Existence and stability of multidimensional shock fronts in the vanishing viscosity limit

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Abstract

In this paper we present a new approach to the study of linear and nonlinear stability of inviscid multidimensional shock waves under small viscosity perturbation, yielding optimal estimates and eventually an extension to the viscous case of the celebrated theorem of Majda on existence and stability of multidimensional shock waves. More precisely, given a curved Lax shock solution u^0 to a hyperbolic system of conservation laws, we construct nearby viscous shock solutions u^ϵ to a parabolic viscous perturbation of the hyperbolic system which converge to u^0 as viscosity $\epsilon \to 0$ and satisfy an appropriate (conormal) version of Majda's stability estimate.

The main new feature of the paper is the derivation of maximal and optimal estimates for the linearization of the parabolic problem about a highly singular approximate solution. These estimates are more robust than the singular estimates obtained in our previous work, and permit us to remove an earlier assumption limiting how much the inviscid shock we start with can deviate from flatness.

The key to the new approach is to work with the full linearization of the parabolic problem, that is, the linearization with respect to both u^ϵ and the unknown viscous front, and to allow variation of the front at all stages - not only in the construction of the approximate solution as done in previous work, but also in the final error equation. After reformulating the problem as a transmission problem, we show that the linearized problem can be desingularized and optimal estimates obtained by adding an appropriate extra boundary condition involving the front. The extra condition determines a local evolution rule for the viscous front.

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Contents

1 Introduction

This paper presents a new approach to the study of the linear and nonlinear stability of inviscid multidimensional shock waves under small viscosity perturbation. Our goal is to revisit the plane wave analysis and the associated Evans' functions conditions in order to obtain maximal and optimal estimates. As a consequence of these estimates, we obtain a viscous version of the celebrated theorem of Majda in the inviscid case, asserting existence and stability of curved multidimensional viscous shock fronts for sufficiently small, strictly parabolic viscosity. More, we establish an asymptotic expansion, with rigorous error bounds, to arbitrary order about the inviscid solution in powers of viscosity strength ϵ. With some elaboration, the method may be applied also to the physical case of "real", or partially parabolic viscosity; see [GMWZ3]. The method would also apply to the analogous problem of long time stability for a fixed viscosity, perhaps yielding new results in this context as well; see the discussions of small viscosity vs. long time stability in [GMWZ1, GMWZ2].

Consider an $N \times N$ hyperbolic system of conservation laws in $\mathbb{R} \times \mathbb{R}^d$:

$$\partial_t u + \sum_{j=1}^{d} \partial_j f_j(u) = 0, \tag{1.1}$$

and a given uniformly stable Lax shock u_0 with front $x = \psi(t, y)$. Below, we denote by $(y, x) \in \mathbb{R}^{d-1} \times \mathbb{R}$ the space variables and $y = (y_1, \ldots, y_{d-1})$. The shock wave solution is $u_0(t, y, x) = u_0^{\pm}(t, y, x)$ on $\pm(x - \psi(t, y)) > 0$, where $u_0^{\pm}$ are solutions of (1.1), smooth up to the boundary and satisfying the Rankine-Hugoniot jump conditions on $\{x = \psi(t, y)\}$:

$$\partial_t \psi \big[u\big] + \sum_{j=1}^{d-1} \partial_j \psi \big[f_j(u)\big] = \big[f_d(u)\big]\,. \tag{1.2}$$

Such solutions have been constructed by A.Majda [Maj] assuming that a *uniform stability condition* is satisfied. This condition is recalled in Section two below.

Next consider a parabolic viscous perturbation of (1.1):

$$\partial_t u + \sum_{j=1}^{d} \partial_j f_j(u) - \varepsilon \sum_{j,k=1}^{d} \partial_j \big(B_{j,k}(u) \partial_k u \big) = 0 \,. \tag{1.3}$$

The problem is to show, under "natural" assumptions, that, first, the given solution u_0 of (1.1) is the limit as ε tends to zero of solutions u_ε of (1.3), and, second, the solutions u_ε satisfy uniform bounded-time stability estimates recovering in the $\varepsilon \to 0$ limit those obtained by Majda in the inviscid case. The assumptions are twofold: first, there are conditions on the nature of hyperbolicity of (1.1) and parabolicity of (1.3) and their compatibility; second, there are (planar) stability conditions. The sharp criterion of stability is expressed by an Evans function hypothesis, which implies the uniform stability of the inviscid shock (see [ZS], [Zu1]).

