On the Boyd-Kadomstev System for a three-wave coupling problem and its asymptotic limit.

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Abstract

We consider the Boyd-Kadomstev system modeling laser plasma interaction. This Brillouin interaction couples the propagation of two laser beams, the incoming and the backscattered waves, with an ion acoustic wave which propagates at a much slower speed. The ratio ε between the plasma sound velocity and the (group) velocity of light is small, with typical value of order 10^{-3}. In this paper, we make a rigorous analysis of the behavior of solutions as $\varepsilon \to 0$. This problem can be cast in the general framework of fast singular limits for hyperbolic systems. The main new point which is addressed in our analysis is that the singular relaxation term present in the equation is a nonlinear first order system.

1 Introduction

We are concerned with the following non linear hyperbolic system

$$\left\{ \begin{array}{rcl} \varepsilon\partial_t u + \partial_x u &=& -wv \\ \varepsilon\partial_t v - \partial_x v &=& \overline{w}u \\ \partial_t w + \partial_x w &=& u\overline{v} \end{array} \right. \tag{1.1}$$

on a one-dimension spatial domain $[0, L]$, with initial data

$$u(0,x) = u_0(x), \quad v(0,x) = 0, \quad w(0,x) = w_0(x), \tag{1.2}$$

and boundary data

$$u(t,0) = u^{in}(t), \quad v(t,L) = 0, \qquad w(t,0) = 0, \tag{1.3}$$

This system, called the the Boyd-Kadomtsev system, has been addressed to model the wave interaction in plasmas, [8] and [3]. It was first introduced to describe weak plasma turbulence and next used in the framework of laser-plasma interaction, in which case the three unknown $u = u(t,x), v = v(t,x)$ and $w = w(t,x)$ correspond to the space-time envelope of the main laser wave, the backscattered laser wave due to Brillouin instability and the ion acoustic wave respectively.

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The boundary value $u^{in}(t)$ corresponds to the incoming laser field. In this framework, the observation time scale T is smaller than the characteristic time of variation of the incoming laser energy $|u^{in}(t)|^2$. This would yield to consider that the incoming energy is almost constant with respect to time t. However, in the present paper, we carry out the analysis assuming only that $u^{in} \in L^\infty(0,T)$ and $\partial_t u^{in} \in L^\infty([0,T])$. On the other hand, the parameter ε corresponds to the ratio between the plasma sound velocity and the (group) velocity of light. It assumed to be very small with rexpect to 1, its typical value in applications is in the order of 10^{-3}. The main subject of this paper is to make a detailed analysis of the behavior of the solutions of (1.1) (1.2) (1.3) as ε tends to 0.

Of course when dealing with realistic simulations one has to address three-dimension geometry and to account for diffraction, refraction phenomena as well as macroscopic hydrodynamic effects (see [2], [15], [17], for such models). We recall in appendix a glance of the derivation of the models which are handled for such simulations (see also [19], [4], [5] for mathematical justifications and [6], [7], [13] for systematic and rigorous derivation of geometric optics models). For example in the *HERA* code (cf. [12]) or the PF3D code (cf. [2]), the modelling is based on a system of the following type

$$\text{(1.4)} \qquad \begin{cases} \varepsilon\partial_t u + \partial_x u + i\alpha\Delta_\perp u = -\Gamma w v + i\beta_0(1-\Gamma)u \\ \varepsilon\partial_t v - \partial_x v + i\alpha\Delta_\perp v = \Gamma \overline{w} u + i\beta_0(1-\Gamma)v \\ \partial_t w + \partial_x w + (\eta + i\omega)w = \Gamma u\overline{v} - w\partial_x \log\Gamma \end{cases}$$

where $\Delta_\perp$ is the Laplace operator in the direction transverse to x and where α, ω, η and β_0 are real constants and Γ a smooth real function which close to 1 and related to the macroscopic variation of the electron density. Moreover, in realistic models, the initial value of the ion acoustic wave is a small random noise; here we assume that this initial value is a known quantity w_0 which does not depend on the parameter ε. It may be a crude approximation, nevertheless it is a first stage in order to understand the mathematical structure of the problem and to give ideas for efficient numerical schemes for solving system (1.4).

Therefore, we mean the Boyd-Kadomtsev system is sufficient to exhibit most of the difficulties of the three-wave coupling phenomena. For this system, notice first that if $w_0 = 0$, we get a trivial solution which is $v = w = 0$ and u solution of the simple advection equation $\varepsilon\partial_t u + \partial_x u = 0$. This trivial solution is of course unstable; this fact is related to the Brillouin instability.

Notice that the following balance relations are crucial for expressing the physical energy conservation:

$$\text{(1.5)} \qquad \begin{cases} \text{i)} \quad \varepsilon\partial_t(|u|^2 + |v|^2) + \partial_x(|u|^2 - |v|^2) = 0, \\ \text{ii)} \quad \partial_t(|w|^2 + \varepsilon|u|^2) + \partial_x(|w|^2 + |u|^2) = 0, \\ \text{iii)} \quad \partial_t(|w|^2 - \varepsilon|v|^2) + \partial_x(|w|^2 + |v|^2) = 0, \end{cases}$$

Up to our knowledge, except for the work [16] on solitons (on the full space) there is no convincing published mathematical work related to this system.

We first show that for a fixed value of ε (satisfying $0 < \varepsilon \leq 1$), this semi-linear hyperbolic initial-boundary value problem is well-posed in L^2 and in L^∞ (see theorem 2.1).

By proving this result, one checks that the backscattered energy $|v(t,0)|^2$ is such that $\varepsilon|v(t,0)|^2$ is bounded by a constant (depending on the final time T). But from a physical point of view, it is natural that the backscattered energy cannot be larger than the incoming one. In order to get such a bound, we are led to study the natural asymptotic problem obtained by setting ε equal to zero, which corresponds to an infinite speed of light.

This limiting system reads as

$$\partial_x u_* = -w_* v_*, \qquad -\partial_x v_* = \overline{w_*} u_* \qquad \partial_t w_* + \partial_x w_* = u_* \overline{v_*} \tag{1.6}$$

with boundary data

$$u_*(t,0) = u^{in}(t), \quad v_*(t,L) = 0, \quad w_*(t,0) = 0. \tag{1.7}$$

and an initial condition on w only

$$w_*(0,x) = w_0(x). \tag{1.8}$$

We will show (cf. Theorem 3.2) that this system is well-posed in L^∞ and that the backscattered energy $|v_*(t,0)|^2$ satisfies the natural bound

$$|v_*(t,0)|^2 \le |u^{in}(t)|^2 \tag{1.9}$$

The main objective of the paper is to prove the convergence of the solutions $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ of the full system (1.1)(1.2)(1.3) to the solutions of the limiting problem. This is a singular perturbation problem, with quadratic coupling terms.

Notice that, in general, the solution of the limit problem, does not satisfy the initial condition $(u_*, v_*)_{|t=0} = (u_0, 0)$ at $t = 0$; instead $(u_*, v_*)_{|t=0}$ satisfy

$$\begin{cases} \partial_x u_{*|t=0} = -w_0 v_{*|t=0}, & -\partial_x v_{*|t=0} = \overline{w}_0 u_{*|t=0} \\ u_{*|t=0}(0) = u^{in}_{|t=0}, & v_{*|t=0}(L) = 0. \end{cases}$$

This indicates that the solution with $\varepsilon > 0$ has a small initial layer in the variables (u, v) to match the initial condition (1.3) to the functions $(u_{*|t=0}, v_{*|t=0})$. The main result we want to prove is the following

Theorem 1.1 (cf.Theorem 4.1)**.** *Suppose that the initial data* u_0, w_0 are in $H^1(0,L)$ *and satisfy the corner conditions (2.1). Then the solutions* $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ *of* (1.1) (1.2) (1.3) *converges in* $[L^2([0,T] \times [0,L])]^3$ *to* (u_*, v_*, w_*) *the solution of* (1.6) (1.7) (1.8).

From a physical point of view this result means, that the backscattered laser intensity $|v^\varepsilon(t,x)|^2$ of the initial problem may be approximated (after a small initial layer) by the backscattered laser intensity corresponding to an infinite speed of light which satisfies the natural bound (1.9).

Another motivation for this study comes from numerical issues. Indeed, for three-dimension parallel numerical codes dealing with laser-plasma interaction and based on system like (1.4), the time discretization is performed up to now in an explicit way, using a classical upwind difference scheme to solve the propagation equations for u and v. This leads to a sub-cycling technique with a time step δt constrained by the criterium $\delta t \le \varepsilon \delta x$; that is to say δt is very small if compared to the characteristic time of the Brillouin instability growing and of course of the characteristic time of the macroscopic hydrodynamic evolution which is in the order of $t_{\text{hydro}} = \delta x$. So it would be interesting to propose a time implicit method for the solution of the equations for u and v, handling the time derivative $\varepsilon\partial_t u$ and $\varepsilon\partial_t v$ as perturbative terms (as it is done for classical propagation equations without coupling terms $\varepsilon\partial_t u + \partial_x u + i\alpha\Delta_\perp u$ $= i\beta(1-\Gamma)u$, cf. [1] for example). Our analysis validates this approach and we refer to [20] for a proposition of implementation of this idea.

The equation (1.1) can be recast in the form

$$\partial_t \mathcal{U} + L_0(\mathcal{U}) + \frac{1}{\varepsilon} L_1(\mathcal{U}) = 0 \tag{1.10}$$

with

$$\mathcal{U} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \qquad L_0(\mathcal{U}) = \begin{pmatrix} 0 \\ 0 \\ \partial_x w - u\overline{v} \end{pmatrix}, \qquad L_1(\mathcal{U}) = \begin{pmatrix} \partial_x u + wv \\ -\partial_x v - \overline{w}v \\ 0 \end{pmatrix}.$$

Written in this form, the problem falls into the category of fast singular limits or relaxation problems, see e.g. [9, 21, 14] and the references therein. Compared to the mentioned papers, the main new difficulty is that the relaxation term $L_1(\mathcal{U})$ is a nonlinear differential system. Using the conservations (1.5), one easily gets uniform bounds for the solutions (see the following section). With them, one can extract subsequences which converge weakly. To prove that the limit satisfy the expected limiting system (1.6), the difficulty is to pass to the weak limit in the quadratic terms. With a bit of compensated compactness, there is no difficulty for the terms wv and $\overline{w}u$. But the term $\overline{v}u$ is highly nontrivial. In addition, the easy first bounds are not sufficient to pass to the limit in the initial conditions, reflecting again the presence of an initial layer.

The heuristic argument for the proof of the main theorem is general to relaxation problems:

- The fast evolution $\partial_t \mathcal{U} + \frac{1}{\varepsilon} L_1(\mathcal{U}) = 0$ brings the initial data to a stationary (with respect to the fast time t/ε) solution of $L_1(\mathcal{U}) = 0$. After reducing the problem to homogeneous boundary conditions, the main point is to analyze the linear problem

$$\partial_s \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \partial_x u + wv \\ -\partial_x v - \overline{w}v \end{pmatrix} = 0. \tag{1.11}$$

where w is given and independent of the fast time $s = t/\varepsilon$. One shows that the energy of solutions of this system (with homogeneous boundary conditions) decays exponentially with respect to s.

- Next, the decay of energy result is extended to solutions of (1.11) when w is slowly varying with s. This proves that the invariant manifold $\mathcal{M}$ of solutions of $L_1(\mathcal{U}) = 0$ is attractive for the fast evolution.

- Using sufficiently good uniform *a priori* bounds for the solutions, we can use the properties above to prove that the dynamics of solutions of (1.1) is close to the projected dynamic on $\mathcal{M}$, that is (1.6).

