STABILITY AND INSTABILITY RESULTS FOR STANDING WAVES OF QUASI-LINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. We study a class of quasi-linear Schrödinger equations arising in the theory of superfluid film in plasma physics. First, using gauge transforms and a derivation process we solve, under some regularity assumptions, the Cauchy problem. Then, by means of variational methods, we study the existence, the orbital stability and instability of standing waves which minimize some associated energy.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

Several physical situations are described by generic quasi-linear equations of the form

$$
\begin{cases} \mathrm{i}\phi_t + \Delta\phi + \phi\ell'(|\phi|^2)\Delta\ell(|\phi|^2) + f(|\phi|^2)\phi = 0 & \text{in } (0,\infty)\times\mathbb{R}^N, \\ \phi(0,x) = a_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}
$$

where ℓ and f are given functions. Here i is the imaginary unit, $N \geq 1$, $\phi : \mathbb{R}^N \to \mathbb{C}$ is a complex valued function. For example, the particular case $\ell(s) = \sqrt{1+s}$ models the self-channeling of a high-power ultra short laser in matter (see [6, 13, 36]) whereas if $\ell(s) = \sqrt{s}$, equation (1.1) appears in dissipative quantum mechanics ([16]). It is also used in plasma physics and fluid mechanics ([14, 28]), in the theory of Heisenberg ferromagnets and magnons ([2]) and in condensed matter theory ([32]). The dynamical features are closely related to the two functions ℓ and f. Only few intents have been

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done to develop general theories for the Cauchy problem (see nevertheless [10, 20, 34]). In this article we focus on the particular case $\ell(s) = s$, that is

$$\begin{cases} \mathrm{i}\phi_t + \Delta\phi + \phi\Delta|\phi|^2 + f(|\phi|^2)\phi = 0 & \text{in } (0,\infty) \times \mathbb{R}^N, \\ \phi(0,x) = a_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.2}$$

Our first result concerns the Cauchy problem. Due to the quasi-linear term, it seems difficult to exhibit a well-posedness result in the natural energy space

$$X_{\mathbb{C}} = \Big\{ u \in H^1(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} |u|^2 |\nabla |u||^2 dx < \infty \Big\}.$$

The local and global well-posedness of the Cauchy problem (1.1) have been studied by Poppenberg in [34] in any dimension $N \geq 1$ and for smooth initial data, precisely belonging to the space H^∞. In [10], equation (1.1) is solved locally in the function space $L^\infty(0,T; H^{s+2}(\mathbb{R}^N)) \cap C([0,T); H^s(\mathbb{R}^N))$, where $s = 2E(\frac{N}{2}) + 2$ (here $E(a)$ denotes the integer part of a) for any initial data and smooth nonlinearities ℓ and f such that there exists a positive constant C_ℓ with

$$1 - 4\sigma \ell'^2(\sigma) > C_\ell \ell'^2(\sigma), \quad \text{for all } \sigma \in \mathbb{R}_+. \tag{1.3}$$

Note that the function $\ell(\sigma) = \sigma$ does not satisfied (1.3) and, then, it is not possible to apply [10, Theorem 1.1] to problem (1.2). Before stating our result, we introduce the energy functional $\mathcal{E}$ associated with (1.2), by setting

$$\mathcal{E}(\phi) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \phi|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} |\nabla |\phi|^2|^2 dx - \int_{\mathbb{R}^N} F(|\phi|^2) dx,$$

for all $\phi \in X_{\mathbb{C}}$, where $F(\sigma) = \int_0^\sigma f(u)du$. Note that $\mathcal{E}(\phi)$ can also be written

$$\mathcal{E}(\phi) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \phi|^2 dx + \int_{\mathbb{R}^N} |\phi|^2 |\nabla |\phi||^2 dx - \int_{\mathbb{R}^N} F(|\phi|^2) dx.$$

We prove the following

Theorem 1.1. *Let $N \geq 1$, $s = 2E(\frac{N}{2}) + 2$ and assume that $a_0 \in H^{s+2}(\mathbb{R}^N)$ and $f \in C^{s+2}(\mathbb{R}^N)$. Then there exists a positive T and a unique solution to the Cauchy problem (1.2) satisfying*

$$\phi(0,x) = a_0(x),$$

$$\phi \in L^\infty(0,T; H^{s+2}(\mathbb{R}^N)) \cap C([0,T]; H^s(\mathbb{R}^N)),$$

and the conservation laws

$$\|\phi(t)\|_2 = \|a_0\|_2, \tag{1.4}$$

$$\mathcal{E}(\phi(t)) = \mathcal{E}(a_0), \tag{1.5}$$

for all $t \in [0,T]$.

The proof of Theorem 1.1 follows the approach developed in [10]. It is based on energy methods and to overcome the loss of derivatives induced by the quasi-linear term, gauge transforms are used. We rewrite equation (1.1) as a system in $(\phi, \overline{\phi})$ where $\overline{z}$ denotes the complex conjugate of z. Then, we differentiate the resulting equation with respect to space and time in order to linearize the quasi-linear part and we introduce a set of new unknowns (see (2.2)). A fixed-point procedure is then applied on the linearized version. Since (1.3) does not hold we need, with respect to [10], to modify the linearized version and to perform different energy estimates on the Schrödinger part of the equation.

Next, equipped with Theorem 1.1 and motivated by [1] we investigate some questions of existence, stability and instability of standing waves solutions of (1.2), when f is the power nonlinearity $f(s) = s^{\frac{p-1}{2}}$, with $p > 1$. In this case (1.2) becomes

$$\begin{cases} \mathrm{i}\phi_t + \Delta\phi + \phi\Delta|\phi|^2 + |\phi|^{p-1}\phi = 0 & \text{in } (0,\infty)\times\mathbb{R}^N, \\ \phi(0,x) = a_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.6}$$

If $p > 1$ is an odd integer or $p > 4E(\frac{N}{2}) + 9$ then $f(s) = s^{\frac{p-1}{2}}$ belongs to $C^{2+s}(\mathbb{R}^N)$ and thus Theorem 1.1 applied. In the remaining cases, when we state our stability or instability results, we shall always assume that there exists a solution to the Cauchy problem (1.6) for our nonlinearity and our initial data $a_0 \in X_{\mathbb{C}}$.

By standing waves, we mean solutions of the form $\phi_\omega(t,x) = u_\omega(x)e^{-i\omega t}$. Here ω is a fixed parameter and $\phi_\omega(t,x)$ satisfies problem (1.2) if and only if u_ω is a solution of the equation

$$-\Delta u - u\Delta|u|^2 + \omega u = |u|^{p-1}u, \qquad \text{in } \mathbb{R}^N. \tag{1.7}$$

Throughout the paper we assume that $1 < p < \frac{3N+2}{N-2}$ if $N \geq 3$ and $p > 1$ if $N = 1, 2$. A function $u \in X_{\mathbb{C}}$ is called a (complex) weak solution of equation (1.7) if

$$\Re\int_{\mathbb{R}^N}(1+|u|^2)\nabla u\cdot\nabla\overline{\phi}dx + \Re\int_{\mathbb{R}^N}\nabla|u|^2\cdot\nabla(u\overline{\phi})dx + \omega\Re\int_{\mathbb{R}^N}u\overline{\phi}dx = \Re\int_{\mathbb{R}^N}|u|^{p-1}u\overline{\phi}dx,$$

for all $\phi \in C_0^\infty(\mathbb{R}^N,\mathbb{C})$ (here $\Re(z)$ is the real part of $z \in \mathbb{C}$). We say that a weak solution of (1.7) is a ground state if it satisfies

$$\mathcal{E}_\omega(u) = m_\omega, \tag{1.8}$$

where

$$m_\omega = \inf\{\mathcal{E}_\omega(u) : u \text{ is a nontrivial solution of (1.7)}\}.$$

Here, $\mathcal{E}_\omega$ is the action associated with (1.7) and reads

$$\mathcal{E}_\omega(u) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx + \frac{1}{4}\int_{\mathbb{R}^N}|\nabla|u|^2|^2dx + \frac{\omega}{2}\int_{\mathbb{R}^N}|u|^2dx - \frac{1}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}dx.$$

We denote by $\mathcal{G}_\omega$ the set of weak solutions to (1.7) satisfying (1.8). It is easy to check that u is a weak solution of equation (1.7) if, and only if,

$$\mathcal{E}_\omega'(u)\phi := \lim_{t\to 0^+} \frac{\mathcal{E}_\omega(u+t\phi)-\mathcal{E}_\omega(u)}{t} = 0,$$

for every direction $\phi \in C_0^\infty(\mathbb{R}^N, \mathbb{C})$. By [30, Section 6. Appendix] it follows that any weak solution of (1.7) is $L^\infty_{\rm loc}(\mathbb{R}^N)$ and, in turn, it enjoys the C^2 regularity (see [22]).

First we establish the existence of ground states to (1.7) and we derive some qualitative properties of any element of $\mathcal{G}_\omega$. Our existence result complement the ones of [12, 29, 30, 31, 35].

Theorem 1.2. *For all $\omega > 0$, $\mathcal{G}_\omega$ is non void and any $u \in \mathcal{G}_\omega$ is of the form*

$$u(x) = e^{i\theta}|u(x)|, \quad x \in \mathbb{R}^N,$$

for some $\theta \in \mathbb{S}^1$. In particular, the elements of $\mathcal{G}_\omega$ are, up to a constant complex phase, real-valued and non-negative. Furthermore any real non-negative ground state $u \in \mathcal{G}_\omega$ satisfies the following properties

i) *$u > 0$ in $\mathbb{R}^N$,*
ii) *u is a radially symmetric decreasing function with respect to some point,*
iii) *$u \in C^2(\mathbb{R}^N)$,*
iv) *for all $\alpha \in \mathbb{N}^N$ with $|\alpha| \leq 2$, there exists $(c_\alpha, \delta_\alpha) \in (\mathbb{R}_+^*)^2$ such that*

$$|D^\alpha u(x)| \leq C_\alpha e^{-\delta_\alpha |x|}, \quad \text{for all } x \in \mathbb{R}^N.$$

Moreover, in the case $N = 1$, there exists a unique positive ground state to (1.7) up to translations.

Observe that if $u \in \mathcal{G}_\omega$ is real and positive that any $v(x) = e^{i\theta}u(x-y)$ for $\theta \in \mathbb{S}^1$ and $y \in \mathbb{R}^N$ belongs to $\mathcal{G}_\omega$. Except when $N = 1$ we do not know if there exists a unique real positive ground state up to translation.

Next we establish, for $p > 1$ sufficiently large, a result of instability by blow-up.

Theorem 1.3. *Assume that $\omega > 0$,*

$$p > 3 + \frac{4}{N},$$

and that the conclusions of Theorem 1.1 hold when $f(s) = s^{\frac{p-1}{2}}$. Let $u \in X_{\mathbb{C}}$ be a ground state solution of

$$-\Delta u + u\Delta |u|^2 + \omega u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N. \tag{1.9}$$

Then, for all $\varepsilon > 0$, there exists $a_0 \in H^{s+2}(\mathbb{R}^N)$ such that $\|a_0 - u\|_{H^1(\mathbb{R}^N)} < \varepsilon$ and the solution $\phi(t)$ of (1.6) with $\phi(0) = a_0$ satisfies

$$\lim_{t\to T_0^-} \|\nabla\phi(t)\|_2 = \infty,$$

for some final time $T_0 = T(a_0) < \infty$.

Observe that the assumptions of Theorems 1.1 and 1.3 intersect for $p \geq 9$ when $N = 1$, $p = 7, 9, 11$ or $p \geq 13$ if $N = 2$, $p = 5, 7, 9$ if $N = 3$ and $p = 5$ if $N = 4$.

To prove Theorem 1.3 we first establish a virial type identity. Then, we introduce some sets which are invariant under the flow, in the spirit of [3]. At this point we take advantage of ideas of [25]. Namely, by introducing a constrained approach and playing between various characterization of the ground states, we are able to derive the blow up result without having to solve directly a minimization problem, in contrast to [3]. When $1 < p < 3 + \frac{4}{N}$, we conjecture that the ground states solutions of (1.7) are orbitally stable. However, for the reasons that we shall discuss in Remark 5.2, we did not manage to prove this result. Instead, we consider the stability issue for the minimizers of the problem

$$m(c) = \inf\{\mathcal{E}(u) : \, u \in X, \; \|u\|_2^2 = c\}, \tag{1.10}$$

where the energy $\mathcal{E}$ reads as

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^N} |\nabla |u|^2|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx.$$

Concerning the constrained minimization problem (1.10) we will show that, if $p < 3 + \frac{4}{N}$ then $m(c) > -\infty$ for any $c > 0$. On the contrary, when $p > 3 + \frac{4}{N}$, we have $m(c) = -\infty$ for any $c > 0$.

Denote by $\mathcal{G}(c)$ the set of solutions to (1.10). Our result of orbital stability is the following

Theorem 1.4. *Assume that*

$$1 < p < 3 + \frac{4}{N},$$

and let $c > 0$ be such that $m(c) < 0$. Then $\mathcal{G}(c)$ is non void and orbitally stable. Furthermore, in the two following cases

$$\text{i) } 1 < p < 1 + \frac{4}{N} \textit{ and } c > 0,$$

$$\text{ii) } p > 1 \textit{ and } c > 0 \textit{ is sufficiently large},$$

we have $m(c) < 0$.

Here also notice that, if $u \in \mathcal{G}(c)$, then any $v(x) = e^{i\theta}u(x-y)$ for $\theta \in \mathbb{S}^1$ and $y \in \mathbb{R}^N$ belongs to $\mathcal{G}(c)$.

In Theorem 1.4 when we say that $\mathcal{G}(c)$ is orbitally stable we mean the following: For every $\varepsilon > 0$, there exists $\delta > 0$ such that, for any initial data $a_0 \in X_{\mathbb{C}}$ such that the conclusions of Theorem 1.1 hold when $f(s) = s^{\frac{p-1}{2}}$, if $\inf_{u \in \mathcal{G}(c)} \|a_0 - u\|_{H^1} < \delta$ then the solution $\phi(t, \cdot)$ of (1.2) with initial condition a_0 satisfies

$$\sup_{t \geq 0} \inf_{u \in \mathcal{G}(c)} \|\phi(t, \cdot) - u\|_{H^1} < \varepsilon.$$

The proof of Theorem 1.4 is based on variational methods. Assuming that $m(c) < 0$ we obtain the convergence of any minimizing sequence for (1.10) using concentration compactness arguments and taking advantage of the autonomous nature of (1.10). This convergence result being established, the proof of orbital stability follows in a standard fashion.

Notice that the fact that condition $m(c) < 0$ holds for all $c > 0$ is standard when $1 < p < 1 + \frac{4}{N}$ (see [37]). On the contrary, when $1 + \frac{4}{N} \leq p \leq 3 + \frac{4}{N}$ and $c > 0$ is small we expect that $m(c) = 0$ and that there is no minimizer. In that direction we prove the following.

Theorem 1.5. *Assume that $1+\frac{4}{N} \leq p \leq 3+\frac{4}{N}$ for $N = 1, 2, 3, 4$ and $1+\frac{4}{N} \leq p \leq \frac{N+6}{N-2}$ for $N \geq 5$. Then there exists $c(p, N) > 0$ such that $m(c) = 0$ and $m(c)$ do not admit a minimizer if $0 < c < c(p, N)$.*

We conjecture, see Remark 4.7, that the restriction $p \leq \frac{N+6}{N-2}$ for $N \geq 5$ is not necessary. Namely that the non-existence of a minimizer for all $1 + \frac{4}{N} \leq p \leq 3 + \frac{4}{N}$ hold in any dimension when $c > 0$ gets small.

As we have already mentioned, the paper [1] has motivated the present work. However, we stress that [1] is partially incorrect and it only deals with the orbital stability issue, no results on the Cauchy problem nor of instability are presented. In Remark 5.3 we compare our results to the ones of [1]. Apart from [1], to the best of our knowledge, we are not aware of any other results comparable to those of our paper.

Notations.

