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Abstract :

The propagation of high-power ultra-short laser pulses in gas gives rise to many phenomenas and leads in particular to the creation of a plasma column where electromagnetic waves propagate. These phenomenas are described by the general fluid mechanics equations relative to relativistic plasmas combined with Maxwell equations. Particularly, the amplitude of the electromagnetic field satisfies a nonlinear relativistic Schrödinger equation.

In this thesis, we study this relativistic Schrödinger equation. We begin by solving the Cauchy problem in arbitrary space dimension in Sobolev spaces $H^s(\mathbb{R}^N)$ with finite regularity without any smallness assumption on the initial data by energy methods.

Afterwards, we proved rigourously that in a sens the relativistic Schrödinger equation is, in dimension two, well-approached by a Klein-Gordon equation.

Finally, in the two-dimensional case, we showed the existence, the uniqueness up to translations and rotations and the orbital stability in a weaker sens than the usual one of solitary waves solutions of the relativistic Schrödinger equation, using the minimization method of T. Cazenave and P.L. Lions.

Key-words : Schrödinger equations - Cauchy problem - Energy methods - Klein-Gordon equations - Solitary waves - Orbital stability.

AMS 1991 Mathematics Subject Classification : 35A07, 35A15, 35B35, 35Q55, 35Q75.

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Chapitre 1

Introduction

"Adieu dit le renard. Voici mon secret
Il est très simple : on ne voit bien qu'avec le coeur.
L'essentiel est invisible à nos yeux."

Antoine de Saint-Exupéry.

Le but de cette thèse est d'étudier quelques propriétés mathématiques d'une équation de Schrödinger quasilinéaire intervenant en physique des plasmas. Tout d'abord, on se propose de résoudre le problème de Cauchy associé à cette équation en dimension quelconque d'espace et sans hypothèse de petitesse sur la donnée initiale. Ensuite, on montrera que, dans un certain sens, cette équation de Schrödinger est bien approchée par une équation d'onde de type Klein-Gordon. Enfin, on étudiera l'existence et la stabilité d'ondes stationnaires en dimension 2 d'espace toujours pour cette même équation .

1.1 De Maxwell à Schrödinger Relativiste: une approche physique

Lorsqu'une impulsion laser haute-énergie (de durée $\tau < 1$ picoseconde, P$= 10^{10}$W) est envoyée dans un milieu gazeux (par exemple H_e, H_2, CO_2, N_2), il se produit une ionisation du gaz qui se situe sur les contours du laser. Même dans les régions où l'intensité est faible, l'impulsion arrache de nombreux électrons aux atomes et molécules de gaz présents créant ainsi une colonne de plasma dans laquelle les plus hautes intensités du laser se propagent.

La présence de ce plasma "abaisse" l'indice de réfraction du gaz (voir exemple) et la colonne de plasma possède alors des propriétés défocalisantes. Cependant, il existe un certain nombre de phénomènes physiques qui font "augmenter" cet indice de réfraction. En premier lieu, il y a l'effet Kerr: les radiations electromagnétiques déforment le nuage électronique des atomes et des ions, créant ainsi des dipôles induits. Ensuite, il y a l'augmentation de masse des électrons libres oscillants dans le champ electromagnétique intense avec des vitesses proches de celle de la lumière et enfin il y a la répulsion des électrons depuis la région occupée par le champ qui aboutit à la création d'un tunnel rempli d'ions. Lorsqu'un seuil critique est atteint, on obtient un régime particulier dans lequel des impulsions lasers autofocalisées se propagent.

Les interactions entre une impulsion laser relativiste et un milieu gazeux ont fait l'objet de nombreuses recherches en physique ainsi qu'en mathématiques. On peut citer de manière non-exhaustive les travaux de A.I. Akhiezer et R.V. Polovin [2], de A.B. Borisov et *al* [9], de A.V. Borovskii et A.L. Galkin [10], de S.V. Bulanov, V.I. Kirsanov et A.S. Sakharov [15], de X.L. Chen et R.N. Sudan [19], de T. Lehner [40], de G-Z Sun, E. Ott, Y.C. Lee et P. Gudzar [58] pour la physique et de A. de Bouard, N. Hayashi et J.C. Saut [12], de C-H. Bruneau, L. Di Menza et T. Lehner [14], de L. Di Menza [22] ou encore de M. Poppenberg [52] pour les mathématiques.

Pour décrire la propagation, on va utiliser l'approche suivante (voir[10]): puisque les atomes et les ions subissent une rapide ionisation sur les contours de l'impulsion laser, on considère que la plupart des impulsions se propagent dans un plasma existant déjà. En conséquence, dans ce qui suit, on va étudier le cas d'une propagation d'onde dans un plasma. D'après les expérimentations, les électrons ne sont soumis qu'à des énergies thermiques E_{ther} de l'ordre de quelques kilo-électronvolts pendant le temps durant lequel l'intensité agit sur le matériel ([9], [44]). D'un autre coté, l'énergie moyenne d'oscillation

E_{osc} des électrons dans de tels champs dépasse les 100 kilo-électronvolts. En conséquence, puisque $E_{ther} \ll E_{osc}$, on peut considérer que le plasma est froid.

Un tel plasma plongé dans un champ électromagnétique ne répond qu'aux forces électromagnétiques. La pression dynamique et les phénomènes qui y conduisent peuvent être négligés. Par contre, du fait de leur grande vitesse oscillante, l'augmentation de masse des électrons doit être prise en compte. Ainsi on va utiliser, pour décrire le modèle, la physique de l'électrodynamique des plasmas froids relativistes ([9]).

On considère donc la propagation d'une impulsion laser ultra-courte dans un plasma constitué initialement par une densité d'électrons inhomogène, radiale, décrite par la fonction $f(r)$ de telle sorte que

$$N_e^{(0)} = N_{e,0} f(r), \quad \max f(r) = 1.$$

$N_e^{(0)}$ désigne la densité de charges initiale. On note

- $\vec{A}$ et ϕ le vecteur et le potentiel du champ électromagnétique
- E le champ électrique correspondant
- $\vec{p_e}$ le moment des électrons
- $\vec{j}$ la densité d'électrons
- ρ la densité de charges
- $N_e = N_e^{(0)} + N_{e,0}\delta n$ la densité de charge dynamique
- $m_{e,0}$ la masse d'électrons restante.

On considère qu'initialement, les électrons sont figés dans le plasma. On peut alors écrire les équations du sytème. On couple les équations de Maxwell avec les équations issues de la dynamique des plasmas:

$$\Box \vec{A} = \frac{1}{c}\vec{\nabla}\frac{\partial \phi}{\partial t} - \frac{4\pi}{c}\vec{j} \tag{1.1}$$

$$\Delta\phi = -4\pi\rho \tag{1.2}$$

$$\nabla \cdot \vec{A} = 0 \tag{1.3}$$

$$(\frac{\partial}{\partial t} + \vec{v}_e \cdot \vec{\nabla})\vec{p}_e = -e[-\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi + \frac{1}{c}\vec{v}_e \times (\vec{\nabla} \times \vec{A})] \tag{1.4}$$

$$\vec{j} = -eN_e\vec{v}_e, \;\; \rho = e(N_e^0 - N_e) \tag{1.5}$$

$$\vec{v}_e = \frac{\vec{p}_e}{m_e}, \;\; m_e = m_{e,0}\gamma \tag{1.6}$$

$$\gamma = \sqrt{1 + \frac{|p_e|^2}{(m_{e,0}c)^2}} \tag{1.7}$$

Les équations (1.1) et (1.2) sont les équations de Maxwell. L'équation (1.3) représente une condition de jauge de Coulomb alors que (1.4) est l'équation de mouvement des électrons. Les équations (1.5) définissent les densités de courant et de charge. (1.6) représente les relations relativistes entre la vitesse v_e des électrons et leurs moments. Enfin, (1.7) définit le facteur de Lorentz γ. Comme d'habitude, $\Box$ désigne l'opérateur d'Alembertien

$$\Box = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}.$$

Il est bien sûr utile de normaliser toutes les grandeurs afin de faire disparaître les constantes liées à la masse. On pose donc

$$\vec{A}\,' = \frac{e}{m_{e,0}c}\vec{A}, \;\; \phi' = \frac{e}{m_{e,0}c}\phi, \;\; E' = \frac{e}{m_{e,0}c}E,$$

$$\vec{p}_e\,' = \frac{\vec{p}_e}{m_{e,0}c}, \;\; \vec{v}_e\,' = \frac{\vec{v}_e}{c}, \;\; N_e' = \frac{N_e}{N_{e,0}}.$$

Dans la suite, on supprimera le signe $'$ pour des commodités d'écriture. D'après les relations

$$\begin{aligned}(\vec{p}_e \cdot \vec{\nabla})\vec{p}_e &= \frac{1}{2}\vec{\nabla}|\vec{p}_e|^2 - \vec{p}_e \times (\vec{\nabla} \times \vec{p}_e)\\ \vec{\nabla}\gamma &= \frac{\nabla|\vec{p}_e|^2}{2\gamma},\end{aligned}$$

on peut réécrire l'équation (1.4) sous la forme

$$\frac{\partial}{\partial t}(\vec{p}_e - \vec{A}) - \vec{v}_e \times [\vec{\nabla} \times (\vec{p}_e - \vec{A})] = \vec{\nabla}(\phi - \gamma). \quad (1.8)$$

On note $\omega_p = (4\pi e^2 N_e/m_e)^{\frac{1}{2}}$ la fréquence du plasma et τ la durée de l'impulsion. En supposant que $\tau >> \frac{2\pi}{\omega_p}$ (on dira que l'on est en présence d'un pulse long), on peut alors écrire $\vec{p}_e = \vec{A}$. Cette hypothèse signifie que l'on peut regarder le système formé par les électrons comme étant "adiabatique". Le système d'équation (1.1) – (1.8) peut ainsi être réécrit sous la forme

$$\begin{aligned}\Box\vec{A} &= k_p^2 N_e \gamma^{-1}\vec{A} && (1.9)\\ \Delta\phi &= k_p^2(N_e - f(r)) && (1.10)\\ \vec{\nabla}(\phi - \gamma) &= 0 && (1.11)\\ \gamma &= \sqrt{1 + |p_e|^2}, \;\; \vec{p}_e = \vec{A} && (1.12)\end{aligned}$$

où $\omega_{p,0}$ est la fréquence du plasma non perturbé

$$\omega_{p,0} = \frac{4\pi e^2 N_{e,0}}{m_{e,0}} \quad \text{et} \quad k_p = \frac{\omega_{p,0}}{c}.$$

On a supprimé le terme $\frac{1}{c}\vec{\nabla}\frac{\partial\phi}{\partial t}$ dans l'équation (1.9) car on suppose que ϕ et γ varient lentement au cours du temps. L'équation (1.10) résulte de l'identité vérifiée par la densité de charge dynamique

$$N_e = \max[0, f(r) + k_p^{-2}\Delta\gamma]. \quad (1.13)$$

En effet, il est possible de montrer que l'équation vérifiée par la charge δn peut s'écrire sous la forme

$$\frac{\partial^2 \delta n}{\partial t^2} + \frac{\omega_{p,0}^2}{\gamma}(1+\delta n)\delta n = c^2(\vec{\nabla} \cdot (N_e \frac{\vec{\nabla}\gamma}{\gamma}) + \vec{\nabla}(\frac{N_e}{\gamma}) \cdot \vec{E} - \vec{\nabla} \cdot (\frac{\partial}{\partial t}(\frac{N_e}{\gamma})(\vec{P}_e - \vec{A})). \quad (1.14)$$

En supposant que la répartition de charge est uniforme dans l'espace et en ne gardant dans (1.14) que les termes d'ordre le plus bas en $q = \frac{E}{E_c}$, on obtient, puisque $\vec{P}_e = \vec{A}$,

$$\frac{\partial^2 \delta n}{\partial t^2} + \frac{\omega_{p,0}^2}{\gamma}\delta n = c^2 \frac{\Delta \gamma}{\gamma}. \quad (1.15)$$

On effectue maintenant une hypothèse de type quasi-statique qui consiste à négliger la variation temporelle de δn devant $\omega_{p,0}$, ce qui permet finalement d'aboutir à

$$\delta n = k_p^{-2} \Delta \gamma. \quad (1.16)$$

En tenant compte du fait que N_e est une quantité positive, la relation (1.13) découle directement de (1.16).

En combinant les équations (1.9), (1.10) et (1.11), on aboutit finalement à l'équation suivante

$$\Box \vec{A} = k_p^2 \left(\frac{\max[0, f(r) + k_p^{-2}\Delta\gamma]}{\gamma} \right) \vec{A}. \quad (1.17)$$

Dans la suite, on supprimera les flèches des différents vecteurs. On regarde maintenant un champ polarisé linéairement suivant l'axe des z et on suppose que l'on dispose de deux échelles spatio-temporelles différentes, c'est-à-dire que l'on suppose A de la forme

$$A(x,y,z,t) = u_\varepsilon(x,y,\varepsilon z, \varepsilon t) e^{i(kz - \omega t)}, \quad (1.18)$$

où le paramètre ε s'écrit

$$\varepsilon = \frac{\omega_{p,0}}{\omega} = \frac{k_p}{k_0}. \quad (1.19)$$

En tenant compte de la relation de dispersion

$$k^2 = k_0^2 - k_p^2 \quad \text{avec} \quad k_0 = \frac{\omega}{c}, \quad (1.20)$$

un calcul rapide montre que u_ε est solution de

$$\begin{aligned} &\Delta_\perp u_\varepsilon + \varepsilon^2 \frac{\partial^2 u_\varepsilon}{\partial z^2} - \varepsilon^2 \frac{1}{c^2}\frac{\partial^2 u_\varepsilon}{\partial t^2} + 2i\varepsilon k \frac{\partial u_\varepsilon}{\partial z} + 2i\varepsilon \frac{\omega}{c^2}\frac{\partial u_\varepsilon}{\partial t} + k_p^2 u_\varepsilon \\ &= k_p^2 \left(\frac{1 + k_p^{-2}\Delta_\perp \gamma_\varepsilon}{\gamma_\varepsilon} \right) u_\varepsilon + \varepsilon^2 \frac{u_\varepsilon}{2\gamma_\varepsilon^2} \frac{\partial^2 |u_\varepsilon|^2}{\partial z^2} - \varepsilon^2 \frac{u_\varepsilon}{4\gamma_\varepsilon^4} \left(\frac{\partial |u_\varepsilon|^2}{\partial z} \right)^2, \end{aligned} \quad (1.21)$$

où $\gamma_\varepsilon = \sqrt{1+|u_\varepsilon|^2}$. Grâce aux relations

$$\varepsilon k = k_p\sqrt{1-\varepsilon^2}, \quad \varepsilon\frac{\omega}{c} = k_p$$

et après une renormalisation avec des constantes fixes par rapport à ε des variables x, y, z et t, on obtient finalement

$$\begin{aligned} &\Delta_\perp u_\varepsilon + \varepsilon^2\frac{\partial^2 u_\varepsilon}{\partial z^2} - \varepsilon^2\frac{\partial^2 u_\varepsilon}{\partial t^2} + 2i\sqrt{1-\varepsilon^2}\frac{\partial u_\varepsilon}{\partial z} + 2i\frac{\partial u_\varepsilon}{\partial t} \\ &= \left(\frac{1}{\gamma_\varepsilon} - 1\right)u_\varepsilon + \frac{\Delta_\perp \gamma_\varepsilon}{\gamma_\varepsilon}u_\varepsilon + \varepsilon^2\frac{u_\varepsilon}{2\gamma_\varepsilon^2}\frac{\partial^2|u_\varepsilon|^2}{\partial z^2} - \varepsilon^2\frac{u_\varepsilon}{4\gamma_\varepsilon^4}\left(\frac{\partial|u_\varepsilon|^2}{\partial z}\right)^2. \end{aligned} \tag{1.22}$$

L'hypothèse $\varepsilon \ll 1$ signifie que la fonction $u_\varepsilon(x, y, \varepsilon z, \varepsilon t)$ varie lentement suivant les variables z et t par rapport à l'onde plane $e^{i(kz-\omega t)}$. Elle revient à supposer que dans l'équation (1.22), on peut négliger les termes $\varepsilon^2\frac{\partial^2 u_\varepsilon}{\partial_t^2}$ et $\frac{\partial^2 u_\varepsilon}{\partial_z^2}$. Physiquement, cela veut dire que l'on est en présence d'un plasma sous-dense. Ainsi, en prenant formellement $\varepsilon = 0$ dans l'équation (1.22), on aboutit à l'équation de Schrödinger relativiste suivante

$$2i\frac{\partial u}{\partial t} + \Delta_\perp u - \frac{u}{\sqrt{1+|u|^2}}\Delta_\perp(\sqrt{1+|u|^2}) + (1 - \frac{1}{\sqrt{1+|u|^2}})u = 0. \tag{1.23}$$

1.2 Le problème de Cauchy pour des équations de Schrödinger quasilinéaires

Le deuxième chapitre de cette thèse traite du problème de Cauchy pour l'équation de Schrödinger quasilinéaire (1.23)

$$\begin{cases} 2i\frac{\partial u}{\partial t} + \Delta u - \frac{u}{\sqrt{1+|u|^2}}\Delta(\sqrt{1+|u|^2}) + u(1 - \frac{1}{\sqrt{1+|u|^2}}) = 0 \\ u(x,0) = a_0(x) \quad x \in \mathbb{R}^N; \ \ 0 \le t \le T \end{cases} \tag{1.24}$$

où la dimension d'espace N est quelconque et a_0 représente la donnée initiale. Cette équation a été déjà étudiée par A. de Bouard, N. Hayashi et J.C. Saut (voir [11], [12]). Dans [12], les auteurs montrent que le problème de Cauchy (1.24) est bien posé dans $H^6(\mathbb{R}^N)$ si $N = 1, 2, 3$ et si la donnée initiale a_0 est suffisamment petite. De plus, ils prouvent qu'en dimension $N = 2, 3$, il existe une unique solution globale au problème (1.24) dans $C([0, +\infty[; H^8(\mathbb{R}^N) \cap W^{6,6}(\mathbb{R}^N))$ si a_0 est suffisament petite. Enfin, en dimension 1, ils montrent que (1.24) est localement bien posée sans hypothèse de petitesse sur la donnée initiale.

Après ces travaux, M. Poppenberg (voir [52]) a étudié l'équation de Schrödinger suivante

$$iu_t + \Delta u + uh'(|u|^2)\Delta(h(|u|^2) - V(x)u - ug(|u|^2) = 0 \tag{1.25}$$

où V est un potentiel donné. Dans [52], l'auteur montre, moyennant certaines hypothèses sur h, g et V que le problème de Cauchy (1.25) est bien posé dans $H^\infty(\mathbb{R}^N)$ quelle que soit la dimension N sans supposer aucune petitesse sur la donnée initiale. Le principal ingrédient de sa preuve est un théorème de fonctions implicites de type Nash-Moser combiné avec la théorie des semi-groupes hyperboliques.

Dans cette partie, nous montrons que le problème (1.24) est bien posé quel que soit la dimension d'espace N dans des espaces de Sobolev de régularité finie et sans restriction de taille sur la donnée initiale. Plus précisément, nous montrons le théorème suivant

Théorème 1.2.1. *Soient $N \geq 1$, $s = 3$ si $N = 1$ et $s = 2[\frac{N}{2}] + 2$ si $N \geq 2$ et supposons que $a_0 \in H^{s+2}(\mathbb{R}^N)$. Alors, il existe un temps $T > 0$ et une unique solution u au problème de Cauchy* (1.24) *telle que*

$$u \in L^\infty(0, T; H^{s+2}(\mathbb{R}^N)) \cap C([0, T]; H^s(\mathbb{R}^N)).$$

On peut étendre ce résultat à la classe d'équation plus générale de la forme suivante

$$2i\frac{\partial u}{\partial t} + \Delta u - 2uh'(|u|^2)\Delta(h(|u|^2)) + ug(|u|^2) = 0, \tag{1.26}$$

où g et h sont des fonctions à valeurs réelles régulières et où h vérifie la condition

$$\exists\, C_h > 0,\ \ \forall s \in \mathbb{R}_+,\ \ 1 - 4sh'^2(s) > C_h h'^2(s). \tag{1.27}$$

La condition (1.27) assure en particulier que le terme $\Delta u - 2uh'(|u|^2)\Delta(h(|u|^2))$ est elliptique et permet de mettre en oeuvre la méthode d'énergie utilisée. Il est à noter que ces hypothèses sont aussi utilisées par M. Poppenberg dans [52].

Pour montrer ce résultat, on utilise la méthode introduite dans [12]. On commence par réécrire l'équation (1.24) en un système en $(u, \overline{u})^t$. Ensuite, pour linéariser le terme quadratique en ∇u, on dérive (1.24) par rapport à l'espace et au temps pour obtenir un système en $u_0, ..., u_{N+2}$ où

$$\forall\ 1 \leq j \leq N,\ \ u_j = \frac{\partial u}{\partial x_j},\ \ u_{N+1} = e^{f(|u_0|^2)}\frac{\partial u}{\partial t},\ \ u_{N+2} = e^{q(|u_0|^2)}\Delta u.$$

Ici, f et q jouent le rôle de transformation de jauge (voir [33]). Ces transformations sont utiles pour pouvoir effectuer des intégrations par parties dans les estimations d'énergies. Enfin, la méthode consiste à appliquer un théorème de point fixe au nouveau système obtenu. Le fait de travailler avec une seule dérivée en temps nous permet de considérer des espaces de Sobolev plus simples que dans [12] ce qui nous permet de supprimer les hypothèses de petitesse sur les données initiales.

1.3 Approximation de l'équation de Schrödinger nonlinéaire relativiste par une équation de Klein-Gordon

Le but de cette partie est de regarder en détail l'approximation $\varepsilon \ll 1$ faite sur l'onde électromagnétique A, à savoir que si l'on introduit deux échelles spacio-temporelles différentes, alors on peut négliger les dérivées secondes suivant z et t de u. Plus précisément, on aimerait pouvoir justifier le passage de l'équation (1.22) à l'équation (1.23) en procédant de la manière suivante: trouver des solutions u_ε de (1.22) qui convergent vers de solutions de (1.23) dans un espace $L^\infty(0,T;H^s)$. Les solutions u_ε considérées devront alors avoir un temps d'existence indépendant de ε et devront avoir des données initiales $u_\varepsilon(\cdot,0)$ de taille elles aussi indépendante de ε.

Cependant, il apparait difficile pour le moment de montrer l'existence de telles solutions. Les raisons en sont multiples. Une des principales est dûe à la présence du paramètre ε^2 devant les dérivées secondes suivant z de u_ε. En effet, cette présence oblige, pour réaliser des estimations d'énergie, à utiliser la norme suivante $||(1-\Delta_\varepsilon)^{\frac{s}{2}} \cdot ||_{L^2}$ où $\Delta_\varepsilon = \partial_x^2 + \partial_y^2 + \varepsilon^2\partial_z^2$. Cette norme dépend clairement du paramètre ε et ne permet pas d'obtenir une borne H^s sur les solutions qui ne tende pas vers zéro avec ε. Pour cette raison, on va étudier une équation modifiée en dimension 2

$$-\varepsilon^2\frac{\partial^2 u_\varepsilon}{\partial t^2} + 2i\frac{\partial u_\varepsilon}{\partial t} + \Delta u_\varepsilon = \left(\frac{1}{\gamma_\varepsilon} - 1\right)u_\varepsilon + \frac{\Delta\gamma_\varepsilon}{\gamma_\varepsilon}u_\varepsilon. \tag{1.28}$$

En prenant formellement $\varepsilon = 0$ dans (1.28), on tombe à nouveau sur l'équation de Schrödinger (1.23).

On va montrer ici que l'équation (1.28) possède des solutions u_ε quelle que soit la valeur du paramètre ε et que ces solutions convergent dans un espace approprié vers des solutions de l'équation (1.23) lorsque ε tend vers zéro, sous certaines restrictions sur leurs données initiales.

Tout d'abord, l'équation (1.28) a la forme d'une équation de Klein-Gordon. En effet, si on pose

$$v_\varepsilon(x,y,t) = u_\varepsilon(x,y,t)e^{i\frac{1}{\varepsilon^2}t}$$

alors on peut montrer que v_ε vérifie l'équation de Klein-Gordon suivante

$$-\varepsilon^2\frac{\partial^2 v_\varepsilon}{\partial t^2} + \Delta v_\varepsilon - \frac{1}{\varepsilon^2}v_\varepsilon = \left(\frac{1}{\gamma_\varepsilon} - 1\right)v_\varepsilon + \frac{\Delta\gamma_\varepsilon}{\gamma_\varepsilon}v_\varepsilon \tag{1.29}$$

avec les conditions initiales

$$\begin{aligned} v_\varepsilon(x,y,0) &= u_0^\varepsilon(x,y) \\ \partial_t v_\varepsilon(x,y,0) &= u_1^\varepsilon(x,y,z) + i\frac{1}{\varepsilon^2}u_0^\varepsilon(x,y). \end{aligned}$$

On se retrouve ici dans les conditions d'application du Théorème C de S. Klainerman et G. Ponce [37], à savoir que si les données initiales v_0^ε et v_1^ε sont petites dans des espaces adéquats, alors le problème de Cauchy (1.29) est globalement bien posé dans

$C([0,+\infty[;H^s(\mathbb{R}^3))\cap C^1([0,+\infty[;H^{s-2}(\mathbb{R}^3))$ pour un certain entier s. En revenant à u_ε, on peut alors montrer que (1.28) admet une unique solution u_ε. Cependant, ce résultat ne sera pas d'une grande utilité ici. En effet, la présence du terme $\frac{1}{\varepsilon^2}v_\varepsilon$ dans (1.29) impose à la solution v_ε de tendre vers zéro avec ε dans les espaces $L^\infty(0,T;H^s(\mathbb{R}^2))$. Il faut donc obtenir un autre théorème d'existence.

Ensuite, en utilisant les estimations d'énergies classiques concernant les équations d'ondes, il est possible de montrer les deux assertions suivantes:

1) le problème de Cauchy (1.28) est localement bien posé dans $L^\infty(0,T_\varepsilon,H^s(\mathbb{R}^2))$ sans supposer aucune restriction de taille sur les données initiales avec un temps d'existence $T_\varepsilon=\frac{\varepsilon^2}{(||u_0^\varepsilon||_{H^s(\mathbb{R}^2)}+||u_1^\varepsilon||_{H^{s-1}(\mathbb{R}^2)})}$.

2) le problème de Cauchy (1.28) est localement bien posé dans $L^\infty(0,T,H^s(\mathbb{R}^2))$ avec un temps d'existence $T=1$ mais pour des données initiales vérifiant $||u_0^\varepsilon||_{H^s(\mathbb{R}^2)}+||u_1^\varepsilon||_{H^{s-1}(\mathbb{R}^2)}\leq h(\varepsilon)$ où $h(\varepsilon)$ tends vers zéro avec ε.

Dans la première assertion, le temps d'existence tend vers zéro avec ε alors que dans la seconde, on demande aux données initiales d'être de l'ordre de ε^2. Au contraire, il nous faudrait disposer d'un théorème d'existence locale où premièrement, le temps d'existence T est indépendant de ε, où deuxièmement la donnée initiale u_0^ε a une taille quelconque et où enfin la taille des solutions reste bornée indépendamment de ε. Il apparait cependant difficile de s'affranchir d'une condition de petitesse sur u_1^ε. Ainsi, dans cette partie, on commence par montrer le résultat suivant.

Theorem 1.3.1. *Supposons que $u_0^\varepsilon\in H^8(\mathbb{R}^3)$ et que $u_1^\varepsilon\in H^7(\mathbb{R}^3)$. Alors il existe deux constantes δ_ε ($\delta_\varepsilon\sim 2^{-\frac{1}{\varepsilon^2}}$) et $M>0$ indépendante de ε telles que si $||u_1^\varepsilon||_{H^7(\mathbb{R}^2)}\leq\delta_\varepsilon$, alors il existe un temps $T>0$ indépendant de ε tel que (1.28) admet une unique solution u_ε vérifiant*

$$u_\varepsilon\in L^\infty(0,T;H^8(\mathbb{R}^2))\cap C([0,T];L^2(\mathbb{R}^2)).$$

De plus,

$$\sup_{t\in[0,T]}\Big(||u_\varepsilon||^2_{H^8(\mathbb{R}^2)}+\varepsilon^2||\partial_t u_\varepsilon||^2_{H^6(\mathbb{R}^2)}+\varepsilon^2||\partial_t^2 u_\varepsilon||^2_{H^6(\mathbb{R}^2)}\Big)\leq M.$$

Pour montrer le Théorème (1.3.1), on va considérer l'équation (1.28) comme une perturbation de l'équation de Schrödinger relativiste (1.23), c'est-à-dire que dans (1.28), le terme $\varepsilon^2\partial_t^2u_\varepsilon$ sera traité comme un terme perturbatif. C'est pourquoi, on va utiliser la méthode introduite dans le Chapitre 2, à savoir que l'on va commencer par transformer (1.28) en un système en $(u_\varepsilon,\overline{u}_\varepsilon)$. Ensuite, on dérive l'équation obtenue par rapport à l'espace et au temps pour obtenir un nouveau système où les inconnues s'écrivent

$$u_0=u_\varepsilon,\ \ u_j=\partial_j u_\varepsilon,\ \ \forall 1\leq j\leq 2,\ \ u_3=e^{q(|u_0|^2)}\Delta u_\varepsilon,\ \ u_4=\partial_t\Delta u_\varepsilon,\ \ u_5=\partial_t^2\Delta u_\varepsilon.$$

Comme dans le Chapitre 2, la fonction q joue ici le rôle de transformation de jauge et permet de traiter les équations vérifiées par u_0, u_1, u_2, et u_3 exactement comme on a traité les équations (2.9), (2.10) et (2.12) du Chapitre 2. En ce qui concerne les équations vérifiées par u_4 et u_5, on n'utilise pas de transformation de jauge car ces équations seront traitées d'une manière différente. En effet, on utilisera les estimations d'énergie relatives aux équations d'ondes qui ne demandent pas d'utiliser des intégrations par parties, c'est-à-dire qu'après avoir multipliées ces deux équations par respectivement $(\overline{u}_4)_t$ et $(\overline{u}_5)_t$, on intégre sur $\mathbb{R}^2$ la partie réelle de chacune d'elles. Finalement, la preuve du Théorème découlera d'un théorème de point fixe.

Une fois montré le Théorème 1.3.1, il est possible d'étudier la convergence de u_ε vers la solution u de (1.23) ayant pour condition initiale $u(x, y, 0) = u_0(x, y)$ sous les conditions que u_0^ε et u_1^ε vérifient les hypothèses du Théorme 1.3.1, que u_0 vérifie les hypothèses du Théorème 1.2.1 et que u_0^ε converge vers u_0 dans l'espace de Sobolev $H^6(\mathbb{R}^2)$. Plus précisément, on montre

Theorem 1.3.2. *Soient (u_0^ε) et (u_1^ε) deux suites bornées respectivement dans $H^8(\mathbb{R}^2)$ et $H^7(\mathbb{R}^2)$ vérifiant*

$$u_0^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u_0 \quad dans \quad H^6(\mathbb{R}^2) \quad et \quad u_1^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad dans \quad H^7(\mathbb{R}^2).$$

On suppose de plus que $||u_1^\varepsilon||_{H^7(\mathbb{R}^2)} \leq \delta_\varepsilon$ où δ_ε est donné par le Théorème 1.3.1. *Soient alors u_ε la solution de* (1.28) *donnée par* 1.3.1 *et u la solution de* (1.23) *donnée par* 1.2.1. *Alors il existe un temps T commun d'existence à u_ε et u tel que*

$$u_\varepsilon - u \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad dans \quad L^\infty(0, T; H^6(\mathbb{R}^2)).$$

La preuve du Théorème 1.3.2 repose sur l'estimation d'énergie utilisée dans le Chapitre 2. Les deux points essentiels sont ici d'utiliser le fait que les solutions u_ε de (1.28) possèdent un temps commun d'existence et sont bornées dans $L^\infty(0, T; H^8(\mathbb{R}^2))$ avec une borne indépendante de ε.

1.4 Stabilité des ondes stationnaires en dimension deux d'espace pour des équations de Schrödinger quasilinéaires

Dans ce chapitre, on s'intéresse à l'existence et à la stabilité d'ondes stationnaires de la forme $u_\omega = e^{i\omega t}\phi_\omega(x)$ où $x \in \mathbb{R}^2$, $t \geq 0$ qui sont solutions de l'équation (1.24). On suppose ici que la dimension de l'espace est égale à 2. En effet, en dimension 1, le problème de l'existence et de la stabilité d'ondes stationnaires pour l'équation (1.24) à déjà été résolu par I.D. Iliev et K.P. Kirchev dans [34]. En dimension supérieure où égale à 3, on ne s'attend pas à trouver de telles ondes solutions de (1.24) (on expliquera un peu plus loin pourquoi). On demande à la fonction ϕ_ω d'être un état fondamental de l'équation

$$\Delta\phi - \frac{\phi}{\sqrt{1+|\phi|^2}}\Delta(\sqrt{1+|\phi|^2}) + \phi(1 - \frac{1}{\sqrt{1+|\phi|^2}}) - 2\omega\phi = 0, \tag{1.30_ω}$$

c'est-à-dire une solution qui minimise l'action S_ω

$$S_\omega(\phi) = \frac{1}{2}\int_{\mathbb{R}^2}\Big(|\nabla\phi|^2 - |\nabla(\sqrt{1+|\phi|^2})|^2 - (\sqrt{1+|\phi|^2} - 1)^2\Big)\,dx + \omega\int_{\mathbb{R}^2}|\phi|^2 dx$$

parmi toutes les solutions de (1.30_ω). Il apparait difficile de travailler directement sur la fonctionnelle S_ω car celle-ci ne contrôle pas la norme H^1, c'est-à-dire qu'une borne sur $S_\omega(\phi)$ ne fournit pas de borne H^1 sur ϕ. Par contre, on peut remarquer que si l'on effectue le changement de fonction $\chi = \text{argsinh}(\phi)$, on se ramène à l'équation elliptique

$$-\Delta\chi = (1-2\omega)\sinh(\chi)\cosh(\chi)-\sinh(\chi), \tag{1.31_ω}$$

à laquelle on peut associer l'action suivante

$$T_\omega = \frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla\chi|^2 - (\cosh(\chi)-1)^2\right)dx + \omega\int_{\mathbb{R}^2}|\sinh(\chi)|^2 dx,$$

(les fonctions sinh, cosh et argsinh désignent respectivement le sinus hyperbolique, le cosinus hyperbolique et la fonction réciproque du sinus hyperbolique). Il existe un lien très fort entre ces deux équations: on peut effectivement montrer qu'une fonction ϕ_ω est un état fondamental de (1.30_ω) si et seulement si $\chi_\omega = \text{argsinh}(\phi_\omega)$ est un état fondamental de (1.31_ω). De plus, en dimension 2, l'inégalité de Trudinger démontrée dans [50] permet d'estimer la norme L^1 de $(e^{|\chi|^2}-1)$ en fonction de la norme H^1 de χ. En conséquence, la fonctionnelle T_ω contrôle cette fois la norme H^1.

On va donc plutôt travailler sur l'équation (1.31_ω), tout en sachant que de nombreux résultats relatifs à l'équation (1.30_ω) seront déduits des résultats obtenus sur (1.31_ω). Par contre, on peut noter qu'en dimension supérieure où égale à trois, les termes $(\cosh(\chi)-1)^2$ et $\sinh^2(\chi)$ ne sont pas a priori dans l'espace $L^1(\mathbb{R}^2)$, ce qui rend l'étude de l'existence des ondes u_ω beaucoup plus délicate.

En utilisant des résultats de H. Berestycki, T. Gallouët et O. Kavian ([5]), on peut alors montrer

Theorem 1.4.1. *Il existe une constante ω_{max} telle que pour tout ω dans $]0, \omega_{max}[$, il existe une solution ϕ_ω de l'équation (1.30_ω) vérifiant*

i) $\phi_\omega > 0$ *dans* $\mathbb{R}^2$,

ii) ϕ_ω *est radiale, symétrique et décroissante par rapport à* $r = |x|$,

iii) $\phi_\omega \in C^2(\mathbb{R}^2)$,

iv) $\forall\alpha \in \mathbb{N}^2,\ |\alpha| \leq 2,\ \exists(c_\alpha, \delta_\alpha) \in (\mathbb{R}_+^*)^2,$ *tels que pour tout* $x \in \mathbb{R}^2$, $|D^\alpha\phi_\omega(x)| \leq C_\alpha e^{-\delta_\alpha|x|}$,

v) Pour tout ϕ *solution de* (1.30_ω), *on a* $S_\omega(\phi_\omega) \leq S_\omega(\phi)$

Ensuite, il apparait naturel d'étudier l'unicité des états fondamentaux ϕ_ω, cette propriété étant par ailleurs utile pour l'analyse de la stabilité. Là encore, le résultat découlera de l'unicité des états fondamentaux de l'équation (1.31_ω). Ce genre de problème a déjà fait l'objet de nombreuses études. On peut citer en particulier les travaux de Adimurthi ([1]), L. Erb et M. Tang ([23]), K. McLeod ([47]), L.A. Peletier ([51]) ou encore de M.K.

Kwong ([39]). Récemment, P. Pucci et J. Serrin ([54]) ont étudié le cas d'une nonlinéarité qui croit exponentiellement à l'infini, ce qui est bien sûr le cas pour l'équation (1.31_ω). Ainsi, des travaux de P. Pucci et J. Serrin, on peut facilement déduire le théorème suivant:

Theorem 1.4.2. *Pour tout $\omega \in]0, \omega_{max}[$, l'équation ($1.30_\omega$) possède un unique état fondamental, aux translations et multiplications par $e^{i\theta}$ ($\theta \in \mathbb{R}$) près.*

Une fois montré les Théorèmes 1.4.1 et 1.4.2, nous possédons tous les ingrédients pour étudier la stabilité des ondes stationnaires u_ω. En dimension 1, dans [34], les auteurs ont montré la stabilité orbitale de l'onde u_ω en utilisant les méthodes spectrales introduites par M. Grillakis, J. Shatah et W. Strauss ([31]).

En dimension 2, cette méthode parait peu adaptée à cause de la difficulté à calculer le spectre de l'opérateur linéarisé autour de ϕ_ω. C'est pour cela que l'on a choisi la méthode de minimization de T. Cazenave et P.L. Lions ([18]). Introduisons l'énergie E associée à (1.30_ω)

$$E(\phi) = S_\omega(\phi) - \omega \int_{\mathbb{R}^2} |\phi|^2 dx.$$

La méthode consiste à étudier les minima de l'énergie E sous la contrainte d'un invariant de l'équation, ici la norme L^2. Elle repose sur le principe de concentration-compacité introduit par P.L. Lions dans [41]. Cependant, comme on l'a déjà remarqué, la fonctionnelle E ne contrôle pas la norme H^1. En d'autres termes, il est possible de construire une suite $(\phi_n)_{n\in\mathbb{N}}$ telle que pour chaque n, ϕ_n est dans l'espace $H^1(\mathbb{R}^2)$, la suite $(E(\phi_n))_{n\in\mathbb{N}}$ est bornée mais dont la norme L^2 du gradient tend vers $+\infty$. De plus, la fonctionnelle E n'est pas continue sur $H^1(\mathbb{R}^2)$.

Pour ces raisons, on est obligé de travailler avec l'équation (1.31_ω) et surtout avec son énergie F

$$F(\chi) = T_\omega(\chi) - \omega \int_{\mathbb{R}^2} |\sinh(\chi)|^2 dx.$$

En conséquence, on ne montrera pas un résultat de stabilité orbitale pour les ondes u_ω mais un résultat un peu plus faible portant sur $\mathrm{argsinh}(|u_\omega|)$. Le résultat est le suivant

Théorème 1.4.3. *Pour $\omega \in]0, \omega_{max}[$, on définit $u_\omega(x,t) = e^{i\omega t}\phi_\omega(x)$ où ϕ_ω est un état fondamental de (1.30_ω). Alors, il existe un sous-ensemble infini Ω de $]0, \omega_{max}[$ tel que pour tout $\omega \in \Omega$, u_ω vérifie la propriété suivante: quel que soit $\varepsilon > 0$, $\exists \delta > 0$ tel que si $a_0 \in H^6(\mathbb{R}^2)$, $||a_0 - \phi_\omega||_{L^2(\mathbb{R}^2)} \leq \delta$ et $|E(a_0) - E(\phi_\omega)| \leq \delta$, alors la solution $u(x,t)$ de l'équation (1.24) avec pour donnée initiale a_0 définie sur $[0,T]$ vérifie*

$$\sup_{t\in[0,T]} \inf_{y\in\mathbb{R}^2} ||argsinh(|u(\cdot,t)|) - argsinh(|\phi_\omega(\cdot - y)|)||_{H^1(\mathbb{R}^2)} \leq \varepsilon$$

et

$$\sup_{t\in[0,T]} |E(u(\cdot,t)) - E(|u(\cdot,t)|)| \leq \varepsilon.$$

Le théorème ci-dessus n'est pas valable pour toutes les valeurs possibles du paramètre ω dans $]0, \omega_{max}[$ pour lesquelles il existe des ondes stationnaires u_ω.

En effet, une condition nécessaire pour que l'onde u_ω soit stable est que $\chi_\omega = \text{argsinh}(\phi_\omega)$ soit un minimum de l'énergie F. Un des problèmes importants pour ce genre d'étude est de pouvoir relier ω et $\mu = \int_{\mathbb{R}^2} \sinh^2(\chi_\omega) dx$. Tout d'abord, grâce au Théorème 1.4.2, il est possible de définir μ comme une fonction de ω. Cependant, on aimerait faire l'inverse, à savoir définir ω comme une fonction unique de μ. Cela revient à montrer que le problème de minimisation suivant

$$\inf_{\chi \in M_\mu} F(\chi) \quad \text{où} \quad M_\mu = \left\{ \chi \in H^1(\mathbb{R}^2) \ / \ \int_{\mathbb{R}^2} \sinh^2(\chi) dx = \mu \right\}$$

possède une unique solution (aux rotations et translations près).

Par exemple, dans le cas de l'équation de Schrödinger cubique, il existe un changement d'échelle qui permet de le faire (voir [16]). Lorsqu'il n'existe pas de tel changement, ce qui est bien sûr le cas pour l'équation qui nous intéresse, le problème apparait beaucoup plus délicat.

En d'autre terme, il est possible que pour une valeur de μ donnée, il existe deux états fondamentaux χ_{ω_1} et χ_{ω_2} avec $\omega_1 \neq \omega_2$ tels que

$$\int_{\mathbb{R}^2} \sinh^2(\chi_{\omega_1}) dx = \int_{\mathbb{R}^2} \sinh^2(\chi_{\omega_2}) dx.$$

Des simulations numériques ont été effectuées sur l'équation (1.31_ω) par L. Di Menza: il a notament tracé la courbe représentant la norme L^2 des fonctions $\sinh(\chi_\omega)$ en fonction du paramètre ω. Il a obtenu une courbe strictement croissante, ce qui tendrait à prouver, qu'au moins dans le cas de l'équation (1.31_ω), il soit possible de relier de manière unique ω et μ. C'est-à-dire, il apparaitrait raisonnable de montrer qu'il existe une valeur critique ω_c telle que le théorème de stabilité 1.4.3 soit vrai pour toutes les valeurs de ω comprises dans l'intervalle $]\omega_c, \omega_{\max}[$. Ceci reste un problème ouvert.

Nous montrons seulement ici qu'il existe un ensemble infini Ω inclus dans $]0, \omega_{\max}[$ pour lequel on a effectivement stabilité des ondes u_ω.

Il reste à noter que d'une manière générale, il n'est pas toujours vrai que les ondes solitaires u_ω soient stables pour toutes les valeurs du paramètre ω. Par exemple, considérons l'équation

$$iu_t + \partial_x^2 u + a|u|^{p-1}u + b|u|^{q-1}u = 0 \tag{1.32}$$

où a, b sont des réels, $x \in \mathbb{R}$ et $1 < p < q < +\infty$. Dans [49], M. Ohta a montré que si a et b sont positifs et si $q = 2p - 1$, alors u_ω est stable pour ω proche de 0 et instable pour de grandes valeurs de ω.

Chapitre 2

Le problème de Cauchy pour des équations de Schrödinger quasilinéaires

"Je fais souvent ce rêve étrange et pénétrant
D'une femme inconnue, et que j'aime, et qui m'aime,
Et qui n'est, chaque fois, ni tout à fait la même
Ni tout à fait une autre, et m'aime et me comprend."

Paul Verlaine

On the local well-posedness of quasilinear Schrödinger equations in arbitrary space dimension

(to appear in Communications in Partial Differential Equations)

Abstract

We study the Cauchy Problem associated to a class of quasilinear Schrödinger equations which have been derived as models of several physical phenomenas. We prove local existence in arbitrary space dimension N without any smallness condition on the initial data.

Résumé

On étudie le problème de Cauchy pour une classe d'équation de Schrödinger quasilinéaires qui modélisent de nombreux phénomènes physiques. On prouve l'existence locale de solutions pour ce type d'équations sans aucune hypothèse se petitesse sur la donnée initiale.

2.1 Introduction

We consider the quasilinear Schrödinger equation (1.24)

$$\begin{cases} 2iu_t + \Delta u - 2uh'(|u|^2)\Delta(h(|u|^2)) + ug(|u|^2) = 0 \\ u(x,0) = a_0(x) \quad x \in \mathbb{R}^N;\ \ 0 \le t \le T \end{cases}$$

where h, g are given smooth functions. The aim of the paper is to prove the well-posedness of the Cauchy problem for equation (1.24) in arbitrary space dimension without any smallness assumptions on the initial value. Equations of the form (1.24) appear in many domains of physics like plasma physics and fluid mechanics [29], [42], dissipative quantum mechanics [32], condensed matter theory [45] and in the theory of Heisenberg ferromagnets and magnons [3]. The case $h(s) = \sqrt{1+s}$, $g(s) = 1 - \frac{1}{\sqrt{1+s}}$ is studied in [12]. This equation models the self-channeling of a high power, ultra-short laser pulse in matter (see [10], [55]). In [12], A. de Bouard, N. Hayashi and J.C. Saut have proved the local well-posedness of the Cauchy problem in $H^6(\mathbb{R}^N)$ for the space dimension $N = 1, 2, 3$ if the initial data is small enough. They also proved in dimension $N = 2, 3$ that there is a unique global solution $u \in C([0, +\infty[; H^8(\mathbb{R}^N) \cap W^{6,6}(\mathbb{R}^N))$ if $a_0 \in H^8(\mathbb{R}^N) \cap W^{6,6/5}(\mathbb{R}^N)$ is small enough and in dimension $N = 1$ that there is a unique local solution $u \in C([0,T]; H^6(\mathbb{R}))$ if $a_0 \in H^6(\mathbb{R})$. In [52], M. Poppenberg has studied the Shrödinger equation (1.25)

$$iu_t + \Delta u + uh'(|u|^2)\Delta(h(|u|^2)) - V(x)u - ug(|u|^2) = 0$$

where V is a given potential. He has proved, under certain conditions on h, g and V, that there is a unique solution $u \in C^1([0,T]; H^\infty(\mathbb{R}^N))$ to the Cauchy Problem (1.25) for any space dimension N. The main ingredient of the proof is a Nash-Moser type implicit function theorem combined with hyperbolic semigroup theory. Up to now, this is the only result which seems to be available in arbitrary space dimension without requiring smallness assumptions on the initial data. In this paper, we solve the Cauchy Problem in Sobolev spaces with finite regularity without supposing any smallness assumption on the initial data.

Our motivations are the followings. In chapter 4, we show that equation (1.24) admits in dimension $N = 2$, stationary waves of the form $u_\omega(x,t) = e^{i\omega t}\phi_\omega(x)$ where ω is a parameter in $]0, \omega_{max}[$ (ω_{max} is a constant depending on the function h) and ϕ_ω is a ground state of the stationary equation associated with (1.24). In order to study the stability properties of u_ω, we have to prove that the Cauchy problem (1.24) is locally well-posed around ϕ_ω and so that it is locally well-posed without any smallness condition on the initial data.