The outline of the paper is the following. First we give a priori estimates for the solutions of the Boyd-Kadomtsev system(1.1) (1.2) (1.3) and we show that it is well-posed. The second section is devoted to prove the existence and uniqueness of solution of the limiting system. In the third one, we prove the main result.

2 The full system

In the sequel of the paper, we assume that the corner compatibility conditions holds

$$u_0(0) = u^{in}_{|t=0}, \quad w_0(0) = 0. \tag{2.1}$$

According to (1.5) we get the basic estimates

$$\begin{aligned} \varepsilon\partial_t(\|u\|^2_{L^2_x} + \|v\|^2_{L^2_x}) + (|u(t,L)|^2 + |v(t,0)|^2) &= |u^{in}|^2, \\ \partial_t(\|w\|^2_{L^2_x} + \varepsilon\,\|u\|^2_{L^2_x}) + (|u(t,L)|^2 + |w(t,L)|^2) &= |u^{in}|^2, \\ \partial_t(\|w\|^2_{L^2_x} - \varepsilon\,\|v\|^2_{L^2_x}) + |w(t,L)|^2 &= |v(t,0)|^2. \end{aligned} \tag{2.2}$$

Here and in the sequel we set $L^2_x = L^2(0,L)$ and $L^\infty_x = L^\infty(0,L)$. Let $\Omega = [0,T]\times[0,L]$. We introduce the three velocities $c_1 = \varepsilon^{-1}, c_2 = -\varepsilon^{-1}, c_3 = 1$ and define the operators

$$K_i = \partial_t + c_i\partial_x.$$

The first main result is the following

Theorem 2.1. *If the data u_0, w_0 are in L^∞_x and satisfy the corner conditions (2.1), the IBV problem* (1.1) (1.2) (1.3) *has a unique solution $(u,v,w) = (u^\varepsilon, v^\varepsilon, w^\varepsilon)$ such that K_1u, K_2v, K_3w are in $L^2(\Omega)$. Moreover it belongs to $[C^0(0,T;L^2_x)]^3$ and $(L^\infty(\Omega))^3$.*

The first subsection is devoted to the existence of a solution $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ of system (1.1)(1.2)(1.3) in a L^2 framework. In the second one, if the initial data are in L^∞_x, we prove bounds of the solutions in $L^2(\Omega), L^\infty(\Omega)$ and $L^2(0,T;L^\infty_x)$.In the last one, with the hypothesis $u_0, w_0 \in H^1(0,L)$, we show that the derivatives $\partial_x u^\varepsilon, \partial_x v^\varepsilon, \partial_x w^\varepsilon, \partial_t w^\varepsilon$ are bounded in $L^2(\Omega)$ uniformely with respect to ε which will be usefull for the asymptotic analysis (cf theo. 2.7).

2.1 Existence and uniqueness in L^2 framework.

Denote $a_1 = -\varepsilon^{-1}$, $a_2 = \varepsilon^{-1}$, $a_3 = 1$. Let us address the original system with general initial values (u_0, v_0, w_0) in $(L^2_x)^3$; it reads as

$$\begin{aligned} K_1u &= a_1vw, \qquad u(0,.) = u_0, \qquad u(t,0) = u^{in} \\ K_2v &= a_2u\overline{w}, \qquad v(0,.) = v_0, \qquad v(t,L) = 0 \\ K_3w &= a_3\overline{v}u, \qquad w(0,.) = w_0, \qquad w(t,0) = 0 \end{aligned} \tag{2.3}$$

In this subsection, ε is fixed and the constants may depend on ε. For a final time denoted by τ, let us introduce the following spaces

$$\mathcal{V}^\tau_1 = \{u = u(t,x), \text{ s. t. } K_1u \in L^2(0,\tau;L^2_x), \quad u(0,.) \in L^2_x, \quad u(.,0) \in L^2(0,\tau)\},$$

endowed with the norms

$$\|u\|_{\mathcal{V}_1^\tau} = \|K_1 u\|_{L^2(0,\tau;L^2_x)} + \|u(0,.)\|_{L^2_x} + \|u(.,0)\|_{L^2(0,\tau)}$$

and the analogous for $\mathcal{V}_2^\tau$ and $\mathcal{V}_3^\tau$, with K_1 replaced by K_2 and K_3 (and the corresponding boundary condition in L for $\mathcal{V}_2^\tau$).

We first have the following local existence result.

Proposition 2.2. *Assume that the initial values (u_0, v_0, w_0) are in L^2_x. Then, there is a unique solution (u, v, w) in $\mathcal{V}_1^T \times \mathcal{V}_2^T \times \mathcal{V}_3^T$ of problem (2.3). Moreover (u, v, w) belongs to $(C(0, T; L^2_x))^3$.*

The proof is based the following general result inspired by the one stated in L^1 framework in [23].

Lemma. *(Compensated integrability) There exists a constant C_0 such that for all τ and for all functions u, v in $\mathcal{V}_1^\tau \times \mathcal{V}_2^\tau$ we get*

$$\|uv\|^2_{L^2(0,\tau;L^2_x)} \leq C_0 \left[\alpha_u + \tau \|K_1 u\|^2_{L^2(0,\tau;L^2_x)}\right] \left[\alpha_v + \tau \|K_2 v\|^2_{L^2(0,\tau;L^2_x)}\right]$$

with $\alpha_u = \|u(.,0)\|^2_{L^2(0,\tau)} + \|u_0\|^2_{L^2_x}$ and $\alpha_v = \|v(.,L)\|^2_{L^2(0,\tau)} + \|v_0\|^2_{L^2_x}$.

The same result holds for the other products uw and vw (and for the products $u\overline{v}, \overline{w}v$). Of course the constant C_0 depends on the velocities c_i occuring in the operators K_i, that is to say on ε.

Proof.

Denote $f = K_1 u$, we have

$$u(t,x) = u_0(x - c_1 t)\mathbf{1}_{x>c_1 t} + u^{in}(\frac{c_1 t - x}{c_1})\mathbf{1}_{x<c_1 t} + \int_0^t f(t-s, x - c_1 s)ds$$

Then we get for all $t < \tau$,

$$|u(t,x)|^2 \leq 2\left(|u_0(x - c_1 t)|\mathbf{1}_{x>c_1 t} + |u^{in}(\frac{c_1 t - x}{c_1})|\mathbf{1}_{x<c_1 t}\right)^2 + 2\tau F(x - c_1 t),$$

with $F(y) = \int_0^t |f(y + c_1 s, s)|^2 ds$, that is to say $|u(t,x)|^2 \leq \phi_u(x - c_1 t)$ where the function ϕ_u defined on $[-c_1\tau, L]$ is given by

$$\phi_u(\sigma) = 2|u_0(\sigma)|^2\mathbf{1}_{\sigma>0} + 2|u^{in}(-\frac{\sigma}{c_1})|^2\mathbf{1}_{\sigma<0} + 2\tau F(\sigma)$$

Moreover we see that

$$\|\phi_u\|_{L^1_x} \leq 2\alpha_u + 2\tau \|K_1 u\|^2_{L^2(0,\tau,L^2_x)}, \qquad \alpha_u = \|u_0\|^2_{L^2_x} + \|u(.,0)\|^2_{L^2_t}.$$

By the same way, we get $|v(t,x)|^2 \leq \phi_v(x - c_2 t)$ with ϕ_v such that for all $t \leq \tau$

$$\|\phi_v\|_{L^1_x} \leq 2\alpha_v + 2\tau \|K_2 v\|^2_{L^2(0,\tau,L^2_x)}, \qquad \alpha_u = \|v_0\|^2_{L^2_x} + \|u(.,L)\|^2_{L^2_t}.$$

Now with the new variables $y = (x - c_1 t)$ and $y' = (x - c_2 t)$ using the fact that $dxdt = |c_1 - c_2|^{-1}dydy'$, we get

$$\int\int |u(t,x)|^2|v(t,x)|^2 dxdt \leq \int\int |c_1-c_2|^{-1}\phi_u(y)\phi_v(y')dydy' \leq |c_1-c_2|^{-1}\left\|\phi_u\right\|_{L^1_x}\left\|\phi_v\right\|_{L^1_x}$$

and the result follows. □

Proof of the proposition.

Denote in the sequel $\mathcal{U}=\{u,v,w\}$. Define the space $\mathcal{L}^{2,\tau}=(L^2(0,\tau,L^2_x))^3$ endowed by the norm $\|\mathcal{U}\|_{\mathcal{L}^{2,\tau}}=\sup_i\|\mathcal{U}_i\|_{L^2(0,\tau,L^2_x)}$. Denote $\mathcal{F}(\mathcal{U})=\{a_1vw,a_2u\overline{v},a_3u\overline{v}\}$ and $\mathcal{K}\mathcal{U}=\{K_1\mathcal{U}_1,K_2\mathcal{U}_2,K_3\mathcal{U}_3\}$.

Existence for τ small enough,.

It is based on a fixed point algorithm. Let us denote by $\mathcal{U}^0=(u^0,v^0,w^0)$ the solution of problem (2.3) without the quadratic right hand side terms and define the sequence $\mathcal{U}^{n+1}=(u^{n+1},v^{n+1},w^{n+1})$ by

$$\begin{aligned}
\mathcal{K}\mathcal{U}^{n+1} &= \mathcal{F}(\mathcal{U}^n)\\
\mathcal{U}^{n+1}(0,.) &= \{u_0,v_0,w_0\}\\
u^{n+1}(t,0) &= u^{in}, \qquad v^{n+1}(t,L)=0, \qquad w^{n+1}(t,0)=0.
\end{aligned}$$

According to the previous lemma, we see that

$$\|\mathcal{F}(\mathcal{U})\|^2_{\mathcal{L}^{2,\tau}} \leq C_1^2\left[A+\tau\|\mathcal{K}\mathcal{U}\|^2_{\mathcal{L}^{2,\tau}}\right]^2$$

with $A=\sup\left(\|u_0\|^2_{L^2_x},\|v_0\|^2_{L^2_x},\|w_0\|^2_{L^2_x}\right)+\|u(.,0)\|^2_{L^2(0,T)}$. Since $\mathcal{K}\mathcal{U}^{n+1}=\mathcal{F}(\mathcal{U}^n)$, the sequence $q_n=\|\mathcal{K}\mathcal{U}^n\|_{\mathcal{L}^{2,\tau}}$ satisfies $q_0\leq A$ and

$$C_1^{-1}q_{n+1}\leq A+\tau q_n^2$$

But for τ small enough, the equation $\tau q^2-C_1^{-1}q+A=0$ admits positive roots; denote by C_2 the smallest one. Therefore we get $q_n=\|\mathcal{K}\mathcal{U}^n\|_{\mathcal{L}^{2,\tau}}\leq C_2$ for all n.

Now, for fixed initial and boundary values, address the mapping $\mathcal{K}\mathcal{U}^n\mapsto\mathcal{K}\mathcal{U}^{n+1}$. The previous lemma says that, for τ small enough, it is a strictly contraction mapping for the norm $\|\cdot\|_{\mathcal{L}^{2,\tau}}$, so the sequence $\mathcal{K}\mathcal{U}^n$ converges to some element $\mathcal{G}$ in $\mathcal{L}^{2,\tau}$. We may define $\mathcal{U}$ the solution of $K_i\mathcal{U}_i=\mathcal{G}_i$ whith the same initial and boundary as above and we have $K_i\mathcal{U}_i^n\to K_i\mathcal{U}_i$ in $\mathcal{L}^{2,\tau}$. So the previous lemma implies $\lim_n\mathcal{F}(\mathcal{U}^n)=\mathcal{F}(\mathcal{U})$, therefore $\mathcal{G}=\mathcal{F}(\mathcal{U})$ and $\mathcal{K}\mathcal{U}=\mathcal{F}(\mathcal{U})$, i.e. $\mathcal{U}=\{u,v,w\}$ is solution to (2.3).