(1) For a function $f : \mathbb{R}^N \to \mathbb{R}^N$ and $1 \leq j \leq N$, we denote by $\partial_j f$ the partial derivative with respect to the j^{th} coordinate.
(2) $M(\mathbb{R}^N)$ is the set of measurable functions in $\mathbb{R}^N$. For any $p > 1$ we denote by $L^p(\mathbb{R}^N)$ the space of f in $M(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} |f|^p dx < \infty$.
(3) The norm $(\int_{\mathbb{R}^N} |f|^p dx)^{1/p}$ in $L^p(\mathbb{R}^N)$ is denoted by $\| \cdot \|_p$.
(4) For $s \in \mathbb{N}$, we denote by $H^s(\mathbb{R}^N)$ the Sobolev space of functions f in $L^2(\mathbb{R}^N)$ having generalized partial derivatives $\partial_i^k f$ in $L^2(\mathbb{R}^N)$, for $i = 1, \dots, N$ and $0 \leq k \leq s$.
(5) The norm $(\int_{\mathbb{R}^N} |f|^2 dx + \int_{\mathbb{R}^N} |\nabla f|^2 dx)^{1/2}$ in $H^1(\mathbb{R}^N)$ is denoted by $\| \cdot \|$ and more generally, the norm in H^s is denoted by $\| \cdot \|_{H^s}$.
(6) $\mathcal{L}^N(E)$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}^N$.
(7) For $R > 0$, $B(0, R)$ is the ball in $\mathbb{R}^N$ centered at zero with radius R.
(8) $\Re(z)$ (resp. $\Im(z)$) denotes the real part (resp. the imaginary part) of a complex number z.
(9) For a real number r, we denote by $E(r)$ the integer part of r.
(10) X denotes the restriction of $X_{\mathbb{C}}$ to real functions.

Organization of the paper.

In Section 2, we prove Theorem 1.1 concerning the well-posedness result for equation (1.2). In Section 3 we establish the existence and properties of the ground states solutions of (1.7), Theorem 1.2 and we prove the instability result, Theorem 1.3. In Section 4 we study the minimization problem (1.10). In particular we prove the existence of a minimizer and study whether the condition $m(c) < 0$ holds. Finally, in Section 5, we prove the convergence of all the minimizing sequences of (1.10) and thus derive the stability result, Theorem 1.4.

2. The Cauchy problem

This section is fully devoted to the proof of Theorem 1.1.

We first rewrite equation (1.2) into a system involving ϕ and $\overline{\phi}$ in the following way

$$2i\begin{pmatrix} \phi_t \\ \overline{\phi}_t \end{pmatrix} + \mathcal{A}(\phi)\begin{pmatrix} \Delta\phi \\ \Delta\overline{\phi} \end{pmatrix} - \begin{pmatrix} 2\phi|\nabla\phi|^2 + \phi f(|\phi|^2) \\ -2\overline{\phi}|\nabla\phi|^2 - \overline{\phi} f(|\phi|^2) \end{pmatrix} = 0, \tag{2.1}$$

where

$$\mathcal{A}(\phi) = \begin{pmatrix} 1+|\phi|^2 & \phi^2 \\ -\overline{\phi}^2 & -(1+|\phi|^2) \end{pmatrix}.$$

A direct calculation shows that $\mathcal{A}(\phi)$ is invertible and that

$$\mathcal{A}^{-1}(\phi) = \frac{1}{1+2|\phi|^2}\mathcal{A}(\phi).$$

In order to overcome the loss of derivatives and to linearize the quadratic term involving $\nabla\phi$, we differentiate the equation with respect to space and time variables to obtain a new system in $\phi_0, \dots, \phi_{N+2}$ where $\phi_0 = \phi$ and

$$\forall 1 \le j \le N,\ \ \phi_j = \partial_j\phi,\ \ \phi_{N+1} = e^{g(|\phi|^2)}\phi_t,\ \ \phi_{N+2} = e^{q(|\phi|^2)}\Delta\phi. \tag{2.2}$$

The functions g and q are used as gauge transforms and their role will be explain later. We also set $\Phi^* = (\phi_j)_{j=0}^N$ and $\Phi = (\phi_j)_{j=0}^{N+2}$. Equation (2.1) can be rewritten as

$$2i\begin{pmatrix} (\phi_0)_t \\ (\overline{\phi}_0)_t \end{pmatrix} + \mathcal{A}(\phi_0)\begin{pmatrix} \Delta\phi_0 \\ \Delta\overline{\phi}_0 \end{pmatrix} + \mathcal{F}_0(\Phi^*) = 0, \tag{2.3}$$

where $\mathcal{F}_0$ is a smooth function depending only on Φ^*. Differentiating equation (2.3) with respect to x_j for $j = 1, \dots, N$, we obtain

$$\begin{aligned} 2i\begin{pmatrix} (\phi_j)_t \\ (\overline{\phi}_j)_t \end{pmatrix} + \mathcal{A}(\phi_0)\begin{pmatrix} \Delta\phi_j \\ \Delta\overline{\phi}_j \end{pmatrix} + \sum_{k=1}^N B(\phi_0,\phi_k)\begin{pmatrix} T_{kj}\phi_{N+2} \\ T_{kj}\overline{\phi}_{N+2} \end{pmatrix} \\ + C(\phi_0,\phi_j)\begin{pmatrix} e^{-q(|\phi_0|^2)}\phi_{N+2} \\ e^{-q(|\phi_0|^2)}\overline{\phi}_{N+2} \end{pmatrix} + \begin{pmatrix} F(\Phi^*,\phi_j) \\ -\overline{F}(\Phi^*,\phi_j) \end{pmatrix} = 0, \end{aligned}$$

where B, C and F are smooth functions of their arguments and especially

$$C(\phi_0,\phi_j)=\partial_j\mathcal{A}(\phi_0)=\begin{pmatrix}\phi_0\overline{\phi_j}+\overline{\phi_0}\phi_j & 2\phi_0\phi_j\\ -2\overline{\phi}_0\overline{\phi}_j & -\phi_0\overline{\phi_j}-\overline{\phi_0}\phi_j\end{pmatrix}.$$

For $i,j=1,\dots,N$, T_{ij} is the following operator of order 0

$$T_{ij}\phi=\partial_i\partial_j\Delta^{-1}(e^{-q(|u_0|^2)}\phi).$$

We can rewrite these equations as follows

$$2i\begin{pmatrix}(\phi_j)_t\\ (\overline{\phi}_j)_t\end{pmatrix}+\mathcal{A}(\phi_0)\begin{pmatrix}\Delta\phi_j\\ \Delta\overline{\phi}_j\end{pmatrix}+\mathcal{F}_j(\Phi^*,\phi_{N+2},T\phi_{N+2})=0, \tag{2.4}$$

where $\mathcal{F}_j$ is a smooth function of its arguments. Differentiating equation (2.3) with respect to t, we derive

$$\begin{aligned}&2i\begin{pmatrix}(e^{-f(|\phi_0|^2)}\phi_{N+1})_t\\ (e^{-f(|\phi_0|^2)}\overline{\phi}_{N+1})_t\end{pmatrix}+C(\phi_0,e^{-f(|\phi_0|^2)}\phi_{N+1})\begin{pmatrix}e^{-q(|\phi_0|^2)}\phi_{N+2}\\ e^{-q(|\phi_0|^2)}\overline{\phi}_{N+2}\end{pmatrix}\\ &+\mathcal{A}(\phi_0)\begin{pmatrix}\Delta(e^{-f(|\phi_0|^2)}\phi_{N+1})\\ \Delta(e^{-f(|\phi_0|^2)}\overline{\phi}_{N+1})\end{pmatrix}+\sum_{k=1}^{N}B(\phi_0,\phi_k)\begin{pmatrix}\partial_k(e^{-f(|\phi_0|^2)}\phi_{N+1})\\ \partial_k(e^{-f(|\phi_0|^2)}\overline{\phi}_{N+1})\end{pmatrix}\\ &+\begin{pmatrix}F(\Phi^*,e^{-f(|\phi_0|^2)}\phi_{N+1})\\ -\overline{F}(\Phi^*,e^{-f(|\phi_0|^2)}\phi_{N+1})\end{pmatrix}=0,\end{aligned} \tag{2.5}$$

which can be rewritten as

$$\begin{aligned}&2i\begin{pmatrix}(\phi_{N+1})_t\\ (\overline{\phi}_{N+1})_t\end{pmatrix}+\mathcal{A}(\phi_0)\begin{pmatrix}\Delta\phi_{N+1}\\ \Delta\overline{\phi}_{N+1}\end{pmatrix}+\sum_{k=1}^{N}\mathcal{D}(\phi_0,\phi_k)\begin{pmatrix}\partial_k\phi_{N+1}\\ \partial_k\overline{\phi}_{N+1}\end{pmatrix}\\ &+\mathcal{G}(\Phi,T\phi_{N+2})=0,\end{aligned} \tag{2.6}$$

where $\mathcal{D}$ and $\mathcal{G}$ are smooth functions of their arguments. By applying the operator Δ on equation (2.3), we obtain

$$\begin{aligned}&2i\begin{pmatrix}(\phi_{N+2})_t\\ (\overline{\phi}_{N+2})_t\end{pmatrix}+\mathcal{A}(\phi_0)\begin{pmatrix}\Delta\phi_{N+2}\\ \Delta\overline{\phi}_{N+2}\end{pmatrix}+\sum_{k=1}^{N}\mathcal{E}(\phi_0,\phi_k)\begin{pmatrix}\partial_k\phi_{N+2}\\ \partial_k\overline{\phi}_{N+2}\end{pmatrix}\\ &+\mathcal{I}(\Phi,T\phi_{N+2})=0,\end{aligned} \tag{2.7}$$

where $\mathcal{E}$ and $\mathcal{I}$ are also smooth functions of their arguments. At this point, we need to make more precise the matrices B, $\mathcal{D}$ and $\mathcal{E}$ since they represent the quasi-linear part of the equations. A direct computation gives

$$B(\phi_0,\phi_k)=\begin{pmatrix}2\phi_0\overline{\phi}_k & 2\phi_0\phi_k\\ -2\overline{\phi}_0\overline{\phi}_k & -2\overline{\phi}_0\phi_k\end{pmatrix},$$

$$\mathcal{D}(\phi_0,\phi_k)=B(\phi_0,\phi_k)-2f'(|\phi_0|^2)\mathcal{A}(\phi_0)\begin{pmatrix}\phi_0\overline{\phi}_k+\overline{\phi}_0\phi_k & 0\\ 0 & \phi_0\overline{\phi}_k+\overline{\phi}_0\phi_k\end{pmatrix},$$

$$\begin{aligned}\mathcal{E}(\phi_0,\phi_k) =& B(\phi_0,\phi_k)+2C(\phi_0,\phi_k)\\ &-2\mathcal{A}(\phi_0)q^{'}(|\phi_0|^2)\left(\begin{array}{cc}\phi_0\overline{\phi}_k+\overline{\phi}_0\phi_k & 0\\ 0 & \phi_0\overline{\phi}_k+\overline{\phi}_0\phi_k\end{array}\right).\end{aligned}$$

Usual energy estimates for Schrödinger equations requires that the diagonal coefficients of $\mathcal{D}$ and $\mathcal{E}$ in equations (2.6) and (2.7) are purely imaginary. Roughly speaking, this allows to integrate by parts the bad terms including first order derivatives of the unknown. This is why we make use of gauge transforms g and q. Finally, in order to avoid any smallness assumption on the initial data, we need to transform slightly equation (2.3) in the following way. We multiply the equation by $\mathcal{A}^{-1}(\phi_0)$ and we split the matrix in front of the time derivatives of ϕ_0 into

$$\mathcal{A}^{-1}(\phi_0)=Id+\Big(\mathcal{A}^{-1}(\phi_0)-Id\Big),$$

where Id is the 2×2 identity matrix. Then recalling that $\partial_t\phi_0=e^{-g(|\phi_0|^2)}\phi_{N+1}$, we rewrite equation (2.3) in

$$2i\left(\begin{array}{c}(\phi_0)_t\\ (\overline{\phi}_0)_t\end{array}\right)+\left(\begin{array}{c}\Delta\phi_0\\ \Delta\overline{\phi}_0\end{array}\right)+\mathcal{G}_0(\Phi)=0, \tag{2.8}$$

where

$$\mathcal{G}_0(\Phi)=\mathcal{A}^{-1}(\phi_0)\mathcal{F}_0(\Phi^*)+ie^{-g(|\phi_0|^2)}\Big(\mathcal{A}^{-1}(\phi_0)-Id\Big)\left(\begin{array}{c}\phi_{N+1}\\ \overline{\phi}_{N+1}\end{array}\right).$$

We then have transformed equation (1.2) into the following system

$$2i\left(\begin{array}{c}(\phi_0)_t\\ (\overline{\phi}_0)_t\end{array}\right)+\left(\begin{array}{c}\Delta\phi_0\\ \Delta\overline{\phi}_0\end{array}\right)+\mathcal{G}_0(\Phi)=0, \tag{2.9}$$

for $j=1,\dots,N$

$$2i\left(\begin{array}{c}(\phi_j)_t\\ (\overline{\phi}_j)_t\end{array}\right)+\mathcal{A}(\phi_0)\left(\begin{array}{c}\Delta\phi_j\\ \Delta\overline{\phi}_j\end{array}\right)+\mathcal{F}_j(\Phi^*,\phi_{N+2},T\phi_{N+2})=0, \tag{2.10}$$

$$\begin{aligned}&2i\left(\begin{array}{c}(\phi_{N+1})_t\\ (\overline{\phi}_{N+1})_t\end{array}\right)+\mathcal{A}(\phi_0)\left(\begin{array}{c}\Delta\phi_{N+1}\\ \Delta\overline{\phi}_{N+1}\end{array}\right)+\sum_{k=1}^{N}\mathcal{D}(\phi_0,\phi_k)\left(\begin{array}{c}\partial_k\phi_{N+1}\\ \partial_k\overline{\phi}_{N+1}\end{array}\right)\\ &+\mathcal{G}(\Phi,T\phi_{N+2})=0,\end{aligned} \tag{2.11}$$

$$\begin{aligned}&2i\left(\begin{array}{c}(\phi_{N+2})_t\\ (\overline{\phi}_{N+2})_t\end{array}\right)+\mathcal{A}(\phi_0)\left(\begin{array}{c}\Delta\phi_{N+2}\\ \Delta\overline{\phi}_{N+2}\end{array}\right)+\sum_{k=1}^{N}\mathcal{E}(\phi_0,\phi_k)\left(\begin{array}{c}\partial_k\phi_{N+2}\\ \partial_k\overline{\phi}_{N+2}\end{array}\right)\\ &+\mathcal{I}(\Phi,T\phi_{N+2})=0.\end{aligned} \tag{2.12}$$

We now apply a fixed point theorem to system (2.9)-(2.12). Let s be as in Theorem 1.1 and introduce the function space

$$\mathcal{X}_T=\left\{\begin{array}{l}\Phi=(\phi_j)_{j=0}^{N+2}:\phi_j\in C([0,T];L^2(\mathbb{R}^N))\cap L^\infty(0,T;H^s(\mathbb{R}^N)),\\ \|\Phi\|_{X_T}=\sum_{j=0}^{N+2}\sup_{0\leq t\leq T}\|\phi_j(t)\|_{H^s(\mathbb{R}^N)}<\infty\end{array}\right\}.$$

For $M=(m_j)_{j=0}^{N+2}\in(\mathbb{R}_+^*)^{N+3}$ and $r\in\mathbb{R}_+^*$, we denote

$$\mathcal{X}_T(M,r)=\left\{\begin{array}{l}\Phi=(\phi_j)_{j=0}^{N+2}\in\mathcal{X}_T:\forall j=0,..,N+2\ \ \|\phi_j\|_{L^\infty(0,T;H^s(\mathbb{R}^N))}\le m_j\\ \|(\phi_0)_t\|_{L^\infty(0,T;H^{E(\frac{N}{2})+1}(\mathbb{R}^N))}\le r \text{ and } \phi_0(0,x)=a_0(x)\end{array}\right\},$$

and let $\Psi=(\psi_j)_{j=0}^{N+2}\in\mathcal{X}_T(M,r)$. Denote $\Psi^*=(\psi_j)_{j=0}^{N}$ and consider the linearized version of system (2.9)-(2.12) as follows

$$2i\begin{pmatrix}(\phi_0)_t\\ (\overline{\phi}_0)_t\end{pmatrix}+\begin{pmatrix}\Delta\phi_0\\ \Delta\overline{\phi}_0\end{pmatrix}+\mathcal{G}_0(\Psi)=0, \tag{2.13}$$

for $j=1,\dots,N$

$$2i\begin{pmatrix}(\phi_j)_t\\ (\overline{\phi}_j)_t\end{pmatrix}+\mathcal{A}(\psi_0)\begin{pmatrix}\Delta\phi_j\\ \Delta\overline{\phi}_j\end{pmatrix}+\mathcal{F}_j(\Psi^*,\psi_{N+2},T\psi_{N+2})=0, \tag{2.14}$$

$$\begin{aligned}&2i\begin{pmatrix}(\phi_{N+1})_t\\ (\overline{\phi}_{N+1})_t\end{pmatrix}+\mathcal{A}(\psi_0)\begin{pmatrix}\Delta\phi_{N+1}\\ \Delta\overline{\phi}_{N+1}\end{pmatrix}+\sum_{k=1}^{N}\mathcal{D}(\psi_0,\psi_k)\begin{pmatrix}\partial_k\phi_{N+1}\\ \partial_k\overline{\phi}_{N+1}\end{pmatrix}\\ &+\mathcal{G}(\Psi,T\psi_{N+2})=0,\end{aligned} \tag{2.15}$$

$$\begin{aligned}&2i\begin{pmatrix}(\phi_{N+2})_t\\ (\overline{\phi}_{N+2})_t\end{pmatrix}+\mathcal{A}(\psi_0)\begin{pmatrix}\Delta\phi_{N+2}\\ \Delta\overline{\phi}_{N+2}\end{pmatrix}+\sum_{k=1}^{N}\mathcal{E}(\psi_0,\psi_k)\begin{pmatrix}\partial_k\phi_{N+2}\\ \partial_k\overline{\phi}_{N+2}\end{pmatrix}\\ &+\mathcal{I}(\Psi,T\psi_{N+2})=0.\end{aligned} \tag{2.16}$$

