \nabla\overline{u}\,dx \\
&= \int_{\mathbb{R}^2} S(u)\psi_n x \cdot \nabla\left(\frac{|u|^2}{2}\right) dx \\
&= -\int_{\mathbb{R}^2} S(u)\frac{|u|^2}{2} x \cdot \nabla\psi_n\,dx - \int_{\mathbb{R}^2} S(u)|u|^2\psi_n\,dx - \int_{\mathbb{R}^2} \frac{|u|^2}{2}\psi_n x \cdot \nabla S(u)\,dx,
\end{aligned} \tag{3.11}$$

by integration by parts. Now, a direct computation furnishes for $j = 1, 2$,

$$\frac{\partial S(u)}{\partial x_j} = -\frac{1}{4\pi}\int_{\mathbb{R}^2} \frac{x_j - y_j}{|x-y|^2}|u(y)|^2\,dy,$$

from which we deduce that

$$x \cdot \nabla S(u) = -\frac{1}{4\pi}\int_{\mathbb{R}^2} \frac{|x|^2 - x\cdot y}{|x-y|^2}|u(y)|^2\,dy. \tag{3.12}$$

We multiply (3.12) by $\psi_n \frac{|u(x)|^2}{2}$ and integrate again on $\mathbb{R}^2$ to obtain

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{|u(x)|^2}{2}\psi_n x\cdot\nabla S(u)\,dx &= -\frac{1}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \psi_n \frac{|x|^2 - x\cdot y}{|x-y|^2}|u(y)|^2|u(x)|^2\,dy\,dx\\ &= -\frac{1}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \psi_n \left(1 - \frac{|y|^2 - x\cdot y}{|x-y|^2}\right)|u(y)|^2|u(x)|^2\,dy\,dx,\end{aligned}$$

from which we deduce by Fubini's theorem that

$$\int_{\mathbb{R}^2} \frac{|u(x)|^2}{2}\psi_n x\cdot\nabla S(u)\,dx = -\frac{1}{16\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \psi_n |u(y)|^2|u(x)|^2\,dy\,dx. \tag{3.13}$$

Collecting (3.11) and (3.13) and using the Lebesgue dominated convergence theorem and the fact that $\nabla\psi_n(x) \underset{n\to+\infty}{\longrightarrow} 0$, we get the expected equality. We also refer to [12, 19] for the derivation of the Pohozaev identity. □

The next lemma is concerned with the sign of $c_-(\mu)$.

Lemma 3.5. *Suppose that $1 < p < 3$ and $\kappa \in (0, 1]$. Then there exists $\mu_- = \mu_-(e, \kappa) > 0$ such that $c_-(\mu) < 0$ for $\mu > \mu_-$.*

Proof. Let us choose $u(x) = \sqrt{\frac{\mu}{\pi}}e^{-|x|^2}$ and put $u_\lambda(x) := \lambda u(\lambda x)$ for $\lambda > 0$. Then it follows that $\|u_\lambda\|_2^2 = \|u\|_2^2 = \mu$ for all $\lambda > 0$.

Now by Lemma 2.2, we have

$$E_-(u_\lambda) = \frac{\lambda^2}{2}\|\nabla u\|_2^2 + \frac{\kappa\lambda^{-2}}{2}V(u) - \frac{e^2\mu^2}{16\pi}\log\lambda - e^2 A(u) - \frac{\lambda^{p-1}}{p+1}\|u\|_{p+1}^{p+1}.$$

Moreover by direct computations, one finds that $\|\nabla u\|_2^2 = 4\mu$, $V(u) = \mu$ and

$$\begin{aligned}A(u) &= -\frac{\mu^2}{16\pi^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y| e^{-|x|^2-|y|^2}\,dx\,dy\\ &\overset{x-y=\sqrt{2}s,\,x+y=\sqrt{2}t}{=} -\frac{\mu^2}{16\pi^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(\sqrt{2}|s|)e^{-|s|^2-|t|^2}\,ds\,dt\\ &= -\frac{\mu^2}{32\pi}(\log 2 - \gamma),\end{aligned}$$

where γ is the Euler constant. Thus it follows that

$$E_-(u_\lambda) = 2\mu\lambda^2 + \frac{\kappa\mu\lambda^{-2}}{2} - \frac{e^2\mu^2}{16\pi}\log\lambda + \frac{e^2\mu^2}{32\pi}(\log 2 - \gamma) - \frac{\mu^{\frac{p+1}{2}}\lambda^{p-1}}{p+1}\int_{\mathbb{R}^2} e^{-(p+1)|x|^2}\,dx.$$

Taking $\lambda = \mu^{\frac{1}{2}}$, we deduce that

$$\begin{aligned} c_-(\mu) \le E_-(u_\lambda) &= 2\mu^2 + \frac{\kappa}{2} - \frac{e^2\mu^2\log\mu}{32\pi} + \frac{e^2\mu^2}{32\pi}(\log 2 - \gamma) - C\mu^p \\ &\le -\mu^2\left(\frac{e^2}{32\pi}(\log\mu - \log 2 + \gamma) - 2\right) + \frac{\kappa}{2} \to -\infty \quad \text{as } \mu \to \infty. \end{aligned} \tag{3.14}$$

Thus for every $\kappa > 0$, there exists $\mu_- = \mu_-(e,\kappa) > 0$ such that $c_-(\mu) < 0$ for $\mu > \mu_-$. □

Remark 3.6. *When $\kappa = 0$, we have from (3.14) that*

$$c_{0,-}(\mu) \le -\mu^2\left(\frac{e^2}{32\pi}(\log\mu - \log 2 + \gamma) - 2\right), \tag{3.15}$$

yielding that $c_{0,-}(\mu) < 0$ for $\mu > 2\exp\left(\frac{64\pi}{e^2} - \gamma\right)$.