Uniqueness.

Assume that there exist two solution $\mathcal{U},\widehat{\mathcal{U}}$; they satisfy $\mathcal{K}\mathcal{U}=\mathcal{F}(\mathcal{U})$ and $\mathcal{K}\widehat{\mathcal{U}}=\mathcal{F}(\widehat{\mathcal{U}})$. Then, setting $\widetilde{\mathcal{U}}=\widehat{\mathcal{U}}-\mathcal{U}$, we get

$$|K_i\widetilde{\mathcal{U}}_i|\leq|a_i|\left(|\mathcal{U}_j\widetilde{\mathcal{U}}_{j'}|+|\widetilde{\mathcal{U}}_j\widehat{\mathcal{U}}_{j'}|\right), \qquad \text{with } (j,j')\neq i$$

Acording to previous lemma, since the initial and boundary values of $\widetilde{\mathcal{U}}$ are zero, we get

$$\left\|K_i\widetilde{\mathcal{U}}_i\right\|^2_{L^2_{t,x}}\leq|a_i|C_0\left[\tau\left\|\mathcal{K}\widetilde{\mathcal{U}}\right\|^2_{\mathcal{L}^{2,\tau}}(\alpha_u+\tau\|\mathcal{K}\mathcal{U}\|^2_{\mathcal{L}^{2,\tau}})+\tau\left\|\mathcal{K}\widetilde{\mathcal{U}}\right\|^2_{\mathcal{L}^{2,\tau}}(\alpha_u+\tau\left\|\mathcal{K}\widehat{\mathcal{U}}\right\|^2_{\mathcal{L}^{2,\tau}})\right].$$

Thus $\widetilde{\mathcal{U}} = 0$ for τ small enough.

Global existence

Indeed, according to the conservation law (2.2), we see that the solution (u, v, w) defined on a local time interval may be extented up to the final time $\tau = T$. Since the right hand sides $\mathcal{F}(\mathcal{U})_i$ belong to $L^2(0, T; L^2_x)$, classical semi-group arguments imply that the solution $\mathcal{U}$ is in $(C(0, T; L^2_x))^3$. $\square$

With the help of the point i) of proposition 2.6 below, this achieves the proof of theorem 2.1.

2.2 Estimates in L^∞

We now make a stronger assumption on the initial data: u_0 and w_0 belong to L^∞_x. Let (u, v, w) the solution to the system (1.1) (1.2) (1.3), it is in $(C(0, T; L^2_x))^3$.

Proposition 2.3. *There is a constant C independent of ε, such that*

$$\|w\|_{L^\infty(\Omega)} \le C. \tag{2.4}$$

For proving this bound, we first look for estimates of u and v along the characteristics of $\partial_t + \partial_x$. Consider a point $P = (t, x) \in \Omega$. The backward characteristics hit the boundary at points P_u, P_v and P_w. From now on, the various constants C do not depend on ε.

Lemma 2.4. *If $\min\{\varepsilon x, \varepsilon(L - x)\} \le t \le T$, then*

$$\int_{[P_w P]} (1 - \varepsilon)|u|^2 dt \le C$$

$$\int_{[P_w P]} (1 + \varepsilon)|v|^2 dt \le C$$

Proof (i) Define the points $O = (0, 0)$ and $L = (0, L)$. It $x \le t$, integrate the balance relation (1.5-ii) over the quadrangle $OP_w PP_u$; we get

$$\int_{[P_w P]} (1 - \varepsilon)|u|^2 dt + \int_{[P_u P]} (1 - \varepsilon)|w|^2 dx \le C$$

Integrate the first conservation law on the triangle $P_w P_v P$ or the quadrangle $P_w L P_v P$:

$$\int_{[P_w P]} (1 + \varepsilon)|v|^2 dt + \varepsilon \int_{[P_v P]} 2|u|^2 dx \le \int_{[P_w P]} (1 - \varepsilon)|u|^2 dt + C$$

(ii) When $t \le x$, this is similar. $\square$

Proof of Proposition 2.3.

Integrating along the characteristics of $\partial_t + \partial_x$

$$w(P) = w_0(P_w) + \int_{[P_w P]} u\bar{v} dt'$$

thus

$$|w(P)| \le C. \qquad \square$$

Lemma 2.5. *There is C depending only on the L^∞ norms of the data, such that*

$$\sup_{x_0\in[0,L]} \|u(\,\cdot,\,x_0)\|_{L^2_t} + \|v(\,\cdot,\,x_0)\|_{L^2_t} \leq C. \tag{2.5}$$

In particular, there is another constant C such that

$$\|u\|_{L^2(\Omega)} \leq C, \qquad \|v\|_{L^2(\Omega)} \leq C \tag{2.6}$$

In addition, $\sqrt{\varepsilon}u$ and $\sqrt{\varepsilon}v$ are uniformly bounded in $C^0(0,T;L^2([0,L]))$: there is a constant C such that

$$\sup_{t\in[0,T]} \|u(t,\,\cdot\,)\|_{L^2_x} + \|v(t,\,\cdot\,)\|_{L^2_x} \leq C\varepsilon^{-\frac{1}{2}}. \tag{2.7}$$

Proof. Repeat the proof of Lemma 2.4 integrating the same conservation laws over the rectangles $\{0 \leq t \leq T, 0 \leq x \leq x_0\}$ and $\{0 \leq t \leq T, x_0 \leq x \leq L\}$ respectively to obtain bounds for $\|u(\cdot,x_0)\|^2_{L^2}$ and $\|w(\cdot,x_0)\|^2_{L^2}$ first, and next of $\|v(\cdot,x_0)\|^2_{L^2}$.

Similarly, integrating over $[0,t]\times[0,L]$ the conservation laws give control of $w(t,\cdot)$, $\sqrt{\varepsilon}u(t,\cdot)$ and $\sqrt{\varepsilon}v(t,\cdot)$ in L^2_x. □

The next result improve the estimates above.

Proposition 2.6. *The solution of problem (u,v,w) of system* (1.1) (1.2) (1.3) *is such that*

i) $\sqrt{\varepsilon}u$ and $\sqrt{\varepsilon}v$ are bounded in $L^\infty(\Omega)$.

ii) u and v are uniformly bounded in $L^2(0,T;L^\infty_x)$, i.e. there is a constant C

$$\int_0^T \big(\|u(t,\,\cdot\,)\|^2_{L^\infty_x} + \|v(t,\,\cdot\,)\|^2_{L^\infty_x}\big)dt \leq C.$$

Proof. Consider the transport equation $\varepsilon\partial_t u + \partial_x u = wv$. As above, integrating along characteristic we find that

$$|u(t,x)| \leq \mathbf{1}_{x<t/\varepsilon}|u^{in}(t-\varepsilon x)| + \mathbf{1}_{x>t/\varepsilon}\,\|u_0\|_\infty + |\int_0^{t/\varepsilon} wv(t-\varepsilon s, x-s)ds| \leq M(t-\varepsilon x)$$

where

$$M(\sigma) = \mathbf{1}_{\sigma>0}|u^{in}(\sigma)| + \mathbf{1}_{\sigma<0}\|u_0\|_{L^\infty} + \|w\|_{L^\infty}\int_0^L |\mathbf{1}_\Omega v(\sigma+\varepsilon y, y)|dy$$

Note that $M(\sigma)=0$ for $\sigma \geq T$ and $\sigma \leq -\varepsilon L$. The integral of $|1_\Omega f|$ over the characteristic is launched from the boundary point $(\sigma,0)$ (truncated to the rectangle if necessary). In particular, according to (2.7) we get

$$M(\sigma) \leq C_0 + \|w\|_{L^\infty}\sup_t \|v(t,\,\cdot\,)\|_{L^2_x} \leq C + \|w\|_{L^\infty}C_1\varepsilon^{-1/2}$$

Then point *i)* follows.

Moreover $M \in L^2$ with norm

$$\|M\|_{L^2(\mathbb{R})} \leq \|u^{in}\|_{L^2([0,T])} + \sqrt{\varepsilon L}\|u_0\|_{L^\infty} + \sqrt{L}\|w\|_{L^\infty}\|v\|_{L^2(\Omega)}.$$

Similarly

$$|v(t,x)| \le N(t - \varepsilon(L-x))$$

with $N \in L^2(\mathbb{R})$, supported in $[-\varepsilon L, T]$. Integrating once more along the characteristic, yields

$$\begin{array}{c} |u(t,x)| \le C_0 + \|w\|_{L^\infty} \int_0^x |\mathbf{1}_\Omega v(t - \varepsilon x + \varepsilon y, y)| dy \\ \le C_0 + \|w\|_{L^\infty} \int_0^x N(t - \varepsilon x + 2\varepsilon y - \varepsilon L) dy. \end{array}$$

Therefore, to get the point *ii),* it suffices to show that the function ϕ defined by $\phi(t) = \sup_x \int_{-x}^x N(t - \varepsilon L + \varepsilon z) dz$ is in $L^2(0,T)$ (and the analogous for v). But we have

$$\phi(t+\varepsilon L) = \sup_x \int_{-x}^x N(t+\varepsilon z) dz \le \frac{1}{\varepsilon} \int_{-L\varepsilon}^{L\varepsilon} N(t+s) ds \le 2LN^*(t). \tag{2.8}$$

where we have introduced the maximal function N^* of N:

$$N^*(t) = \max_\rho \frac{1}{2\rho} \int_{-\rho}^{\rho} N(t+s) ds, \tag{2.9}$$

By a classical harmonic analysis result (see for example [10]) we have

$$\|N^*\|_{L^2} \le C_4 \|N\|_{L^2}, \tag{2.10}$$

implies that $\phi(.)$ is in $L^2(0,T)$. □

2.3 Estimates for derivatives

Theorem 2.7. *Assume that u_0 and w_0 are in $H^1([0,L])$, $u^{in} \in H^1([0,T])$ and satisfy the corner conditions (2.1). Then the solutions given by Theorem 2.1 belong to $[C^0(0,T;H^1([0,L])) \cap C^1(0,T;L^2([0,L]))]^3$.*

Moreover, there are uniform bounds for $\partial_x u$, $\partial_x v$, $\partial_x w$, $\varepsilon\partial_t u$, $\varepsilon\partial_t v$ in $L^\infty_x([0,L];L^2_t([0,T]))$ thus in $L^2(\Omega)$. Moreover there are uniform bounds for $\partial_t w$ in $L^\infty_x([0,L];L^1_t([0,T]))$ and in $L^1_t(0,T;L^2_x([0,L]))$.