Let $\mathcal{Z}=\left[L^\infty(0,T;H^s(\mathbb{R}^N))\cap C([0,T];L^2(\mathbb{R}^N))\right]^{N+3}$. Then the Cauchy problem (2.13)-(2.16) with initial condition

$$\begin{aligned}&\phi_0(0,x)=a_0(x),\ \text{for } j=1,\dots,N,\ \phi_j(0,x)=\partial_j a_0(x),\\ &\phi_{N+1}(0,x)=\frac{1}{2i}e^{g(|a_0(x)|^2)}\left(-\mathcal{A}(\phi_0(0))\Delta a_0(x)-\mathcal{F}_0(\Psi^*(0))\right),\\ &\phi_{N+2}(0,x)=e^{q(|a_0(x)|^2)}\Delta a_0(x),\end{aligned}$$

defines a mapping $\mathcal{S}$

$$\begin{aligned}\mathcal{S}:\ \mathcal{Z}&\longrightarrow\mathcal{Z}\\ \Psi&\longmapsto\Phi.\end{aligned}$$

For more details on the existence result for system (2.13)-(2.16), we refer to [10] and [34]. In order to prove Theorem 1.1, we have to find a time $T>0$ and constants $M\in(\mathbb{R}_+^*)^{N+3}$ and $r\in\mathbb{R}_+^*$ such that $\mathcal{S}$ maps the closed ball $\mathcal{X}_T(M,r)$ into itself and is a contraction mapping under the constraint that it acts on $\mathcal{X}_T(M,r)$ in the norm $\sum_{j=0}^{N+2}\sup_{t\in[0,T]}\|\phi_j\|_{L^2}$. We begin with equation (2.16) and perform an H^s-estimate.

Following [10], we apply the operator $(1-\Delta)^{\frac{s}{2}}$ on equation (2.16) and multiply the resulting equation by $\mathcal{A}^{-1}(\phi_0)$ to obtain, denoting $\chi = (1-\Delta)^{\frac{s}{2}}\phi_{N+2}$

$$\begin{aligned} &2i\mathcal{A}^{-1}(\psi_0)\begin{pmatrix} (\chi)_t \\ (\overline{\chi})_t \end{pmatrix} + \begin{pmatrix} \Delta\chi \\ \Delta\overline{\chi} \end{pmatrix} + \sum_{k=1}^{N} \mathcal{L}(\psi_0, \psi_k, \partial_k\psi_0)\begin{pmatrix} \partial_k\chi \\ \partial_k\overline{\chi}_{N+2} \end{pmatrix} \\ &+\mathcal{J}^s_{j=0}(D^j\Psi, D^j\phi_{N+2}, T\psi_{N+2}) = 0 \end{aligned} \tag{2.17}$$

where D^j denotes any space derivation of order less or equal to s with respect to the j^{th} space coordinate. The matrix $\mathcal{L}$ reads

$$\mathcal{L}(\psi_0, \psi_k, \partial_k\psi_0) = \mathcal{A}^{-1}(\psi_0)\Big(\mathcal{E}(\psi_0, \psi_k) + s\partial_k\mathcal{A}(\psi_0)\Big).$$

We notice here that the dependence of $\mathcal{J}$ in ϕ_{N+2} and its derivatives is affine. We are now able to choose the gauge transform q. Recall that

$$\begin{aligned} \mathcal{E}(\psi_0, \psi_k) =& B(\psi_0, \psi_k) + 2C(\psi_0, \psi_k) \\ & - 2\mathcal{A}(\psi_0)q'(|\psi_0|^2)\begin{pmatrix} \psi_0\overline{\psi}_k + \overline{\psi}_0\psi_k & 0 \\ 0 & \psi_0\overline{\psi}_k + \overline{\psi}_0\psi_k \end{pmatrix}, \end{aligned}$$

a direct calculation shows that for $j = 1, 2$ (denoting by b^{11} and b^{22} the diagonal coefficients of a 2x2 matrix b),

$$\Re\Big(\mathcal{A}^{-1}(\psi_0)\big(B(\psi_0, \psi_k) + 2C(\psi_0, \psi_k)\big)\Big)^{jj} = \frac{3}{1+2|\psi|^2}\big(\psi_0\overline{\psi}_k + \overline{\psi}_0\psi_k\big).$$

Then choosing

$$q(\sigma) = \frac{3}{4}\ln(1+2\sigma)$$

gives

$$\Re\Big(\mathcal{A}^{-1}(\psi_0)\mathcal{E}(\psi_0, \psi_k)\Big)^{jj} = 0.$$

Furthermore, by differentiating equation (2.17) s times in space, we add in matrix $\mathcal{L}$ the term $s\mathcal{A}^{-1}(\psi_0)\partial_k\mathcal{A}(\psi_0)$ which is not eliminated by q. As a consequence we have to use a second gauge transform by putting $\kappa = e^{b(|\psi_0|^2)}\chi$ solution to

$$\begin{aligned} &2i\mathcal{A}^{-1}(\psi_0)\begin{pmatrix} (\kappa)_t \\ (\overline{\kappa})_t \end{pmatrix} + \begin{pmatrix} \Delta\kappa \\ \Delta\overline{\kappa} \end{pmatrix} + \sum_{k=1}^{N} \mathcal{M}(\psi_0, \psi_k, \partial_k\psi_0)\begin{pmatrix} \partial_k\kappa \\ \partial_k\overline{\kappa}_{N+2} \end{pmatrix} \\ &+\mathcal{K}^s_{j=0}(D^j\Psi, D^j\phi_{N+2}, T\psi_{N+2}, (\psi_0)_t) = 0, \end{aligned} \tag{2.18}$$

where

$$\mathcal{M}(\psi_0, \psi_k, \partial_k\psi_0) = \mathcal{L}(\psi_0, \psi_k, \partial_k\psi_0) - 2\begin{pmatrix} \partial_k b(|\psi_0|^2) & 0 \\ 0 & \partial_k b(|\psi_0|^2) \end{pmatrix}.$$

Note that the matrix $\mathcal{K}$ depends also on $(\psi_0)_t$. Once again, an easy calculation shows that if we choose b such that

$$b(\sigma) = \frac{s}{4}\ln(1+2\sigma),$$

then for $j = 1, 2$

$$\Re\Big(s\mathcal{A}^{-1}(\psi_0)\partial_k\mathcal{A}(\psi_0) - 2\begin{pmatrix} \partial_k b(|\psi_0|^2) & 0 \\ 0 & \partial_k b(|\psi_0|^2) \end{pmatrix}\Big)^{jj} = 0.$$

We are now able to perform the suitable energy estimate on equation (2.18). Multiplying equation (2.18) by $\overline{\kappa}$, integrate over $\mathbb{R}^N$ and taking the first line of the resulting system leads to

$$\begin{aligned} & i\int_{\mathbb{R}^N} \frac{1+|\psi_0|^2}{1+2|\psi_0|^2}\kappa_t\overline{\kappa}dx + i\int_{\mathbb{R}^N} \frac{\psi_0^2}{1+2|\psi_0|^2}\overline{\kappa}_t\overline{\kappa}dx + \int_{\mathbb{R}^N} \Delta\kappa\overline{\kappa}dx \\ & + \sum_{k=1}^N \int_{\mathbb{R}^N} \mathcal{M}^{11}(\psi_0, \psi_k, \partial_k\psi_0)(\partial_k\kappa)\overline{\kappa}dx \\ & + \int_{\mathbb{R}^N} \mathcal{M}^{12}(\psi_0, \psi_k, \partial_k\psi_0)(\partial_k\overline{\kappa})\overline{\kappa}dx \\ & + \int_{\mathbb{R}^N} \mathcal{K}_{j=0}^s(D^j\Psi, D^j\phi_{N+2}, T\psi_{N+2}, (\psi_0)_t)\overline{\kappa}dx. \end{aligned} \tag{2.19}$$

We take the imaginary part of equation (2.19). We have

$$\begin{aligned} & \Im\Big(i\int_{\mathbb{R}^N} \frac{1+|\psi_0|^2}{1+2|\psi_0|^2}\kappa_t\overline{\kappa}dx + i\int_{\mathbb{R}^N} \frac{\psi_0^2}{1+2|\psi_0|^2}\overline{\kappa}_t\overline{\kappa}dx\Big) \\ & = \int_{\mathbb{R}^N} \frac{1+|\psi_0|^2}{2+4|\psi_0|^2}|\kappa|_t^2 dx + \int_{\mathbb{R}^N} \Big(\frac{\psi_0^2}{4(1+2|\psi_0|^2)}(\overline{\kappa}^2)_t + \frac{\overline{\psi}_0^2}{4(1+2|\psi_0|^2)}(\kappa^2)_t\Big)dx \\ & = \frac{d}{dt}\Big(\int_{\mathbb{R}^N} \frac{1+|\psi_0|^2}{2+4|\psi_0|^2}|\kappa|^2 dx + \int_{\mathbb{R}^N} \Big(\frac{\psi_0^2}{4(1+2|\psi_0|^2)}\overline{\kappa}^2 + \frac{\overline{\psi}_0^2}{4(1+2|\psi_0|^2)}\kappa^2\Big)dx\Big) \\ & - \int_{\mathbb{R}^N} \Big(\frac{1+|\psi_0|^2}{2+4|\psi_0|^2}\Big)_t|\kappa|^2 dx - \int_{\mathbb{R}^N} \Big(\Big(\frac{\psi_0^2}{4(1+2|\psi_0|^2)}\Big)_t\overline{\kappa}^2 + \Big(\frac{\overline{\psi}_0^2}{4(1+2|\psi_0|^2)}\Big)_t\kappa^2\Big)dx. \end{aligned}$$

The other terms in equation (2.19) are classical and can be treated exactly as in [10]. The important point to notice is that since the diagonal coefficients of $\mathcal{M}$ are pure imaginary, one has for $k = 1, \ldots, N$

$$\begin{aligned} \Im\Big(\int_{\mathbb{R}^N} \mathcal{M}^{11}(\psi_0, \psi_k, \partial_k\psi_0)(\partial_k\kappa)\overline{\kappa}dx\Big) & = \int_{\mathbb{R}^N} \mathcal{I}m\big(\mathcal{M}^{11}(\psi_0, \psi_k, \partial_k\psi_0)\big)\partial_k\frac{|\kappa|^2}{2}dx, \\ & = -\int_{\mathbb{R}^N} \partial_k\Big(\mathcal{I}m\big(\mathcal{M}^{11}(\psi_0, \psi_k, \partial_k\psi_0)\big)\Big)\frac{|\kappa|^2}{2}dx, \end{aligned}$$

by integration by parts. This allows to overcome the loss of derivatives of this quasi-linear Schrödinger equation and brings the following estimate

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{\mathbb{R}^N}\frac{1+|\psi_0|^2}{2+4|\psi_0|^2}|\kappa|^2dx+\int_{\mathbb{R}^N}\Big(\frac{\psi_0^2}{4(1+2|\psi_0|^2)}\overline{\kappa}^2+\frac{\overline{\psi}_0^2}{4(1+2|\psi_0|^2)}\kappa^2\Big)dx\Big)\\ &\le 4\int_{\mathbb{R}^N}(|\psi_0|^2)_t|\kappa|^2dx+C_1(M,r)\|\kappa\|_2^2,\end{aligned}\tag{2.20}$$

where $C_1(M,r)$ is a constant depending only on M and r. To derive inequality (2.20), we have used the fact that

$$\Big(\frac{1+|\psi_0|^2}{2+4|\psi_0|^2}\Big)_t=\frac{(|\psi_0|^2)_t}{2+4|\psi_0|^2}-4\Big(\frac{1+|\psi_0|^2}{(2+4|\psi_0|^2)^2}\Big)(|\psi_0|^2)_t$$

$$\Big(\frac{\psi_0^2}{4(1+2|\psi_0|^2)}\Big)_t=\frac{(\psi_0^2)_t}{4(1+2|\psi_0|^2)}-\Big(\frac{\psi_0^2}{2(1+2|\psi_0|^2)^2}\Big)(|\psi_0|^2)_t$$

$$\Big(\frac{\overline{\psi}_0^2}{4(1+2|\psi_0|^2)}\Big)_t=\frac{(\overline{\psi}_0^2)_t}{4(1+2|\psi_0|^2)}-\Big(\frac{\overline{\psi}_0^2}{2(1+2|\psi_0|^2)^2}\Big)(|\psi_0|^2)_t$$

and then

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N}\Big(\frac{1+|\psi_0|^2}{2+4|\psi_0|^2}\Big)_t|\kappa|^2dx-\int_{\mathbb{R}^N}\Big(\Big(\frac{\psi_0^2}{4(1+2|\psi_0|^2)}\Big)_t\overline{\kappa}^2+\Big(\frac{\overline{\psi}_0^2}{4(1+2|\psi_0|^2)}\Big)_t\kappa^2\Big)dx\Big|\\ &\le 4\int_{\mathbb{R}^N}(|\psi_0|^2)_t|\kappa|^2dx.\end{aligned}$$

Using the fact that

$$\sup_{t\in[0,T]}\|(\psi_0)_t\|_{H^{E(\frac{N}{2})+1}(\mathbb{R}^N)}\le r$$

and the continuous embedding $H^{E(\frac{N}{2})+1}(\mathbb{R}^N)\hookrightarrow L^\infty(\mathbb{R}^N)$, we can find a constant $C_2(M,r)$ such that

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{\mathbb{R}^N}\frac{1+|\psi_0|^2}{2+4|\psi_0|^2}|\kappa|^2dx+\int_{\mathbb{R}^N}\Big(\frac{\psi_0^2}{4(1+2|\psi_0|^2)}\overline{\kappa}^2+\frac{\overline{\psi}_0^2}{4(1+2|\psi_0|^2)}\kappa^2\Big)dx\Big)\\ &\le C_2(M,r)\|\kappa\|_2^2.\end{aligned}\tag{2.21}$$

Integrating inequality (2.21) from 0 to t gives

$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{1+|\psi_0(t)|^2}{2+4|\psi_0(t)|^2}|\kappa(t)|^2dx+\int_{\mathbb{R}^N}\Big(\frac{\psi_0^2(t)}{4(1+2|\psi_0(t)|^2)}\overline{\kappa}^2(t)+\frac{\overline{\psi}_0^2(t)}{4(1+2|\psi_0(t)|^2)}\kappa^2(t)\Big)dx\\ &\le\int_{\mathbb{R}^N}\frac{1+|\psi_0(0)|^2}{2+4|\psi_0(0)|^2}|\kappa(0)|^2dx+\int_{\mathbb{R}^N}\Big(\frac{\psi_0^2(0)}{4(1+2|\psi_0(0)|^2)}\overline{\kappa}^2(0)+\frac{\overline{\psi}_0^2(0)}{4(1+2|\psi_0(0)|^2)}\kappa^2(0)\Big)dx\\ &+C_2(M,r)\int_0^t\|\kappa(s)\|_2^2ds.\end{aligned}$$

For all $t \in [0,T]$, we have

$$\frac{1+|\psi_0(t)|^2}{2+4|\psi_0(t)|^2}|\kappa(t)|^2+\frac{\psi_0^2(t)}{4(1+2|\psi_0(t)|^2)}\overline{\kappa}^2(t)+\frac{\overline{\psi}_0^2(t)}{4(1+2|\psi_0(t)|^2)}\kappa^2(t) \geq \frac{1}{2+4|\psi_0(t)|^2}|\kappa(t)|^2.$$