Using Lemma 3.5, we are able to obtain the positivity of the Lagrange multiplier ω_- which corresponds to the minimizer u_-. It is worth mentioning that the next lemma holds even if $\kappa = 0$.

Lemma 3.7. *Suppose that $2 \le p < 3$, $\kappa \in [0,1]$ and assume that $\mu > \mu_-$, where $\mu_- > 0$ is the constant in Lemma 3.5. Then the Lagrange multiplier $\omega_- = \omega_-(\mu,\kappa)$ satisfies*

$$\omega_- > \frac{(3-p)e^2\mu}{16\pi(p-1)} > 0.$$

Proof. By Lemma 3.4, it follows that

$$\begin{aligned} \frac{1}{p+1}\|u_-\|_{p+1}^{p+1} &= \frac{1}{p-1}\|\nabla u_-\|_2^2 - \frac{\kappa}{p-1}V(u_-) - \frac{e^2\mu^2}{16\pi(p-1)}, \\ e^2A(u_-) &= -\frac{1}{2(p-1)}\|\nabla u_+\|_2^2 - \frac{\omega_-}{4}\mu + \frac{p\kappa}{2(p-1)}V(u_-) + \frac{(p+1)e^2\mu^2}{64\pi(p-1)}. \end{aligned}$$

Thus by Lemma 3.5 and from $2 \le p < 3$, we obtain

$$\begin{aligned} 0 > c_-(\mu) &= \frac{1}{2}\|\nabla u_-\|_2^2 + \frac{\kappa}{2}V(u_-) - e^2A(u_-) - \frac{1}{p+1}\|u_-\|_{p+1}^{p+1} \\ &= \frac{p-2}{2(p-1)}\|\nabla u_-\|_2^2 + \frac{\kappa}{2(p-1)}V(u_-) - \frac{\omega_-}{4}\mu + \frac{(3-p)e^2\mu^2}{64\pi(p-1)} \\ &> -\frac{\omega_-}{4}\mu + \frac{(3-p)e^2\mu^2}{64\pi(p-1)}. \end{aligned}$$

from which we conclude. $\square$

3.2 Link between L^2-constraint minimizers and ground state solutions

In this subsection, we investigate the link between minimizers for $c_-(\mu)$ and ground state solutions of (3.1). Indeed the positivity of ω_- enables us to establish the following result.

Proposition 3.8. *Suppose that $2 \le p < 3$, $\kappa \in [0,1]$ and assume that $\mu > \mu_-$. Then any minimizer $u_- \in \mathcal{B}_\mu \cap \mathcal{X}$ is a ground state solution of (3.1) with $\omega = \omega_-$.*

To prove Proposition 3.8, let us define a functional:

$$\begin{aligned} J_-(u) &:= 2N_-(u) - P_-(u) \\ &= 2\|\nabla u\|_2^2 + \omega\|u\|_2^2 - 4e^2 A(u) - \frac{e^2}{16\pi}\|u\|_2^4 - \frac{2p}{p+1}\|u\|_{p+1}^{p+1}. \end{aligned}$$

By Lemma 3.4, we know that $J_-(u) = 0$ for any nontrivial solution $u \in \mathcal{X}$ of (3.1). Moreover we have the following lemma.

Lemma 3.9 (Energy inequality)**.** *Suppose that $1 < p < 3$, $\kappa \in [0,1]$ and $\omega > 0$. Let $u \in \mathcal{X}$ be arbitrarily given and put $u^\lambda(x) := \lambda^2 u(\lambda x)$ for $\lambda > 0$. Then for all $\lambda > 0$, u satisfies*

$$\begin{aligned} &I_-(u) - I_-(u^\lambda) - \frac{1}{4}(1-\lambda^4)J_-(u) \\ &= \frac{\omega}{4}\|u\|_2^2(\lambda^2-1)^2 + \frac{e^2}{64\pi}\|u\|_2^4\left(4\lambda^4\log\lambda - \lambda^4 + 1\right) + \frac{1}{2(p+1)}\|u\|_{p+1}^{p+1}\left(2\lambda^{2p} - p\lambda^4 + p - 2\right). \end{aligned}$$

Especially if $2 \le p < 3$, it holds that

$$I_-(u) - I_-(u^\lambda) - \frac{1}{4}(1-\lambda^4)J_-(u) \ge \frac{\omega}{4}\|u\|_2^2(\lambda^2-1)^2 \quad \text{for all } \lambda > 0.$$

Proof. First by Lemma 2.2, we have

$$I_-(u^\lambda) = \frac{\lambda^4}{2}\|\nabla u\|_2^2 + \frac{\omega\lambda^2}{2}\|u\|_2^2 + \frac{\kappa}{2}V(u) - \frac{e^2\lambda^4\log\lambda}{16\pi}\|u\|_2^4 - e^2\lambda^4 A(u) - \frac{\lambda^{2p}}{p+1}\|u\|_{p+1}^{p+1}.$$

Thus by a direct calculation, one deduces that

$$\begin{aligned} I_-(u) - I_-(u^\lambda) - \frac{1}{4}(1-\lambda^4)J_-(u) &= \frac{\omega}{4}\|u\|_2^2\left(\lambda^4 - 2\lambda^2 + 1\right) + \frac{e^2}{64\pi}\|u\|_2^4\left(4\lambda^4\log\lambda - \lambda^4 + 1\right) \\ &\quad + \frac{1}{2(p+1)}\|u\|_{p+1}^{p+1}\left(2\lambda^{2p} - p\lambda^4 + p - 2\right). \end{aligned}$$