The study of the Cauchy problem (1.24) with $h(s) = 0$ that is for semilinear Schrödinger equation was initiated by J. Ginibre and G. Velo. In [27], [28], the authors proved under certain conditions on g and on the initial data that the Cauchy problem is globally well-posed in the energy space $H^1(\mathbb{R}^N)$ using Strichartz inequalities. Since the works of J. Ginibre and G. Velo, many authors have studied this kind of equations. One can see in particular [35] and [59]. If h is not identically equal to zero, this method can not be applied, the reason being the presence of nonlinearities including derivatives of the unknown function.

In this chapter, we will follow the general strategy introduced in [11], [12]. We first rewrite equation (1.24) as a system in $(u,\overline{u})^t$. In order to "linearize" the quadratic term involving ∇u, we differentiate the equation with respect to x and t to obtain a new system in $u_0,..,u_{N+2}$ (see (2.9), (2.10), (2.11) and (2.12)) where

$$\forall 1 \leq j \leq N,\ \ u_j = \partial_j u,\ \ u_{N+1} = e^{f(|u_0|^2)} u_t,\ \ u_{N+2} = e^{q(|u_0|^2)}\Delta u$$

(we will explain later the role of f and q). Thanks to this transformation, the nonlinear term of the system depends now only on the first derivatives of the vector $U = (u_0,..,u_{N+2})$. In compensation, a nonlinear term appears in front of the space derivatives ΔU. The linearized version of the last system then defines a mapping $\mathcal{S}$

$$\begin{aligned} \mathcal{S} :\ \mathcal{Z} &\longrightarrow \mathcal{Z} \\ V &\longmapsto U \end{aligned}$$

where $\mathcal{Z} = \left[L^\infty(0,T;H^s(\mathbb{R}^N)) \cap C([0,T];L^2(\mathbb{R}^N))\right]^{N+3}$ for some s. The idea is to apply a fixed point theorem to $\mathcal{S}$ by getting estimates on U through the use of energy methods. For this purpose, setting $U = \mathcal{S}V$, we have to estimate each functions u_0,...,u_{N+2} in $H^s(\mathbb{R}^N)$, the main difficulty being to estimate u_{N+2} due to the presence of terms of the form

$$\sum_{k=1}^N \mathcal{E}(u_0,u_k) \begin{pmatrix} \partial_k u_{N+2} \\ \partial_k \overline{u}_{N+2} \end{pmatrix}$$

in the equation satisfied by u_{N+2} and of the same kind for u_{N+1}. The energy method works if the diagonal coefficients of the highest order term of equation (2.48) are purely imaginary, and this is why we introduce the gauge transform $u_{N+2} = e^{q(|u_0|^2)}\Delta u$ (see [33] for more details). Furthermore, the differentiation of the equation satisfied by u_{N+2}, which reduces the H^s-estimate to an L^2-estimate, adds in this equation new terms depending only on v_0. Then, a second gauge transform is used to cancel these terms. In [12], the authors used several order time derivatives for the equation (1.24). As a consequence, they worked in function spaces involving time derivatives of u. In this article, we only use one time derivative, which allows us to work in much simpler function spaces and so remove the smallness condition on the initial data.

Notation and function spaces: As usual, we denote by $L^p(\mathbb{R}^N)$ the Lebesgue space

$$L^p(\mathbb{R}^N) = \left\{ u \in \mathcal{S}'(\mathbb{R}^N)\ \ /\ \ ||u||_p < +\infty \right\}$$

where

$$||u||_p = \left(\int_{\mathbb{R}^N} |u(x)|^p dx\right)^{\frac{1}{p}} \text{ if } 1 \leq p < +\infty$$

and

$$||u||_\infty = \text{ess.sup}\left\{|u(x)|; x \in \mathbb{R}^N\right\}.$$

We define the Sobolev space $H^s(\mathbb{R}^N)$ as follows

$$H^s(\mathbb{R}^N) = \left\{ u \in \mathcal{S}'(\mathbb{R}^N)\ \ /\ \ ||u||^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi < +\infty \right\}$$

where $\widehat{u}(\xi)$ is the Fourier transform of u. Let $C(I,E)$ be the space of continuous functions from an interval I of $\mathbb{R}$ to a Banach space E. For simplicity, for $T>0$, we denote for $2\leq q\leq\infty$ and for $u\in L^\infty(0,T,L^q(\mathbb{R}^N))$

$$||u||_{T,q}=\sup_{t\in[0,T]}||u(t)||_q.$$

For $1\leq j\leq N$, we set $\partial_j=\partial_{x_j}$. For $k\in\mathbb{N}^N$, $k=(k_1,...,k_N)$, we denote $\partial^k u=\partial_1^{k_1}...\,\partial_N^{k_N}u$ and $|k|=k_1+...+k_N$. As usual, the coefficients of a matrix $\mathcal{D}$ will be denoted by $\mathcal{D}^{ij}$. The notation $\mathcal{D}(D^rV)$ means that the matrix $\mathcal{D}$ depends on ∂^kV for $|k|\leq r$. Different positive constants might be denoted by the same letter C. We also denote by $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$ the real part and the imaginary part of u and $[a]$ is used to denote the integer part of a.

We assume that $h,g\in C^{s+2}(\mathbb{R}^+)$ where $s=2[\frac{N}{2}]+2$. In this chapter, for the sake of clarity, we will treat the particular case $h(s)=\sqrt{1+s}$. However, it will be made clear in section 1.5 that all the results can be extended to a larger class of functions h, provided that h satisfies condition (1.27)

$$\exists\, C_h>0,\ \ \forall s\in\mathbb{R}_+,\ \ 1-4sh^{'2}(s)>C_hh^{'2}(s).$$

Under this assumption, we can recall Theorem (1.2.1)

Theorem. *Let $N\geq 1$, $s=3$ if $N=1$ and $s=2[\frac{N}{2}]+2$ if $N\geq 2$ and assume that $a_0\in H^{s+2}(\mathbb{R}^N)$ and that h satisfies (H). Then there is a positive T and a unique local solution to the Cauchy problem* (1.24) *such that*

$$u\in L^\infty(0,T;H^{s+2}(\mathbb{R}^N))\cap C([0,T];H^s(\mathbb{R}^N)).$$

Condition (1.27) ensures in particular that the term $\Delta u-2uh'(|u|^2)\Delta(h(|u|^2))$ is elliptic, i.e., equation (1.24) is of Schrödinger type. It is also necessary for the use of energy methods and for the gauge transforms (see section 1.5).

The chapter is organized as follows: in section 2.2, we transform equation (1.24) into a system to which we can apply an energy method. In section 2.3, we prove theorem 1.2.1 with $N=2$ considering the case $h(s)=\sqrt{1+s}$ in order to simplify the proof. Section 2.4 is devoted to the general space dimension N and section 2.5 ends this paper with final remarks and particularly we discuss the generalization of Theorem 1.2.1 to an arbitrary function h.

2.2 Transformation of the Equation

In this section, we transform equation (1.24) into a system to which we can apply an energy method. For the sake of clarity, we consider equation (1.24) with $h(s)=\sqrt{1+s}$. The function h is in $C^\infty(\mathbb{R}_+)$. We will discuss the general case in section 1.5. We fix in this section $a_0(x)$ in $H^{s+2}(\mathbb{R}^N)$. Equation (1.24) becomes

$$\begin{cases} 2iu_t+\Delta u-\frac{u}{\sqrt{1+|u|^2}}\Delta(\sqrt{1+|u|^2})+ug(|u|^2)=0 \\ u(x,0)=a_0(x)\quad x\in\mathbb{R}^N;\ \ 0\leq t\leq T. \end{cases} \tag{2.1}$$

We define $\gamma = \gamma(u) = \sqrt{1+|u|^2}$. A direct computation of the nonlinear term in (2.1) gives the following equation

$$2iu_t + \frac{2+|u|^2}{2\gamma^2}\Delta u - \frac{u^2}{2\gamma^2}\Delta \overline{u} - \frac{u}{\gamma^2}|\nabla u|^2 + \frac{u}{4\gamma^4}|\nabla |u|^2|^2 + ug(|u|^2) = 0.$$

The equation can then be rewritten as a system involving u and $\overline{u}$

$$2i\begin{pmatrix} u_t \\ \overline{u}_t \end{pmatrix} + \mathcal{A}(u)\begin{pmatrix} \Delta u \\ \Delta \overline{u} \end{pmatrix} - \begin{pmatrix} \frac{u}{\gamma^2}|\nabla u|^2 - \frac{u}{4\gamma^4}|\nabla |u|^2|^2 + ug(|u|^2) \\ \frac{-\overline{u}}{\gamma^2}|\nabla u|^2 + \frac{\overline{u}}{4\gamma^4}|\nabla |u|^2|^2 - \overline{u}g(|u|^2) \end{pmatrix} = 0 \tag{2.2}$$

where

$$\mathcal{A}(u) = \frac{1}{2\gamma^2}\begin{pmatrix} 2+|u|^2 & -u^2 \\ \overline{u}^2 & -(2+|u|^2) \end{pmatrix}.$$

Let us set $u_0 = u$, $u_j = \partial_j u$ for $j = 1, ..., N$, $u_{N+1} = e^{f(|u|^2)}u_t$ and $u_{N+2} = e^{q(|u|^2)}\Delta u$ where f and q are real functions which will be chosen later. We also set $U^* = (u_j)_{j=0}^N$ and $U = (u_j)_{j=0}^{N+2}$. Then equation (2.2) can be rewritten as

$$2i\begin{pmatrix} (u_0)_t \\ (\overline{u}_0)_t \end{pmatrix} + \mathcal{A}(u_0)\begin{pmatrix} \Delta u_0 \\ \Delta \overline{u}_0 \end{pmatrix} + \mathcal{F}_0(U^*) = 0 \tag{2.3}$$

where

$$\mathcal{F}_0(U^*) = -\begin{pmatrix} \frac{u_0}{\gamma^2(u_0)}(\sum_{k=1}^N |u_k|^2) - \frac{u_0}{4\gamma^4(u_0)}\sum_{k=1}^N (u_0\overline{u}_k + \overline{u}_0 u_k)^2 \\ \frac{-\overline{u}_0}{\gamma^2(u_0)}(\sum_{k=1}^N |u_k|^2) + \frac{\overline{u}_0}{4\gamma^4(u_0)}\sum_{k=1}^N (u_0\overline{u}_k + \overline{u}_0 u_k)^2 \end{pmatrix} + \begin{pmatrix} u_0 g(|u_0|^2) \\ -\overline{u}_0 g(|u_0|^2) \end{pmatrix}.$$

Differentiating equation (2.2) with respect to x_j, we obtain for $j = 1, ..., N$

$$2i\begin{pmatrix} (u_j)_t \\ (\overline{u}_j)_t \end{pmatrix} + \mathcal{A}(u_0)\begin{pmatrix} \Delta u_j \\ \Delta \overline{u}_j \end{pmatrix} + \sum_{k=1}^N B(u_0, u_k)\begin{pmatrix} T_{kj}u_{N+2} \\ T_{kj}\overline{u}_{N+2} \end{pmatrix}$$
$$+ C(u_0, u_j)\begin{pmatrix} e^{-q(|u_0|^2)}u_{N+2} \\ e^{-q(|u_0|^2)}\overline{u}_{N+2} \end{pmatrix} + \begin{pmatrix} F(U^*, u_j) \\ -\overline{F}(U^*, u_j) \end{pmatrix} = 0$$

where

$$C(v,w) = \begin{pmatrix} (\frac{1}{2\gamma^2(v)} - \frac{2+|v|^2}{2\gamma^4(v)})(v\overline{w} + \overline{v}w) & -\frac{vw}{\gamma^2(v)} + \frac{v^2}{4\gamma^4(v)}(v\overline{w} + \overline{v}w) \\ \frac{\overline{vw}}{\gamma^2(v)} - \frac{\overline{v}^2}{2\gamma^4(v)}(v\overline{w} + \overline{v}w) & (\frac{-1}{2\gamma^2(v)} + \frac{2+|v|^2}{2\gamma^4(v)})(v\overline{w} + \overline{v}w) \end{pmatrix}$$

$$B(v,w) = \begin{pmatrix} \frac{v}{4\gamma^4(v)}(2|v|^2\overline{w} + 2\overline{v}^2 w) - \frac{v}{\gamma^2(v)}\overline{w} & \frac{v}{4\gamma^4(v)}(2|v|^2 w + 2v^2\overline{w}) - \frac{v}{\gamma^2(v)}w \\ \frac{-\overline{v}}{4\gamma^4(v)}(2|v|^2\overline{w} + 2\overline{v}^2 w) + \frac{\overline{v}}{\gamma^2(v)}\overline{w} & \frac{-\overline{v}}{4\gamma^4(v)}(2|v|^2 w + 2v^2\overline{w}) + \frac{\overline{v}}{\gamma^2(v)}w \end{pmatrix}$$

$$
\begin{aligned}
F(U^*,w) = & \left(\frac{u_0\overline{w}+\overline{u}_0 w}{\gamma^4(u_0)} - \frac{w}{\gamma^2(u_0)}\right)\left(\sum_{k=1}^N |u_k|^2\right) + u_0 g^{'}(|u_0|^2)(u_0\overline{w}+\overline{u}_0 w) \\
& + \left(\frac{w}{4\gamma^4(u_0)} - \frac{(u_0\overline{w}+\overline{u}_0 w)}{2\gamma^6(u_0)}\right)\left(\sum_{k=1}^N (u_0\overline{u}_k+\overline{u}_0 u_k)^2 + \right) \\
& + \frac{u_0}{4\gamma^4(u_0)}\left(2(u_0\overline{w}+\overline{u}_0 w)(\sum_{k=1}^N |u_k|^2) + 2\overline{u}_0\overline{w}(\sum_{k=1} u_k^2)\right. \\
& \left. +2u_0 w(\sum_{k=1}^N \overline{u}_k^2)\right) + wg(|u_0|^2)
\end{aligned}
$$

and for $i,j=1,...,N$, T_{ij} is the following operator of order 0

$$T_{ij}u_{N+2} = \partial_i\partial_j\Delta^{-1}(e^{-q(|u_0|^2)}u_{N+2}).$$

We can rewrite these equations as follows

$$2i\begin{pmatrix} (u_j)_t \\ (\overline{u}_j)_t \end{pmatrix} + \mathcal{A}(u_0)\begin{pmatrix} \Delta u_j \\ \Delta\overline{u}_j \end{pmatrix} + \mathcal{F}_j(U^*,u_{N+2},Tu_{N+2}) = 0 \tag{2.4}$$

where $\mathcal{F}_j$ depends on U^*, u_{N+2} and on $T_{ij}u_{N+2}$ for $i,j=1,..,N$

$$
\begin{aligned}
\mathcal{F}_j(U^*,u_{N+2},Tu_{N+2}) = & \begin{pmatrix} F(U^*,u_j) \\ -\overline{F}(U^*,u_j) \end{pmatrix} + C(u_0,u_j)\begin{pmatrix} e^{-q(|u_0|^2)}u_{N+2} \\ e^{-q(|u_0|^2)}\overline{u}_{N+2} \end{pmatrix} \\
& + \sum_{k=1}^N B(u_0,u_k)\begin{pmatrix} T_{kj}u_{N+2} \\ T_{kj}\overline{u}_{N+2} \end{pmatrix}.
\end{aligned}
$$

Differentiating equation (2.2) with respect to t, we obtain

$$
\begin{aligned}
& 2i\begin{pmatrix} (e^{-f(|u_0|^2)}u_{N+1})_t \\ (e^{-f(|u_0|^2)}\overline{u}_{N+1})_t \end{pmatrix} + C(u_0, e^{-f(|u_0|^2)}u_{N+1})\begin{pmatrix} e^{-q(|u_0|^2)}u_{N+2} \\ e^{-q(|u_0|^2)}\overline{u}_{N+2} \end{pmatrix} \\
& +\mathcal{A}(u_0)\begin{pmatrix} \Delta(e^{-f(|u_0|^2)}u_{N+1}) \\ \Delta(e^{-f(|u_0|^2)}\overline{u}_{N+1}) \end{pmatrix} + \sum_{k=1}^N B(u_0,u_k)\begin{pmatrix} \partial_k(e^{-f(|u_0|^2)}u_{N+1}) \\ \partial_k(e^{-f(|u_0|^2)}\overline{u}_{N+1}) \end{pmatrix} \\
& + \begin{pmatrix} F(U^*, e^{-f(|u_0|^2)}u_{N+1}) \\ -\overline{F}(U^*, e^{-f(|u_0|^2)}u_{N+1}) \end{pmatrix} = 0.
\end{aligned} \tag{2.5}
$$

so that we can rewrite this equation in the following way

$$
\begin{aligned}
& 2i\begin{pmatrix} (u_{N+1})_t \\ (\overline{u}_{N+1})_t \end{pmatrix} + \mathcal{A}(u_0)\begin{pmatrix} \Delta u_{N+1} \\ \Delta\overline{u}_{N+1} \end{pmatrix} + \sum_{k=1}^N \mathcal{D}(u_0,u_k)\begin{pmatrix} \partial_k u_{N+1} \\ \partial_k\overline{u}_{N+1} \end{pmatrix} \\
& +\mathcal{G}(U,Tu_{N+2}) = 0
\end{aligned} \tag{2.6}
$$

where

$$\mathcal{D}(u_0, u_k) = B(u_0, u_k) - 2f'(|u_0|^2)\mathcal{A}(u_0)\begin{pmatrix} u_0\overline{u}_k + \overline{u}_0 u_k & 0 \\ 0 & u_0\overline{u}_k + \overline{u}_0 u_k \end{pmatrix}$$

and $\mathcal{G}$ is a matrix depending on U and $T_{ij}u_{N+2}$ for $i, j = 1, .., N$

$$\begin{aligned}
&\mathcal{G}(U, Tu_{N+2}) = \\
&e^{f(|u_0|^2)}\begin{pmatrix} F(U^*, e^{-f(|u_0|^2)}u_{N+1}) \\ -\overline{F}(U^*, e^{-f(|u_0|^2)}u_{N+1}) \end{pmatrix} \\
&- 2if'(|u_0|^2)e^{-f(|u_0|^2)}\begin{pmatrix} u_0\overline{u}_{N+1}u_{N+1} + \overline{u}_0 u_{N+1}^2 \\ u_0\overline{u}_{N+1}^2 + \overline{u_0}u_{N+1}\overline{u}_{N+1} \end{pmatrix} \\
&+ \mathcal{A}(u_0)\begin{pmatrix} (f'^2(|u_0|^2) - f''(|u_0|^2))(\sum_{k=1}^N(\overline{u}_0 u_k + u_0\overline{u}_k)^2)u_{N+1} \\ (f'^2(|u_0|^2) - f''(|u_0|^2))(\sum_{k=1}^N(\overline{u}_0 u_k + u_0\overline{u}_k)^2)\overline{u}_{N+1} \end{pmatrix} \\
&- \mathcal{A}(u_0)\begin{pmatrix} f'(|u_0|^2)(2\sum_{k=1}^N |u_k|^2 + \overline{u}_0(\sum_{k=1}^2 T_{kk}u_{N+2}))u_{N+1} \\ f'(|u_0|^2)(2\sum_{k=1}^N |u_k|^2 + \overline{u}_0(\sum_{k=1}^N T_{kk}u_{N+2}))\overline{u}_{N+1} \end{pmatrix} \\
&- \mathcal{A}(u_0)\begin{pmatrix} f'(|u_0|^2)u_0(\sum_{k=1}^N T_{kk}\overline{u}_{N+2})u_{N+1} \\ f'(|u_0|^2)u_0(\sum_{k=1}^N T_{kk}\overline{u}_{N+2})\overline{u}_{N+1} \end{pmatrix} \\
&+ C(u_0, e^{-f(|u_0|^2)}u_{N+1})\begin{pmatrix} e^{-q(|u_0|^2)+f(|u_0|^2)}u_{N+2} \\ e^{-q(|u_0|^2)+f(|u_0|^2)}\overline{u}_{N+2} \end{pmatrix} \\
&- \sum_{k=1}^N f'(|u_0|^2)B(u_0, u_k)\begin{pmatrix} u_0\overline{u}_k u_{N+1} + \overline{u}_0 u_k u_{N+1} \\ u_0\overline{u}_k\overline{u}_{N+1} + \overline{u}_0 u_k\overline{u}_{N+1} \end{pmatrix}.
\end{aligned}$$

In order to derive an equation for u_{N+2}, we differentiate equation (2.2) twice with respect to x_j to obtain

$$\begin{aligned}
&2i\begin{pmatrix} (\partial_j^2 u_0)_t \\ (\partial_j^2\overline{u}_0)_t \end{pmatrix} + \mathcal{A}(u_0)\begin{pmatrix} \partial_j^2(e^{-q(|u_0|^2)}u_{N+2}) \\ \partial_j^2(e^{-q(|u_0|^2)}\overline{u}_{N+2}) \end{pmatrix} \\
&+ \sum_{k=1}^N B(u_0, u_k)\begin{pmatrix} \partial_k\partial_j^2 u_0 \\ \partial_k\partial_j^2\overline{u}_0 \end{pmatrix} + \sum_{k=1}^N \partial_j B(u_0, u_k)\begin{pmatrix} \partial_k\partial_j u_0 \\ \partial_k\partial_j\overline{u}_0 \end{pmatrix} \\
&+2C(u_0, u_j)\begin{pmatrix} \partial_j(e^{-q(|u_0|^2)}u_{N+2}) \\ \partial_j(e^{-q(|u_0|^2)}\overline{u}_{N+2}) \end{pmatrix} + \partial_j C(u_0, u_j)\begin{pmatrix} e^{-q(|u_0|^2)}u_{N+2} \\ e^{-q(|u_0|^2)}\overline{u}_{N+2} \end{pmatrix} \\
&+ \begin{pmatrix} \partial_j F(U^*, u_j) \\ -\partial_j\overline{F}(U^*, u_j) \end{pmatrix} = 0.
\end{aligned} \tag{2.7}$$

We have

$$\partial_j C(u_0, u_j) = \frac{\partial C}{\partial v}(u_0, u_j)u_j + \frac{\partial C}{\partial w}(u_0, u_j)T_{jj}u_{N+2}$$

$$\partial_j B(u_0, u_j) = \frac{\partial B}{\partial v}(u_0, u_j)u_j + \frac{\partial B}{\partial w}(u_0, u_j)T_{jj}u_{N+2}$$

$$\partial_j F(U^*, u_j) = \frac{\partial F}{\partial u_0}u_j + \sum_{k=1}^N \frac{\partial F}{\partial u_k}T_{jk}u_{N+2} + \frac{\partial F}{\partial w}T_{jj}u_{N+2}.$$

Summing equations (2.7) from $j = 1$ to N, we obtain multiplying by $e^{q(|u_0|^2)}$

$$\begin{aligned} &2i\begin{pmatrix} (u_{N+2})_t \\ (\overline{u}_{N+2})_t \end{pmatrix} + \mathcal{A}(u_0)\begin{pmatrix} \Delta u_{N+2} \\ \Delta \overline{u}_{N+2} \end{pmatrix} + \sum_{k=1}^{N} \mathcal{E}(u_0, u_k)\begin{pmatrix} \partial_k u_{N+2} \\ \partial_k \overline{u}_{N+2} \end{pmatrix} \\ &+\mathcal{I}(U, Tu_{N+2}) = 0 \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} \mathcal{E}(u_0, u_k) =& B(u_0, u_k) + 2C(u_0, u_k) \\ &- 2\mathcal{A}(u_0)q^{'}(|u_0|^2)\begin{pmatrix} u_0\overline{u}_k + \overline{u}_0 u_k & 0 \\ 0 & u_0\overline{u}_k + \overline{u}_0 u_k \end{pmatrix} \end{aligned}$$

and $\mathcal{I}$ is a matrix depending on U and on $T_{ij}u_{N+2}$ for $i, j = 1, .., N$

$$\begin{aligned} &\mathcal{I}(U, Tu_{N+2}) = \\ &\sum_{m=1}^{N} e^{q(|u_0|^2)}\begin{pmatrix} \partial_m F(U^*, u_j) \\ -\partial_m \overline{F}(U^*, u_j) \end{pmatrix} \\ &- 2ie^{-f(|u_0|^2)}q^{'}(|u_0|^2)\begin{pmatrix} u_0\overline{u}_{N+1}u_{N+2} + \overline{u}_0 u_{N+1}u_{N+2} \\ u_0\overline{u}_{N+1}\overline{u}_{N+2} + \overline{u}_0 u_{N+1}\overline{u}_{N+2} \end{pmatrix} \\ &- \sum_{k=1}^{N}(2C(u_0, u_k) + B(u_0, u_k))q^{'}(|u_0|^2)\begin{pmatrix} (u_0\overline{u}_k + \overline{u}_0 u_k)u_{N+2} \\ (u_0\overline{u}_k + \overline{u}_0 u_k)\overline{u}_{N+2} \end{pmatrix} \\ &+ \sum_{k=1}^{N} \partial_k C(u_0, u_k)\begin{pmatrix} u_{N+2} \\ \overline{u}_{N+2} \end{pmatrix} + \sum_{j=1}^{N}\sum_{k=1}^{N} e^{q(|u_0|^2)}\partial_k B(u_0, u_k)\begin{pmatrix} T_{kj}u_{N+2} \\ T_{kj}\overline{u}_{N+2} \end{pmatrix} \\ &+ \mathcal{A}(u_0)\begin{pmatrix} (q^{'2}(|u_0|^2) - q^{''}(|u_0|^2))(\sum_{k=1}^{N}(\overline{u}_0 u_k + u_0\overline{u}_k)^2)u_{N+2} \\ (q^{'2}(|u_0|^2) - q^{''}(|u_0|^2))(\sum_{k=1}^{N}(\overline{u}_0 u_k + u_0\overline{u}_k)^2)\overline{u}_{N+2} \end{pmatrix} \\ &- \mathcal{A}(u_0)\begin{pmatrix} q^{'}(|u_0|^2)(2\sum_{k=1}^{N}|u_k|^2 + \overline{u}_0(\sum_{k=1}^{N} T_{kk}u_{N+2}))u_{N+2} \\ q^{'}(|u_0|^2)(2\sum_{k=1}^{N}|u_k|^2 + \overline{u}_0(\sum_{k=1}^{N} T_{kk}u_{N+2}))\overline{u}_{N+2} \end{pmatrix} \\ &- \mathcal{A}(u_0)\begin{pmatrix} q^{'}(|u_0|^2)u_0(\sum_{k=1}^{N} T_{kk}\overline{u}_{N+2}))u_{N+2} \\ q^{'}(|u_0|^2)u_0(\sum_{k=1}^{N} T_{kk}\overline{u}_{N+2}))\overline{u}_{N+2} \end{pmatrix} \end{aligned}$$

As a conclusion, we have transformed equation (1.24) into the following system

$$2i\begin{pmatrix} (u_0)_t \\ (\overline{u}_0)_t \end{pmatrix} + \mathcal{A}(u_0)\begin{pmatrix} \Delta u_0 \\ \Delta \overline{u}_0 \end{pmatrix} + \mathcal{F}_0(U^*) = 0, \tag{2.9}$$

for $j = 1, ..., N$

$$2i\begin{pmatrix} (u_j)_t \\ (\overline{u}_j)_t \end{pmatrix} + \mathcal{A}(u_0)\begin{pmatrix} \Delta u_j \\ \Delta \overline{u}_j \end{pmatrix} + \mathcal{F}_j(U^*, u_{N+2}, Tu_{N+2}) = 0, \tag{2.10}$$

$$\begin{aligned} &2i\begin{pmatrix} (u_{N+1})_t \\ (\overline{u}_{N+1})_t \end{pmatrix} + \mathcal{A}(u_0)\begin{pmatrix} \Delta u_{N+1} \\ \Delta \overline{u}_{N+1} \end{pmatrix} + \sum_{k=1}^{N} \mathcal{D}(u_0, u_k)\begin{pmatrix} \partial_k u_{N+1} \\ \partial_k \overline{u}_{N+1} \end{pmatrix} \\ &+\mathcal{G}(U, Tu_{N+2}) = 0 \end{aligned} \tag{2.11}$$

$$2i\begin{pmatrix}(u_{N+2})_t\\(\overline{u}_{N+2})_t\end{pmatrix}+\mathcal{A}(u_0)\begin{pmatrix}\Delta u_{N+2}\\\Delta\overline{u}_{N+2}\end{pmatrix}+\sum_{k=1}^{N}\mathcal{E}(u_0,u_k)\begin{pmatrix}\partial_k u_{N+2}\\\partial_k\overline{u}_{N+2}\end{pmatrix}\\+\mathcal{I}(U,Tu_{N+2})=0. \tag{2.12}$$

2.3 Proof of theorem 1.2.1 in the case N=2

In this section N is equal to 2. We will now apply the usual energy method (see [11]). Therefore, we study the system of equations (2.9), (2.10), (2.11) and (2.12) in the function space

$$X_T=\left\{\begin{array}{l}U=(u_j)_{j=0}^4 : u_j\in C([0,T];L^2(\mathbb{R}^2))\cap L^\infty(0,T;H^4(\mathbb{R}^2)),\\ ||U||_{X_T}=\displaystyle\sum_{j=0}^{4}\sup_{0\le t\le T}||u_j(t)||_{H^4(\mathbb{R}^2)}<\infty\end{array}\right\}.$$

For $M=(m_j)_{j=0}^4\in\mathbb{R}_+^5$ and $r\in\mathbb{R}_+$, we denote

$$X_T(M,r)=\left\{\begin{array}{l}U=(u_j)_{j=0}^4\in X_T:\forall j=0,..,4\ \ ||u_j||_{L^\infty(0,T;H^4(\mathbb{R}^2))}\le m_j\\ ||(u_0)_t||_{L^\infty(0,T;H^2(\mathbb{R}^2))}\le r \text{ and } u_0(0,x)=a_0(x)\end{array}\right\}$$

and we let $V=(v_j)_{j=0}^4\in X_T(M,r)$. We also set $V^*=(v_j)_{j=0}^2$. Consider the linearized version of equations (2.9), (2.10), (2.11) and (2.12)

$$2i\begin{pmatrix}(u_0)_t\\(\overline{u}_0)_t\end{pmatrix}+\mathcal{A}(v_0)\begin{pmatrix}\Delta u_0\\\Delta\overline{u}_0\end{pmatrix}+\mathcal{F}_0(V^*)=0, \tag{2.13}$$

$$2i\begin{pmatrix}(u_1)_t\\(\overline{u}_1)_t\end{pmatrix}+\mathcal{A}(v_0)\begin{pmatrix}\Delta u_1\\\Delta\overline{u}_1\end{pmatrix}+\mathcal{F}_1(V^*,v_4,Tv_4)=0, \tag{2.14}$$

$$2i\begin{pmatrix}(u_2)_t\\(\overline{u}_2)_t\end{pmatrix}+\mathcal{A}(v_0)\begin{pmatrix}\Delta u_2\\\Delta\overline{u}_2\end{pmatrix}+\mathcal{F}_2(V^*,v_4,Tv_4)=0, \tag{2.15}$$

$$2i\begin{pmatrix}(u_3)_t\\(\overline{u}_3)_t\end{pmatrix}+\mathcal{A}(v_0)\begin{pmatrix}\Delta u_3\\\Delta\overline{u}_3\end{pmatrix}+\sum_{k=1}^{2}\mathcal{D}(v_0,v_k)\begin{pmatrix}\partial_k u_3\\\partial_k\overline{u}_3\end{pmatrix}\\+\mathcal{G}(V,Tv_4)=0, \tag{2.16}$$

$$2i\begin{pmatrix}(u_4)_t\\(\overline{u}_4)_t\end{pmatrix}+\mathcal{A}(v_0)\begin{pmatrix}\Delta u_4\\\Delta\overline{u}_4\end{pmatrix}+\sum_{k=1}^{2}\mathcal{E}(v_0,v_k)\begin{pmatrix}\partial_k u_4\\\partial\overline{u}_4\end{pmatrix}\\+\mathcal{I}(V,Tv_4)=0. \tag{2.17}$$

Let $\mathcal{Z} = \left[L^\infty(0,T;H^4(\mathbb{R}^2)) \cap C([0,T];L^2(\mathbb{R}^2))\right]^5$. Then the Cauchy problem (2.13), (2.14), (2.15), (2.16) and (2.17) with initial condition

$$\begin{aligned}
&u_0(0,x) = a_0(x), \;\; u_1(0,x) = \partial_1 a_0(x), \;\; u_2(0,x) = \partial_2 a_0(x),\\
&u_3(0,x) = \frac{1}{2i} e^{f(|a_0(x)|^2)} \left(-\mathcal{A}(v_0(0))\Delta a_0(x) - \mathcal{F}_0(V^*(0))\right),\\
&u_4(0,x) = e^{q(|a_0(x)|^2)} \Delta a_0(x)
\end{aligned}$$

defines a mapping $\mathcal{S}$

$$\begin{aligned}
\mathcal{S} :\; &\mathcal{Z} \longrightarrow \mathcal{Z}\\
&V \longmapsto U.
\end{aligned}$$

This result can be proved by first regularizing the functions g and V, applying theorem 4.9 of [52] and taking the limit.

We want to show that there exist a time $T > 0$ and constants M and r such that the mapping $\mathcal{S}$ maps the closed ball $X_T(M,r)$ into itself. After that we prove that we can find a time $T > 0$ such that $\mathcal{S}$ is a contraction mapping under the constraint that it acts on $X_T(M,r)$ in the norm $\sum_{j=0}^4 ||v_j||_{T,2}$. Let V be fixed in the ball $X_T(M,r)$ and $U = \mathcal{S}(V)$. The first thing to do is to estimate u_4 in $H^4(\mathbb{R}^2)$. Differentiating equation (2.17) twice with respect to x_j for $j = 1,2$, we obtain

$$\begin{aligned}
&2i \begin{pmatrix} \partial_j^2 (u_4)_t \\ \partial_j^2 (\overline{u}_4)_t \end{pmatrix} + \mathcal{A}(v_0) \begin{pmatrix} \Delta \partial_j^2 u_4 \\ \Delta \partial_j^2 \overline{u}_4 \end{pmatrix} + \sum_{k=1}^2 \mathcal{E}(v_0, v_k) \begin{pmatrix} \partial_k \partial_j^2 u_4 \\ \partial_k \partial_j^2 \overline{u}_4 \end{pmatrix}\\
&+2\partial_j \mathcal{A}(v_0) \begin{pmatrix} \partial_j \Delta u_4 \\ \partial_j \Delta \overline{u}_4 \end{pmatrix} + \mathcal{J}_j(D^2V, D^2u_4, Tv_4) = 0
\end{aligned} \tag{2.18}$$

where $\mathcal{J}_j$ is a matrix depending on $\partial^{l_1} V$ for $|l_1| \leq 2$, on $\partial^{l_2} u_4$ for $|l_2| \leq 2$ and on $T_{ij} v_4$ for $i,j = 1,2$

$$\begin{aligned}
\mathcal{J}_j(D^2V, D^2u_4, Tv_4) =& \partial_j^2 \mathcal{A}(v_0) \begin{pmatrix} \Delta u_4 \\ \Delta \overline{u}_4 \end{pmatrix} + 2 \sum_{k=1}^2 \partial_j \mathcal{E}(v_0, v_k) \begin{pmatrix} \partial_k \partial_j u_4 \\ \partial_k \partial_j \overline{u}_4 \end{pmatrix}\\
&+ \sum_{k=1}^2 \partial_j^2 \mathcal{E}(v_0, v_k) \begin{pmatrix} \partial_k u_4 \\ \partial_k \overline{u}_4 \end{pmatrix} + \partial_j^2 \mathcal{I}(V, Tv_4).
\end{aligned}$$

Define $\phi = (1-\Delta)u_4$, we obtain substracting equation (2.18) from equation (2.17)

$$\begin{aligned}
&2i \begin{pmatrix} \phi_t \\ \overline{\phi}_t \end{pmatrix} + \mathcal{A}(v_0) \begin{pmatrix} \Delta \phi \\ \Delta \overline{\phi} \end{pmatrix} + \sum_{k=1}^2 \mathcal{H}(v_0, v_k, \partial_k v_0) \begin{pmatrix} \partial_k \phi \\ \partial_k \overline{\phi} \end{pmatrix}\\
&+\mathcal{K}(D^2V, D^2u_4, Tv_4) = 0
\end{aligned} \tag{2.19}$$

where

$$\mathcal{H}(v_0, v_k, \partial_k v_0) = \mathcal{E}(v_0, v_k) + 2\partial_k \mathcal{A}(v_0)$$

$$\begin{aligned}\mathcal{K}(D^2V, D^2u_4, Tv_4) =& \mathcal{I}(V, Tv_4) - 2\sum_{k=1}^{2} \partial_k \mathcal{A}(v_0) \begin{pmatrix} \partial_k u_4 \\ \partial_k \overline{u}_4 \end{pmatrix} \\ & - \sum_{k=1}^{2} \mathcal{J}_k(D^2V, D^2u_4, Tv_4).\end{aligned}$$

Differentiating again equation (2.19) with respect to x_j for $j = 1, 2$ and putting $\chi = (1 - \Delta)\phi$, we obtain multiplying by $\mathcal{A}^{-1}(v_0) = \gamma^2(v_0)\mathcal{A}(v_0)$

$$\begin{aligned}& 2i\gamma^2(v_0)\mathcal{A}(v_0) \begin{pmatrix} \chi_t \\ \overline{\chi}_t \end{pmatrix} + \begin{pmatrix} \Delta\chi \\ \Delta\overline{\chi} \end{pmatrix} + \sum_{k=1}^{2} \mathcal{L}(v_0, v_k, \partial_k v_0) \begin{pmatrix} \partial_k \chi \\ \partial_k \overline{\chi} \end{pmatrix} \\ & + \mathcal{M}(D^4V, D^4u_4, Tv_4) = 0\end{aligned} \tag{2.20}$$

where

$$\mathcal{L}(v_0, v_k, \partial_k v_0) = \gamma^2(v_0)\mathcal{A}(v_0)\left(\mathcal{H}(v_0, v_k, \partial_k v_0) + 2\partial_k \mathcal{A}(v_0)\right)$$

$$\begin{aligned}\mathcal{M}(D^4V, D^4u_4, Tv_4) =& \gamma^2(v_0)\mathcal{A}(v_0)\Big(\mathcal{K}(D^2V, D^2u_4, Tv_4) \\ & - 2\sum_{k=1}^{2} \partial_k^2 \mathcal{K}(D^2V, D^2u_4, Tv_4) \\ & - \sum_{k=1}^{2} \partial_k^2 \mathcal{A}(v_0) \begin{pmatrix} (1-\Delta)u_4 \\ (1-\Delta)\overline{u}_4 \end{pmatrix} \\ & - 2\sum_{k=1}^{2}\sum_{j=1}^{2} \partial_j \mathcal{H}(v_0, v_k, \partial_k v_0) \begin{pmatrix} \partial_k\partial_j(1-\Delta)u_4 \\ \partial_k\partial_j(1-\Delta)\overline{u}_4 \end{pmatrix} \\ & - \sum_{k=1}^{2}\sum_{j=1}^{2} \partial_j^2 \mathcal{H}(v_0, v_k, \partial_k v_0) \begin{pmatrix} \partial_k(1-\Delta)u_4 \\ \partial_k(1-\Delta)\overline{u}_4 \end{pmatrix} \\ & -2\sum_{k=1}^{2} \partial_k \mathcal{A}(v_0) \begin{pmatrix} \partial_k u_4 \\ \partial_k \overline{u}_4 \end{pmatrix}\Big).\end{aligned}$$

We can notice here that the dependence of the matrix $\mathcal{M}$ in u_4 and its derivatives is affine. We want to make an energy estimate on equation (2.20). In this perspective, the diagonal coefficients of the matrix $\mathcal{L}$ must be pure imaginary. We have

$$\mathcal{E}(v_0, v_k) = B(v_0, v_k) + 2C(v_0, v_k) - 2\mathcal{A}(v_0)q'(|v_0|^2) \begin{pmatrix} v_0\overline{v}_k + \overline{v}_0 v_k & 0 \\ 0 & v_0\overline{v}_k + \overline{v}_0 v_k \end{pmatrix}.$$

An easy calculation shows that we have for all $j = 1, 2$ and $k = 1, 2$

$$\mathrm{Re}(\gamma^2(v_0)\mathcal{A}(v_0)(B(v_0, v_k) + 2C(v_0, v_k)))^{jj} = -\frac{3}{2\gamma^2(v_0)}(v_0\overline{v}_k + \overline{v}_0 v_k).$$

Then choosing

$$q(s) = \ln(1+s)^{-\frac{3}{2}} \tag{2.21}$$

leads to the identity

$$\mathrm{Re}(\gamma^2(v_0)\mathcal{A}(v_0)\mathcal{E}(v_0,v_k))^{jj}=0$$

for all $j=1,2$ and $k=1,2$. By differentiating equation (2.17) four times in space, we add in the system terms of the form

$$4\sum_{k=1}^{2}\partial_k\mathcal{A}(v_0)\begin{pmatrix}\partial_k\chi\\ \partial_k\overline{\chi}\end{pmatrix}$$

which cannot be eliminated by the function q. Thus we have to use a second gauge transform to cancel these terms. Let $\psi=e^{p(v_o)}\chi$. We obtain the following equation in ψ

$$\begin{aligned}&2i\gamma^2(v_0)\mathcal{A}(v_0)\begin{pmatrix}\psi_t\\ \overline{\psi_t}\end{pmatrix}+\begin{pmatrix}\Delta\psi\\ \Delta\overline{\psi}\end{pmatrix}+\sum_{k=1}^{2}\mathcal{N}(v_0,v_k,\partial_kv_0)\begin{pmatrix}\partial_k\psi\\ \partial_k\overline{\psi}\end{pmatrix}\\&+\mathcal{O}(D^4V,D^4u_4,(v_o)_t,Tv_4)=0\end{aligned}\tag{2.22}$$

where

$$\mathcal{N}(v_0,v_k,\partial_kv_0)=\mathcal{L}(v_0,v_k,\partial_kv_0)-2\begin{pmatrix}\partial_kp(v_0)&0\\0&\partial_kp(v_0)\end{pmatrix}$$

and $\mathcal{O}$ is a matrix depending on $\partial^{l_1}V$ for $|l_1|\le 4$, on $\partial^{l_2}u_4$ for $|l_2|\le 4$, on $(v_0)_t$ and on $T_{ij}v_4$ for $i,j=1,2$

$$\begin{aligned}\mathcal{O}(D^4V,D^4u_4,(v_o)_t,Tv_4)=&e^{p(v_0)}\mathcal{M}(D^4V,D^4u_4,Tv_4)+\begin{pmatrix}\psi\Delta p(v_0)\\ \overline{\psi\Delta p(v_0)}\end{pmatrix}\\&-2\gamma^2(v_0)\mathcal{A}(v_0)(p(v_0))_t\begin{pmatrix}\psi\\ \overline{\psi}\end{pmatrix}\\&-\sum_{k=1}^{2}\mathcal{L}(v_0,v_k,\partial_kv_0)\begin{pmatrix}\psi\partial_kp(v_0)\\ \overline{\psi\partial_kp(v_0)}\end{pmatrix}\end{aligned}$$

Once again, an easy calculation shows that it is possible to chose p such that for $k=1,2$ and $j=1,2$

$$\mathrm{Re}\left(4\gamma^2(v_0)\mathcal{A}(v_0)\partial_k\mathcal{A}(v_0)-2\begin{pmatrix}\partial_kp(v_0)&0\\0&\partial_kp(v_0)\end{pmatrix}\right)^{jj}=0.$$

For such functions p and q, one has for $k=1,2$ and $j=1,2$

$$\mathrm{Re}(\mathcal{N}(v_0,v_k,\partial_kv_0))^{jj}=0.$$

It is now possible to make the usual estimate for equation (2.22). Multiplying equation

(2.22) by $\overline{\psi}$, taking the first line of the system and integrating over $\mathbb{R}^2$ leads to

$$\begin{aligned}
&i\int_{\mathbb{R}^2}(2+|v_0|^2)\psi_t\overline{\psi}dx - i\int_{\mathbb{R}^2} v_0^2\overline{\psi}_t\overline{\psi}dx + \int_{\mathbb{R}^2}\Delta\psi\overline{\psi}dx\\
&+\sum_{k=1}^{2}\int_{\mathbb{R}^2}\mathcal{N}^{11}(v_0,v_k,\partial_k v_0)(\partial_k\psi)\overline{\psi}dx\\
&+\sum_{k=1}^{2}\int_{\mathbb{R}^2}\mathcal{N}^{12}(v_0,v_k,\partial_k v_0)(\partial_k\overline{\psi})\overline{\psi}dx\\
&+\int_{\mathbb{R}^2}\mathcal{O}^1(D^4V,D^4u_4,(v_0)_t,Tv_4)\overline{\psi}dx = 0.
\end{aligned}\tag{2.23}$$

We take the imaginary part of equation (2.23). We have

$$\begin{aligned}
&\mathrm{Im}\left(i\int_{\mathbb{R}^2}(2+|v_0|^2)\psi_t\overline{\psi}dx - i\int_{\mathbb{R}^2} v_0^2\overline{\psi}_t\overline{\psi}dx\right)\\
&=\int_{\mathbb{R}^2}(\frac{2+|v_0|^2}{2})|\psi|_t^2 - \int_{\mathbb{R}^2}(\frac{v_0^2}{4}(\overline{\psi}^2)_t + \frac{\overline{v_0}^2}{4}(\psi^2)_t)dx\\
&=\frac{d}{dt}\left(\int_{\mathbb{R}^2}(\frac{2+|v_0|^2}{2})|\psi|^2 - \int_{\mathbb{R}^2}(\frac{v_0^2}{4}(\overline{\psi}^2) + \frac{\overline{v_0}^2}{4}(\psi^2))dx\right)\\
&-\int_{\mathbb{R}^2}\frac{|v_0|_t^2}{2}|\psi|^2dx + \int_{\mathbb{R}^2}(\frac{(v_0^2)_t}{4}(\overline{\psi}^2) + \frac{(\overline{v_0}^2)_t}{4}(\psi^2))dx.
\end{aligned}$$

$$\mathrm{Im}\left(\int_{\mathbb{R}^2}\Delta\psi\overline{\psi}dx\right) = -\mathrm{Im}\left(\int_{\mathbb{R}^2}|\nabla\psi|^2dx\right) = 0.$$

Since $\mathcal{N}^{11}(v_0,v_k,\partial_k v_0)\in i\mathbb{R}$, we have

$$\begin{aligned}
&\mathrm{Im}\left(\sum_{k=1}^{2}\int_{\mathbb{R}^2}\mathcal{N}^{11}(v_0,v_k,\partial_k v_0)(\partial_k\psi)\overline{\psi}dx\right)\\
&=\sum_{k=1}^{2}\int_{\mathbb{R}^2}\mathrm{Im}(\mathcal{N}^{11}(v_0,v_k,\partial_k v_0))\partial_k\frac{|\psi|^2}{2}dx\\
&=-\sum_{k=1}^{2}\int_{\mathbb{R}^2}\partial_k(\mathrm{Im}(\mathcal{N}^{11}(v_0,v_k,\partial_k v_0)))\frac{|\psi|^2}{2}dx
\end{aligned}$$

Since the embedding $H^2(\mathbb{R}^2)\hookrightarrow L^\infty(\mathbb{R}^2)$ is continuous and because $V=(v_j)_{j=0}^4\in X_T(M,r)$, there exists a constant $C(M)$ depending only on M such that

$$||\partial_k(\mathrm{Im}(\mathcal{N}^{11}(v_0,v_k,\partial_k v_0)))||_{T,\infty}\leq 2C(M).$$

Then

$$\left|\mathrm{Im}\left(\sum_{k=1}^{2}\int_{\mathbb{R}^2}\mathcal{N}^{11}(v_0,v_k,\partial_k v_0)(\partial_k\psi)\overline{\psi}dx\right)\right|\leq C(M)||\psi||_2^2.$$

In the same way, since $\mathcal{N}^{12}(v_0, v_k, \partial_k v_0)$ depends only on v_0, $\partial_k v_0$ and v_k, there exists a constant $C(M)$ depending only on M such that

$$||\partial_k \mathcal{N}^{12}(v_0, v_k, \partial_k v_0)||_{T,\infty} \leq C(M).$$