Denoting by

$$CI_{N+2}(0) = \int_{\mathbb{R}^N} \frac{1+|\psi_0(0)|^2}{2+4|\psi_0(0)|^2}|\kappa(0)|^2dx+\int_{\mathbb{R}^N} \Big(\frac{\psi_0^2(0)}{4(1+2|\psi_0(0)|^2)}\overline{\kappa}^2(0)+\frac{\overline{\psi}_0^2(0)}{4(1+2|\psi_0(0)|^2)}\kappa^2(0)\Big)dx,$$

we derive

$$\int_{\mathbb{R}^N} \frac{1}{2+4|\psi_0(t)|^2}|\kappa(t)|^2dx \leq CI_{N+2}(0) + C_2(M,r)\int_0^t \|\kappa(s)\|_2^2 ds. \tag{2.22}$$

Recalling that $\psi_0 \in L^\infty(0,T;H^s(\mathbb{R}^N))$ and the continuous embedding $H^s(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ and denote by C_b the best constant of this embedding, we have

$$\|\psi_0(t)\|_{L^\infty(\mathbb{R}^N)} \leq C_b m_0.$$

This provides

$$\int_{\mathbb{R}^N} \frac{1}{2+4|\psi_0(t)|^2}|\kappa(t)|^2dx \geq \frac{1}{2+4C_b^2m_0^2}\int_{\mathbb{R}^N} |\kappa(t)|^2dx$$

which gives

$$\int_{\mathbb{R}^N} |\kappa(t)|^2dx \leq (2+4C_b^2m_0^2)\Big(CI_{N+2}(0) + C_2(M,r)\int_0^t \|\kappa(s)\|_2^2 ds\Big). \tag{2.23}$$

Since the gauge transform b does not depend of ψ_0 and for all $t \in [0,T]$, $\|\psi_0(t)\|_{L^\infty(\mathbb{R}^N)} \leq m_0$, there is a constant $C(m_0)$ depending only on m_0 such that

$$\sup_{t\in[0,T]} \|e^{-b(|\psi_0(t)|^2)}\|^2_{L^\infty(\mathbb{R}^N)} \leq C(m_0).$$

Recalling that $\kappa(0) = e^{p(|a_0|^2)}(1-\Delta)^{\frac{s}{2}}(e^{q(|a_0|^2)}\Delta a_0)$ and choosing m_{N+2} such that

$$m_{N+2}^2 \geq 2C(m_0)(2+4C_b^2m_0^2)CI_{N+2}(0)+1, \tag{2.24}$$

one can find a positive T such that for this choice of m_{N+2}

$$\sup_{t\in[0,T]} \|\phi_{N+2}\|_{H^s(\mathbb{R}^N)} \leq m_{N+2}. \tag{2.25}$$

Note that m_{N+2} depends only on the initial data a_0 and m_0. Performing the same kind of estimates on equations (2.16), one can find a positive T and constant m_{N+1} depending only on a_0 and m_0 satisfying

$$m_{N+1}^2 \geq 2C(m_0)(2+4C_b^2m_0^2)CI_{N+1}(0)+1, \tag{2.26}$$

where

$$CI_{N+1}(0) = \int_{\mathbb{R}^N} \frac{1+|\psi_0(0)|^2}{2+4|\psi_0(0)|^2}|\nu(0)|^2 dx \\ + \int_{\mathbb{R}^N} \Big(\frac{\psi_0^2(0)}{4(1+2|\psi_0(0)|^2)}\overline{\nu}^2(0) + \frac{\overline{\psi}_0^2(0)}{4(1+2|\psi_0(0)|^2)}\nu^2(0)\Big) dx$$

with

$$\nu(0) = e^{p(|a_0|^2)}(1-\Delta)^{\frac{s}{2}}(e^{g(|a_0|^2)}\partial_t a_0),$$

such that

$$\sup_{t\in[0,T]} \|\phi_{N+1}\|_{H^s(\mathbb{R}^N)} \leq m_{N+1}. \tag{2.27}$$

Dealing with equation (2.14), we introduce for $j = 1, \dots, N$

$$\mu_j(0) = (1-\Delta)^{\frac{s}{2}}\partial_j a_0$$

and

$$CI_j(0) = \int_{\mathbb{R}^N} \frac{1+|\psi_0(0)|^2}{2+4|\psi_0(0)|^2}|\mu_j(0)|^2 dx \\ + \int_{\mathbb{R}^N} \Big(\frac{\psi_0^2(0)}{4(1+2|\psi_0(0)|^2)}\overline{\mu}_j^2(0) + \frac{\overline{\psi}_0^2(0)}{4(1+2|\psi_0(0)|^2)}\mu_j^2(0)\Big) dx.$$

Choosing m_j depending only on a_0 and m_0 such that

$$m_j^2 \geq 2(2+4C_b^2 m_0^2)CI_j(0) + 1, \tag{2.28}$$

we derive

$$\text{for } j = 1, \dots, N, \sup_{t\in[0,T]} \|\phi_j\|_{H^s(\mathbb{R}^N)} \leq m_j. \tag{2.29}$$

Treating now equation (2.13), we introduce

$$\xi(0) = (1-\Delta)^{\frac{s}{2}} a_0$$

and

$$CI_0(0) = \int_{\mathbb{R}^N} |\xi(0)|^2 dx.$$

It is crucial to remark here that equation (2.13) is not quasi-linear. As a consequence, we can perform a classical energy estimate on it and choose the constant m_0 such that

$$m_0^2 \geq 2CI_0(0) + 1. \tag{2.30}$$

The choice of m_0 depends only on the initial data a_0.

Remark 2.1. If we work with equation (2.3) instead of equation (2.9) and perform the energy estimates of equation (2.16) for example, we have to choose m_0 such that

$$m_0^2 \geq 2(2+4C_b^2 m_0^2) CI_0(0)+1$$

where

$$\begin{aligned}\widetilde{C}I_0(0) &= \int_{\mathbb{R}^N} \frac{1+|\psi_0(0)|^2}{2+4|\psi_0(0)|^2}|\xi(0)|^2 dx \\ &\quad + \int_{\mathbb{R}^N} \Big(\frac{\psi_0^2(0)}{4(1+2|\psi_0(0)|^2)}\overline{\xi}^2(0) + \frac{\overline{\psi}_0^2(0)}{4(1+2|\psi_0(0)|^2)}\xi^2(0)\Big) dx.\end{aligned}$$

Such a choice requires of course a smallness assumption on the initial data a_0.

Let us take m_0 as in (2.30). Then one can find also a positive T such that

$$\sup_{t\in[0,T]} \|\phi_0\|_{H^s(\mathbb{R}^N)} \leq m_0. \tag{2.31}$$

We refer to [10] for the technical details. Due to the structure of the space $\mathcal{X}_T$, it remains to estimate $(\psi_0)_t$ in $H^{E(\frac{N}{2})+1}(\mathbb{R}^N)$. This is done directly on equation (2.13) and provides that there exists a constant $C_0(M)$ depending only on M such that

$$\sup_{t\in[0,T]} \|(\phi_0)_t\|_{H^{E(\frac{N}{2})+1}(\mathbb{R}^N)} \leq C_0(M). \tag{2.32}$$

As a conclusion, we choose constants M, r and T as follows. We first fix m_0 depending only on a_0 such that (2.30) holds. Then we take (m_j), m_{N+1} and m_{N+2} depending only on a_0 and m_0 satisfying respectively (2.28), (2.26) and (2.24). Finally take r such that

$$r \geq C_0(M),$$

and T sufficiently small such that

$$C_4(M,r)T \leq \frac{1}{2},$$

and similar conditions to take into account the equations on ϕ_0, ϕ_j and ϕ_{N+1}. For such a choice of parameter, we have showed

$$\mathcal{S}\Big(\mathcal{X}_T(M,r)\Big) \subset \mathcal{X}_T(M,r).$$

The fact that the mapping $\mathcal{S}$ is a contraction for the suitable norm is very standard and we refer once again to [10] since the proof reads exactly the same. By the contraction mapping principle, there exists a unique solution

$$\Phi = \Big(\phi_0, (\phi_j)_{j=0}^N, \phi_{N+1}, \phi_{N+2}\Big)$$

to system (2.13)-(2.16). Furthermore, for each $0 \leq j \leq N+2$, the function ϕ_j satisfies

$$\phi_j \in L^\infty(0,T; H^s(\mathbb{R}^N)) \cap C([0,T]; L^2(\mathbb{R}^N)).$$

To conclude the proof, we have to show that the solution Φ solves system (2.9)-(2.12) and has the following regularity

$$\Phi \in \Big(L^\infty(0,T;H^{s+2}(\mathbb{R}^N)) \cap C([0,T];H^s(\mathbb{R}^N))\Big)^{N+3}.$$

This can be done exactly as in [10]. The proof of the conservation laws (1.4)-(1.5) is very standard once we have proved that ϕ is regular and so we omit it. At this point the proof of Theorem 1.1 is completed.

3. Existence of ground states and orbital instability

In this section we derive the existence, as well as some qualitative properties, of the ground states solutions of (1.7). When $p > 3 + \frac{4}{N}$ we shall also prove that the ground states are instable by blow-up.

We begin with the following Pohozaev-type identity.

Lemma 3.1. *Any $u \in X_{\mathbb{C}}$ solution of (1.7) satisfies $P(u) = 0$ where $P : X_{\mathbb{C}} \to \mathbb{R}$ is the function defined by*

$$P(u) = \frac{N-2}{N}\Big(\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx + \int_{\mathbb{R}^N}|u|^2|\nabla|u||^2dx\Big) + \frac{\omega}{2}\int_{\mathbb{R}^N}|u|^2dx - \frac{1}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}dx.$$

Proof. Since the proof only uses classical arguments, we shall just sketch it and refer to [11] for further details. Let $u \in X_{\mathbb{C}}$ be a solution to equation (1.7). From [30, Section 6. Appendix] we learn that $u \in L^\infty_{\rm loc}(\mathbb{R}^N)$. We are then able to pursue as in [11, Proposition 2.1]. Let $\psi \in C_0^\infty(\mathbb{R}^N)$ be such that $\psi \geq 0$, $\mathrm{supp}(\psi) \subset B(0,2)$ and $\psi \equiv 1$ on $B(0,1)$. For all $j \in \mathbb{N}^*$, we set $\psi_j(x) = \psi(\frac{x}{j})$. Now let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of even positive functions in $L^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}\rho_n dx = 1$ such that, for all $\kappa \in L^q(\mathbb{R}^N)$, $\rho_n * \kappa$ tends to κ in $L^q(\mathbb{R}^N)$, as $n \to \infty$, for all $1 \leq q < \infty$. First, we take the convolution of (1.7) with ρ_n. Then, we multiply the resulting equation by $\psi_j \;\, x\cdot\nabla(\overline{u}*\rho_n)$, integrate over $\mathbb{R}^N$ and consider the real part of the equality. From that point, the calculus are standard consisting in various integrations by parts. Hence, we omit the details and we refer the reader to [11]. In order to conclude the proof, it is sufficient to apply the Lebesgue dominated convergence theorem. □

Proof of Theorem 1.2. We shall distinguish between the cases $N = 1$ and $N \geq 2$, which require a separate treatment.

- Case $N \geq 2$. We divide the proof into four steps.

Step I (existence of a solution to (1.7)). We prove the existence of a ground state solution $u_\omega \in X_{\mathbb{C}}$ to (1.7) satisfying conditions i)-iv) of Theorem 1.2. Following the arguments of [12], we perform a change of unknown by setting $v = r^{-1}(u)$, where the function $r : \mathbb{R} \to \mathbb{R}$ is the unique solution to the Cauchy problem

$$r'(s) = \frac{1}{\sqrt{1+2r^2(s)}}, \qquad r(0) = 0. \tag{3.1}$$

Here $u \in X_{\mathbb{C}}$ is assumed to be real valued. Then, in [12] it is proved that, if $v \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ is a real solution to

$$-\Delta v = \frac{1}{\sqrt{1+2r^2(v)}}\Big(|r(v)|^{p-1}r(v) - \omega r(v)\Big), \tag{3.2}$$

then $u = r(v) \in X_{\mathbb{C}} \cap C^2(\mathbb{R}^N)$ and it is a real solution of (1.7). Let us set

$$k(v) := \frac{1}{\sqrt{1+2r^2(v)}}\Big(|r(v)|^{p-1}r(v) - \omega r(v)\Big) = r'(v)\Big(|r(v)|^{p-1}r(v) - \omega r(v)\Big),$$

and denote by $\mathcal{T}_\omega : H^1(\mathbb{R}^N) \to \mathbb{R}$ the action associated with equation (3.2), namely

$$\begin{aligned}
\mathcal{T}_\omega(v) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx - \int_{\mathbb{R}^N} K(v)dx, \\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |r(v)|^{p+1}dx + \frac{\omega}{2}\int_{\mathbb{R}^N} |r(v)|^2 dx,
\end{aligned}$$

where we have set $K(t) = \int_0^t k(s)ds$. Now, it is straightforward to check that k satisfies assumptions (g0)-(g3) of [12]. Thus, from [12] (see also [4, 5]) we deduce the existence of a ground state v_ω of (3.2) satisfying conditions i)-iv) of Theorem 1.2, that is v_ω solves (3.2) and minimizes the action $\mathcal{T}_\omega$ among all nontrivial solutions to (3.2). Therefore, setting $u_\omega = r(v_\omega)$, we get that u_ω solves (1.7) and satisfies conditions i)-iv) of Theorem 1.2 (see [12, Theorem 1.2]).

Step II (Existence of a ground state to (1.7)). In this step we prove that u_ω minimizes the action $\mathcal{E}_\omega$, over the set of nontrivial solutions to the original equation (1.7). To achieve this goal, we make the following observations. Notice first that, if $u = r(v)$ with $u \in X_{\mathbb{C}}$ real, then $\mathcal{E}_\omega(u) = \mathcal{T}_\omega(v)$. Indeed, we have

$$\begin{aligned}
\mathcal{E}_\omega(r(v)) &= \frac{1}{2}\int_{\mathbb{R}^N} r'^2(v)|\nabla v|^2 dx + \int_{\mathbb{R}^N} |r(v)|^2 r'^2(v)|\nabla|v||^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |r(v)|^{p+1}dx \\
&\quad + \frac{\omega}{2}\int_{\mathbb{R}^N} |r(v)|^2 dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N} \frac{1}{1+2r^2(v)}|\nabla v|^2 dx + \int_{\mathbb{R}^N} \frac{1}{1+2r^2(v)} r(v)^2|\nabla v|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |r(v)|^{p+1}dx \\
&\quad + \frac{\omega}{2}\int_{\mathbb{R}^N} |r(v)|^2 dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |r(v)|^{p+1}dx + \frac{\omega}{2}\int_{\mathbb{R}^N} |r(v)|^2 dx \\
&= \mathcal{T}_\omega(v),
\end{aligned}$$

thanks to the Cauchy problem (3.1). Also, if $u \in X_{\mathbb{C}}$ is a solution to (1.7) we have, in light of Lemma 3.1, that

$$\mathcal{E}_\omega(u) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^2 + 2|u|^2|\nabla|u||^2 dx. \tag{3.3}$$

Once this facts have been observed, take any solution $u \in X_{\mathbb{C}}$ to (1.7) (notice that u can be a complex valued function) and set $v = r^{-1}(|u|)$. Due to the well-known point-wise inequality $|\nabla|u(x)|| \le |\nabla u(x)|$ for a.e. $x \in \mathbb{R}^N$, it holds

$$\int_{\mathbb{R}^N} |\nabla|u(x)||^2 dx \le \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx, \tag{3.4}$$

so that $\mathcal{E}_\omega(|u|) \le \mathcal{E}_\omega(u)$ (notice that all the other terms in the functional $\mathcal{E}_\omega$ are invariant to the modulus). Thus, in turn, we have

$$\mathcal{E}_\omega(u) \ge \mathcal{E}_\omega(|u|) = \mathcal{E}_\omega(r(v)) = \mathcal{T}_\omega(v). \tag{3.5}$$

Now, let us set

$$A = \{v \in H^1(\mathbb{R}^N) : \tilde{P}(v) = 0\},$$

where $\tilde{P} : H^1(\mathbb{R}^N) \to \mathbb{R}$ is the functional defined as

$$\tilde{P}(v) = (N-2)\int_{\mathbb{R}^N} |\nabla v|^2 dx - 2N \int_{\mathbb{R}^N} K(v) dx.$$

Clearly, for any $v \in A$, we have

$$\mathcal{T}_\omega(v) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v|^2 dx. \tag{3.6}$$

Also, as for the proof that $\mathcal{E}_\omega(u) = \mathcal{T}_\omega(v)$, it is readily checked that, if $v = r^{-1}(u)$ with $u \in X_{\mathbb{C}}$ real, then $\tilde{P}(v) = P(u)$. Finally, it is well known (see e.g. [4, 5]) that v_ω satisfies

$$v_\omega \in A, \qquad \mathcal{T}_\omega(v_\omega) = \inf_{v\in A} \mathcal{T}_\omega(v). \tag{3.7}$$

Now, if $N = 2$, it follows from the definition of P in Lemma 3.1 that $P(|u|) = 0$. Thus, in turn, $\tilde{P}(v) = 0$ and, using (3.5) and (3.7), it follows that

$$\mathcal{E}_\omega(u) \ge \mathcal{T}_\omega(v) \ge \mathcal{T}_\omega(v_\omega) = \mathcal{E}_\omega(u_\omega), \tag{3.8}$$

proving the desired claim. If $N \ge 3$, one of the following possibilities occurs.

i) $P(|u|) = 0$. In this case inequality (3.8) holds exactly as in the case $N = 2$.

ii) $P(|u|) = \tilde{P}(v) < 0$. In this case there exists a number $\theta \in (0,1)$ such that, setting $v_\theta(x) = v(x/\theta)$, we have $\tilde{P}(v_\theta) = 0$. Now, since $v_\theta \in A$, using (3.3), (3.4), (3.6), (3.7), it follows that

$$\begin{aligned}
\mathcal{T}_\omega(v_\theta) &= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v_\theta|^2 dx = \frac{\theta^{N-2}}{N}\int_{\mathbb{R}^N} |\nabla v|^2 dx \\
&= \frac{\theta^{N-2}}{N}\int_{\mathbb{R}^N} |\nabla|u||^2 + 2|u|^2|\nabla|u||^2 dx \\
&\le \frac{\theta^{N-2}}{N}\int_{\mathbb{R}^N} |\nabla u|^2 + 2|u|^2|\nabla|u||^2 dx \\
&= \frac{\theta^{N-2}}{N} N\mathcal{E}_\omega(u) = \theta^{N-2}\mathcal{E}_\omega(u) < \mathcal{E}_\omega(u).
\end{aligned}$$

Thus, we get

$$\mathcal{E}_\omega(u) > \mathcal{T}_\omega(v_\theta) \geq \mathcal{T}_\omega(v_\omega) = \mathcal{E}_\omega(u_\omega).$$

Then, in conclusion, we proved that for both the cases $N = 2$ and $N \geq 3$, $u_\omega \in X_{\mathbb{C}}$ indeed minimizes the action $\mathcal{E}_\omega$ over the set of nontrivial solutions to (1.7).

