# Counterexamples to the well posedness of the Cauchy problem for hyperbolic systems

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September 10, 2014

**Abstract**

This paper is concerned with the well posedness of the Cauchy problem for first order symmetric hyperbolic systems in the sense of Friedrichs. The classical theory says that if the coefficients of the system and if the coefficients of the symmetrizer are Lipschitz continuous, then the Cauchy problem is well posed in $L^2$. When the symmetrizer is Log-Lipschtiz or when the coefficients are analytic or quasi-analytic, the Cauchy problem is well posed $C^\infty$. In this paper we give counterexamples which show that these results are sharp. We discuss both the smoothness of the symmetrizer and of the coefficients.

## Contents



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# 1 Introduction

This paper is concerned with the well posedness of the Cauchy problem for first order symmetric hyperbolic systems in the sense of Friedrichs [Fr1], who proved that if the coefficients of the system *and if* the coefficients of the symmetrizer are Lipschitz continuous, then the Cauchy problem is well posed in $L^2$. This has been extended to hyperbolic systems which admits Lipschitzean microlocal symmetrizers (see [Me]).

The main objective of this paper is to discuss the necessity of these smoothness assumptions and to provide new counterexamples to the well posedness. In the spirit of [CS2, CoNi], we make a detailed analysis of systems in space dimension one, with coefficients which depend only on time. Even more, we concentrate our analysis on $2 \times 2$ system

$$Lu := \partial_t u + \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \partial_x u = \partial_t u + A(t)u. \tag{1.1}$$

The symbol is assumed to be strongly hyperbolic or uniformly diagonalizabe, which means that there is a bounded symmetrizer $S(t)$, with $S^{-1}$ is bounded, which is a definite positive and such that $S(t)A(t)$ is symmetric. This is equivalent to the condition that there is $\delta > 0$ such that

$$\delta\big((a-d)^2 + b^2 + c^2\big) \leq \frac{1}{4}(a-d)^2 + bc. \tag{1.2}$$

If the symmetrizer $S$ and the coefficients are Lipschitz continuous then the Cauchy problem is well posed in $L^2$. Indeed, in this case, solutions on $[0,T] \times \mathbb{R}$ of $Lu = f$ satisfy

$$\big\|u(t)\big\|_{L^2} \leq C\Big(\big\|u(0)\big\|_{L^2} + \big\|Lu\big\|_{L^2}\Big) \tag{1.3}$$

with

$$C = C_0 \exp\Big(\int_0^T |\partial_t S(s)| ds\Big).$$

Lipschitz smoothness of the symmetrizer is almost necessary for the well posedness in $L^2$, even for very smooth coefficients:

**Theorem 1.1.** *For all modulus of continuity $\omega$ such that $t^{-1}\omega(t) \to +\infty$ as $t \to 0$, there is a system* (1.1) *with coefficients in $\cap_{s>1} G^s([0,T])$, with a symmetrizer satisfying*

$$\big|S(t) - S(t')\big| \leq C\omega(|t-t'|) \tag{1.4}$$

*such that the Cauchy problem is ill posed in $L^2$ in the sense that there is no constant $C$ such that the estimate* (1.3) *is satisfied.*

Here and below, we denote by $G^s([0,T]]$ the Gevrey class of functions of order $s$. They are $C^\infty$ functions $f$ such that, for some constant $C$ which depends on $f$, there holds

$$\forall j \in \mathbb{N}, \qquad \big\|\partial_t^j f\big\|_{L^\infty} \leq C^{j+1}(j!)^s.$$

This theorem extends to systems a similar result obtained in [CiCo] for the strictly hyperbolic wave equation

$$\partial_t^2 u - a(t)\partial_x^2 u = f. \tag{1.5}$$

Indeed, there is a close parallel between the energy $|\partial_t u|^2 + a(t)|\partial_x u|^2$ for the wave equation and $(S(t)u, u)$ for the system, and in this case, the smoothness of $S(t)$ plays a role analogue to the smoothness of $a$. For the wave equation, when $a$ is Log-Lipschitz, i.e. admits the modulus of continuity $\omega(t) = t|\ln t|$, the Cauchy problem is well posed in $C^\infty$ with a loss of derivatives proportional to time ([CDGS]). An intermediate cases between Lipschitz and Log-Lipschitz, that is when $(t|\ln t|)^{-1}\omega(t) \to 0$ and $t^{-1}\omega(t) \to +\infty$, then the loss of derivative is effective but is arbitrarily small on any interval ([CiCo]). The proof of these results extends immediately to systems (1.1) where the smoothness of the symmetrizer plays the role of the smoothness of the coefficient $a$.