Moreover it is straightforward to see that $4\lambda^4 \log \lambda - \lambda^4 + 1 \geq 0$ for all $\lambda > 0$. When $2 \leq p < 3$, we also have $2\lambda^{2p} - p\lambda^4 + p - 2 \geq 0$ for any $\lambda > 0$, from which we conclude. $\square$

Proof of Proposition 3.8. Let $u_- \in \mathcal{B}_\mu \cap \mathcal{X}$ be a minimizer for $c_-(\mu)$ and ω_- be the associated Lagrange multiplier. By Lemma 3.7, we know that $\omega_- > 0$. Let $v \in \mathcal{X}$ be a nontrivial solution of (3.1) with $\omega = \omega_-$. Then it suffices to show that $I_-(u_-) \leq I_-(v)$.

Now by Lemma 3.4, it follows that $J_-(v) = 0$. Then Lemma 3.9 yields that

$$I_-(v) - I_-(v^\lambda) \geq \frac{1}{4}\left(1 - \lambda^4\right) J_-(v) = 0 \quad \text{for all } \lambda > 0. \tag{3.16}$$

We choose $\lambda = \frac{\|u_-\|_2}{\|v\|_2}$ so that

$$\|v^\lambda\|_2^2 = \lambda^2 \|v\|_2^2 = \|u_-\|_2^2 = \mu.$$

Since u_- is a minimizer of $c_-(\mu)$, it holds that $E_-(u_-) \leq E_-(v^\lambda)$. Then from (3.16), we get

$$\begin{aligned} I_-(u_-) &= E_-(u_-) + \frac{\omega_-}{2}\|u_-\|_2^2 \leq E_-(v^\lambda) + \frac{\omega_-}{2}\|v^\lambda\|_2^2 \\ &= I_-(v^\lambda) \leq I_-(v). \end{aligned}$$

This implies that u_- is a ground state solution of (3.1) with $\omega = \omega_-$. $\square$

We now investigate the link between ground state solutions of (3.1) and energy minimizers. To this end, let us denote by $\Omega_-(\mu)$ the set of Lagrange multipliers associated with minimizers for $c_-(\mu)$, that is,

$$\Omega_-(\mu) := \Big\{\omega_-(\mu) > 0 \mid \omega_-(\mu) \text{ is a Lagrange multiplier} \\ \text{associated with a minimizer for } c_-(\mu)\Big\}.$$

By Lemma 3.7, it follows that $\Omega_-(\mu) \neq \emptyset$ for $\mu > \mu_-$. Moreover for any $\omega_-(\mu) \in \Omega(\mu)$, there exists a ground state solution $w_\mu \in \mathcal{X}$ of (3.1) with $\omega = \omega_-(\mu)$ by Proposition 3.8.

Proposition 3.10. *Suppose that $2 \leq p < 3$, $\kappa \in [0, 1]$ and assume that $\mu > \mu_-$. Then any ground state solution $w_\mu \in \mathcal{X}$ of (3.1) with $\omega = \omega_-(\mu) \in \Omega_-(\mu)$ is a minimizer for $c_-(\mu)$. This means in particular that all ground state solutions for $\omega \in \Omega(\mu)$ share the same L^2-norm.*

Proof. First by Lemma 3.4 and Lemma 3.9, we infer that

$$\begin{aligned} I_-(w_\mu) - I_-\left((w_\mu)^\lambda\right) &\geq \frac{1-\lambda^4}{4} J_-(w_\mu) + \frac{\omega_-(\mu)}{4}\|w_\mu\|_2^2\left(\lambda^2-1\right)^2 \\ &= \frac{\omega_-(\mu)}{4}\|w_\mu\|_2^2\left(\lambda^2-1\right)^2 \geq 0 \quad \text{for all } \lambda > 1. \end{aligned} \tag{3.17}$$

Let $u_- \in \mathcal{B}_\mu \cap \mathcal{X}$ be a minimizer for $c_-(\mu)$ whose Lagrange multiplier coincides with $\omega_-(\mu)$. Then it holds that

$$I_-(w_\mu) \leq I_-(u_-) \quad \text{and} \quad E_-(u_-) \leq E_-(u) \quad \text{for any } u \in \mathcal{B}_\mu \cap \mathcal{X}. \tag{3.18}$$

Putting $\lambda_\mu = \frac{\sqrt{\mu}}{\|w_\mu\|_2}$, we deduce that

$$\left\|(w_\mu)^{\lambda_\mu}\right\|_2^2 = \lambda_\mu^2\|w_\mu\|_2^2 = \mu = \|u_-\|_2^2 \tag{3.19}$$

and hence $E_-(u_-) \leq E_-\left((w_\mu)^{\lambda_\mu}\right)$. Thus from (3.17)-(3.19), one finds that

$$\begin{aligned} I_-(w_\mu) \leq I_-(u_-) &= E_-(u_-) + \frac{\omega_-(\mu)}{2}\|u_-\|_2^2 \\ &\leq E_-\left((w_\mu)^{\lambda_\mu}\right) + \frac{\omega_-(\mu)}{2}\left\|(w_\mu)^{\lambda_\mu}\right\|_2^2 \\ &= I_-\left((w_\mu)^{\lambda_\mu}\right) \leq I_-(w_\mu), \end{aligned} \tag{3.20}$$

yielding that $I_-(w_\mu) = I_-\left((w_\mu)^{\lambda_\mu}\right)$. Going back to (3.17), we arrive at

$$0 \geq \frac{\omega_-(\mu)}{4}\|w_\mu\|_2^2\left(\lambda_\mu^2-1\right)^2 \geq 0.$$

This implies that $\lambda_\mu = 1$ and hence $\|w_\mu\|_2^2 = \mu$. Then from (3.20), we get

$$E_-(w_\mu) = E_-(u_-) = c_-(\mu),$$

which ends the proof. $\square$

Next we investigate the asymptotic behavior of the Lagrange multiplier with respect to μ. In the case $\kappa = 0$, we have the following result.