The first, existence problem was solved in 1-D by J.Goodman and Z.Xin [GX] for sufficiently weak shocks (see also [GW]) and by F.Rousset [Ro2] for shocks of arbitrary strength satisfying the Evans function hypothesis. An analogous result in multi-D has been obtained in [GMWZ2], under the additional technical assumption that the shock front does not deviate too much from an hyperplane. Note that the recent works by Freistühler and Szmolyan [FS] and Plaza and Zumbrun [PZ] show that the Evans condition holds for sufficiently weak Lax shocks.

However, the "natural" Evans' function introduced in [ZS], [Zu1] and used in [Ro1], [GMWZ2] has a singularity at the origin. This reflects the existence of a pole for the Green function and induces only a weak type of stability with losses of epsilons in the estimates. As a result, the second, stability problem has not been solved in any dimension (though see the related and in some respects stronger results of S.-H. Yu for small-amplitude shocks in one dimension [Yu]). This is quite unnatural and unsatisfactory, since the expected estimates should recover the inviscid estimates in the limit ε tends to zero. In this paper, we propose a *modified* but equivalent problem, such that the associated Evans function *is not singular* at the origin, inducing the desired maximal and sharp estimates. These not only resolve the stability problem, but also permit a greatly improved and simplified treatment of existence, in particular allowing us to drop the artificial assumption made in [GMWZ2] of approximate flatness of the background inviscid front. The estimates so obtained are in a class of conormal spaces introduced in [MZ1] that is natural for the study of singular perturbation problems. As $\varepsilon \to 0$, they reduce to the Sobolev estimates of Majda; see Remark 5.7.

Besides their mathematical interest, these results have physical implications for continuum mechanics and modeling of flow in compressible media. Both existence of and stability about viscous shock fronts are required to validate "physicality" of shock wave solutions in the sense of their presence as persistent features of the flow. The bounded-time version of stability considered here agrees with the common-sense notion of a coherent "shock-like" structure that is observable for small but finite time, hence in this sense is quite satisfactory.

Let us now explain the main features of our approach. The basic idea is not new. In the inviscid case, we know from [Maj] that the equations must determine $u_0^\pm$ *and* the front ψ. The equations (1.1) (1.2) for $u_0^\pm$ form a free boundary transmission problem. The front is fixed by introducing the unknown change of variables:

$$\tilde{x} = x - \psi(t, y), \tag{1.4}$$

which transforms (1.1) into

$$\partial_t u + \sum_{j=1}^{d-1} A_j(u)\partial_j u + A_\nu(u, d\psi)\partial_{\tilde{x}} u = 0\,, \tag{1.5}$$

where $A_j(u) := f_j'(u)$ is the Jacobian matrix of f_j and

$$A_\nu(u, d\psi) := A_d(u) - \sum_{j=1}^{d-1} \partial_j\psi A_j(u) - \partial_t\psi \mathrm{Id} \tag{1.6}$$

is the boundary matrix. The equation (1.5) is solved separately on $\{\tilde{x} > 0\}$ and $\{\tilde{x} < 0\}$, together with the transmission conditions, deduced from (1.2):

$$\partial_t\psi\big[u\big] + \sum_{j=1}^{d-1} \partial_j\psi\big[f_j(u)\big] = \big[f_d(u)\big] \qquad \text{on} \quad \{\tilde{x} = 0\}\,. \tag{1.7}$$