Lemma 2.8. *$u_1 = (\partial_x - \varepsilon\partial_t)u$ and $w_1 = (\partial_x - \varepsilon\partial_t)w$ are bounded in $L^\infty_x([0,L];L^2_t([0,T]))$. Moreover, w_1 is also bounded in $L^\infty_t([0,T];L^2_x([0,L]))$.*

Proof. Differentiate the first and third equations with respect to $\partial_x - \varepsilon\partial_t$:

$$\left\{ \begin{array}{l} \varepsilon\partial_t u_1 + \partial_x u_1 = -v w_1 + |w|^2 u, \\ \partial_t w_1 + \partial_x w_1 = \overline{v} u_1 - |u|^2 w. \end{array} \right. \tag{2.11}$$

The initial-boundary values

$$\left\{ \begin{array}{lcl} u_1(0,x) & = & 2\partial_x u(0,x) + wv(0,x) = 0, \\ u_1(t,0) & = & -2\varepsilon\partial_t u^{in}(t) - wv(t,0) = -2\varepsilon\partial_t u^{in}(t) \\ w_1(0,x) & = & (1+\varepsilon)\partial_x w_0(x) - \varepsilon u\overline{v}(0,x) \\ w_1(t,0) & = & -(1+\varepsilon)\partial_t w^{in}(t) + u^{in}\overline{v}(t,0) \end{array} \right.$$

are uniformly bounded in L^2 by the assumptions on the data and the L^2 bound for $v_{|x=0}$ in given by Lemma 2.5. The equations imply that

$$\partial_t(\varepsilon|u_1|^2 + |w_1|^2) + \partial_x(|u_1|^2 + |w_1|^2) = |w|^2(\partial_x - \varepsilon\partial_t)|u|^2 - |u|^2(\partial_x - \varepsilon\partial_t|w|^2),$$

hence:

$$\partial_t(\varepsilon|u_1|^2 + |w_1|^2) + \partial_x(|u_1|^2 + |w_1|^2) + (\partial_x - \varepsilon\partial_t)(|u|^2|w|^2) = 4\mathrm{Re}\,|w|^2 u\overline{u}_1.$$

Integrate this identity over the quadrangle $OP'PQ$, where $P = (t, x)$, $P' = (0, x)$, $Q = (t, 0)$ and $O = (0, 0)$. Then

$$\begin{aligned}
&\int_0^t (|u_1|^2 + |w_1|^2)(t', x)dt' + \int_0^x \big(\varepsilon|u_1|^2 + |w_1|^2\big)(t, x')dx' + \int_0^t |u|^2|w|^2(t', x)dt' \\
&+ \int_0^x \varepsilon|u|^2|w|^2(0, x')dx' = \int_0^t (|u_1|^2 + |w_1|^2)(t', 0)dt' + \int_0^x (\varepsilon|u_1|^2 + |w_1|^2)(0, x')dx' \\
&+ \int_0^t |u|^2|w|^2(t', 0)dt' + \int_0^x \varepsilon|u|^2|w|^2(t, x')dx' + \iint 4\mathrm{Re}\,|w|^2 u\overline{u}_1(t', x')dt'dx'
\end{aligned}$$

The boundary terms in the right hand side are bounded. Moreover, w is bounded in L^∞. Therefore

$$\begin{aligned}
&\int_0^t (|u_1|2 + |w_1|^2)(t', x)dt' + \int_0^x |w_1|^2(t, x')dx' \\
&\qquad\qquad \le C + C\varepsilon \int_0^x |u(t, x')|^2 dx' + C \iint |u||u_1|(t', x')dt'dx'.
\end{aligned}$$

By Lemma 2.5, the second term in the right hand side is bounded. We already have a bound for u in $L^\infty_x(L^2_t)$, thus we can absorb the double integral from the right to the left using Gronwall's lemma, and conclude that

$$\int_0^t (|u_1|^2 + |w_1|^2)(t', x)dt' + \int_0^x |w_1(t, x')|^2 dx' \le C.$$

This proves the lemma. □

Lemma 2.9. *$v_2 = (\partial_x + \varepsilon\partial_t)v$ and $w_2 = (\partial_x + \varepsilon\partial_t)\overline{w}$ are bounded in $L^\infty_x([0, L]; L^2_t([0, T])$, thus in $L^2(\Omega)$.*

Proof. Differentiate the second and third equations with respect to $\partial_x + \varepsilon\partial_t$:

$$\text{(2.12)} \qquad \begin{cases} (\partial_x - \varepsilon\partial_t)v_2 = -uw_2 + |w|^2 v, \\ (\partial_x + \partial_t)w_2 = \overline{u}v_2 - |v|^2\overline{w}. \end{cases}$$

The initial-boundary values are

$$\text{(2.13)} \qquad \begin{cases} v_2(0, x) = 2\partial_x v(0, x) + \overline{w}u(0, x) = \overline{w}_0 u_0, \\ v_2(t, L) = 2\varepsilon\partial_t v(t, L) - \overline{w}u(t, L) = -\overline{w}u(t, L) \\ w_2(0, x) = (1 - \varepsilon)\partial_x\overline{w}(0, x) + \varepsilon\overline{u}v(0, x) = (1 - \varepsilon)\partial_x\overline{w}_0(x) \\ w_2(t, L) = (\varepsilon - 1)\partial_t\overline{w}(t, L) - \overline{u}v(t, L) = \frac{1+\varepsilon}{1-\varepsilon}\overline{w}_1(t, L) \end{cases}$$

are bounded in L^2 since we know that $w \in L^\infty$, $u_{|x=L} \in L^2$ and $w_{1|x=L} =\in L^2$.

The equations imply that

$$\partial_t(|w_2|^2 - \varepsilon|v_2|^2) + \partial_x(|v_2|^2 + |w_2|^2) = |w|^2(\partial_x + \varepsilon\partial_t)|v|^2 - |v|^2(\partial_x + \varepsilon\partial_t|w|^2)$$

Hence:

$$\partial_t(|w_2|^2 - \varepsilon|v_2|^2) + \partial_x(|v_2|^2 + |w_2|^2) + (\partial_x + \varepsilon\partial_t)(|v|^2|w|^2) = 4\mathrm{Re}\,|w|^2 v\overline{v}_2$$

Given $P = (t,x) \in \Omega$, integrate this identity over the rectangle $P'O'Q'P$, where $P' = (0,x)$, $Q' = (t,L)$ and $O' = (0,L)$. Then

$$\begin{aligned}
&\int_0^t (|v_2|^2 + |w_2|^2)(t',x)dt' + \int_x^L \varepsilon|v_2(t,x')|^2dx' + \int_x^L |w_2(0,x')|^2dx' \\
&+ \int_0^t |v|^2|w|^2(t',x)dt' + \int_x^L \varepsilon|v|^2|w|^2(0,x')dx' \\
&= \int_0^t (|v_2|^2 + |w_2|^2)(t',L)dt' + \int_x^L |w_2(t,x')|^2dx' + \int_x^L \varepsilon|v_2(0,x')|^2dx' \\
&+ \int_0^t |v|^2|w|^2(t',L)dt' + \int_x^L \varepsilon|v|^2|w|^2(t,x')dx' - \iint 4\mathrm{Re}\,|w|^2 v\overline{v}_2(t',x')dt'dx'
\end{aligned}$$

In the right hand side, the second boundary integral is controlled by (2.13). The third vanishes, the fourth is bounded since we know that w is bounded in L^∞ and that $\sqrt{\varepsilon}v(t,\cdot)$ is bounded in L^2. Moreover,

$$w_2 = \frac{1-\varepsilon}{1+\varepsilon}\overline{w}_1 + \frac{2\varepsilon}{1+\varepsilon}\overline{u}v$$

By Lemma 2.8 we have a bound for $\overline{w}_1(t,\cdot)$ in L^2, and by Lemma 2.5 and Proposition 2.6 we have a uniform bound for the L^2 norm of $\varepsilon uv(t,\cdot)$. Therefore, the integrals

$$\int_x^L |w_2(t,x')|^2dx'$$

are uniformly bounded. Therefore, we end up with an inequality of the form

$$\int_0^t (|v_2|^2 + |w_2|^2)(t',x)dt' \le C + M\int_0^t\int_x^L |v||v_2|(t',x')dt'dx'$$

We already have a bound for v in $L^\infty_x(L^2_t)$. We conclude using Gronwall's lemma. □

3 The limiting problem

For a given function $W \in L^\infty([0,L])$ and for a given constant $u^{in} \in \mathbb{C}$, we first address the following stationary system on $[0,L]$ satisfied by (u,v)

(3.1) $$\partial_x u + Wv = 0, \qquad \partial_x v + \overline{W}u = 0$$

with the boundary conditions

(3.2) $$u(0) = u^{in}, \qquad v(L) = 0.$$

Since this problem is linear, it suffices to consider the case where $u^{in} = 1$.

Proposition 3.1. *For all $W \in L^\infty([0,L])$, the problem (3.1) (3.2) with $u^{in} = 1$ has a unique solution $(u,v) = \mathcal{Y}(W)$ in $[L^\infty_x]^2$. It satisfies*

$$\|u\|_{L^\infty_x} \leq \exp(\sqrt{L}\,\|W\|_{L^2_x}), \qquad \|v\|_{L^\infty_x} \leq \exp(\sqrt{L}\,\|W\|_{L^2_x}) \tag{3.3}$$

$$|v(t,0)| \leq 1 \tag{3.4}$$

Moreover, for all $w_\infty > 0$, there exists $C(w_\infty)$ such that for two functions W, $\widetilde{W}$ whose norms in $L^\infty(0,L)$ are smaller than w_∞, we have

$$\big\|\mathcal{Y}(W) - \mathcal{Y}(\widetilde{W})\big\|_{[L^\infty_x]^2} \leq C(w_\infty)\big\|W - \widetilde{W}\big\|_{L^2_x} \tag{3.5}$$

Notice that

$$\partial_x(|u(x)|^2 - |v(x)|^2) = 0$$

so that there is a conserved quantity

$$\mu = |u(x)|^2 - |v(x)|^2. \tag{3.6}$$

is constant. From the boundary values, we see that $0 \leq \mu \leq |u^{in}|^2$ and (3.4) holds. The proposition may be proved by using the auxiliary function $z = u\overline{v}$ which satisfies $\partial_x z = W\sqrt{\mu^2 + 4|z|^2}$, but we present a more direct proof (following an idea of [22]).

Proof.

Let us denote

$$b(x) = -\int_0^x W(s)ds, \qquad B(x) = |b(x)|$$

If $\mathbf{A}(x)$ denote the matrix

$$\mathbf{A}(x) = \begin{pmatrix} 0 & b(x) \\ \overline{b}(x) & 0 \end{pmatrix},$$

we have $(\mathbf{A}(x))^2 = \mathbf{I}B(x)^2$. Moreover, we see that the matrix $e^{\mathbf{A}(x)}$ defined by

$$e^{\mathbf{A}} = \mathbf{1} + \mathbf{A} + ..\frac{1}{n!}\mathbf{A}^n ... = (1 + \frac{1}{2}B^2 + ..\frac{1}{(2p)!}B^{2p} + ...)\mathbf{1} + (1 + \frac{1}{3!}B^2 + ..\frac{1}{(2p+1)!}B^{2p} + ..)\mathbf{A}$$

Then the system

$$\partial_x \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & W \\ \overline{W} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0$$

has a solution which reads as

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \cosh(B(x))\begin{pmatrix} 1 \\ v^0 \end{pmatrix} + \frac{\sinh(B(x))}{B(x)}\begin{pmatrix} v^0 b(x) \\ \overline{b}(x) \end{pmatrix}$$

To get $v(L) = 0$, we must have $v^0 = v^0_W = -\tanh(B(L))\overline{b}(L)/B(L)$, thus, the solution of the system (3.1)(3.2) is given by

$$\begin{aligned} u(x) &= \cosh(B(x)) + v^0_W b(x)\sinh(B(x))/B(x), \\ v(x) &= \cosh(B(x))v^0_W + \overline{b}(x)\sinh(B(x))/B(x). \end{aligned}$$

They satisfy (3.3). Moreover we check that if $W, \widetilde{W} < w_\infty$, the functions $b(x)$, $\sinh(B(x))/B(x)$, $\cosh(B(x))$ and $\tanh(B(x))$ are Lipschitz continuous from L^2 into L^∞ with respect to W with a coefficient depending only on w_∞. Thus v_W^0 is also Lipschitz continuous and u and v are Lipschitz continuous from L^2 into L^∞.□

In the sequel, for all bounded function W, if $(u, v) = \mathcal{Y}(W)$, we denote

$$\Lambda(W) = u\overline{v}$$

By linearity, the solution of (3.1) (3.2) is $(u, v) = u^{in}\mathcal{Y}(W)$. Thus, for bounded functions (u_*, v_*, w_*) on $[0, T] \times [0, L]$, problem (1.6) (1.7) (1.8) may read as

$$(u_*(t), v_*(t)) = u^{in}(t)\mathcal{Y}(w_*(t)), \tag{3.7}$$

$$(\partial_t + \partial_x)w_* = |u^{in}|^2\Lambda(w_*), \qquad w_{*|t=0} = w_0, \qquad w_*(t, 0) = 0. \tag{3.8}$$

Thus, it remains to solve (3.8).