The next result extends to systems the result in [CDGS, CS2] showing that the Log-Lipschitz smoothness of the symmetrizer is a sharp condition for the well posedness in $C^\infty$, even for $C^\infty$ coefficients:

**Theorem 1.2.** *For all modulus of continuity $\omega$ satisfying $(t|\ln t|)^{-1}\omega(t) \to +\infty$ as $t \to 0$, there are systems* (1.1)*, with $C^\infty$ coefficients, with symmetrizers which satisfy the estimate* (1.4) *such that the Cauchy problem is ill posed in $C^\infty$, meaning that for all $n$ and all $T > 0$, there is no constant $C$ such that the estimate*

$$\|u\|_{L^2} \leq C\|Lu\|_{H^n} \tag{1.6}$$

*is satisfied for all $u \in C_0^\infty([0,T] \times \mathbb{R})$.*

In [CoNi] the question of the well posedness of the Cauchy problem is considered under the angle of the smoothness of the coefficients alone. In this aspect, the analysis is related to the analysis of the weakly hyperbolic wave equation (1.5) (see citeCJS). If the coefficients are $C^\infty$, the problem is well posed in all Gevrey classes $G^s$, but the well posedness in $C^\infty$ is obtained only when the coefficients are analytic or belong to a quasi-analytic class. Indeed, the next theorem shows that this is sharp.

**Theorem 1.3.** *There are example of systems* (1.1) *on* $[0, T] \times \mathbb{R}$*, with uniformly hyperbolic symbols and coefficients in the intersection of the Gevrey classes* $\cap G^s$ *for* $s > 1$*, admitting continuous symmetrizers, such that the Cauchy problem is ill posed in* $C^\infty$*.*

This theorem improves the similar result obtained in [CoNi] where the counterexample had coefficients in $\cap G^s$ for $s > 2$. The same construction can be used to provide a similar improvement to the known result in [CS1] for the wave equation:

**Theorem 1.4.** *There are nonnegative functions* $a \in \cap_{s>1} G^s([0, T])$*, such that the Cauchy problem for the weakly hyperbolic wave equation* (1.5) *is ill posed in* $C^\infty$*.*

The theorems above show that the smoothness of *both* the coefficients *and* the symmetrizer play a role in the well posedness in $C^\infty$. The next theorem is a kind of interpolation between the two extreme cases of Theorem 1.2 and Theorem 1.3:

**Theorem 1.5.** *For all* $s > 1$ *and* $\mu < 1 - 1/s$*, there are example of systems* (1.1) *on* $[0, T] \times \mathbb{R}$*, with uniformly hyperbolic symbols, coefficients in the Gevrey classes* $G^s$*, symmetrizer in the Hölder space* $C^\mu$*, and such that the Cauchy problem is ill posed in* $C^\infty$*.*

This leaves open the question of the well posedness in $C^\infty$ when the coefficients belong to $G^s$ and the symmetrizer to $C^\mu$ when $\mu + 1/s \geq 1$.

We end this introduction by several remarks about symmetrizers or $2 \times 2$ system (1.1). For simplicity, we assume that the coefficients are real. Write

$$A(t) = \frac{1}{2}\mathrm{tr} A(t) \mathrm{Id} + A_1(t).$$

Then $A_1^2 = h\mathrm{Id}$ with $h = \frac{1}{4}(a - d)^2 + bc$ satisfying (1.2). In particular,

$$\Sigma(t) = A_1^*(t) A_1(t) + h(t) \mathrm{Id}$$

is a symmetrizer for $A$ in the sense that $\Sigma$ and $\Sigma A = \frac{1}{2}(\mathrm{tr}A)\Sigma + hA_1^* + hA_1$ are symmetric. In general, $\Sigma$ is *not* a symmetrizer in the sense of Friedrichs, since it is not uniformly positive definite, unless $h > 0$, which means that the system is strictly hyperbolic. More precisely, $\Sigma \approx h\mathrm{Id}$. But $\Sigma$ has the same smoothness as the coefficients of $A$.

On the other hand, since the system is uniformly diagonalizable, there are bounded symmetrizers $\Sigma_1(t)$ which are uniformly positive definite. For instance $h^{-1}\Sigma$ is a bounded symmetrizer. More generally, writing

$$\frac{1}{2}(a-d) = h^{\frac{1}{2}}a_1, \quad b = b_1 h^{\frac{1}{2}}, \quad c = c_1 h^{\frac{1}{2}}, \tag{1.7}$$

one has $a_1^2 + b_1c_1 \geq \delta(a_1^2 + b_1^2 + c_1^2) \geq \delta > 0$ and the symmetrizer are of the form

$$\Sigma_1 = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \qquad \text{with} \qquad 2a_1\beta = b_1\alpha - c_1\gamma. \tag{1.8}$$

Therefore there is a cone of positive symmetrizers of dimension 2. Their smoothness depend on the smoothness of $a_1, b_1, c_1$, that is of $h^{-\frac{1}{2}}A_1$. There might be better choices than others. For instance, if the system is symmetric, $\Sigma_1 = \mathrm{Id}$ is a very smooth symmetrizer. Our discussion below concerns the smoothness of both $\Sigma$ and $\Sigma_1$ and their possible interplay.