So

$$\begin{aligned}
\left|\mathrm{Im}\sum_{k=1}^{2}\int_{\mathbb{R}^2}\mathcal{N}^{12}(v_0, v_k, \partial_k v_0)(\partial_k\overline{\psi})\overline{\psi}dx\right| &= \left|\sum_{k=1}^{2}\int_{\mathbb{R}^2}\mathcal{N}^{12}(v_0, v_k, \partial_k v_0)\frac{\partial_k(\overline{\psi}^2)}{2}dx\right| \\
&= \left|\sum_{k=1}^{2}\int_{\mathbb{R}^2}\partial_k(\mathcal{N}^{12}(v_0, v_k, \partial_k v_0))\frac{\overline{\psi}^2}{2}dx\right| \\
&\leq C(M)||\psi||_2^2.
\end{aligned}$$

We have to make more precise the structure of the matrix $\mathcal{O}$. It is important to point out the derivatives of u_4. In fact the matrix $\mathcal{O}$ depends on the fourth order derivatives of u_4 at most, the fourth order derivatives of V at most and on $(u_0)_t$ in the following way:

$$\begin{aligned}
\mathcal{O} = &\sum_{|l_1|\leq 2, |l_2|\leq 4}\mathcal{P}_{l_1,l_2}(\partial^{l_1}V)\partial^{l_2}u_4 \\
&+\sum_{k,l=0}^{2}\sum_{j=0}^{4}\sum_{|l_1|,|l_2|\leq 2}\sum_{|l_3|\leq 4}\mathcal{Q}^{k,l,j}_{l_1,l_2,l_3}(\partial^{l_1}V)\partial^{l_2}T_{kl}u_4\partial^{l_3}v_j \\
&+\sum_{j=0}^{4}\sum_{|l_1|\leq 2}\sum_{|l_2|,|l_3|=3}\mathcal{R}^{j}_{l_1,l_2,l_3}(\partial^{l_1}V)\partial^{l_2}u_4\partial^{l_3}v_j \\
&+\mathcal{V}(V, Tv_4) + \mathcal{T}(v_0)(v_0)_t\psi
\end{aligned}$$

where the matrices $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ depend only on the second order derivatives of the vector V, $\mathcal{V}$ depends only on V and $T_{ij}v_4$ for $i, j = 1, 2$ and $\mathcal{T}$ depends only on v_0. Then recalling that $V \in X_T(M, r)$ we have

$$\begin{aligned}
&\left|\mathrm{Im}\int_{\mathbb{R}^2}\mathcal{O}^1(D^4V, D^4u_4, (v_0)_t, Tv_4)\overline{\psi}dx\right| \\
&\leq C(M)\sum_{|l_2|\leq 4}\int_{\mathbb{R}^2}|\partial^{l_2}u_4||\psi|dx \\
&+C(M)\sum_{k,l=0}^{2}\sum_{j=0}^{4}\sum_{|l_2|\leq 2, |l_3|\leq 4}\int_{\mathbb{R}^2}|\partial^{l_2}T_{kl}u_4||\partial^{l_3}v_j||\psi|dx \\
&+C(M)\sum_{j=0}^{4}\sum_{|l_2|,|l_3|=3}\int_{\mathbb{R}^2}|\partial^{l_2}u_4||\partial^{l_3}v_j||\psi|dx \\
&+C(M)||\psi||_2 + C(M)||(v_0)_t||_{T,\infty}||\psi||_2^2.
\end{aligned}$$

Thanks to the Cauchy-Schwarz inequality we have

$$\begin{aligned}\sum_{|l_2|\leq 4}\int_{\mathbb{R}^2}|\partial^{l_2}u_4||\psi|dx &\leq \sum_{|l_2|\leq 4}||\partial^{l_2}u_4||_2||\psi||_2\\ &\leq C(M)||u_4||_{H^4(\mathbb{R}^2)}||\psi||_2\\ &\leq C(M)||\psi||_2^2.\end{aligned}$$

Furthermore

$$\begin{aligned}&\sum_{k,l=0}^{2}\sum_{j=0}^{4}\sum_{|l_2|\leq 2,|l_3|\leq 4}\int_{\mathbb{R}^2}|\partial^{l_2}T_{kl}u_4||\partial^{l_3}v_j||\psi|dx\\ &\leq \sum_{k,l=0}^{2}\sum_{j=0}^{4}\sum_{|l_2|\leq 2,|l_3|\leq 4}||\partial^{l_2}T_{kl}u_4||_{L^\infty(\mathbb{R}^2)}||\partial^{l_3}v_j||_2||\psi||_2\\ &\leq C(M)||u_4||_{H^4(\mathbb{R}^2)}||\psi||_2\\ &\leq C(M)||\psi||_2^2\end{aligned}$$

since T_{kl} is continuous from $H^s(\mathbb{R}^2)$ into $H^s(\mathbb{R}^2)$ for $s=1,..,4$. For the next term, we have to use an $L^4-L^4-L^2$ estimate

$$\begin{aligned}&\sum_{j=0}^{4}\sum_{|l_2|,|l_3|=3}\int_{\mathbb{R}^2}|\partial^{l_2}u_4||\partial^{l_3}v_j||\psi|dx\\ &\leq \sum_{j=0}^{4}\sum_{|l_2|,|l_3|=3}||\partial^{l_2}u_4||_4||\partial^{l_3}v_j||_4||\psi||_2\\ &\leq \sum_{j=0}^{4}\sum_{|l_2|,|l_3|=3}||\partial^{l_2}u_4||_{H^1(\mathbb{R}^2)}||\partial^{l_3}v_j||_{H^1(\mathbb{R}^2)}||\psi||_2\\ &\leq C(M)||\psi||_2^2.\end{aligned}$$

Then

$$\left|\mathrm{Im}\int_{\mathbb{R}^2}\mathcal{O}^1(D^4V,D^4u_4,(v_0)_t,Tv_4)\overline{\psi}dx\right| \leq C(M,r)||\psi||_2^2+C(M)||\psi||_2$$

using the fact that

$$\sup_{t\in[0,T]}||(v_0)_t||_{H^2(\mathbb{R}^2)}\leq r.$$

From these inequalities, we deduce that one can find a constant $C_4(M,r)$ such that

$$\begin{aligned}&\frac{d}{dt}\left(\int_{\mathbb{R}^2}(\frac{2+|v_0|^2}{2})|\psi|^2-\int_{\mathbb{R}^2}(\frac{v_0^2}{4}(\overline{\psi}^2)+\frac{\overline{v_0}^2}{4}(\psi^2))dx\right)\\ &\leq \int_{\mathbb{R}^2}|v_0|_t^2|\psi|^2dx+C(M,r)||\psi||_2^2+C(M)||\psi||_2\\ &\leq C_4(M,r)||\psi||_2^2+C(M)||\psi||_2.\end{aligned}$$

Integrating this last inequality from 0 to t, one obtains

$$\begin{aligned}
&||\psi(t)||_2^2+\int_{\mathbb{R}^2}(\frac{|v_0(t)|^2|\psi(t)|^2}{2}-\frac{v_0^2(t)\overline{\psi}^2(t)+\overline{v}_0^2(t)\psi^2(t)}{4})dx\\
&\leq ||\psi(0)||_2^2+\int_{\mathbb{R}^2}|v_0(0)|^2|\psi(0)|^2dx+C_4(M,r)\int_0^t||\psi(s)||_2^2ds\\
&+C(M)\int_0^t||\psi(s)||_2ds.
\end{aligned}$$

Since for all $t\in[0,T]$, $\frac{|v_0(t)|^2|\psi(t)|^2}{2}-\frac{v_0^2(t)\overline{\psi}^2(t)+\overline{v}_0^2(t)\psi^2(t)}{4}\geq 0$, we have

$$\begin{aligned}
||\psi(t)||_2^2\leq&||\psi(0)||_2^2+\int_{\mathbb{R}^2}|v_0(0)|^2|\psi(0)|^2dx+C_4(M,r)\int_0^t||\psi(s)||_2^2ds\\
&+C(M)\int_0^t||\psi(s)||_2ds.
\end{aligned}$$

This implies

$$||\psi||_{T,2}^2\leq ||\psi(0)||_2^2+\int_{\mathbb{R}^2}|v_0(0)|^2|\psi(0)|^2dx+C_4(M,r)T(||\psi||_{T,2}^2+1) \tag{2.24}$$

Since p does not depend of v_0 and $||v_0||_{T,\infty}\leq m_0$, there is a contant $C(m_0)$ depending only on m_0 such that

$$||e^{-p(v_0)}||_{T,\infty}^2\leq C(m_0).$$

Recalling that $\psi(0)=e^{p(a_0)}(1-\Delta)^2(e^{q(|a_0|^2)}\Delta a_0)$ and choosing m_4 such that

$$m_4^2\geq 2C(m_0)\left(||\psi(0)||_2^2+\int_{\mathbb{R}^2}|a_0|^2|\psi(0)|^2dx\right)+1, \tag{2.25}$$

one can find a positive T such that for such m_4, which depends only on $||a_0||_{H^6(\mathbb{R}^2)}$ and m_0, and a fixed r (r to be chosen later),

$$\begin{aligned}
||\chi||_{T,2}^2&\leq C(m_0)||\psi||_{T,2}^2\\
&\leq 2C(m_0)\left(||\psi(0)||_2^2+\int_{\mathbb{R}^2}|v_0(0)|^2|\psi(0)|^2dx\right)+2C_4(M,r)T\\
&\leq m_4^2.
\end{aligned}$$

This implies that

$$\sup_{t\in[0,T]}||u_4(t)||_{H^4(\mathbb{R}^2)}\leq m_4. \tag{2.26}$$

We now treat equation (2.16) in the same way. Differentiating equation (2.16) four times with respect to x_j for $j=1,2$ and multiplying by $\gamma^2(v_0)\mathcal{A}(v_0)$, we obtain, denoting

$\zeta = e^{l(v_0)}(1-\Delta)^2 u_3$,

$$
\begin{aligned}
&2i\gamma^2(v_0)\mathcal{A}(v_0)\begin{pmatrix} \zeta_t \\ \overline{\zeta_t} \end{pmatrix} + \begin{pmatrix} \Delta\zeta \\ \Delta\overline{\zeta} \end{pmatrix} + \sum_{k=1}^{2} \mathcal{T}(v_0, v_k, \partial_k v_0)\begin{pmatrix} \partial_k\zeta \\ \partial_k\overline{\zeta} \end{pmatrix} \\
&+\mathcal{U}(D^4V, D^4u_3, (v_0)_t, Tv_4) = 0
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{T}(v_0, v_k, \partial_k v_0) =& \gamma^2(v_0)\mathcal{A}(v_0)\left(\mathcal{D}(v_0, v_k) + 4\partial_k\mathcal{A}(v_0)\right) \\
&- 2f'(|v_0|^2)\begin{pmatrix} v_0\overline{v}_k + \overline{v}_0 v_k & 0 \\ 0 & v_0\overline{v}_k + \overline{v}_0 v_k \end{pmatrix} \\
&- 2\begin{pmatrix} \partial_k l(v_0) & 0 \\ 0 & \partial_k l(v_0) \end{pmatrix}
\end{aligned}
$$

and $\mathcal{U}(D^4V, D^4u_3, (v_0)_t, Tv_4)$ is a matrix which depends only on the fourth order derivatives of V and u_3, on $(v_0)_t$ and on $T_{ij}v_4$ for $i,j=1,2$. An easy calculation shows that we can choose the functions f and l such that for $j=1,2$ and $k=1,2$, $\mathrm{Re}(\mathcal{T}(v_0,v_k,\partial_k v_0)^{jj}) = 0$. Using the same method as above, one can show the following estimate: there exists a constant $C_3(M,r)$ such that

$$
||\zeta||_{T,2}^2 \leq ||\zeta(0)||_2^2 + \int_{\mathbb{R}^2} |v_0(0)|^2|\zeta(0)|^2 dx + C_3(M,r)T(||\zeta||_{T,2}^2 + 1).
$$

Since l does not depend on v_0 and $||v_0||_{T,\infty} \leq m_0$, there is a constant $C(m_0)$ such that

$$
||e^{-l(v_0)}||_{T,\infty}^2 \leq C(m_0).
$$

Since $\zeta(0)$ depends only on a_0, choosing m_3 such that

$$
m_3^2 \geq 2C(m_0)\left(||\zeta(0)||_2^2 + \int_{\mathbb{R}^2} |a_0|^2|\zeta(0)|^2 dx\right) + 1, \tag{2.27}
$$

one can find a positive T such that for a fixed r

$$
\sup_{t\in[0,T]} ||u_3(t)||_{H^4(\mathbb{R}^2)} \leq m_3. \tag{2.28}
$$

It is important to notice that m_3 depends only on $||a_0||_{H^6(\mathbb{R}^2)}$ and m_0.

We now estimate u_2. Differentiating equation (2.15) four times with respect to x_j for $j=1,2$ we obtain, denoting $\kappa = (1-\Delta)^2 u_2$,

$$
2i\begin{pmatrix} \kappa_t \\ \overline{\kappa}_t \end{pmatrix} + \mathcal{A}(v_0)\begin{pmatrix} \Delta\kappa \\ \Delta\overline{\kappa} \end{pmatrix} + \mathcal{X}(D^4V, D^4Tv_4) = 0
$$

where $\mathcal{X}$ is a matrix depending on $\partial^{l_1}V$ for $|l_1| \leq 4$ and on $\partial^{l_2}T_{kl}v_4$ for $|l_2| \leq 4$ and $k,l = 0,..,2$. Multiplying the last equation by $\gamma^2(v_0)\overline{\kappa}\mathcal{A}(v_0)$, integrating over $\mathbb{R}^2$ the imaginary part of the first equation and treating the first two terms as we have already done, one can find a constant $C_2(M,r)$ such that

$$||\kappa||_{T,2}^2 \le ||\kappa(0)||_2^2 + \int_{\mathbb{R}^2} |v_0(0)|^2|\kappa(0)|^2 dx + C_2(M,r)T||\kappa||_{T,2}^2.$$

Recalling that $\kappa(0)$ depends only on a_0 and choosing m_2 such that

$$m_2^2 \ge 2\left(||\kappa(0)||_2^2 + \int_{\mathbb{R}^2} |a_0|^2|\kappa(0)|^2 dx\right), \tag{2.29}$$

one can find a positive T such that

$$\sup_{t\in[0,T]} ||u_2||_{H^4(\mathbb{R}^2)} \le m_2. \tag{2.30}$$

In the same way, there is a constant $C_1(M,r) > 0$ such that

$$\begin{aligned}\sup_{t\in[0,T]} ||u_1||_{H^4(\mathbb{R}^2)}^2 \le & ||(1-\Delta)^2 u_1(0)||_2^2 + \int_{\mathbb{R}^2} |v_0(0)|^2|(1-\Delta)^2 u_1(0)|^2 dx \\ & + C_1(M,r)T \sup_{t\in[0,T]} ||u_1||_{H^4(\mathbb{R}^2)}^2.\end{aligned}$$

Then choosing m_1 such that

$$m_1^2 \ge 2\left(||(1-\Delta)^2 u_1(0)||_2^2 + \int_{\mathbb{R}^2} |a_0|^2|(1-\Delta)^2 u_1(0)|^2 dx\right), \tag{2.31}$$

there is a positive $T > 0$ such that

$$\sup_{t\in[0,T]} ||u_1||_{H^4(\mathbb{R}^2)} \le m_1. \tag{2.32}$$

Once again, we can notice that m_1 and m_2 depend only on $||a_0||_{H^6(\mathbb{R}^2)}$.

Differentiating equation (2.13) four times with respect to x_j for $j = 1, 2$ and making the same estimate, one can prove the following: there is a constant $C_0(M,r)$ such that

$$\begin{aligned}\sup_{t\in[0,T]} ||u_0||_{H^4(\mathbb{R}^2)}^2 \le & ||(1-\Delta)^2 u_0(0)||_2^2 + \int_{\mathbb{R}^2} |v_0(0)|^2|(1-\Delta)^2 u_0(0)|^2 dx \\ & + C_0(M,r)T \sup_{t\in[0,T]} ||u_0||_{H^4(\mathbb{R}^2)}^2.\end{aligned}$$

Recalling that $u_0(0) = v_0(0) = a_0$ and choosing m_0 such that

$$m_0^2 \ge 2\left(||(1-\Delta)^2 a_0||_2^2 + \int_{\mathbb{R}^2} |a_0|^2|(1-\Delta)^2 a_0|^2 dx\right), \tag{2.33}$$

one can find a positive T such that

$$\sup_{t\in[0,T]} ||u_0||_{H^4(\mathbb{R}^2)} \le m_0. \tag{2.34}$$

It is crucial to notice here that m_0 depends only on $||a_0||_{H^4(\mathbb{R}^2)}$.

We have now to estimate $(u_0)_t$ in $H^2(\mathbb{R}^2)$. Differentiating equation (2.13) twice with respect to x_j for $j = 1, 2$ we obtain, denoting $\sigma = (1 - \Delta)u_0$

$$2i \begin{pmatrix} \sigma_t \\ \overline{\sigma}_t \end{pmatrix} + \mathcal{A}(v_0) \begin{pmatrix} \Delta\sigma \\ \Delta\overline{\sigma} \end{pmatrix} + \mathcal{Y}(D^2 V^*) = 0$$

where $\mathcal{Y}$ is a matrix depending on $\partial^l V^*$ for $|l| \leq 2$. A direct estimate in $L^2(\mathbb{R}^2)$ gives

$$||\sigma_t||_{T,2} \leq C(M)||\Delta\sigma||_{T,2} + C(M) \leq C(M) \sup_{t\in[0,T]} ||u_0||_{H^4(\mathbb{R}^2)} + C(M).$$

Then there exists a constant $C_0(M)$ depending only on M such that

$$\sup_{t\in[0,T]} ||(u_0)_t||_{H^2(\mathbb{R}^2)} \leq C_0(M).$$

Choosing r such that

$$r \geq C_0(M), \tag{2.35}$$

we obtain

$$\sup_{t\in[0,T]} ||(u_0)_t||_{H^2(\mathbb{R}^2)} \leq r. \tag{2.36}$$

As a conclusion we choose M, r and T as follows. Let m_0 satisfying (2.33). Then we choose m_1, m_2, m_3 and m_4 satisfying (2.31), (2.29), (2.27) and (2.25), r such that $r \geq C_0(M)$ and T such that for all $j = 0, ..., 4$

$$TC_j(M, r) \leq \frac{1}{2}.$$

Finally, for this choice of constants M, r and T, one has for $j = 0, .., 4$

$$\sup_{t\in[0,T]} ||u_j||_{H^4(\mathbb{R}^2)} \leq m_j$$

and

$$\sup_{t\in[0,T]} ||(u_0)_t||_{H^2(\mathbb{R}^2)} \leq r.$$

Hence $\mathcal{S}(X_T(M, r)) \subset X_T(M, r)$.

We now prove that the mapping $\mathcal{S}$ is a contraction mapping in the ball $X_T(M, r)$ endowed with the metric of Y_T where

$$Y_T = \left\{ \begin{array}{l} U = (u_0, u_1, u_2, u_3, u_4) : u_j \in C([0, T]; L^2(\mathbb{R}^2)), \;\; j = 0, ..., 4 \\ ||U||_{Y_T} = \sum_{j=0}^4 ||u_j(t)||_{T,2} < \infty \end{array} \right\}.$$

Let $V^1 = (v_0^1, v_1^1, v_2^1, v_3^1, v_4^1) \in X_T(M, r)$ and $V^2 = (v_0^2, v_1^2, v_2^2, v_3^2, v_4^2) \in X_T(M, r)$. We put $U^1 = (u_0^1, u_1^1, u_2^1, u_3^1, u_4^1) = \mathcal{S}V_1$ and $U^2 = (u_0^2, u_1^2, u_2^2, u_3^2, u_4^2) = \mathcal{S}V_2$, the solutions of (2.13), (2.14), (2.15), (2.16) and (2.17) with the same initial condition. For $j = 0, ..., 4$, we put $w_j = u_j^1 - u_j^2$. The equation satisfied by w_4 is

$$\begin{aligned} &2i \begin{pmatrix} (w_4)_t \\ (\overline{w}_4)_t \end{pmatrix} + \mathcal{A}(v_0^1) \begin{pmatrix} \Delta w_4 \\ \Delta \overline{w}_4 \end{pmatrix} + \left(\mathcal{A}(v_0^1) - \mathcal{A}(v_0^2)\right) \begin{pmatrix} \Delta u_4^2 \\ \Delta \overline{u}_4^2 \end{pmatrix} \\ &+ \sum_{k=1}^{2} \mathcal{E}(v_0^1, v_k^1) \begin{pmatrix} \partial_k w_4 \\ \partial_k \overline{w}_4 \end{pmatrix} + \sum_{k=1}^{2} \left(\mathcal{E}(v_0^1, v_k^1) - \mathcal{E}(v_0^2, v_k^2)\right) \begin{pmatrix} \partial_k u_4^2 \\ \partial_k \overline{u}_4^2 \end{pmatrix} \\ &+ \mathcal{I}(V^1, Tv_4^1) - \mathcal{I}(V^2, Tv_4^2) = 0. \end{aligned}$$

The matrices $\mathcal{A}$, $\mathcal{E}$ and $\mathcal{I}$ depend only on V and not on its derivatives. Moreover, they are of class C^1 with respect to their arguments. As a consequence, there exist constants $C(M)$ such that for $i, j = 1, 2$

$$\left|\left|\left(\mathcal{A}(v_0^1) - \mathcal{A}(v_0^2)\right)^{ij}\right|\right|_{T,2} \leq C(M)\, ||V_1 - V_2||_{Y_T} \tag{2.37}$$

$$\left|\left|\left(\mathcal{E}(v_0^1, v_k^1) - \mathcal{E}(v_0^2, v_k^2)\right)^{ij}\right|\right|_{T,2} \leq C(M)\, ||V_1 - V_2||_{Y_T} \tag{2.38}$$

$$\left|\left|\left(\mathcal{I}(V^1, Tv_4^1) - \mathcal{I}(V^2, Tv_4^2)\right)^{j}\right|\right|_{T,2} \leq C(M)\, ||V_1 - V_2||_{Y_T}\,. \tag{2.39}$$

Treating the last equation in the same way as we did for u_4 and using inequalities (2.37), (2.38) and (2.39), one can find a constant $C_5(M)$ such that

$$||u_4^1 - u_4^2||_{T,2} \leq C_5(M) T ||v_4^1 - v_4^2||_{T,2}.$$

With the same arguments, one can find a constant $C_6(M)$ such that for all $j = 0, ..., 3$,

$$||u_j^1 - u_j^2||_{T,2} \leq C_6(M) T ||v_j^1 - v_j^2||_{T,2}.$$

This implies that there exists a time $T > 0$ such that

$$||U_1 - U_2||_{Y_T} \leq \frac{1}{2}\, ||V_1 - V_2||_{Y_T}\,. \tag{2.40}$$

Then the mapping $\mathcal{S}$ is a contraction mapping on $X_T(M, r)$ in the norm $||\cdot||_{Y_T}$. The ball $X_T(M, r)$, endowed with the metric of Y_T, is closed. By the contraction mapping principle, there exists a unique solution $U = (u_0, u_1, u_2, u_3, u_4)$ of (2.13), (2.14), (2.15), (2.16) and (2.17) defined in the interval $[0, T]$ satisfying

$$u_j \in C([0, T]; L^2(\mathbb{R}^2)) \cap L^\infty(0, T; H^4(\mathbb{R}^2)), \;\; j = 0, ..., 4. \tag{2.41}$$

Let us set

$$\tilde{u}_1 = \partial_1 u_0, \;\; \tilde{u}_2 = \partial_2 u_0, \;\; \tilde{u}_3 = e^{f(|u_0|^2)} (u_0)_t, \;\; \tilde{u}_4 = e^{q(|u_0|^2)} \Delta u_0 \quad \text{on} \quad [0, T].$$

Then $\widetilde{U} = (u_0, \widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{u}_4)$ satisfies the Cauchy problem (2.13), (2.14), (2.15), (2.16) and (2.17) with $V = U$. Hence by (2.40), one has

$$||\widetilde{U} - U||_{Y_T} \leq \frac{1}{2}||U - U||_{Y_T} = 0.$$

As a consequence, we have

$$\widetilde{u}_1 = u_1, \ \ \widetilde{u}_2 = u_2, \ \ \widetilde{u}_3 = u_3 \text{ and } \widetilde{u}_4 = u_4. \tag{2.42}$$

Then (2.41) and (2.42) imply that the unique solution of (2.1) given by $u = u_0$ satisfies

$$u \in L^\infty(0, T; H^6(\mathbb{R}^2)). \tag{2.43}$$

Furthermore, using the integral equation associated with (2.1) and (2.43), one proves

$$u \in C([0, T]; H^4(\mathbb{R}^2)),$$

which ends the proof of Theorem 1.2.1. $\square$

2.4 Proof of Theorem 1.2.1 in the cases N $\geq$ 3 and N= 1

We now briefly discuss the case $N \geq 3$. Let $a_0 \in H^{s+2}(\mathbb{R}^N)$. Following section 3, we work in the Banach space

$$X_T = \left\{ \begin{array}{l} U = (u_j)_{j=0}^{N+2} : u_j \in C([0,T]; L^2(\mathbb{R}^N)) \cap L^\infty(0,T; H^s(\mathbb{R}^N)), \\ ||U||_{X_T} = \sum_{j=0}^{N+2} \sup_{t \in [0,T]} ||u_j(t)||_{H^s(\mathbb{R}^N)} < \infty \end{array} \right\}$$

where $s = 2[\frac{N}{2}] + 2$ is determined by the usual Sobolev embeddings. For $M = (m_j)_{j=0}^{N+2} \in \mathbb{R}_+^{N+3}$ and $r \in \mathbb{R}_+$, we denote

$$X_T(M, r) = \left\{ \begin{array}{l} U \in X_T : \forall j = 0, .., N+2, \ \ ||u_j||_{L^\infty(0,T;H^s(\mathbb{R}^N))} \leq m_j, \\ ||(u_0)_t||_{L^\infty(0,T;H^{E(\frac{N}{2})+1}(\mathbb{R}^N)} \leq r \text{ and } u_0(0) = a_0 \end{array} \right\}.$$

Let $V = (v_j)_{j=0}^{N+2} \in X_T(M, r)$. We consider the following linearized system

$$2i \begin{pmatrix} (u_0)_t \\ (\overline{u}_0)_t \end{pmatrix} + \mathcal{A}(v_0) \begin{pmatrix} \Delta u_0 \\ \Delta \overline{u}_0 \end{pmatrix} + \mathcal{F}_0(V^*) = 0, \tag{2.44}$$

for $j = 1, ..., N$

$$2i \begin{pmatrix} (u_j)_t \\ (\overline{u}_j)_t \end{pmatrix} + \mathcal{A}(v_0) \begin{pmatrix} \Delta u_j \\ \Delta \overline{u}_j \end{pmatrix} + \mathcal{F}_j(V^*, v_{N+2}, T v_{N+2}) = 0, \tag{2.45}$$

$$\begin{aligned} &2i \begin{pmatrix} (u_{N+1})_t \\ (\overline{u}_{N+1})_t \end{pmatrix} + \mathcal{A}(v_0) \begin{pmatrix} \Delta u_{N+1} \\ \Delta \overline{u}_{N+1} \end{pmatrix} + \sum_{k=1}^{N} \mathcal{D}(v_0, v_k) \begin{pmatrix} \partial_k u_{N+1} \\ \partial_k \overline{u}_{N+1} \end{pmatrix} \\ &+ \mathcal{G}(V, T v_{N+2}) = 0, \end{aligned} \tag{2.46}$$

$$\begin{aligned}&2i\begin{pmatrix}(u_{N+2})_t\\(\overline{u}_{N+2})_t\end{pmatrix}+\mathcal{A}(v_0)\begin{pmatrix}\Delta u_{N+2}\\\Delta\overline{u}_{N+2}\end{pmatrix}+\sum_{k=1}^{N}\mathcal{E}(v_0,v_k)\begin{pmatrix}\partial_k u_{N+2}\\\partial_k\overline{u}_{N+2}\end{pmatrix}\\&+\mathcal{I}(V,Tv_{N+2})=0.\end{aligned}\tag{2.47}$$

The proof is similar to the one of section 3. The Cauchy problem (2.44), (2.45), (2.46) and (2.47) defines a mapping $\mathcal{S}$ from $\mathcal{Z}$ into itself (see [52]) where

$$\mathcal{Z}=\left[L^\infty(0,T;H^s(\mathbb{R}^N))\cap C([0,T];L^2(\mathbb{R}^N))\right]^{N+3}.$$

We show that $\mathcal{S}$ maps the ball $X_T(M,r)$ into itself and that it is a contraction mapping in the norm

$$||u||_{Y_T}=\sum_{j=0}^{N+2}||u_j(t)||_{T,2}.$$

We have to estimate each of the functions u_j for $j=0,...,N+2$ in the space $L^\infty(0,T;H^s(\mathbb{R}^N))$. The first thing to do is to estimate u_{N+2}. Let $r'=\frac{s}{2}$ and $\chi=(1-\Delta)^{r'}u_{N+2}$. Applying the operator $(1-\Delta)^{r'}$ to equation (2.47), we obtain

$$\begin{aligned}&2i\begin{pmatrix}\chi_t\\\overline{\chi}_t\end{pmatrix}+\mathcal{A}(v_0)\begin{pmatrix}\Delta\chi\\\Delta\overline{\chi}\end{pmatrix}+\sum_{|\alpha|\le 2r'}\mathcal{G}_\alpha(D^{2r'-\alpha}V^*)\partial^\alpha u_{N+2}\\&+\sum_{k=1}^{N}\mathcal{H}(v_0,v_k,\partial_k v_0)\begin{pmatrix}\partial_k\chi\\\partial_k\overline{\chi}\end{pmatrix}+(1-\Delta)^{r'}\mathcal{I}(V,Tv_{N+2})=0\end{aligned}\tag{2.48}$$

where $\mathcal{G}_\alpha$ is a matrix depending on $\partial^l V$ for $|l|\le 2r'-\alpha$. It is then possible to make the same energy estimate as in section 3. Denote $\psi=e^{p(v_0)}\chi$. For suitable q and p, and using the fact that the following embeddings are continuous:

$$H^r(\mathbb{R}^N)\hookrightarrow L^\infty(\mathbb{R}^N),\quad H^{E(\frac{N}{4})+1}(\mathbb{R}^N)\hookrightarrow L^4(\mathbb{R}^N)$$

one obtains

$$\sup_{t\in[0,T]}||u_{N+2}||_{H^s(\mathbb{R}^N)}\le m_{N+2}$$

provided that m_{N+2} is large enough and T is chosen according to m_{N+2} and r. With the same arguments, one can also prove that

$$\sup_{t\in[0,T]}||u_{N+1}||_{H^s(\mathbb{R}^N)}\le m_{N+1}$$

and for $j=0,...,N$

$$\sup_{t\in[0,T]}||u_j||_{H^s(\mathbb{R}^N)}\le m_j.$$

Furthermore, it is not difficult to see that we have from equation (2.44)

$$\sup_{t\in[0,T]}||(u_0)_t||_{H^{E(\frac{N}{2})+1}(\mathbb{R}^N)}\le r$$

if r is large enough. Hence $\mathcal{S}$ maps the ball $X_T(M,r)$ into itself provided that M is chosen according to the initial data, r is chosen according to M and T is chosen according to M and r. Once again, with the same arguments as in section 3, one easily shows that $\mathcal{S}$ is a contraction mapping in the norm $||\cdot||_{Y_T}$ if T is small enough. We end the proof of Theorem 1.2.1 as we did in section 3. □

In the case $N=1$, we use the fact that the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ is continuous. Then, starting from system (2.9), (2.10), (2.11) and (2.12), it is possible to work in the space $L^\infty(0,T;H^3(\mathbb{R})) \cap C([0,T];L^2(\mathbb{R}))$ and so to prove that the Cauchy problem (1.24) is locally well-posed in $H^5(\mathbb{R})$.

2.5 Final Remarks

We have solved the Cauchy problem (2.1) without any smallness condition on the initial data and we have obtained a local solution $u \in H^{s+2}(\mathbb{R}^N)$ in dimension N. But if we assume that the initial data is small enough in the right space, we can prove that the Cauchy problem is locally well-posed in $H^s(\mathbb{R}^N)$. In fact, in that case, we do not have to transform equation (2.1) into a system in u_0,...,u_{N+2}. We can directly work on (2.1) and make the energy estimate. The smallness condition of the initial data ensures that we can find a ball $X_T(M,r)$ which is set into itself by the mapping $\mathcal{S}$ where $\mathcal{S}$ is defined by the linearized version of equation (2.1). More precisely, we have the following theorem.

Theorem 2.5.1. *Let $N \in \mathbb{N}$, $r = \frac{s}{2}$ and assume that $g, h \in C^s(\mathbb{R}_+)$ with $s = 2[\frac{N}{2}] + 2$ and that h satisfies (H). There exists a constant $\varepsilon > 0$ such that for all $a_0 \in H^s(\mathbb{R}^N)$ with $||(1-\Delta)^r a_0||_2 \leq \varepsilon$, the Cauchy Problem (2.1) admits a unique local solution $u \in L^\infty(0,T;H^s(\mathbb{R}^N))$. Furthermore, the existence time T tends to $+\infty$ as $||(1-\Delta)^r a_0||_2$ tends to zero.*

Proof. We give here a sketch of the proof. If a_0 is small, we don't have to differentiate equation (2.2) with respect to t and to introduce the gauge transforms f and q. We only have to make a gauge transform just before doing the energy estimates. As a consequence, we don't have to introduce the system (2.9), (2.10), (2.11) and (2.12). Following [11], we differentiate equation (2.2) twice with respect to space coordinates to obtain an equation in $\phi = (1-\Delta)u$

$$2i\begin{pmatrix} \phi_t \\ \overline{\phi_t} \end{pmatrix} + \mathcal{A}(u)\begin{pmatrix} \Delta\phi \\ \Delta\overline{\phi} \end{pmatrix} + \sum_{k=1}^{N} \delta(u, \partial_k u)\begin{pmatrix} \partial_k\phi \\ \partial_k\overline{\phi} \end{pmatrix} + \mathcal{P}(Du, \phi) = 0 \tag{2.49}$$

where $\mathcal{P}$ is a matrix depending on the second derivatives at most of u. We then linearize equation (2.49). We fix $\psi \in H^s(\mathbb{R}^N)$ and we consider

$$2i\begin{pmatrix} \phi_t \\ \overline{\phi_t} \end{pmatrix} + \mathcal{A}(v)\begin{pmatrix} \Delta\phi \\ \Delta\overline{\phi} \end{pmatrix} + \sum_{k=1}^{N} \delta(v, \partial_k v)\begin{pmatrix} \partial_k\phi \\ \partial_k\overline{\phi} \end{pmatrix} + \mathcal{P}(Dv, \psi) = 0 \tag{2.50}$$

where $v = (1-\Delta)^{-1}\psi$. Let $\mathcal{Z} = L^\infty(0,T;H^{s-2}(\mathbb{R}^N)) \cap C([0,T];L^2(\mathbb{R}^N))$. Then the Cauchy problem (2.50) with initial condition

$$\phi_0(x) = (1-\Delta)a_0(x)$$

defines a mapping $\mathcal{S}$

$$\begin{aligned} \mathcal{S}:\ \mathcal{Z} &\longrightarrow \mathcal{Z} \\ \psi &\longmapsto \phi. \end{aligned}$$

To repeat the energy estimate (2.24), we have to bound the term

$$\delta(v, \partial_k v) = \delta((1-\Delta)^{-1}\psi, \partial_k(1-\Delta)^{-1}\psi)$$

in $L^\infty(0,T;L^\infty(\mathbb{R}^N))$. Recalling that the embedding

$$H^r(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$$

is continuous, we can see that we can work in $L^\infty(0,T;H^{s-2}(\mathbb{R}^N))$ instead of $L^\infty(0,T;H^s(\mathbb{R}^N))$. As a consequence, for $M \in \mathbb{R}_+$ and $R \in \mathbb{R}_+$, we introduce

$$X_T(M,R) = \left\{ \begin{array}{l} \psi \in \mathcal{Z} \ / \ ||\psi||_{L^\infty(0,T;H^{s-2}(\mathbb{R}^2))} \leq M \\ ||((1-\Delta)^{-1}\psi))_t||_{L^\infty(0,T;H^r(\mathbb{R}^2))} \leq R \text{ and } u_0(0,x) = a_0(x) \end{array} \right\}$$

We have to show that there exists $T > 0$, M and R such that $\mathcal{S}$ maps $X_T(M,R)$ into itself and is a contraction mapping for the norm $||\cdot||_{T,2}$. Differentiating equation (2.50) $s-2$ times with respect to the space coordinates, we obtain denoting $\chi = (1-\Delta)^{\frac{s-2}{2}}\phi$

$$\begin{aligned} &2i\begin{pmatrix} \chi_t \\ \overline{\chi_t} \end{pmatrix} + \mathcal{A}(v)\begin{pmatrix} \Delta\chi \\ \Delta\overline{\chi} \end{pmatrix} + \sum_{k=1}^N \mu(v,\partial_k v)\begin{pmatrix} \partial_k\chi \\ \partial_k\overline{\chi} \end{pmatrix} \\ &+\mathcal{Q}(D^{s+2}v, D^s\psi, D^s\phi) = 0 \end{aligned} \tag{2.51}$$

Following section 1.3, in order to apply an energy method to (2.51), we have to use a gauge transform so that the diagonal terms of μ become pure imaginary. A direct calculation shows that the real parts of these coefficients are the gradient of a fonction h. Then denoting $\kappa = e^{h(|u|^2)}\chi$, we can derive

$$\begin{aligned} &2i\begin{pmatrix} \kappa_t \\ \overline{\kappa_t} \end{pmatrix} + \mathcal{A}(v)\begin{pmatrix} \Delta\kappa \\ \Delta\overline{\kappa} \end{pmatrix} + \sum_{k=1}^N \alpha(v,\partial_k v)\begin{pmatrix} \partial_k\kappa \\ \partial_k\overline{\kappa} \end{pmatrix} \\ &+\mathcal{R}(D^{s+2}v, D^s\psi, D^s\phi, u_t) = 0 \end{aligned} \tag{2.52}$$

Then, applying to equation (2.52), estimate (2.24) takes the form

$$||\kappa||_{T,2}^2 \leq ||\kappa(0)||_2^2 + \int_{\mathbb{R}^2} |v(0)|^2|\kappa(0)|^2 dx + C(M,R)T(||\kappa||_{T,2}^2 + 1) \tag{2.53}$$

which implies if T is small enough that

$$||\kappa||_{T,2} \leq C\left(||\kappa(0)||_2^2 + \int_{\mathbb{R}^2} |v(0)|^2|\kappa(0)|^2 dx\right). \tag{2.54}$$

Furthermore, there exists a constant $C(M)$ depending only on M such that

$$||e^{h(|v|^2}) ||_{T,\infty} \leq C(M). \tag{2.55}$$

Then, back to χ, we can write

$$||\chi||_{T,2} \leq C(M)\left(||e^{-h(|v(0)|^2)}\chi(0)||_2^2 + \int_{\mathbb{R}^2} |v(0)|^2|e^{-h(|v(0)|^2)}\chi(0)|^2dx\right). \tag{2.56}$$

As a consequence, if $||e^{-h(|v(0)|^2)}\chi(0)||_2^2 + \int_{\mathbb{R}^2} |v(0)|^2|e^{-h(|v(0)|^2)}\chi(0)|^2dx$ is small enough, one has

$$||\chi||_{T,2} \leq M. \tag{2.57}$$

Exactly as in section 1.3, we end the proof of Theorem 2.5.1 by showing that if T and a_0 are small enough, one can find R such that

$$||(1-\Delta)^{-1}\phi||_{L^\infty(0,T;H^r(\mathbb{R}^N))} \leq R$$

which proves that $\mathcal{S}$ maps $X_T(M,R)$ into itself. The fact that $\mathcal{S}$ is a contraction does not bring any new difficulties. □

For the sake of clarity, we have proved all the results of this chapter in the case $h(s) = \sqrt{1+s}$. It is clear that the results also hold in dimension N with a general function h satisfying the following assumptions

$$\begin{aligned}
&{}_*h \in C^{s+2}(\mathbb{R}^N)\\
&{}_*\exists C_h > 0,\ \ \forall s \in \mathbb{R}_+,\ \ 1 - 4sh'^2(s) > C_h h'^2(s).
\end{aligned}$$

The first condition allows to use space derivatives of equation (2.1). The second condition is needed for the gauge transform and for the energy estimate. In fact, for a general function h, the gauge transform (2.21) should be replaced by

$$q(s) = -\int_0^s \frac{12th'(t)h''(t) + 6h'^2(t)}{1 - 4th'^2(t)}dt \quad \text{and} \quad p(s) = \frac{2^{r'}}{3}q(s).$$

We have to notice here that the definition of q and p is valid thanks to hypothesis (H). Furthermore, inequality (2.24) becomes

$$\begin{aligned}
||\psi||_{T,2}^2 \leq & ||\psi(0)||_2^2 + \int_{\mathbb{R}^2} \frac{2h'^2(|v_0(0)|^2)}{1 - 4|v_0(0)|^2h'^2(|v_0(0)|^2)}|v_0(0)|^2|\psi(0)|^2dx\\
&+ C_4(M,r)T(||\psi||_{T,2}^2 + 1).
\end{aligned}$$

Thanks to (1.27), we then obtain

$$||\psi||_{T,2}^2 \leq ||\psi(0)||_2^2 + \frac{2}{C_h}\int_{\mathbb{R}^2} |v_0(0)|^2|\psi(0)|^2dx + C_4(M,r)T(||\psi||_{T,2}^2 + 1)$$

and we conclude the proof as in section 3.

Chapitre 3

Approximation de l'équation de Schrödinger nonlinéaire relativiste par une équation de Klein-Gordon

"Comment s'étaient-ils rencontrés?
Par hasard comme tout le monde.
Comment s'appelaient-ils? Que vous importe?
D'où venaient-ils ? Du lieu le plus prochain.
Où allaient-ils? Est-ce que l'on sait où l'on va?"

Denis Diderot.

Approximation of a nonlinear Schrödinger equation by a Klein-Gordon equation

Abstract

In this chapter, we study the nonlinear plasma wave equation

$$-\varepsilon^2 \frac{\partial^2 u_\varepsilon}{\partial t^2} + 2i \frac{\partial u_\varepsilon}{\partial t} + \Delta u_\varepsilon = \left(\frac{1}{\sqrt{1+|u_\varepsilon|^2}} - 1 \right) u_\varepsilon + \frac{\Delta(\sqrt{1+|u_\varepsilon|^2})}{\sqrt{1+|u_\varepsilon|^2}} u_\varepsilon$$

with initial data $u_\varepsilon(\cdot, 0) = u_0^\varepsilon(\cdot) \in H^8(\mathbb{R}^2)$, $\partial_t u_\varepsilon(\cdot, 0) = u_1^\varepsilon(\cdot) \in H^7(\mathbb{R}^2)$. We show that the Cauchy problem is locally well-posed on an interval $[0, T]$ where the time T is independent of ε if u_1^ε is small enough. Then, we demonstrate the strong convergence of u_ε towards the solution u of the nonlinear Schrödinger equation (1.23) as ε goes to 0.

Résumé

Dans ce chapitre, on étudie l'équation d'onde

$$-\varepsilon^2 \frac{\partial^2 u_\varepsilon}{\partial t^2} + 2i \frac{\partial u_\varepsilon}{\partial t} + \Delta u_\varepsilon = \left(\frac{1}{\sqrt{1+|u_\varepsilon|^2}} - 1 \right) u_\varepsilon + \frac{\Delta(\sqrt{1+|u_\varepsilon|^2})}{\sqrt{1+|u_\varepsilon|^2}} u_\varepsilon$$

où les conditions initiales s'écrivent $u_\varepsilon(\cdot, 0) = u_0^\varepsilon(\cdot) \in H^8(\mathbb{R}^2)$, $\partial_t u_\varepsilon(\cdot, 0) = u_1^\varepsilon(\cdot) \in H^7(\mathbb{R}^2)$. On montre que le problème de Cauchy est localement bien posé sur un intervalle de temps $[0, T]$ où le temps T est indépendant de ε. Ensuite, on montre la convergence forte de u_ε vers une solution u de l'équation de Schrödinger relativiste (1.23).

3.1 Introduction

The nonlinear Schrödinger equation (1.23)

$$2i\frac{\partial u}{\partial t} + \Delta_\perp u - \frac{u}{\sqrt{1+|u|^2}}\Delta_\perp(\sqrt{1+|u|^2}) + (1 - \frac{1}{\sqrt{1+|u|^2}})u = 0.$$

is known to described properly the self-channeling of a high power ultra-short laser pulse in matter. In other words, it appears that the Maxwell system (1.9)-(1.12) is correctly approximated by equation (1.23). More precisely, it is shown in the introduction that the Maxwell system describing the propagation of laser pulses in underdense plasma can be reduced to the wave equation (1.17)

$$\Box A = k_p^2\left(\frac{\max[0, f(r) + k_p^{-2}\Delta\gamma]}{\gamma}\right)A,$$

where $\gamma = \sqrt{1+|u|^2}$. Then taking $f(r) \equiv 1$ and supposing that A takes the form

$$A(x,y,z,t) = u_\varepsilon(x,y,\varepsilon z,\varepsilon t)e^{i(kz-\omega t)}$$

where

$$\varepsilon = \frac{\omega_{p,0}}{\omega} = \frac{k_p}{k_0}.$$

leads, after a rescaling in space, to equation (1.22)

$$\Delta_\perp u_\varepsilon + \varepsilon^2\frac{\partial^2 u_\varepsilon}{\partial z^2} - \varepsilon^2\frac{\partial^2 u_\varepsilon}{\partial t^2} + 2i\sqrt{1-\varepsilon^2}\frac{\partial u_\varepsilon}{\partial z} + 2i\frac{\partial u_\varepsilon}{\partial t} = \left(\frac{1}{\gamma_\varepsilon} - 1\right)u_\varepsilon + \frac{(\Delta_\perp + \varepsilon^2\partial_z^2)\gamma_\varepsilon}{\gamma_\varepsilon}u_\varepsilon$$

where $\gamma_\varepsilon = \sqrt{1+|u_\varepsilon|^2}$. We suppose here that $\varepsilon \ll 1$ which caracterize the fact that we work with underdense plasmas. Now, taking formally $\varepsilon = 0$ in (1.22) gives the nonlinear Schrödinger equation (1.23). It is then natural to think that the solutions u_ε of (1.22) converge towards the solutions u of equation (1.23).

This kind of problem constitute an active domain of research in Mathematics. For example, in [8], T. Colin and L. Bergé investigate the so-called Langmuir wave envelope approximation which consists in taking the limit $\varepsilon \longrightarrow 0$ in the nonlinear plasma wave equation

$$\varepsilon^2\frac{\partial^2 E_\varepsilon}{\partial t^2} - 2i\frac{\partial E_\varepsilon}{\partial t} - \Delta E_\varepsilon = f(|E_\varepsilon|^2)E_\varepsilon.$$

Under specific assumptions on f, the authors show the convergence of E_ε towards the solution E of the corresponding Schrödinger equation. In [7], the authors prove the same kind of results for a set of equations describing the inertial regime of the strong Langmuir turbulence, namely

$$\varepsilon^2\frac{\partial^2 E_\varepsilon}{\partial t^2} - 2i\frac{\partial E_\varepsilon}{\partial t} - \Delta E_\varepsilon = -n_\varepsilon E_\varepsilon$$
$$\frac{1}{c^2}\frac{\partial^2 n_\varepsilon}{\partial t^2} - \Delta n_\varepsilon = \Delta|E_\varepsilon|^2.$$

In [46], the authors show that the solutions for the nonlinear Klein-Gordon equation can be described by using a system of two coupled nonlinear Schrödinger equations.