Proposition 3.11. *Suppose that $2 \leq p < 3$ and let $\omega_{0,-}(\mu)$ be the Lagrange multiplier associated with a minimizer for $c_{0,-}(\mu)$. Then it holds that*

$$\omega_{0,-}(\mu) \to \infty \text{ as } \mu \to \infty \quad \text{and} \quad \omega_{0,-}(\mu) \to 0 \text{ as } \mu \to 0.$$

Proof. First by Lemma 3.7, we readily see that $\omega_{0,-}(\mu) \to \infty$ as $\mu \to \infty$. Let $u_0 \in \mathcal{B}_\mu \cap \mathcal{X}_0$ be a minimizer for $c_{0,-}(\mu)$. Then using the Nehari identity and applying the Gagliardo-Nirenberg inequality as in Lemma 3.1, we deduce that

$$\begin{aligned}
0 = N_-(u_0) &\geq \|\nabla u_0\|_2^2 + \omega_{0,-}(\mu)\|u_0\|_2^2 - 4e^2 A_2(u_0) - \|u_0\|_{p+1}^{p+1} \\
&\geq \omega_{0,-}(\mu)\mu - Ce^4\mu^3 - C\mu^{\frac{2}{3-p}}.
\end{aligned}$$

This implies that $\limsup_{\mu \to 0} \omega_{0,-}(\mu) \leq 0$.

Next by Lemma 3.4, we have

$$\begin{aligned}
\frac{1}{2}\|\nabla u_0\|_2^2 &= \frac{e^2}{32\pi}\|u_0\|_2^4 + \frac{p-1}{2(p+1)}\|u_0\|_{p+1}^{p+1}, \\
e^2 A(u_0) &= \frac{\omega_{0,-}(\mu)}{4}\|u_0\|_2^2 + \frac{e^2}{64\pi}\|u_0\|_2^4 - \frac{1}{2(p+1)}\|u_0\|_{p+1}^{p+1},
\end{aligned}$$

from which one finds that

$$\begin{aligned}
c_{0,-}(\mu) &= \frac{1}{2}\|\nabla u_0\|_2^2 - e^2 A(u_0) - \frac{1}{p+1}\|u_0\|_{p+1}^{p+1} \\
&= -\frac{\omega_{0,-}(\mu)}{4}\|u_0\|_2^2 + \frac{e^2}{64\pi}\|u_0\|_2^4 + \frac{p-2}{2(p+1)}\|u_0\|_{p+1}^{p+1}.
\end{aligned}$$

Thus we obtain

$$\omega_{0,-}(\mu) \geq -\frac{4c_{0,-}(\mu)}{\mu} + \frac{e^2\mu}{16\pi}. \tag{3.21}$$

Furthermore by (3.15), it follows that

$$\frac{c_{0,-}(\mu)}{\mu} \leq -\frac{e^2}{32\pi}\mu\log\mu + (\log 2 - \gamma)\mu + 2\mu \to 0 \quad \text{as } \mu \to 0.$$

Thus from (3.21), we infer that $\liminf_{\mu \to 0} \omega_{0,-}(\mu) \geq 0$ and hence $\lim_{\mu \to 0} \omega_{0,-}(\mu) = 0$. □

Remark 3.12. (i) *By Lemma 3.7, it holds that $\omega_-(\mu) \to \infty$ as $\mu \to \infty$ even if $0 < \kappa \leq 1$.*

(ii) *By Proposition 3.11, it is natural to expect that the set of Lagrange multipliers $\Omega_{0,-}(\mu)$ for $c_{0,-}(\mu)$ satisfies*

$$\bigcup_{\mu > 0} \Omega_{0,-}(\mu) = (0, \infty).$$

3.3 Asymptotic behavior of minimizers as $\kappa \to 0$

In this subsection, we study the asymptotic behavior of minimizers for $c_{\kappa,-}(\mu)$ as $\kappa \to 0$. To emphasize the dependence with respect to κ, we write $E_-(u) = E_{\kappa,-}(u)$ and $c_-(\mu) = c_{\kappa,-}(\mu)$. Then we are able to prove the following result.

Proposition 3.13. *Suppose that $2 \le p < 3$, $\kappa \in (0,1]$ and assume that $\mu > \mu_-$. Let $u_\kappa = u_{\kappa,-} \in \mathcal{B}_\mu \cap \mathcal{X}$ be a minimizer for $c_{\kappa,-}(\mu)$. Then there exist $\kappa_j \to 0$, $\{y_j\} \subset \mathbb{R}^2$ and $u_0 \in \mathcal{B}_\mu \cap \mathcal{X}_0$ such that*

$$u_{\kappa_j}(\cdot - y_j) \to u_0 \ \textit{in } \mathcal{X}_0.$$

Moreover u_0 is a minimizer for $c_{0,-}(\mu)$.