## 2 The counterexamples

We consider systems of the form

$$LU := \partial_t U + \begin{pmatrix} 0 & a(t) \\ b(t) & 0 \end{pmatrix} \partial_x U. \tag{2.1}$$

with $a$ and $b$ real. We always assume that it is uniformly strongly hyperbolic, that is that $\sigma = a/b > 0$ and $1/\sigma$ are bounded. Our goal is to contradict the estimates (1.3) and (1.6). We contradict the analogous estimates which are obtained by Fourier transform in $x$, and more precisely, we construct sequences of functions $u_k, v_k$ and $f_k$ in $C^\infty([0,T])$, vanishing near $t = 0$, satisfying

$$\partial_t u_k + ih_k a(t) v_k = f_k, \qquad \partial_t v_k + ih_k b(t) u_k = 0 \tag{2.2}$$

with $h_k \to +\infty$ and such that

$$\|f_k\|_{L^2} / \|(u_k, v_k)\|_{L^2} \to 0 \qquad as\ k \to \infty \tag{2.3}$$

in the first case or, for all $j$ and $l$ ,

$$\big\|h_k^j\partial_t^l f_k\big\|_{L^2}/\big\|(u_k,v_k)\big\|_{L^2} \;\to\; 0 \qquad as\ k\to\infty \tag{2.4}$$

in the second case. Moreover, the support of these function is contained in an interval $I_k=[t_k,t'_k]$ with $0<t_k<t'_k$ and $t'_k\to 0$, showing that the problem is ill posed on any interval $[0,T]$ with $T>0$.

## 2.1 Exponentially amplified solutions of the wave equation

In this section we review and adapt the construction of [CS2]. The key remark is that the function $\underline{w}_\varepsilon(t)=e^{-\varepsilon\phi(t)}\cos t$ satisfies

$$\partial_t^2\underline{w}_\varepsilon+\underline{\alpha}_\varepsilon\underline{w}_\varepsilon=0 \tag{2.5}$$

if

$$\phi(t)=\int_0^t(\cos s)^2ds,\qquad \underline{\alpha}_\varepsilon(t)=1+2\varepsilon\sin 2t-\varepsilon^2(\cos t)^4. \tag{2.6}$$

The important property of the $\underline{w}_\varepsilon$ is their exponential decay at $+\infty$. More precisely

$$e^{\frac12\varepsilon t}\underline{w}_\varepsilon(t)=e^{\frac14\varepsilon\sin 2t}\cos t\quad \text{is } 2\pi\text{periodic}$$

and

$$\underline{w}_\varepsilon(t+2\pi)=e^{-\varepsilon\pi}\underline{w}_\varepsilon(t). \tag{2.7}$$

Next, one symmetrizes and localizes this solution. More precisely, consider $\chi\in C^\infty(\mathbb{R})$, supported in $]-7\pi,7\pi[$, odd, equal to 1 on $[-6\pi,-2\pi]$ and thus equal to $-1$ on $[2\pi,6\pi]$, and such that for all $t$, $0\le\chi(t)\le 1$ and $|\partial_t\chi(t)|\le 1$. For $\nu\in\mathbb{N}$, let

$$\Phi_\nu(t)=\int_0^t\chi_\nu(s)(\cos s)^2ds,\qquad \chi_\nu(t)=\chi(t/\nu). \tag{2.8}$$

For $\varepsilon>0$, $w_{\varepsilon,\nu}(t)=e^{\varepsilon\Phi_\nu(t)}\cos t$ satisfies

$$\partial_t^2 w_{\varepsilon,\nu}+\alpha_{\varepsilon,\nu}w_{\varepsilon,\nu}=0 \tag{2.9}$$

where

$$\begin{aligned}\alpha_{\varepsilon,\nu}(t)&=1+\varepsilon\chi_\nu\sin 2t-\varepsilon\Phi''_\nu-(\varepsilon\Phi'_\nu)^2\\&=1+2\varepsilon\chi_\nu\sin 2t-\varepsilon\chi'_\nu(\cos t)^2-\varepsilon^2\chi_\nu^2(\cos t)^4.\end{aligned} \tag{2.10}$$

For $\varepsilon \le \varepsilon_0 = 1/10$ and for all $\nu$

$$|\alpha_{\varepsilon,\nu} - 1| \le \frac{1}{2} \tag{2.11}$$

and we always assume below that the condition $\varepsilon \le \varepsilon_0$ is satisfied. We note also that $\alpha_{\varepsilon,\nu} = 1$ for $|t| \ge 7\nu\pi$ since $\chi_\nu$ vanishes there.

The final step is to localize the solution in $[-6\nu\pi, 6\nu\pi]$. Introduce an odd cut off function $\zeta(t)$ supported in $]-6\pi, 6\pi[$ and equal to 1 for $|t| \le 4\pi$ and let

$$\tilde{w}_{\varepsilon,\nu}(t) = \zeta(t/\nu) w_{\varepsilon,\nu}(t). \tag{2.12}$$

It is supported in $[-6\nu\pi, 6\nu\pi]$ and equal to $w_{\varepsilon,\nu}$ on $[-4\nu\pi, 4\nu\pi]$. Then

$$f_{\varepsilon,\nu} = \partial_t^2 \tilde{w}_{\varepsilon,\nu} + \alpha_{\varepsilon,\nu}\tilde{w}_{\varepsilon,\nu} = 2\nu^{-1}\zeta'(t/\nu)\partial_t w_{\varepsilon,\nu} + \nu^{-2}\zeta''(t/\nu) w_{\varepsilon,\nu} \tag{2.13}$$

is supported in $[-6\nu\pi, -4\nu\pi] \cup [4\nu\pi, 6\nu\pi]$.