In [24] , T. Gallay and G. Schneider describe the KP equation

$$2\frac{\partial^2 A}{\partial x \partial t} + \frac{\partial^2 A^2}{\partial x^2} + \frac{\partial^4 A}{\partial x^4} + \frac{\partial^2 A}{\partial y^2} = 0$$

as an approximation for the Boussinesq equation

$$\frac{\partial^2 \phi}{\partial t^2} = \Delta\phi + \Delta(\phi^2) + \Delta\left(\frac{\partial^2 \phi}{\partial t^2}\right).$$

In this chapter, we will not study equation (1.22) due to the difficulty in proving that this equation admits solutions in $H^s(\mathbb{R}^3)$ for suitable s with an H^s-bound independent of ε. Indeed, the presence of ε^2 in front ∂_z^2 gives rise to many problems. For example, in order to derive an energy estimate on equation (1.22), we have to consider the norm $||(1-\Delta_\varepsilon)^{\frac{s}{2}}||_{L^2(\mathbb{R}^3)}$ where $\Delta_\varepsilon = \partial_x^2 + \partial_y^2 + \varepsilon^2\partial_z^2$, which prevents from injecting the functions u_ε in $L^\infty(\mathbb{R}^3)$ with a constant independent of ε. This is why we will study a modified equation in dimension 2 where all the terms depending on z have been dropped, which reads as

$$-\varepsilon^2\frac{\partial^2 u_\varepsilon}{\partial t^2} + 2i\frac{\partial u_\varepsilon}{\partial t} + \Delta u_\varepsilon = \left(\frac{1}{\gamma_\varepsilon} - 1\right)u_\varepsilon + \frac{\Delta\gamma_\varepsilon}{\gamma_\varepsilon}u_\varepsilon,$$

where $\gamma_\varepsilon = \sqrt{1+|u_\varepsilon|^2}$. Formally, taking $\varepsilon = 0$ in (2.4) gives the relativistic Schrödinger equation (1.23).

We first solve the Cauchy problem (1.28) with initial condition

$$u_\varepsilon(x,y,0) = u_0^\varepsilon(x,y), \;\; \partial_t u_\varepsilon(x,y,0) = u_1^\varepsilon(x,y). \tag{3.1}$$

A first approach could be to treat (1.28) as a nonlinear Klein-Gordon equation. Indeed, if we denote

$$v_\varepsilon(x,y,t) = u_\varepsilon(x,y,t)e^{i\frac{1}{\varepsilon^2}t}$$

an easy calculation shows that v_ε is a solution of

$$-\varepsilon^2\frac{\partial^2 v_\varepsilon}{\partial t^2} + \Delta v_\varepsilon - \frac{1}{\varepsilon^2}v_\varepsilon = \left(\frac{1}{\gamma_\varepsilon} - 1\right)v_\varepsilon + \frac{\Delta\gamma_\varepsilon}{\gamma_\varepsilon}v_\varepsilon \tag{3.2}$$

with initial condition

$$\begin{aligned} &v_\varepsilon(x,y,0) = u_0^\varepsilon(x,y)\\ &\partial_t v_\varepsilon(x,y,0) = u_1^\varepsilon(x,y,z) + i\frac{1}{\varepsilon^2}u_0^\varepsilon(x,y). \end{aligned}$$

Equation (3.2) satisfies the hypothesis of Klainerman and Ponce [37]. We can deduce that if the initial data are small enough in an adequate space, the Cauchy problem (3.2) admits a unique global solution $v_\varepsilon \in C([0,+\infty[; H^s(\mathbb{R}^2)) \cap C^1([0,+\infty[; H^{s-2}(\mathbb{R}^2))$ where s is an integer sufficiently large. Back to u_ε, we can show that (1.28) admits a unique solution $u_\varepsilon \in C([0,+\infty[; H^s(\mathbb{R}^2)) \cap C^1([0,+\infty[; H^{s-2}(\mathbb{R}^2))$ if the initial data is small enough. The problem is that, in doing so, it is easy to see that the presence of the term $\frac{1}{\varepsilon^2}v_\varepsilon$ imposes to v_ε and so to u_ε to tend to 0 in the space $L^\infty(0,T;H^s(\mathbb{R}^2))$ as ε goes to 0. Thus, we have to solve the Cauchy problem (1.28) with initial condition (3.1) in an other way.

A second approach could be to apply the usual energy estimates (see [37]) to equation (1.28). Indeed, it is possible to prove the two following assertions

- The Cauchy problem (1.28) is locally well posed in $L^\infty(0,T_\varepsilon;H^s(\mathbb{R}^3))$ for any initial data u_0^ε, u_1^ε with $T_\varepsilon = \frac{\varepsilon^2}{(||u_0^\varepsilon||_{H^s(\mathbb{R}^3)}+||u_1^\varepsilon||_{H^{s-1}(\mathbb{R}^3)})}$.
- The Cauchy problem (1.28) is locally well posed in $L^\infty(0,T_\varepsilon;H^s(\mathbb{R}^3))$ with $T=1$ if $||u_0^\varepsilon||_{H^s(\mathbb{R}^3)} + ||u_1^\varepsilon||_{H^{s-1}(\mathbb{R}^3)} \leq h(\varepsilon)$ where $h(\varepsilon)$ tends to zero with ε.

The first assertion is not very useful to our problem. Indeed, it is clear that the time T_ε tends to 0 with ε which is catastrophic: we really need that u_ε exists on an interval with a size independent of ε. Concerning the second one, we see that we impose to the initial data u_0^ε to tend to zero with ε. Once again, it is not very useful to prove the convergence of u_ε to u since the only solution of (1.23) with initial data equal to 0 is $u \equiv 0$. This is why we first prove the following theorem

Theorem. *Assume that $u_0^\varepsilon \in H^8(\mathbb{R}^2)$ and $u_1^\varepsilon \in H^7(\mathbb{R}^2)$. Then there exists $\delta_\varepsilon > 0$ ($\delta_\varepsilon \sim 2^{-\frac{1}{\varepsilon^2}}$) and $M > 0$ independent of ε and depending only on the initial datas such that if $||u_1^\varepsilon||_{H^7(\mathbb{R}^2)} \leq \delta_\varepsilon$, there exists a time T independent of ε such that (1.28) admits a unique solution u_ε satisfying*

$$u_\varepsilon \in L^\infty(0,T;H^8(\mathbb{R}^2)) \cap C([0,T];L^2(\mathbb{R}^2)).$$

Furthermore, we have

$$\sup_{t\in[0,T]} \Big(||u_\varepsilon||^2_{H^8(\mathbb{R}^2)} + \varepsilon^2||\partial_t u_\varepsilon||^2_{H^6(\mathbb{R}^2)} + \varepsilon^2||\partial_t^2 u_\varepsilon||^2_{H^6(\mathbb{R}^2)} \Big) \leq M.$$

We have to notice that we suppose that u_1^ε tends to 0 very rapidly with ε. We will consider (1.28) not as a wave equation but as a perturbation of a quasilinear Schrödinger equation. Indeed, in equation (1.23), the term $\varepsilon^2\partial_t^2 u_\varepsilon$ will be treated as a perturbative term. That is why we will use the method introduced in Chapter 2, namely we rewrite equation (1.28) as a system in $(u_\varepsilon, \overline{u}_\varepsilon)^t$. Then, we differentiate the equation with respect to space and time to obtain a new system in u_0,..,u_5 (we drop for convenience the index ε) where

$$u_0 = u_\varepsilon,\ \ u_j = \partial_j u_\varepsilon\ \ \forall 1 \leq j \leq 2,\ \ u_3 = e^{q(|u_0|^2)}\Delta u_\varepsilon,\ \ u_4 = \partial_t \Delta u_\varepsilon,\ \ u_5 = \partial_t^2 \Delta u_\varepsilon.$$

The function q plays the role of a gauge transform which allows to treat the equations satisfied by u_0, u_1, u_2, and u_3 exactly as we have treated equations (2.9), (2.10) and (2.12) of section 2. Concerning u_4 and u_5 we do not use gauge transforms because their respective equations will be treated in a different way, which does not suppose to integrate by parts the terms including the first derivatives of u_4 and u_5. Indeed, to estimate u_4 and u_5, we use the classical energy estimates on wave equations, namely we multiply the equations respectively by $(\overline{u}_4)_t$ and $(\overline{u}_5)_t$ and we integrate the real part over $\mathbb{R}^2$. Finally, the proof of Theorem 1.3.1 will follow from a classical fixed-point Theorem.

Equipped with Theorem (1.3.1), we can study the convergence of u_ε to a solution u of (1.23) with initial data $u(x,y,0) = u_0(x,y)$ under the condition that

$$u_0^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u_0 \quad \text{in} \quad H^6(\mathbb{R}^2) \quad \text{and} \quad u_1^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{in} \quad H^7(\mathbb{R}^2)$$

More precisely, we have the following

Theorem. *Let (u_0^ε) and (u_1^ε) be two bounded sequences respectively in $H^8(\mathbb{R}^2)$ and $H^7(\mathbb{R}^2)$ satisfying*

$$u_0^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u_0 \quad \textit{in} \quad H^6(\mathbb{R}^2) \quad \textit{and} \quad u_1^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \textit{in}\ H^7(\mathbb{R}^2).$$

We suppose furthermore that $||u_1^\varepsilon||_{H^7(\mathbb{R}^2)} \leq 2^{-\frac{1}{\varepsilon^2}}$. Let u_ε be the solution of (1.28) *with initial condition $u_\varepsilon(\cdot,0) = u_0^\varepsilon(\cdot)$, $\partial_t u_\varepsilon(\cdot,0) = u_1^\varepsilon(\cdot)$ given by Theorem* 1.3.1*. Let u the solution of* (1.23) *with initial condition $u(\cdot,0) = u_0(\cdot)$ given by Theorem* 1.2.1*. Then, there exists a time $T > 0$ independent of ε such that u_ε and u exist on $[0,T]$ for any ε and*

$$u_\varepsilon - u \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \textit{in} \quad L^\infty(0,T;H^6(\mathbb{R}^2)).$$

The proof uses an energy estimate which reads exactly as the one of chapter 2. We use furthermore that because u_0^ε and u_1^ε are bounded sequences respectively in $H^8(\mathbb{R}^2)$ and $H^7(\mathbb{R}^2)$, there exists a constant M independent of ε such that

$$\sup_{t\in[0,T]} ||u_\varepsilon||_{H^8(\mathbb{R}^2)} \leq M.$$

In this chapter, we will use the notations of Chapter 2. Furthermore, by convention, we denote $\partial_0 u = u$. For $k \in \mathbb{N}^2$, $k = (k_1,k_2)$, we denote $|k| = k_1 + k_2$.

In section 3.2, we will transform equation (1.28) into a system to which we can apply the appropriate energy method. Then, in section 3.3, we prove Theorem 1.3.1 concerning the Cauchy problem (1.28). Finally, in section 3.4, we prove Theorem 1.3.2.

3.2 Transformation of equation (1.28)

In this section, we transform equation (1.28) into a system to which we can apply our energy method. We fix $a_0^\varepsilon \in H^8(\mathbb{R}^2)$ and $a_1^\varepsilon \in H^7(\mathbb{R}^2)$. Following Section 2.2, equation (1.28) can be rewritten as a system in $(u_\varepsilon,\overline{u}_\varepsilon)^t$

$$\begin{aligned} &-\varepsilon^2 \begin{pmatrix} (u_\varepsilon)_{tt} \\ (\overline{u}_\varepsilon)_{tt} \end{pmatrix} + 2i \begin{pmatrix} (u_\varepsilon)_t \\ (\overline{u}_\varepsilon)_t \end{pmatrix} + \mathcal{A}(u_\varepsilon) \begin{pmatrix} \Delta u_\varepsilon \\ \Delta \overline{u}_\varepsilon \end{pmatrix} \\ &- \begin{pmatrix} \frac{u_\varepsilon}{\gamma_\varepsilon^2}|\nabla u_\varepsilon|^2 - \frac{u_\varepsilon}{4\gamma_\varepsilon^4}|\nabla|u_\varepsilon|^2|^2 + u_\varepsilon g(|u_\varepsilon|^2) \\ \frac{-\overline{u}_\varepsilon}{\gamma_\varepsilon^2}|\nabla u_\varepsilon|^2 + \frac{\overline{u}_\varepsilon}{4\gamma_\varepsilon^4}|\nabla|u_\varepsilon|^2|^2 - \overline{u}_\varepsilon g(|u_\varepsilon|^2) \end{pmatrix} = 0 \end{aligned} \tag{3.3}$$

where

$$g(s) = \frac{1}{\sqrt{1+s}} - 1, \ \ \gamma_\varepsilon = \sqrt{1+|u_\varepsilon|^2},$$

$$\mathcal{A}(u_\varepsilon) = \frac{1}{2\gamma_\varepsilon^2}\begin{pmatrix} 2+|u_\varepsilon|^2 & -u_\varepsilon^2 \\ \overline{u_\varepsilon}^2 & -(2+|u_\varepsilon|^2) \end{pmatrix}.$$

We now drop for convenience the parameter ε. We set $u_0 = u_\varepsilon$, $u_1 = \partial_1 u_\varepsilon$, $u_2 = \partial_2 u_\varepsilon$, $u_3 = e^{q(|u_0|^2)}\Delta u_\varepsilon$, $u_4 = \partial_t \Delta u_\varepsilon$ and $u_5 = \partial_t^2 \Delta u_\varepsilon$. Here, u_0 does not denote an initial condition. The function q will play the role of a gauge transform and will be chosen later (see 2.3 for more details). Following Section 2.2, we also set $U^* = (u_j)_{j=0}^2$ and $U = (u_j)_{j=0}^5$. Then, equation (3.3) can be rewritten as

$$2i\begin{pmatrix} \partial_t u_0 \\ \partial_t \overline{u}_0 \end{pmatrix} + \mathcal{A}(u_0)\begin{pmatrix} \Delta u_0 \\ \Delta \overline{u}_0 \end{pmatrix} - \varepsilon^2 \begin{pmatrix} \Delta^{-1} u_5 \\ \Delta^{-1}\overline{u}_5 \end{pmatrix} + \mathcal{F}_0(U^*) = 0 \tag{3.4}$$

where

$$\mathcal{F}_0(U^*) = \begin{pmatrix} \dfrac{-u_0}{\gamma^2(u_0)}(\sum_{k=1}^2 |u_k|^2) + \dfrac{u_0}{4\gamma^4(u_0)}(\sum_{k=1}^2 (u_0\overline{u}_k + \overline{u}_0 u_k)^2) + u_0 g(|u_0|^2) \\ \dfrac{\overline{u}_0}{\gamma^2(u_0)}(\sum_{k=1}^2 |u_k|^2) - \dfrac{\overline{u}_0}{4\gamma^4(u_0)}(\sum_{k=1}^2 (u_0\overline{u}_k + \overline{u}_0 u_k)^2) - \overline{u}_0 g(|u_0|^2) \end{pmatrix}.$$

Differentiating equation (3.3) with respect to x_j for $j = 1, 2$, we obtain

$$\begin{aligned} 2i\begin{pmatrix} (u_j)_t \\ (\overline{u}_j)_t \end{pmatrix} + \mathcal{A}(u_0)\begin{pmatrix} \Delta u_j \\ \Delta \overline{u}_j \end{pmatrix} + \sum_{k=1}^2 B(u_0,u_k)\begin{pmatrix} T_{kj}u_3 \\ T_{kj}\overline{u}_3 \end{pmatrix} \\ + C(u_0,u_j)\begin{pmatrix} (e^{-q(|u_0|^2)}u_3) \\ (e^{-q(|u_0|^2)}\overline{u}_3) \end{pmatrix} - \varepsilon^2\begin{pmatrix} \partial_j \Delta^{-1} u_5 \\ \partial_j \Delta^{-1}\overline{u}_5 \end{pmatrix} + \begin{pmatrix} F(U^*,u_j) \\ -\overline{F}(U^*,u_j) \end{pmatrix} = 0 \end{aligned}$$

where the matrices $B(u_0, u_k)$ and $C(u_0, u_j)$ are the same as in Section 2.2 and for $i, j = 1, 2$, $T_{i,j}$ is the following operator of order 0

$$T_{i,j} v = \partial_i \partial_j \Delta^{-1}(e^{-q(|u_0|^2)} v).$$

Furthermore, this operator is continuous from L^2 to L^2 with a norm independent of ε. We can rewrite these equations as follows

$$2i\begin{pmatrix} (u_j)_t \\ (\overline{u}_j)_t \end{pmatrix} + \mathcal{A}(u_0)\begin{pmatrix} \Delta u_j \\ \Delta \overline{u}_j \end{pmatrix} - \varepsilon^2\begin{pmatrix} \partial_j \Delta^{-1} u_5 \\ \partial_j \Delta^{-1}\overline{u}_5 \end{pmatrix} + \mathcal{F}_j(U^*, u_3, Tu_3) = 0 \tag{3.5}$$

where, as in Section 2.2

$$\begin{aligned} \mathcal{F}_j(U^*, u_3, Tu_3) = \begin{pmatrix} F(U^*,u_j) \\ -\overline{F}(U^*,u_j) \end{pmatrix} + C(u_0,u_j)\begin{pmatrix} (e^{-q(|u_0|^2)}u_3) \\ (e^{-q(|u_0|^2)}\overline{u}_3) \end{pmatrix} \\ + \sum_{k=1}^2 B(u_0,u_k)\begin{pmatrix} T_{kj}u_3 \\ T_{kj}\overline{u}_3 \end{pmatrix}. \end{aligned}$$

In order to derive an equation for u_3, we differentiate equation (3.3) twice with respect to x_j to obtain

$$\begin{aligned}
&2i\begin{pmatrix} (\partial_j^2 u_0)_t \\ (\partial_j^2 \overline{u}_0)_t \end{pmatrix} + \mathcal{A}(u_0)\begin{pmatrix} \Delta\partial_j^2 u_0 \\ \Delta\partial_j^2 \overline{u}_0 \end{pmatrix} + \sum_{k=1}^{2} B(u_0,u_k)\begin{pmatrix} \partial_k\partial_j^2 u_0 \\ \partial_k\partial_j^2 \overline{u}_0 \end{pmatrix} \\
&+2C(u_0,u_j)\begin{pmatrix} \partial_j\Delta u_0 \\ \partial_j\Delta \overline{u}_0 \end{pmatrix} + \partial_j C(u_0,u_j)\begin{pmatrix} \Delta u_0 \\ \Delta \overline{u}_0 \end{pmatrix} - \varepsilon^2\begin{pmatrix} (\partial_j^2 u_0)_{tt} \\ (\partial_j^2 \overline{u}_0)_{tt} \end{pmatrix} \\
&+\sum_{k=1}^{2} \partial_j B(u_0,u_k)\begin{pmatrix} \partial_k\partial_j u_0 \\ \partial_k\partial_j \overline{u}_0 \end{pmatrix} + \begin{pmatrix} \partial_j F(U^*,u_j) \\ -\partial_j \overline{F}(U^*,u_j) \end{pmatrix} = 0.
\end{aligned} \tag{3.6}$$

Summing the two equations and multiplying by $\mathcal{A}^{-1}(u_0)$, we obtain

$$\begin{aligned}
&2i\mathcal{A}^{-1}(u_0)\begin{pmatrix} (\Delta u_0)_t \\ (\Delta \overline{u}_0)_t \end{pmatrix} + \begin{pmatrix} \Delta\Delta u_0 \\ \Delta\Delta \overline{u}_0 \end{pmatrix} + \sum_{k=1}^{2} \mathcal{A}^{-1}(u_0)B(u_0,u_k)\begin{pmatrix} \partial_k\Delta u_0 \\ \partial_k\Delta \overline{u}_0 \end{pmatrix} \\
&+2\sum_{j=1}^{2} \mathcal{A}^{-1}(u_0)C(u_0,u_j)\begin{pmatrix} \partial_j\Delta u_0 \\ \partial_j\Delta \overline{u}_0 \end{pmatrix} + \sum_{j=1}^{2} \mathcal{A}^{-1}(u_0)\partial_j C(u_0,u_j)\begin{pmatrix} \Delta u_0 \\ \Delta \overline{u}_0 \end{pmatrix} \\
&-\varepsilon^2\mathcal{A}^{-1}(u_0)\begin{pmatrix} (\Delta u_0)_{tt} \\ (\Delta \overline{u}_0)_{tt} \end{pmatrix} + \sum_{j=1}^{2}\sum_{k=1}^{2} \mathcal{A}^{-1}(u_0)\partial_j B(u_0,u_k)\begin{pmatrix} \partial_k\partial_j u_0 \\ \partial_k\partial_j \overline{u}_0 \end{pmatrix} \\
&+\sum_{j=1}^{2} \mathcal{A}^{-1}(u_0)\begin{pmatrix} \partial_j F(U^*,u_j) \\ -\partial_j \overline{F}(U^*,u_j) \end{pmatrix} = 0.
\end{aligned} \tag{3.7}$$

which can be rewritten as in Section 2.2, after the multiplication by $e^{q(|u_0|^2)}$, in the following way

$$\begin{aligned}
&2i\begin{pmatrix} (u_3)_t \\ (\overline{u}_3)_t \end{pmatrix} + \mathcal{A}(u_0)\begin{pmatrix} \Delta u_3 \\ \Delta \overline{u}_3 \end{pmatrix} + \sum_{k=1}^{2} \mathcal{E}(u_0,u_k)\begin{pmatrix} \partial_k u_3 \\ \partial_k \overline{u}_3 \end{pmatrix} - \varepsilon^2\begin{pmatrix} u_5 \\ \overline{u}_5 \end{pmatrix} \\
&+\mathcal{I}(U,Tu_3,\Delta^{-1}u_4) = 0
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
\mathcal{E}(u_0,u_k) =& B(u_0,u_k) + 2C(u_0,u_k) \\
&- 2\mathcal{A}(u_0)q^{'}(|u_0|^2)\begin{pmatrix} u_0\overline{u}_k + \overline{u}_0 u_k & 0 \\ 0 & u_0\overline{u}_k + \overline{u}_0 u_k \end{pmatrix}
\end{aligned}$$

and $\mathcal{I}$ is a matrix depending on U, $T_{ij}u_3$ for $i,j=1,2$ and $\Delta^{-1}u_4$ in the following way

$$
\begin{aligned}
&\mathcal{I}(U, Tu_3, \Delta^{-1}u_4) = \\
&\sum_{m=1}^{2} e^{q(|u_0|^2)} \begin{pmatrix} \partial_m F(U^*, u_j) \\ -\partial_m \overline{F}(U^*, u_j) \end{pmatrix} - 2iq'(|u_0|^2) \begin{pmatrix} u_0(\Delta^{-1}\overline{u}_4)u_3 + \overline{u}_0(\Delta^{-1}u_4)u_3 \\ u_0(\Delta^{-1}\overline{u}_4)\overline{u}_3 + \overline{u}_0(\Delta^{-1}u_4)\overline{u}_3 \end{pmatrix} \\
&- \sum_{k=1}^{2} (2C(u_0,u_k) + B(u_0,u_k))q'(|u_0|^2) \begin{pmatrix} (u_0\overline{u}_k + \overline{u}_0 u_k)u_3 \\ (u_0\overline{u}_k + \overline{u}_0 u_k)\overline{u}_3 \end{pmatrix} \\
&+ \sum_{k=1}^{2} \partial_k C(u_0,u_k) \begin{pmatrix} u_3 \\ \overline{u}_3 \end{pmatrix} + \sum_{j=1}^{2}\sum_{k=1}^{2} e^{q(|u_0|^2)} \partial_k B(u_0,u_k) \begin{pmatrix} T_{kj}u_3 \\ T_{kj}\overline{u}_3 \end{pmatrix} \\
&+ \mathcal{A}(u_0) \begin{pmatrix} (q'^2(|u_0|^2) - q''(|u_0|^2))(\sum_{k=1}^{2}(\overline{u}_0 u_k + u_0\overline{u}_k)^2)u_3 \\ (q'^2(|u_0|^2) - q''(|u_0|^2))(\sum_{k=1}^{2}(\overline{u}_0 u_k + u_0\overline{u}_k)^2)\overline{u}_3 \end{pmatrix} \\
&- \mathcal{A}(u_0) \begin{pmatrix} q'(|u_0|^2)(2\sum_{k=1}^{2}|u_k|^2 + \overline{u}_0(\sum_{k=1}^{2} T_{kk}u_3)u_3 \\ q'(|u_0|^2)(2\sum_{k=1}^{2}|u_k|^2 + \overline{u}_0(\sum_{k=1}^{2} T_{kk}u_3))\overline{u}_3 \end{pmatrix} - \mathcal{A}(u_0) \begin{pmatrix} q'(|u_0|^2)u_0(\sum_{k=1}^{2} T_{kk}\overline{u}_3)u_3 \\ q'(|u_0|^2)u_0(\sum_{k=1}^{2} T_{kk}\overline{u}_3)\overline{u}_3 \end{pmatrix}
\end{aligned}
$$

We have to notice here that the matrix $\mathcal{I}$ depends on $\Delta^{-1}u_4$, which is due to the gauge transform $u_3 = e^{q(|u_0|^2)}\Delta u_0$. Indeed, in equation (3.7), we have to replace Δu_0 by $e^{q(|u_0|^2)}u_3$ in the term $(\Delta u_0)_t$ which gives the contribution $e^{q(|u_0|^2)}(q(|u_0|^2))_t u_3$. Then, recalling that $(u_0)_t = \Delta^{-1}u_4$, we include this last term in $\mathcal{I}$.

Differentiating equation (3.7) with respect to t, we obtain

$$
\begin{aligned}
&-\varepsilon^2 \mathcal{A}^{-1}(u_0) \begin{pmatrix} (u_4)_{tt} \\ (\overline{u}_4)_{tt} \end{pmatrix} - \varepsilon^2 \partial_t\left(\mathcal{A}^{-1}(u_0)\right) \begin{pmatrix} u_5 \\ \overline{u}_5 \end{pmatrix} + 2i\mathcal{A}^{-1}(u_0) \begin{pmatrix} (u_4)_t \\ (\overline{u}_4)_t \end{pmatrix} + 2i\partial_t\left(\mathcal{A}^{-1}(u_0)\right) \begin{pmatrix} u_4 \\ \overline{u}_4 \end{pmatrix} \\
&+ \begin{pmatrix} \Delta u_4 \\ \Delta \overline{u}_4 \end{pmatrix} + \sum_{k=1}^{2} \mathcal{A}^{-1}(u_0)B(u_0,u_k) \begin{pmatrix} \partial_k u_4 \\ \partial_k \overline{u}_4 \end{pmatrix} + \sum_{k=1}^{2} \partial_t\left(\mathcal{A}^{-1}(u_0)B(u_0,u_k)\right) \begin{pmatrix} \partial_k(e^{-q(|u_0|^2)}u_3) \\ \partial_k(e^{-q(|u_0|^2)}\overline{u}_3) \end{pmatrix} \\
&+2\sum_{k=1}^{2} \mathcal{A}^{-1}(u_0)C(u_0,u_k) \begin{pmatrix} \partial_k u_4 \\ \partial_k \overline{u}_4 \end{pmatrix} + 2\sum_{j=1}^{2} \partial_t\left(\mathcal{A}^{-1}(u_0)C(u_0,u_j)\right) \begin{pmatrix} \partial_j(e^{-q(|u_0|^2)}u_3) \\ \partial_j(e^{-q(|u_0|^2)}\overline{u}_3) \end{pmatrix} \\
&+ \sum_{j=1}^{2} \mathcal{A}^{-1}(u_0)\partial_j C(u_0,u_j) \begin{pmatrix} u_4 \\ \overline{u}_4 \end{pmatrix} + \sum_{j=1}^{2} \partial_t\left(\mathcal{A}^{-1}(u_0)\partial_j C(u_0,u_j)\right) \begin{pmatrix} e^{-q(|u_0|^2)}u_3 \\ e^{-q(|u_0|^2)}\overline{u}_3 \end{pmatrix} \\
&+ \sum_{j=1}^{2}\sum_{k=1}^{2} \mathcal{A}^{-1}(u_0)\partial_j B(u_0,u_k) \begin{pmatrix} \partial_k\partial_j\Delta^{-1}u_4 \\ \partial_k\partial_j\Delta^{-1}\overline{u}_4 \end{pmatrix} + \sum_{j=1}^{2}\sum_{k=1}^{2} \partial_t\left(\mathcal{A}^{-1}(u_0)\partial_j B(u_0,u_k)\right) \begin{pmatrix} T_{kj}u_3 \\ T_{kj}\overline{u}_3 \end{pmatrix} \\
&+ \sum_{j=1}^{2} \partial_t\left(\mathcal{A}^{-1}(u_0) \begin{pmatrix} \partial_j F(U^*,u_j) \\ -\partial_j \overline{F}(U^*,u_j) \end{pmatrix}\right) = 0.
\end{aligned}
\tag{3.9}
$$

Keeping in view that we have $\partial_t u_0 = \Delta^{-1} u_4$, we can rewrite the last equation in the following way

$$
\begin{aligned}
&-\varepsilon^2 \mathcal{A}^{-1}(u_0) \begin{pmatrix} (u_4)_{tt} \\ (\overline{u}_4)_{tt} \end{pmatrix} + 2i\mathcal{A}^{-1}(u_0) \begin{pmatrix} (u_4)_t \\ (\overline{u}_4)_t \end{pmatrix} + \begin{pmatrix} \Delta u_4 \\ \Delta \overline{u}_4 \end{pmatrix} + \sum_{k=1}^{2} \mathcal{D}_k(u_0, u_k) \begin{pmatrix} \partial_k u_4 \\ \partial_k \overline{u}_4 \end{pmatrix} \\
&+\mathcal{G}(DU, Tu_3) \begin{pmatrix} u_4 \\ \overline{u}_4 \end{pmatrix} + \sum_{j=0}^{2} \sum_{k=0}^{2} \mathcal{H}_{jk}(DU, Tu_3) \begin{pmatrix} \partial_j \partial_k \Delta^{-1} u_4 \\ \partial_j \partial_k \Delta^{-1} \overline{u}_4 \end{pmatrix} = 0
\end{aligned}
\tag{3.10}
$$

where for $k = 1, 2$

$$
\mathcal{D}_k(u_0, u_k) = \mathcal{A}^{-1}(u_0) \left(B(u_0, u_k) + 2C(u_0, u_k) \right),
$$

$C(u_0, u_k)$ is the same as in Section 2.2 and for $j, k = 1, 2$, $\mathcal{H}_{jk}$ is a matrix depending on U and its first derivatives and on $T_{mn} u_3$ $(m, n = 1, 2)$. We have to notice here that in equation (3.10), all the terms of equation (3.9) which contains $\partial_t u_0$ and its first and second space derivatives are included in the last terms

$$
\sum_{j=0}^{2} \sum_{k=0}^{2} \mathcal{H}_{jk}(DU, Tu_3) \begin{pmatrix} \partial_j \partial_k \Delta^{-1} u_4 \\ \partial_j \partial_k \Delta^{-1} \overline{u}_4 \end{pmatrix}.
$$

Indeed, this sum include the values $j = 0$ and $k = 0$. Recalling that we have adopted the notation $\partial_0 u = u$, then it is possible to write for $j = 0, .., 2$ and $k = 0, .., 2$

$$
\partial_t \partial_j \partial_k u_0 = \partial_j \partial_k \Delta^{-1} u_4.
$$

Finally, in order to obtain an equation satisfied by u_5, we differentiate twice with respect to t equation (3.7) to obtain

$$
\begin{aligned}
&-\varepsilon^2\mathcal{A}^{-1}(u_0)\begin{pmatrix}(u_5)_{tt}\\(\overline{u}_5)_{tt}\end{pmatrix}-2\varepsilon^2\partial_t\left(\mathcal{A}^{-1}(u_0)\right)\begin{pmatrix}(u_5)_t\\(\overline{u}_5)_t\end{pmatrix}-\varepsilon^2\partial_t^2\left(\mathcal{A}^{-1}(u_0)\right)\begin{pmatrix}u_5\\\overline{u}_5\end{pmatrix}\\
&+2i\mathcal{A}^{-1}(u_0)\begin{pmatrix}(u_5)_t\\(\overline{u}_5)_t\end{pmatrix}+4i\partial_t\left(\mathcal{A}^{-1}(u_0)\right)\begin{pmatrix}u_5\\\overline{u}_5\end{pmatrix}+2i\partial_t^2\left(\mathcal{A}^{-1}(u_0)\right)\begin{pmatrix}u_4\\\overline{u}_4\end{pmatrix}\\
&+\begin{pmatrix}\Delta u_5\\\Delta\overline{u}_5\end{pmatrix}+\sum_{k=1}^{2}\mathcal{A}^{-1}(u_0)B(u_0,u_k)\begin{pmatrix}\partial_k u_5\\\partial_k\overline{u}_5\end{pmatrix}+2\sum_{k=1}^{2}\partial_t\left(\mathcal{A}^{-1}(u_0)B(u_0,u_k)\right)\begin{pmatrix}\partial_k u_4\\\partial_k\overline{u}_4\end{pmatrix}\\
&+\sum_{k=1}^{2}\partial_t^2\left(\mathcal{A}^{-1}(u_0)B(u_0,u_k)\right)\begin{pmatrix}\partial_k(e^{-q(|u_0|^2)}u_3)\\\partial_k(e^{-q(|u_0|^2)}\overline{u}_3)\end{pmatrix}+2\sum_{j=1}^{2}\mathcal{A}^{-1}(u_0)C(u_0,u_j)\begin{pmatrix}\partial_j u_5\\\partial_j\overline{u}_5\end{pmatrix}\\
&+4\sum_{j=1}^{2}\partial_t\left(\mathcal{A}^{-1}(u_0)C(u_0,u_j)\right)\begin{pmatrix}\partial_j u_4\\\partial_j\overline{u}_4\end{pmatrix}+2\sum_{j=1}^{2}\partial_t^2\left(\mathcal{A}^{-1}(u_0)C(u_0,u_j)\right)\begin{pmatrix}\partial_j(e^{-q(|u_0|^2)}u_3)\\\partial_j(e^{-q(|u_0|^2)}\overline{u}_3)\end{pmatrix}\\
&+\sum_{j=1}^{2}\mathcal{A}^{-1}(u_0)\partial_jC(u_0,u_j)\begin{pmatrix}u_5\\\overline{u}_5\end{pmatrix}+2\sum_{j=1}^{2}\partial_t\left(\mathcal{A}^{-1}(u_0)\partial_jC(u_0,u_j)\right)\begin{pmatrix}u_4\\\overline{u}_4\end{pmatrix}\\
&+\sum_{j=1}^{2}\partial_t^2\left(\mathcal{A}^{-1}(u_0)\partial_jC(u_0,u_j)\right)\begin{pmatrix}(e^{-q(|u_0|^2)}u_3)\\(e^{-q(|u_0|^2)}\overline{u}_3)\end{pmatrix}+\sum_{j=1}^{2}\sum_{k=1}^{2}\mathcal{A}^{-1}(u_0)\partial_jB(u_0,u_k)\begin{pmatrix}\partial_k\partial_j\Delta^{-1}u_5\\\partial_k\partial_j\Delta^{-1}\overline{u}_5\end{pmatrix}\\
&+2\sum_{j=1}^{2}\sum_{k=1}^{2}\partial_t\left(\mathcal{A}^{-1}(u_0)\partial_jB(u_0,u_k)\right)\begin{pmatrix}\partial_k\partial_j\Delta^{-1}u_4\\\partial_k\partial_j\Delta^{-1}\overline{u}_4\end{pmatrix}+\sum_{j=1}^{2}\sum_{k=1}^{2}\partial_t^2\left(\mathcal{A}^{-1}(u_0)\partial_jB(u_0,u_k)\right)\begin{pmatrix}T_{kj}u_3\\T_{kj}\overline{u}_3\end{pmatrix}\\
&+\partial_t^2\left(\sum_{j=1}^{2}\mathcal{A}^{-1}(u_0)\begin{pmatrix}\partial_jF(U^*,u_j)\\\partial_j\overline{F}(U^*,u_j)\end{pmatrix}\right)=0.
\end{aligned}
$$

This equation takes the form

$$
\begin{aligned}
&-\varepsilon^2\mathcal{A}^{-1}(u_0)\begin{pmatrix}(u_5)_{tt}\\(\overline{u}_5)_{tt}\end{pmatrix}+\mathcal{C}(u_0,\Delta^{-1}u_4)\begin{pmatrix}(u_5)_t\\(\overline{u}_5)_t\end{pmatrix}+\begin{pmatrix}\Delta u_5\\\Delta\overline{u}_5\end{pmatrix}+\sum_{k=1}^{2}\mathcal{D}_k(u_0,u_k)\begin{pmatrix}\partial_k u_5\\\partial_k\overline{u}_5\end{pmatrix}\\
&+\mathcal{I}(DU,Tu_3)\begin{pmatrix}u_5\\\overline{u}_5\end{pmatrix}+\sum_{j=0}^{2}\sum_{k=0}^{2}\mathcal{J}_{jk}(DU,Tu_3)\begin{pmatrix}\partial_j\partial_k\Delta^{-1}u_5\\\partial_j\partial_k\Delta^{-1}\overline{u}_5\end{pmatrix}\\
&+\sum_{j=0}^{2}\sum_{k=0}^{2}\mathcal{K}_{jk}(DU,Tu_3,D^2\Delta^{-1}u_4)\begin{pmatrix}\partial_j\partial_k\Delta^{-1}u_4\\\partial_j\partial_k\Delta^{-1}\overline{u}_4\end{pmatrix}=0.
\end{aligned}
\tag{3.11}
$$

where

$$\mathcal{C}(u_0,\Delta^{-1}u_4)=2i\mathcal{A}^{-1}(u_0)-2\varepsilon^2\partial_t\left(\mathcal{A}^{-1}(u_0)\right).$$

For $j,k=0,..,2$, the matrices $\mathcal{J}_{jk}$ depend on j, k, on U and its first derivatives and on $T_{mn}u_3$ for $m,m=1,2$. For $j,k=0,..,2$, the matrices $\mathcal{K}_{jk}$ depend on j, k, on U and its first derivatives, on $T_{mn}u_3$ for $m,m=1,2$ and on $\Delta^{-1}u_4$ and its first and second derivatives. In order to simplify the notations, we now introduce for $j,k=0,..,2$ the operator R_{jk}

$$R_{jk} = \partial_j \partial_k \Delta^{-1}.$$

Once again, we have included the terms which contain $\partial_t u_0$ and its first and second space derivatives in

$$\sum_{j=0}^{2} \sum_{k=0}^{2} \mathcal{K}_{jk}(DU, Tu_3, D^2\Delta^{-1}u_4) \begin{pmatrix} \partial_j \partial_k \Delta^{-1} u_4 \\ \partial_j \partial_k \Delta^{-1} \overline{u}_4 \end{pmatrix}$$

and the terms which contain $\partial_t^2 u_0$ and its first and second space derivatives in

$$\sum_{j=0}^{2} \sum_{k=0}^{2} \mathcal{J}_{jk}(DU, Tu_3) \begin{pmatrix} \partial_j \partial_k \Delta^{-1} u_5 \\ \partial_j \partial_k \Delta^{-1} \overline{u}_5 \end{pmatrix}.$$

In conclusion, we have transformed equation (3.3) into the following system

$$2i \begin{pmatrix} \partial_t u_0 \\ \partial_t \overline{u}_0 \end{pmatrix} + \mathcal{A}(u_0) \begin{pmatrix} \Delta u_0 \\ \Delta \overline{u}_0 \end{pmatrix} - \varepsilon^2 \begin{pmatrix} \Delta^{-1} u_5 \\ \Delta^{-1} \overline{u}_5 \end{pmatrix} + \mathcal{F}_0(U^*) = 0 \tag{3.12}$$

for $j = 1, 2$,

$$2i \begin{pmatrix} (u_j)_t \\ (\overline{u}_j)_t \end{pmatrix} + \mathcal{A}(u_0) \begin{pmatrix} \Delta u_j \\ \Delta \overline{u}_j \end{pmatrix} - \varepsilon^2 \begin{pmatrix} \partial_j \Delta^{-1} u_5 \\ \partial_j \Delta^{-1} \overline{u}_5 \end{pmatrix} + \mathcal{F}_j(U^*, u_3, Tu_3) = 0 \tag{3.13}$$

$$\begin{aligned} &2i \begin{pmatrix} (u_3)_t \\ (\overline{u}_3)_t \end{pmatrix} + \mathcal{A}(u_0) \begin{pmatrix} \Delta u_3 \\ \Delta \overline{u}_3 \end{pmatrix} + \sum_{k=1}^{2} \mathcal{E}(u_0, u_k) \begin{pmatrix} \partial_k u_3 \\ \partial_k \overline{u}_3 \end{pmatrix} - \varepsilon^2 \begin{pmatrix} u_5 \\ \overline{u}_5 \end{pmatrix} \\ &+ \mathcal{I}(U, Tu_3, \Delta^{-1}u_4) = 0 \end{aligned} \tag{3.14}$$

$$\begin{aligned} &-\varepsilon^2 \mathcal{A}^{-1}(u_0) \begin{pmatrix} (u_4)_{tt} \\ (\overline{u}_4)_{tt} \end{pmatrix} + 2i\mathcal{A}^{-1}(u_0) \begin{pmatrix} (u_4)_t \\ (\overline{u}_4)_t \end{pmatrix} + \begin{pmatrix} \Delta u_4 \\ \Delta \overline{u}_4 \end{pmatrix} + \sum_{k=1}^{2} \mathcal{D}_k(u_0, u_k) \begin{pmatrix} \partial_k u_4 \\ \partial_k \overline{u}_4 \end{pmatrix} \\ &+ \sum_{j=0}^{2} \sum_{k=0}^{2} \mathcal{H}_{jk}(DU, Tu_3) \begin{pmatrix} R_{jk} u_4 \\ R_{jk} \overline{u}_4 \end{pmatrix} + \mathcal{G}(DU, Tu_3) \begin{pmatrix} u_4 \\ \overline{u}_4 \end{pmatrix} = 0 \end{aligned} \tag{3.15}$$

$$\begin{aligned} &-\varepsilon^2 \mathcal{A}^{-1}(u_0) \begin{pmatrix} (u_5)_{tt} \\ (\overline{u}_5)_{tt} \end{pmatrix} + \mathcal{C}(u_0, \Delta^{-1}u_4) \begin{pmatrix} (u_5)_t \\ (\overline{u}_5)_t \end{pmatrix} + \begin{pmatrix} \Delta u_5 \\ \Delta \overline{u}_5 \end{pmatrix} + \sum_{k=1}^{2} \mathcal{D}_k(u_0, u_k) \begin{pmatrix} \partial_k u_5 \\ \partial_k \overline{u}_5 \end{pmatrix} \\ &+ \sum_{j=0}^{2} \sum_{k=0}^{2} \mathcal{J}_{jk}(DU, Tu_3) \begin{pmatrix} R_{jk} u_5 \\ R_{jk} \overline{u}_5 \end{pmatrix} + \sum_{j=0}^{2} \sum_{k=0}^{2} \mathcal{K}_{jk}(DU, Tu_3, D^2\Delta^{-1}u_4) \begin{pmatrix} R_{jk} u_4 \\ R_{jk} \overline{u}_4 \end{pmatrix} \\ &+ \mathcal{I}(DU, Tu_3) \begin{pmatrix} u_5 \\ \overline{u}_5 \end{pmatrix} = 0. \end{aligned} \tag{3.16}$$

3.3 Proof of Theorem 1.3.1

In this section, we give a proof of Theorem 1.3.1. We study the system (3.12), (3.13), (3.14), (3.15) and (3.16) in the following function space

$$X_T = \left\{ \begin{array}{l} U = (u_j)_{j=0}^5 : u_j \in C([0,T]; L^2(\mathbb{R}^2)) \cap L^\infty(0,T; H^4(\mathbb{R}^2)), \\ ||U||_{X_T} = \sum_{j=0}^5 \sup_{0\le t\le T} ||u_j(t)||_{H^4(\mathbb{R}^2)} < \infty \end{array} \right\}$$

For $M = (m_j)_{j=0}^5 \in \mathbb{R}_+^6$ and $r \in \mathbb{R}_+$, we denote

$$X_T(M,r) = \left\{ \begin{array}{l} U = (u_j)_{j=0}^5 \in X_T : \forall j = 0,..,5 \ ||u_j||_{L^\infty(0,T;H^4(\mathbb{R}^2))} \le m_j \\ ||(u_0)_t||_{L^\infty(0,T;H^2(\mathbb{R}^2))} \le r \text{ and } u_0(\cdot,0) = u_0^\varepsilon(\cdot) \end{array} \right\}$$

and we let $V = (v_j)_{j=0}^5 \in X_T(M,r)$. We also set $V^* = (v_j)_{j=0}^2$. Consider the linearized version of equations (3.12), (3.13), (3.14), (3.15) and (3.16)

$$2i \begin{pmatrix} \partial_t u_0 \\ \partial_t \overline{u}_0 \end{pmatrix} + \mathcal{A}(v_0) \begin{pmatrix} \Delta u_0 \\ \Delta \overline{u}_0 \end{pmatrix} - \varepsilon^2 \begin{pmatrix} \Delta^{-1} v_5 \\ \Delta^{-1} \overline{v}_5 \end{pmatrix} + \mathcal{F}_0(V^*) = 0 \tag{3.17}$$

for $j = 1, 2$,

$$2i \begin{pmatrix} (u_j)_t \\ (\overline{u}_j)_t \end{pmatrix} + \mathcal{A}(v_0) \begin{pmatrix} \Delta u_j \\ \Delta \overline{u}_j \end{pmatrix} - \varepsilon^2 \begin{pmatrix} \partial_j \Delta^{-1} v_5 \\ \partial_j \Delta^{-1} \overline{v}_5 \end{pmatrix} + \mathcal{F}_j(V^*, v_3, Tv_3) = 0 \tag{3.18}$$

$$\begin{aligned} &2i \begin{pmatrix} (u_3)_t \\ (\overline{u}_3)_t \end{pmatrix} + \mathcal{A}(v_0) \begin{pmatrix} \Delta u_3 \\ \Delta \overline{u}_3 \end{pmatrix} + \sum_{k=1}^2 \mathcal{E}(v_0, v_k) \begin{pmatrix} \partial_k u_3 \\ \partial_k \overline{u}_3 \end{pmatrix} - \varepsilon^2 \begin{pmatrix} v_5 \\ \overline{v}_5 \end{pmatrix} \\ &+\mathcal{I}(V, Tv_3, \Delta^{-1} v_4) = 0 \end{aligned} \tag{3.19}$$

$$\begin{aligned} &-\varepsilon^2 \mathcal{A}^{-1}(v_0) \begin{pmatrix} (u_4)_{tt} \\ (\overline{u}_4)_{tt} \end{pmatrix} + 2i\mathcal{A}^{-1}(v_0) \begin{pmatrix} (u_4)_t \\ (\overline{u}_4)_t \end{pmatrix} + \begin{pmatrix} \Delta u_4 \\ \Delta \overline{u}_4 \end{pmatrix} + \sum_{k=1}^2 \mathcal{D}_k(v_0, v_k) \begin{pmatrix} \partial_k u_4 \\ \partial_k \overline{u}_4 \end{pmatrix} \\ &+ \sum_{j=0}^2 \sum_{k=0}^2 \mathcal{H}_{jk}(DV, Tv_3) \begin{pmatrix} R_{jk} u_4 \\ R_{jk} \overline{u}_4 \end{pmatrix} + \mathcal{G}(DV, Tv_3) \begin{pmatrix} u_4 \\ \overline{u}_4 \end{pmatrix} = 0 \end{aligned} \tag{3.20}$$

$$\begin{aligned} &-\varepsilon^2 \mathcal{A}^{-1}(v_0) \begin{pmatrix} (u_5)_{tt} \\ (\overline{u}_5)_{tt} \end{pmatrix} + \mathcal{C}(v_0, \Delta^{-1} v_4) \begin{pmatrix} (u_5)_t \\ (\overline{u}_5)_t \end{pmatrix} + \begin{pmatrix} \Delta u_5 \\ \Delta \overline{u}_5 \end{pmatrix} + \sum_{k=1}^2 \mathcal{D}_k(v_0, v_k) \begin{pmatrix} \partial_k u_5 \\ \partial_k \overline{u}_5 \end{pmatrix} \\ &+ \sum_{j=0}^2 \sum_{k=0}^2 \mathcal{J}_{jk}(DV, Tv_3) \begin{pmatrix} R_{jk} u_5 \\ R_{jk} \overline{u}_5 \end{pmatrix} + \sum_{j=0}^2 \sum_{k=0}^2 \mathcal{K}_{jk}(DV, Tv_3, D^2 \Delta^{-1} v_4) \begin{pmatrix} R_{jk} u_4 \\ R_{jk} \overline{u}_4 \end{pmatrix} \\ &+\mathcal{I}(DV, Tv_3) \begin{pmatrix} u_5 \\ \overline{u}_5 \end{pmatrix} = 0. \end{aligned} \tag{3.21}$$