Step III (real character of solutions). First we prove that, if $u \in X_{\mathbb{C}}$ is a ground state solution to (1.7), then $|u| \in X$ is also a ground state. We set $v = r^{-1}(|u|)$. Observe that it holds

$$m_\omega = \mathcal{E}_\omega(u) \geq \mathcal{E}_\omega(|u|) \geq \mathcal{T}_\omega(v). \tag{3.9}$$

In the case $N = 2$, we have $\tilde{P}(v) = P(|u|) = 0$ and, thus, we conclude $\mathcal{E}_\omega(|u|) = m_\omega$ by using (3.7), (3.9) and recalling that $\mathcal{T}_\omega(v_\omega) = \mathcal{E}_\omega(u_\omega) = m_\omega$. If $N \geq 3$, and $\tilde{P}(v) = P(|u|) < 0$ we introduce, as before, the rescaling v_θ such that $\tilde{P}(v_\theta) = 0$. Then, we get

$$\mathcal{T}_\omega(v_\theta) < \mathcal{E}_\omega(u) = m_\omega,$$

and we immediately reach a contradiction by arguing as before. Now, let $u \in X_{\mathbb{C}}$ be a ground state solution of (1.7) and assume that

$$\mathcal{L}^N(\{x \in \mathbb{R}^N : |\nabla|u|(x)| < |\nabla u(x)|\}) > 0.$$

Then we get

$$\begin{aligned} m_\omega &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla|u||^2 dx + \int_{\mathbb{R}^N} |u|^2|\nabla|u||^2 dx + \frac{\omega}{2}\int_{\mathbb{R}^N} |u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx \\ &< \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |u|^2|\nabla|u||^2 dx + \frac{\omega}{2}\int_{\mathbb{R}^N} |u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx = m_\omega. \end{aligned}$$

This is obviously not possible and, hence, we have $|\nabla|u(x)|| = |\nabla u(x)|$, for a.e. $x \in \mathbb{R}^N$. But this is true if, and only if, $\Re\, u \nabla(\Im\, u) = \Im\, u \nabla(\Re\, u)$. Whence, if this last condition holds, we get

$$\bar{u}\nabla u = \Re\, u \nabla(\Re\, u) + \Im\, u \nabla(\Im\, u), \quad \text{a.e. in } \mathbb{R}^N,$$

which implies that $\Re\,(i\bar{u}(x)\nabla u(x)) = 0$ a.e. in $\mathbb{R}^N$. This last identity immediately gives the existence of $\theta \in \mathbb{S}^1$ such that $u(x) = e^{i\theta}|u(x)|$.

Step IV (properties i)-iv) for any real non negative ground state). In light of some recent achievements [33, 7], we can prove that any real ground state solution to (1.7) is radially symmetric and radially decreasing about some point. In fact we observe first that for any given solution u of (1.7), by [30, Section 6. Appendix], $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and in turn $u \in C^2(\mathbb{R}^N)$ (cf. [22]). Considering now the strictly increasing function $\mu : \mathbb{R} \to \mathbb{R}$ such that

$$\mu'(s) = \sqrt{1 + 2s^2}, \qquad \mu(0) = 0, \tag{3.10}$$

it is easy to see that $v = \mu(u)$ is a solution of (3.2). Notice that μ is precisely the inverse function of the function r introduced in Step II, $r \circ \mu = \mu \circ r = \mathrm{Id}$. Furthermore, we claim that if u is any given ground state of (1.7), then $v = \mu(u) = r^{-1}(u)$ is a ground state of (3.2). In fact, taking into account the computations in Step II of the proof, for any nontrivial solution w of (3.2), $r(w)$ is a (nontrivial) solution of (1.7), and we have

$$\mathcal{T}_\omega(w) = \mathcal{E}_\omega(r(w)) \geq m_\omega = \mathcal{E}_\omega(u) = \mathcal{E}_\omega(r(v)) = \mathcal{T}_\omega(v),$$

which yields the desired conclusion. At this point the fact that any ground state solution is radially symmetric and radially decreasing about some point is a consequence of the results of [7] (see also [19]) applied to equation (3.2). Here let us point out that the radial symmetry (plus radial decrease) could have also been proved by arguing directly on equation (1.7) which, in fact, satisfies a scaling property being the essence of the results of [7]. Now let $u \in \mathcal{G}_\omega$ be such that $u \geq 0$ in $\mathbb{R}^N$. Since $u \in C^2(\mathbb{R}^N)$ we have by the maximum principle (applies to $v = \mu(u)$) that $u > 0$ on $\mathbb{R}^N$. Finally using [5, Lemma 2] on equation (3.2) we immediately derive the exponential decays indicated in the statement of Theorem 1.2.

• Case $N = 1$.

By taking advantage of the transformation of problem (1.7), via the dual approach, into the semi-linear equation (3.2), we know that equation (1.7) admits a unique positive and even solution (see [5, Theorem 5, Remark 6.3]). Thus it just remains to prove that any solution u of (1.7) is of the form $u = e^{i\theta}\phi$, where $\theta \in \mathbb{R}$ and $\phi > 0$ is a solution to (1.7). In fact $|u| > 0$, otherwise we would get a contradiction with the identity

$$\frac{1}{2}|u'|^2 + \frac{1}{4}|(|u|^2)'|^2 - \frac{\omega}{2}|u|^2 + \frac{1}{p+1}|u|^{p+1} = 0.$$

This identity is obtained multiplying (1.7) by the conjugate of u' and by performing standard manipulations. Then, we can write down the solution in polar form, $u = \rho e^{\mathrm{i}\theta}$, where $\rho, \theta \in C^2(\mathbb{R})$. By direct computation, it holds $u'' = \big[\rho\theta'' + 2\rho'\theta'\big]e^{\mathrm{i}\theta}\mathrm{i} + \big[\rho'' - \rho(\theta')^2\big]e^{\mathrm{i}\theta}$. Then, by dropping this formula into equation (1.7), exactly as in [8, proof of Theorem 8.1.7(iii)], one immediately reaches (by comparison of real and imaginary parts) the following identity

$$\rho\theta'' + 2\rho'\theta' = 0, \tag{3.11}$$

namely $\theta' = \frac{K}{\rho^2}$, for some $K \geq 0$. At this point it is sufficient to follow the argument of [8, proof of Theorem 8.1.7(iii)] to prove that $K = 0$ and get the desired property. Thus, when $N = 1$, Theorem 1.2 holds true and the set of solutions of (1.7) is essentially unique. □

In the rest of this section we prove the instability result, Theorem 1.3. We start with two preliminary results. We define the variance $\mathbb{V}(t)$, by

$$\mathbb{V}(t) = \int_{\mathbb{R}^N} |x|^2|\phi(t,x)|^2\,dx, \quad t \in [0,\infty) \tag{3.12}$$

and derive a so-called virial identity in the following lemma.

Lemma 3.2. *Let ϕ be a solution of (1.6) on an interval $I = (-t_1, t_1)$. Then,*

$$\mathbb{V}''(t) = 8Q(\phi(t)), \quad t \in I, \tag{3.13}$$

where we have set

$$Q(\phi) = \int_{\mathbb{R}^N} |\nabla\phi|^2 dx + (N+2)\int_{\mathbb{R}^N} |\phi|^2 |\nabla|\phi||^2 dx - \frac{N(p-1)}{2(p+1)}\int_{\mathbb{R}^N} |\phi|^{p+1} dx, \tag{3.14}$$

for all $\phi \in X_{\mathbb{C}}$.

Proof. We introduce the following notations:

$$z = (z^1, \dots, z^n) \in \mathbb{C}^N; \quad z \cdot w = \sum_{i=1}^N z^i w^i, \qquad z, w \in \mathbb{C}^N;$$

$$\phi_i = \frac{\partial \phi}{\partial x_i}, \qquad \phi : \mathbb{R}^N \to \mathbb{C}.$$

Let us first prove that

$$\mathbb{V}'(t) = 4\Im \int_{\mathbb{R}^N} \big(x \cdot \nabla\phi\big)\overline{\phi}\, dx, \quad t \in I. \tag{3.15}$$

By multiplying equation (1.6) by $2\overline{\phi}$ and taking the imaginary parts, yields

$$\frac{\partial}{\partial t}|\phi|^2 = -2\Im(\overline{\phi}\Delta\phi) = -2\nabla \cdot (\Im\overline{\phi}\nabla\phi), \tag{3.16}$$

Now, multiplying (3.16) by $|x|^2$, and integrating by parts in space, we get (3.15). In order to prove (3.13), let us multiply equation (1.6) by $2x \cdot \nabla\overline{\phi}$, integrate in space on $\mathbb{R}^N$ and, finally, take the real parts, yielding

$$\begin{aligned} 0 = {} & 2\Re \int_{\mathbb{R}^N} \mathrm{i}(x \cdot \nabla\overline{\phi})\phi_t\, dx + 2\Re \int_{\mathbb{R}^N} (x \cdot \nabla\overline{\phi})\Delta\phi\, dx \\ & + 2\Re \int_{\mathbb{R}^N} (x \cdot \nabla\overline{\phi})\phi\Delta|\phi|^2\, dx + 2\Re \int_{\mathbb{R}^N} (x \cdot \nabla\overline{\phi})|\phi|^{p-1}\phi\, dx. \end{aligned}$$

We rewrite the last identity in the form

$$\mathrm{I} = \mathrm{II} + \mathrm{III}, \tag{3.17}$$

where

$$\begin{aligned} \mathrm{I} = {} & 2\Re \int_{\mathbb{R}^N} \mathrm{i}(x \cdot \nabla\overline{\phi})\phi_t\, dx, \\ \mathrm{II} = {} & -2\Re \int_{\mathbb{R}^N} (x \cdot \nabla\overline{\phi})\Delta\phi\, dx - 2\Re \int_{\mathbb{R}^N} (x \cdot \nabla\overline{\phi})\phi\Delta|\phi|^2\, dx, \\ \mathrm{III} = {} & -2\Re \int_{\mathbb{R}^N} (x \cdot \nabla\overline{\phi})|\phi|^{p-1}\phi\, dx. \end{aligned}$$

For the first term, recalling formula (3.15) for $\mathbb{V}'$, we have

$$
\begin{aligned}
\mathrm{I} &= \Re \int_{\mathbb{R}^N} \mathrm{i} \sum_{j=1}^N \left(x^j \overline{\phi}_j \phi_t - x^j \phi_j \overline{\phi}_t \right) dx = \Re \int_{\mathbb{R}^N} \mathrm{i} \sum_{j=1}^N x^j \left[(\overline{\phi}_j \phi)_t - (\phi \overline{\phi}_t)_j \right] dx \\
&= \frac{d}{dt} \Re \int_{\mathbb{R}^N} \mathrm{i}(x \cdot \nabla \overline{\phi}) \phi \, dx + N \Re \int_{\mathbb{R}^N} \mathrm{i} \phi \overline{\phi}_t \, dx \\
&= \frac{d}{dt} \Im \int_{\mathbb{R}^N} (x \cdot \nabla \phi) \overline{\phi} \, dx - N \int_{\mathbb{R}^N} |\nabla \phi|^2 \, dx \\
&+ N \int_{\mathbb{R}^N} |\phi|^2 \Delta |\phi|^2 \, dx + N \int_{\mathbb{R}^N} |\phi|^{p+1} \, dx. \\
&= \frac{1}{4} \frac{d}{dt} \mathbb{V}(t) - N \int_{\mathbb{R}^N} |\nabla \phi|^2 \, dx - 4N \int_{\mathbb{R}^N} |\phi|^2 |\nabla |\phi||^2 \, dx + N \int_{\mathbb{R}^N} |\phi|^{p+1} \, dx.
\end{aligned} \tag{3.18}
$$

A multiple integration by parts in formula II gives

$$
\mathrm{II} = (2-N) \int_{\mathbb{R}^N} |\nabla \phi|^2 dx + 2(2-N) \int_{\mathbb{R}^N} |\phi|^2 |\nabla |\phi||^2 dx. \tag{3.19}
$$

As for the term III, we write it by components

$$
\begin{aligned}
\mathrm{III} &= -\sum_{j=1}^N \int_{\mathbb{R}^N} x^j |\phi|^{p-1} (2\Re \overline{\phi}_j \phi) \, dx \\
&= -2 \sum_{j=1}^N \int_{\mathbb{R}^N} x^j \frac{\partial_j |\phi|^{p+1}}{p+1} dx = \frac{2N}{p+1} \int_{\mathbb{R}^N} |\phi|^{p+1} \, dx.
\end{aligned} \tag{3.20}
$$

Finally, recollecting (3.17), (3.18), (3.19), (3.20) and (3.15), and taking into account the definition of Q, the proof of (3.13) is complete. □

In our next preliminary result we establish some qualitative properties of a class of L^2-invariant rescaling.

Lemma 3.3. *Let $\psi \in X_{\mathbb{C}}$ and $Q(\psi) \leq 0$ and assume that*

$$
p > 3 + \frac{4}{N}. \tag{3.21}
$$

Let $\sigma > 0$ and define the rescaling $\psi^\sigma(x) = \sigma^{N/2} \psi(\sigma x)$. Then there exists $\sigma_0 \in (0, 1]$ such that following facts hold

1. $Q(\psi^{\sigma_0}) = 0$;
2. $\sigma_0 = 1$ *if and only if* $Q(\psi) = 0$;
3. $\frac{\partial}{\partial \sigma} \mathcal{E}_\omega(\psi^\sigma) > 0$ *for* $\sigma \in (0, \sigma_0)$, *and* $\frac{\partial}{\partial \sigma} \mathcal{E}_\omega(\psi^\sigma) < 0$ *for* $\sigma \in (\sigma_0, \infty)$;
4. $\sigma \mapsto \mathcal{E}_\omega(\psi^\sigma)$ *is concave on* (σ_0, ∞);
5. $\frac{\partial}{\partial \sigma} \mathcal{E}_\omega(\psi^\sigma) = \frac{Q(\psi^\sigma)}{\sigma}$.