**Lemma 2.1.** *For all $j$, there is a constant $C_j$ such that for all $\varepsilon \le \varepsilon_0$ and all $\nu \ge 1$*

$$\big\|\partial_t^j f_{\varepsilon,\nu}\big\|_{L^2} \le C_j \nu^{-1} e^{-\varepsilon\nu\pi} \big\|\tilde{w}_{\varepsilon,\nu}\big\|_{L^2}. \tag{2.14}$$

*Proof.* By symmetry, it is sufficient to estimate $f_{\varepsilon,\nu}$ for $t \ge 0$, that is on $[4\nu\pi, 6\nu\pi]$. On $[2\nu\pi, 6\nu\pi]$, $\chi_\nu = -1$, hence $\Phi_\nu - \phi$ is constant and

$$w_{\varepsilon,\nu}(t) = c_{\nu,\varepsilon}\underline{w}_\varepsilon(t), \qquad c_{\nu,\varepsilon} = e^{\varepsilon\Phi_\nu(2\nu\pi)}.$$

Moreover, on this interval $\alpha_{\varepsilon,\nu} = \underline{\alpha}_\varepsilon$ is bounded with derivatives bounded independently of $\varepsilon$ , and hence

$$\big\|\partial_t^j f_{\varepsilon,\nu}\big\|_{L^2} \le C_j \nu^{-1} c_{\nu,\varepsilon} \big\|(\underline{w}_\varepsilon, \partial_t \underline{w}_\varepsilon)\big\|_{L^2([4\nu\pi, 6\nu\pi])}.$$

By (2.7), this implies

$$\big\|\partial_t^j f_{\varepsilon,\nu}\big\|_{L^2} \le C_j \nu^{-1} c_{\nu,\varepsilon} e^{-\varepsilon\nu\pi} \big\|(\underline{w}_\varepsilon, \partial_t \underline{w}_\varepsilon)\big\|_{L^2([2\nu\pi, 4\nu\pi])}.$$

On the other hand

$$\big\|w_{\varepsilon,\nu}\big\|_{L^2} \ge c_{\nu,\varepsilon} \big\|\underline{w}_\varepsilon\big\|_{L^2([2\nu\pi, 4\nu\pi])}.$$

Therefore it is sufficient to prove that there is a constant $C$ such that for all $\nu$ and $\varepsilon$:

$$\big\|(\underline{w}_\varepsilon, \partial_t \underline{w}_\varepsilon)\big\|_{L^2([2\nu\pi, 4\nu\pi])} \le C \big\|\underline{w}_\varepsilon\big\|_{L^2([2\nu\pi, 4\nu\pi])}.$$

Using again (2.7), one has

$$\left\|(\underline{w}_\varepsilon, \partial_t \underline{w}_\varepsilon)\right\|^2_{L^2([2\nu\pi, 4\nu\pi])} = \sum_{k=0}^{\nu-1} e^{-2(\varepsilon k+\nu)\pi} \left\|(\underline{w}_\varepsilon, \partial_t \underline{w}_\varepsilon)\right\|^2_{L^2([0,2\pi])}$$

and

$$\left\|\underline{w}_\varepsilon\right\|^2_{L^2([2\nu\pi, 4\nu\pi])} = \sum_{k=0}^{\nu-1} e^{-2(\varepsilon k+\nu)\pi} \left\|\underline{w}_\varepsilon\right\|^2_{L^2([0,2\pi])}.$$

On $[0, 2\pi]$ the $H^1$ norm of the $\underline{w}_\varepsilon$ are uniformly bounded while their $L^2$ norm remain larger than a positive constant. □

## 2.2 Construction of the coefficients and of the solutions

For $k \geq 1$, let $\rho_k = k^{-2}$. We consider intervals $I_k = [t_k, t'_k]$ and $J_k = [t'_k, t_{k-1}]$ of the same length $\rho_k = t'_k - t_t = t_{k-1} - t'_k$, starting at $t_0 = 2\sum_{k=1}^{\infty} \rho_k$, and thus such that $t_k \to 0$.

The functions $a$ and $b$ are defined on $]0, t_0]$ as follows: we fix a function $\beta \in C^\infty(\mathbb{R})$ supported in $]0, 1[$ and with sequences $\varepsilon_k$, $\nu_k$ and $\delta_k$ to be chosen later on,

$$\text{(2.15)} \qquad \begin{aligned} &\text{on } I_k : \quad \begin{cases} a(t) = \delta_k \alpha_{\varepsilon_k,\nu_k}\big(-8\pi\nu_k + 16\pi(t-t_k)\nu_k/\rho_k\big), \\ b(t) = \delta_k \end{cases} . \\ &\text{on } J_k : \quad a(t) = b(t) = \delta_k + (\delta_{k-1} - \delta_k)\beta\big((t - t'_k)/\rho_k\big) \end{aligned}$$

Because $\alpha_{\varepsilon,\nu} = 1$ for $|t| \geq 7\nu\pi$, the coefficient $a = \delta_k$ near the end points of $I_k$. The use of the function $\beta$ on $J_k$ makes a smooth transition between $\delta_k$ and $\delta_{k-1}$. Therefore, $a$ and $b$ are $C^\infty$ on $]0, t_0]$. The coefficients will be chosen so that they extend smoothly up to $t = 0$.