In order to establish Proposition 3.13, we first prepare the following asymptotic result for the minimum of the energy.

Lemma 3.14. *Suppose that $2 \le p < 3$ and assume that $\mu > \mu_-$. Then it holds that*

$$\lim_{\kappa \to 0} c_{\kappa,-}(\mu) = c_{0,-}(\mu).$$

Proof. Let $u_0 \in \mathcal{B}_\mu \cap \mathcal{X}_0$ be a minimizer for $c_{0,-}(\mu)$. By Lemma 3.7, the associated Lagrange multiplier $\omega_0(\mu)$ is positive. Then by Proposition 2.7, it follows that u_0 decays exponentially at infinity and hence $u_0 \in \mathcal{X}$. Thus one finds that

$$c_{\kappa,-}(\mu) \le E_{\kappa,-}(u_0) = E_{0,-}(u_0) + \frac{\kappa}{2}V(u_0) = c_{0,-}(\mu) + \frac{\kappa}{2}V(u_0)$$

and $\limsup_{\kappa \to 0} c_{\kappa,-}(\mu) \le c_{0,-}(\mu)$.

On the other hand since $E_{\kappa,-}(u) \ge E_{0,-}(u)$ for any $u \in \mathcal{B}_\mu \cap \mathcal{X}$ and $c_{\kappa,-}(\mu)$ admits a minimizer, we have $c_{\kappa,-}(\mu) \ge c_{0,-}(\mu)$. Thus we arrive at

$$c_{0,-}(\mu) \le \liminf_{\kappa \to 0} c_{\kappa,-}(\mu) \le \limsup_{\kappa \to 0} c_{\kappa,-}(\mu) \le c_{0,-}(\mu)$$

from which we conclude. □

Now we are ready to prove Proposition 3.13.

Proof of Proposition 3.13. The proof consists of three steps.

Step 1: We prove that there exist $\kappa_j \to 0$, $\{y_j\} \subset \mathbb{R}^2$ and $u_0 \in \mathcal{B}_\mu \cap \mathcal{X}_0$ such that $u_{\kappa_j}(\cdot - y_j) \rightharpoonup u_0$ in $\mathcal{X}_0$ as $j \to \infty$. First we observe that

$$E_{\kappa,-}(u) = E_{0,-}(u) + \frac{\kappa}{2}V(u) \le E_{0,-}(u) + \frac{1}{2}V(u) = E_{1,-}(u) \quad \text{for all } 0 < \kappa \le 1 \text{ and } u \in \mathcal{X}.$$

Since $c_{\kappa,-}(\mu)$ is attained by Proposition 3.3, it follows that $c_{\kappa,-}(\mu) \le c_{1,-}(\mu)$ for any $0 < \kappa \le 1$ and $\mu > 0$. Then by the Gagliardo-Nirenberg inequality, one deduces that

$$\frac{1}{8}\|\nabla u_\kappa\|_2^2+\frac{\kappa}{2}V(u_\kappa)+e^2A_1(u_\kappa) \le c_{\kappa,-}(\mu)+C\left(e^4\mu^3+\mu^{\frac{2}{3-p}}\right) \le c_{1,-}(\mu)+C\left(e^4\mu^3+\mu^{\frac{2}{3-p}}\right),$$

from which we conclude that

$$\|u_\kappa\|_{H^1} \quad \text{and} \quad A_1(u_\kappa) \quad \text{are bounded.}$$

Then by Lemma 2.4, there exist $\kappa_j \to 0$, $\{y_j\} \subset \mathbb{R}^2$ and $u_0 \in \mathcal{B}_\mu$ such that $v_j := u_{\kappa_j}(\cdot - y_j) \to u_0$ in $L^2(\mathbb{R}^2)$. Moreover since $\|u_0\|_2^2 = \mu$, applying the Egorov theorem, we infer that there exist $j_0 \in \mathbb{N}$, $\delta_0 > 0$, $R_0 > 0$ and a measurable subset $A_0 \subset B(0, R_0)$ such that $|v_j(x)|^2 \ge \delta_0$ for all $x \in A_0$ and $j \ge j_0$. Thus we are able to apply Lemma 2.5 to obtain

$$V_0(v_j) \le C\left(A_1(v_j) + \|v_j\|_2^2\right) \le C \quad \text{for } j \ge j_0.$$

This implies that, passing to a subsequence if necessary, $v_j \rightharpoonup u_0$ in $\mathcal{X}_0$ and $u_0 \in \mathcal{B}_\mu \cap \mathcal{X}_0$.

Here we note that

$$v_j \to u_0 \quad \text{in } L^q(\mathbb{R}^2) \text{ for } q \in [2,\infty) \tag{3.22}$$

by Lemma 2.1.

Step 2: We claim that u_0 is a minimizer for $c_{0,-}(\mu)$. By the weak lower semicontinuity of $\|\nabla \cdot\|_2$, (3.22), Lemma 2.3 (iv) and Lemma 3.14, we obtain

$$\begin{aligned} c_{0,-}(\mu) \le E_{0,-}(u_0) &\le \liminf_{j\to\infty} E_{0,-}(v_j) = \liminf_{j\to\infty} E_{0,-}(u_{\kappa_j}) \\ &\le \liminf_{j\to\infty} E_{\kappa_j,-}(u_{\kappa_j}) = \liminf_{j\to\infty} c_{\kappa_j,-}(\mu) = c_{0,-}(\mu) \end{aligned}$$

and hence $E_{0,-}(u_0) = c_{0,-}(\mu)$ as claimed.