Proof. By direct computation, we have

$$\mathcal{E}_\omega(\psi^\sigma) = \frac{\sigma^2}{2}\int_{\mathbb{R}^N} |\nabla\psi|^2 dx + \sigma^{N+2}\int_{\mathbb{R}^N} |\psi|^2|\nabla|\psi||^2 dx + \frac{\omega}{2}\int_{\mathbb{R}^N} |u|^2 dx - \frac{\sigma^{\frac{N(p-1)}{2}}}{p+1}\int_{\mathbb{R}^N} |\psi|^{p+1} dx,$$

so that, using the functional Q defined by (3.14), for all $\sigma > 0$, we get

$$\begin{aligned}\frac{\partial}{\partial\sigma}\mathcal{E}_\omega(\psi^\sigma) &= \sigma\int_{\mathbb{R}^N} |\nabla\psi|^2 dx + (N+2)\sigma^{N+1}\int_{\mathbb{R}^N} |\psi|^2|\nabla|\psi||^2 dx \\ &\quad - \frac{N(p-1)}{2(p+1)}\sigma^{\frac{N(p-1)}{2}-1}\int_{\mathbb{R}^N} |\psi|^{p+1} dx = \frac{1}{\sigma}Q(\psi^\sigma).\end{aligned}$$

Then, taking into account (3.21), it is readily seen that there exists $\sigma_0 \in (0,1]$ such that

$$Q(\psi^{\sigma_0}) = \sigma_0\frac{\partial}{\partial\sigma}\mathcal{E}_\omega(\psi^\sigma)_{|\sigma=\sigma_0} = 0,$$

as well as $\frac{\partial}{\partial\sigma}\mathcal{E}_\omega(\psi^\sigma) > 0$ for $\sigma \in (0,\sigma_0)$ and $\frac{\partial}{\partial\sigma}\mathcal{E}_\omega(\psi^\sigma) < 0$ for $\sigma \in (\sigma_0,\infty)$. Furthermore, writing $\sigma = t\sigma_0$ we have

$$\begin{aligned}\frac{\partial^2}{\partial\sigma^2}\mathcal{E}_\omega(\psi^\sigma) &= \int_{\mathbb{R}^N} |\nabla\psi|^2 dx + (N+2)(N+1)t^N\sigma_0^N\int_{\mathbb{R}^N} |\psi|^2|\nabla|\psi||^2 dx \\ &\quad - \frac{N(p-1)}{2(p+1)}\Big(\frac{N(p-1)}{2} - 1\Big)t^{\frac{N(p-1)}{2}-2}\sigma_0^{\frac{N(p-1)}{2}-2}\int_{\mathbb{R}^N} |\psi|^{p+1} dx, \\ &= t^N\Big(\frac{1}{t^N}\int_{\mathbb{R}^N} |\nabla\psi|^2 dx + (N+2)(N+1)\sigma_0^N\int_{\mathbb{R}^N} |\psi|^2|\nabla|\psi||^2 dx \\ &\quad - \frac{N(p-1)}{2(p+1)}\Big(\frac{N(p-1)}{2} - 1\Big)t^{\frac{Np-3N-4}{2}}\sigma_0^{\frac{N(p-1)}{2}-2}\int_{\mathbb{R}^N} |\psi|^{p+1} dx\Big).\end{aligned}$$

Since, of course, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla\psi|^2 dx + (N+2)(N+1)\sigma_0^N\int_{\mathbb{R}^N} |\psi|^2|\nabla|\psi||^2 dx \\ &-\frac{N(p-1)}{2(p+1)}\Big(\frac{N(p-1)}{2} - 1\Big)\sigma_0^{\frac{N(p-1)}{2}-2}\int_{\mathbb{R}^N} |\psi|^{p+1} dx \le 0\end{aligned}$$

and $t > 1$, it follows that the quantity inside the parenthesis is negative. Hence the map $\sigma \mapsto \mathcal{E}_\omega(\psi^\sigma)$ is concave on (σ_0,∞), concluding the proof. □

In order to establish the instability of ground states we now show, in the spirit of [25] that they enjoy two additional variational characterizations. First, we have the following

Lemma 3.4. *Assume that $\omega > 0$ and $3 \le p \le \frac{3N+2}{N-2}$ if $N \ge 3$ and $3 \le p$ if $N = 1,2$. Then the set of minimizers of*

$$d_\omega = \inf\{\mathcal{E}_\omega(u) : \mathcal{I}_\omega(u) = 0\}, \tag{3.22}$$

where

$$\mathcal{I}_\omega(u) = \int_{\mathbb{R}^N} |\nabla \phi|^2 dx + \omega \int_{\mathbb{R}^N} |\phi|^2 dx + 4 \int_{\mathbb{R}^N} |\phi|^2 |\nabla |\phi||^2 dx - \int_{\mathbb{R}^N} |\phi|^{p+1} dx.$$

is exactly the set of ground state $\mathcal{G}_\omega$. In addition the value of the two infimums are equal.

Proof. First we show that if $u \in X_{\mathbb{C}}$ is a minimizer of d_ω then $|u| \in X$ is also a minimizer of d_ω. Let $u \in X_{\mathbb{C}}$ with $\mathcal{I}_\omega(u) = 0$. Then $\mathcal{E}_\omega(|u|) \le \mathcal{E}_\omega(u)$ as well as $\mathcal{I}_\omega(|u|) \le \mathcal{I}_\omega(u) = 0$. In particular and since $P \ge 3$, there exists $t \in (0, 1]$ such that $\mathcal{I}_\omega(t|u|) = 0$. Observe now that, for all $v \in X_{\mathbb{C}}$ such that $\mathcal{I}_\omega(v) = 0$, it holds

$$\mathcal{E}_\omega(v) = \frac{p-1}{2(p+1)} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{p-3}{p+1} \int_{\mathbb{R}^N} |v|^2 |\nabla |v||^2 dx + \omega \frac{p-1}{2(p+1)} \int_{\mathbb{R}^N} |v|^2 dx.$$

Thus, since $p \ge 3$, it is readily seen that

$$0 < \mathcal{E}_\omega(t|u|) \le t^2 \mathcal{E}_\omega(u).$$

In particular, if $u \in X_{\mathbb{C}}$ is a complex minimizer of d_ω, then we have

$$\mathcal{E}_\omega(u) = d_\omega = \inf_{\mathcal{I}_\omega = 0} \mathcal{E}_\omega(\phi) \le \mathcal{E}_\omega(t|u|) \le t^2 \mathcal{E}_\omega(u).$$

Now, recalling that $\mathcal{E}_\omega(u) > 0$ and $t \le 1$, we immediately get $t = 1$. Thus $\mathcal{I}_\omega(|u|) = \mathcal{I}_\omega(u)$ and in turn $\mathcal{E}_\omega(|u|) = \mathcal{E}_\omega(u)$ proving that $|u| \in X$ is also a minimizer. Obviously it is only possible if the set $\{x \in \mathbb{R}^N : |\nabla |u|(x)| \ne |\nabla u(x)|\}$ has zero Lebesgue measure, which in turn implies that $u = |u| e^{\mathrm{i}\theta}$, for some $\theta \in \mathbb{S}^1$ (see e.g. Step III of the proof of Theorem 1.2). Now, when $\mathcal{E}_\omega$ is considered over X, in [30, Theorem 1.1] it is established that there exists a nontrivial solution to the minimization problem (3.22) and that this minimizer is a solution to equation (1.7) (cf. [30, Lemma 2.5]). Clearly, since any minimizer is of the form $u = |u| e^{\mathrm{i}\theta}$ it is also solution to equation (1.7). Now, any element $u \in X$ of $\mathcal{G}_\omega$ must satisfy $\mathcal{I}_\omega(u) = 0$ and thus we deduce that the set of ground states $\mathcal{G}_\omega$ and the set of minimizer of (3.22) coincide and that the values of the two infimum values are equal. $\square$

We also have the following

Lemma 3.5. *Let us set*

$$c_\omega = \inf\{\mathcal{E}_\omega(\phi) : \phi \in \mathcal{M}\} \text{ where } \mathcal{M} = \{\phi \in X \setminus \{0\} : \; Q(\phi) = 0, \; \mathcal{I}_\omega(\phi) \le 0\}.$$

Then $c_\omega = d_\omega$ ($= m_\omega$).

Proof. Let $u \in X_{\mathbb{C}}$ be a solution to (3.22). By Lemma 3.4 it is a ground state solution of (1.7) and applying the virial identity (3.13) to a standing wave solution we immediately deduce that $Q(u) = 0$. By definition $\mathcal{I}_\omega(u) = 0$ and thus we have $u \in \mathcal{M}$. Hence $c_\omega \le d_\omega$, since $\mathcal{E}_\omega(u) = d_\omega$. On the other hand, given $\phi \in \mathcal{M}$, either $\mathcal{I}_\omega(\phi) = 0$

(so that $\mathcal{E}_\omega(\phi) \geq d_\omega$) or $\mathcal{I}_\omega(\phi) < 0$. In this second case, if $\sigma > 0$ and we consider the rescaling $\phi^\sigma(x) = \sigma^{N/2}\phi(\sigma x)$, we have $\mathcal{I}_\omega(\phi^1) < 0$ and

$$\lim_{\sigma\to 0^+} \mathcal{I}_\omega(\phi^\sigma) = \lim_{\sigma\to 0^+} \Big(\sigma^2 \int_{\mathbb{R}^N} |\nabla\phi|^2 dx + \omega \int_{\mathbb{R}^N} |\phi|^2 dx + 4\sigma^{N+2} \int_{\mathbb{R}^N} |\phi|^2 |\nabla|\phi||^2 dx - \sigma^{\frac{N(p-1)}{2}} \int_{\mathbb{R}^N} |\phi|^{p+1} dx\Big) > 0.$$

In turn, one can find $\hat\sigma \in (0,1)$ such that $\mathcal{I}_\omega(\phi^{\hat\sigma}) = 0$. Then, we get $\mathcal{E}_\omega(\phi^{\hat\sigma}) \geq d_\omega$. Since $Q(\phi) = 0$ and $\|\phi\|_2 = \|\phi^{\hat\sigma}\|_2$, from Lemma 3.3 we obtain $\mathcal{E}_\omega(\phi) \geq \mathcal{E}_\omega(\phi^{\hat\sigma}) \geq d_\omega$. Whence $\mathcal{E}_\omega(\phi) \geq d_\omega$ holds true for any $\phi \in \mathcal{M}$, which yields $c_\omega \geq d_\omega$, proving the claim. □

Proof of Theorem 1.3. Let $\varepsilon > 0$ be fixed and consider $u^\sigma(x) = \sigma^{N/2}u(\sigma x)$ for the ground state solution u. We have $\|u\|_2 = \|u^\sigma\|_2$ and by the continuity of the mapping $\sigma \longrightarrow \sigma^{N/2}u(\sigma x)$, it is clear that, for σ sufficiently close to 1, $\|u - u^\sigma\|_{H^1(\mathbb{R}^N)} \leq \varepsilon$. Furthermore,

$$\mathcal{E}_\omega(u^\sigma) < \mathcal{E}_\omega(u), \quad Q(u^\sigma) < 0, \quad \mathcal{I}_\omega(u^\sigma) < 0, \tag{3.23}$$

provided that $\sigma > 1$ is sufficiently close to 1. The first two inequalities just follow by Lemma (3.3). Concerning the last one, it holds

$$\begin{aligned}
\mathcal{I}_\omega(u^\sigma) &= 2\mathcal{E}_\omega(u^\sigma) + \frac{2}{N}Q(u^\sigma) - \frac{4}{N}\int_{\mathbb{R}^N} |u^\sigma|^2|\nabla u^\sigma|^2 dx - \frac{2}{N}\int_{\mathbb{R}^N} |\nabla u^\sigma|^2 dx \\
&\leq 2\mathcal{E}_\omega(u) + \frac{2}{N}Q(u) - \mathcal{I}_\omega(u) - \frac{4\sigma^{N+2}}{N}\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx - \frac{2\sigma^2}{N}\int_{\mathbb{R}^N} |\nabla u|^2 dx \\
&= \frac{4}{N}\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx - \frac{2}{N}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{4\sigma^{N+2}}{N}\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx - \frac{2\sigma^2}{N}\int_{\mathbb{R}^N} |\nabla u|^2 dx \\
&= \frac{4}{N}(1-\sigma^{N+2})\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx + \frac{2}{N}(1-\sigma^2)\int_{\mathbb{R}^N} |\nabla u|^2 dx < 0.
\end{aligned}$$

Now fixing a $\sigma > 1$ such that (3.23) hold we approximate $u^\sigma \in X_{\mathbb{C}}$ by a function $v \in C_0^\infty(\mathbb{R}^N) \subset H^{s+2}(\mathbb{R}^N)$ in such a way that we still have $\|v\|_2 = \|u\|_2$, $\|v - u\|_{H^1(\mathbb{R}^N)} \leq \varepsilon$ and

$$\mathcal{E}_\omega(v) < \mathcal{E}_\omega(u), \quad Q(v) < 0, \quad \mathcal{I}_\omega(v) < 0. \tag{3.24}$$

This comes from the fact that by Theorem 1.2, the ground state u and then u^σ are bounded as well as their derivatives up to order 2. Then direct estimates on $\mathcal{E}_\omega$, Q and $\mathcal{I}_\omega$ give the desired inequalities (3.24). Assume that $\phi(t)$ is the solution of (1.6) with initial data $\phi(0) = v$. Then, we claim

$$\mathcal{E}_\omega(\phi(t)) < \mathcal{E}_\omega(u), \quad Q(\phi(t)) < 0, \quad \mathcal{I}_\omega(\phi(t)) < 0, \quad \text{for all } t \in [0, T_{\max}), \tag{3.25}$$

$T_{\max} \in (0, \infty]$ being the maximal existence time. First, due to the conservation of the energy and (3.23), we get

$$\mathcal{E}_\omega(\phi(t)) = \mathcal{E}_\omega(v) < \mathcal{E}_\omega(u), \quad \text{for all } t \in [0, T_{\max}).$$

In turn, it follows immediately that $\mathcal{I}_\omega(\phi(t)) \neq 0$ for all $t \in [0, T_{\max})$. Hence $\mathcal{I}_\omega(\phi(t)) < 0$ for all $t \in [0, T_{\max})$ since it is negative for $t = 0$. Similarly, $Q(\phi(t)) \neq 0$ for all $t \in [0, T_{\max})$, otherwise if $Q(\phi(t_0)) = 0$ for some $t_0 \in [0, T_{\max})$, we would have $\phi(t_0) \in \mathcal{M}$, yielding $\mathcal{E}_\omega(\phi(t_0)) \geq \mathcal{E}_\omega(u)$ which contradicts the first inequality of (3.25). Hence $Q(\phi(t)) < 0$ for all $t \in [0, T_{\max})$ as it is negative for $t = 0$, concluding the proof of (3.25).

Let now $\psi = \phi(t)$ be the solution to (1.6) at a fixed time $t \in (0, T_{\max})$ and let ψ^σ be the usual L^2-invariant rescaling. We know that $Q(\psi) < 0$. Hence there exists $\tilde{\sigma} \in (0, 1)$ such that $Q(\psi^{\tilde{\sigma}}) = 0$. If $\mathcal{I}_\omega(\psi^{\tilde{\sigma}}) \leq 0$ we do not change the value of $\tilde{\sigma}$, otherwise we pick $\hat{\sigma} \in (\tilde{\sigma}, 1)$ such that $\mathcal{I}_\omega(\psi^{\hat{\sigma}}) = 0$. In any case, one obtains $\mathcal{E}_\omega(\psi^{\tilde{\sigma}}) \geq d_\omega$ and $Q(\psi^{\tilde{\sigma}}) \leq 0$. Therefore, by Lemma 3.3

$$\begin{aligned} \mathcal{E}_\omega(v) = \mathcal{E}_\omega(\psi) &\geq \mathcal{E}_\omega(\psi^{\tilde{\sigma}}) + (1 - \tilde{\sigma}) \frac{\partial}{\partial \sigma} \mathcal{E}_\omega(\psi^\sigma)_{|\sigma=1} \\ &= \mathcal{E}_\omega(\psi^{\tilde{\sigma}}) + (1 - \tilde{\sigma}) Q(\psi) > d_\omega + Q(\psi). \end{aligned}$$

Putting $\varrho_0 := d_\omega - \mathcal{E}_\omega(v) > 0$, concluding we have

$$Q(\phi(t)) \leq -\varrho_0, \quad \text{for all } t \in [0, T_{\max}).$$

Finally, assuming that $T_{\max} = +\infty$ and using the virial identity of Lemma 3.2, we obtain

$$0 < \mathbb{V}(t) \leq \mathbb{V}(0) + \mathbb{V}'(0)t - 4\varrho_0 t^2$$

which yields a contradiction taking t sufficiently large. Then $0 < T_{\max} < +\infty$ and the solution blows-up in finite time. This concludes the proof.