This is quite similar to the choice in [CoNi], except that we introduce a new sequence $\varepsilon_k$, which is crucial to control the Hölder continuity of $\sigma = a/b$.

We use the family (2.12) to construct solutions of the system supported in $I_k$, for $k$ large. On $I_k$, $b$ is constant and the equation (2.2) reads

$$\text{(2.16)} \qquad \partial_t u_k + i h_k \delta_k \alpha_k v_k = f_k, \qquad \partial_t v_k + i h_k \delta_k u_k = 0,$$

with

$$\alpha_k(t) = \alpha_{\varepsilon_k,\nu_k}\big(-8\pi\nu_k + 16\pi(t-t_k)\nu_k/\rho_k\big).$$

Therefore, a $C^\infty$ solution of (2.2) supported in $I_k$ is

$$\text{(2.17)} \qquad \begin{aligned} u_k(t) &= i\partial_t \tilde{w}_{\varepsilon_k,\nu_k}\big(-8\pi\nu_k + 16\pi(t-t_k)\nu_k/\rho_k\big) \\ v_k(t) &= \tilde{w}_{\varepsilon_k,\nu_k}\big(-8\pi\nu_k + 16\pi(t-t_k)\nu_k/\rho_k\big) \end{aligned}$$

with

$$f_k(t) = 16i\pi(\nu_k/\rho_k) f_{\varepsilon_k,\nu_k}\big( - 8\pi\nu_k + 16\pi(t - t_k)\nu_k/\rho_k\big) \tag{2.18}$$

provided that

$$h_k = 16\pi\nu_k/\rho_k\delta_k. \tag{2.19}$$

# 3 Properties of the coefficients

We always assume that

$$\varepsilon_k \le \varepsilon_0, \quad \varepsilon_k\nu_k \to +\infty, \quad \delta_k \to 0. \tag{3.1}$$

## 3.1 Conditions for blow up

**Lemma 3.1.** *If*

$$(\rho_k)^{-1} e^{-\varepsilon_k\nu_k\pi} \to 0, \tag{3.2}$$

*then the blow up property in $L^2$ (2.3) is satisfied.*

*Proof.* By Lemma 2.1

$$\big\|f_k\big\|_{L^2} \le C\rho_k^{-1} e^{-\varepsilon_k\nu_k\pi}\big\|v_k\big\|_{L^2}.$$

□

**Lemma 3.2.** *If*

$$\frac{1}{\varepsilon_k\nu_k} \ln(h_k\nu_k/\rho_k) \to 0, \tag{3.3}$$

*then the blow up property in $C^\infty$ (2.4) is satisfied.*

*Proof.* By Lemma 2.1 one has

$$\big\|\partial_t^l h_k^j f_k\big\|_{L^2}/\big\|(u_k, v_k)\big\|_{L^2} \le C_l \nu_k^{-1} h_k^j (16\pi\nu_k/\rho_k)^{l+1} e^{-\varepsilon_k\nu_k\pi}.$$

This tends to 0 if

$$\varepsilon_k\nu_k\pi - j\ln h_k - (l+1)\ln(\nu_k/\rho_k) \to +\infty.$$

If (3.3) is satisfied, this is true for all $j$ and $l$. □

## 3.2 Smoothness of the coefficients

**Lemma 3.3.** *If*

$$\ln(\nu_k/\rho_k)/|\ln(\delta_k\varepsilon_k)| \to 0 \tag{3.4}$$

*then the functions $a$ and $b$ are $C^\infty$ up to $t = 0$.*

*Proof.* $a$ and $b$ are $O(\delta_k)$ and thus converge to 0 when $t \to 0$. Moreover, for $j \geq 1$,

$$|\partial_t^j a| \leq C_j \begin{cases} \delta_k\varepsilon_k(\nu_k/\rho_k)^j & \text{on } I_k, \\ \delta_k\rho_k^{-j} & \text{on } J_k. \end{cases}$$

The worst situation occurs on $I_k$ and the right hand side is bounded if

$$j\ln(\nu_k/\rho_k) - |\ln(\delta_k\varepsilon_k)|$$

is bounded from above. This is true for all $j$ under the assumtion (3.4), implying that $a$ is $C^\infty$ on $[0, t_0]$. The proof for $b$ is similar and easier. □

Next, we investigate the possible Gevrey regularity of the coefficients. For that we need make a special choice of the cut-off functions $\chi$ and $\beta$ which occur in the construction of $a$ and $b$. We can choose them in a class contained in $\cap_{s>1}G^s$ and containing compactly supported function, (see e.g. [Ma]). We choose them such that there is a constant $C$ such that for all $j$

$$\sup_t \left(\left|\partial_t^j\chi(t)\right| + \left|\partial_t^j\beta(t)\right|\right) \leq C^{j+1}j!(\ln j)^{2j}. \tag{3.5}$$