Step 3: We show hat $v_j \to u_0$ in $\mathcal{X}_0$ as $j \to \infty$. First by Lemma 3.14, we can see that

$$c_{0,-}(\mu) \le E_{0,-}(u_{\kappa_j}) \le E_{0,-}(u_{\kappa_j})+\frac{\kappa_j}{2}V(u_{\kappa_j}) = E_{\kappa_j,-}(u_{\kappa_j}) = c_{\kappa_j,-}(\mu) \to c_{0,-}(\mu)$$

and hence

$$E_{0,-}(v_j) = E_{0,-}(u_{\kappa_j}) \to c_{0,-}(\mu) = E_{0,-}(u_0) \quad \text{and} \quad \kappa_j V(u_{\kappa_j}) \to 0 \text{ as } j \to \infty. \tag{3.23}$$

By Lemma 3.4, one also has

$$
\begin{aligned}
0 = N_-(u_{\kappa_j}) - P_-(u_{\kappa_j}) &= \|\nabla u_{\kappa_j}\|_2^2 - \kappa_j V(u_{\kappa_j}) - \frac{e^2}{16\pi}\|u_{\kappa_j}\|_2^4 - \frac{p-1}{p+1}\|u_{\kappa_j}\|_{p+1}^{p+1} \\
&= \|\nabla v_j\|_2^2 - \kappa_j V(u_{\kappa_j}) - \frac{e^2}{16\pi}\|v_j\|_2^4 - \frac{p+1}{p+1}\|v_j\|_{p+1}^{p+1}.
\end{aligned}
\tag{3.24}
$$

Moreover since u_0 is a minimizer for $c_{0,-}(\mu)$, u_0 is a nontrivial solution of

$$
-\Delta u + \omega_{0,-}(\mu)u - e^2 S(u)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^2,
$$

where $\omega_{0,-}(\mu)$ is the corresponding Lagrange multiplier which is positive by Lemma 3.7. Thus we are able to apply Lemma 3.4 to obtain

$$
0 = \|\nabla u_0\|_2^2 - \frac{e^2}{16\pi}\|u_0\|_2^4 - \frac{p-1}{p+1}\|u_0\|_{p+1}^{p+1}. \tag{3.25}
$$

Thus from (3.22), (3.23), (3.24) and (3.25), we infer that

$$
\nabla v_j \to \nabla u_0 \quad \text{in } L^2(\mathbb{R}^2). \tag{3.26}
$$

Next by (3.22), (3.23) and (3.26), one finds that

$$
\begin{aligned}
E_{0,-}(u_0) + o(1) = E_{0,-}(v_j) &= \frac{1}{2}\|\nabla v_j\|_2^2 - e^2 A(v_j) - \frac{1}{p+1}\|v_j\|_{p+1}^{p+1} \\
&= \frac{1}{2}\|\nabla v_j\|_2^2 + e^2 A_1(v_j) - e^2 A_2(v_j) - \frac{1}{p+1}\|v_j\|_{p+1}^{p+1} \\
&= \frac{1}{2}\|\nabla u_0\|_2^2 + e^2 A_1(v_j) - e^2 A_2(u_0) - \frac{1}{p+1}\|u_0\|_{p+1}^{p+1} + o(1) \\
&= E_{0,-}(u_0) + e^2 A_1(v_j) - e^2 A_1(u_0) + o(1),
\end{aligned}
$$

which yields that

$$
A_1(v_j) \to A_1(u_0) \quad \text{as } j \to \infty. \tag{3.27}
$$

Moreover since

$$
A_1(u) = \frac{1}{16\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|)|u(x)|^2|u(y)|^2\,dx\,dy = B_1(|u|^2, |u|^2),
$$

a direct calculation yields that

$$
B_1(|v_j|^2, |v_j-u_0|^2) = A_1(v_j) - B_1(|v_j|^2, |u_0|^2) + 2B_1\big(|v_j|^2, u_0(u_0-v_j)\big). \tag{3.28}
$$

By Proposition 2.6 (i), we know that

$$B_1\big(|v_j|^2, u_0(u_0 - v_j)\big) \to 0 \quad \text{as } j \to \infty. \tag{3.29}$$

Furthermore by the Fatou lemma, one gets

$$\liminf_{j\to\infty} B_1(|v_j|^2, |u_0|^2) \geq B_1(|u_0|^2, |u_0|^2) = A_1(u_0).$$

Thus from (3.27), (3.28) and (3.29), we obtain

$$\begin{aligned} 0 &\leq \liminf_{j\to\infty} B_1(|v_j|^2, |v_j - u_0|^2) \leq \limsup_{j\to\infty} B_1(|v_j|^2, |v_j - u_0|^2) \\ &\leq \limsup_{j\to\infty} A_1(v_j) - \liminf_{j\to\infty} B_1(|v_j|^2, |u_0|^2) \\ &\leq A_1(u_0) - A_1(u_0) = 0 \end{aligned}$$

and hence $B_1(|v_j|^2, |v_j - u_0|^2) \to 0$ as $j \to \infty$. Then by Proposition 2.6 (ii), we conclude that $V_0(v_j - u_0) \to 0$ and thus $v_j \to u_0$ in $\mathcal{X}_0$. □

4 L^2-constraint minimizer for the nonlinear Schrödinger-Poisson system

In this section, we study the existence of a L^2-constraint minimizer of the following nonlocal elliptic problem:

$$-\Delta u + \omega u + \kappa |x|^2 u + e^2 S(u) u = |u|^{p-1} u \quad \text{in } \mathbb{R}^2, \tag{4.1}$$

which corresponds to the nonlinear Schrödinger-Poisson system.