In the viscous case, the front is not defined at all, since the jump is smoothed out. However, in the limit $\varepsilon \to 0$, the front must become apparent. The idea is to introduce it by force in the equation, performing the change of variables (1.4) in the viscous equation too. The new equations read:

$$\partial_t u + \sum_{j=1}^{d-1} A_j(u)\partial_j u + A_\nu(u, d\psi)\partial_{\tilde{x}} u - \varepsilon \sum_{j,k=1}^{d} D_j\Big(B_{j,k}(u)D_k u\Big) = 0\,, \tag{1.8}$$

with $D_j = \partial_j - (\partial_j \psi)\partial_{\tilde{x}}$ when $j < d$ and $D_d = \partial_{\tilde{x}}$. In the viscous case, the solutions are smooth and have no jumps on $\tilde{x} = 0$. Thus the only reasonable transmission conditions are

$$\big[u\big] = 0\,, \quad \big[\partial_{\tilde{x}} u\big] = 0 \qquad \text{on} \quad \{\tilde{x} = 0\}. \tag{1.9}$$

For instance, when u is a planar shock, i.e. when $u_0^\pm$ and $d\psi$ are constants, exact stationary solutions of (1.8) are

$$u^\varepsilon(t, y, x) = w(\tilde{x}/\varepsilon)\,, \qquad \psi = \sigma t + \theta y, \tag{1.10}$$

with w solving the profile equation

$$\partial_z f_\nu(w, d\psi) = \partial_z\big(B_\nu(w, d\psi)\partial_z w\big), \quad \lim_{z\to-\infty} = u_0^-\,, \quad \lim_{z\to+\infty} = u_0^+, \tag{1.11}$$

where the normal flux and the normal viscosity are respectively

$$f_\nu(u, d\psi) = f_d(u) - \sum_{j=1}^{d-1} \partial_j\psi f_j(u) - \partial_t \psi u$$

$$B_\nu(u, d\psi) = \sum_{j,k}^{d} \nu_j \nu_k B_{j,k}(u), \qquad \nu = (-\partial_1\psi, \ldots, -\partial_{d-1}\psi, 1).$$

The formulation (1.8) is used in [GW] to construct asymptotic solutions:

$$u^\varepsilon_{app} \sim \sum_{n\geq 0} \varepsilon^n U_n(t, y, \tilde{x}, \tilde{x}/\varepsilon)\,, \qquad \psi^\varepsilon_{app} \sim \sum_{n\geq 0} \varepsilon^n \psi_n(t, y)\,. \tag{1.12}$$

In this expansion, the first term ψ_0 is the inviscid shock front and U_0 as a function of $z = \tilde{x}/\varepsilon$ is a solution of (1.11) converging to the inviscid solutions $u_0^\pm$ as z tends to $\pm\infty$. Since the only physical front is the inviscid one ψ_0, the first try would be solve (1.8) with $\psi = \psi_0$. A striking fact is that it does not work, and that introducing a corrector ψ_1 is *necessary* to find U_1, even in 1-D. This indicates strongly the importance of the unknown ψ in the problem.

The next step is to analyze the linear (and nonlinear) stability of the planar solutions (1.10) or of the approximate solutions (1.12). In the literature mentioned above, ([GX], [Ro2], [ZS], [Zu1], [GMWZ1], [GMWZ2]), the stability analysis concerns only the stability for perturbations in u. This is the most natural approach since for any fixed ψ and $\varepsilon > 0$, (1.8) (1.9) is a well posed parabolic problem. Denoting by $\mathcal{E}(u, \psi)$ the left hand side of (1.8), the *partially* linearized equation has the form

$$\mathcal{E}'_u(u^\varepsilon_{app}, \psi^\varepsilon_{app})\dot{u} = \dot{f}\,, \qquad [\dot{u}(0)] = [\partial_{\tilde{x}}\dot{u}(0)] = 0\,. \tag{1.13}$$

The stability criterion involves an Evans' function $D(p,\zeta)$, where the $\zeta = (\tau - i\gamma, \eta)$ denote the (rescaled) Fourier-Laplace frequencies and p the relevant parameters (see section 2 below; for instance, if u^ε_{app}, ψ^ε_{app} are given by (1.10), $p = (u^\pm, \sigma, \theta)$). A major result in [ZS] is that for small frequencies ζ (which play a fundamental role due to the rescaling), one has in polar coordinates: $\zeta = \rho\check{\zeta}$:

$$D(p,\zeta) = \rho\Big(\beta(p)\Delta(p,\check{\zeta}) + o(1)\Big), \tag{1.14}$$