**Lemma 3.4.** *If* (3.5) *is satisfied, then for $j \geq 1$*

$$\begin{aligned} &\sup_{t\in I_k\cup J_k} \left(\left|\partial_t^j a(t)\right| + \left|\partial_t^j b(t)\right|\right) \leq \\ &\qquad K^{j+1}\delta_k\varepsilon_k\Big((\nu_k/\rho_k))^j + (1/\rho_k)^j j!(\ln j)^{2j}\Big). \end{aligned} \tag{3.6}$$

*Proof.* On $I_k$ we take advantage of the explicit form (2.10) of $\alpha_{\varepsilon,\nu}$: it is a finite sum of sin and cos with coefficients of the form $\chi(t/\nu)$. Scaled on $I_k$, each derivative of the trigonometric functions yields a factor $\nu_k/\rho_k$, while the derivatives of $\chi_{\nu_k}$ have only a factor $1/\rho_k$. Since $\chi'$ and $\chi^2$ satisfy estimates similar to (3.5), we conclude that $a$ satisfies

$$\left|\partial_t^j a(t)\right| \leq \varepsilon_k\delta_k K^j \sum_{l\leq j}(\nu_k/\rho_k)^{j-l}C^{l+1}l!(\ln l)^{2l}$$

implying the estimate (3.6) on $I_k$. On $I_k$, $b$ is constant. On $J_k$ things are clear by scaling since the coefficients are functions of $\beta((t - t'_k)/\rho_k)$. □

To estimate quantities such as $\delta_k(\nu_k/\rho_k)^j$ we use the following inequalities for $a > 0$ and $x \geq 1$

$$e^{-x}x^a \leq a^a \tag{3.7}$$

and

$$e^{-e^x}x^a \leq \begin{cases} |\ln a|^a & \text{when } \; a \geq e \\ 1 & \text{when } \; a \leq e. \end{cases} \tag{3.8}$$

**Corollary 3.5.** *Suppose that $\delta_k = e^{-\eta_k}$ and that for $s > s' > 1$*

$$(\nu_k/\rho_k) \leq C\eta_k^s, \qquad (1/\rho_k)^j \leq C\eta_k^{s'-1}. \tag{3.9}$$

*Then the coefficients belong to the Gevrey class $G^s$.*

*If for some $p > 0$ and $q > 0$,*

$$\eta_k \geq e^{k^q}, \qquad (\nu_k/\rho_k) \leq Ck^p\eta_k \tag{3.10}$$

*then the coefficients belong to $\cap_{s>1}G^s$.*

*Proof.* We neglect $\varepsilon_k$ and only use the bound $\varepsilon_k \leq \varepsilon_0$. In the first case, we obtain from (3.7) that

$$\delta_k(\nu_k/\rho_k)^j \leq e^{-\eta_k}(C\eta_k)^{sj} \leq (C'j)^{js}, \qquad \delta_k(1/\rho_k)^j \leq (C''j)^{j(s'-1)}$$

implying that

$$|\partial_t^j(a,b)| \leq K^{j+1}j^{sj}.$$

In the second case, combining (3.7) and (3.8)

$$e^{-\eta_k}(\nu_k/\rho_k)^j \leq C'^j j^j \; k^{pj}e^{-\frac{1}{2}\eta_k} \leq C''^j j^j(1+\ln j)^{pj/q}.$$

Using again (3.8) for the second term, we obtain that

$$|\partial_t^j(a,b)| \leq K^{j+1}j^j(\ln j)^{rj}$$

with $r = \max\{p, 4\}/q$. In particular, the right hand side is estimated by $K_s^{k+1}j^{js}$ for all $s > 1$, proving that the functions $a$ and $b$ belong to $\cap_{s>1}G^s$. □

### 3.3 Smoothness of the symmetrizer

**Lemma 3.6.** *Suppose that $\omega$ is a continuous and increasing function on $[0,1]$ such that $t^{-1}\omega(t)$ is decreasing. If*

$$\varepsilon_k \le \omega(\rho_k/\nu_k) \tag{3.11}$$

*then $\sigma = a/b$ satisfies*

$$\big|\sigma(t) - \sigma(t')\big| \le C\omega(|t-t'|). \tag{3.12}$$

*In particular, if $\mu \le 1$ and*

$$\limsup_k \varepsilon_k (\nu_k/\rho_k)^\mu < +\infty \tag{3.13}$$

*then $\sigma$ is Hölder continuous of exponent $\mu$. If*

$$\varepsilon_k(\nu_k/\rho_k) \le C\ln(\nu_k/\rho_k)^\theta \tag{3.14}$$

*then $\omega(t) = t|\ln t|^\theta$ is a modulus of continuity for $\sigma$.*

*Proof.* On $J_k$, $\tilde\sigma = \sigma - 1$ vanishes and on $I_k$

$$\tilde\sigma = \varepsilon_k \alpha_{\varepsilon_k,\nu_k}(-8\pi\nu_k + 16\pi(t-t_k)\nu_k/\rho_k),$$

and thus

$$|\tilde\sigma| \le C\varepsilon_k, \qquad |\partial_t\tilde\sigma| \le C\varepsilon_k\nu_k/\rho_k. \tag{3.15}$$