Theorem 3.2. *For w_0 in L^∞_x, the problem* (1.6) (1.7) (1.8) *has a unique solution* (u_*, v_*, w_*) *in $[L^\infty([0, T] \times [0, L])]^3$. It satisfies for all t*

$$|v_*(t, 0)| \leq |u^{in}(t)|$$

We first prove a local in time existence theorem and next conclude using uniform a priori bounds.

Lemma 3.3. *For any w_0 in L^∞_x, there exists a time t_f depending only on $\|w_0\|_{L^\infty}$, $\|u^{in}\|_{L^\infty}$ and L such that the equation*

$$(\partial_t + \partial_x)w = |u^{in}|^2\Lambda(w), \qquad w|_{x=0} = 0, \qquad w|_{t=0} = w_0. \tag{3.9}$$

has a solution w belonging to $L^\infty([0, t_f] \times [0, L])$.

Proof. Denote $\mathbf{T}_t$ the semi-group corresponding to the advection $(\partial_t + \partial_x)$ with the homogeneous boundary condition in $x = 0$. It is a contraction in L^p for all p. One solves the equation

$$W(t) = \mathbf{T}_t w_0 + \int_0^t \mathbf{T}_{t-s}\big(|u^{in}|^2\Lambda(W(s))\big)ds := \mathbf{T}_t w_0 + \mathcal{T}(W)(t) \tag{3.10}$$

by Picard's fixed point theorem.

By (3.3),

$$\big\|\Lambda(W)(t)\big\|_{L^\infty_x} \leq \exp(2\sqrt{L}\|W(t)\|_{L^2_x})|u^{in}|,$$

therefore, if $Y(t) = 2\sqrt{L}\sup_{s<t}\|W(s)\|_{L^\infty_x}$, we get $\big\|\mathcal{T}(W)\big\|_{L^2_x} \leq t\sqrt{L}\|u^{in}\|^2_{L^\infty}\exp(Y(t))$; and there exists C such that

$$Y(t) \leq C + 2tL\|u^{in}\|^2_{L^\infty}\exp(Y(t)).$$

So for t small enough, $Y(t)$ is bounded and there exists aconstant w_∞, such that $\sup_{s<t}\|W(s)\|_{L^\infty_x} \leq w_\infty$. Moreover, Proposition 3.1 implies that for W and $\widetilde{W}$ with L^∞ norm bounded by w_∞, there is $C(w_\infty)$ such that

$$\big\|\mathcal{T}(W)(t) - \mathcal{T}(\widetilde{W})(t)\big\|_{L^\infty_x} \leq t\|u^{in}\|^2_{L^\infty}C(w_\infty)\|W - \widetilde{W}\|_{L^\infty([0,t]\times[0,L])}. \tag{3.11}$$

From here, it is clear that if t_f is small enough, the mapping $W \mapsto \mathbf{T}_t w_0 + \mathcal{T}(W)(t)$ is Lipschitz continuous from the ball of center $\mathbf{T}_t w_0$ and radius 1 in $L^\infty([0,t_f] \times [0,L])$ into itself, with Lipschitz constant < 1, implying the existence of a fixed point in this ball. □

Proof of Theorem 3.2 .

A priori estimates. Assume that there exists a solution w_* of (3.8), then multiplying (3.8) by $\overline{w}$ and integrating with respect to the space variable, we have

$$\partial_t \|w_*(t)\|_{L^2}^2 + |w_*(t,L)|^2 = 2\int \mathrm{Re}(u_* \overline{v_* w_*}) dx$$

But, integrating (1.6) over $[0,L]$, we get $2\mathrm{Re}\int (u_* \overline{v_* w_*}) dx = -\int \partial_x |u_*|^2 \le |u^{in}|^2$; so we get

$$\|w_*(t)\|_{L^2} \le \|w_0\|_{L^2} + T\|u^{in}\|_{L^\infty}^2, \tag{3.12}$$

Next, according to (3.3) we have for all $t \le T$

$$\|u_*(t)\|_{L^\infty} \le \exp(\sqrt{L}(\|w_0\|_{L^2} + T\|u^{in}\|_{L^\infty}^2)), \qquad \|v_*(t)\|_{L^\infty} \le \exp(\sqrt{L}(\|w_0\|_{L^2} + T\|u^{in}\|_{L^\infty}^2)).$$

Therefore there exists w_∞ depending only on $\|w_0\|_{L^2}$ and L such that

$$\|w_*(t)\|_{L^\infty} \le w_\infty. \tag{3.13}$$

Uniqueness. Assume that there exist two solutions w and $\widetilde{w}$ satisfying $\|w(t)\|_{L^\infty}, \|\widetilde{w}(t)\|_{L^\infty} \le w_\infty$ and

$$(\partial_t + \partial_x) w = |u^{in}|^2 \Lambda(w), \quad (\partial_t + \partial_x)\widetilde{w} = |u^{in}|^2 \Lambda(\widetilde{w}),$$

with vanishing boundary condition at $x = 0$. Using that $(w - \widetilde{w})(t) = \int_0^t \mathbf{T}_{t-\tau}(\Lambda(\widetilde{w} + \zeta) - \Lambda(\widetilde{w}))(\tau) d\tau$, we get

$$|(w - \widetilde{w})(t,x)| \le \|u^{in}\|_{L^\infty}^2 \int_0^t \|\Lambda(w) - \Lambda(\widetilde{w}))\|_{L^\infty} d\tau$$

The Lipschitz continuity of Λ implies that

$$\|(w - \widetilde{w})(t)\|_{L^2} \le \sqrt{L} C(w_\infty) \int_0^t \|(w - \widetilde{w})(s)\|_{L^2} ds, \tag{3.14}$$

and uniqueness follows from Gronwall's lemma.

Existence. By Lemma 3.3 one knows that for t_f small enough, but depending only on the L^∞ norms of the data and L, there exists a solution $w_*(t) \in L^\infty([0,t_f] \times [0,L])$. By the a priori bound (3.13), the norm of $w(t_f)$ in $L^\infty([0,L])$ is bounded by a constant, which depends only on the data, so that the solution can be continued to $[0,T]$.

The bound of $|v_*(t)|$ comes from (3.4). □

4 Asymptotic analysis

In the sequel we use the following generic notations for the sake of conciseness: $U(t,x) = \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix}$ whose value is in $\mathbb{C}^2$; for w in $\mathbb{C}$ and U in $\mathbb{C}^2$. Let $w\perp U$ denote the vector

$$w\perp U = w\perp \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} wv \\ -\overline{w}u \end{pmatrix}.$$

For all bounded function w, consider the operator

$$M_w U = \begin{pmatrix} \partial_x u + wv \\ -\partial_x v - \overline{w}u \end{pmatrix}$$

Therefore the original system (1.1) (1.2) (1.3) is equivalent to find w^ε and $U^\varepsilon = (u^\varepsilon, v^\varepsilon)$ satisfying

$$\varepsilon\partial_t U^\varepsilon + M_{w^\varepsilon} U^\varepsilon = 0 \tag{4.1}$$

with initial and boundary conditions

$$u^\varepsilon_{|t=0} = u_0, \quad v^\varepsilon_{|t=0} = 0; \quad u^\varepsilon(t,0) = u^{in}, \quad v^\varepsilon(t,L) = 0. \tag{4.2}$$

and

$$\partial_t w^\varepsilon + \partial_x w^\varepsilon = u^\varepsilon\overline{v^\varepsilon}, \quad w^\varepsilon_{|t=0} = w_0, \quad w^\varepsilon(t,0) = 0. \tag{4.3}$$

The aim of this section is to prove the

Theorem 4.1 (Main result). *Suppose that the initial data u_0, w_0 are in $H^1(0,L)$ and satisfy the corner conditions (2.1). Let $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ be solutions of* (1.1) (1.2) (1.3) .

(i) The $(u^\varepsilon, v^\varepsilon)$ are bounded in $L^\infty_t([0,T]; L^2_x([0,L]))$ and in $L^2_t([0,T]; L^\infty_x([0,L]))$. Moreover, $(\partial_x u^\varepsilon, \partial_x v^\varepsilon)$ and $(\varepsilon\partial_t u^\varepsilon, \varepsilon\partial_t v^\varepsilon)$ are bounded in $L^2([0,T]\times[0,L])$;

(ii) The w^ε are bounded in $L^\infty([0,T]\times[0,L]) \cap H^1([0,T]\times[0,L])$;

(iii) $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ converges in $[L^2([0,T]\times[0,L])]^3$ to (u_*, v_*, w_*) the solution of (1.6) (1.7) (1.8).

The proof has several ingredients:

1) We first prove a preliminary result concerning systems like (4.1). Resolving the boundary conditions, we are led to consider systems of the form

$$\varepsilon\partial_t U + M_W U = 0, \qquad u(0) = v(L) = 0, \tag{4.4}$$

where W is slowly varying in time. The main result is the exponential decay of the energy of solutions, that is of order $O(e^{-\gamma t/\varepsilon})$.

2) Next we apply this result to (4.1) to prove uniform L^∞ bounds in time for $U^\varepsilon = (u^\varepsilon, v^\varepsilon)$. Using the equation (4.3) and the known estimates for w^ε, this implies that the family $\{w^\varepsilon\}$ is bounded in $H^1([0,T]\times[0,T])$.

3) Therefore, there are subsequences which converge strongly to w_* in $L^2(\Omega)$. Using the decay of energy once more, we show that U^ε converges strongly to U_* which satisfies $\varepsilon\partial_t U_* + M_{w_*} U_* = 0$, with the right boundary conditions. Hence (U_*, w_*) is a solution of the limiting problem. By uniqueness of this solution (U_*, w_*), this implies that the full family $(U^\varepsilon, w^\varepsilon)$ converges.

4.1 The fast system; exponential decay of energy.

For a given potential $W \in C([0,T]; L^2_x([0,L]))$, denote by $\mathcal{E}^\varepsilon_W(t,s)V$ the value $U(t)$ of the solution at time t (larger than or equal to s) of the equation

$$\varepsilon \partial_t U + M_W U = 0, \qquad U|_{\tau=s} = V \tag{4.5}$$

with initial data $V \in [L^2_x]^2$ and boundary conditions $u(t,0) = 0, \quad v(t,L) = 0$. The conservation $\partial_t(|u|^2 + |v|^2) + \varepsilon^{-1}\partial_x(|u|^2 - |v|^2) = 0$ and the boundary conditions immediately imply that

$$\|\mathcal{E}^\varepsilon_W(t,s)V\|_{L^2([0,L])} \le \|V\|_{L^2([0,L])}. \tag{4.6}$$

The linear mapping $\mathcal{E}^\varepsilon_W(t,s)$ satisfies the group property $\mathcal{E}^\varepsilon_W(t,s')\mathcal{E}^\varepsilon_W(s',s) = \mathcal{E}^\varepsilon_W(t,s)$. In particular, if W is a bounded potential independant of time, $\mathcal{E}^\varepsilon_W(t,0)$ is a continuous contraction semi-group on $L^2([0,L])$.