Let $\mathcal{Z} = [L^\infty(0,T;H^4(\mathbb{R}^2)) \cap C([0,T];L^2(\mathbb{R}^2))]^6$. Then, it is not difficult to see that the linear inhomogeneous Cauchy problem (3.17), (3.18), (3.19), (3.20) and (3.21) with initial condition

$$\begin{aligned}
&u_0(\cdot,0) = u_0^\varepsilon(\cdot),\;\; u_1(\cdot,0) = \partial_1 u_0^\varepsilon(\cdot),\;\; u_2(\cdot,0) = \partial_2 u_0^\varepsilon(\cdot),\\
&u_3(\cdot,0) = e^{q(|u_0^\varepsilon|^2)}\Delta u_0^\varepsilon(\cdot),\;\; u_4(\cdot,0) = \Delta u_1^\varepsilon(\cdot),\;\; u_5(\cdot,0) = \partial_t u_4(\cdot,0),\\
&\partial_t u_4(\cdot,0) = \frac{1}{\varepsilon^2}\Delta \left(\Delta u_0^\varepsilon + 2iu_1^\varepsilon - (\tfrac{1}{\sqrt{1+|u_0^\varepsilon|^2}} - 1)u_0^\varepsilon \right.\\
&\qquad\qquad \left. - \tfrac{1}{\sqrt{1+|u_0^\varepsilon|^2}}\Delta(\sqrt{1+|u_0^\varepsilon|^2})u_0^\varepsilon\right),\\
&\partial_t u_5(\cdot,0) = \tfrac{1}{\varepsilon^2}\Delta \left(\Delta u_1^\varepsilon - (\tfrac{1}{\sqrt{1+|u_0^\varepsilon|^2}} - 1)u_1^\varepsilon + \tfrac{u_0^\varepsilon\overline{u}_1^\varepsilon + \overline{u}_0^\varepsilon u_1^\varepsilon}{2(1+|u_0^\varepsilon|^2)^{\frac{3}{2}}}u_0^\varepsilon \right.\\
&\qquad\qquad - \tfrac{1}{\sqrt{1+|u_0^\varepsilon|^2}}\Delta(\sqrt{1+|u_0^\varepsilon|^2})u_1^\varepsilon + \tfrac{u_0^\varepsilon\overline{u}_1^\varepsilon + \overline{u}_0^\varepsilon u_1^\varepsilon}{2(1+|u_0^\varepsilon|^2)^{\frac{3}{2}}}\Delta(\sqrt{1+|u_0^\varepsilon|^2})u_0^\varepsilon\\
&\qquad\qquad + \tfrac{u_0^\varepsilon}{\sqrt{1+|u_0^\varepsilon|^2}}\Delta\left(\tfrac{u_0^\varepsilon\overline{u}_1^\varepsilon + \overline{u}_0^\varepsilon u_1^\varepsilon}{2(1+|u_0^\varepsilon|^2)^{\frac{3}{2}}}\right) + \tfrac{2i}{\varepsilon^2}\left(\Delta u_0^\varepsilon + 2iu_1^\varepsilon\right.\\
&\qquad\qquad \left.\left. - (\tfrac{1}{\sqrt{1+|u_0^\varepsilon|^2}} - 1)u_0^\varepsilon - \tfrac{1}{\sqrt{1+|u_0^\varepsilon|^2}}\Delta(\sqrt{1+|u_0^\varepsilon|^2})u_0^\varepsilon\right)\right).
\end{aligned} \tag{3.22}$$

defines a mapping $\mathcal{S}$

$$\begin{aligned}\mathcal{S}:\; \mathcal{Z} &\longrightarrow \mathcal{Z}\\ V &\longmapsto U.\end{aligned}$$

The initial conditions (3.22) are derived from equations (3.12), (3.13), (3.14), (3.15) and (3.16). To prove Theorem 1.3.1, we have to find a time T and constants M and r such that $\mathcal{S}$ maps the closed ball $X_T(M,r)$ into itself and is a contracting mapping in the norm

$$\sup_{t\in[0,T]} \sum_{j=0}^{5} ||v_j||_2.$$

The first thing to do is to estimate u_4 by using equation (3.20). As we have already said in the introduction of this chapter, we will treat (3.20) as a wave equation. Multiplying equation (3.20) by $\mathcal{A}(v_0)$ and applying the operator $(1-\Delta)$, we obtain denoting $\phi_4 = (1-\Delta)u_4$

$$\begin{aligned}
&-\varepsilon^2\begin{pmatrix}(\phi_4)_{tt}\\(\overline{\phi}_4)_{tt}\end{pmatrix} + 2i\begin{pmatrix}(\phi_4)_t\\(\overline{\phi}_4)_t\end{pmatrix} + \mathcal{A}(v_0)\begin{pmatrix}\Delta\phi_4\\\Delta\overline{\phi}_4\end{pmatrix} + \sum_{k=1}^{2}\mathcal{L}(v_0,v_k,\partial_k v_0)\begin{pmatrix}\partial_k\phi_4\\\partial_k\overline{\phi}_4\end{pmatrix}\\
&\sum_{|\alpha|\le 2}\mathcal{M}_\alpha(D^{h_1-1}V, D^{h_2-1}Tv_3)\begin{pmatrix}\partial_1^{\alpha_1}\partial_2^{\alpha_2}u_4\\\partial_1^{\alpha_1}\partial_2^{\alpha_2}\overline{u}_4\end{pmatrix}\\
&+\sum_{|\alpha|\le 2}\sum_{j,k=0}^{2}\mathcal{N}_\alpha(D^{h_2}V, D^{h_2-1}Tv_3)\begin{pmatrix}\partial_1^{\alpha_1}\partial_2^{\alpha_2}R_{jk}u_4\\\partial_1^{\alpha_1}\partial_2^{\alpha_2}R_{jk}\overline{u}_4\end{pmatrix} = 0.
\end{aligned} \tag{3.23}$$

where the matrices $\mathcal{M}_\alpha$ depend on the multi-index $\alpha \in \mathbb{N}^2$, $\mathcal{N}_\alpha$ on α, j and k

$$\mathcal{L}(v_0,v_k,\partial_k v_0) = \mathcal{A}(v_0)\mathcal{D}_k(v_0,v_k) + 2\partial_k\mathcal{A}(v_0),$$

$$h_1 = 3 \quad \text{if } |\alpha| = 1, 2, \quad h_1 = 4 \quad \text{if } |\alpha| = 0,$$

$$h_2 = 3 - |\alpha|.$$

Finally, differentiating (3.23) with respect to x_m for $m = 1, 2$ and denoting $\phi_4^m = \partial_m \phi_4$, we obtain

$$\begin{aligned} &-\varepsilon^2 \begin{pmatrix} (\phi_4^m)_{tt} \\ (\overline{\phi}_4^m)_{tt} \end{pmatrix} + 2i \begin{pmatrix} (\phi_4^m)_t \\ (\overline{\phi}_4^m)_t \end{pmatrix} + \mathcal{A}(v_0) \begin{pmatrix} \Delta \phi_4^m \\ \Delta \overline{\phi}_4^m \end{pmatrix} + \sum_{k=1}^{2} \mathcal{L}(v_0, v_k, \partial_k v_0) \begin{pmatrix} \partial_k \phi_4^m \\ \partial_k \overline{\phi}_4^m \end{pmatrix} \\ &+ \sum_{k=1}^{2} \partial_m \mathcal{A}(v_0) \begin{pmatrix} \partial_k \phi_4^k \\ \partial_k \overline{\phi}_4^k \end{pmatrix} + \sum_{|\alpha| \le 3} \widetilde{\mathcal{M}}_\alpha (D^{h_1} V, D^{h_2} T v_3) \begin{pmatrix} \partial_1^{\alpha_1} \partial_2^{\alpha_2} u_4 \\ \partial_1^{\alpha_1} \partial_2^{\alpha_2} \overline{u}_4 \end{pmatrix} \\ &+ \sum_{|\alpha| \le 3} \sum_{j,k=0}^{2} \widetilde{\mathcal{N}}_\alpha (D^{h_2+1} V, D^{h_2} T v_3) \begin{pmatrix} \partial_1^{\alpha_1} \partial_2^{\alpha_2} R_{jk} u_4 \\ \partial_1^{\alpha_1} \partial_2^{\alpha_2} R_{jk} \overline{u}_4 \end{pmatrix} = 0. \end{aligned} \tag{3.24}$$

where

$$h_1 = 2, \quad h_2 = 0 \quad \text{if } |\alpha| = 3.$$

The matrices $\widetilde{\mathcal{M}}_\alpha$ and $\widetilde{\mathcal{N}}_\alpha$ depend on α for $|\alpha| \le 3$. They also depend on respectively $D^{h_1} V$, $D^{h_2} T v_3$ and $D^{h_2+1} V$, $D^{h_2} T v_3$. We have to notice that in (3.23), the differentiation of

$$\mathcal{A}(v_0) \begin{pmatrix} \Delta \phi_4 \\ \Delta \overline{\phi}_4 \end{pmatrix}$$

with respect to x_m for a fixed m gives the term

$$\partial_m \left(\mathcal{A}(v_0) \right) \begin{pmatrix} \Delta \phi_4 \\ \Delta \overline{\phi}_4 \end{pmatrix}$$

which is rewritten in (3.24) as

$$\sum_{k=1}^{2} \partial_m \mathcal{A}(v_0) \begin{pmatrix} \partial_k \phi_4^k \\ \partial_k \overline{\phi}_4^k \end{pmatrix}.$$

We also recall here the expression of $\mathcal{A}^{-1}(v_0)$

$$\mathcal{A}^{-1}(v_0) = \frac{1}{2} \begin{pmatrix} 2 + |v_0|^2 & -v_0^2 \\ v_0^2 & -(2 + |v_0|^2) \end{pmatrix}.$$

We multiply equation (3.24) by $\mathcal{A}^{-1}(v_0)$ and we consider the first line of the resulting equation. Furthermore, we add to both side of the equation the term $-\phi_4^m$. We can now perform the usual energy estimate for wave equations, namely we multiply the equation by $(\overline{\phi}_4^m)_t$ and we integrate over $\mathbb{R}^2$.

$$
\begin{aligned}
&\varepsilon^2 \int_{\mathbb{R}^2} \frac{(2+|v_0|^2)}{2} (\phi_4^m)_{tt} (\overline{\phi}_4^m)_t dx - \varepsilon^2 \int_{\mathbb{R}^2} \frac{v_0^2}{2} (\overline{\phi}_4^m)_{tt} (\overline{\phi}_4^m)_t dx \\
&-i \int_{\mathbb{R}^2} (2+|v_0|^2) |(\phi_4^m)_t|^2 dx + i \int_{\mathbb{R}^2} v_0^2 (\overline{\phi}_4^m)_t (\overline{\phi}_4^m)_t dx \\
&- \int_{\mathbb{R}^2} \Delta \phi_4^m (\overline{\phi}_4^m)_t dx - \sum_{k=1}^2 \int_{\mathbb{R}^2} \mathcal{O}^{11}(v_0, v_k, \partial_k v_0) \partial_k \phi_4^m (\overline{\phi}_4^m)_t dx \\
&- \sum_{k=1}^2 \int_{\mathbb{R}^2} \mathcal{O}^{12}(v_0, v_k, \partial_k v_0) \partial_k \overline{\phi}_4^m (\overline{\phi}_4^m)_t dx + \int_{\mathbb{R}^2} \phi_4^m (\overline{\phi}_4^m)_t dx \\
&- \sum_{k=1}^2 \int_{\mathbb{R}^2} \left(\mathcal{A}^{-1}(v_0) \partial_m \mathcal{A}(v_0)\right)^{11} \partial_k \phi_4^k (\overline{\phi}_4^m)_t dx - \int_{\mathbb{R}^2} \phi_4^m (\overline{\phi}_4^m)_t dx \\
&- \sum_{k=1}^2 \int_{\mathbb{R}^2} \left(\mathcal{A}^{-1}(v_0) \partial_m \mathcal{A}(v_0)\right)^{12} \partial_k \overline{\phi}_4^k (\overline{\phi}_4^m)_t dx \\
&- \sum_{|\alpha| \le 3} \int_{\mathbb{R}^2} \mathcal{P}_\alpha^{11}(D^{h_1} V, D^{h_2} T v_3)(\partial_1^{\alpha_1} \partial_2^{\alpha_2} u_4)(\overline{\phi}_4^m)_t dx \\
&- \sum_{|\alpha| \le 3} \int_{\mathbb{R}^2} \mathcal{P}_\alpha^{12}(D^{h_1} V, D^{h_2} T v_3)(\partial_1^{\alpha_1} \partial_2^{\alpha_2} \overline{u}_4)(\overline{\phi}_4^m)_t dx \\
&- \sum_{|\alpha| \le 3} \sum_{j,k=0}^2 \int_{\mathbb{R}^2} \mathcal{Q}_\alpha^{11}(D^{h_2+1} V, D^{h_2} T v_3) \partial_1^{\alpha_1} \partial_2^{\alpha_2} R_{jk} u_4 (\overline{\phi}_4^m)_t dx \\
&- \sum_{|\alpha| \le 3} \sum_{j,k=0}^2 \int_{\mathbb{R}^2} \mathcal{Q}_\alpha^{12}(D^{h_2+1} V, D^{h_2} T v_3) \partial_1^{\alpha_1} \partial_2^{\alpha_2} R_{jk} \overline{u}_4 (\overline{\phi}_4^m)_t dx = 0.
\end{aligned} \tag{3.25}
$$

where $\mathcal{O} = \mathcal{A}^{-1}(v_0)\mathcal{L}$, $\mathcal{P}_\alpha = \mathcal{A}^{-1}(v_0)\widetilde{\mathcal{M}}_\alpha$ and $\mathcal{Q}_\alpha = \mathcal{A}^{-1}(v_0)\widetilde{\mathcal{N}}_\alpha$.

We now take the real part of equation (3.25). The first line of the resulting expression gives

$$
\begin{aligned}
&\mathrm{Re}\left(\varepsilon^2 \int_{\mathbb{R}^2} \frac{(2+|v_0|^2)}{2} (\phi_4^m)_{tt} (\overline{\phi}_4^m)_t dx - \varepsilon^2 \int_{\mathbb{R}^2} \frac{v_0^2}{2} (\overline{\phi}_4^m)_{tt} (\overline{\phi}_4^m)_t dx\right) \\
&= \varepsilon^2 \int_{\mathbb{R}^2} \frac{(2+|v_0|^2)}{4} |(\phi_4^m)_t|_t^2 dx - \varepsilon^2 \int_{\mathbb{R}^2} \left(\frac{v_0^2}{8} \frac{d}{dt} (\overline{\phi}_4^m)_t^2 + \frac{\overline{v}_0^2}{8} \frac{d}{dt} (\phi_4^m)_t^2\right) dx \\
&= \frac{d}{dt}\left(\varepsilon^2 \int_{\mathbb{R}^2} \frac{(2+|v_0|^2)}{4} |(\phi_4^m)_t|^2 dx - \varepsilon^2 \int_{\mathbb{R}^2} \left(\frac{v_0^2}{8} (\overline{\phi}_4^m)_t^2 + \frac{\overline{v}_0^2}{8} (\phi_4^m)_t^2\right) dx\right) \\
&- \varepsilon^2 \int_{\mathbb{R}^2} \frac{(2+|v_0|^2)_t}{4} |(\phi_4^m)_t|^2 dx + \varepsilon^2 \int_{\mathbb{R}^2} \left(\frac{(v_0^2)_t}{8} (\overline{\phi}_4^m)_t^2 + \frac{(\overline{v}_0^2)_t}{8} (\phi_4^m)_t^2\right) dx.
\end{aligned}
$$

For the second line, since $V \in X_T(M, r)$ we have, using the continuous embedding of

$H^2(\mathbb{R}^2)$ into $L^\infty(\mathbb{R}^2)$

$$\left| -i \int_{\mathbb{R}^2} (2+|v_0|^2)(\phi_4^m)_t(\overline{\phi}_4^m)_t dx + i \int_{\mathbb{R}^2} v_0^2 (\overline{\phi}_4^m)_t (\overline{\phi}_4^m)_t dx \right| \leq C(M) \int_{\mathbb{R}^2} |(\phi_4^m)_t|^2 dx.$$

The first term of the third line is,

$$-\mathrm{Re}\left(\int_{\mathbb{R}^2} \Delta \phi_4^m (\overline{\phi}_4^m)_t dx \right) = \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^2} |\nabla \phi_4^m|^2 dx.$$

Using again the fact that $V \in X_T(M,r)$ and that the matrix $\mathcal{O}_m$ depends only on V and $\partial_k v_0$, we obtain by Cauchy-Schwarz inequality

$$\begin{aligned}
&\left| -\sum_{k=1}^{2} \left(\int_{\mathbb{R}^2} \mathcal{O}^{11}(v_0, v_k, \partial_k v_0) \partial_k \phi_4^m (\overline{\phi}_4^m)_t dx + \int_{\mathbb{R}^2} \mathcal{O}^{12}(v_0, v_k, \partial_k v_0) \partial_k \overline{\phi}_4^m (\overline{\phi}_4^m)_t dx \right) \right| \\
&\leq C(M) \sum_{k=1}^{2} ||\partial_k \phi_4^m||_2 ||(\phi_4^m)_t||_2 \\
&\leq C(M) \left(\sum_{k=1}^{2} ||\partial_k \phi_4^m||_2^2 + ||(\phi_4^m)_t||_2^2 \right).
\end{aligned}$$

In the same way, we have

$$\left| \sum_{k=1}^{2} \int_{\mathbb{R}^2} \left(\mathcal{A}^{-1}(v_0) \partial_m \mathcal{A}(v_0) \right)^{11} \partial_k \phi_4^k (\overline{\phi}_4^m)_t dx \right| \leq C(M) \left(\sum_{k=1}^{2} ||\partial_k \phi_4^k||_2^2 + ||(\phi_4^m)_t||_2^2 \right),$$

$$\left| \sum_{k=1}^{2} \int_{\mathbb{R}^2} \left(\mathcal{A}^{-1}(v_0) \partial_m \mathcal{A}(v_0) \right)^{12} \partial_k \overline{\phi}_4^k (\overline{\phi}_4^m)_t dx \right| \leq C(M) \left(\sum_{k=1}^{2} ||\partial_k \phi_4^k||_2^2 + ||(\phi_4^m)_t||_2^2 \right).$$

We now treat the two terms containing $\phi_4^m (\overline{\phi}_4^m)_t$ by two different methods. On one hand, one can write

$$\mathrm{Re}\left(\int_{\mathbb{R}^2} \phi_4^m (\overline{\phi}_4^m)_t dx \right) = \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^2} |\phi_4^m|^2 dx.$$

On the other hand, we derive from Cauchy-Schwarz inequality

$$\left| \int_{\mathbb{R}^2} \phi_4^m (\overline{\phi}_4^m)_t dx \right| \leq C(M) \left(||\phi_4^m||_2^2 + ||(\phi_4^m)_t||_2^2 \right).$$

For the next term, we have to be more careful. If $|\alpha| = 3$, $\mathcal{P}_\alpha$ depends on derivatives of V of order less than or equal to 2 and on Tv_3 and its first order derivatives. Thus, it can be estimated in $L^\infty(\mathbb{R}^2)$. The two other terms are estimated by Cauchy-Schwarz inequality. As a consequence, one can find a positive $C(M)$ such that

$$\begin{aligned}\left|\sum_{|\alpha|=3}\int_{\mathbb{R}^2}\mathcal{P}_\alpha^{11}(D^{h_1}V, D^{h_2}Tv_3)(\partial_1^{\alpha_1}\partial_2^{\alpha_2}u_4)(\overline{\phi}_4^m)_t dx\right| &\leq C(M)\sum_{|\alpha|=3}||\partial_1^{\alpha_1}\partial_2^{\alpha_2}u_4||_2||(\phi_4^m)_t||_2\\ &\leq C(M)\left(\sum_{k=1}^{2}||\phi_4^k||_2^2 + ||(\phi_4^m)_t||_2^2\right).\end{aligned}$$

If $|\alpha| = 0,..,2$, $\mathcal{P}_\alpha$ depends on derivatives of V of order less or equal to 4 and on derivatives of Tv_3 of order less or equal to 3. Thus it has to be estimated in $L^2(\mathbb{R}^2)$ together with $(\overline{\phi}_4^m)_t$ by Cauchy-Schwarz inequality whereas $\partial_1^\alpha\partial_2^\beta u_4$ is estimated in $L^\infty(\mathbb{R}^2)$.

$$\begin{aligned}&\left|\sum_{|\alpha|=0}^{2}\int_{\mathbb{R}^2}\mathcal{P}_\alpha^{11}(D^{h_1}V, D^{h_2}Tv_3)(\partial_1^{\alpha_1}\partial_2^{\alpha_2}u_4)(\overline{\phi}_4^m)_t dx\right|\\ &\leq C(M)\sum_{|\alpha|=0}^{2}||\partial_1^{\alpha_1}\partial_2^{\alpha_2}u_4||_{L^\infty(\mathbb{R}^2)}||(\phi_4^m)_t||_2\\ &\leq C(M)\left(||\phi_4||_2^2 + \sum_{k=1}^{2}(||\partial_k\phi_4^k||_2^2 + ||\phi_4^k||_2^2) + ||(\phi_4^m)_t||_2^2\right).\end{aligned}$$

In definitive, we have proved that

$$\begin{aligned}&\left|\sum_{|\alpha|\leq 3}\int_{\mathbb{R}^2}\mathcal{P}_\alpha^{11}(D^{h_1}V, D^{h_2}Tv_3)(\partial_1^{\alpha_1}\partial_2^{\alpha_2}u_4)(\overline{\phi}_4^m)_t dx\right|\\ &\leq C(M)\left(||\phi_4||_2^2 + \sum_{k=1}^{2}(||\phi_4^k||_2^2 + ||\partial_k\phi_4^k||_2^2) + ||(\phi_4^m)_t||_2^2\right).\end{aligned}$$

In the same way, one can also obtain

$$\begin{aligned}&\left|\sum_{|\alpha|\leq 3}\int_{\mathbb{R}^2}\mathcal{P}_\alpha^{12}(D^{h_1}V, D^{h_2}Tv_3)(\partial_1^{\alpha_1}\partial_2^{\alpha_2}\overline{u}_4)(\overline{\phi}_4^m)_t dx\right|\\ &\leq C(M)\left(||\phi_4||_2^2 + \sum_{k=1}^{2}(||\phi_4^k||_2^2 + ||\partial_k\phi_4^k||_2^2) + ||(\phi_4^m)_t||_2^2\right).\end{aligned}$$

The last two terms are treated exactly in the same way, the only thing to notice is that the operator R_{jk} is continuous from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$. Thus we can write

$$\begin{aligned}&\left|\sum_{|\alpha|\leq 3}\sum_{j,k=0}^{2}\int_{\mathbb{R}^2}\mathcal{Q}^{11}_{\alpha}(D^{h_2+1}V, D^{h_2}Tv_3)\partial_1^{\alpha_1}\partial_2^{\alpha_2}R_{jk}u_4(\overline{\phi}_4^m)_t dx\right|\\ &\leq\left(||\phi_4||_2^2+\sum_{k=1}^{2}(||\phi_4^k||_2^2+||\partial_k\phi_4^k||_2^2)+||(\phi_4^m)_t||_2^2\right),\end{aligned}$$

$$\begin{aligned}&\left|\sum_{|\alpha|\leq 3}\sum_{j,k=0}^{2}\int_{\mathbb{R}^2}\mathcal{Q}^{12}_{\alpha}(D^{h_2+1}V, D^{h_2}Tv_3)\partial_1^{\alpha_1}\partial_2^{\alpha_2}R_{jk}\overline{u}_4(\overline{\phi}_4^m)_t dx\right|\\ &\leq\left(||\phi_4||_2^2+\sum_{k=1}^{2}(||\phi_4^k||_2^2+||\partial_k\phi_4^k||_2^2)+||(\phi_4^m)_t||_2^2\right).\end{aligned}$$

Collecting all these results, we derive from (3.25)

$$\begin{aligned}&\frac{d}{dt}\left(\varepsilon^2\int_{\mathbb{R}^2}\frac{(2+|v_0|^2)}{4}|(\phi_4^m)_t|^2dx-\varepsilon^2\int_{\mathbb{R}^2}\left(\frac{v_0^2}{8}(\overline{\phi}_4^m)_t^2+\frac{\overline{v}_0^2}{8}(\phi_4^m)_t^2\right)dx+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla\phi_4^m|^2dx\right)\\ &+\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}^2}|\phi_4^m|^2dx\right)-\varepsilon^2\int_{\mathbb{R}^2}\frac{|v_0|_t^2}{4}|(\phi_4^m)_t|^2dx+\varepsilon^2\int_{\mathbb{R}^2}\left(\frac{(v_0^2)_t}{8}(\overline{\phi}_4^m)_t^2+\frac{(\overline{v}_0^2)_t}{8}(\phi_4^m)_t^2\right)dx \qquad (3.26)\\ &\leq C(M)\left(||\phi_4||_2^2+\sum_{k=1}^{2}\left(||\partial_k\phi_4^k||_2^2+||\phi_4^k||_2^2\right)+||(\phi_4^m)_t||_2^2\right).\end{aligned}$$

Since $\phi_4=(1-\Delta)u_4$, it is not difficult to derive from (3.20)

$$||\phi_4||_2^2\leq C(M)\left(\sum_{k=1}^{2}\left(||\partial_k\phi_4^k||_2^2+||\phi_4^k||_2^2\right)+||(\phi_4^m)_t||_2^2\right).$$

Then, integrating inequality (3.26) from 0 to t, one obtains

$$
\begin{aligned}
&\frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}|(\phi_4^m(t))_t|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla\phi_4^m(t)|^2dx+\frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}\frac{|v_0(t)|^2}{2}|(\phi_4^m(t))_t|^2dx\\
&+\frac{1}{2}\int_{\mathbb{R}^2}|\phi_4^m|^2dx-\frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}\left(\frac{v_0^2(t)}{4}(\overline{\phi}_4^m(t))_t^2-\frac{\overline{v}_0^2(t)}{4}(\phi_4^m(t))_t^2\right)dx\\
&\leq \varepsilon^2\int_{\mathbb{R}^2}\frac{(2+|v_0(0)|^2)}{4}|(\phi_4^m)_t(0)|^2dx-\varepsilon^2\int_{\mathbb{R}^2}\left(\frac{v_0^2(0)}{8}(\overline{\phi}_4^m)_t^2(0)+\frac{\overline{v}_0^2(0)}{8}(\phi_4^m)_t^2(0)\right)dx\\
&+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla\phi_4^m(0)|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\phi_4^m(0)|^2dx+\varepsilon^2\int_0^t\int_{\mathbb{R}^2}\frac{(|v_0(s)|^2)_s}{4}|(\phi_4^m(s))_s|^2dxds\\
&+\varepsilon^2\int_0^t\int_{\mathbb{R}^2}\left(\frac{|(v_0^2(s))_s|}{8}|(\overline{\phi}_4^m(s))_s^2|+\frac{|(\overline{v}_0^2(s))_s|}{8}|(\phi_4^m(s))_s^2|\right)dxds\\
&+C(M)\int_0^t\left(\sum_{k=1}^2\left(||\partial_k\phi_4^k(s)||_2^2+||\phi_4^k(s)||_2^2\right)+||(\phi_4^m)_s(s)||_2^2\right)ds.
\end{aligned}
\tag{3.27}
$$

Using the fact for all $s\in[0,t]$, $\frac{|v_0(s)|^2}{2}|(\phi_4(s))_s|^2-\frac{v_0^2(s)}{4}(\overline{\phi}_4^m(s))_s^2-\frac{\overline{v}_0^2(s)}{4}(\phi_4^m(s))_s^2\geq 0$, that $||(u_0)_t||_{L^\infty(\mathbb{R}^2)}\leq r$, and denoting

$$
\begin{aligned}
f^m(0)=&\frac{1}{2}\int_{\mathbb{R}^2}|\nabla_\varepsilon\phi_4^m(0)|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\phi_4^m(0)|^2dx\\
&+\varepsilon^2\int_{\mathbb{R}^2}\frac{(2+|v_0(0)|^2)}{4}|(\phi_4^m)_t(0)|^2dx\\
&-\varepsilon^2\int_{\mathbb{R}^2}\left(\frac{v_0^2(0)}{8}(\overline{\phi}_4^m)_t^2(0)+\frac{\overline{v}_0^2(0)}{8}(\phi_4^m)_t^2(0)\right)dx,
\end{aligned}
$$

we obtain from (3.27)

$$
\begin{aligned}
&\sum_{m=1}^2\left(\frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}|(\phi_4^m(t))_t|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla\phi_4^m(t)|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\phi_4^m(t)|^2dx\right)\\
&\leq\sum_{m=1}^2f^m(0)+C(M,r)\sum_{m=1}^2\int_0^t\left(\sum_{k=1}^2\left(||\partial_k\phi_4^m(s)||_2^2+||\phi_4^k(s)||_2^2\right)+||(\phi_4^m(s))_s||_2^2\right)ds.
\end{aligned}
\tag{3.28}
$$

This implies

$$
\begin{aligned}
&\sum_{m=1}^2\left(\frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}|(\phi_4^m(t))_t|^2dx+\sum_{k=1}^2\int_{\mathbb{R}^2}|\partial_k\phi_4^m(t)|^2dx+\int_{\mathbb{R}^2}|\phi_4^m(t)|^2dx\right)\\
&\leq\sum_{m=1}^2f^m(0)+C(M,r)\sum_{m=1}^2\int_0^t\left(\sum_{k=1}^2\left(||\partial_k\phi_4^m(s)||_2^2+||\phi_4^k(s)||_2^2\right)+||(\phi_4^m(s))_s||_2^2\right)ds.
\end{aligned}
\tag{3.29}
$$

Applying Gronwall lemma to (3.29), we get

$$\begin{aligned}&\sup_{t\in[0,T]}\sum_{m=1}^{2}\left(\frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}|(\phi_4^m(t))_t|^2dx+\sum_{k=1}^{2}\int_{\mathbb{R}^2}|\partial_k\phi_4^m(t)|^2dx+\int_{\mathbb{R}^2}|\phi_4^m(t)|^2dx\right)\\&\leq e^{C(M,r)\frac{T}{\varepsilon^2}}\sum_{m=1}^{2}f^m(0).\end{aligned}\tag{3.30}$$

Then, choosing $C(M,r)T\leq 1$ and $u_4(0)$ such that

$$\sum_{m=1}^{2}f^m(0)\leq e^{-\frac{T}{\varepsilon^2}}m_4\tag{3.31}$$

gives

$$\sup_{t\in[0,T]}||u_4(t)||_{H^4(\mathbb{R}^2)}\leq m_4.\tag{3.32}$$

It is important to notice here that (3.31) implies that

$$||u_4(\cdot,0)||_{H^4(\mathbb{R}^2)}\leq e^{-\frac{T}{\varepsilon^2}}m_4.$$

Furthermore, we also have

$$\sup_{t\in[0,T]}\varepsilon||\partial_t u_4(t)||_{H^3(\mathbb{R}^2)}\leq m_4.\tag{3.33}$$

We now study equation (3.21) satisfied by u_5. This equation is of the same type as equation (3.20) and it will be treated with the same method; we multiply (3.21) by $\mathcal{A}(v_0)$ and we apply the operator $\partial_m(1-\Delta)$ for $m=1,2$. Denoting $\phi_5=(1-\Delta)u_5$ and $\phi_5^m=\partial_m(1-\Delta)u_5$, we obtain

$$\begin{aligned}&-\varepsilon^2\begin{pmatrix}(\phi_5^m)_{tt}\\(\overline{\phi}_5^m)_{tt}\end{pmatrix}+C(v_0,\Delta^{-1}v_4)\begin{pmatrix}(\phi_5^m)_t\\(\overline{\phi}_5^m)_t\end{pmatrix}+\mathcal{A}(u_0)\begin{pmatrix}\Delta\phi_5^m\\\Delta\overline{\phi}_5^m\end{pmatrix}+\sum_{k=1}^{2}\mathcal{L}_k(v_0,v_k,\partial_kv_0)\begin{pmatrix}\partial_k\phi_5^m\\\partial_k\overline{\phi}_5^m\end{pmatrix}\\&+\sum_{k=1}^{2}\partial_m\mathcal{A}(v_0)\begin{pmatrix}\partial_k\phi_5^k\\\partial_k\overline{\phi}_5^k\end{pmatrix}+\sum_{|\alpha|\leq3}\mathcal{R}_\alpha(D^{h_1}V,D^{h_2}Tv_3)\begin{pmatrix}\partial_1^{\alpha_1}\partial_2^{\alpha_2}u_5\\\partial_1^{\alpha_1}\partial_2^{\alpha_2}\overline{u}_5\end{pmatrix}\\&+\sum_{|\alpha|\leq3}\sum_{j,k=0}^{2}\mathcal{T}_\alpha(D^{h_2+1}V,D^{h_2}Tv_3)\begin{pmatrix}\partial_1^{\alpha_1}\partial_2^{\alpha_2}R_{jk}u_5\\\partial_1^{\alpha_1}\partial_2^{\alpha_2}R_{jk}\overline{u}_5\end{pmatrix}\\&+\sum_{|\alpha|\leq3}\sum_{j,k=0}^{2}\mathcal{U}_\alpha(D^{h_2+1}V,D^{h_2}Tv_3)\begin{pmatrix}\partial_1^{\alpha_1}\partial_2^{\alpha_2}R_{jk}u_4\\\partial_1^{\alpha_1}\partial_2^{\alpha_2}R_{jk}\overline{u}_4\end{pmatrix}\\&+\sum_{|\alpha|\leq2}\mathcal{V}_\alpha(D^{h_2}V,D^{h_2}\Delta^{-1}u_4)\begin{pmatrix}\partial_1^{\alpha_1}\partial_2^{\alpha_2}(u_5)_t\\\partial_1^{\alpha_1}\partial_2^{\alpha_2}(\overline{u}_5)_t\end{pmatrix}=0.\end{aligned}\tag{3.34}$$

We now multiply equation (3.34) by $\mathcal{A}^{-1}(v_0)$, and then we take the real part of the first line of the equation multiplied by $(\overline{\phi}_5^m)_t$. The difference with equation (3.25) is due to the presence of the terms

$$\sum_{|\alpha|\leq 3}\sum_{j,k=0}^{2}\mathcal{U}_\alpha(D^{h_2+1}V, D^{h_2}Tv_3)\begin{pmatrix}\partial_1^{\alpha_1}\partial_2^{\alpha_2}R_{jk}u_4\\ \partial_1^{\alpha_1}\partial_2^{\alpha_2}R_{jk}\overline{u}_4\end{pmatrix}$$

and

$$\sum_{|\alpha|\leq 2}\mathcal{V}_\alpha(D^{h_2}V, D^{h_2}\Delta^{-1}u_4)\begin{pmatrix}\partial_1^{\alpha_1}\partial_2^{\alpha_2}(u_5)_t\\ \partial_1^{\alpha_1}\partial_2^{\alpha_2}(\overline{u}_5)_t\end{pmatrix}.$$

Anyway, these terms do not yield any new difficulty. Indeed, each term of the first sum above can be estimated in the following way

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\mathcal{A}^{-1}(v_0)\mathcal{U}_\alpha(D^{h_2+1}V, D^{h_2}Tv_3)\partial_1^\alpha\partial_2^\beta R_{jk}u_4(\overline{\phi}_5^m)_t dx\right|\\ &\leq C(M)\left(||\phi_4||_2^2+\sum_{k=1}^{2}(||\phi_4^k||_2^2+||\partial_k\phi_4^k||_2^2)+||(\phi_5^m)_t||_2^2\right)\end{aligned}$$

and the terms of the second sum

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\mathcal{V}_\alpha(D^{h_2}V, D^{h_2}\Delta^{-1}u_4)\partial_1^\alpha\partial_2^\beta(u_5)_t(\overline{\phi}_5^m)_t dx\right|\\ &\leq C(M)\left(||(\phi_5)_t||_2^2+\sum_{k=1}^{2}||(\phi_5^k)_t||_2^2+||(\phi_5^m)_t||_2^2\right).\end{aligned}$$

It is then possible to prove, using the same estimate as above, that there exists a positive $C(M,r)$ such that denoting

$$\begin{aligned}g^m(0)=&\frac{1}{2}\int_{\mathbb{R}^2}|\nabla\phi_5^m(0)|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\phi_5^m(0)|^2dx\\ &+\varepsilon^2\int_{\mathbb{R}^2}\frac{(2+|v_0(0)|^2)}{4}|(\phi_5^m)_t(0)|^2dx\\ &-\varepsilon^2\int_{\mathbb{R}^2}\left(\frac{v_0^2(0)}{8}(\overline{\phi}_5^m)_t^2(0)+\frac{\overline{v}_0^2(0)}{8}(\phi_5^m)_t^2(0)\right)dx.\end{aligned}$$

we have

$$\sum_{m=1}^{2}\left(\frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}|(\phi_5^m(t))_t|^2dx+\frac{1}{2}\sum_{k=1}^{2}\int_{\mathbb{R}^2}|\partial_k\phi_5^m(t)|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\phi_5^m(t)|^2dx\right)$$
$$\leq \sum_{m=1}^{2}g^m(0)+\int_0^t\sum_{m=1}^{2}\left(||\phi_4^m(s)||_2^2+\sum_{k=1}^{2}||\partial_k\phi_4^4(s)||_2^2\right)ds$$
$$+C(M,r)\int_0^t\sum_{m=1}^{2}\left(\frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}|(\phi_5^m(s))_s|^2dx+\frac{1}{2}\sum_{k=1}^{2}\int_{\mathbb{R}^2}|\partial_k\phi_5^m(s)|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\phi_5^m(s)|^2dx\right).$$

Then, using again a Gronwall lemma, we obtain

$$\sup_{t\in[0,T]}||u_5(t)||_{H^4(\mathbb{R}^2)}\leq e^{C(M,r)\frac{T}{\varepsilon^2}}\sum_{m=1}^{2}(g^m(0)+Tf^m(0)). \tag{3.35}$$

Finally, if $C(M,r)T\leq 1$ and

$$\sum_{m=1}^{2}g^m(0)\leq e^{-\frac{T}{\varepsilon^2}}m_5-T\sum_{m=1}^{2}f^m(0)\quad\text{and}\quad\sum_{m=1}^{2}f^m(0)\leq\frac{e^{-\frac{T}{\varepsilon^2}}}{T}m_5, \tag{3.36}$$

we obtain

$$\sup_{t\in[0,T]}||u_5(t)||_{H^4(\mathbb{R}^2)}\leq m_5. \tag{3.37}$$

We can notice that (3.36) implies that

$$||u_5(\cdot,0)||_{H^4(\mathbb{R}^2)}\leq e^{-\frac{T}{\varepsilon^2}}m_5.$$

We have also proved

$$\sup_{t\in[0,T]}\varepsilon||\partial_tu_5(t)||_{H^3(\mathbb{R}^2)}\leq m_5. \tag{3.38}$$

In order to obtain an estimate on u_3, we now work on equation (3.19). As we have already mentionned in the introduction, (3.19) can be considered as a Schrödinger equation. Indeed, since no terms of the form $(u_3)_{tt}$ appear in (3.19), it is possible to estimate u_3 in the same way as we did in chapter 2 to estimate (2.17). For this purpose, we set $\chi_3=(1-\Delta)u_3$ and we first apply the operateur $(1-\Delta)$ to (3.19)

$$\begin{aligned}&2i\begin{pmatrix}(\chi_3)_t\\(\overline{\chi}_3)_t\end{pmatrix}+\mathcal{A}(v_0)\begin{pmatrix}\Delta\chi_3\\\Delta\overline{\chi}_3\end{pmatrix}+\sum_{k=1}^{2}H(v_0,v_k,\partial_kv_0)\begin{pmatrix}\partial_k\chi_3\\\partial_k\overline{\chi}_3\end{pmatrix}\\&-\varepsilon^2\begin{pmatrix}(1-\Delta)v_5\\(1-\Delta)\overline{v}_5\end{pmatrix}+K(D^2V,D^2u_3,D^2Tv_3D^2\Delta^{-1}v_4)=0\end{aligned} \tag{3.39}$$

where

$$H(v_0, v_k, \partial_k) = \mathcal{E}(v_0, v_k) + 2\partial_k \mathcal{A}(v_0),$$

$$\begin{aligned}
K(D^2V, D^2u_3, D^2Tv_3, D^2\Delta^{-1}v_4) =& \mathcal{I}(V, Tv_3, \Delta^{-1}v_4) - 2\sum_{k=1}^{2} \partial_k \mathcal{A}(v_0) \begin{pmatrix} \partial_k u_3 \\ \partial_k \overline{u}_3 \end{pmatrix} \\
& - \sum_{j=1}^{2} \left(\partial_j^2 \mathcal{I}(V, Tv_3, \Delta^{-1}v_4) + \partial_j^2 \mathcal{A}(v_0) \begin{pmatrix} \Delta u_3 \\ \Delta \overline{u}_3 \end{pmatrix} \right) \\
& - 2\sum_{j=1}^{2}\sum_{k=1}^{2} \partial_j \mathcal{E}(v_0, v_k) \begin{pmatrix} \partial_k \partial_j u_3 \\ \partial_k \partial_j \overline{u}_3 \end{pmatrix} \\
& - \sum_{j=1}^{2}\sum_{k=1}^{2} \partial_j^2 \mathcal{E}(v_0, v_k) \begin{pmatrix} \partial_k u_3 \\ \partial_k u_3 \end{pmatrix}
\end{aligned}$$

Applying again the operator $(1-\Delta)$ to (3.39) and setting $\phi_3 = (1-\Delta)\chi_3$, we get

$$\begin{aligned}
& 2i \begin{pmatrix} (\phi_3)_t \\ (\overline{\phi}_3)_t \end{pmatrix} + \mathcal{A}(v_0) \begin{pmatrix} \Delta\phi_3 \\ \Delta\overline{\phi}_3 \end{pmatrix} + \sum_{k=1}^{2} \mathcal{W}(v_0, v_k, \partial_k v_0) \begin{pmatrix} \partial_k \phi_3 \\ \partial_k \overline{\phi}_3 \end{pmatrix} \\
& -\varepsilon^2 \begin{pmatrix} (1-\Delta)^2 v_5 \\ (1-\Delta)^2 \overline{v}_5 \end{pmatrix} + \mathcal{X}(D^4V, D^4u_3, D^4Tv_3, D^4\Delta^{-1}v_4) = 0
\end{aligned} \tag{3.40}$$

where

$$\mathcal{W}(v_0, v_k, \partial_k) = H(v_0, v_k) + 2\partial_k \mathcal{A}(v_0),$$

$$\begin{aligned}
& \mathcal{X}(D^4V, D^4u_3, D^4Tv_3, D^4\Delta^{-1}v_4) \\
& = K(D^2V, D^2u_3, D^2Tv_3, D^2\Delta^{-1}v_4) - 2\sum_{k=1}^{2} \partial_k \mathcal{A}(v_0) \begin{pmatrix} \partial_k \chi_3 \\ \partial_k \overline{\chi}_3 \end{pmatrix} \\
& - \sum_{j=1}^{2} \left(\partial_j^2 K(D^2V, D^2u_3, D^2Tv_3, D^2\Delta^{-1}v_4) + \partial_j^2 \mathcal{A}(v_0) \begin{pmatrix} \Delta\chi_3 \\ \Delta\overline{\chi}_3 \end{pmatrix} \right) \\
& - 2\sum_{j=1}^{2}\sum_{k=1}^{2} \partial_j H(v_0, v_k, \partial_k v_0) \begin{pmatrix} \partial_j \partial_k \chi_3 \\ \partial_j \partial_k \overline{\chi}_3 \end{pmatrix} \\
& - \sum_{j=1}^{2}\sum_{k=1}^{2} \partial_j^2 H(v_0, v_k, \partial_k v_0) \begin{pmatrix} \partial_k \chi_3 \\ \partial_k \overline{\chi}_3 \end{pmatrix}
\end{aligned}$$

Following the energy method intoduced in [11], we first choose q such that $\mathrm{Re}(\mathcal{E}^{11}(v_0, v_k)) = 0$ (see section 2.3). The energy method supposes to integrate by parts the terms involving the first space derivatives of ϕ_3. This is why we need the diagonal terms of the matrices

$\mathcal{W}$ to be purely imaginary. For this purpose, we have to use a second gauge transform (see Chapter 2 for more details). Following this idea, we denote $\psi_3 = e^{p(v_0)}\phi_3$ and we derive from (3.40)

$$\begin{aligned}
&2i\begin{pmatrix} (\psi_3)_t \\ (\overline{\psi}_3)_t \end{pmatrix} + \mathcal{A}(v_0)\begin{pmatrix} \Delta\psi_3 \\ \Delta\overline{\psi}_3 \end{pmatrix} + \sum_{k=1}^{2}\mathcal{Y}(v_0, v_k, \partial_k v_0)\begin{pmatrix} \partial_k\psi_3 \\ \partial_k\overline{\psi}_3 \end{pmatrix} \\
&-\varepsilon^2\begin{pmatrix} (1-\Delta)^2 v_5 \\ (1-\Delta)^2\overline{v}_5 \end{pmatrix} + \widetilde{\mathcal{X}}(D^4V, D^5u_3, D^4Tv_3, D^4\Delta^{-1}v_4, (v_0)_t) = 0
\end{aligned} \tag{3.41}$$

where

$$\mathcal{Y}(v_0, v_k, \partial_k v_0) = \mathcal{W}(v_0, v_k, \partial_k v_0) - 2\begin{pmatrix} \partial_k p(v_0) & 0 \\ 0 & \partial_k p(v_0) \end{pmatrix}$$

and

$$\begin{aligned}
\widetilde{\mathcal{X}}(D^4V, D^4u_3, D^4Tv_3, D^4\Delta^{-1}v_4, (v_0)_t) =&\mathcal{X}(D^4V, D^4u_4, D^4Tv_3, D^4\Delta^{-1}v_4) \\
&+ 2i(e^{-p(v_0)})_t\begin{pmatrix} \psi_3 \\ \overline{\psi}_3 \end{pmatrix} + \mathcal{A}(v_0)(\Delta e^{-p(v_0)})\begin{pmatrix} \psi_3 \\ \overline{\psi}_3 \end{pmatrix} \\
&+ \sum_{k=1}^{2}\mathcal{W}(v_0, v_k, \partial_k v_0)\partial_k(e^{-p(v_0)})\begin{pmatrix} \phi_3 \\ \overline{\phi}_3 \end{pmatrix}.
\end{aligned}$$

Once again, an easy calculation shows that there exists a function p such that

$$\mathrm{Re}^{11}\left(\mathcal{Y}(v_0, v_k, \partial_k v_0)\right) = 0.$$

With such q and p, it is possible to make the energy estimates of section 2.3. We multiply (3.41) by $\mathcal{A}^{-1}(v_0)$ and we integrate over $\mathbb{R}^2$ the first line after having multiplied by $\overline{\psi}_3$

$$\begin{aligned}
&i\int_{\mathbb{R}^2}(2+|v_0|^2)(\psi_3)_t\overline{\psi}_3 dx - i\int_{\mathbb{R}^2} v_0^2(\overline{\psi}_3)_t\overline{\psi}_3 dx + \int_{\mathbb{R}^2}\Delta\psi_3\overline{\psi}_3 dx \\
&+\sum_{k=1}^{2}\int_{\mathbb{R}^2}\mathcal{Y}^{11}(v_0, v_k, \partial_k v_0)(\partial_k\psi_3)\overline{\psi}_3 dx + \sum_{k=1}^{2}\int_{\mathbb{R}^2}\mathcal{Y}^{12}(v_0, v_k, \partial_k v_0)(\partial_k\overline{\psi}_3)\overline{\psi}_3 dx \\
&+\int_{\mathbb{R}^2}\widetilde{\mathcal{X}}^1(D^4V, D^4u_3, D^4Tv_3, D^4\Delta^{-1}v_4, (v_0)_t)\overline{\psi}_3 dx - \varepsilon^2\int_{\mathbb{R}^2}\frac{(2+|v_0|^2)}{2}(1-\Delta)^2 v_5\overline{\psi}_3 dx \\
&-\varepsilon^2\int_{\mathbb{R}^2}\frac{v_0^2}{2}(1-\Delta)^2\overline{v}_5\overline{\psi}_3 dx = 0
\end{aligned} \tag{3.42}$$

where all the matrices $\mathcal{Y}$, $\mathcal{V}$ and $\mathcal{X}$ have been multiplied by $\mathcal{A}^{-1}(v_0)$. We take the imaginary part of equation (3.42).