where β measures the stability of the profile equation (1.11) and Δ is the Lopatinski determinant of the inviscid problem (1.5) (1.7). The stability condition reads (see e.g. [Ro2], [Zu1] [GMWZ2]):

$$\begin{array}{lll} D(p,\zeta) \neq 0\,, & \text{for} & \zeta \neq 0, \gamma \geq 0, \\ |D(p,\zeta)| \geq c|\zeta|\,, & \text{for} & \zeta \text{ small}\,, \gamma \geq 0, \end{array} \tag{1.15}$$

for some constant $c > 0$. Note that D does vanish at $\rho = 0$ by (1.14). Thus the condition (1.15) only implies a *weak* kind of stability and the typical estimates in [GMWZ2] (see Theorem 9.1 therein) have the form

$$\sqrt{\varepsilon}\|\dot{u}\|_{L^2} \lesssim \|\dot{f}\|_{L^2}. \tag{1.16}$$

This was sufficient to prove the nonlinear stability in [GMWZ2] since we applied this estimate to source terms $\dot{f}$ which were the sum of a very small error $O(\varepsilon^m)$ and a quadratic term in $\dot{u}$. The balance is correct and yields typically $\dot{u} = O(\varepsilon^{m-1/2})$ provided that m is large enough. However, this estimate is not satisfactory: the estimates for the viscous perturbation should improve the inviscid estimates, while (1.16) does not even recover the inviscid estimate.

The main idea of this paper is to continue to take advantage of the unknown ψ and to discuss the stability of the approximate solutions (1.12) with respect to perturbations in u *and* ψ. This leads to consider the *fully linearized* equations from (1.8)(1.9):

$$\mathcal{E}'_u(u^\varepsilon_{app}, \psi^\varepsilon_{app})\dot{u} + \mathcal{E}'_\psi(u^\varepsilon_{app}, \psi^\varepsilon_{app})\dot{\psi} = \dot{f}\,, \qquad [\dot{u}(0)] = [\partial_d \dot{u}(0)] = 0\,. \tag{1.17}$$

Since the problem (1.13) is well posed, for $\varepsilon > 0$, this new problem is underdetermined. This simply reflects that for $\varepsilon > 0$, the "front" is not uniquely determined, and indeed there is no front at all. In order to determine a unique solution, the main idea is to add *an extra boundary condition*

$$\partial_t \dot{\psi} - \varepsilon\Delta_y \dot{\psi} + \ell \cdot \dot{u}_{|\tilde{x}=0} = 0\,. \tag{1.18}$$

The idea of making a good choice of ψ is not new (see Remark 1.1 below). However, an important feature of our approach is that this extra equation is local (differential) and simple, while in previous literature the choice was more hidden and often nonlocal. We also want to emphasize that there is a large flexibility in the choice of this additional condition, this is discussed in section 2 below. For instance, the choice of the Laplacian is almost completely arbitrary. Any second order elliptic with the correct sign and with the appropriate strength ε in front of it would do. . Similarly, there are many choices for the coefficient ℓ in the right hand side. For instance, when u^ε_{app}, ψ^ε_{app} are given by (1.10), the only restriction is that

$$\ell \cdot \partial_z w(0) > 0 \,. \tag{1.19}$$

The main objective of this paper is to show that the singular limit of (1.17) (1.18) is the linearized equation from the inviscid free boundary problem (1.5) (1.7). In this direction, we give three main results.

• First, in sections 2 and 3 we show that the Evans' function for (1.17) (1.18), denoted by $D_m(p,\zeta)$ satisfies for low frequencies

$$D_m(p,\zeta) = \beta(p)\Delta(p,\check{\zeta}) + o(1) \tag{1.20}$$

and for ζ away from zero

$$D_m(p,\zeta) \approx D(p,\zeta) \,. \tag{1.21}$$

Therefore, *the stability condition* (1.15) *is equivalent to*

$$\begin{array}{lll} D_m(p,\zeta) \neq 0 \,, & \text{for} & \zeta \neq 0, \gamma \geq 0, \\ |D(p,\zeta)| \geq c \,, & \text{for} & \zeta \text{ small}\,, \gamma \geq 0. \end{array} \tag{1.22}$$

• Following the analysis of [MZ1] for boundary layers, we show in section 5 that this condition implies *optimal estimates* for (1.17)(1.18). The precise estimate is given in Theorem 5.5 below. In particular, we show that:

$$\|\dot u\|_{L^2} + \|\dot\nabla\psi\|_{L^2} \lesssim \|\dot f\|_{L^2} \,. \tag{1.23}$$

Thus, we recover, as expected, estimates for the viscous solutions which are at least as good as the inviscid estimates of Majda.