Theorem 4.2. *Given constants C_0 and C_1, there are C and $\gamma > 0$ such that for all W satisfying*

$$\|W\|_{L^\infty([0,T]\times[0,L])} \le C_0 \qquad \|W\|_{H^1([0,T]\times[0,L])} \le C_1, \tag{4.7}$$

then for all $\varepsilon \in]0,1]$, all $0 \le s \le t \le T$:

$$\|\mathcal{E}^\varepsilon_W(t,s)\|_{\mathcal{L}(L^2_x;L^2_x)} \le Ce^{-\gamma(t-s)/\varepsilon}$$

To prove the result, it is sufficient to show that there is $\delta < 1$, such that

$$\|\mathcal{E}_W(s + 2\varepsilon L, s)\|_{\mathcal{L}(L^2;L^2)} \le \delta \tag{4.8}$$

for all $s \in [0, T - 2\varepsilon L]$ and all W in the given bounded subset of $L^\infty \cap H^1(\Omega)$. To prove this estimate, stretch and change time $t = s+\varepsilon\tau$, so that the function $\widetilde{U}(\tau) = \mathcal{E}_W(s+\varepsilon\tau, s)V$, satisfies

$$\partial_\tau \widetilde{U} + M_a \widetilde{U} = 0, \qquad \widetilde{U}_{|\tau=0} = V \tag{4.9}$$

with unchanged homogeneous boundary conditions and

$$a(\tau, x) = W(s + \varepsilon\tau, x)$$

Note that when W satisfies (4.7) then a remains bounded in $L^\infty \cap H^1(\Omega_0)$ where $\Omega_0 = [0, 2L] \times [0, L]$. Therefore, the Theorem follows from the next lemma where one takes $s = 0$.

Lemma 4.3. *Given a bounded subset in $L^\infty \cap H^1(\Omega_0)$, there is $\delta < 1$ such that for all a in this bounded set , the solution $\widetilde{U}$ of (4.9) satisfies*

$$\|\widetilde{U}(2L)\|_{L^2([0,L])} \le \delta\|\widetilde{U}(0)\|_{L^2([0,L])}.$$

Proof. The conservation $\partial_\tau(|\widetilde{u}|^2 + |\widetilde{v}|^2) + \partial_x(|\widetilde{u}|^2 - |\widetilde{v}|^2) = 0$ implies that

$$\|\widetilde{U}(T)\|^2_{L^2([0,L])} = \|\widetilde{U}_{|\tau=0}\|^2_{L^2([0,L])} - \|\widetilde{u}(\cdot, L)\|^2_{L^2([0,2L])} - \|\widetilde{v}(\cdot, 0)\|^2_{L^2([0,2L])}.$$

Hence, it is sufficient to prove that there is C such that for all a in the given bounded susbset of $L^\infty \cap H^1(\Omega_0)$, the solution $\widetilde{U}$ of (4.9) satisfies:

$$\|\widetilde{U}_{|\tau=0}(\ .\ )\|^2_{L^2([0,L])} \le C\Big(\|\widetilde{u}(\cdot, L)\|^2_{L^2([0,2L])} + \|\widetilde{v}(\cdot, 0)\|^2_{L^2([0,2L])}\Big).$$

If not, there are sequences a^n, U^n such that

$$\|{U^n}_{|t=0}\|_{L^2} = 1, \tag{4.10}$$

$$\|u^n(\cdot, L)|_{L^2([0,T])} + \|v^n(\cdot, 0)\|_{L^2([0,T])} \ \to\ 0. \tag{4.11}$$

Extracting a subsequence if necessary, we can assume that U^n converges weakly to a limit U in L^2 and that a^n converges weakly to a in $L^\infty \cap H^1$ and strongly in H^σ for all $\sigma < 1$. In particular, this implies the following strong convergence:

$$a^n \ \to\ a \quad \text{in} \ \ L^\infty_t([0, 2L]; L^2_x([0, L]). \tag{4.12}$$

Therefore, $a^n v^n$ and $\overline{a^n} u^n$ converge weakly to av and $\overline{a} u$ respectively. Hence, we can pass to the weak limit in the equations

$$\begin{cases} \partial_\tau u^n + \partial_x u^n = -a^n v^n, \\ \partial_\tau v^n - \partial_x v^n = \overline{a^n} u^n, \end{cases} \tag{4.13}$$

to find that U satisfies

$$\begin{cases} \partial_\tau u + \partial_x u = -av \\ \partial_\tau v - \partial_x v = \overline{a} u \end{cases} \tag{4.14}$$

The right hand sides in (4.13) are bounded in L^2, implying that the traces on the boundary are well defined and moreover that ${U^n}_{|x=0} \rightharpoonup U_{|x=0}$ weakly, with similar results for the traces on $x = L$ and $t = 0$. With (4.11) this implies the strong convergence of the traces on the lateral boundaries:

$${U^n}_{|x=0} \ \to U_{|x=0} = 0 \quad \textit{and} \quad {U^n}_{|x=L} \ \to U_{|x=L} = 0 \quad \textit{in}\ L^2([0, 2L]). \tag{4.15}$$

Thus, U and U^n are weak=strong solutions of (4.9) with potential a and a^n respectively. This implies the conservation law

$$(\partial_\tau - \partial_x)|v^n - v|^2 = f^n := 2\mathrm{Re}\, \overline{a}(u^n - u)(\overline{v}^n - \overline{v}) + (\overline{a}^n - \overline{a})u^n(\overline{v}^n - \overline{v}).$$

Integrating over the the triangle $\Delta := \{0 \le t, 0 \le x, t + x \le L\}$ yields:

$$\|(v^n - v)_{|t=0}\|^2_{L^2([0,L])} = \|(v^n - v)_{|x=0}\|^2_{L^2([0,L])} - \int_\Delta f^n dt dx. \tag{4.16}$$

The classical energy estimates for (4.9) imply that U^n, U and thus $U^n - U$ are bounded in $L^\infty_t(L^2_x)$. By Proposition 2.6 they are also bounded in $L^2_t(L^\infty_x)$. Thus $u^n(\overline{v}^n - \overline{v})$ is bounded in $L^1_t(L^\infty_x)$ and by (4.12), this implies that

$$(\overline{a}^n - \overline{a})u^n(\overline{v}^n - \overline{v}) \to 0 \qquad \text{in} \ \ L^1(\Omega). \tag{4.17}$$

Moreover, $(u^n - u)(\overline{v}^n - \overline{v})$ is bounded in L^1_{tx}. Since $(\partial_t + \partial_x)(u^n - u)$ and $(\partial_t - \partial_x)(v^n - v)$ are bounded in $L^2_{t,x}$ by basic compensated compactness, the product $(u^n - u)(\overline{v}^n - \overline{v})$ converges in the sense of distributions to 0. This implies that

$$\int_\Delta \overline{a}(u^n - u)(\overline{v}^n - \overline{v})dtdx \to 0, \qquad \int_\Delta f^n dtdx \to 0.$$

With (4.15) and (4.16), this implies that ${v^n}_{|t=0}$ converges strongly in L^2. Similarly, ${u^n}_{|t=0}$ converges strongly in L^2. Hence, by (4.10)

$$\|U_{|t=0}\|_{L^2} = 1.$$

Therefore the contradiction which implies Lemma 4.3 and Theorem 4.2 is a consequence of the next Lemma 4.4. □

Lemma 4.4. *Suppose that $W \in L^\infty([0,T] \times [0,L])$ and $U \in L^2([0,T] \times [0,L])$ satisfies $\partial_\tau U + M_W U = 0$, $U_{|x=0} = 0$ and $U_{|x=L} = 0$. If $T > L$, then $U = 0$.*

Proof. By hyperbolicity in the x direction, and local uniqueness of the Cauchy problem with data on $x = 0$, $U = 0$ on the triangle $\Delta_1 = \{0 \le x \le L, x \le t \le T - x\}$. Similarly, $U = 0$ on the triangle $\Delta_2 = \{0 \le x \le L, L - x \le t \le T - L + x\}$. Because $T > L$, the two triangles intersect, and their union contain a neighborhood of the line $t = T/2$. Therefore, by uniqueness of the Cauchy problem in time for the boundary value problem (4.5), $U = 0$. □

4.2 The initial layer and uniform L^∞ bounds for u and v.

Theorem 4.5. *Suppose that the initial data u_0, w_0 are in $H^1(0,L)$ and satisfy the corner conditions (2.1). Let $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ be solutions of* (1.1) (1.2) (1.3) *. Then u^ε and v^ε are bounded in $L^\infty(0,T; L^2([0,L]))$ for $\varepsilon \in]0, \frac{1}{2}]$.*

Proof. We study the fast system (4.1) considering w^ε as known and using the estimates proved in Sections 2 and 3. The idea is that after a small initial layer, $U^\varepsilon(t)$ is close to $\underline{U}^\varepsilon(t) = u^{in}(t)\mathcal{Y}(w^\varepsilon(t))$ which, for all t, is the stationary solution of

$$M_{w^\varepsilon(t)}\underline{U}^\varepsilon(t) = 0 \tag{4.18}$$

with the boundary conditions

$$\underline{u}(t,0) = u^{in}(t), \quad \underline{v}(t,L) = 0. \tag{4.19}$$

According to Proposition 3.1, $\|\underline{U}^\varepsilon(t)\|_{L^2_x}$ is uniformely bounded.

Denote now $\widetilde{U}$ the solution of

$$\varepsilon\partial_t\widetilde{U} + M_{w_0}\widetilde{U} = 0 \tag{4.20}$$

with *homogeneous* boundary conditions $\tilde{u}^\varepsilon(t,0) = \tilde{v}^\varepsilon(t,L) = 0$ and initial conditions

$$\widetilde{U}^\varepsilon(0) = -u^{in}\mathcal{Y}(w_0) = (-\underline{u}(0,.), -\underline{v}(0,.))$$

($\widetilde{U}$ is like an initial layer).According to Theorem 4.2 we get

$$\|\widetilde{U}(t)\|_{L^2_x} = \|u^{in}\|_{L^\infty}\|\mathcal{E}^\varepsilon_{w_0}(t,0)\mathcal{Y}(w_0)\|_{L^2_x} \leq Ce^{-\gamma t/\varepsilon}. \tag{4.21}$$

Therefore, to prove the theorem, it is sufficient to show that $U^\sharp = U^\varepsilon - \underline{U}^\varepsilon - \widetilde{U}$ is bounded in $L^\infty(0,T;L^2_x))$. $U^\sharp$ satisfies

$$\varepsilon\partial_t U^\sharp + M_{w^\varepsilon}U^\sharp = F^\varepsilon, \tag{4.22}$$

with *homogneneous* boundary conditions $u^\sharp(t,0) = v^\sharp(t,L) = 0$ and initial conditions $u^\sharp(0,x) = u_0(x)$, $v^\sharp(0,x) = 0$, and

$$F^\varepsilon = -\varepsilon\partial_t\underline{U}^\varepsilon - (w^\varepsilon - w_0)\perp\widetilde{U}^\varepsilon = \begin{pmatrix} -\varepsilon\partial_t\underline{u}^\varepsilon - (w^\varepsilon - w_0)\widetilde{v} \\ -\varepsilon\partial_t\underline{v}^\varepsilon + (\overline{w^\varepsilon} - \overline{w}_0)\widetilde{u} \end{pmatrix}. \tag{4.23}$$