4. Stationary solutions with prescribed L^2 norm

In this section we study the minimization problem (1.10). We prove the existence of a minimizer when $1 < p < 3 + \frac{4}{N}$ and $m(c) < 0$. We also discuss the condition $m(c) < 0$. Consider the (complex) minimization problem

$$\text{minimize} \quad \mathcal{E} \quad \text{on} \quad \|u\|_2^2 = c, \tag{4.1}$$

where c is a positive number. We have the following result.

Proposition 4.1. *Let v be a solution to the minimization problem (4.1). Then*

$$v(x) = e^{i\theta} |v(|x|)|, \quad x \in \mathbb{R}^N,$$

for some $\theta \in \mathbb{S}^1$. In particular, the solutions of problem (4.1) are, up to a constant complex phase, real-valued positive and radially symmetric.

Proof. The proof has some similarities with the final part of the proof of Theorem 1.2 so we will be brief here. Let X denote again the restriction of $X_{\mathbb{C}}$ to real-valued functions. We set

$$\sigma_{\mathbb{C}} = \inf \{\mathcal{E}(v) : v \in X_{\mathbb{C}},\, \|v\|_2^2 = c\}, \quad \sigma_{\mathbb{R}} = \inf \{\mathcal{E}(v) : v \in X, \|v\|_2^2 = c\}.$$

Let us prove that $\sigma_{\mathbb{C}} = \sigma_{\mathbb{R}}$. Trivially one has $\sigma_{\mathbb{C}} \leq \sigma_{\mathbb{R}}$, since $X \subset X_{\mathbb{C}}$. Moreover, if $v \in X_{\mathbb{C}}$, we see using (3.4) that $\mathcal{E}(|v|) \leq \mathcal{E}(v)$. In particular, we conclude that $\sigma_{\mathbb{R}} \leq \sigma_{\mathbb{C}}$, yielding the desired equality $\sigma_{\mathbb{C}} = \sigma_{\mathbb{R}}$. Now let $v \in X_{\mathbb{C}}$ be a solution to $\sigma_{\mathbb{C}}$ and assume by contradiction that the Lebesgue measure $\mathcal{L}^N$ of the set $\{x \in \mathbb{R}^N : |\nabla |v|(x)| < |\nabla v(x)|\}$ is positive. Then, of course, $\||v|\|_2^2 = \|v\|_2^2 = c$, and

$$\begin{aligned} \sigma_{\mathbb{R}} &\leq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla |v||^2 dx + \int_{\mathbb{R}^N} |v|^2 |\nabla |v||^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |v|^{p+1} dx \\ &< \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} |v|^2 |\nabla |v||^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |v|^{p+1} dx = \sigma_{\mathbb{C}}, \end{aligned}$$

contradicting equality $\sigma_{\mathbb{C}} = \sigma_{\mathbb{R}}$. Hence, we have $|\nabla |v(x)|| = |\nabla v(x)|$ for a.e. $x \in \mathbb{R}^N$ and as in the proof of Theorem 1.2 this gives the existence of $\theta \in \mathbb{S}^1$ such that $v = e^{i\theta}|v|$. Finally the result of radial symmetry is a direct consequence of [33, Theorem 2]. □

From Proposition 4.1 we deduce that it is sufficient to study the (real) minimization problem

$$\text{minimize} \quad \mathcal{E} \quad \text{on} \quad \|u\|_2^2 = c \quad \text{with } u \in X. \tag{4.2}$$

for a positive number c. We set

$$m(c) = \inf\{\mathcal{E}(u) :\, u \in X,\ \|u\|_2^2 = c\}. \tag{4.3}$$

Lemma 4.2. *We have*

(1) *Assume that $1 < p < 3 + \frac{4}{N}$. Then $m(c) > -\infty$ for any $c > 0$. In addition if $(u_n) \subset X$ is any minimizing sequence for problem (4.2) then (u_n) is bounded in X and the sequence*

$$\int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx = \frac{1}{4}\int_{\mathbb{R}^N} |\nabla (u_n^2)|^2 dx \tag{4.4}$$

is bounded in $\mathbb{R}$.

(2) *In the case $p = 3 + \frac{4}{N}$ the same conclusions hold provided that $c > 0$ is sufficiently small.*

(3) *Assume that $3 + \frac{4}{N} < p < \frac{4N}{N-2}$. Then $m(c) = -\infty$ for any $c > 0$.*

Proof. Notice that, using Hölder and Sobolev inequalities we have for

$$\theta = \frac{(p-1)(N-2)}{2(N+2)}$$

and some $K > 0$ depending only on N, that for any $u \in X$

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^{p+1}dx &\le \Big(\int_{\mathbb{R}^N} |u|^2 dx\Big)^{1-\theta}\Big(\int_{\mathbb{R}^N} |u|^{\frac{4N}{N-2}}dx\Big)^{\theta} \\ &\le K\Big(\int_{\mathbb{R}^N} |u|^2 dx\Big)^{1-\theta}\Big(\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx\Big)^{\frac{\theta N}{N-2}}. \end{aligned} \tag{4.5}$$

Here we have used the fact that

$$\int_{\mathbb{R}^N} |u|^{\frac{4N}{N-2}}dx = \int_{\mathbb{R}^N} |u^2|^{\frac{2N}{N-2}}dx, \qquad \int_{\mathbb{R}^N} |\nabla(u^2)|^2 dx = 4\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx.$$

From (4.5) we get that

$$\mathcal{E}(u) \ge \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx - \frac{1}{p+1}Kc^{1-\theta}\Big(\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx\Big)^{\frac{\theta N}{N-2}}.$$

If we assume that $p < 3 + \frac{4}{N}$, we see that $\frac{\theta N}{N-2} < 1$ and thus the sequence (4.4) is bounded in $\mathbb{R}$. From (4.5) we then got that $(\|u_n\|_{p+1})$ is bounded and thus also that $(\|\nabla u_n\|_2)$ is bounded. This proves Point (1). In the limit case $p = 3+\frac{4}{N}$ we still reach the boundedness result for any positive c such that $Kc^{1-\theta} < p+1$, where $K, \theta > 0$ are the numbers introduced in the proof. Now for point (3) we fix $c > 0$ and take $u \in X$ such that $\|u\|_2^2 = c$. Then, considering the scaling,

$$\sigma \mapsto u^{\sigma}(x) = \sigma^{\frac{N}{2}}u(\sigma x),$$

we get, for all $\sigma > 0$,

$$\int_{\mathbb{R}^N} |u^{\sigma}|^2 dx = \int_{\mathbb{R}^N} |u|^2 dx = c, \qquad \int_{\mathbb{R}^N} |\nabla u^{\sigma}|^2 dx = \sigma^2\int_{\mathbb{R}^N} |\nabla u|^2 dx,$$

$$\int_{\mathbb{R}^N} |u^{\sigma}|^{p+1}dx = \sigma^{\frac{N(p-1)}{2}}\int_{\mathbb{R}^N} |u|^{p+1}dx, \qquad \int_{\mathbb{R}^N} |u^{\sigma}|^2|\nabla u^{\sigma}|^2 dx = \sigma^{(N+2)}\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx.$$

Thus $\|u^{\sigma}\|_2^2 = c$ for all $\sigma > 0$ and

$$\mathcal{E}(u^{\sigma}) = \frac{\sigma^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \sigma^{(N+2)}\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx - \frac{\sigma^{\frac{N(p-1)}{2}}}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}dx.$$

Now just notice that, in the range $3 + \frac{4}{N} < p < \frac{4N}{N-2}$ the dominant term is

$$\frac{\sigma^{\frac{N(p-1)}{2}}}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}dx.$$

Thus $\mathcal{E}(u^{\sigma}) \to -\infty$ as $\sigma \to +\infty$. This concludes the proof of (3). $\square$

Lemma 4.3. *We have*

(1) *Assume that $1 < p < 1 + \frac{4}{N}$. Then $m(c) < 0$ for any $c > 0$.*

(2) *Assume that $1 + \frac{4}{N} \le p < 3 + \frac{4}{N}$. Then there exists a $c > 0$, sufficiently large, such that $m(c) < 0$.*

Proof. We use the scaling introduced in the proof of Lemma 4.2, Point (3). As $\sigma \to 0^+$, and since $p < 1 + \frac{4}{N}$, we see that the dominant term is

$$\frac{\sigma^{\frac{N(p-1)}{2}}}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx,$$

concluding the proof of Point (1). Now for Point (2) we consider, for a fixed $R > 0$, the radial function $w_R \in H^1(\mathbb{R}^N)$ defined by

$$w_R(r) := \begin{cases} 1 & \text{if } r \leq R, \\ 1 + R - r & \text{if } R \leq r \leq R+1, \\ 0 & \text{if } r \geq R+1. \end{cases}$$

Integrating in radial coordinates, we have

$$\int_{\mathbb{R}^N} |w_R(|x|)|^2 dx = C_N R^N + \varepsilon_1(R^{N-1}),$$

where $\varepsilon_1(R^{N-1})/R^N \to 0$, as $R \to \infty$. Also

$$\int_{\mathbb{R}^N} |w_R(|x|)|^{p+1} dx = C_N R^N + \varepsilon_2(R^{N-1}), \qquad \int_{\mathbb{R}^N} |\nabla w_R(|x|)|^2 dx = \varepsilon_3(R^{N-1}),$$

and

$$\int_{\mathbb{R}^N} |w_R(|x|)|^2 |\nabla w_R(|x|)|^2 dx = \varepsilon_4(R^{N-1}),$$

where $\varepsilon_i(R^{N-1})/R^N \to 0$, as $R \to \infty$, for any $i = 2, 3, 4$. Thus letting $R \to \infty$ we have $\|w_R\|_2^2 \to +\infty$ and $\mathcal{E}(w_R) \to -\infty$. This proves the claim. $\square$

Concerning the existence of a minimizer we first show

Lemma 4.4. *Assume that $1 < p < 3 + \frac{4N}{N-2}$. The following facts hold.*

(1) *If $u_n \rightharpoonup u$ in X then setting*

$$T(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 dx,$$

we have

$$T(u) \leq \liminf_{n \to \infty} T(u_n).$$

(2) *For any $u \in X$ there exists a Schwarz symmetric function $u^* \in X$ satisfying*

$$T(u^*) \leq T(u), \quad \int_{\mathbb{R}^N} |u^*|^2 dx = \int_{\mathbb{R}^N} |u|^2 dx, \quad \int_{\mathbb{R}^N} |u^*|^{p+1} dx = \int_{\mathbb{R}^N} |u|^{p+1} dx.$$

(3) *Let $(u_n) \subset X$ be a minimizing sequence for (4.2) of Schwartz symmetric functions satisfying $u_n \rightharpoonup u$ in X. Then we have*

$$\mathcal{E}(u) \leq \liminf_{n \to \infty} \mathcal{E}(u_n) = m(c). \tag{4.6}$$

Proof. Point (1) is standard. Defining $j : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $j(s,\xi) = \frac{1}{2}\xi^2 + s^2\xi^2$, then $\{\xi \to j(s,\xi)\}$ is convex and thus the result follows from classical results of A. Ioffe (see e.g. [17, 18]). Concerning assertion (2) all we need to prove is $T(u^*) \leq T(u)$, which follows from [15, Corollary 3.3]. For Point (3), we claim that

$$\int_{\mathbb{R}^N} |u_n|^{p+1} dx \to \int_{\mathbb{R}^N} |u|^{p+1} dx \tag{4.7}$$

as $n \to \infty$. In fact, since $(u_n) \subset X$ is minimizing we have, by Lemma 4.2, Point (1) that $\nabla(u_n^2)$ is uniformly bounded in $L^2(\mathbb{R}^N)$ and thus by the Sobolev embedding $\sup_{n\in\mathbb{N}} \|u_n^2\|_{\frac{2N}{N-2}} < \infty$, which gives $\sup_{n\in\mathbb{N}} \|u_n\|_{\frac{4N}{N-2}} < \infty$. Now, using the fact that $(u_n) \subset X$ consists of radial decreasing functions, from the radial Lemma A.IV of [5], we deduce that (u_n) has a uniform decay at infinity (with respect to both $n \in \mathbb{N}$ and $|x|$) and this shows, by standard argument, that (4.7) hold. Now we conclude observing that, from point (1), $T(u) \leq \liminf_{n\to\infty} T(u_n)$. $\square$

We now prove the existence of a minimizer for problem (4.2).

Lemma 4.5. *Assume that $1 < p < 3 + \frac{4}{N}$ and $c > 0$ is such that $m(c) < 0$. Then the problem (4.2) admits a minimizer which is Schwartz symmetric.*

Proof. Let (u_n) be a minimizing sequence for (4.2). By Lemma 4.4 we know that $(u_n) \subset X$ can be replaced by a minimizing sequence $(u_n^*) \subset X$ of Schwarz symmetric functions such that $u_n^* \rightharpoonup u^*$ and

$$\mathcal{E}(u^*) \leq \liminf_{n\to\infty} \mathcal{E}(u_n^*) = m(c). \tag{4.8}$$

We still denote u^* by u. To conclude we just need to prove that $\|u\|_2^2 = c$. Since $\mathcal{E}(u) \leq m(c) < 0$ necessarily $u \neq 0$. Assume thus that $0 < \|u\|_2^2 = \lambda < c$ and consider the scaling $v(x) = u(\sigma^{-\frac{1}{N}}x)$ for $\sigma > 1$. Then $\|v\|_2^2 = \sigma\lambda$ and for $\sigma = \frac{c}{\lambda}$ we have $\|v\|_2^2 = c$. Now we also get that

$$\mathcal{E}(v) = \sigma^{1-\frac{2}{N}} \left[\int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 + |\nabla u|^2|u|^2 dx \right] - \frac{\sigma}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx.$$

Thus, since $\sigma > 1$ and $\mathcal{E}(u) < 0$ we conclude that $\mathcal{E}(v) < \mathcal{E}(u)$, which is a contradiction. This proves that $\|u\|_2^2 = c$ and thus (4.2) admits a minimizer. Finally, observe that, since $\|u_n^*\|_{p+1} \to \|u^*\|_{p+1}$ as $n \to \infty$, necessarily $\|\nabla u_n^*\|_2 \to \|\nabla u^*\|_2$ as $n \to \infty$ and we deduce that the Schwarz symmetric sequence strongly converges to $u^* \in X$. $\square$

Lemma 4.6. *The following facts hold:*

(1) *Assume that $1 + \frac{4}{N} \leq p < 3 + \frac{4}{N}$. Then $-\infty < m(c) \leq 0$ for any $c > 0$. These inequalities also hold if $p = 3 + \frac{4}{N}$ and $c > 0$ is small.*
(2) *There exists $0 \leq c_0 < \infty$ such that $m(c) = 0$ for $c \in (0, c_0)$ and $\{c \to m(c)\}$ is strictly decreasing for $c > c_0$.*

Proof. For the proof of assertion (1), we consider again the scaling u^σ introduced in Lemma 4.2, Point (3). Since $\mathcal{E}(u^\sigma) \to 0$ as $\sigma \to 0^+$, we have that $m(c) \leq 0$ for any $c > 0$. The other statements where already proved in Lemma 4.2, Points (1) and (2). For the proof of assertion (2) we know from Lemma 4.3 that there exists a $c > 0$ such that $m(c) < 0$. Now let $d > 0$ be such that $m(d) < 0$ and $u \in X$ be an associated minimizer. We consider now the scaling $v(x) = u(\sigma^{-\frac{1}{N}} x)$ used in the proof of Lemma 4.5. For $\sigma > 1$ we have $\|v\|_2^2 > d$ and $\mathcal{E}(v) < \mathcal{E}(u)$. This proves the claim. We also point out that very likely the function $\{c \to m(c)\}$ is continuous for $c > 0$ so that also $m(c_0) = 0$. However we did not pursued in that direction. $\square$

Proof of Theorem 1.5. We assume by contradiction that there exist a sequence $(c_n) \subset \mathbb{R}^+$ with $c_n \to 0$ as $n \to \infty$ and $(u_n) \subset X$ such that $m(c_n)$ is reached by $u_n \in X$. On one hand, we know from assertion (2) that $\mathcal{E}(u_n) \leq 0$, for all $n \in \mathbb{N}$. Thus using inequality (4.5), we get

$$\int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx \leq K \Big(\int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx \Big)^{\frac{\theta N}{N-2}} c_n^{1-\theta}. \tag{4.9}$$

If $p = 3 + \frac{4}{N}$ we have $\frac{\theta N}{N-2} = 1$ and $1 - \theta = \frac{2}{N} > 0$. Thus, since $c_n \to 0$, we immediately get a contradiction from (4.9). Now if $p < 3 + \frac{4}{N}$, we have that $\frac{\theta N}{N-2} < 1$ and thus,

$$\int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx \to 0, \quad \text{as } n \to \infty.$$