$$\begin{aligned}
&\mathrm{Im}\left(i\int_{\mathbb{R}^2}(2+|v_0|^2)(\psi_3)_t\overline{\psi}_3dx - i\int_{\mathbb{R}^2}v_0^2(\overline{\psi}_3)_t\overline{\psi}_3dx\right)\\
&=\int_{\mathbb{R}^2}\left(\frac{2+|v_0|^2}{2}\right)|\psi_3|_t^2 - \int_{\mathbb{R}^2}\left(\frac{v_0^2}{4}(\overline{\psi}_3^2)_t + \frac{\overline{v_0}^2}{4}(\psi_3^2)_t\right)dx\\
&=\frac{d}{dt}\left(\int_{\mathbb{R}^2}\left(\frac{2+|v_0|^2}{2}\right)|\psi_3|^2 - \int_{\mathbb{R}^2}\left(\frac{v_0^2}{4}(\overline{\psi_3}^2) + \frac{\overline{v_0}^2}{4}(\psi_3^2)\right)dx\right)\\
&-\int_{\mathbb{R}^2}\frac{|v_0|_t^2}{2}|\psi_3|^2dx + \int_{\mathbb{R}^2}\left(\frac{(v_0^2)_t}{4}(\overline{\psi}_3^2) + \frac{(\overline{v_0}^2)_t}{4}(\psi_3^2)\right)dx.
\end{aligned}$$

$$\mathrm{Im}\left(\int_{\mathbb{R}^2}\Delta\psi_3\overline{\psi}_3dx\right) = -\mathrm{Im}\left(\int_{\mathbb{R}^2}|\nabla\psi_3|^2dx\right) = 0.$$

Since $\mathcal{Y}^{11}(v_0, v_k, \partial_k v_0) \in i\mathbb{R}$, we have

$$\begin{aligned}
\mathrm{Im}\left(\sum_{k=1}^2\int_{\mathbb{R}^2}\mathcal{Y}^{11}(v_0, v_k, \partial_k v_0)(\partial_k\psi_3)\overline{\psi}_3dx\right) &= \sum_{k=1}^2\int_{\mathbb{R}^2}\mathrm{Im}(\mathcal{Y}^{11}(v_0, v_k, \partial_k v_0))\partial_k\frac{|\psi_3|^2}{2}dx\\
&= -\sum_{k=1}^2\int_{\mathbb{R}^2}\partial_k(\mathrm{Im}(\mathcal{Y}^{11}(v_0, v_k, \partial_k v_0)))\frac{|\psi_3|^2}{2}dx
\end{aligned}$$

Using the continuous embedding of $H^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$, we obtain

$$\left|\mathrm{Im}\left(\sum_{k=1}^2\int_{\mathbb{R}^2}\mathcal{Y}^{11}(v_0, v_k, \partial_k v_0)(\partial_k\psi_3)\overline{\psi}_3dx\right)\right| \le C(M)||\psi_3||_2^2.$$

Since $\partial_k\overline{\psi}_3\overline{\psi}_3 = \frac{1}{2}\partial_k(\overline{\psi}_3^2)$ an integration by parts gives directly

$$\left|\mathrm{Im}\left(\sum_{k=1}^2\int_{\mathbb{R}^2}\mathcal{Y}^{12}(v_0, v_k, \partial_k v_0)(\partial_k\overline{\psi}_3)\overline{\psi}_3dx\right)\right| \le C(M)||\psi_3||_2^2.$$

The last terms do not involve space derivatives of ϕ_3. As a consequence, they can be estimated directly by the Cauchy-Schwarz inequality and by the preceding continuous embedding. The only thing to notice is that since $V \in X_T(M, r)$

$$\int_{\mathbb{R}^2}|(1-\Delta)^2 v_5|^2dx \le m_5.$$

Thus, recalling that $||(v_0)_t||_{H^2(\mathbb{R}^2)} \le r$ and using the same computations we did in section 2.3, one can find two constants $C(M, r)$ and $C(M)$ such that

$$\left|\mathrm{Im}\left(\int_{\mathbb{R}^3}\tilde{\mathcal{X}}^1(D^4V, D^4u_3, D^4Tv_3, D^4\Delta^{-1}v_4, (v_0)_t)\overline{\psi}_3dx\right)\right| \le C(M, r)||\psi_3||_2,$$

$$\left|\mathrm{Im}\left(\varepsilon^2\left(\int_{\mathbb{R}^3}\frac{(2+|v_0|^2)}{2}(1-\Delta)^3 v_5\overline{\psi}_3 dx-\int_{\mathbb{R}^2}\frac{v_0^2}{2}(1-\Delta)^2\overline{v}_5\overline{\psi}_3 dx\right)\right)\right|\le C(M)||\psi_3||_2.$$

Collecting all these inequalities, we derive from (3.42)

$$\frac{d}{dt}\left(\int_{\mathbb{R}^2}\left(\frac{2+|v_0|^2}{2}\right)|\psi_3|^2-\int_{\mathbb{R}^2}\left(\frac{v_0^2}{4}(\overline{\psi}_3^2)+\frac{\overline{v_0}^2}{4}(\psi_3^2)\right)dx\right)$$

$$\le\int_{\mathbb{R}^2}\frac{|v_0|_t^2}{2}|\psi_3|^2dx-\int_{\mathbb{R}^2}\left(\frac{(v_0^2)_t}{4}(\overline{\psi}_3^2)+\frac{(\overline{v_0}^2)_t}{4}(\psi_2^2)\right)dx.$$

$$+C(M)||\psi_3||_2^2+C(M,r)||\psi_3||_2.$$

Integrating this last inequality from 0 to t, one obtains keeping in view that $||(v_0)_t||_{L^\infty(\mathbb{R}^2)}\le r$,

$$\int_{\mathbb{R}^2}\left(\frac{2+|v_0(t)|^2}{2}\right)|\psi_3(t)|^2-\int_{\mathbb{R}^2}\left(\frac{v_0^2(t)}{4}(\overline{\psi}_3^2(t))+\frac{\overline{v_0}^2(t)}{4}(\psi_3^2(t))\right)dx$$
$$\le\int_{\mathbb{R}^2}\left(\frac{2+|v_0(0)|^2}{2}\right)|\psi_3(0)|^2-\int_{\mathbb{R}^2}\left(\frac{v_0^2(0)}{4}(\overline{\psi}_3^2(0))+\frac{\overline{v_0}^2(0)}{4}(\psi_3^2(0))\right)dx$$
$$+C(M,r)T\left(\sup_{t\in[0,T]}||\psi_3(t)||_2^2+1\right).$$

Since for all $t\in[0,T]$,

$$\int_{\mathbb{R}^2}\frac{|v_0(t)|^2}{2}|\psi_3(t)|^2-\int_{\mathbb{R}^2}\left(\frac{v_0^2(t)}{4}(\overline{\psi}_3^2(t))+\frac{\overline{v_0}^2(t)}{4}(\psi_3^2(t))\right)dx\ge 0,$$

we obtain

$$\sup_{t\in[0,T]}||\psi_3(t)||_2^2\le\int_{\mathbb{R}^2}(1+|v_0(0)|^2)|\psi_3(0)|^2dx+C(M,r)T\left(\sup_{t\in[0,T]}||\psi_3(t)||_2^2+1\right),$$

which implies if $C(M,r)T\le\frac{1}{2}$ that

$$\sup_{t\in[0,T]}||\psi_3(t)||_2^2\le 2\int_{\mathbb{R}^2}(1+|v_0(0)|^2)|\psi_3(0)|^2dx+1.$$

Recalling that p only depends on v_0, there is a constant $C(m_0)$ depending only on m_0 such that

$$||e^{-p(v_0)}||^2_{L^\infty(0,T;H^4(\mathbb{R}^2))}\le C(m_0).$$

Then, since $\phi_3=e^{-p(v_0)}\psi_3$, one has

$$\sup_{t\in[0,T]}||\phi_3(t)||_2^2\le C(m_0)\sup_{t\in[0,T]}||\psi_3(t)||_2^2$$
$$\le 2C(m_0)\int_{\mathbb{R}^2}(1+|v_0(0)|^2)|\psi_3(0)|^2dx+C(m_0).$$

Choosing m_3 such that

$$m_3^2 \geq 4C(m_0)\int_{\mathbb{R}^2}(1+|v_0(0)|^2)|e^{p(v_0(0))}\phi_3(0)|^2 dx + 1, \tag{3.43}$$

gives finally

$$\sup_{t\in[0,T]} ||\phi_3||^2_{H^4(\mathbb{R}^2)} \leq m_3^2. \tag{3.44}$$

We now have to prove that there exists a time T and constant m_0, m_1 and m_2 such that for all $j = 0, .., 2$

$$\sup_{t\in[0,T]} ||u_j(t)||_{H^4(\mathbb{R}^2)} \leq m_j, \tag{3.45}$$

which is done exactly as in Section 2.3.

It remains to estimate $(u_0)_t$ in $H^2(\mathbb{R}^2)$ which is made, as in Section 2.3, by applying the operator $(1-\Delta)$ to (3.17) and by doing a direct estimate on the resulting equation. Indeed, doing so, we get

$$\sup_{t\in[0,T]} ||(u_0)_t||_{H^2(\mathbb{R}^2)} \leq C(M).$$

It is important to notice that this constant $C(M)$ does not depend on T if we make the estimate before estimating u_3, u_2, u_1 and u_0. Doing so, it is then enough to choose $r \geq C(M)$ to obtain

$$\sup_{t\in[0,T]} ||(u_0)_t||_{H^2(\mathbb{R}^2)} \leq r. \tag{3.46}$$

In conclusion, we have proved that $\mathcal{S}(X_T(M,r)) \subset X_T(M,r)$.

We now show that the mapping $\mathcal{S}$ is a contraction mapping in the ball $X_T(M,r)$ endowed with the metric of Y_T where

$$Y_T = \left\{ \begin{array}{l} U = (u_0, ..., u_5) : u_j \in C([0,T]; L^2(\mathbb{R}^2)),\ \ j = 0, .., 5 \\ ||U||_{Y_T} = \displaystyle\sum_{j=0}^{5} \sup_{t\in[0,T]} ||u_j(t)||_2 < \infty \end{array} \right\}.$$

Let $V^1 = (v_0^1, ..., v_5^1) \in X_T(M,r)$ and $V^2 = (v_0^2, ..., v_5^2) \in X_T(M,r)$. We put

$$U^1 = (u_0^1, ..., u_5^1) = \mathcal{S}V_1$$

and

$$U^2 = (u_0^2, ..., u_5^2) = \mathcal{S}V_2,$$

the solutions of (3.17), (3.18), (3.19), (3.20) and (3.21) with initial condition (3.22). For $j = 0, ..., 5$, we put $w_j = u_j^1 - u_j^2$. The equation satisfied by w_4 is

$$
\begin{aligned}
&-\varepsilon^2 \begin{pmatrix} (w_4)_{tt} \\ (\overline{w}_4)_{tt} \end{pmatrix} + 2i \begin{pmatrix} (w_4)_t \\ (\overline{w}_4)_t \end{pmatrix} + \mathcal{A}(v_0^1) \begin{pmatrix} \Delta w_4 \\ \Delta \overline{w}_4 \end{pmatrix} + \left(\mathcal{A}(v_0^1) - \mathcal{A}(v_0^2)\right) \begin{pmatrix} \Delta u_4^2 \\ \Delta \overline{u}_4^2 \end{pmatrix} \\
&+ \sum_{k=1}^{2} \mathcal{D}_k(v_0^1, v_k^1) \begin{pmatrix} \partial_k w_4 \\ \partial_k \overline{w}_4 \end{pmatrix} + \sum_{k=1}^{2} \left(\mathcal{D}_k(v_0^1, v_k^1) - \mathcal{D}_k(v_0^2, v_k^2)\right) \begin{pmatrix} \partial_k u_4^2 \\ \partial_k \overline{u}_4^2 \end{pmatrix} \\
&+ \sum_{j,k=0}^{2} \mathcal{H}_{jk}(DV^1, Tv_3^1) \begin{pmatrix} R_{jk} w_4 \\ R_{jk} \overline{w}_4 \end{pmatrix} + \sum_{j,k=0}^{2} \left(\mathcal{H}_{jk}(DV^1, Tv_3^1) - \mathcal{H}_{jk}(DV^2, Tv_4^2)\right) \begin{pmatrix} R_{jk} u_4^2 \\ R_{jk} \overline{u}_4^2 \end{pmatrix} \\
&+ \mathcal{G}(DV^1, Tv_3^1) \begin{pmatrix} w_4 \\ \overline{w}_4 \end{pmatrix} + \left(\mathcal{G}(DV^1, Tv_3^1) - \mathcal{G}(DV^2, Tv_3^2)\right) \begin{pmatrix} u_4^2 \\ \overline{u}_4^2 \end{pmatrix} = 0
\end{aligned}
\tag{3.47}
$$

Treating (3.47) exactly as we have treated (3.20) and using the fact that the matrices $\mathcal{D}$, $\mathcal{H}$ and $\mathcal{G}$ are of class C^1 with respect to their arguments, we easily get if T is small enough

$$
\sup_{t\in[0,T]} ||w_4||_2 \leq \frac{1}{2} \sup_{t\in[0,T]} \sum_{j=0}^{5} ||v_j^1 - v_j^2||_2. \tag{3.48}
$$

Using the same kind of arguments, it is clear that the following is also true

$$
\sup_{t\in[0,T]} ||w_5||_2 \leq \frac{1}{2} \sup_{t\in[0,T]} \sum_{j=0}^{5} ||v_j^1 - v_j^2||_2. \tag{3.49}
$$

Next we write the equation involving w_3

$$
\begin{aligned}
&2i \begin{pmatrix} (w_3)_t \\ (\overline{w}_3)_t \end{pmatrix} + \mathcal{A}(v_0^1) \begin{pmatrix} \Delta w_3 \\ \Delta \overline{w}_3 \end{pmatrix} + \left(\mathcal{A}(v_0^1) - \mathcal{A}(v_0^2)\right) \begin{pmatrix} \Delta u_3^2 \\ \Delta \overline{u}_3^2 \end{pmatrix} \\
&+ \sum_{k=1}^{2} \mathcal{E}(v_0^1, v_k^1) \begin{pmatrix} \partial_k w_3 \\ \partial_k \overline{w}_3 \end{pmatrix} + \sum_{k=1}^{2} \left(\mathcal{E}(v_0^1, v_k^1) - \mathcal{E}(v_0^2, v_k^2)\right) \begin{pmatrix} \partial_k u_3^2 \\ \partial_k \overline{u}_3^2 \end{pmatrix} \\
&-\varepsilon^2 \begin{pmatrix} w_5 \\ \overline{w}_5 \end{pmatrix} + \mathcal{I}(V^1, Tv_3^1, \Delta^{-1} v_4) - \mathcal{I}(V^2, Tv_3^2, \Delta^{-1} v_4) = 0
\end{aligned}
\tag{3.50}
$$

and we treat it as we have treated (3.19). We then obtain

$$
\sup_{t\in[0,T]} ||w_3||_2 \leq \frac{1}{2} \sup_{t\in[0,T]} \sum_{j=0}^{5} ||v_j^1 - v_j^2||_2 \tag{3.51}
$$

from which follows that for all $j = 0, .., 2$

$$
\sup_{t\in[0,T]} ||w_j||_2 \leq \frac{1}{2} \sup_{t\in[0,T]} \sum_{j=0}^{5} ||v_j^1 - v_j^2||_2. \tag{3.52}
$$

Then the mapping $\mathcal{S}$ is a contraction mapping on $X_T(M, r)$ in the norm $|| \cdot ||_{Y_T}$. The ball $X_T(M, r)$, endowed with the metric of Y_T, is closed. By the contraction mapping principle, there exists a unique solution $U = (u_0, ..., u_5)$ of (3.12), (3.13), (3.14), (3.15) and (3.16) defined in the interval $[0, T]$ satisfying

$$u_j \in C([0,T]; L^2(\mathbb{R}^2)) \cap L^\infty(0,T; H^4(\mathbb{R}^2)), \;\; j = 0, ..., 5. \tag{3.53}$$

Now, let us set for $j = 1, .., 2$

$$\widetilde{u}_j = \partial_j u_0$$

and

$$\widetilde{u}_3 = e^{q(|u_0|^2)} \Delta u_0, \;\; \widetilde{u}_4 = \partial_t \Delta u_0 \text{ and } \widetilde{u}_5 = \partial_t^2 \Delta u_0.$$

Then, it is clear that $\widetilde{U} = (u_0, \widetilde{u}_1, ..., \widetilde{u}_5)$ satisfies the linear inhomogeneous Cauchy problem (3.17), (3.18), (3.19), (3.20) and (3.21) with $V = U$. Hence since $\mathcal{S}$ is a contraction mapping, one has

$$||\widetilde{U} - U||_{Y_T} \leq \frac{1}{2} ||U - U||_{Y_T} = 0.$$

As a consequence, we have for $j = 1, ..., 5$, $\widetilde{u}_j = u_j$ and so

$$u_1 = \partial_1 u_0, \;\; u_2 = \partial_2 u_0, \;\; u_3 = e^{q(|u|^2)} \Delta u_0, \;\; u_4 = \partial_t \Delta u_0, \;\; u_5 = \partial_t^2 \Delta u_0. \tag{3.54}$$

This implies that u_0 is solution of equation (3.3) with initial condition (3.1).

Moreover (3.32), (3.37), (3.44) and (3.45) imply that the unique solution of (3.3) given by $u = u_0$ satisfies

$$u \in L^\infty(0, T; H^8(\mathbb{R}^2)). \tag{3.55}$$

Indeed, (3.32) and (3.37) show that there is a positive M such that

$$\varepsilon^2 \sup_{t \in [0,T]} ||\partial_t \Delta u||^2_{H^4(\mathbb{R}^2)} \leq M,$$

$$\varepsilon^2 \sup_{t \in [0,T]} ||\partial_t^2 \Delta u||^2_{H^4(\mathbb{R}^2)} \leq M,$$

Thus, since u satisfies (3.3), there exists a positive constant M independent of ε such that

$$\sup_{t \in [0,T]} ||u||_{H^8(\mathbb{R}^2)} \leq M \tag{3.56}$$

(to obtain (3.56), we first apply the operator $(1 - \Delta)$ to (3.3) which allows to include the term $\Delta(\sqrt{1 + |u|^2})$ in the term $\mathcal{A}(u)\Delta u$). $\square$

3.4 Convergence of u_ε towards a solution u of (1.23)

In this section, we prove Theorem 1.3.2, namely we prove that the solutions of (1.28) converge towards the solutions of (1.23). The estimates obtained on u_ε in Section 3.3 are independent of ε. Equipped with such inequalities, the proof of the convergence of u_ε towards u is very classical. More precisely, let $u_0 \in H^6(\mathbb{R}^2)$ and denote by $u(\cdot, t)$ the solution of (1.23) with initial condition $u(\cdot,0) = u_0(\cdot)$ which exists at least on a finite interval $[0,T_0]$. According to section 2, we have furthermore

$$u \in L^\infty(0,T_0;H^6(\mathbb{R}^2)) \cap C([0,T_0];H^4(\mathbb{R}^2)).$$

Now let $u_0^\varepsilon \in H^8(\mathbb{R}^2)$ and $u_1^\varepsilon \in H^7(\mathbb{R}^2)$ be two sequences of initial data satisfying

$$u_0^\varepsilon \underset{\varepsilon\to+\infty}{\longrightarrow} u_0 \quad \text{in} \quad H^6(\mathbb{R}^2), \;\; u_1^\varepsilon \underset{\varepsilon\to+\infty}{\longrightarrow} 0 \quad \text{in} \quad H^7(\mathbb{R}^2). \tag{3.57}$$

Assume furthermore that u_0^ε and u_1^ε satisfy the hypothesis of Theorem 1.3.1. Then, there exists a time T_1 independent of ε such that for all $\varepsilon > 0$, there is a unique solution u_ε to the Cauchy problem (1.28) with initial condition

$$u_\varepsilon(\cdot,0) = u_0^\varepsilon(\cdot), \;\; \partial_t u_\varepsilon(\cdot,0) = u_1^\varepsilon(\cdot).$$

Furthermore, we have

$$u_\varepsilon \in L^\infty(0,T_1;H^8(\mathbb{R}^2)) \cap C([0,T_1];H^4(\mathbb{R}^2)),$$

and there exists a constant M independent of ε such that for all $\varepsilon > 0$,

$$\sup_{t\in[0,T_1]} \Big(||u_\varepsilon||^2_{H^8(\mathbb{R}^2)} + \varepsilon^2 ||\partial_t u_\varepsilon||^2_{H^6(\mathbb{R}^2)} + \varepsilon^2 ||\partial_t^2 u_\varepsilon||^2_{H^6(\mathbb{R}^2)} \Big) \le M^2. \tag{3.58}$$

We denote by $T = \inf(T_0,T_1)$. Our goal is to show that u_ε tends to u as ε goes to zero in the space

$$L^\infty(0,T;H^6(\mathbb{R}^2)).$$

The proof uses an energy estimate of the same type as (3.42). The equation satisfied by u_ε is

$$\begin{aligned} &-\varepsilon^2 \begin{pmatrix} (u_\varepsilon)_{tt} \\ (\overline{u}_\varepsilon)_{tt} \end{pmatrix} + 2i \begin{pmatrix} (u_\varepsilon)_t \\ (\overline{u}_\varepsilon)_t \end{pmatrix} + \mathcal{A}(u_\varepsilon) \begin{pmatrix} \Delta u_\varepsilon \\ \Delta \overline{u}_\varepsilon \end{pmatrix} \\ &- \begin{pmatrix} \frac{u_\varepsilon}{\gamma_\varepsilon^2}|\nabla u_\varepsilon|^2 - \frac{u_\varepsilon}{4\gamma_\varepsilon^4}|\nabla|u_\varepsilon|^2|^2 + u_\varepsilon g(|u_\varepsilon|^2) \\ \frac{-\overline{u}_\varepsilon}{\gamma_\varepsilon^2}|\nabla u_\varepsilon|^2 + \frac{\overline{u}_\varepsilon}{4\gamma_\varepsilon^4}|\nabla|u_\varepsilon|^2|^2 - \overline{u}_\varepsilon g(|u_\varepsilon|^2) \end{pmatrix} = 0, \end{aligned} \tag{3.59}$$

while u satisfies

$$2i \begin{pmatrix} (u)_t \\ (\overline{u})_t \end{pmatrix} + \mathcal{A}(u) \begin{pmatrix} \Delta u \\ \Delta \overline{u} \end{pmatrix} - \begin{pmatrix} \frac{u}{\gamma^2}|\nabla u|^2 - \frac{u}{4\gamma^4}|\nabla|u|^2|^2 + ug(|u|^2) \\ \frac{-\overline{u}}{\gamma^2}|\nabla u|^2 + \frac{\overline{u}}{4\gamma^4}|\nabla|u|^2|^2 - \overline{u}g(|u|^2) \end{pmatrix} = 0. \tag{3.60}$$

As usual, we apply to (3.59) the operator $(1-\Delta)$ and we set $\phi_\varepsilon^1 = (1-\Delta)u_\varepsilon$ to obtain

$$\begin{aligned} &-\varepsilon^2 \begin{pmatrix} (\phi_\varepsilon^1)_{tt} \\ (\overline{\phi}_\varepsilon^1)_{tt} \end{pmatrix} + 2i \begin{pmatrix} (\phi_\varepsilon^1)_t \\ (\overline{\phi}_\varepsilon^1)_t \end{pmatrix} + \mathcal{A}(u_\varepsilon) \begin{pmatrix} \Delta \phi_\varepsilon^1 \\ \Delta \overline{\phi}_\varepsilon^1 \end{pmatrix} + \sum_{k=1}^2 \mathcal{E}(u_\varepsilon, \partial_k u_\varepsilon) \begin{pmatrix} \partial_k \phi_\varepsilon^1 \\ \partial_k \overline{\phi}_\varepsilon^1 \end{pmatrix} \\ &+\mathcal{I}_1(D^2 u_\varepsilon) = 0. \end{aligned} \tag{3.61}$$

where

$$
\begin{aligned}
\mathcal{I}_1(D^2u_\varepsilon) = & \sum_{j=1}^{2} \partial_j C(u_\varepsilon, \partial_j u_\varepsilon) \begin{pmatrix} \Delta u_\varepsilon \\ \Delta \overline{u}_\varepsilon \end{pmatrix} \\
& + \sum_{k=1}^{2} \partial_j B(u_\varepsilon, \partial_k u_\varepsilon) \begin{pmatrix} \partial_j \partial_k u_\varepsilon \\ \partial_j \partial_k \overline{u}_\varepsilon \end{pmatrix} - \sum_{j=1}^{2} \begin{pmatrix} \partial_j F(Du_\varepsilon) \\ -\partial_j \overline{F}(Du_\varepsilon) \end{pmatrix}.
\end{aligned}
$$

The matrices $\mathcal{E}$, B, C and F are the same as in section 2.2. The only thing to notice is that in F, we formally replace the argument u_j by $\partial_j u_\varepsilon$.

Applying again the operator $(1-\Delta)$ to (3.61) and setting $\phi_\varepsilon^2 = (1-\Delta)\phi_\varepsilon^1$, we can write

$$
\begin{aligned}
& -\varepsilon^2 \begin{pmatrix} (\phi_\varepsilon^2)_{tt} \\ (\overline{\phi}_\varepsilon^2)_{tt} \end{pmatrix} + 2i \begin{pmatrix} (\phi_\varepsilon^2)_t \\ (\overline{\phi}_\varepsilon^2)_t \end{pmatrix} + \mathcal{A}(u_\varepsilon) \begin{pmatrix} \Delta \phi_\varepsilon^2 \\ \Delta \overline{\phi}_\varepsilon^2 \end{pmatrix} + \sum_{k=1}^{2} \mathcal{H}(u_\varepsilon, \partial_k u_\varepsilon) \begin{pmatrix} \partial_k \phi_\varepsilon^2 \\ \partial_k \overline{\phi}_\varepsilon^2 \end{pmatrix} \\
& + \mathcal{I}_2(D^4 u_\varepsilon) = 0.
\end{aligned}
\tag{3.62}
$$

where

$$
\mathcal{H}(u_\varepsilon, \partial_k u_\varepsilon) = \mathcal{E}(u_\varepsilon, \partial_k u_\varepsilon) + 2\partial_k \mathcal{A}(u_\varepsilon),
$$

$$
\begin{aligned}
\mathcal{I}_2(D^4u_\varepsilon) = & \mathcal{I}_1(D^2u_\varepsilon) - 2\sum_{k=1}^{2} \partial_k \mathcal{A}(u_\varepsilon) \begin{pmatrix} \partial_k \phi_\varepsilon^1 \\ \partial_k \overline{\phi}_\varepsilon^1 \end{pmatrix} \\
& - \sum_{j=1}^{2} \left(\partial_j^2 \mathcal{I}_1(D^2 u_\varepsilon) + \partial_j^2 \mathcal{A}(u_\varepsilon) \begin{pmatrix} \Delta \phi_\varepsilon^1 \\ \Delta \overline{\phi}_\varepsilon^1 \end{pmatrix} \right) \\
& - 2\sum_{j=1}^{2} \sum_{k=1}^{2} \partial_j \mathcal{E}(u_\varepsilon, \partial_k u_\varepsilon) \begin{pmatrix} \partial_k \partial_j \phi_\varepsilon^1 \\ \partial_k \partial_j \overline{\phi}_\varepsilon^1 \end{pmatrix} \\
& - \sum_{j=1}^{2} \sum_{k=1}^{2} \partial_j^2 \mathcal{E}(u_\varepsilon, \partial_k u_\varepsilon) \begin{pmatrix} \partial_k \phi_\varepsilon^1 \\ \partial_k \phi_\varepsilon^1 \end{pmatrix}
\end{aligned}
$$

Finally, we can derive the following equation for $\phi_\varepsilon = (1-\Delta)\phi_\varepsilon^2 = (1-\Delta)^3 u_\varepsilon$

$$
\begin{aligned}
& -\varepsilon^2 \begin{pmatrix} (\phi_\varepsilon)_{tt} \\ (\overline{\phi}_\varepsilon)_{tt} \end{pmatrix} + 2i \begin{pmatrix} (\phi_\varepsilon)_t \\ (\overline{\phi}_\varepsilon)_t \end{pmatrix} + \mathcal{A}(u_\varepsilon) \begin{pmatrix} \Delta \phi_\varepsilon \\ \Delta \overline{\phi}_\varepsilon \end{pmatrix} + \sum_{k=1}^{2} \mathcal{W}(u_\varepsilon, \partial_k u_\varepsilon) \begin{pmatrix} \partial_k \phi_\varepsilon \\ \partial_k \overline{\phi}_\varepsilon \end{pmatrix} \\
& + \mathcal{I}_3(D^6 u_\varepsilon) = 0.
\end{aligned}
\tag{3.63}
$$

$$
\mathcal{W}(u_\varepsilon, \partial_k u_\varepsilon) = \mathcal{H}(u_\varepsilon, \partial_k u_\varepsilon) + 2\partial_k \mathcal{A}(u_\varepsilon),
$$

$$\begin{aligned}
\mathcal{I}_3(D^6u_\varepsilon) =& \mathcal{I}_2(D^4u_\varepsilon) - 2\sum_{k=1}^{2} \partial_k \mathcal{A}(u_\varepsilon) \begin{pmatrix} \partial_k \phi_\varepsilon^2 \\ \partial_k \overline{\phi}_\varepsilon^2 \end{pmatrix} \\
& - \sum_{j=1}^{2} \left(\partial_j^2 \mathcal{I}_2(D^4u_\varepsilon) + \partial_j^2 \mathcal{A}(u_\varepsilon) \begin{pmatrix} \Delta\phi_\varepsilon^2 \\ \Delta\overline{\phi}_\varepsilon^2 \end{pmatrix} \right) \\
& - 2\sum_{j=1}^{2}\sum_{k=1}^{2} \partial_j \mathcal{H}(u_\varepsilon, \partial_k u_\varepsilon) \begin{pmatrix} \partial_j\partial_k \phi_\varepsilon^2 \\ \partial_j\partial_k \overline{\phi}_\varepsilon^2 \end{pmatrix} \\
& - \sum_{j=1}^{2}\sum_{k=1}^{2} \partial_j^2 \mathcal{H}(u_\varepsilon, \partial_k u_\varepsilon) \begin{pmatrix} \partial_k \phi_\varepsilon^2 \\ \partial_k \overline{\phi}_\varepsilon^2 \end{pmatrix}
\end{aligned}$$

Keeping the same notations, we can directly write the equation satisfied by $\phi = (1-\Delta)^3 u$

$$\begin{aligned}
&-\varepsilon^2 \begin{pmatrix} (\phi)_{tt} \\ (\overline{\phi})_{tt} \end{pmatrix} + 2i \begin{pmatrix} (\phi)_t \\ (\overline{\phi})_t \end{pmatrix} + \mathcal{A}(u) \begin{pmatrix} \Delta\phi \\ \Delta\overline{\phi} \end{pmatrix} + \sum_{k=1}^{2} \mathcal{W}(u, \partial_k) \begin{pmatrix} \partial_k\phi \\ \partial_k\overline{\phi} \end{pmatrix} \\
&+\mathcal{I}_3(D^6u) = 0.
\end{aligned} \tag{3.64}$$

We are now able to write the equation satisfied by the difference $\chi_\varepsilon = \phi_\varepsilon - \phi$

$$\begin{aligned}
&-\varepsilon^2 \begin{pmatrix} (\phi_\varepsilon)_{tt} \\ (\overline{\phi}_\varepsilon)_{tt} \end{pmatrix} + 2i \begin{pmatrix} (\chi_\varepsilon)_t \\ (\overline{\chi}_\varepsilon)_t \end{pmatrix} + \mathcal{A}(u) \begin{pmatrix} \Delta\chi_\varepsilon \\ \Delta\overline{\chi}_\varepsilon \end{pmatrix} + (\mathcal{A}(u_\varepsilon) - \mathcal{A}(u)) \begin{pmatrix} \Delta\phi_\varepsilon \\ \Delta\overline{\phi}_\varepsilon \end{pmatrix} \\
&+\sum_{k=1}^{2} \mathcal{W}(u, \partial_k u) \begin{pmatrix} \partial_k\chi_\varepsilon \\ \partial_k\overline{\chi}_\varepsilon \end{pmatrix} + \sum_{k=1}^{2} (\mathcal{W}(u_\varepsilon, \partial_k u_\varepsilon) - \mathcal{W}(u, \partial_k u)) \begin{pmatrix} \partial_k\phi_\varepsilon \\ \partial_k\overline{\phi}_\varepsilon \end{pmatrix} \\
&+\mathcal{I}_3(D^6u_\varepsilon) - \mathcal{I}_3(D^6u) = 0.
\end{aligned} \tag{3.65}$$

The last thing to do is to use a gauge transform in order to integrate by parts the terms including the first derivatives of χ_ε. Thus, we set $\psi_\varepsilon = e^{p(u)}\chi_\varepsilon$ and we derive

$$\begin{aligned}
&-\varepsilon^2 e^{p(u)} \begin{pmatrix} (\phi_\varepsilon)_{tt} \\ (\overline{\phi}_\varepsilon)_{tt} \end{pmatrix} + 2i \begin{pmatrix} (\psi_\varepsilon)_t \\ (\overline{\psi}_\varepsilon)_t \end{pmatrix} + \mathcal{A}(u) \begin{pmatrix} \Delta\psi_\varepsilon \\ \Delta\overline{\psi}_\varepsilon \end{pmatrix} + e^{p(u)} (\mathcal{A}(u_\varepsilon) - \mathcal{A}(u)) \begin{pmatrix} \Delta\phi_\varepsilon \\ \Delta\overline{\phi}_\varepsilon \end{pmatrix} \\
&+\sum_{k=1}^{2} \mathcal{X}(u, \partial_k u) \begin{pmatrix} \partial_k\psi_\varepsilon \\ \partial_k\overline{\psi}_\varepsilon \end{pmatrix} + \sum_{k=1}^{2} e^{p(u)} (\mathcal{W}(u_\varepsilon, \partial_k u_\varepsilon) - \mathcal{W}(u, \partial_k u)) \begin{pmatrix} \partial_k\phi_\varepsilon \\ \partial_k\overline{\phi}_\varepsilon \end{pmatrix} \\
&+e^{p(u)} \left(\mathcal{I}_3(D^6u_\varepsilon) - \mathcal{I}_3(D^6u) \right) + \left(2i\partial_t(e^{-p(u)}) + \Delta(-e^{p(u)}) \right) \begin{pmatrix} \psi_\varepsilon \\ \overline{\psi}_\varepsilon \end{pmatrix} = 0,
\end{aligned} \tag{3.66}$$

where

$$\mathcal{X}(u, \partial_k u) = \mathcal{W}(u, \partial_k u) - 2 \begin{pmatrix} \partial_k p(u) & 0 \\ 0 & \partial_k p(u) \end{pmatrix}.$$

Multiplying equation (3.66) by $\mathcal{A}^{-1}(u)$, multiplying by $\overline{\psi}_\varepsilon$ and integrating over $\mathbb{R}^2$ the first line of the resulting equation, we obtain

$$\begin{aligned}
&-\varepsilon^2 \int_{\mathbb{R}^2} e^{p(u)} \frac{(2+|u|^2)}{2} (\phi_\varepsilon)_{tt} \overline{\psi}_\varepsilon dx + \varepsilon^2 \int_{\mathbb{R}^2} e^{p(u)} \frac{u^2}{2} (\overline{\phi}_\varepsilon)_{tt} \overline{\psi}_\varepsilon dx + i \int_{\mathbb{R}^2} (2+|u|^2)(\psi_\varepsilon)_t \overline{\psi}_\varepsilon dx \\
&-i \int_{\mathbb{R}^2} u^2 (\overline{\psi}_\varepsilon)_t \overline{\psi}_\varepsilon dx + \int_{\mathbb{R}^2} \Delta \psi_\varepsilon \overline{\psi}_\varepsilon dx + \sum_{k=1}^2 \int_{\mathbb{R}^2} \mathcal{X}^{11}(u, \partial_k u)(\partial_k \psi_\varepsilon) \overline{\psi}_\varepsilon dx \\
&+ \sum_{k=1}^2 \int_{\mathbb{R}^2} \mathcal{X}^{12}(u, \partial_k u)(\partial_k \overline{\psi}_\varepsilon) \overline{\psi}_\varepsilon dx + \int_{\mathbb{R}^2} \mathcal{B}^{11}(u_\varepsilon, u) \Delta \phi_\varepsilon \overline{\psi}_\varepsilon dx + \int_{\mathbb{R}^2} \mathcal{B}^{12}(u_\varepsilon, u) \Delta \overline{\phi}_\varepsilon \overline{\psi}_\varepsilon dx \quad (3.67) \\
&+ \sum_{k=1}^2 \int_{\mathbb{R}^2} \mathcal{Y}^{11}(Du_\varepsilon, Du) \partial_k \phi_\varepsilon \overline{\psi}_\varepsilon dx + \sum_{k=1}^2 \int_{\mathbb{R}^2} \mathcal{Y}^{12}(Du_\varepsilon, Du) \partial_k \overline{\phi}_\varepsilon \overline{\psi}_\varepsilon dx \\
&+ \int_{\mathbb{R}^2} J(D^6 u_\varepsilon, D^6 u) \overline{\psi}_\varepsilon dx = 0,
\end{aligned}$$

In equation (3.67) we have setted

$$\begin{aligned}
\mathcal{B}(u_\varepsilon, u) &= e^{p(u)} \mathcal{A}^{-1}(u) \left(\mathcal{A}(u_\varepsilon) - \mathcal{A}(u) \right) \\
\mathcal{Y}(Du_\varepsilon, Du) &= e^{p(u)} \mathcal{A}^{-1}(u) \left(\mathcal{W}(u_\varepsilon, \partial_k u_\varepsilon) - \mathcal{W}(u, \partial_k u) \right) \\
J(D^6 u_\varepsilon, D^6 u) &= e^{p(u)} \mathcal{A}^{-1}(u) \left(\mathcal{I}_3(D^6 u_\varepsilon) - \mathcal{I}_3(D^6 u) \right),
\end{aligned}$$

and we still denote by $\mathcal{X}$ the product $\mathcal{A}^{-1}(u)\mathcal{X}$. We then take the imaginary part of (3.67). Since

$$\varepsilon \sup_{t \in [0,T]} ||\partial_t^2 u_\varepsilon||_{H^6(\mathbb{R}^2)} \leq M,$$

we can write thanks to Cauchy-Schwarz inequality

$$\begin{aligned}
\left| \varepsilon^2 \int_{\mathbb{R}^2} \frac{(2+|u|^2)}{2} (\phi_\varepsilon)_{tt} \overline{\psi}_\varepsilon dx \right| &\leq \varepsilon\, C(M) \varepsilon ||\partial_t^2 \phi_\varepsilon||_2 ||\psi_\varepsilon||_2 \\
&\leq \varepsilon\, C(M) ||\psi_\varepsilon||_2
\end{aligned}$$

and

$$\left| \varepsilon^2 \int_{\mathbb{R}^2} \frac{u^2}{2} (\overline{\phi}_\varepsilon)_{tt} \overline{\psi}_\varepsilon dx \right| \leq \varepsilon\, C(M) ||\psi_\varepsilon||_2.$$

Furthermore,

$$\begin{aligned}
&\mathrm{Im} \left(i \int_{\mathbb{R}^2} (2+|u|^2)(\psi_\varepsilon)_t \overline{\psi}_\varepsilon dx - i \int_{\mathbb{R}^2} u^2 (\overline{\psi}_\varepsilon)_t \overline{\psi}_\varepsilon dx \right) \\
&= \int_{\mathbb{R}^2} \left(\frac{2+|u|^2}{2} \right) |\psi_\varepsilon|_t^2 - \int_{\mathbb{R}^2} \left(\frac{u^2}{4} (\overline{\psi}_\varepsilon^2)_t + \frac{\overline{u}^2}{4} (\psi_\varepsilon^2)_t \right) dx \\
&= \frac{d}{dt} \left(\int_{\mathbb{R}^2} \left(\frac{2+|u|^2}{2} \right) |\psi_\varepsilon|^2 - \int_{\mathbb{R}^2} \left(\frac{u^2}{4} (\overline{\psi_\varepsilon}^2) + \frac{\overline{u}^2}{4} (\psi_\varepsilon^2) \right) dx \right) \\
&- \int_{\mathbb{R}^2} \frac{|u|_t^2}{2} |\psi_\varepsilon|^2 dx + \int_{\mathbb{R}^2} \left(\frac{(u^2)_t}{4} (\overline{\psi}_\varepsilon^2) + \frac{(\overline{u}^2)_t}{4} (\psi_\varepsilon^2) \right) dx.
\end{aligned}$$

$$\mathrm{Im}\left(\int_{\mathbb{R}^2}\Delta\psi_\varepsilon\overline{\psi}_\varepsilon dx\right) = -\mathrm{Im}\left(\int_{\mathbb{R}^2}|\nabla\psi_\varepsilon|^2 dx\right) = 0.$$

We choose now p such that $\mathrm{Re}(\mathcal{X}^{11}) = 0$. With such a p, we get

$$\begin{aligned}\mathrm{Im}\left(\sum_{k=1}^{2}\int_{\mathbb{R}^2}\mathcal{X}^{11}(u,\partial_k u)(\partial_k\psi_\varepsilon)\overline{\psi}_\varepsilon dx\right) &= \sum_{k=1}^{2}\int_{\mathbb{R}^2}\mathrm{Im}(\mathcal{X}^{11}(u,\partial_k u))\partial_k\frac{|\psi_\varepsilon|^2}{2}dx\\ &= -\sum_{k=1}^{2}\int_{\mathbb{R}^2}\partial_k(\mathrm{Im}(\mathcal{X}^{11}(u,\partial_k u)))\frac{|\psi_\varepsilon|^2}{2}dx\end{aligned}$$

Using the continuous embedding of $H^2(\mathbb{R}^2)$ into $L^\infty(\mathbb{R}^2)$, we obtain

$$\left|\mathrm{Im}\left(\sum_{k=1}^{2}\int_{\mathbb{R}^2}\mathcal{X}^{11}(u,\partial_k u)(\partial_k\psi_\varepsilon)\overline{\psi}_\varepsilon dx\right)\right| \leq C(M)||\psi_\varepsilon||_2^2.$$

Since $\partial_k\overline{\psi}_\varepsilon\overline{\psi}_\varepsilon = \frac{1}{2}\partial_k(\overline{\psi}_\varepsilon^2)$ an integration by parts gives directly

$$\left|\mathrm{Im}\left(\sum_{k=1}^{2}\int_{\mathbb{R}^2}\mathcal{X}^{12}(u,\partial_k u)(\partial_k\overline{\psi}_\varepsilon)\overline{\psi}_\varepsilon dx\right)\right| \leq C(M)||\psi_\varepsilon||_2^2.$$

Since the matrices $\mathcal{A}$, $\mathcal{W}$ and $\mathcal{I}_3$ are of class C^1 with respect to their arguments, there exists a positive $C(M)$ such that for $i,j = 1,2$

$$\begin{aligned}\left|\left|\mathcal{B}^{ij}(u_\varepsilon,u)\right|\right|_{L^\infty(\mathbb{R}^2)} &\leq C(M)||u_\varepsilon - u||_{L^\infty(\mathbb{R}^2)}\\ &\leq C(M)||\psi_\varepsilon||_2,\end{aligned}$$

$$\begin{aligned}\left|\left|\mathcal{Y}^{ij}(Du_\varepsilon,Du)\right|\right|_{L^\infty(\mathbb{R}^2)} &\leq C(M)\left(||u_\varepsilon - u||_{L^\infty(\mathbb{R}^2)} + ||\nabla(u_\varepsilon - u)||_{L^\infty(\mathbb{R}^2)}\right)\\ &\leq C(M)||\psi_\varepsilon||_2,\end{aligned}$$

$$\left|\left|J^i(D^6u_\varepsilon,D^6u)\right|\right|_{L^2(\mathbb{R}^2)} \leq C(M)||\psi_\varepsilon||_2.$$

This implies using the fact that

$$||u_\varepsilon||_{H^8(\mathbb{R}^2)} \leq M$$

and the Cauchy-Schwarz inequality

$$\left|\int_{\mathbb{R}^2}\mathcal{B}^{11}(u_\varepsilon,u)\Delta\phi_\varepsilon\overline{\psi}_\varepsilon dx\right| \leq C(M)||\psi_\varepsilon||_2^2,$$

$$\left|\int_{\mathbb{R}^2}\mathcal{B}^{12}(u_\varepsilon,u)\Delta\overline{\phi}_\varepsilon\overline{\psi}_\varepsilon dx\right| \leq C(M)||\psi_\varepsilon||_2^2,$$

$$\left|\sum_{k=1}^{2}\int_{\mathbb{R}^2}\mathcal{Y}^{11}(Du_\varepsilon, Du)\partial_k\phi_\varepsilon\overline{\psi}_\varepsilon dx\right| \leq C(M)||\psi_\varepsilon||_2^2,$$

$$\left|\sum_{k=1}^{2}\int_{\mathbb{R}^2}\mathcal{Y}^{12}(Du_\varepsilon, Du)\partial_k\overline{\phi}_\varepsilon\overline{\psi}_\varepsilon dx\right| \leq C(M)||\psi_\varepsilon||_2^2,$$

and

$$\left|\int_{\mathbb{R}^2} J(D^6u_\varepsilon, D^6u)\overline{\psi}_\varepsilon dx\right| \leq C(M)||\psi_\varepsilon||_2^2.$$

Collecting again all these results, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{\mathbb{R}^2}\left(\frac{2+|u|^2}{2}\right)|\psi_\varepsilon|^2 - \int_{\mathbb{R}^2}\left(\frac{u^2}{4}(\overline{\psi_\varepsilon}^2) + \frac{\overline{u}^2}{4}(\psi_\varepsilon^2)\right)dx\right)\\
&-\int_{\mathbb{R}^2}\frac{|u|_t^2}{2}|\psi_\varepsilon|^2dx + \int_{\mathbb{R}^2}\left(\frac{(u^2)_t}{4}(\overline{\psi}_\varepsilon^2) + \frac{(\overline{u}^2)_t}{4}(\psi_\varepsilon^2)\right)dx\\
&\leq \varepsilon\, C(M)||\psi_\varepsilon||_2 + C(M)||\psi_\varepsilon||_2^2.
\end{aligned}\tag{3.68}$$

Integrating (3.68) from 0 to t gives, recalling that

$$\sup_{t\in[0,T]}||u_t||_{L^\infty(\mathbb{R}^2)} \leq M,$$

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left(\frac{2+|u(t)|^2}{2}\right)|\psi_\varepsilon(t)|^2 - \int_{\mathbb{R}^2}\left(\frac{u^2(t)}{4}(\overline{\psi_\varepsilon(t)}^2) + \frac{\overline{u(t)}^2}{4}(\psi_\varepsilon^2)\right)dx\\
&\leq \int_{\mathbb{R}^2}\left(\frac{2+|u(0)|^2}{2}\right)|\psi_\varepsilon(0)|^2 - \int_{\mathbb{R}^2}\left(\frac{u^2(0)}{4}(\overline{\psi}_\varepsilon(0)^2) + \frac{\overline{u}^2}{4}(\psi_\varepsilon^2(0))\right)dx\\
&+C(M)\int_0^t\left(||\psi_\varepsilon(s)||_2^2 + \varepsilon^2\right)ds
\end{aligned}\tag{3.69}$$

Denoting

$$\ell(0) = \int_{\mathbb{R}^2}\left(\frac{2+|u(0)|^2}{2}\right)|\psi_\varepsilon(0)|^2 - \int_{\mathbb{R}^2}\left(\frac{u^2(0)}{4}(\overline{\psi}_\varepsilon(0)^2) + \frac{\overline{u}^2}{4}(\psi_\varepsilon^2(0))\right)dx,$$

we classically obtain by Gronwall lemma,

$$\sup_{t\in[0,T]}||\psi_\varepsilon||_2^2 \leq C(T)(\ell(0) + 2\varepsilon^2).\tag{3.70}$$

This implies

$$\sup_{t\in[0,T]} ||\phi_\varepsilon||_2^2 \leq C(M)\left(\ell(0)+2\varepsilon^2\right). \tag{3.71}$$

Thanks to hypothesis (3.57), it is easy to see that

$$\ell(0) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Thus, we finally obtain

$$u_\varepsilon - u \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{in } L^\infty(0,T;H^6(\mathbb{R}^2)),$$

which ends the proof of Theorem 1.3.2 □

Chapitre 4

Stabilité des ondes stationnaires en dimension deux d'espace pour des équations de Schrödinger quasilinéaires

"We are such stuff as dreams are made on."