Classical energy estimates for solutions of (4.22) with homogeneous boundary conditions, imply that for all $t_0 \leq T$

$$\|U^\sharp(t_0)\|_{L^2_x} \leq \|u_0\|_{L^2_x} + \frac{1}{\varepsilon}\int_0^{t_0}\|F^\varepsilon(t)\|_{L^2_x}dt$$

Therefore, it remains to show that there is C such that for all $\varepsilon \in]0,\frac{1}{2}]$

$$\int_0^T \|F^\varepsilon(t)\|_{L^2_x}dt \leq C\varepsilon \tag{4.24}$$

The first term in F^ε is

$$\varepsilon\partial_t\mathcal{Y}(w^\varepsilon(t)) = \varepsilon\lim_h\frac{1}{h}\big[u^{in}(t+h)\mathcal{Y}(w^\varepsilon(t+h)) - u^{in}(t)\mathcal{Y}(w^\varepsilon(t))\big].$$

By Theorem 2.1, the w^ε are uniformly bounded in L^∞ and therefore Proposition 3.1 implies that for some constant C independent of ε:

$$\varepsilon\|\partial_t\underline{U}^\varepsilon(t)\|_{L^2_x} \leq \varepsilon C\Big(\|\partial_t w^\varepsilon(t)\|_{L^2_x} + |\partial_t u^{in}(t)|\Big).$$

The second term in F^ε is $(w^\varepsilon - w_0)\perp\widetilde{U}$. By (4.21) and the L^∞ bounds for w^ε, it satisfies

$$\|(w^\varepsilon - w_0)\perp\widetilde{U}t)\|_{L^2} \leq Ce^{-\gamma t/\varepsilon}.$$

Integrating these two estimates over $[0,T]$ imply (4.24) and the theorem follows. □

Corollary 4.6. *Under the assumptions of the theorem, $u^\varepsilon\overline{v^\varepsilon}$ belongs to a bounded set in $L^2([0,T]\times[0,L])$. Moreover $\partial_t w^\varepsilon$ is bounded in $L^2([0,T]\times[0,L])$ and w^ε is bounded in $H^1([0,T]\times[0,L])$.*

Proof. By Proposition 2.6, u^ε is bounded in $L^2_t(L^\infty_x)$. The theorem asserts that v^ε is bounded in $L^\infty_t(L^2_x)$. Thus $u^\varepsilon\overline{v^\varepsilon}$ is bounded in $L^2([0,T]\times[0,L])$.

By Theorem 2.7, $\partial_x w^\varepsilon$ is bounded in $L^2(\Omega)$ and therefore $\partial_t w^\varepsilon = -\partial_x w^\varepsilon + u^\varepsilon\overline{v^\varepsilon}$ is bounded in $L^2(\Omega)$. □

4.3 Strong convergence. End of proof of Theorem 4.1

Therefore the products $u^\varepsilon\overline{v^\varepsilon}$, $v^\varepsilon w^\varepsilon$ and $u^\varepsilon\overline{w^\varepsilon}$ are uniformely bounded in $L^2(\Omega)$. To finish the proof of Theroem 4.1 it suffices to prove the convergence of a subsequence of the solutions $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ of (4.1)(4.2)(4.3):

$$(u^\varepsilon, v^\varepsilon, w^\varepsilon) \to (u_*, v_*, w_*), \quad \text{in } [L^2([0,T]\times[0,L])]^3. \tag{4.25}$$

Indeed, knowing this strong convergence, we can pass to the limit in the quadratic terms in the equations and therefore (u_*, v_*, w_*) is a solution of (1.6)(1.7) (1.8). By uniqueness of the solution of the limit problem, this implies the convergence of the full family.

Proof of (4.25)**.** The w^ε are bounded in $L^\infty([0,T]\times[0,L]) \cap H^1([0,T]\times[0,L])$, thus there is a subsequence which converges to a function w_* strongly in $C([0,T], L^2([0,L]))$. In particular $w_{*|t=0} = w_0$. Denote $U_*(t)$ the vector function which solves for all t the system

$$M_{w_*(t)}U_* = 0, \qquad u_*(0) = u^{in}, \qquad v_*(L) = 0.$$

The proof of the strong convergence of U^ε is parallel to the proof of Theorem 4.5. We use the same vector function $\underline{U}^\varepsilon(t) = u^{in}(t)\mathcal{Y}(w^\varepsilon(t))$ as above, it satisfies:

$$M_{w^\varepsilon(t)}\underline{U}^\varepsilon = 0.$$

According to Proposition 3.1, there exists C such that

$$\|\mathcal{Y}(w^\varepsilon(t)) - \mathcal{Y}(w_*(t))\|_{[L^2_x]^2} \le C\|w^\varepsilon(t) - w_*(t)\|_{L^2_x} \qquad \forall t \le T,$$

and hence $\underline{U}^\varepsilon(t)$ converges strongly to U_* in $L^2_{t,x}$.

To describe the initial layer, introduce the solution $\widetilde{U}^\varepsilon = \mathcal{E}^\varepsilon_{w^\varepsilon}(t,0)\left(U^\varepsilon_{|t=0} - \underline{U}^\varepsilon_{|t=0}\right)$ of

$$\varepsilon\partial_t\widetilde{U}^\varepsilon + M_{w^\varepsilon}\widetilde{U}^\varepsilon = 0, \qquad \widetilde{U}^\varepsilon_{|t=0} = U^\varepsilon_{|t=0} - \underline{U}^\varepsilon_{|t=0}, \qquad \tilde{u}(t,0) = \tilde{v}(L,t) = 0.$$

Theorem 4.2 and Corollary 4.6 imply that there exist C and $\gamma > 0$ such that for all ε and all $t \in [0,T]$:

$$\|\widetilde{U}^\varepsilon(t)\|_{[L^2_x]^2} \le Ce^{-\gamma t/\varepsilon}$$

Let $U^\varepsilon_\sharp = U^\varepsilon - \underline{U}^\varepsilon - \widetilde{U}^\varepsilon$. It satisfies

$$\varepsilon\partial_t U^\varepsilon_\sharp + M_{w^\varepsilon}U^\varepsilon_\sharp = -\varepsilon\partial_t\underline{U}^\varepsilon \tag{4.26}$$

with *homogneneous* boundary and initial conditions. Therefore, by Duhamel's principle,

$$U^\varepsilon_\sharp(t) = -\int_0^t \mathcal{E}_W(t,s)(\partial_t\underline{U}^\varepsilon)(s)ds = -\int_0^t \mathcal{E}_W(t,s)\partial_t\left[u^{in}\mathcal{Y}(w^\varepsilon(s))\right]ds$$

Since $\partial_t\mathcal{Y}(w^\varepsilon(t)) = \lim_{h\to 0}\frac{1}{h}[\mathcal{Y}(w^\varepsilon(t+h)) - \mathcal{Y}(w^\varepsilon(t))]$, according to uniform bound of $\|w^\varepsilon\|_{L^\infty_{t,x}}$

and to Proposition 3.1, we have

$$\|\partial_t \mathcal{Y}(w^\varepsilon(t))\|_{L^2_x} \le C(w_\infty)\|\partial_t w^\varepsilon(t)\|_{L^2_x}$$

By corollary 4.6, one knows that $\|\partial_t w^\varepsilon(t)\|_{L^2_x}$ is in $L^2(0,T)$. Thus, thanks to theorem 4.2, there exists a positive constant γ such that

$$\begin{aligned}\|U^\varepsilon_\sharp(t)\|_{L^2_x} &\le C\int_0^t e^{-\gamma(t-s)/\varepsilon}\left(\|\partial_t w^\varepsilon(s)\|_{L^2_x} + |\partial_t u^{in}(s)|\right)ds \\ &\le C\left(\int_0^t e^{-2\gamma s/\varepsilon}ds\right)^{1/2}\left(\|\partial_t w^\varepsilon\|_{L^2_{t,x}} + \|\partial_t u^{in}\|_{L^2_t}\right) = \sqrt{\varepsilon}C\left(\|\partial_t w^\varepsilon\|_{L^2_{t,x}} + C'\right).\end{aligned}$$

Therefore, $U^\varepsilon(t)$ converges strongly to $U_*(t) = u^{in}(t)\mathcal{Y}(w_*)$ in L^2_x for all t and (4.25) holds. The proof of the main theorem is now complete. □

5 Appendix. A physical glance on the three-wave coupling.

Several physical phenomena may occur when high power laser beams propagates in hot plasmas. Different typical lengths may be found which are related to these different phenomena: the typical length L_{pl} of variation of the mean density of the plasma; the typical length L_l of variation of the amplitude of the laser intensity, $2\pi L_l$ is in the order of the width of the speckles (which are hot spots of light in the laser beam) and of course the laser wave length in the vacuum $2\pi/k_0$.

Generally one has $2\pi/k_0 \ll 2\pi L_l \ll L_{pl}$; for example, for the high power intensity laser beam, the wave length is equal to a fraction of one micron, L_l is typically of order of one micron (then $k_0 L_l \approx 10$) and L_{pl} is larger than 100 microns.

Beside the absorption by the plasma, since the refraction index of the plasma depends on its density, there is refraction of light at the macroscopic level (characterized by L_{pl}); there is also refraction of light at the scale of the width of the speckles (characterized by L_l) which produces a self-focusing of these speckles; at the scale of the width of the speckles, the diffraction of the laser light has to be taken into account. Here we address only the coupling between the main laser wave and an ion acoustic wave which creates the so called *stimulated Brillouin backscattered* laser wave. Moreover in these plasmas, the ion acoustic waves propagates with a speed which is in the order of 10^{-3} of the speed of light.

The derivation of the coupling model has been performed for a long time but it is quite tricky; for a recent physical introduction to this three-wave coupling modeling, see for example [2],[12],[18] (see also [19] for an mathematical introduction to this derivation). We only give the outlines of this derivation.

So, we first explain how from a basic detailled model (based on (5.1)) we may derive a model where three waves are accounted for: an ion acoustic wave (which is a perturbation of ion density N), the main laser wave which travels forwards, the back-scattered laser wave which travels backwards. Secondly, we focus on simplifications of this system in order to obtain the so-called *standard decay model* which is the three-wave coupling system we address in this paper.

First coupling model.