Now, again in light of inequality (4.5), we learn that $\|u_n\|_{p+1} \to 0$ as $n \to \infty$. In turn, also

$$\|\nabla u_n\|_2 \to 0, \quad \text{as } n \to \infty, \tag{4.10}$$

since $\mathcal{E}(u_n) \leq 0$ implies that $\|\nabla u_n\|_2^2 \leq \frac{2}{p+1} \|u_n\|_{p+1}^{p+1}$, for all $n \in \mathbb{N}$. On the other hand, one can use the Gagliardo-Nirenberg inequality

$$\|u\|_q \leq C(N, q) \|\nabla u\|_2^\gamma \|u\|_2^{1-\gamma}, \tag{4.11}$$

where $q \in [2, \frac{2N}{N-2}]$ if $N \geq 3$, $2 \leq q < \infty$ in the cases $N = 1, 2$ and $\gamma = N(\frac{1}{2} - \frac{1}{q})$. In our setting, choosing $q = p + 1$, this yields

$$\|u_n\|_{p+1}^{p+1} \leq C(p, N) \|\nabla u_n\|_2^{\frac{N}{2}(p-1)} c_n^{\frac{1}{2}(p+1) - \frac{N}{4}(p-1)}.$$

Now since $\mathcal{E}(u_n) \leq 0$ we have

$$\|\nabla u_n\|_2^2 \leq \frac{2}{p+1} \int_{\mathbb{R}^N} |u_n|^{p+1} dx \leq C(p, N) \|\nabla u_n\|_2^{\frac{N}{2}(p-1)} c_n^{\frac{1}{2}(p+1) - \frac{N}{4}(p-1)}. \tag{4.12}$$

If $p = 1 + \frac{4}{N}$ we have $\frac{N}{2}(p-1) = 2$ and $\frac{1}{2}(p+1) - \frac{N}{4}(p-1) > 0$. Thus we get directly a contradiction since $c_n \to 0$. If $1 + \frac{4}{N} < p \leq \frac{N+2}{N-2}$ for $N = 3$ and $1 + \frac{4}{N} < p$ if $N = 1, 2$ we have $\frac{N}{2}(p-1) > 2$ and $\frac{1}{2}(p+1) - \frac{N}{4}(p-1) \geq 0$. Thus there exists a $d > 0$ such that $||\nabla u_n||_2 \geq d$ for all $n \in \mathbb{N}$ yielding a contradiction with (4.10). If $\frac{N+2}{N-2} \leq p \leq \frac{N+6}{N-2}$ we proceed as follows. We have for any $\frac{N+2}{N-2} \leq p \leq \frac{3N+2}{N-2}$ by interpolation,

$$\|u_n\|_{p+1}^{p+1} \leq \|u_n\|_{\frac{2N}{N-2}}^{\frac{4N}{N-2}-(p+1)} \|u_n\|_{\frac{4N}{N-2}}^{2(p+1)-\frac{4N}{N-2}}. \tag{4.13}$$

It is possible to write (4.13) since $(u_n) \subset L^{\frac{2N}{N-2}} \cap L^{\frac{4N}{N-2}}$. Then using again the fact that $\mathcal{E}(u_n) \leq 0$ and the Sobolev inequality we have

$$\|\nabla u_n\|_2^2 \leq \frac{2}{p+1} \int_{\mathbb{R}^N} |u_n|^{p+1} dx \leq C(p,N) \|\nabla u_n\|_2^{\frac{4N}{N-2}-(p+1)} \|u_n\|_{\frac{4N}{N-2}}^{2(p+1)-\frac{4N}{N-2}}. \tag{4.14}$$

Since $p \geq \frac{N+2}{N-2}$, we have $2(p+1) - \frac{4N}{N-2} \geq 0$ and, since $p \leq \frac{N+6}{N-2}$, we have $\frac{4N}{N-2} - (p+1) \geq 2$. Thus we get a contradiction with (4.10) as $n \to \infty$. At this point we observe that, for $N = 1, 2, 3, 4$, we have covered the entire range $1 + \frac{4}{N} \leq p \leq 3 + \frac{4}{N}$, as in these cases $3 + \frac{4}{N} < \frac{N+6}{N-2}$.

Remark 4.7. Considering the sequence $(c_n) \subset \mathbb{R}^+$ with $c_n \to 0$ introduced in the proof of Theorem 1.5 we suspect that assuming, by contradiction, the existence of a minimizer $u_n \in X$, it is possible to prove that the sequence $(\|u_n\|_\infty)$ remains bounded, as $n \to \infty$. If this was true then, by interpolation

$$\|u_n\|_{p+1}^{p+1} \leq \|\nabla u_n\|_2^{\frac{2N}{N-2}} \|u_n\|_\infty^{(p+1)-\frac{2N}{N-2}}. \tag{4.15}$$

Thus, arguing, as in (4.14) and since $\frac{2N}{N-2} > 2$, we have the desired contradiction.

We end this section by showing

Lemma 4.8. *Assume that $u_c \in X$ is a minimizer of (4.2) for some $c > 0$. Then the associated Lagrange multiplier is strictly negative.*

Proof. Let $\lambda_c \in \mathbb{R}$ be the Lagrange multiplier associated to u_c, i.e.

$$\mathcal{E}'(u_c) = \lambda_c u_c. \tag{4.16}$$

Applying Pohozaev identity to (4.16) yields

$$\frac{1}{p+1} \int_{\mathbb{R}^N} |u_c|^{p+1} dx = \frac{N-2}{N} \left[\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_c|^2 dx + \int_{\mathbb{R}^N} |u_c|^2 |\nabla u_c|^2 dx \right] - \frac{\lambda_c}{2} \int_{\mathbb{R}^N} |u_c|^2 dx.$$

Thus, we obtain

$$\mathcal{E}(u_c) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u_c|^2 + 2|u_c|^2 |\nabla u_c|^2 dx + \frac{\lambda_c}{2} \int_{\mathbb{R}^N} |u_c|^2 dx.$$

Since $\mathcal{E}(u_c) \leq 0$ we deduce that $\lambda_c < 0$. □

5. Orbital stability

In this section we prove the orbital stability result, Theorem 1.4. The key point is to prove the following

Lemma 5.1. *Assume that $1 < p < 3 + \frac{4}{N}$ and that $c > 0$ is such that $m(c) < 0$. Then for any real minimizing sequence of (4.2), there exists a subsequence that is strongly converging in X, up to a translation in $\mathbb{R}^N$.*

Proof. Let $(u_n) \subset X$ be any minimizing sequence for problem (4.2). We will prove the assertion by means of Lions's Compactness-Concentration Principle (cf. [26, 27]), applied to the sequence

$$\rho_n(x) = u_n^2(x), \qquad n \in \mathbb{N}.$$

Let us first proves that the vanishing, namely

$$\sup_{y\in\mathbb{R}^N} \int_{y+B_R} |u_n|^2 dx \to 0 \quad \text{for all } R > 0,$$

cannot occur. By Lemma 4.2, we know that

$$(u_n) \subset X \text{ is bounded and } \int_{\mathbb{R}^N} |\nabla(u_n^2)| dx \text{ is bounded in } \mathbb{R}. \tag{5.1}$$

We apply [27, Lemma I.1] to the sequence ρ_n. Indeed, ρ_n is bounded in $L^1(\mathbb{R}^N)$ and $\nabla\rho_n$ is bounded in $L^2(\mathbb{R}^N)$. Then for every α such that $1 \le \alpha \le \frac{2N}{N-2}$, $\rho_n \to 0$ in $L^\alpha(\mathbb{R}^N)$, as n goes to ∞. Taking $\alpha = \frac{p+1}{2}$ (this choice is valid since $1 < p < 3 + \frac{4}{N}$) provides

$$\|\rho_n\|_{\frac{p+1}{2}} = \|u_n\|_{p+1}^2 \to 0 \quad \text{as } n \to \infty,$$

and then $\liminf_{n\to\infty} \mathcal{E}(u_n) \ge 0$, which contradicts the fact that $m(c) < 0$. Now, by following the lines of the proof of [26, Lemma III.1], one can show that there exists a subsequence u_{n_k} (that we will still denote by (u_n)) such that either compactness occurs or dichotomy occurs in the following sense: there exists $\alpha \in (0, c)$ such that, for all $\varepsilon > 0$, there exists $k_0 \ge 1$ and two sequences $(u_n^1), (u_n^2)$ bounded in X such that, for all $k \ge k_0$,

$$\|u_n - (u_n^1 + u_n^2)\|_{L^{p+1}} \le \delta(\varepsilon), \quad 1 < p < 3 + \frac{4}{N}, \quad \text{with } \delta(\varepsilon) \to 0 \text{ for } \varepsilon \to 0, \tag{5.2}$$

$$\left| \int_{\mathbb{R}^N} (u_n^1)^2 dx - \alpha \right| \le \varepsilon, \quad \left| \int_{\mathbb{R}^N} (u_n^1)^2 dx - (c - \alpha) \right| \le \varepsilon,$$

$$\operatorname{dist}(\operatorname{supp} u_n^1, \operatorname{supp} u_n^2) \to \infty, \quad \text{as } n \to \infty,$$

$$\liminf_{k\to\infty} \int_{\mathbb{R}^N} (|\nabla u_n|^2 - |\nabla u_n^1|^2 - |\nabla u_n^2|^2) dx \ge 0, \tag{5.3}$$

$$\liminf_{k\to\infty} \int_{\mathbb{R}^N} (|\nabla (u_n)^2|^2 - |\nabla (u_n^1)^2|^2 - |\nabla (u_n^2)^2|^2) dx \ge 0. \tag{5.4}$$

We point out that, only inequalities (5.2) and (5.4) have to be proved, the other inequalities are already contained in [26, Lemma III.1]. Because of (5.1) and taking into account inequality (4.5) we learn that there exists a positive constant K such that, for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N} |u_n|^{p+1} dx \leq K\Big(\int_{\mathbb{R}^N} |u_n|^2 dx\Big)^{1-\theta}, \qquad \theta = \frac{(p-1)(N-2)}{2(N+2)}. \tag{5.5}$$

Thus, inequality (5.2) follows from the corresponding inequality for the L^2 norm which is contained in the proof of [26, Lemma III.1]. Now inequality (5.4) can be obtained by arguing as for the proof of (5.3). Indeed, notice that, if φ_R is a given smooth cut-off function, $0 \leq \varphi_R \leq 1$, $\varphi_R = 1$ on $B(0,R)$, $\varphi_R = 0$ outside $B(0,2R)$ and $|\nabla \varphi_R| \leq \frac{1}{R}$, and v_n is a sequence in X satisfying the boundedness condition (5.5), then we have

$$\begin{aligned} |\nabla(\varphi_R v_n)^2|^2 - \varphi_R^4 |\nabla v_n^2|^2 &= 4\varphi_R^3 v_n^2 \nabla\varphi_R \cdot \nabla v_n^2 + 4\varphi_R^2 |\nabla \varphi_R|^2 v_n^4 \\ &\leq 2\varphi_R^3 |\nabla \varphi_R| v_n^4 + 2\varphi_R^3 |\nabla\varphi_R| |\nabla v_n^2|^2 + 4\varphi_R^2 |\nabla\varphi_R|^2 v_n^4, \end{aligned}$$

for all $n \geq 1$, yielding

$$\Big|\int_{\mathbb{R}^N} |\nabla(\varphi_R v_n)^2|^2 dx - \int_{\mathbb{R}^N} \varphi_R^4 |\nabla v_n^2|^2 dx\Big| \leq \frac{C}{R}, \quad \text{for all } n \geq 1,$$

for some positive constant C independent of n. This last inequality is therefore sufficient to obtain Inequality (5.4).

Now, it is standard to see that if the dichotomy property holds (with the inequalities indicated above), then sending ε to zero, the following inequality holds true

$$m(c) \geq m(\alpha) + m(c-\alpha).$$

To conclude we now prove that instead we have, for any $c_1, c_2 > 0$ such that $c_1 + c_2 = c$,

$$m(c) < m(c_1) + m(c_2). \tag{5.6}$$

In light of [26, Lemma II.1], to show that (5.6) holds, it is sufficient to prove that, for any $d > 0$ such that $m(d) < 0$,

$$m(\lambda d) < \lambda m(d), \quad \text{for any } \lambda > 1. \tag{5.7}$$

To prove inequality (5.7) we observe that, if $u_d \in X$ is a minimizer of $m(d)$, then setting $v(x) = u_d(\lambda^{-\frac{1}{N}} x)$ we have $\|v\|_2^2 = \lambda d$ and

$$\begin{aligned} \mathcal{E}(v) &= \lambda^{1-\frac{2}{N}} \Big[\int_{\mathbb{R}^N} \frac{1}{2}|\nabla u_d|^2 + |u_d|^2 |\nabla u_d|^2 dx\Big] - \lambda \int_{\mathbb{R}^N} |u_d|^{p+1} dx \\ &= \lambda\Big[\lambda^{-\frac{2}{N}} \int_{\mathbb{R}^N} \big(\frac{1}{2}|\nabla u_d|^2 + |u_d|^2|\nabla u_d|^2 dx\big) - \int_{\mathbb{R}^N} |u_d|^{p+1} dx\Big] \\ &< \lambda m(d). \end{aligned}$$

Thus $\mathcal{E}(v) < \lambda m(d)$ which lead to $m(\lambda d) < \lambda m(d)$, proving the claim.

Since we ruled out both vanishing and dichotomy, we have compactness for ρ_n, namely we know that there exists a sequence $(y_n) \subset \mathbb{R}^N$ such that, for any $\varepsilon > 0$, there is $R > 0$ with

$$\int_{y_n+B_R} |u_n|^2 dx \geq c - \varepsilon. \tag{5.8}$$

We then denote $\tilde{u}_n = u_n(\cdot + y_n)$ and clearly from inequality (5.8) we have $\tilde{u}_n \to \tilde{u}$ strongly in $L^2(\mathbb{R}^N)$, as $n \to \infty$. By (5.5) we then see that $\tilde{u}_n \to \tilde{u}$ strongly in $L^p(\mathbb{R}^N)$. At this point, taking into account point 1) of Lemma 4.4, and since $\tilde{u}_n \rightharpoonup \tilde{u}$ in X, we get that $\mathcal{E}(\tilde{u}) \leq \liminf \mathcal{E}(\tilde{u_n}) = m(c)$. This proves that $\tilde{u} \in X$ minimize (4.2) and then, necessarily, $\nabla \tilde{u}_n \to \nabla u$ in $L^2(\mathbb{R}^N)$, as $n \to \infty$, proving the strong convergence of $\tilde{u}_n$ to $\tilde{u}$ in X. This concludes the proof. ☐

Now we can give the

Proof of Theorem 1.4. First note that if (u_n) is a minimizing sequence for (4.2), then $(|u_n|)$ is also a minimizing sequence and is real. Then by Lemma 5.1, there exists a subsequence $(|u_{n_k}|)$ of $(|u_n|)$ and a sequence $(y_{n_k}) \subset \mathbb{R}^N$ such that $(|u_{n_k}(\cdot - y_{n_k})|)$ converges strongly in $H^1(\mathbb{R}^N)$ toward u where u is real and solves (4.2). Then the result follows by standard considerations (see, for example, [9]).

Remark 5.2. Take any solution u to problem (1.10), namely $\|u\|_2^2 = c$ and $\mathcal{E}(u) = m(c)$. Then it is a classical fact that there exists a parameter ω^*, depending on c and u, such that u solves equation (1.7) with $\omega = \omega^*$ (see Lemma 4.8 in that direction). If aim to study the orbital stability issue of the ground states of (1.7) via the constrained approach (as it is the case in the classical paper of Cazenave-Lions [9]) we need to have more informations on the ground states of (1.7). In particular we need to know that they share the same L^2 norm. Except when $N = 1$ where we have the uniqueness of the ground states, this information is not available to us. Now, when $N = 1$ we still need to know if, when u_1 and u_2 are two distinct solutions to the minimization problem (1.10), then we have $\omega_1^* = \omega_2^*$. We did not manage to show this.

Remark 5.3. In [1] two results about orbital stability are presented. When $N \geq 2$, in Theorem 3.3, assuming that $1 < p < 1 + \frac{4}{N}$ a result of orbital stability within the class of radial functions, is announced for the minimizers of (1.10). However, the proof is incorrect and, we guess, can be fixed only assuming in addition that $p \geq 3$. When $N = 1$, $4 \leq p < 6$, Proposition 4.7 guarantees the orbital stability of the minimizers of (1.10) but Theorem 4.8, which establish the connection with the problem where $\omega > 0$ is fixed, is false because the uniqueness of the Lagrange parameter is not known (see Remark 5.2).

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