Hence, for $t$ and $t'$ in $I_k$,

$$\big|\tilde\sigma(t) - \tilde\sigma(t')\big| \le C\varepsilon_k \min\{1, |t-t'|\nu_k/\rho_k\}.$$

If $\rho_k/\nu_k \le |t-t'|$ we use the first estimate and

$$\big|\tilde\sigma(t) - \tilde\sigma(t')\big| \le C\varepsilon_k \le C\omega(\rho_k/\nu_k) \le C\omega(|t-t'|).$$

If $|t-t'| \le \rho_k/\nu_k$ we use the second estimate and the monotonicity of $t^{-1}\omega(t)$

$$\big|\tilde\sigma(t) - \tilde\sigma(t')\big| \le C\varepsilon_k(\nu_k/\rho_k)|t-t'| \le C(\nu_k/\rho_k)\omega(\rho_k/\nu_k)|t-t'| \le C\omega(|t-t'|).$$

This shows that (3.12) is satisfied when $t$ and $t'$ belong to the same interval $I_k$.

If $t$ belong to $I_k$ and $t' \in J_k$, then $\tilde\sigma(t') = \tilde\sigma(t'_k) = 0$ and

$$|\tilde\sigma(t) - \tilde\sigma(t')| \le C\omega(|t-t'_k|) \le C\omega(|t-t'|).$$

Similarly, if $t < t'$ and $t$ and $t'$ do not belong to the same $I_k \cup J_k$, there are end points $t_j$ and $t_l$ such that $t_j \le t \le t_{j-1} \le t_l \le t' \le t_{l-1}$. Since $\tilde\sigma$ vanishes at the endpoints of $I_k$ and on $J_k$,

$$\begin{aligned}|\tilde\sigma(t) - \tilde\sigma(t')| &\le C|\tilde\sigma(t) - \tilde\sigma(t_j)| + |\tilde\sigma(t) - \tilde\sigma(t_j)| \\ &\le C\omega(|t - t_{j-1}) + C\omega(|t_l - t'|) \le C\omega(|t - t'|)\end{aligned}$$

and the lemma is proved. □

# 4 Proof of the theorems

We now adapt the choice of the the parameters $\varepsilon_k$, $\nu_k$ and $\delta_k$ so that the coefficients and the symmetrizer satisfy the properties stated in the different theorems. We will choose two increasing functions, $f$ and $g$, on $\{x \ge 1\}$ and define $\varepsilon_k$ and $\delta_k$ in terms of $\nu_k$ through the relations:

$$\varepsilon_k \nu_k/\rho_k = f(\nu_k/\rho_k), \qquad \delta_k = e^{-\eta_k}, \quad \eta_k = g(\nu_k/\rho_k). \tag{4.1}$$

Recall that $\rho_k = k^{-2}$. The sequence of integers $\nu_k$ will be chosen to converge to $+\infty$ and thus $\nu_k/\rho_k \to +\infty$. The conditions (3.1) are satisfied if at $+\infty$:

$$f(x) \ll x, \qquad g(x) \to +\infty. \tag{4.2}$$

Here $\phi(x) \ll \psi(x)$ means that $\psi(x)/\phi(x) \to \infty$. In particular, the first condition implies that $\varepsilon_k \to 0$ so that the condition $\varepsilon \le \varepsilon_0$ is certainly satisfied if $k$ is large enough.

One has

$$|\ln(\delta_k\varepsilon_k)| = \eta_k + \ln(\nu_k/\rho_k) + \ln f(\nu_k/\rho_k)$$

Hence, by Lemma 3.3, the coefficients $a$ and $b$ are $C^\infty$ when

$$\ln x \ll g(x) \ll x. \tag{4.3}$$

since with (4.2) it implies that $|\ln(\delta_k\varepsilon_k)| \sim \eta_k \gg \ln(\nu_k/\rho_k)$.

## 4.1 Proof of Theorem 1.1

Given the modulus of continuity $\omega$, we choose $f(x) = x\omega(x^{-1})$. The assumption on $\omega$ is that $f$ is increasing and $f(x) \to +\infty$ at infinity. The spirit of the theorem is that $f$ can grow to infinity as slowly as one wants. Lemma 3.6 implies that $\omega$ is a modulus of continuity for $\sigma = a/b$. By Lemma 3.1, the blow up property (2.3) occurs when

$$k^2 e^{-k^{-2} f(k^2\nu_k)\pi} \to 0.$$

This condition is satisfied if $\nu_k$ satisfies

$$f(k^2\nu_k) \geq k^3, \tag{4.4}$$

Let $f_1(x) = \min\{f(x), \ln x\}$. We choose $g(x) = x/f_1(x)$ and $\nu_k$ such that

$$2k^3 \leq f_1(k^2\nu_k) \leq 4k^3.$$

Note that this implies (4.4). We show that the conditions (3.10) are satisfied with $p = q = 3$ and $C = 4$ and a suitable choice of $\nu_k$, so that by Corollary 3.5 the coefficients belong to $\cap_{s>1} G^s$ and the theorem is proved.