Denote by c, the speed of light. The laser pulsation is $\omega_0 = ck_0$. It is classical to use the dimensionless electron density normalized with the critical density (defined by $q_e^{-2}{\omega_0}^2\varepsilon_0 m_e$ where q_e, m_e denote the electron charge and mass and ε_0 the vacuum permeability). One assumes here that the electron density it quite constant that is to say it satisfies (where the mean value N_{ref} is stricly smaller than 1 and in pratice less than 0.5)

$$N(t, \mathbf{x}) = N_{\text{ref}}(1 + n(t, \mathbf{x})), \qquad n \ll 1.$$

The laser wave is represented by the electromagnetic field $\Psi = \Psi(t, \mathbf{x})$ satisfying the following wave equation (derived from the full Maxwell ones)

$$\frac{\partial^2}{\partial t^2}\Psi - c^2\Delta\Psi + \omega_p^2(1+n)\Psi = 0 \tag{5.1}$$

where

$$\omega_p^2 = \omega_0^2 N_{\text{ref}}$$

The laser beam is assumed to travel in the fixed direction characterized by the unit vector $\mathbf{e}_b$, denote $z = \mathbf{x}.\mathbf{e}_b$. One introduces $E = E(t, \mathbf{x})$ the space-time envelope of the field corresponding to the main wave traveling in the direction $\mathbf{e}_b$ and $\Phi(t, \mathbf{x})$ the one corresponding to the backscattered wave; they are complex functions and are slowly varying with respect to the time and space variables $\mathbf{x}$

$$\Psi(t, \mathbf{x}) = \left[E(t, \mathbf{x})e^{ik_p z - i\omega_0 t} + c.c\right] + \left[\Phi(t, \mathbf{x})e^{-ik_p z - i\omega_0 t} + c.c.\right],$$

where k_p solves the dispersion relation (obtained without the perturbation n), that is to say

$$k_p = \frac{1}{c}\sqrt{\omega_0^2 - \omega_p^2} = k_0\sqrt{1 - N_{\text{ref}}}.$$

Denote $c_g = c\sqrt{1 - N_{\text{ref}}}$ the so-called group velocity and $\beta_0 = \frac{\omega_p^2}{2\omega_0}$. The two waves satisfiy the paraxial equations obtained by assuming that k_p^{-1} and ω_0^{-1} are small enough compared to the characteristic values of the space and time variation of E and Φ

$$\begin{aligned}
\frac{\partial}{\partial t}E + c_g\partial_z E - \frac{ic}{2k_0}\Delta_\perp E &= -i\beta_0 n e^{-2ik_p z}\Phi \\
\frac{\partial}{\partial t}\Phi - c_g\partial_z \Phi - \frac{ic}{2k_0}\Delta_\perp \Phi &= -i\beta_0 n e^{2ik_p z}E
\end{aligned}$$

where the diffraction terms $i\Delta_\perp E$ and $i\Delta_\perp\Phi$ correspond to a diffusion in the directions transverse to the main propagation z; in the sequel we does not account for these transverse effects. Notice that the quantity n is highly oscillating with respect to the space variable (see below) thus terms like $\beta_0 nE$ and $\beta_0 n\Phi$ which appear theoreticaly in the expansion of the first and second equation respectively are also highly oscillating with respect to the spacee and so they have to be withdrawn.

The laser field produce a ponderomotive force in the plasma which is proportional to $\nabla|\Psi|^2$,so neglecting the transverse phenomena, it reduces to a term proportional to $2ik_pE\overline{\Phi}e^{2ik_pz}$; it generates a wave, called ion acoustic wave corresponding to the wave number $k_s = 2k_p$. Therefore for plasma response, the simplest model is the following (qc_s corresponds to the plasma velocity of the ion acoustic wave)

$$\frac{\partial}{\partial t}n + c_s\partial_z q = 0, \qquad \frac{\partial}{\partial t}q + c_s\partial_z n + 2\nu_L q = -\gamma_p c_s^{-1}(ik_sE\overline{\Phi}e^{ik_sz} + c.c.).$$

the term of the form $\nu_L q$ is related to the Landau damping effect and γ_p is a constant depeding only of the characteristic of the plasma ions. Then, neglecting as above the term $\nabla\Phi$ with respect to $ik_s\Phi$, we check that the density fluctuation n satisfies

$$\frac{\partial^2}{\partial t^2}n - c_s^2\partial_z^2 n + 2\nu_L\frac{\partial}{\partial t}n = \gamma_p k_s^2(e^{ik_sz}E\overline{\Phi} + c.c.)$$

Notice, in the propagation equation for the laser field E, when one evaluates the coupling term $ne^{-ik_sz}\Phi$, the component of n corresponding to e^{-ik_sz} is highly oscillating (with respect to the space variable) thus it has to be neglected. Therefore we only adress the other component corresponding to e^{ik_sz} and in sequel we address the system

$$\frac{\partial}{\partial t}n + c_s\partial_z q = 0, \tag{5.2}$$

$$\frac{\partial}{\partial t}q + c_s\partial_z n + 2\nu_L q = -\gamma_p c_s^{-1}ik_sE\overline{\Phi}e^{ik_sz} \tag{5.3}$$

We now take the space envelope M (which is not highly oscillating with respect to space), i.e. we set

$$M(t,z) = n(t,z)e^{-ik_sz}$$

thus the previous system is equivalent to the following equation for M

$$\frac{\partial^2}{\partial t^2}M - c_s^2(\partial_z + ik_s)^2M + 2\nu_L\frac{\partial}{\partial t}M = -\gamma_p k_s^2 E\overline{\Phi} \tag{5.4}$$

After neglecting also the term nE which is highly oscillating, we get for the propagation equation for E

$$\frac{\partial}{\partial t}E + c_g\partial_z E = -i\beta_0 M\Phi$$

By the same way, we get for Φ

$$\frac{\partial}{\partial t}\Phi - c_g\partial_z\Phi = -i\beta_0\overline{M}E$$

These two equations coupled with (5.4) make up the so-called *modified decay model* for the Brillouin instability. Of course, it has to be supplemented by initial conditions $E(0,.), \Phi(0,.), M(0,.)$

and boundary conditions on both sides of the simulation interval for M and on the sides $z = 0$ and $z = L$ for E and Φ.

The standard decay model.

The derivation of this model is based on a linearization so we assume first that the field E is constant and in a second step we reintroduce the variation of E.

First step. The field E is assumed to be constant, the previous system consists only in (5.4) and the propagation equation for the backscattered wave Φ

$$\left(\frac{\partial}{\partial t} - c_g \partial_z\right) \Phi = -i\beta_0 \overline{M} E$$

This is a linear system which couples M and Φ; the point is to determine the unstable modes. So we may address this system on the whole space ($z \in \mathbf{R}$). We are aimed at replacing this linear system satisfied by M, Φ by a simpler one where the density fluctuation satisfies a first order one. Thus we introduce

$$m(z) = e^{-ik_s z}\frac{n(z) + q(z)}{2}, \qquad s(z) = e^{-ik_s z}\frac{n(z) - q(z)}{2}$$

One may show the coupling between s and Φ is stable; thus if we neglect s with respect to m,we get the following system for m and Φ

$$\frac{\partial}{\partial t} m + c_s(ik_s + \partial_z)m + \nu_L m = -i\frac{\gamma_p k_s}{2c_s} E\overline{\Phi} \tag{5.5}$$

$$\left(\frac{\partial}{\partial t} - c_g \partial_z\right) \Phi = -i\beta_0 \overline{m} E \tag{5.6}$$

One may show that from the stabilty analysis it is equivalent to the one satisfied by M and Φ. For the density fluctuation, we get the following approximation.

$$n(t, z) = \mathrm{Re}(e^{ik_s z} m(t, z)) \tag{5.7}$$

Second step. We now go back to the general case where E is not constant; then one supplements (5.5)(5.6) with the evolution equation for the wave E and we get the so-called *standard decay model*

$$\begin{aligned} \partial_t E + c_g \partial_z E &= -i\beta_0 m \Phi \\ \partial_t \Phi - c_g \partial_z \Phi &= -i\beta_0 \overline{m} E, \\ \partial_t m + c_s(ik_s + \partial_z)m &= -i\frac{\gamma_p k_s}{2c_s} E\overline{\Phi} \end{aligned} \tag{5.8}$$

Notice that in this system, the two characteristic speeds c_g and c_s occur, which are respectively in the order of the light speed and the sound speed. The quadratic coupling terms in the right hand side correspond to the coupling between the three waves. Of course if the plasma is not homogeneous, there are supplementary terms corresponding to the variation of the plasma density (see the introduction). But in the case when the plasma is homogeneous and all the transverse effects are neglected , it is this system which has to be addressed, see for example [2], [11].

So the main mathematical difficulties may be seen on this system; it is posed on a fixed interval $[0, z_{\max}]$ and it is supplemented with the following natural boundary conditions

$$E(t,0) = \alpha^{in}, \qquad \Phi(t, z_{\max}) = 0, \qquad m(t,0) = 0. \tag{5.9}$$

If the initial values of m and Φ are zero, then $\Phi(t) = m(t) = 0$ is a trivial solution. For getting a non trivial solution for (5.8), it sufficient to have $m(0,.)$ equal to a small random noise (or to address a boundary condition on M by setting $m(t,0)$ equal to a small random noise).

Dimensionless form of the system

Introduce the dimensionless functions as follows. Denoting α_{ref} a charactersitic value of α^{in}, if we define $\widehat{E}, \widehat{\Phi}, \widehat{m}$ by

$$E = \widehat{E}\alpha_{\text{ref}}, \qquad \Phi = \widehat{\Phi}\alpha_{\text{ref}}, \qquad m = -i\widehat{m}\frac{\alpha_{\text{ref}}}{c_s}\sqrt{2\gamma_p\frac{1-N_{\text{ref}}}{N_{\text{ref}}}}.$$

and if we set

$$\gamma_0 = \alpha_{\text{ref}}\frac{k_0}{c_s}\sqrt{\frac{\gamma_p}{2}N_{\text{ref}}}, \qquad \varepsilon = \frac{c_s}{c_g},$$

then, the previous system reads

$$\begin{aligned}
\partial_t\widehat{E} + \frac{c_s}{\varepsilon}\partial_z\widehat{E} &= -\frac{c_s}{\varepsilon}\gamma_0\widehat{m}\widehat{\Phi}, \\
\partial_t\widehat{\Phi} - \frac{c_s}{\varepsilon}\partial_z\widehat{\Phi} &= \frac{c_s}{\varepsilon}\gamma_0\overline{\widehat{m}}\widehat{E_p}, \\
\partial_t\widehat{m} + c_s(ik_s + \partial_z)\widehat{m} + \nu_L\widehat{m} &= c_s\gamma_0\widehat{E_p}\overline{\widehat{\Phi}}.
\end{aligned}$$

One can check that the good characteristic time is given by $1/\gamma_0$, then we define the dimensionless time and space variables

$$t' = \gamma_0 t, \qquad x = z\gamma_0/c_s, \qquad \eta = \nu_L/\gamma_0.$$

So let us perform the change of notations :

$$\widehat{E}(t,z) = E_p(t',x), \qquad \widehat{\Phi}(t,z) = E_m(t',x), \qquad \widehat{m}(t,z) = W(t',x).$$

and we get the following dimensionless system :

$$\begin{aligned}
(\varepsilon\partial_{t'} + \partial_x)E_p &= -E_m W, \\
(\varepsilon\partial_{t'} - \partial_x)E_m &= E_p\overline{W}, \\
\partial_{t'}W + (ik_s + \partial_x)W + \eta W &= E_p\overline{E_m}.
\end{aligned}$$

Lastly we set

$$E_p(t,x) = e^{-i(t'-\varepsilon x)k_s/2}u, \qquad E_m(t,x) = e^{i(t'+\varepsilon x)k_s/2}v, \qquad W(t,x) = w(t,x)e^{-it'k_s}.$$

Then writing t instead of t', the previous system reads as

$$(\varepsilon\partial_t + \partial_x)u = -wv, \tag{5.10}$$

$$(\varepsilon\partial_t - \partial_x)v = \overline{w}v \tag{5.11}$$

$$(\partial_t + \partial_x)w + \eta w = u\overline{v}, \tag{5.12}$$

This system which we call *Boyd-Kadomstev system* is supplemented with the boundary conditions

$$u(T,0) = u^{in}, \qquad v(t,L) = 0, \qquad w(t,0) = 0.$$

where $|u^{in}|$ is in the order of 1. Notice that $w(t,0)$ may be also a function of t which is small compared to 1. Of course initial conditions have also to be prescribed, for instance

$$u(0,.) = u_0, \qquad v(0,.) = 0, \qquad w(0,.) = w_0$$

where u_0 and w_0 are bounded functions; in general the L^∞-norm of u_0 is in the order of 1 and the one of w_0 is much smaller than 1. From a mathematical point of view, the problem with $\eta = 0$ is more difficult, so we have chosen to study the Boyd-Kadomstev system it that case.

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