William Shakespeare.

Stability of stationary waves for a quasilinear Schrödinger equation in space dimension 2

(submitted to Differential and Integral Equations, December 15, 2000)

Abstract

In this chapter, we study the existence and the properties of standing waves of the form $u_\omega(x,t) = \phi_\omega(x)e^{i\omega t}$, where $x \in \mathbb{R}^2$, $t \geq 0$, for a quasilinear Schrödinger equation. Using the minimization method introduced by T. Cazenave and P.L. Lions, we prove a stability theorem for such waves.

Résumé

Dans ce chapitre, on étudie l'existence et les propriétés d'ondes stationnaires de la forme $u_\omega(x,t) = \phi_\omega(x)e^{i\omega t}$, où $x \in \mathbb{R}^2$, $t \geq 0$, solutions d'une classe d'équations de Schrödinger quasilinéaires. En utilisant la méthode de minimization introduite par T. Cazenave et P.L Lions, on montre un résultat de stabilité pour de telles ondes.

4.1 Introduction

We consider the nonlinear Schrödinger equation (1.24)

$$\begin{cases} 2iu_t + \Delta u - 2uh'(|u|^2)\Delta(h(|u|^2)) + ug(|u|^2) = 0 \\ u(x,0) = u_0(x) \quad x \in \mathbb{R}^2;\ \ 0 \leq t \leq T. \end{cases}$$

where h,g are given real functions. This chapter is devoted to the study of the stability properties of standing waves of the form $u_\omega(x,t) = \phi_\omega(x)e^{i\omega t},\ \ x \in \mathbb{R}^2,\ \ t \geq 0$, where ω is a fixed parameter and ϕ_ω is a ground state (i.e. a solution with minimal action) of equation (1.30_ω) depending on ω

$$\Delta\phi - 2\phi h'(|\phi|^2)\Delta(h(|\phi|^2)) + \phi g(|\phi|^2) - 2\omega\phi = 0.$$

Quasilinear Schrödinger equations play an important role in various domains of physics like plasma physics, nonlinear optics and fluid mechanics (see [13]). If $g(s) = as^p,\ \ h(s) = s$, equation (1.24) is known in plasma physics, if $g(s) = -(\alpha + \frac{\beta}{(a+s)^3}),\ \ h(s) = s$, it can be found in superfluid-film physics and the case $g(s) = as,\ \ h(s) = \sqrt{s}$ appears in quantum physics. The particular case $g(s) = 1 - \frac{1}{\sqrt{1+s}},\ \ h(s) = \sqrt{1+s} - 1$ models the self-channeling of a high power ultra-short laser pulse in a plasma (see [10], [12], [55]). In [12], A. de Bouard, N. Hayashi and J.C. Saut have studied the Cauchy problem for equation (1.24) in the model case $g(s) = 1 - \frac{1}{\sqrt{1+s}},\ \ h(s) = \sqrt{1+s} - 1$ in space dimension $n = 1, 2, 3$. They have proved that the Cauchy problem is locally well-posed in the Sobolev space $H^6(\mathbb{R}^n)$ if the initial data is small enough. In Chapter 2, we have studied the Cauchy problem associated with equation (1.24) for more general functions h and g. We have proved that the Cauchy problem is locally well-posed for any space dimension in Sobolev spaces with finite regularity and for any initial data if h and g are sufficiently regular and if h satisfies an ellipticity condition.

This chapter is concerned with the model case $g(s) = 1 - \frac{1}{\sqrt{1+s}},\ \ h(s) = \sqrt{1+s} - 1$. The case $h(s) = s$ has been studied in [53]. We first prove the existence of ground states of equation (1.30_ω). One of the first general studies for equations of the form $-\Delta\phi = p(\phi)$ is due to W. Strauss ([57]) . In [20], S. Coleman, V. Glazer and A. Martin have introduced a constrained minimization problem to solve equations of this type. Using this idea, H. Berestycki and P.-L. Lions (see [6]) proved that, under certain conditions on p, the equation $-\Delta\phi = p(\phi)$ has a positive, spherically symmetric and decreasing solution in dimension $n \geq 3$. In [5], H. Berestycki, T. Gallouët and O. Kavian have treated the case $n = 2$. Using a suitable change of function, we prove that equation (1.30_ω) can be reduced to an equation of the form $-\Delta\phi = p(\phi)$. Then, the existence of ground states of (1.30_ω) in space dimension 2 is a consequence of Theorem 1 of [5]. In this chapter, we limit our attention to space dimension 2. In fact, the one-dimensional case has already been treated in [34]. Furthermore, it seems difficult to prove the existence of ground states of equation (1.30_ω) in dimension $n \geq 3$. One of the main reasons is that, by our change of function, we are reduced to an equation of the form $-\Delta\phi = p(\phi)$ where p grows exponentially at infinity. In dimension $n = 2$, we can control such nonlinearities by using a Trudinger inequality whereas in dimension $n \geq 3$, these kinds of equations are not expected to have nonsingular solutions.

Before proving a stability theorem for standing waves u_ω, one generally needs a uniqueness result for such ground states. Uniqueness of ground states of equations $-\Delta\phi = p(\phi)$ has interested many researchers in the last decade. One may see in particular Adimurthi [1], L. Erbe and M. Tang [23], K. McLeod [47] and L.A. Peletier [51]. One of the first general results was obtained by M.K. Kwong (see [39]) in the case $p(s) = s^\ell - s$. Later, P. Pucci and J. Serrin (see [54]) have solved the problem with very few conditions on p, using the famous identity of Erbe and Tang. Once again, a suitable change of function allows to prove the uniqueness of ϕ_ω up to translations and rotations.

The first result concerning stability of standing waves for a nonlinear Schrödinger equation with a logarithmic nonlinearity was obtained by T. Cazenave (see [17]). Later, T. Cazenave and P.L. Lions have proved a stability result in the case $g(s) = s^{\frac{\ell-1}{2}}$ $(1 < \ell < 1 + \frac{4}{n})$, $h(s) = 0$ (see [18]). In [34], I.D. Iliev and K.P. Kirchev obtained the first stability result concerning a one-dimensional quasilinear Schrödinger equation. They show, using a method introduced by M. Grillakis, J. Shatah and W. Strauss (see [31]) (originating from an idea of T.B. Benjamin [4]) , that u_ω is stable (resp. unstable) if the $L^2(\mathbb{R})$-norm of ϕ_ω is increasing (resp. decreasing) with respect to ω. We have not been able to apply this method in dimension 2, due to the difficulty in computing the spectrum of the linearized operator. So, we propose here a stability result in two dimensions using the minimization method introduced by T. Cazenave and P.-L. Lions, which will however give us a weaker result, due to the fact that the energy associated with (1.24) does not control the H^1-norm. The proof will use the concentration-compactness principle of P.L. Lions (see [41]) and one of the difficulties in this approach is the study of the Lagrange multipliers associated with the minimization problem. The problem is the following. We want to minimize a functional F defined on $H^1(\mathbb{R}^2)$ under a constraint of the form $Q = \mu$ where Q is a functional defined on $H^1(\mathbb{R}^2)$ and μ is a constant. Assume that there exists a solution χ to this problem. Then, there exists a Lagrange multiplier θ such that χ is a solution of $F'(\chi) = \theta Q'(\chi)$. We then have to know wether θ is unique for a fixed μ. For example, for the cubic nonlinear Schrödinger equation, this problem is well-known. Thanks to the scaling of the equation, one can show that the number θ is unique. In our case, considering a general nonlinearity g, the problem is more difficult to solve. We will discuss this point at the end of the paper.

Notations: For $p \in [1, +\infty]$ and $s \in \mathbb{N}^*$, we denote by $L^p(\mathbb{R}^2)$ the usual Lebesgue spaces and by $H^s(\mathbb{R}^2)$ the usual Sobolev spaces of real or complex-valued functions in $\mathbb{R}^2$. Different positive constants might be denoted by the same letter C.

We can now recall our hypothesis on the functions h and g. Let $G(t) = \int_0^t g(s)ds$. We assume that g and h satisfy

$$g, h \in C^6(\mathbb{R}_+), \;\; g(0) = 0, \;\; h(0) = 0 \tag{4.1}$$

$$\forall s \in \mathbb{R}_+, \;\; g(s) \geq 0 \tag{4.2}$$

$$\exists\, C \in \mathbb{R}_+^* \;/\; \forall s \in \mathbb{R}_+, \;\; \frac{g(s)}{\sqrt{G(s)}} \leq C h'(s) \tag{4.3}$$

and we assume that h satisfies (1.27). Let f be defined by $f^{-1}(t) = \int_0^t \sqrt{1-4\rho^2 h'^2(\rho^2)}d\rho$.

Remark 4.1.1. *The function f can be defined on $\mathbb{R}$ by setting for all $t \in \mathbb{R}_+$, $f(-t) = -f(t)$. It is clear that this extended function f is C^1 on $\mathbb{R}$.*

We assume furthermore

$$\exists\,(C,\nu) \in \mathbb{R}_+^2 \;/\; \forall (t_1,t_2) \in \mathbb{R}_+^2,\;\; |f^{'}(t_1) - f^{'}(t_2)| \leq C|e^{\nu t_1} - e^{\nu t_2}|. \tag{4.4}$$

We will see at the end of section 2.1 that all these assumptions are satisfied in the model case $g(s) = 1 - \frac{1}{\sqrt{1+s}}$, $h(s) = \sqrt{1+s} - 1$ (in which case, one has $f(t) = \sinh(t)$). They are not satisfied in the other cases of the introduction. We now derive some estimates on the functions g, h and f which will be useful in the sequel, and which allow to make more precise the nature of our assumptions. First, it is clear that from (1.27), one has

$$\forall s \in \mathbb{R}_+,\;\; 1 - 4sh^{'2}(s) > 0. \tag{4.5}$$

We can derive from (1.27) the following inequality

$$\forall s \in \mathbb{R}_+,\;\; h^{'2}(s) \leq \frac{1}{\beta + 4s},$$

which implies that there exists a constant $C > 0$ such that

$$\forall s \in \mathbb{R}_+,\;\; |h^{'}(s)| \leq C. \tag{4.6}$$

Let us consider now the function G. From (1.27), we obtain by integrating (4.3),

$$\forall s \in \mathbb{R}_+,\;\; G(s) \leq C\left(\sqrt{\beta+4s} - \sqrt{\beta}\right)^2.$$

Hence,

$$\forall s \in \mathbb{R}_+,\;\; G(s) \leq C \inf(s, s^2). \tag{4.7}$$

In particular, we obtain from (4.6), (4.7) and (4.3)

$$\forall s \in \mathbb{R}_+,\;\; g(s) \leq Cs. \tag{4.8}$$

Integrating the relation in (4.3), we have, using (4.6),

$$\forall (s_1,s_2) \in \mathbb{R}_+^2,\;\; \left|\sqrt{G(s_1)} - \sqrt{G(s_2)}\right| \leq C|s_1 - s_2|.$$

This implies that

$$\forall (s_1,s_2) \in \mathbb{R}_+^2,\;\; |G(s_1) - G(s_2)| \leq C|s_1 - s_2|\left|\sqrt{G(s_1)} + \sqrt{G(s_2)}\right| \tag{4.9}$$

Taking $t_2 = 0$ in (4.4) and integrating the relation, we obtain

$$\forall (t_1,t_2) \in \mathbb{R},\;\; |f(t_1) - f(t_2)| \leq C\, e^{\nu(|t_1|+|t_2|)}|t_1 - t_2|. \tag{4.10}$$

In particular, there exists $\nu' > 0$ such that

$$\forall t \in \mathbb{R}, \; |f(t)| \leq C \left(e^{\nu' t} - 1 \right). \tag{4.11}$$

We also have

$$\forall t \in \mathbb{R}_+, \; |(f^{-1})'(t)| = |\sqrt{1 - 4t^2 h'^2(t^2)}| \leq 1, \text{ hence } |t| \leq |f(t)|. \tag{4.12}$$

The paper is organized as follows. In section 4.2, we briefly discuss the existence of standing waves of the equation (1) and their properties. In section 4.3, we formulate and prove the stability result concerning the standing waves and we discuss the Euler-Lagrange problem associated with the minimization method of T. Cazenave and P.L. Lions.

4.2 Existence of ground states

In this section, we study the existence and the properties of the ground states of equation (1.30_ω)

$$\Delta\phi - 2\phi h'(|\phi|^2)\Delta(h(|\phi|^2)) + \phi g(|\phi|^2) - 2\omega\phi = 0,$$

where ω is a fixed parameter. By a ground state, we mean a solution in $H^1(\mathbb{R}^2)$, which minimizes the action associated with (1.30_ω) (see below) among non trivial solutions of (1.30_ω).

We define the action S_ω and the energy E associated with (1.30_ω) by

$$S_\omega(\phi) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla\phi|^2 - |\nabla(h(|\phi|^2))|^2 - G(|\phi|^2) \right) dx + \omega \int_{\mathbb{R}^2} |\phi|^2 dx$$

and

$$E(\phi) = S_\omega(\phi) - \omega \int_{\mathbb{R}^2} |\phi|^2 dx.$$

Remark 4.2.1. *We may notice that the functional E does not depend on ω.*

Lemma 4.2.2. *The action S_ω and the energy E are well-defined on $H^1(\mathbb{R}^2)$.*

Proof. let $\phi \in H^1(\mathbb{R}^2)$. By (4.7), we have

$$\int_{\mathbb{R}^2} G(|\phi|^2) dx \leq C \int_{\mathbb{R}^2} |\phi|^2 dx < +\infty.$$

Furthermore,

$$|\nabla h(|\phi|^2)|^2 = |\phi|^2 h'^2(|\phi|^2)|\nabla|\phi||^2 \leq C|\nabla|\phi||^2$$

by (2.1.4). Using the fact that $|\nabla|\phi|| \leq |\nabla\phi|$ *a.e.*, we obtain

$$\int_{\mathbb{R}^2} |\nabla h(|\phi|^2)|^2 dx \leq C \int_{\mathbb{R}^2} |\nabla\phi|^2 dx < +\infty. \; \square$$

Remark 4.2.3. *For all $\phi \in H^1(\mathbb{R}^2)$, we have $S_\omega(\phi) \geq S_\omega(|\phi|)$.*

We give here a Pohozaev-type identity for equation (1.30_ω).

Proposition 4.2.4. *Let $\phi \in H^1(\mathbb{R}^2)$ be a solution of (1.30_ω) and assume that $\phi \in L^\infty_{loc}(\mathbb{R}^2)$. Then ϕ satisfies*

$$\frac{1}{2}\int_{\mathbb{R}^2} G(|\phi|^2)dx = \omega \int_{\mathbb{R}^2} |\phi|^2 dx. \tag{4.13}$$

As a consequence, there exists a constant $\omega_{max} > 0$ such that (1.30_ω) has no nontrivial solution in $H^1(\mathbb{R}^2) \cap L^\infty_{Loc}(\mathbb{R}^2)$ for $\omega \notin]0, \omega_{max}[$.

Proof. Let $\phi \in H^1(\mathbb{R}^2)$ be a solution of (1.30_ω) satisfying the hypothesis of Proposition 4.2.4. By (4.7), $G(|\phi|^2) \in L^1(\mathbb{R}^2)$. Let $\psi \in \mathcal{D}(\mathbb{R}^2)$ such that $\psi \geq 0$, $\mathrm{supp}(\psi) \subset B(0,2)$ and $\psi \equiv 1$ on $B(0,1)$. For all $j \in \mathbb{N}^*$, we set $\psi_j(x) = \psi(\frac{x}{j})$. Multiplying equation (1.30_ω) by $\psi_j \ x \cdot \overline{\nabla\phi}$ and taking the real part, one has

$$\begin{aligned} \mathrm{Re}\Bigg(\int_{\mathbb{R}^2} \Delta\phi\psi_j \ x \cdot \overline{\nabla\phi} dx - \int_{\mathbb{R}^2} 2\phi h'(|\phi|^2)\Delta(h(|\phi|^2))\psi_j \ x \cdot \overline{\nabla\phi} dx \\ + \int_{\mathbb{R}^2} (\phi g(|\phi|^2) - 2\omega\phi)\psi_j \ x \cdot \overline{\nabla\phi} dx\Bigg) = 0. \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} &\mathrm{Re}\left(\int_{\mathbb{R}^2} \Delta\phi\psi_j \ x \cdot \overline{\nabla\phi} dx\right) \\ &= \mathrm{Re}\left(-\int_{\mathbb{R}^2} \nabla\phi \cdot \nabla\psi_j \ x \cdot \overline{\nabla\phi} dx - \int_{\mathbb{R}^2} \psi_j \nabla\phi \cdot \nabla(x \cdot \overline{\nabla\phi}) dx\right). \end{aligned}$$

Using the fact that $\mathrm{Re}\big(\nabla\phi \cdot \nabla(x \cdot \overline{\nabla\phi})\big) = |\nabla\phi|^2 + \frac{1}{2} x \cdot \nabla(|\nabla\phi|^2)$, and making a second integration by parts, one has

$$\begin{aligned} &\mathrm{Re}\left(\int_{\mathbb{R}^2} \Delta\phi\psi_j \ x \cdot \overline{\nabla\phi} dx\right) \\ &= -\mathrm{Re}\left(\int_{\mathbb{R}^2} \nabla\phi \cdot \nabla\psi_j \, x \cdot \overline{\nabla\phi} dx\right) - \int_{\mathbb{R}^2} \psi_j |\nabla\phi|^2 dx \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\phi|^2 x \cdot \nabla\psi_j dx + \frac{n}{2}\int_{\mathbb{R}^2} \psi_j |\nabla\phi|^2 dx \\ &= -\mathrm{Re}\left(\int_{\mathbb{R}^2} \nabla\phi \cdot \nabla\psi_j \, x \cdot \overline{\nabla\phi} dx\right) + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\phi|^2 x \cdot \nabla\psi_j dx \end{aligned}$$

because $n = 2$. Furthermore,

$$
\begin{aligned}
&-\operatorname{Re}\left(\int_{\mathbb{R}^2} 2\phi h'(|\phi|^2)\Delta(h(|\phi|^2))\psi_j \ x\cdot\overline{\nabla\phi}dx\right)\\
&=\int_{\mathbb{R}^2}\nabla h(|\phi|^2)\cdot\nabla(\psi_j \ x\cdot\nabla h(|\phi|^2))dx\\
&=\int_{\mathbb{R}^2}\nabla h(|\phi|^2)\cdot\nabla\psi_j \ x\cdot\nabla h(|\phi|^2)dx+\int_{\mathbb{R}^2}\psi_j\nabla h(|\phi|^2)\cdot\nabla(x\cdot\nabla h(|\phi|^2))dx\\
&=\int_{\mathbb{R}^2}\nabla h(|\phi|^2)\cdot\nabla\psi_j \ x\cdot\nabla h(|\phi|^2)dx+\int_{\mathbb{R}^2}\psi_j|\nabla h(|\phi|^2)|^2dx\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^2}\psi_j x\cdot\nabla(|\nabla h(|\phi|^2)|^2)dx\\
&=\int_{\mathbb{R}^2}\nabla h(|\phi|^2)\cdot\nabla\psi_j \ x\cdot\nabla h(|\phi|^2)dx-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla(|\phi|^2)|^2 \ x\cdot\nabla\psi_j dx
\end{aligned}
$$

using the fact that $n = 2$ and a second integration by parts. Finally,

$$
\begin{aligned}
&\operatorname{Re}\left(\int_{\mathbb{R}^2}(\phi g(|\phi|^2)-2\omega\phi)\psi_j \ x\cdot\overline{\nabla\phi}dx\right)\\
&=\int_{\mathbb{R}^2}(g(|\phi|^2)-2\omega)\psi_j \ x\cdot\frac{\nabla|\phi|^2}{2}dx\\
&=\int_{\mathbb{R}^2}\psi_j \ x\cdot\nabla(\frac{1}{2}G(|\phi|^2)-\omega|\phi|^2)dx\\
&=-\int_{\mathbb{R}^2}(\frac{1}{2}G(|\phi|^2)-\omega|\phi|^2) \ x\cdot\nabla\psi_j dx-2\int_{\mathbb{R}^2}\psi_j(\frac{1}{2}G(|\phi|^2)-\omega|\phi|^2)dx.
\end{aligned}
$$

Using (4.7), the Lebesgue dominated convergence theorem and the fact that $\nabla\psi_j(x) \underset{j\to+\infty}{\longrightarrow} 0$, we get the expected equality.

As a consequence, it is clear that (1.30_ω) has no nontrivial solution in $H^1(\mathbb{R}^2)\cap L^\infty_{Loc}(\mathbb{R}^2)$ for $\omega\leq 0$. Furthermore, from (4.7), we have

$$\forall s\in\mathbb{R}_+, \ \ G(s)\leq Cs. \tag{4.14}$$

Let C_{max} be the best constant in (4.14) and $\omega_{max}=\frac{C_{max}}{2}$ ($c_{max}=1$ in the model case). Then, for all $\omega\geq\omega_{max}$, (1.30_ω) has no nontrivial solution in $H^1(\mathbb{R}^2)\cap L^\infty_{Loc}(\mathbb{R}^2)$, due to (4.7) and (4.14). □

Assume now that ϕ is real. Thanks to the hypothesis (1.27), we can make a change of function by replacing ϕ by the function $\chi = f^{-1}(\phi)$. Let $k(t)=f'(t)f(t)g(f^2(t))$ and $k_\omega(t)=k(t)-2\omega f'(t)f(t)$. On the one hand, an easy computation gives

$$k_\omega(\chi)=f'(\chi)f(\chi)g(f^2(\chi))-2\omega f'(\chi)f(\chi)=f'(\chi)(\phi g(\phi^2)-2\omega\phi).$$

On the other hand,

$$\Delta\chi=\Delta f^{-1}(\phi)=(f^{-1})'(\phi)\Delta\phi+(f^{-1})''(\phi)|\nabla\phi|^2.$$

By the definition of f^{-1}, we have for all $t \in \mathbb{R}_+$,

$$(f^{-1})''(t) = \frac{1}{\sqrt{1-4t^2h'^2(t^2)}}\left(-4th'^2(t^2) - 8t^3h'(t^2)h''(t^2)\right).$$

Hence

$$\begin{aligned}\Delta\chi &= \sqrt{1-4\phi^2h'^2(\phi^2)}\Delta\phi + \frac{-4\phi h'^2(\phi^2) - 8\phi^3h'(\phi^2)h''(\phi^2)}{\sqrt{1-4\phi^2h'^2(\phi^2)}}|\nabla\phi|^2 \\ &= \frac{(1-4\phi^2h'^2(\phi^2))\Delta\phi - (4\phi h'^2(\phi^2) + 8\phi^3h'(\phi^2)h''(\phi^2))|\nabla\phi|^2}{\sqrt{1-4\phi^2h'^2(\phi^2)}} \\ &= f'(\chi)\left(\Delta\phi - 2\phi h'(\phi^2)\Delta h(\phi^2)\right).\end{aligned}$$

Then, since $f'(t) \neq 0$ for $t > 0$, χ is a solution of equation (1.31_ω)

$$\Delta\chi + k_\omega(\chi) = 0.$$

This change of function allows us to deduce some results on equation (1.30_ω) from results on equation (1.31_ω). We can define the action T_ω and the energy F associated with (1.31_ω), for all $\chi \in H^1(\mathbb{R}^2)$, by

$$T_\omega(\chi) = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla\chi|^2dx - \int_{\mathbb{R}^2}K_\omega(\chi)dx$$

and

$$F(\chi) = T_\omega(\chi) - \omega\int_{\mathbb{R}^2}|f(\chi)|^2dx$$

where

$$K_\omega(\zeta) = \int_0^\zeta k_\omega(t)dt = \frac{1}{2}G(f^2(\zeta)) - \omega f^2(\zeta). \tag{4.15}$$

The action T_ω and the energy F are well-defined on $H^1(\mathbb{R}^2)$ thanks to the following Trudinger inequality (see [50]).

Proposition 4.2.5. *There exist positive constants C and η such that for all $\chi \in H^1(\mathbb{R}^2)$ with $||\nabla\chi||_{L^2(\mathbb{R}^2)} \leq 1$,*

$$\int_{\mathbb{R}^2}(exp(\eta|\chi|^2) - 1)dx \leq C||\chi||^2_{L^2(\mathbb{R}^2)}.$$

From (4.7), we can find a constant C such that

$$\forall t \in \mathbb{R}_+,\ \ |K_\omega(t)| \leq Cf^2(t).$$

Using (4.11), we then have

$$\forall t \in \mathbb{R}_+,\ \ |K_\omega(t)| \leq C\left(e^{\nu' t} - 1\right)^2.$$

Let $\chi \in H^1(\mathbb{R}^2)$ and $r = max(||\nabla\chi||_{L^2(\mathbb{R}^2)}, 1)$. We can find a constant $C > 0$ such that for all $t \in \mathbb{R}_+$, $(e^{\nu' t} - 1)^2 \leq C(e^{\frac{\eta}{r}t^2} - 1)$. This implies, applying Proposition 2.2 to the function $\frac{\chi}{\sqrt{r}}$, that

$$\int_{\mathbb{R}^2} |K_\omega(\chi)|\, dx < +\infty.$$

Remark 4.2.6. *We have for all* $\chi \in H^1(\mathbb{R}^2)$, $T_\omega(\chi) \geq T_\omega(|\chi|)$.

Remark 4.2.7. *Let* $\phi \in H^1(\mathbb{R}^2)$ *and* $\chi = f^{-1}(\phi)$. *Then,* $E(\phi) = F(\chi)$.

We also set $Q(\chi) = \int_{\mathbb{R}^2} |f(\chi)|^2 dx$. We have the following proposition.

Proposition 4.2.8. *The functionals* F, T_ω *and* Q *are* C^1 *on* $H^1(\mathbb{R}^2)$.

Proof. This a classical result. One can see for example [5].

4.2.1 Existence Theorems

The two following theorems are adapted from theorems of H. Berestycki, T. Gallouët, and O. Kavian (see [5]). The first one is the existence Theorem (1.4.1)

Theorem. *Assume that* $0 < \omega < \omega_{max}$ *and suppose that g,h satisfy hypothesis* (4.1), (1.27), (4.3) *and* (4.4). *Then, there exists a solution* ϕ_ω *in* $H^1(\mathbb{R}^2)$ *of equation* (1.30_ω) *such that*

> *i)* $\phi_\omega > 0$ *in* $\mathbb{R}^2$,
> *ii)* ϕ_ω *is a radially symmetric decreasing function with respect to* $r = |x|$,
> *iii)* $\phi_\omega \in C^2(\mathbb{R}^2)$,
> *iv)* $\forall \alpha \in \mathbb{N}^2$, $|\alpha| \leq 2$, $\exists\, (c_\alpha, \delta_\alpha) \in (\mathbb{R}_+^*)^2$, *such that for all* $x \in \mathbb{R}^2$, $|D^\alpha \phi_\omega(x)| \leq C_\alpha e^{-\delta_\alpha |x|}$
> *v) For all* ϕ *solution of* (1.30_ω), *we have* $S_\omega(\phi_\omega) \leq S_\omega(\phi)$.

Proof. Let $\omega \in]0, \omega_{max}[$. We have to prove that under the assumptions of Theorem 1.4.1, the function k_ω satisfies

$$(a)\ \exists\, \zeta > 0 \text{ such that } K_\omega(\zeta) = \int_0^\zeta (k_\omega(t))dt > 0.$$
$$(b)\ \lim_{t\to 0} \frac{k_\omega(t)}{t} = -m < 0.$$
$$(c)\ \forall \alpha > 0,\ \exists\, C_\alpha > 0\ /\ \forall t \in \mathbb{R}_+\ |k_\omega(t)| \leq C_\alpha e^{\alpha t^2}$$

By the definition of ω_{max}, there exists $s_0 \in \mathbb{R}_+$ such that $G(s_0^2) > 2\omega s_0^2$. Let $\zeta = f^{-1}(s_0)$. We have $K_\omega(\zeta) = \frac{1}{2}G(f^2(\zeta)) - \omega f^2(\zeta) = \frac{1}{2}G(s_0^2) - \omega s_0^2 > 0$ and so (a) is satisfied. Since $g(0) = 0$, $f'(0) = 1$ and $f(0) = 0$, we have

$$\lim_{t\to 0} \frac{k_\omega(t)}{t} = -2\omega < 0.$$

Hence k_ω satisfies (b). From (4.8), (4.11) and (4.4), we have

$$\forall t \in \mathbb{R}_+,\ \ k_\omega(t) \leq C(e^{\nu' t})^4 \leq C' e^{\nu t^2}.$$

which implies that (c) is satisfied. Applying Theorem 1 of [5], we prove that equation (1.31_ω) has a solution $\chi \in C^2(\mathbb{R}^2)$, positive, radially symmetric, decreasing with respect to $r = |x|$. Furthermore, χ minimizes the action T_ω associated with (1.31_ω) among the solutions of (1.31_ω) and satisfies

$$\forall \alpha \in \mathbb{N}^2,\ \ |\alpha| \leq 2,\ \ \exists\, (c_\alpha, \delta_\alpha) \in \mathbb{R}_+^{*2} \text{ such that for } x \in \mathbb{R}^2,\ \ |D^\alpha \chi(x)| \leq C_\alpha e^{-\delta_\alpha |x|}.$$

Then, $\phi = f(\chi)$ is in $H^1(\mathbb{R}^2)$ and is a solution of (1.30_ω) satisfying clearly conditions $i) - iii)$. Since $f'(0) = 1$, ϕ clearly satisfies $iv)$. It remains to prove that ϕ minimizes the action S_ω over the set of solutions of (1.30_ω). Let ψ be another solution of (1.30_ω) and $\chi_1 = f^{-1}(|\psi|)$. Then

$$S_\omega(\psi) \geq S_\omega(|\psi|) = S_\omega(f(\chi_1)) = T_\omega(\chi_1)$$

thanks to Remark 4.2.7. Let

$$A = \left\{ \kappa \in \vec{H}^1(\mathbb{R}^2)\ \ /\ \ \int_{\mathbb{R}^2} K_\omega(\kappa)dx = 0 \right\}.$$

Since ψ is a solution of (1.30_ω), ψ satisfies the Pohozaev identity given by Proposition 4.2.4 (note that $\psi \in L^\infty_{loc}(\mathbb{R}^2)$ because $\chi_1 = f^{-1}(|\psi|)$, which satisfies (1.31_ω), does, by elliptic regularity). This implies that $\chi_1 \in A$. Recalling that χ is a solution of the following constrained minimization problem (see [5])

$$\left\{ \begin{array}{l} \chi \in A \\ T_\omega(\chi) = \inf\limits_{\kappa \in A} T_\omega(\kappa) \end{array} \right. \tag{4.16}$$

we have

$$S_\omega(\psi) \geq T_\omega(\chi_1) \geq T_\omega(\chi) = S_\omega(\phi) \text{ because } \phi \text{ is nonnegative. } \square$$

Theorem 4.2.9. *Assume that g, h satisfy hypothesis* (4.1), (1.27), (4.3) *and* (4.4) *and that $\omega \in]0, \omega_{max}[$. Then there exist infinitely many solutions (ϕ_k) in $H^1(\mathbb{R}^2)$ of equation (1.30_ω). Furthermore,*

i) $\forall k \geq 1$ ϕ_k *is radially symmetric;* $\phi_k \in C^2(\mathbb{R}^2)$,

ii) $\forall \alpha \in \mathbb{N}^2$, $\exists\, (C_k, \delta_k) \in (\mathbb{R}_+^*)^2$ *such that for* $x \in \mathbb{R}^2$, $|D^\alpha \phi_k(x)| \leq C_k e^{-\delta_k |x|}$

iii) $\forall k \geq 1$, $S_\omega(\phi_k) \leq S_\omega(\phi_{k+1})$ *and* $\lim\limits_{k \to +\infty} S_\omega(\phi_k) = +\infty$.

Proof. Again, this theorem is proved by applying Theorem 2 of [5] to the transformed equation (1.31_ω). The only thing to notice is that the change of function f is well-defined on $\mathbb{R}$. $\square$

We can make more precise here what happens in the model case $g(s) = 1 - \frac{1}{\sqrt{1+s}}$ and $h(s) = \sqrt{1+s} - 1$. In that case, $f(t) = \sinh(t)$ and equation (1.31_ω) becomes

$$\Delta\chi + (1-2\omega)\sinh(\chi)\cosh(\chi) - \sinh(\chi) = 0.$$

Hypothesis (1.27), (4.1) and (4.2) are satisfied. Since $G(s) = \left(\sqrt{1+s} - 1\right)^2$, (4.3) is satisfied with

$$\frac{g(s)}{\sqrt{G(s)}} = \frac{1}{\sqrt{1+s}} = 2h'(s).$$

Moreover, we have $\omega_{max} = \frac{1}{2}$. The fact that $f(t) = \sinh(t)$ implies that (4.4) is satisfied. Finally,

$$S_\omega(\phi) = \frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla\phi|^2 - |\nabla\sqrt{1+|\phi|^2}|^2 - (\sqrt{1+|\phi|^2} - 1)^2\right)dx + \omega\int_{\mathbb{R}^2}|\phi|^2 dx,$$

$$T_\omega(\chi) = \frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla\chi|^2 - (\cosh(\chi) - 1)^2\right)dx + \omega\int_{\mathbb{R}^2}\sinh^2(\chi)dx$$

and

$$Q(\chi) = \int_{\mathbb{R}^2}\sinh^2(\chi)dx.$$

4.2.2 Uniqueness theorem and properties of ground states

In this section, we prove the uniqueness of the ground states ϕ_ω up to translations and rotations. Before stating the main theorem of this section, we need the following proposition which make more precise the nature of the ground states of (1.30_ω).

Proposition 4.2.10. *Let $\psi \in H^1(\mathbb{R}^2)$ be a ground state of (1.30_ω) with $\omega \in]0, \omega_{max}[$. Then there exists $\theta \in \mathbb{R}$ and a real positive function ψ_1 such that for all $x \in \mathbb{R}^2$, $\psi(x) = e^{i\theta}\psi_1(x)$. Furthermore, there exists $x_0 \in \mathbb{R}^2$ such that $\psi_1(\cdot - x_0)$ is radially symmetric.*

Proof. Let $\psi \in H^1(\mathbb{R}^2)$ be a ground state of (1.30_ω). By Remark 4.2.3, we have $S_\omega(\psi) \geq S_\omega(|\psi|)$. We set $\kappa = f^{-1}(|\psi|)$. Then, $\kappa \in H^1(\mathbb{R}^2)$ and

$$S_\omega(|\psi|) = \frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla|\psi||^2 - |\nabla h(|\psi|^2)|^2\right)dx = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla\kappa|^2 dx.$$

Let χ be the ground state of equation (1.31_ω) introduced in the proof of Theorem 2.1. We set $\phi = f(\chi)$. The function ϕ is then a solution of equation (1.30_ω). Furthermore, by the definition of χ, we have

$$S_\omega(\psi) \geq S_\omega(|\psi|) = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla\kappa|^2 dx \geq \frac{1}{2}\int_{\mathbb{R}^2}|\nabla\chi|^2 dx = S_\omega(\phi).$$

Since ψ is a ground state of (1.30_ω), all the inequalities are equalities. This implies, together with the fact that $\int_{\mathbb{R}^2}K_\omega(\kappa)dx = 0$, that

$$\int_{\mathbb{R}^2}|\nabla\psi|^2 dx = \int_{\mathbb{R}^2}|\nabla|\psi||^2 dx \tag{4.17}$$

and that κ is a solution of the minimization problem (4.16). Thus, we can find a constant σ such that $\kappa(\frac{\cdot}{\sigma})$ is a ground state of (1.31_ω). Applying the Harnack principle (see [26]), one can see that for all $x \in \mathbb{R}^2$, $\kappa(x) \neq 0$. We set $\psi = \psi_1 e^{i\psi_2}$ where $\psi_1 = |\psi|$. Since $\psi_1(x) \neq 0$ for $x \in \mathbb{R}^2$ and $\psi, \psi_1 \in H^1(\mathbb{R}^2)$, the gradient of $e^{i\psi_2}$ exists everywhere on $\mathbb{R}^2$ and $\nabla e^{i\psi_2} \in L^2(\mathbb{R}^2)$. By (4.17), we then have for all $x \in \mathbb{R}^2$, $\psi_1(x)\nabla(e^{i\psi_2}) = 0$ and so $\nabla(e^{i\psi_2}) = 0$, which ends the proof of the first part of the lemma.

For the second part, it is enough to prove that there exists $x_0 \in \mathbb{R}^2$ such that $\kappa(\cdot - x_0)$ is radially symmetric, which is done exactly as in [43] (see also [25]). □

The next lemma is concerned with the connection between the ground states of (1.30_ω) and the ground states of (1.31_ω).

Lemma 4.2.11. *Let ϕ_ω be a real function. Then the function $\chi_\omega = f^{-1}(\phi_\omega)$ is a ground state of (1.31_ω) if and only if ϕ_ω is a ground state of (1.30_ω).*

Proof. Let ϕ_ω be a ground state of (1.30_ω). By Proposition 4.2.10, we can suppose that ϕ_ω is positive. The function $\chi_\omega = f^{-1}(\phi_\omega)$ is a solution of (1.31_ω) and $F(\chi_\omega) = E(\phi_\omega)$ by Remark 4.2.7. Let κ_ω be a ground state of (1.31_ω). By the elliptic regularity, one has $\kappa_\omega \in L^\infty(\mathbb{R}^2)$. Hence, the function $\psi_\omega = f(\kappa_\omega)$ belongs to $H^1(\mathbb{R}^2)$ and is a solution of (1.30_ω). So, we have $F(\kappa_\omega) = E(\psi_\omega) \geq E(\phi_\omega) = F(\chi_\omega)$. This implies that χ_ω is a ground state of (1.31_ω). With the same arguments, one can show the converse. □

We can now state the uniqueness Theorem (1.4.2). Following [54], we introduce two hypothesis which are satisfied in the model case (see the end of section 2.2 for more details) and are necessary to apply Theorem 1 of [54],

$$\exists\, a \in \mathbb{R} \ / \ \forall s \in \mathbb{R}_+,\ 0 < s < a \Rightarrow k_\omega(s) < 0 \text{ et } s > a \Rightarrow k_\omega(s) > 0 \tag{4.18}$$

$$\forall s \in \mathbb{R}_+,\ \frac{d}{ds}\left(\frac{K_\omega(s)}{k_\omega(s)}\right) \geq 0 \tag{4.19}$$

Theorem. *Under assumptions (4.18) and (4.19), there exists a unique ground state of equation (1.30_ω), up to translations and rotations.*

Proof. This result follows from Theorem 1 of [54]. Let ϕ_ω^1 and ϕ_ω^2 be two ground states of (1.30_ω). By Proposition 4.2.10, we can suppose that ϕ_ω^1 and ϕ_ω^2 are real and positive. We define $\chi_\omega^1 = f^{-1}(\phi_\omega^1)$ and $\chi_\omega^2 = f^{-1}(\phi_\omega^2)$. Then, by Lemma 4.2.11, the two functions χ_ω^1 and χ_ω^2 are ground states of equation (1.31_ω). Applying Theorem 1 of [54], we obtain $\chi_\omega^1 = \chi_\omega^2$ up to a translation, which ends the proof of Theorem 1.4.2. □

We now have to prove that assumptions (4.18) and (4.19) are satisfied in the model case. Recalling that $k_\omega(s) = (1 - 2\omega)\sinh(s)\cosh(s) - \sinh(s)$, a direct calculation shows that (4.18) is true. Furthermore, for all $s \in \mathbb{R}_+$,

$$\frac{d}{ds}\left(\frac{K_\omega(s)}{k_\omega(s)}\right)$$
$$= \frac{(1-2\omega)\cosh^3(s) + (4\omega^2+4\omega-3)\cosh^2(s)(3-2\omega)\cosh(s) - (4\omega^2+1)}{2k_\omega^2(s)}$$
$$= \frac{(\cosh(s)-1)[(1-2\omega)\cosh^2(s) + (4\omega^2+2\omega-2)\cosh(s) + 4\omega^2+1]}{2k_\omega^2(s)}.$$

Consider the polynomial $P(X) = (1-2\omega)X^2 + (4\omega^2+2\omega-2)X + 4\omega^2+1$. Its discriminant is $\Delta = 8\omega^2(\omega - \frac{1}{2})(2\omega+7)$ and so it is negative for all $\omega \in]0, \frac{1}{2}[$. We can deduce that for $s \in \mathbb{R}_+$ and $\omega \in]0, \frac{1}{2}[$,

$$(1-2\omega)\cosh^2(s) + (4\omega^2+2\omega-1)\cosh(s) + 4\omega^2 + 1 \geq 0,$$

which provides

$$\frac{d}{ds}\left(\frac{K_\omega(s)}{k_\omega(s)}\right) \geq 0.$$

4.3 The stability result

The following theorem is concerned with the solution of the evolution problem (1). We do not prove exactly the orbital stability in $H^1(\mathbb{R}^2)$ of the standing wave $u_\omega(x,t)$, the reason being that a bound on the energy $E(\phi)$ does not provide an H^1-bound on ϕ, and moreover the energy E is not continuous on H^1. This can be easily seen in the model case $g(s) = 1 - \frac{1}{\sqrt{1+s}}$, $h(s) = \sqrt{1+s} - 1$. We prove here that if the initial data u_0 is sufficiently close in $H^1(\mathbb{R}^2)$ to the ground state ϕ_ω of (1.30_ω) and if the corresponding energies are also close enough, then the function $f^{-1}(|u(x,t)|)$ where $u(x,t)$ is the solution of (1) with initial data u_0 (see [21]) stay close to the function $f^{-1}(|u_\omega(x,t)|)$ in $\vec{H}^1(\mathbb{R}^2)$. More precisely, we have Theorem (1.4.3):

Theorem. *Assume that g and h satisfy (1.27), (4.1), (4.2), (4.3), (4.4), (4.18) and (4.19). For $\omega \in]0, \omega_{max}[$, we define $u_\omega(x,t) = e^{i\omega t}\phi_\omega(x)$, where ϕ_ω is a ground state of equation (1.30_ω). Then, there exists an infinite subset Ω of $]0, \omega_{max}[$ which contains at least an interval such that for all $\omega \in \Omega$, the function $u_\omega(x,t)$ satisfies the following property: $\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that if $u_0 \in H^6(\mathbb{R}^2)$, $||u_0 - \phi_\omega||_{L^2(\mathbb{R}^2)} \leq \delta$ and $|E(u_0) - E(\phi_\omega)| \leq \delta$ then the function $u(x,t)$ solution of equation (1) with initial data u_0 defined on $[0,T]$ (see [21]) satisfies*

$$\sup_{t\in[0,T]} \inf_{y\in\mathbb{R}^2} ||f^{-1}(|u(\cdot,t)|) - f^{-1}(|\phi_\omega(\cdot - y)|)||_{H^1(\mathbb{R}^2)} \leq \varepsilon \tag{4.20}$$

and

$$\sup_{t\in[0,T]} |E(u(\cdot,t)) - E(|u(\cdot,t)|)| \leq \varepsilon. \tag{4.21}$$

Remark 4.3.1. (4.20) *and* (4.21) *imply*

$$\sup_{t\in[0,T]}\left|\int_{\mathbb{R}^2}\left(1-4|u|^2h'^2(|u|^2)\right)\left(|\nabla u|^2-|\nabla|u||^2\right)dx\right|\leq\varepsilon,$$

$$\sup_{t\in[0,T]}|E(u(\cdot,t))-E(u_\omega(\cdot,t))|\leq 2\varepsilon.$$

For $\mu>0$, we set

$$M_\mu=\left\{\chi\ /\ \chi\in H^1(\mathbb{R}^2)\text{ and }\int_{\mathbb{R}^2}f^2(\chi)dx=\mu\right\}.$$

The three following lemmas will be useful later.

Lemma 4.3.2. *There exists a constant* $C_0>0$ *such that*

i) $\forall\chi\in H^1(\mathbb{R}^2),\ \int_{\mathbb{R}^2}G(f^2(\chi))dx\leq C_0\left(\int_{\mathbb{R}^2}f^2(\chi)dx\right)\left(\int_{\mathbb{R}^2}|\nabla\chi|^2dx\right).$

ii) $\forall\varepsilon>0$, *there exists a nonnegative function* χ_ε *in* $H^1(\mathbb{R}^2)$ *such that*
$\int_{\mathbb{R}^2}G(f^2(\chi_\varepsilon))dx\geq(C_0-\varepsilon)\left(\int_{\mathbb{R}^2}f^2(\chi_\varepsilon)dx\right)\left(\int_{\mathbb{R}^2}|\nabla\chi_\varepsilon|^2dx\right).$

Proof. By the continuous embedding of $W^{1,1}(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$ and the Gagliardo-Nirenberg inequality, there exists $C_1>0$ such that $\forall\chi\in H^1(\mathbb{R}^2)$

$$\begin{aligned}\int_{\mathbb{R}^2}G(f^2(\chi))dx&\leq C_1\left(\int_{\mathbb{R}^2}|\nabla(\sqrt{G(f^2(\chi))})|dx\right)^2\\&\leq C_1\left(\int_{\mathbb{R}^2}\frac{g(f^2(\chi))}{\sqrt{G(f^2(\chi))}}|f(\chi)||f'(\chi)||\nabla\chi|dx\right)^2\\&\leq C_1C_2\left(\int_{\mathbb{R}^2}h'(f^2(\chi))|f(\chi)||f'(\chi)||\nabla\chi|dx\right)^2\ \text{by (4.3)}\\&\leq C\left(\int_{\mathbb{R}^2}\sqrt{1-4f^2(\chi)h'^2(f^2(\chi))}|f(\chi)||f'(\chi)||\nabla\chi|dx\right)^2\end{aligned}$$

by (1.27). Furthermore, we have for $t\in\mathbb{R}$,

$$f'(t)=\frac{1}{\sqrt{1-4f^2(t)h'^2(f^2(t))}}.$$

This implies

$$\begin{aligned}\int_{\mathbb{R}^2}G(f^2(\chi))dx&\leq C\left(\int_{\mathbb{R}^2}|f(\chi)||\nabla\chi|dx\right)^2\\&\leq C\left(\int_{\mathbb{R}^2}f^2(\chi)dx\right)\left(\int_{\mathbb{R}^2}|\nabla\chi|^2dx\right)\end{aligned}$$

by the Cauchy-Schwarz inequality. To complete the proof, we choose C_0 as the best constant in the last inequality. In $ii)$, we can choose a function $\chi_\varepsilon \geq 0$ thanks to the relation $|\nabla\chi| \geq |\nabla|\chi||$ a.e. and Remark 4.1.1. □

Lemma 4.3.3. *For all $\mu > 0$, F is bounded from below on M_μ.*

Proof. Using (4.7), we have for all $\chi \in M_\mu$,

$$\begin{aligned} F(\chi) = \geq & -\frac{1}{2}\int_{\mathbb{R}^2} G(f^2(\chi))dx \\ \geq & -C\int_{\mathbb{R}^2} f^2(\chi)dx = -C\mu. \ \square \end{aligned}$$

Lemma 4.3.4. *Denote by C_0 the constant of Lemma 4.3.2. Then*

$$\begin{aligned} i)\ \forall\mu \in]0, \frac{1}{C_0}],\ \inf_{\chi\in M_\mu} F(\chi) = 0. \\ ii)\ \forall\mu > \frac{1}{C_0},\ \inf_{\chi\in M_\mu} F(\chi) < 0. \end{aligned}$$

Proof.
$i)$ Let $\mu \in]0, \frac{1}{C_0}]$. From Lemma 4.3.2, we have for all $\chi \in M_\mu$,

$$F(\chi) \geq \frac{(1 - C_0\mu)}{2}\int_{\mathbb{R}^2} |\nabla\chi|^2 dx \geq 0.$$

Let $\chi \in M_\mu \cap L^\infty(\mathbb{R}^2)$. For $\alpha \in \mathbb{R}_+^*$, we set $\kappa_\alpha(\cdot) = f^{-1}(\alpha f(\chi(\alpha\cdot))$. We then have $\kappa_\alpha \in M_\mu$ and

$$0 \leq F(\kappa_\alpha) \leq \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\kappa_\alpha|^2 dx = \frac{\alpha^2}{2}\int_{\mathbb{R}^2} (f^{-1'}(\alpha f(\chi)))^2 f'^2(\chi)|\nabla\chi|^2 dx.$$

Since $\chi \in L^\infty(\mathbb{R}^2)$, the function $f'^2(\chi)$ is in $L^\infty(\mathbb{R}^2)$. Then, it follows from (4.12) that there exists a constant C independent of α such that

$$0 \leq F(\kappa_\alpha) \leq C\frac{\alpha^2}{2}\int_{\mathbb{R}^2} |\nabla\chi|^2 dx, \text{ which implies } F(\kappa_\alpha) \underset{\alpha\to 0}{\longrightarrow} 0$$

and ends the proof of $i)$.