Indeed, since $f_1(k^2\nu_k) \leq 4k^3$, the condition $\nu_k/\rho_k \leq 4k^3\eta_k$ is satisfied. Moreover, since $\ln(k^2\nu_k) \geq 2k^3$,

$$\nu_k \geq k^{-2}e^{2k^3} \geq e^{k^3}.$$

for $k$ large.

## 4.2 Proof of Theorem 1.2

The proof is similar. Given the modulus of continuity $\omega$, we choose $f(x) = x\omega(x^{-1})$. The assumption on $\omega$ is now that

$$\ln x \ll f(x). \tag{4.5}$$

The spirit of the theorem is now that $f(x)/\ln x$ can grow to infinity as slowly as one wants. By Lemma 3.6, $\omega$ is a modulus of continuity for $\sigma = a/b$.

By Lemma 3.2, the blow up property (2.4) is satisfied if

$$\ln h_k = \eta_k + \ln(\nu_k/\rho_k) + \ln(16\pi) \ll \varepsilon_k\nu_k$$

that is if

$$\rho_k f(\nu_k/\rho_k) \gg g(\nu_k/\rho_k) + \ln(\nu_k/\rho_k). \tag{4.6}$$

Let $\psi(x) = f(x)/\ln x$ and $g(x) = \sqrt{\psi(x)}\ln x$. Then

$$\psi(x) \gg 1, \qquad \ln x \ll g(x) \ll f(x).$$

Therefore, the condition (4.6) is satisfied when $\rho_k\sqrt{\psi}(\nu_k/\rho_k) \to +\infty$ and for that it is sufficient to choose $\nu_k$ such that

$$\psi(k^2\nu_k) \geq k^5. \tag{4.7}$$

The condition $g(x) \gg \ln x$ implies that the coefficients are $C^\infty$ and the theorem is proved.

### 4.3 Proof of Theorem 1.5

With $s > 1$ and $0 < \mu < 1 - 1/s$, we choose

$$g(x) = x^{1/s} \ll f(x) = x^{1-\mu}. \tag{4.8}$$

The choice of $f$ implies that $\sigma = a/b \in C^\mu$. The choice of $g$ implies that

$$\nu_k/\rho_k \le \big(g(\nu_k/\rho_k)\big)^s = \eta_k^s.$$

With $s' \in ]1, s[$, the condition

$$\rho_k^{-1} \le \eta_k^{s'-1}$$

is satisfied when $k^2 \le (k^2\nu_k)^{(s'-1)/s}$, that is when

$$\nu_k \ge k^{2p}, \qquad p = (1 + s - s')/(s' - 1). \tag{4.9}$$

In this case, Corollary 3.5 implies that the coefficients $a$ and $b$ belong to the Gevrey class $G^s$.

The blow up property (2.4) is satisfied when (4.6) holds, that is when

$$k^{-2}(k^2\nu_k)^{1-\mu} \gg (k^2\nu_k)^{1/s},$$

which is true if

$$\nu_k \ge k^{2q}, \qquad q = (\mu + 1/s)/(1 - \mu - 1/s).$$

Therefore, if $\nu_k \ge k^{2\max\{p,q\}}$, the system satisfies the conclusions of Theorem 1.5.

### 4.4 Proof of Theorem 1.3

The analysis above shows that if one looks for coefficients in $\cap_{s>1} G^s$, one must choose $g$ and thus $f$, close to $x$. We choose here

$$g(x) = x/(\ln x)^2 \ll f(x) = x/\ln x \ll x$$

Since $f(x)/x \to 0$ at infinity, the symmetrizer is continuous up to $t = 0$ but in no $C^\mu$ for all $\mu > 0$.

The ill posedness in $C^\infty$ is again garantied by the condition (4.6), that is $\ln(k^2\nu_k) \gg k^2$. In particular, it is satisfied when

$$\nu_k \ge e^{k^3}. \tag{4.10}$$

By Corollary 3.5, to finish the proof of Theorem 1.3, is is sufficient to show that one can choose $\nu_k$ satisfying (4.10) and such that $\nu_k/\rho_k \le 4k^6\eta_k$. This condition reads $\ln(k^2\nu_k) \le 2k^3$, or

$$\nu_k \le k^{-2}e^{2k^3}$$

which is compatible with (4.10) if $k$ is large enough.

### 4.5 Proof of Theorem 1.4

Let $a \in \cap_{s>1}G^s$ denote the coefficient constructed for the proof of Theorem (1.3). The definition (2.15) shows that $a \ge 0$ and indeed $a > 0$ for $t > 0$. The functions $v_k$ defined at (2.17) are supported in $I_k$ and are solutions of the wave equation (1.5) with source term $f_k$ and we have shown that

$$\left\|h_k^j\partial_t^l f_k\right\|_{L^2}/\left\|v_k\right\|_{L^2} \;\to\; 0 \qquad as\ k\to\infty.$$

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