• From here, we can repeat the analysis of [MZ1] and deduce the nonlinear stability from the maximal linear stability. This allows to extend the result of [GMWZ2] to the natural framework, dropping the technical assumption on how much the inviscid shock can deviate from flatness. Indeed, since the estimate (1.23) is strictly stronger than the typical estimate (1.16) of [GMWZ2] the analysis is much more robust.

Remark 1.1. 1. The introduction of the front ψ in the estimate of remainders for the viscous equation was already used in the related analysis in the long time stability problem. In the 1-D analysis of [Go1, Zu2], the difference between the exact and the approximate solutions is looked for as

$$v(t,x) = u(t, x+\delta(t)) - u_\varepsilon(t,x)\,.$$

The shift $\delta(t)$ is the exact analogue of our present front ψ. But the choice of δ is nonlocal and hidden in the analysis: derived by least-squares fit in [Go1], by Green-function considerations in [Zu2]. The front ψ has also been introduced in the analysis of multidimensional scalar conservation laws, by Goodman, [Go2] Goodman-Miller [GM] and Hoff-Zumbrun [HoZ], its evolution again prescribed nonlocally. One may think that a simple and direct condition such as (1.18) could work as well.

2. To understand the relation between (1.18) and the various nonlocal evolution rules cited above, it may be helpful to consider a simple example in the one-dimensional case, namely, an initial perturbation consisting of a translate of the unperturbed shock profile $w(x)$ by distance δ. Least-squares fit (independent of the time evolution, so equally appropriate for either long-time or small-viscosity problem) would give the optimal prescription $\psi \equiv \delta$. The specialization

$$\partial_t \dot\psi + \ell \cdot \dot u_{|\tilde x = 0} = 0 \tag{1.24}$$

of (1.18) to one dimension, on the other hand, leads, roughly speaking (i.e., freely exchanging nonlinear and linear perturbations), to the ODE

$$\begin{aligned} \partial_t \dot\psi &= -\ell \cdot (w(x - \delta + \psi) - w(x))_{|\tilde x = 0} \\ &\sim -(\ell \cdot \partial_x w)_{|\tilde x = 0} (\psi - \delta) = -c(\dot\psi - \delta), \end{aligned} \tag{1.25}$$

where $c > 0$ by assumption (1.19), whose solution $\psi(t) = \delta(1 - e^{-ct})$ converges exponentially in time to the optimal value δ. The local evolution scheme (1.18) might thus loosely be described as a *relaxation* of the nonlocal prescription by least squares or other method, all converging time-asymptotically to a unique value. In the small-viscosity context, the profile is of form $w^\varepsilon = w(x/\varepsilon)$, and we obtain instead $\psi(t) = \delta(1 - e^{-ct/\varepsilon})$, which converges exponentially in ϵ^{-1} to δ for each fixed $t > 0$, with an order ε initial layer.

3. The extra boundary condition (1.18) is the one appropriate for the linearized problem. One can also impose the extra boundary condition at the level of the nonlinear problem. This is done in (7.1).

We end this introduction with several remarks on the method of adding unknowns and on the comparison between the partially and fully linearized equations. First we note that these two problems are closely related.