$ii)$ Let $\mu > \frac{1}{C_0}$. We choose $\varepsilon > 0$ such that $\mu > \frac{1}{C_0 - \varepsilon}$. Let χ_ε be the function associated with ε in Lemma 4.3.2, ii). For $\alpha > 0$, we set $\kappa_{\varepsilon,\alpha}(\cdot) = \chi_\varepsilon(\frac{\cdot}{\alpha})$. It is easy to check that $\kappa_{\varepsilon,\alpha}$ also satisfies the relation $ii)$ of Lemma 4.3.2. Furthermore, $||f(\kappa_{\varepsilon,\alpha})||^2_{L^2(\mathbb{R}^2)} = \alpha^2||f(\chi_\varepsilon)||^2_{L^2(\mathbb{R}^2)}$. We choose α so that $||f(\kappa_{\varepsilon,\alpha})||^2_{L^2(\mathbb{R}^2)} = \mu$. Hence,

$$\begin{aligned} F(\kappa_{\varepsilon,\alpha}) &= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\kappa_{\varepsilon,\alpha}|^2 dx - \frac{1}{2}\int_{\mathbb{R}^2} G(f^2(\kappa_{\varepsilon,\alpha}))dx \\ &\leq \frac{1 - (C_0 - \varepsilon)\mu}{2}\int_{\mathbb{R}^2} |\nabla\kappa_{\varepsilon,\alpha}|^2 dx < 0. \end{aligned}$$

This completes the proof of the lemma. □

The proof of Theorem 3.1 is based on the study of the following minimization problem

$$\left\{\begin{array}{l}\chi \in M_\mu \\ F(\chi) = J(\mu) \equiv \inf\limits_{\kappa \in M_\mu} F(\kappa)\end{array}\right. \tag{4.22}$$

with $\mu > \frac{1}{C_0}$. The main point is Proposition (4.3.5)

Proposition 4.3.5. *Let $\mu > \frac{1}{C_0}$. For all sequence $(\chi_n)_{n\in\mathbb{N}} \in H^1(\mathbb{R}^2)$ such that $F(\chi_n)$ tends to $J(\mu)$ and $||f(\chi_n)||^2_{L^2(\mathbb{R}^2)}$ tends to μ as n tends to $+\infty$, there exists a subsequence $(\chi_{n_k})_{k\in\mathbb{N}}$ of $(\chi_n)_{n\in\mathbb{N}}$ and a family $(y_{n_k})_{k\in\mathbb{N}} \in \mathbb{R}^2$ such that $\chi_{n_k}(\cdot - y_{n_k})$ has a strong limit χ in $H^1(\mathbb{R}^2)$. Moreover, χ is a solution of problem (4.22).*

Proof. Let $(\chi_n)_{n\in\mathbb{N}}$ be a sequence in $H^1(\mathbb{R}^2)$ satisfying the hypothesis of Proposition 4.3.5. Since $(f(\chi_n))_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^2)$, then $(\chi_n)_{n\in\mathbb{N}}$ is also bounded in $L^2(\mathbb{R}^2)$ thanks to (4.12). Modulo a subsequence, there exists $\rho \in]0,\mu[$ such that $||\chi_n||^2_{L^2(\mathbb{R}^2)}$ tends to ρ as n tends to $+\infty$. From Lemma 4.3.4 and (4.7), we can deduce that $(\chi_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^2)$. We then apply the concentration-compactness lemma of P.L. Lions (see [41]) to the sequence $(|\chi_n|^2)_{n\in\mathbb{N}}$. There exists a subsequence of $(|\chi_n|^2)_{n\in\mathbb{N}}$, which we still denote by $(|\chi_n|^2)_{n\in\mathbb{N}}$, which satisfies one of the following properties: "compactness", "vanishing" or "dichotomy" (see [16]).

i) Suppose first that vanishing occurs. Then,

$$\forall R < +\infty, \quad \lim_{n\to+\infty} \sup_{y\in\mathbb{R}^2} \int_{y+B_R} |\chi_n|^2 dx = 0. \tag{4.23}$$

We first prove the following lemma.

Lemma 4.3.6. *There exists a constant $C > 0$ such that for all $\chi \in H^1(\mathbb{R}^2)$ with $||\chi||_{H^1(\mathbb{R}^2)} \leq 1$,*

$$\int_{\mathbb{R}^2} G(f^2(\chi))dx \leq C \left(\sup_{y\in\mathbb{R}^2} \int_{|x-y|\leq 1} |\chi|^2 dx \right) ||\nabla\chi||^2_{L^2(\mathbb{R}^2)}.$$

Proof. Cover $\mathbb{R}^2$ by a sequence of unit cubes $(Q_j)_{j\in\mathbb{N}}$ such that $Q_j \cap Q_k = \emptyset$ if $j \neq k$. It follows from the continuous embedding of $W^{1,1}(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$, the Gagliardo-Nirenberg inequality, (1.27) and (4.3) that there exists a positive C_1 such that

$$\int_{Q_j} G(f^2(\chi))dx \leq C_1 \int_{Q_j} f^2(\chi)dx \int_{Q_j} |\nabla\chi|^2 dx. \tag{4.24}$$

By Remark 3 of [50] and using (4.11) , since $||\chi||_{\vec{H}^1(\mathbb{R}^2)} \leq 1$, there is a positive C_2 such that

$$\int_{Q_j} f^2(\chi)dx \leq C_2 \int_{Q_j} |\chi|^2 dx. \tag{4.25}$$

Note that all Q_j's are isometric; and so C_1 and C_2 can be chosen to be independent of j. Summing on j inequality (4.24), we obtain, thanks to (4.25),

$$\int_{\mathbb{R}^2} G(f^2(\chi))dx \leq C \left(\sup_{j\in\mathbb{N}} \int_{Q_j} |\chi|^2 dx \right) \int_{\mathbb{R}^2} |\nabla\chi|^2 dx,$$

from which the result follows. □

Recalling that $(\chi_n)_{n\in\mathbb{N}}$ is a minimizing sequence for (4.22) and is bounded in $H^1(\mathbb{R}^2)$ and applying Lemma 4.3.6, one can find $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$\sup_{y\in\mathbb{R}^2} \left(\int_{|x-y|\leq 1} |\chi_n|^2 dx \right) ||\chi_n||^2_{\dot{H}^1(\mathbb{R}^2)} \geq -\frac{CJ(\mu)}{2} > 0$$

This last inequality contradicts (4.23). So vanishing cannot occur.

ii) Suppose that dichotomy occurs. We first prove a technical lemma on the subadditivity of the function J.

Lemma 4.3.7. *Let C_0 be the constant of Lemma 3.1. Then for all $\mu > \frac{1}{C_0}$, for all $\theta > 1$, we have $J(\theta\mu) < \theta J(\mu)$. Consequently, for $\mu > \frac{1}{C_0}$ and $\gamma \in]0, \mu[$, $J(\mu) < J(\gamma) + J(\mu - \gamma)$.*

Proof. Let σ be a function in M_μ and $\theta > 1$. We set $\kappa(x) = \sigma(\frac{x}{\sqrt{\theta}})$ for all $x \in \mathbb{R}^2$. It is clear that κ is in $M_{\theta\mu}$. Furthermore,

$$F(\kappa) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\sigma|^2 dx - \frac{\theta}{2}\int_{\mathbb{R}^2} G(f^2(\sigma))dx.$$

Let $\varepsilon > 0$ be such that $\theta - \varepsilon > 1$. We have

$$\begin{aligned} J(\theta\mu) \ &\leq F(\kappa) \leq \tfrac{\theta-\varepsilon}{2}\textstyle\int_{\mathbb{R}^2} |\nabla\sigma|^2 dx - \frac{\theta}{2}\int_{\mathbb{R}^2} G(f^2(\sigma))dx \\ &= \theta F(\sigma) - \tfrac{\varepsilon}{2}\textstyle\int_{\mathbb{R}^2} |\nabla\sigma|^2 dx. \end{aligned} \tag{4.26}$$

Let $(\sigma_n)_{n\in\mathbb{N}}$ be a real minimizing sequence for $J(\mu)$, and assume first

$$\int_{\mathbb{R}^2} |\nabla\sigma_n|^2 dx \longrightarrow 0 \quad \text{as } n \text{ tends to } +\infty.$$

By Lemma 3.1, we deduce that, since $(f(\sigma_n))_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} G(f^2(\sigma_n))dx \underset{n\to+\infty}{\longrightarrow} 0.$$

But this implies that $F(\sigma_n) \underset{n\to+\infty}{\longrightarrow} 0$, which contradicts Lemma 3.3. So there exists a $\beta > 0$ and a subsequence $(\sigma_{n_k})_{k\in\mathbb{N}}$ of $(\sigma_n)_{n\in\mathbb{N}}$ such that

$$\liminf_{n_k\to+\infty} \int_{\mathbb{R}^2} |\nabla\sigma_{n_k}|^2 dx \geq \beta.$$

From (4.26) applied to $\sigma_n \in M_\mu$, we deduce

$$J(\theta\mu) \leq \theta \liminf_{n_k \to +\infty} F(\sigma_{n_k}) - \varepsilon\beta = \theta J(\mu) - \varepsilon\beta < \theta J(\mu),$$

which ends the proof of the first part of the lemma. The second part is an easy consequence of the first one (see [41]). $\square$

Assume that the hypothesis of dichotomy is true. Then it is easy to see, by classical arguments, that for some $\gamma \in]0, \rho[$ ($\rho = \lim_{n\to+\infty} ||\chi_n||^2_{L^2(\mathbb{R}^2)}$), one can built two sequences $(v_\ell)_{\ell\in\mathbb{N}^*}$ and $(w_\ell)_{\ell\in\mathbb{N}^*}$ in $H^1(\mathbb{R}^2)$ with disjoint compact supports such that for some subsequence $(\chi_{n(\ell)})_{\ell\in\mathbb{N}^*}$ of $(\chi_n)_{n\in\mathbb{N}}$, one has

$$i) \quad ||\chi_{n(\ell)} - v_\ell - w_\ell||_{L^2(\mathbb{R}^2)} \leq \frac{1}{\ell} \tag{4.27}$$

$$ii) \quad \left| \int_{\mathbb{R}^2} |v_\ell|^2 dx - \gamma \right| \leq \frac{1}{\ell} \tag{4.28}$$

$$iii) \quad \left| \int_{\mathbb{R}^2} |w_\ell|^2 dx - (\rho - \gamma) \right| \leq \frac{1}{\ell} \tag{4.29}$$

$$iv) \quad \int_{\mathbb{R}^2} |\nabla \chi_{n(\ell)}|^2 - |\nabla v_\ell|^2 - |\nabla w_\ell|^2 dx \geq -\frac{1}{\ell}. \tag{4.30}$$

We set

$$\begin{aligned} R_\ell &= F(\chi_{n(\ell)}) - F(v_\ell) - F(w_\ell) \\ &= \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla \chi_{n(\ell)}|^2 - |\nabla v_\ell|^2 |\nabla w_\ell|^2\right) dx \\ &- \frac{1}{2}\int_{\mathbb{R}^2} \left(G(f^2(\chi_{n(\ell)})) - G(f^2(v_\ell + w_\ell))\right) dx. \end{aligned}$$

Since $(\chi_{n(\ell)})_{\ell\in\mathbb{N}^*}$ is bounded in $H^1(\mathbb{R}^2)$ and using (4.27) and (4.30), one can prove that the two sequences $(v_\ell)_{\ell\in\mathbb{N}^*}$ and $(w_\ell)_{\ell\in\mathbb{N}^*}$ are bounded in $H^1(\mathbb{R}^2)$. Then, it follows from Proposition 4.2.5 that $(f(v_\ell))_{\ell\in\mathbb{N}^*}$ and $(f(w_\ell))_{\ell\in\mathbb{N}^*}$ are bounded in $L^2(\mathbb{R}^2)$. Together with the fact that $(f(\chi_n))_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^2)$ and the relation (4.7), we obtain from (4.9)

$$\begin{aligned} &\left| \int_{\mathbb{R}^2} \left(G(f^2(\chi_{n(\ell)})) - G(f^2(v_\ell + w_\ell))\right) dx \right|^2 \\ &\leq C \left| \int_{\mathbb{R}^2} \left|f^2(\chi_{n(\ell)}) - f^2(v_\ell + w_\ell)\right|^2 dx \right| \\ &\leq \left(\int_{\mathbb{R}^2} \left|f(\chi_{n(\ell)}) + f(v_\ell + w_\ell)\right|^4 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \left|f(\chi_{n(\ell)}) - f(v_\ell + w_\ell)\right|^4 dx \right)^{\frac{1}{2}} \end{aligned}$$

by the Cauchy-Schwarz inequality. From (4.10), we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left|f(\chi_{n_0(\ell)}) - f(v_\ell + w_\ell)\right|^4 dx \\
&\leq C \int_{\mathbb{R}^2} e^{4\nu(|\chi_{n(\ell)}|+|v_\ell+w_\ell|)} \left|\chi_{n(\ell)} - v_\ell - w_\ell\right|^4 dx \\
&\leq C \int_{\mathbb{R}^2} \left(e^{4\nu(|\chi_{n(\ell)}|+|v_\ell+w_\ell|)} - 1\right) \left|\chi_{n(\ell)} - v_\ell - w_\ell\right|^4 dx \\
&+ C \int_{\mathbb{R}^2} \left|\chi_{n(\ell)} - v_\ell - w_\ell\right|^4 dx.
\end{aligned}$$

Then, applying Proposition 4.2.5 to $(\chi_{n(\ell)})_{\ell\in\mathbb{N}^*}$ and $(v_\ell + w_\ell)_{\ell\in\mathbb{N}^*}$, which are bounded in $H^1(\mathbb{R}^2)$, one obtains by Cauchy-Schwarz inequality

$$\begin{aligned}
\int_{\mathbb{R}^2} \left|f(\chi_{n(\ell)}) - f(v_\ell + w_\ell)\right|^4 dx \leq& C \left(\int_{\mathbb{R}^2} \left|\chi_{n(\ell)} - v_\ell - w_\ell\right|^8 dx\right)^{\frac{1}{2}} \\
&+ C \int_{\mathbb{R}^2} \left|\chi_{n(\ell)} - v_\ell - w_\ell\right|^4 dx
\end{aligned}$$

which easily implies by (4.27), using Gagliardo-Nirenberg inequality,

$$\int_{\mathbb{R}^2} \left|f(\chi_{n(\ell)}) - f(v_\ell + w_\ell)\right|^4 dx \underset{\ell\to+\infty}{\longrightarrow} 0. \tag{4.31}$$

Then,

$$\int_{\mathbb{R}^2} \left(G(f^2(\chi_{n(\ell)})) - G(f^2(v_\ell + w_\ell))\right) dx \underset{\ell\to+\infty}{\longrightarrow} 0.$$

Together with (4.30), this implies $\liminf\limits_{\ell\to+\infty} R_\ell \geq 0$, hence

$$\liminf_{\ell\to+\infty} F(\chi_{n(\ell)}) \geq \liminf_{\ell\to+\infty} F(v_\ell) + \liminf_{\ell\to+\infty} F(w_\ell). \tag{4.32}$$

Since $(f(v_\ell))_{\ell\in\mathbb{N}^*}$ and $(f(w_\ell))_{\ell\in\mathbb{N}^*}$ are bounded in $L^2(\mathbb{R}^2)$, there exists two constants τ and α such that, modulo a subsequence, $\int_{\mathbb{R}^2} f^2(v_\ell)dx \underset{\ell\to+\infty}{\longrightarrow} \tau$ and $\int_{\mathbb{R}^2} f^2(w_\ell)dx \underset{\ell\to+\infty}{\longrightarrow} \alpha$. Thanks to (4.31), we have $\alpha = \mu - \tau$. Now, it is clear that using a rescaling of v_ℓ and w_ℓ, one can prove that

$$\liminf_{\ell\to+\infty} F(v_\ell) \geq J(\tau) \quad \text{and} \quad \liminf_{\ell\to+\infty} F(w_\ell) \geq J(\alpha).$$

Then using the fact $(\chi_n)_{n\in\mathbb{N}}$ is a minimizing sequence for (4.22), we have from (4.32),

$$J(\mu) = \liminf_{\ell\in\mathbb{N}} F(\chi_{n(\ell)}) \geq J(\tau) + J(\mu - \tau)$$

which contradicts Lemma 4.3.7. So, dichotomy cannot occur.

iii) The only remaining possibility is the compactness of the minimizing sequence modulo translations, that is there exists a sequence $(y_n)_{n\in\mathbb{N}} \in \mathbb{R}^2$ such that

$$\forall \varepsilon > 0,\ \ \exists\, R_\varepsilon < +\infty \ \text{ such that } \forall\, n \in \mathbb{N},\ \int_{|x-y_n|\leq R_\varepsilon} |\chi_n(x)|^2 dx \geq \rho - \varepsilon$$

where $\rho = \lim_{n\to+\infty} ||\chi_n||^2_{L^2(\mathbb{R}^2)}$. Since $(\chi_n(\cdot - y_n))_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^2)$, $\chi_n(\cdot - y_n)$ converges weakly to a function χ in $H^1(\mathbb{R}^2)$ and converges strongly to χ in $L^2_{loc}(\mathbb{R}^2)$ (modulo a subsequence). The preceding inequality then implies $\chi_n(\cdot - y_n) \longrightarrow \chi(\cdot)$ strongly in $L^2(\mathbb{R}^2)$ as $n \to +\infty$. On the one hand, using Proposition 4.2.5 and the fact that $(\chi_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^2)$, one can prove that

$$\left| \int_{R^2} \left(f^2(\chi_n) - f^2(\chi)\right) dx \right| \underset{n\to+\infty}{\longrightarrow} 0$$

which implies that $\chi \in M_\mu$. On the other hand, by the lower semicontinuity of the H^1-norm,

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla \chi_n|^2 dx \geq \int_{\mathbb{R}^2} |\nabla \chi|^2 dx.$$

Thus, χ is a solution of problem (4.22) and so

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla \chi_n|^2 dx = \int_{\mathbb{R}^2} |\nabla \chi|^2 dx.$$

As a consequence, $\chi_n(\cdot - y_n) \underset{n\to+\infty}{\longrightarrow} \chi$ strongly in $H^1(\mathbb{R}^2)$. This achieves the proof of Proposition 4.3.5. □

It is now necessary to connect the ground states of equation (1.31_ω) with the solutions of problem (4.22). For a fixed $\omega \in]0, \omega_{max}[$, we recall that χ_ω denotes the ground state of (1.31_ω) which is positive, radially symmetric and we put $\lambda(\omega) = \int_{\mathbb{R}^2} |f(\chi_\omega)|^2 dx$. Concerning the function λ, we have the following lemma.

Proposition 4.3.8. *The mapping $\omega \longrightarrow \chi_\omega$ defined on $]0, \omega_{max}[$ is a continuous mapping with values in $\vec{H}^1(\mathbb{R}^2)$. As a consequence, λ is a continuous mapping on $]0, \omega_{max}[$.*

Proof. This is a classical result. We give a complete proof in the Appendix A. □

The next lemma introduces the Lagrange multipliers associated with the minimization problem (4.22).

Lemma 4.3.9. *Let $\mu > \frac{1}{C_0}$ and χ be a solution of problem (4.22) with parameter μ. Then, there exists a Lagrange multiplier $\theta(\mu) \in]0, \omega_{max}[$ such that χ is a ground state of equation (1.31_ω) with $\omega = \theta(\mu)$. We will say that $\theta(\mu)$ is a Lagrange multiplier associated with μ.*

Proof. Let χ be a solution of problem (4.22). The existence of a Lagrange multiplier $\theta(\mu)$ such that χ is a solution of equation (1.31_ω) with $\omega = \theta(\mu)$ can be proved by an application of the Implicit Functions Theorem (see [36]). Now, let χ_ω be a ground state of this equation. For $\alpha > 0$, we set $\psi(\cdot) = \chi_\omega(\sqrt{\alpha}\,\cdot)$. We choose α so that $||f(\psi)||^2_{L^2(\mathbb{R}^2)} = \mu$. We then have

$$\begin{aligned} F(\chi) \leq F(\psi) &= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\chi_\omega|^2 dx - \frac{1}{2\alpha}\int_{\mathbb{R}^2} G(f^2(\chi_\omega))dx \\ &= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\chi_\omega|^2 dx - \omega\mu \end{aligned}$$

by Pohozaev identity. Since χ also satisfies Pohozaev identity and moreover $||f(\chi)||^2_{L^2(\mathbb{R}^2)} = \mu$, one has

$$T_\omega(\chi) \leq T_\omega(\chi_\omega),$$

which ends the proof of the lemma. In particular, by Theorem 1.4.2, one has $\lambda(\omega) = \mu$ for any $\omega = \theta(\mu)$. $\square$

Remark 4.3.10. *The Lagrange multiplier $\theta(\mu)$ is not unique in general. It depends on the function χ.*

Remark 4.3.11. *In particular, if for a fixed ω, we have $\lambda(\omega) > \frac{1}{C_0}$ and if there exists a Lagrange multiplier $\theta(\mu)$ such that $\omega = \theta(\mu)$, then the corresponding ground state χ_ω is a solution of problem (4.22) with $\mu = \lambda(\omega)$. Conversely, if $\mu > \frac{1}{C_0}$ and if χ is a solution of problem (4.22) with parameter μ, then we can associate to χ a Lagrange multiplier $\theta(\mu)$ (which a priori depends on χ) such that χ is a ground state of equation (1.31_ω) with $\omega = \theta(\mu)$. Furthermore, we have $\lambda(\theta(\mu)) = \mu$. As a consequence, one can find a sequence $(\omega_n)_{n\in\mathbb{N}} \in]0, \omega_{max}[$ such that $\lambda(\omega_n) \underset{n\to+\infty}{\longrightarrow} +\infty$.*

Concerning the regularity of the function J, we have the following lemma.

Lemma 4.3.12. *The function J is concave on $]\frac{1}{C_0}, +\infty[$. As a consequence, there exists a countable subset $\mathcal{D}$ of $]\frac{1}{C_0}, +\infty[$ such that for all $\mu \in]\frac{1}{C_0}, +\infty[\backslash\mathcal{D}$, $J'(\mu)$ exists. Then, for every $\mu \in]\frac{1}{C_0}, +\infty[\backslash\mathcal{D}$, there exists a unique Lagrange multiplier associated with μ, which implies that (4.22) has a unique solution χ_0 up to translations.*

Proof. Recall that

$$J(\mu) = \inf_{\chi\in M_\mu} F(\chi).$$

Let $\mu \in]\frac{1}{C_0}, +\infty[$ and χ_0 be a solution of (4.22). By Lemma 4.3.9, there exists $\omega_0 \in]0, \omega_{max}[$ such that $\chi_0 = \chi_{\omega_0}$ and $\lambda(\omega_0) = \mu$. Furthermore, by Proposition 4.2.4,

$$J(\mu) = F(\chi_0) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\chi_{\omega_0}|^2 dx - \omega_0\mu. \tag{4.33}$$

If $\omega \in]0, \omega_{max}[$, one can see that $\chi_\omega\left(\sqrt{\frac{\lambda(\omega)}{\mu}}\right) \in M_\mu$, hence

$$J(\mu) \leq F\left(\chi_\omega\left(\sqrt{\frac{\lambda(\omega)}{\mu}}\right)\right) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\chi_\omega|^2 dx - \omega\mu.$$

This implies that

$$J(\mu) = \inf_{\omega \in]0,\omega_{max}[} \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \chi_\omega|^2 dx - \omega\mu. \tag{4.34}$$

Hence, the function J is concave, as a lower envelope of a family of concave functions of μ. It follows that there exists a countable set $\mathcal{D}$ such that for $\mu \in]\frac{1}{C_0}, +\infty[\backslash \mathcal{D}$, $J'(\mu)$ exists.

The second part of the lemma follows from Theorem E (section 1.2) of [56]. In fact, assume that for a fixed $\mu_0 \in]\frac{1}{C_0}, +\infty[\backslash \mathcal{D}$ there exists two different Lagrange multipliers ω_1 and ω_2. Then $A_1(\mu) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \chi_{\omega_1}|^2 dx - \omega_1 \mu$ and $A_2(\mu) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \chi_{\omega_2}|^2 dx - \omega_2 \mu$ are two lines of supports for J at μ_0, contradicting Theorem E (section 1.2) of [56]. $\square$

Remark 4.3.13. *By Remark* 4.3.11 *and Lemma* 4.3.12*, for every* $\mu \in]\frac{1}{C_0}, +\infty[\backslash \mathcal{D}$ *there exists a unique* $\omega = \omega(\mu) \in]0, \omega_{max}[$ *such that* $\chi_{\omega(\mu)}$ *is the unique solution of problem* (4.22) *up to translations.*

It is now possible to define Ω. We set

$$\Omega = \left\{ \omega(\mu) \ / \ \mu \in]\frac{1}{C_0}, +\infty[\backslash \mathcal{D} \right\}.$$

Remark 4.3.14. *For all* $\omega \in \Omega$*, the function* χ_ω *is the unique solution of problem* (4.22) *with* $\mu = \lambda(\omega)$*.*

Remark 4.3.15. *The set* Ω *is infinite thanks to Remark* 4.3.11 *and Lemma* 4.3.12*.*

Proof of Theorem 3.1. Let $\omega \in \Omega$. We argue by contradiction and we suppose first that there exists $\varepsilon > 0$ and two sequences $(u_0^n)_{n\in\mathbb{N}} \in H^6(\mathbb{R}^2)$ and $(t_n)_{n\in\mathbb{N}} \in \mathbb{R}$ such that if we call $u_n(x,t)$ the solution of equation (1) with initial data u_0^n, we have

$$||u_0^n - \phi_\omega||_{L^2(\mathbb{R}^2)} \underset{n\to+\infty}{\longrightarrow} 0 \tag{4.35}$$

$$E(u_0^n) - E(\phi_\omega) \underset{n\to+\infty}{\longrightarrow} 0 \tag{4.36}$$

$$\inf_{y\in\mathbb{R}^2} ||f^{-1}(|u_n(\cdot,t_n)|) - f^{-1}(\phi_\omega(\cdot - y))||_{H^1(\mathbb{R}^2)} = \varepsilon \tag{4.37}$$

According to Theorem 1.4.2, (4.37) is equivalent to

$$\inf_{\phi\in\mathcal{H}} ||f^{-1}(|u_n(\cdot,t_n)|) - f^{-1}(|\phi|)||_{\vec{H}^1(\mathbb{R}^2)} = \varepsilon,$$

where $\mathcal{H} = \{\phi \in H^1(\mathbb{R}^2) \ / \ \phi \text{ is a ground state of } (1.30_\omega)\}$.

Let $\chi_\omega = f^{-1}(\phi_\omega)$. By Lemma 4.2.11, χ_ω is a ground state of (1.31_ω). Hence, since $\omega \in \Omega$, χ_ω is a solution of the minimization problem (4.22) with parameter $\mu = \lambda(\omega)$. This implies, together with (4.36), Remark 4.2.6 and Remark 4.2.7, that

$$F(f^{-1}(|u_0^n|)) - F(\chi_\omega) \underset{n\to+\infty}{\longrightarrow} 0.$$

Let $\chi_n = f^{-1}(|u_n(\cdot, t_n)|)$. By conservation of charge and energy, we have in the same way

$$||u_n(t_n)||^2_{L^2(\mathbb{R}^2)} = ||f(\chi_n)||^2_{L^2(\mathbb{R}^2)} \underset{n\to+\infty}{\longrightarrow} \mu \quad \text{and} \quad F(\chi_n) \underset{n\to+\infty}{\longrightarrow} F(\chi_\omega) = J(\mu).$$

Applying Proposition 4.3.5 to the sequence $(\chi_n)_{n\in\mathbb{N}}$, one can find $\chi_0 \in H^1(\mathbb{R}^2)$, solution of problem (4.22) with parameter μ, and a sequence $(y_n)_{n\in\mathbb{N}} \in \mathbb{R}^2$ such that, modulo a subsequence,

$$\chi_n(\cdot) - \chi_0(\cdot - y_n) \underset{n\to+\infty}{\longrightarrow} 0 \quad \text{in } H^1(\mathbb{R}^2).$$

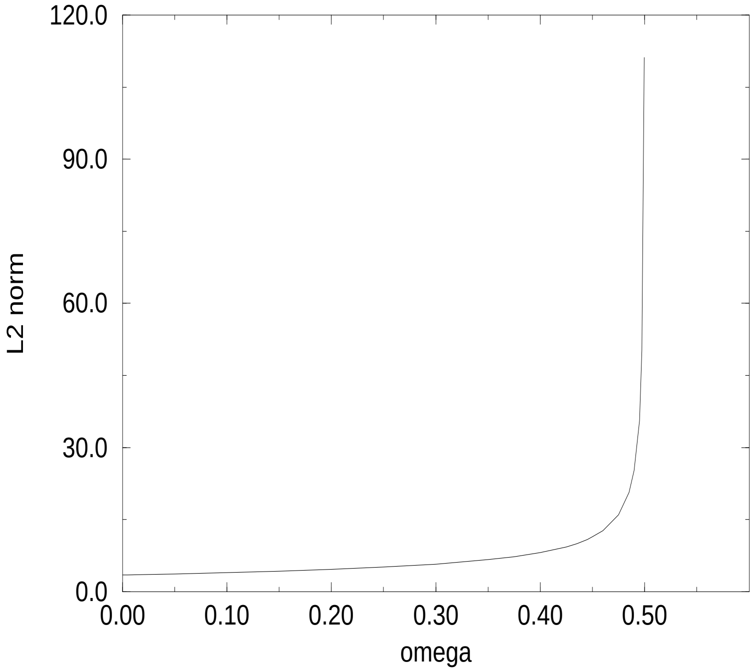
By Remark 4.3.14, χ_0 is a ground state of equation (1.31_ω). It follows that $\phi_0 = f(\chi_0) \in \mathcal{H}$, and so $\phi_0(\cdot - y_n)$ is also in $\mathcal{H}$. This contradicts (4.37).

Now, assume that (4.35), (4.36) hold and that

$$|E(u_n(\cdot, t_n)) - E(|u_n(\cdot, t_n)|)| \geq \varepsilon. \tag{4.38}$$

By conservation of energy, we have $E(u_n(\cdot, t_n)) - E(\phi_\omega) \underset{n\to+\infty}{\longrightarrow} 0$. Furthermore, keeping the same notations as above, we have $E(|u_n(\cdot, t_n)|) = F(\chi_n) \underset{n\to+\infty}{\longrightarrow} F(\chi_\omega) = E(\phi_\omega)$ because the energy F is continuous on $H^1(\mathbb{R}^2)$ (see Proposition 4.2.8). This contradicts (4.38) and achieves the proof of Theorem 1.4.3. $\square$

Remark 4.3.16. *It is expected in the model case $h(s) = \sqrt{1+s} - 1$, $g(s) = 1 - \frac{1}{\sqrt{1+s}}$ that λ is increasing on $]0, \omega_{max}[$ which would lead to the fact that there exists $\omega^* \in]0, \omega_{max}[$ such that $\Omega =]\omega^*, \omega_{max}[$. Indeed, assume that λ is increasing on $]0, \omega_{max}[$. Thanks to the remark 4.3.11, there exists ω^* such that for all $\omega > \omega^*$, $\lambda(\omega) > \frac{1}{C_0}$. Furthermore, it is clear that for $\omega \in]\omega^*, \omega_{max}[$, there exists a unique Lagrange multiplier associated with $\lambda(\omega)$. Numerically, this property seems to be true in the model case. Indeed, L. Di Menza has computed for $\omega \in]0, \omega_{max}[$, the function $\lambda(\omega)$ in this case. He has obtained the following graph:*


120.0
90.0
60.0
30.0
0.0
L2 norm
0.00
0.10
0.20
0.30
0.40
0.50
omega

Appendix A

Proof of Proposition 4.3.8

In this appendix, we give a complete proof of Proposition 4.3.8, that is we prove that the mapping $\omega \longrightarrow \chi_\omega$ is continuous. The proof will be divided into several lemmas. Denote by

$$d(\omega) \equiv \inf_{\chi \in A} T_\omega(\chi) = \inf_{\chi \in H^1(\mathbb{R}^2)} \left\{ \int_{\mathbb{R}^2} |\nabla \chi|^2 dx \ / \ \int_{\mathbb{R}^2} K_\omega(\chi) dx = 0 \right\}$$

where K_ω is defined by (4.15). We first need the following lemma which give an other definition of $d(\omega)$.

Lemma A.1. *We have*

$$d(\omega) = \inf_{\chi \in H^1(\mathbb{R}^2)} \left\{ \int_{\mathbb{R}^2} |\nabla \chi|^2 dx \ / \ \int_{\mathbb{R}^2} K_\omega(\chi) dx \geq 0 \right\}.$$

Proof. Let $\chi \in H^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be a positive function such that $\int_{\mathbb{R}^2} K_\omega(\chi) dx \geq 0$. Assume furthermore that $\int_{\mathbb{R}^2} K_\omega(\chi) dx \neq 0$ and denote

$$f(\rho) = \int_{\mathbb{R}^2} K_\omega(\rho \chi) dx.$$

By classical arguments, $f \in C^1(\mathbb{R}_+)$ and

$$f'(\rho) = \int_{\mathbb{R}^2} k_\omega(\rho \chi) \chi dx.$$

By (4.18), there exists $a > 0$ such that for all $s \in]0, 1[$, $k_\omega(s) < 0$. Since $\chi \in L^\infty(\mathbb{R}^2)$, one can find a constant $\rho_0 \in]0, 1[$ such that $||\rho_0 \chi||_{L^\infty(\mathbb{R}^2)} \leq a$. Then, for all $\rho \in]0, \rho_0[$, $f'(\rho) < 0$. Since $f(0) = 0$ and $f(1) > 0$, there exists $\rho_1 \in]0, 1[$ such that $f(\rho_1) = 0$.

Recalling that $d(\omega) = \int_{\mathbb{R}^2} |\nabla \chi_\omega|^2 dx$ (see section 2.2.1) and by definition of $d(\omega)$, one has

$$d(\omega) = \int_{\mathbb{R}^2} |\nabla \chi_\omega|^2 dx \leq \int_{\mathbb{R}^2} |\nabla (\rho_1 \chi)|^2 dx = \rho_1^2 \int_{\mathbb{R}^2} |\nabla \chi|^2 dx \leq \int_{\mathbb{R}^2} |\nabla \chi|^2 dx,$$

from which the result follows. $\Box$

Lemma A.2. *Let* $(\omega_1, \omega_2) \in]0, \omega_{max}[^2$ *with* $\omega_1 < \omega_2$. *Then* $d(\omega)$ *is bounded on* $[\omega_1, \omega_2]$.

Proof. Assume that there exists a sequence $(\omega_n)_{n\in\mathbb{N}}$ in $[\omega_1, \omega_2]$ such that

$$d(\omega_n) \underset{n\to+\infty}{\longrightarrow} +\infty. \tag{A.1}$$

Modulo a subsequence, one can find $\omega^* \in [\omega_1, \omega_2]$ such that

$$\omega_n \underset{n\to+\infty}{\longrightarrow} \omega^*.$$

Let $\chi \in H^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be such that $\int_{\mathbb{R}^2} K_{\omega^*}(\chi)dx > 0$. By the continuity of the mapping

$$\omega \longrightarrow \int_{\mathbb{R}^2} K_\omega(\cdot)dx$$

on $[\omega_1, \omega_2]$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\int_{\mathbb{R}^2} K_{\omega_n}(\chi)dx > 0.$$

By Lemma $A.1$, one has for all $n > n_0$

$$d(\omega) = \int_{\mathbb{R}^2} |\nabla \chi_{\omega_n}|^2 dx \leq \int_{\mathbb{R}^2} |\nabla \chi|^2 dx,$$

which contradicts (A.1). $\square$

Lemma A.3. *Let* $(\omega_1, \omega_2) \in]0, \omega_{max}[^2$ *with* $\omega_1 < \omega_2$*. Then* $\omega \longrightarrow \int_{\mathbb{R}^2} |\chi_\omega|^2 dx$ *is bounded on* $[\omega_1, \omega_2]$*.*

Proof. Let $\chi \in H^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} |\nabla\chi|^2 dx \leq 1$. By Lemma 4.3.2 i), (4.11) and Proposition 4.2.5, one can find a positive C such that

$$\int_{\mathbb{R}^2} G(f^2(\chi))dx \leq C \left(\int_{\mathbb{R}^2} |\chi|^2 dx \right)^{\frac{1}{2}}. \tag{A.2}$$

Applying Inequality (A.2) to χ_ω for $\omega \in [\omega_1, \omega_2]$, one has by Lemma $A.2$

$$\int_{\mathbb{R}^2} G(f^2(\chi_\omega))dx \leq C \left(\int_{\mathbb{R}^2} |\chi_\omega|^2 dx \right)^{\frac{1}{2}}.$$

By Proposition 4.2.4 and (4.12), one can write

$$\int_{\mathbb{R}^2} G(f^2(\chi_\omega))dx = 2\omega \int_{\mathbb{R}^2} f^2(\chi_\omega)dx \geq 2\omega \int_{\mathbb{R}^2} |\chi_\omega|^2 dx$$

which implies that for all $\omega \in [\omega_1, \omega_2]$

$$\left(\int_{\mathbb{R}^2} |\chi_\omega|^2 dx \right)^{\frac{1}{2}} \leq \frac{C}{2\omega}. \ \square$$

Lemma A.4. *$\omega \longrightarrow d(\omega)$ is a non decreasing mapping on $]0, \omega_{max}[$.*

Proof. Let $(\omega_1, \omega_2) \in]0, \omega_{max}[^2$ with $\omega_1 < \omega_2$. We have

$$\begin{aligned}\int_{\mathbb{R}^2} K_{\omega_1}(\chi_{\omega_2})dx &= \int_{\mathbb{R}^2} G(f^2(\chi_{\omega_2}))dx - \omega_1 \int_{\mathbb{R}^2} f^2(\chi_{\omega_2})dx \\ &= (\omega_2 - \omega_1) \int_{\mathbb{R}^2} f^2(\chi_{\omega_2})dx \quad \text{by Proposition 4.2.4} \\ &\geq 0.\end{aligned}$$

By Lemma $A.1$, one has, since $\chi_{\omega_2} \in L^\infty(\mathbb{R}^2)$,

$$d(\omega_1) = \int_{\mathbb{R}^2} |\nabla \chi_{\omega_1}|^2 dx \geq \int_{\mathbb{R}^2} |\nabla \chi_{\omega_2}|^2 dx = d(\omega_2). \ \square$$

We recall now a classical lemma (see [5] for more details).

Lemma A.5. *Let H be a continuous mapping from $\mathbb{R}$ to $\mathbb{R}$. Assume that H satisfies*

$$\begin{aligned}&i)\ \forall \alpha > 0,\ \exists\, C_\alpha > 0,\ \text{such that } \forall s \in \mathbb{R},\ |H(s)| \leq C_\alpha e^{\alpha s^2} \\ &ii)\ H(s) = s^2 + o(1)\ \text{as } s \to 0.\end{aligned}$$

Then for all radial bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\mathbb{R}^2)$ such that $u_n \underset{n\to+\infty}{\longrightarrow} u$ almost everywhere, we have $H(u_n) \underset{n\to+\infty}{\longrightarrow} H(u)$ in $L^1(\mathbb{R}^2)$.

Proof of Proposition 4.3.8. Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of $]0, \omega_{max}[$ such that $\omega_n \underset{n\to+\infty}{\longrightarrow} \omega^* \in]0, \omega_{max}[$. By Lemmas $A.2$ and $A.3$, $(\chi_{\omega_n})_{n\in\mathbb{N}}$ is a radial bounded sequence in $H^1(\mathbb{R}^2)$. Then, modulo a subsequence which we will still denote by χ_{ω_n}, one has

$$\chi_{\omega_n} \underset{n\to+\infty}{\longrightarrow} v \quad \text{weakly in } H^1(\mathbb{R}^2) \tag{A.3}$$

$$\chi_{\omega_n} \underset{n\to+\infty}{\longrightarrow} v \quad \text{a.e.} \tag{A.4}$$

Since f^2 and $G(f^2)$ satisfy the hypothesis of Lemma $A.5$, one has

$$\begin{aligned}\int_{\mathbb{R}^2} f^2(\chi_{\omega_n})dx &\underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^2} f^2(v)dx \\ \int_{\mathbb{R}^2} G(f^2(\chi_{\omega_n}))dx &\underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^2} G(f^2(v))dx\end{aligned}$$

which implies that

$$\omega_n \int_{\mathbb{R}^2} G(f^2(\chi_{\omega_n}))dx \underset{n\to+\infty}{\longrightarrow} \omega^* \int_{\mathbb{R}^2} G(f^2(v))dx. \tag{A.5}$$

$$\chi_{\omega_n} \underset{n\to+\infty}{\longrightarrow} v \quad \text{strongly in } L^2(\mathbb{R}^2). \tag{A.6}$$

Then, since for all $n \in \mathbb{N}$, $\int_{\mathbb{R}^2} K_{\omega_n}(\chi_{\omega_n})dx = 0$, we deduce that $\int_{\mathbb{R}^2} K_{\omega^*}(v)dx = 0$.

Assume now that for all $n \in \mathbb{N}$, $\omega_n \leq \omega^*$. By Lemma (A.4) and (A.3), we have

$$\int_{\mathbb{R}^2} |\nabla \chi_{\omega^*}|^2 dx \geq \liminf_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla \chi_{\omega_n}|^2 dx \geq \int_{\mathbb{R}^2} |\nabla v|^2 dx$$

which implies since $\int_{\mathbb{R}^2} K_{\omega^*}(v)dx = 0$ that

$$\int_{\mathbb{R}^2} |\nabla v|^2 dx = \int_{\mathbb{R}^2} |\nabla \chi_{\omega^*}|^2 dx.$$

Since v is a radial function and thanks to Theorem 1.4.2, we have $v = \chi_{\omega^*}$. Then,

$$\int_{\mathbb{R}^2} |\nabla \chi_{\omega^*}|^2 dx \geq \limsup_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla \chi_{\omega_n}|^2 dx \geq \liminf_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla \chi_{\omega_n}|^2 dx \geq \int_{\mathbb{R}^2} |\nabla \chi_{\omega^*}|^2 dx.$$

As a consequence, all these inequalities are equalities and so

$$d(\omega_n) \underset{n\to+\infty}{\longrightarrow} d(\omega).$$

Hence, $\nabla \chi_{\omega_n} \underset{n\to+\infty}{\longrightarrow} \nabla \chi_{\omega^*}$ strongly in $L^2(\mathbb{R}^2)$ which implies together with (A.6) that $\chi_{\omega_n} \underset{n\to+\infty}{\longrightarrow} \chi_{\omega^*}$ strongly in $H^1(\mathbb{R}^2)$.

Finally, assume that for all $n \in \mathbb{N}$, $\omega_n \geq \omega^*$. Furthermore, assume that there exists $\delta > 0$ such that

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla \chi_{\omega_n}|^2 dx \geq \int_{\mathbb{R}^2} |\nabla \chi_{\omega^*}|^2 dx + \delta. \tag{A.7}$$

We claim that there exists $\chi \in H^1(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} K_{\omega^*}(\chi)dx > 0 \quad \text{and} \quad \int_{\mathbb{R}^2} |\nabla \chi_{\omega}|^2 dx \leq \int_{\mathbb{R}^2} |\nabla \chi_{\omega^*}|^2 dx + \frac{\delta}{2}. \tag{A.8}$$

Indeed, introduce

$$g(\rho) = \int_{\mathbb{R}^2} K_{\omega^*}(\rho \chi_{\omega^*})dx.$$

Then, g is in $C^1(\mathbb{R})$ and

$$g'(\rho) = \int_{\mathbb{R}^2} k_{\omega^*}(\rho \chi_{\omega^*})\chi_{\omega^*} dx.$$

Now, the multiplication of equation (1.31_ω) with $\omega = \omega^*$ by χ_{ω^*} provides directly

$$g'(1) = \int_{\mathbb{R}^2} |\nabla \chi_{\omega^*}|^2 dx > 0.$$

Since $g(1) = 0$, there exists $\rho_0 > 1$ such that for all $\rho \in]1, \rho_0[$, $g(\rho) > 0$. We just have to choose $\chi = \rho \chi_{\omega^*}$ with $\rho > 1$ sufficiently close to 1 to prove (A.8).

Now, for ω_n sufficiently close to ω^* and such a χ, one has $\int_{\mathbb{R}^2} K_{\omega_n}(\chi)dx > 0$. Then, by Lemma A.1,

$$\int_{\mathbb{R}^2} |\nabla \chi_{\omega_n}|^2 dx \leq \int_{\mathbb{R}^2} |\nabla \chi|^2 dx.$$

Hence

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla \chi_{\omega_n}|^2 dx \leq \int_{\mathbb{R}^2} |\nabla \chi|^2 dx \leq \int_{\mathbb{R}^2} |\nabla \chi_{\omega^*}|^2 dx + \frac{\delta}{2},$$

contradicting (A.7). So

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla \chi_{\omega_n}|^2 dx \leq \int_{\mathbb{R}^2} |\nabla \chi_{\omega^*}|^2 dx,$$

which provides by Lemma (A.4) and the fact that for all $n \in \mathbb{N}$, $\omega_n \geq \omega^*$,

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla \chi_{\omega_n}|^2 dx = \liminf_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla \chi_{\omega_n}|^2 dx = \int_{\mathbb{R}^2} |\nabla \chi_{\omega^*}|^2 dx.$$

We can conclude together with (A.6) that $\chi_{\omega_n} \underset{n\to+\infty}{\longrightarrow} \chi_{\omega^*}$ strongly in $H^1(\mathbb{R}^2)$, wich ends the proof of Proposition 4.3.8. $\square$

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Résumé :

La propagation d'impulsions laser intenses et ultra-courtes dans un milieu gazeux engendre de nombreux phénomènes: en particulier, il se crée une colonne de plasma dans laquelle des ondes autofocalisées se propagent. Ces phénomènes sont décrits par les équations générales de la mécanique des fluides relatives aux plasmas relativistes associées aux équations de Maxwell. Notamment, l'évolution de l'amplitude de l'onde électromagnétique est régie par une équation de Schrödinger relativiste.

Dans cette thèse, nous étudions cette équation de Schrödinger relativiste. On commence par résoudre le problème de Cauchy local en dimension quelconque dans des espaces de Sobolev $H^s(\mathbb{R}^N)$ de régularité finie sans aucune hypothèse de petitesse sur la donnée initiale à l'aide de méthodes d'énergie.

Ensuite, nous montrons rigoureusement que dans un certain sens l'équation de Schrödinger relativiste est, en dimension deux d'espace, bien approchée par une équation d'onde de type Klein-Gordon.

Enfin, nous montrons l'existence, l'unicité aux translations et rotations près et la stabilité orbitale dans un sens un peu plus faible que le sens usuel de solutions de type onde solitaire pour l'équation de Schrödinger en dimension deux d'espace, en utilisant la méthode de minimisation introduite par T. Cazenave et P.L. Lions.

Mots-clés : Equations de Schrödinger - Problème de Cauchy - Méthodes d'énergie - Equations de Klein-Gordon - Ondes solitaires - Stabilité orbitale.

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