First we begin with the following lemma, which shows that the presence of the harmonic potential is essential for the existence of a L^2-constraint minimizer for (4.1).

Lemma 4.1. *Suppose that $1 < p < 3$, $\kappa \in (0, 1]$ and let $\mu > 0$ be given. Then E_+ is bounded from below on $\mathcal{B}_\mu \cap \mathcal{X}$.*

Proof. By Lemma 2.3 (i) and (iii), we have

$$A(u) \geq -A_1(u) \geq -\frac{\mu}{8\pi}\left(C_\kappa \mu + \kappa^{\frac{1}{2}} \mu^{\frac{1}{2}} V(u)^{\frac{1}{2}}\right) \geq -\frac{C_\kappa}{8\pi}\mu^2 - \frac{e^2}{64\pi^2}\mu^3 - \frac{\kappa}{4e^2} V(u).$$

Thus by the Gagliardo-Nirenberg inequality and the Young inequality, we find that

$$\begin{aligned} E_+(u) &= \frac{1}{2}\|\nabla u\|_2^2 + \frac{\kappa}{2}V(u) - \frac{1}{p+1}\|u\|_{p+1}^{p+1} + e^2 A(u) \\ &\geq \frac{1}{4}\|\nabla u\|_2^2 + \frac{\kappa}{4}V(u) - C\mu^{\frac{2}{3-p}} - \frac{C_\kappa e^2}{8\pi}\mu^2 - \frac{e^4}{64\pi^2}\mu^3 \\ &\geq -C\mu^{\frac{2}{3-p}} - \frac{C_\kappa e^2}{8\pi}\mu^2 - \frac{e^4}{64\pi^2}\mu^3 \quad \text{for any } u \in \mathcal{B}_\mu \cap \mathcal{X}. \end{aligned} \tag{4.2}$$

This completes the proof. □

Remark 4.2. *If we work on $\mathcal{B}_\mu \cap \mathcal{X}_0$, we can see that, for any $\mu > 0$,*

$$\inf_{u \in \mathcal{B}_\mu \cap \mathcal{X}_0} E_{0,+}(u) = -\infty.$$

The next lemma can be shown similarly as Lemma 3.2.

Lemma 4.3. *Suppose that $1 < p < 3$, $\kappa \in (0, 1]$ and let $\mu > 0$ be given. Let $\{u_j\} \subset \mathcal{X}$ be a sequence satisfying $\|u_j\|_2^2 \to \mu$ and $E_+(u_j) \to c_+(\mu)$. Then there exist a subsequence of $\{u_j\}$ which is still denoted by the same and $u_+ \in \mathcal{X}$ such that $u_j \to u_+$ in $H^1(\mathbb{R}^3, \mathbb{C})$ and $E_+(u_+) = c_+(\mu)$.*

By Lemma 4.1 and Lemma 4.3, we are able to obtain the following result.

Proposition 4.4. *Suppose that $1 < p < 3$, $\kappa \in (0, 1]$ and let $\mu > 0$ be arbitrarily given. Then $c_+(\mu)$ admits a minimizer $u_+ \in \mathcal{B}_\mu \cap \mathcal{X}$.*

Next we present the Nehari identity and the Pohozaev identity associated with (4.1). Since the proof of Lemma 4.5 is the same than that of Lemma 3.4, we omit it.

Lemma 4.5. *Let $u \in \mathcal{X}$ be a solution of (4.1). Then u satisfies the following identities:*

$$0 = N_+(u) := \|\nabla u\|_2^2 + \omega\|u\|_2^2 + \kappa V(u) + 4e^2 A(u) - \|u\|_{p+1}^{p+1},$$
$$0 = P_+(u) := \omega\|u\|_2^2 + 2\kappa V(u) + 4e^2 A(u) - \frac{e^2}{16\pi}\|u\|_2^4 - \frac{2}{p+1}\|u\|_{p+1}^{p+1}.$$

The following lemma is concerned with the sign of $c_+(\mu)$. We are already mentioned in Remark 4.2 that $c_{0,+}(\mu) = -\infty$. The next lemma gives the precise asymptotic behavior of $c_{\kappa,+}(\mu)$ as $\kappa \to 0$.

Lemma 4.6. *Suppose that $1 < p < 3$ and let $\mu > 0$ be given. Then there exists $\kappa_+ = \kappa_+(e, \mu) \in (0, 1)$ such that for $0 < \kappa < \kappa_+$,*

$$\frac{e^2\mu^2}{8\pi} \log \kappa \le c_+(\mu) \le \frac{e^2\mu^2}{64\pi} \log \kappa$$

Especially $c_+(\mu) < 0$ for $0 < \kappa < \kappa_+$.

Proof. First since $C_\kappa \le -\frac{1}{2} \log \kappa + \kappa^{\frac{1}{2}}$, it follows from (4.2) that

$$c_+(\mu) \ge -C\mu^{\frac{2}{3-p}} + \frac{e^2\mu^2}{16\pi} \log \kappa - \frac{e^2\mu^2\kappa^{\frac{1}{2}}}{8\pi} - \frac{e^4\mu^3}{64\pi^2},$$

and hence

$$\begin{aligned} \frac{c_+(\mu)}{e^2\mu^2 \log \kappa} &\le \frac{1}{16\pi} - \frac{C\mu^{\frac{2(p-2)}{3-p}}}{e^2 \log \kappa} - \frac{\kappa^{\frac{1}{2}}}{8\pi \log \kappa} - \frac{e^2\mu}{64\pi^2 \log \kappa}, \\ &\to \frac{1}{16\pi} \quad \text{as } \kappa \to 0. \end{aligned}$$

Thus we have $\frac{c_+(\mu)}{e^2\mu^2 \log \kappa} \le \frac{1}{8\pi}$ for sufficiently small $\kappa > 0$.

To estimate $c_+(\mu)$ from above, let us consider the test function $u(x) = \sqrt{\frac{\mu}{\pi}} e^{-|x|^2}$ and put $u_\kappa(x) := \sqrt{\kappa} u(\sqrt{\kappa} x)$. Then one finds that $\|u_\kappa\|_2^2 = \mu$ for all $\kappa \in (0, 1)$. Moreover we have

$$\|\nabla u_\kappa\|_2^2 = \kappa \|\nabla u\|_2^2 = 4\mu\kappa, \quad \kappa V(u_\kappa) = V(u) = \mu,$$

$$A(u_\kappa) = \frac{\log \kappa}{32\pi} \|u\|_2^4 + A(u) = \frac{\mu^2}{32\pi} \log \kappa - \frac{\mu^2}{32\pi} (\log 2 - \gamma),$$

where γ is the Euler constant. Thus it holds that

$$\begin{aligned} c_+(\mu) \le E_+(u_\kappa) &= \frac{1}{2} \|\nabla u_\kappa\|_2^2 + \frac{\kappa}{2} V(u_\kappa) + e^2 A(u_\kappa) - \frac{1}{p+1} \|u_\kappa\|_{p+1}^{p+1} \\ &= 2\mu\kappa + \frac{\mu}{2} + \frac{e^2\mu^2}{32\pi} \log \kappa - \frac{e^2\mu^2}{32\pi} (\log 2 - \gamma) - \frac{\mu^{\frac{p+1}{2}} \kappa^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^2} e^{-(p+1)|x|^2} \, dx, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \frac{c_+(\mu)}{e^2\mu^2 \log \kappa} &\ge \frac{1}{32\pi} + \frac{2\kappa}{e^2\mu \log \kappa} + \frac{1}{2e^2\mu \log \kappa} - \frac{\log 2 - \gamma}{32\pi \log \kappa} - \frac{\mu^{-\frac{3-p}{2}} \kappa^{\frac{p-1}{2}}}{(p+1)e^2 \log \kappa} \int_{\mathbb{R}^2} e^{-(p+1)|x|^2} \, dx \\ &\to \frac{1}{32\pi} \quad \text{as } \kappa \to 0. \end{aligned}$$

This yields that there exists $\kappa_+ = \kappa_+(e, \mu) \in (0, 1)$ such that $\frac{c_+(\mu)}{e^2\mu^2 \log \kappa} \ge \frac{1}{64\pi}$ for $0 < \kappa < \kappa_+$, completing the proof. □

Remark 4.7. *Since* $\log 2 - \gamma > 0$*, estimate* (4.3) *shows that for fixed* $\kappa \in (0,1)$*, there exists* $\mu_+ = \mu_+(e,\kappa) > 0$ *such that* $c_+(\mu) < 0$ *for* $\mu > \mu_+$*.*

Now let $\omega_+ = \omega_+(\mu,\kappa)$ be the Lagrange multiplier associated with the minimizer $u_+ \in \mathcal{B}_\mu \cap \mathcal{X}$ obtained in Proposition 4.4.

Lemma 4.8. *Suppose that* $2 \le p < 3$ *and let* $\mu > 0$ *be given. Assume that* $0 < \kappa < \kappa_+$*, where* $\kappa_+ \in (0,1)$ *is the constant in Lemma 4.6. Then the Lagrange multiplier* ω_+ *satisfies*

$$\omega_+ + \frac{(3-p)e^2\mu}{16\pi(p-1)} > 0.$$

Proof. The proof is essentially same than that of Lemma 4.8. Indeed by Lemma 4.5 and Lemma 4.6, we obtain

$$\begin{aligned} 0 > c_+(\mu) &= \frac{1}{2}\|\nabla u_+\|_2^2 + \frac{\kappa}{2}V(u_+) + e^2 A(u_+) - \frac{1}{p+1}\|u_+\|_{p+1}^{p+1} \\ &= \frac{p-2}{2(p-1)}\|\nabla u_+\|_2^2 + \frac{\kappa}{2(p-1)}V(u_+) - \frac{\omega_+}{4}\mu - \frac{(3-p)e^2\mu^2}{64\pi(p-1)} \\ &> -\frac{\omega_+}{4}\mu - \frac{(3-p)e^2\mu^2}{64\pi(p-1)}, \end{aligned}$$

from which we conclude. □

Remark 4.9. *We may expect that similar results as Proposition 3.8 and Proposition 3.10 hold for the nonlinear Schrödinger-Poisson system* (4.1)*. As we have observed in the proof of these propositions, the key ingredients are the energy inequality introduced in Lemma 3.9 and the positivity of the Lagrange multiplier* ω_+*.*

If we compute $I_+(u) - I_+(u^\lambda) - \frac{1}{4}(1-\lambda^4)J_+(u)$ *as in Lemma 3.9, since the sign of the nonlocal term is opposite, we find that the sign of* $\|u\|_2^4$ *becomes opposite and thus a competition occurs. Furthermore since the only information we could know about the Lagrange multiplier* ω_+ *is the estimate obtained in Lemma 4.8, we do not know whether* ω_+ *is positive or not.*

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