Lemma 1.2. *The fully linearized equations from* (1.8) *at* (u, ψ) *is*

$$\mathcal{E}'_u(u,\psi)\dot{u} + \mathcal{E}'_\psi(u,\psi)\dot{\psi} = \mathcal{E}'_u(u,\psi)(\dot{u} - \dot{\psi}\partial_{\tilde{x}}u) + \dot{\psi}\partial_{\tilde{x}}\mathcal{E}(u,\psi)\,. \tag{1.26}$$

Proof. This identity can be checked by direct and elementary computations. It was pointed out by S. Alinhac ([Al]) as well as the role of what he called "the good unknown" $\dot{u} - \dot{\psi}\partial_{\tilde{x}}u$. Denoting by $\mathcal{F}(u)$ the left hand side of the equation (1.3) in the original coordinates, and by * the substitution $u^*(t,y,x) = u(t,y,x-\psi(t,y))$, one has

$$\mathcal{F}(u^*) = \{\mathcal{E}(u,\psi)\}^*\,. \tag{1.27}$$

Through linearization, one has $\delta(u^*) = (\delta u - \delta\psi\partial_{\tilde{x}}u)^*$. Moreover, differentiating in u alone, one checks that $(\mathcal{E}'_u(u,\psi)v)^* = \mathcal{F}'_u(u^*)(v)^*$. Linearizing (1.27) implies (1.26). □

Consider for simplicity the typical example where $u^\varepsilon_{app} = w(\tilde{x}/\varepsilon)$ and $\psi^\varepsilon_{app} = \sigma t + \theta y$ as in (1.10), are exact solutions of (1.8) (1.9). In this case, the error term $\partial_{\tilde{x}}\mathcal{E}(u^\varepsilon_{app}, \psi^\varepsilon_{app})$ is exactly equal to zero in the right hand side of (1.26) and the transmission conditions for $\dot{u}$ and $\dot{u} - \dot{\psi}\partial_{\tilde{x}}u^\varepsilon_{app}$ are equivalent. Hence, the fully linearized equation (1.17) for $(\dot{u}, \dot{\psi})$ is equivalent to the partially linearized equation (1.13) for $\dot{v} = \dot{u} - \dot{\psi}\partial_{\tilde{x}}u^\varepsilon_{app}$. This invariance simply reflects that all the equations (1.8) are equivalent by change of variables. The extra boundary condition (1.18) reads

$$(\partial_t - \varepsilon\Delta_y)\dot{\psi} + \dot{\psi}\,(\ell\cdot\partial_{\tilde{x}}u^\varepsilon_{app|\tilde{x}=0}) = -\ell\cdot\dot{v}$$

and appears as an artificial way to define *a posteriori* $\dot{\psi}$. So the fully linearized equations do not provide new solutions: we use them to understand how we can get better estimates. In situations for which the partially linearized equations are already well-behaved, for example in the medium and high frequency regimes here, we may use the good unknown of Alinhac to work with these simpler equations instead.

Besides providing the maximal estimates (1.23) the consideration of the fully linearized equations (1.17) also gives an interesting insight in the analysis of the plane wave stability i.e. the stability of the planar solutions (1.10). In this case, the coefficients of the linearized equations are constant in (t,y). After a Laplace-Fourier transform in these variables and after rescaling, the equations are reduced to transmission problems for ordinary differential systems, say $\mathcal{L}(p,\zeta,\partial_z)$, depending on parameters (p,ζ). Their well posedness is equivalent to the nonvanishing of the corresponding Evans' function at (p,ζ). When applied to the partially linearized equation, the vanishing of the Evans functions at the origin reflects that $\mathcal{L}^{-1}$ has a pole at $\zeta = 0$, or that $\mathcal{L}$ at $\zeta = 0$ has a kernel. This comes from the invariance by translation of the profile equation (1.11). The key point is that the similar analysis for the fully linearized equations, yields an augmented system $\mathcal{L}_m$, such that $\mathcal{L}_m^{-1}$ has no pole at the origin: it appears as a *desingularization of the pole*. Let us explain this idea on a simple example.

Suppose that $\mathcal{A} = \mathcal{A}_0 + \rho\mathcal{A}_1$ is a family of matrices (or operators), invertible when $\rho > 0$, but such that $\mathcal{A}_0$ has a one dimensional kernel generated by e. Suppose that we look for bounds for the solution of

$$\mathcal{A}v = f\,. \tag{1.28}$$

A pole is expected at $\rho = 0$. Of course, at least in the finite dimensional case, one can solve the equation, projecting v on the kernel of $\mathcal{A}_0$ and on a supplementary space. But the problem can also be analyzed *without explicit computation of the spectral projectors* and reduced to inverting uniformly nonsingular operators. Consider a linear form ℓ such that

$$\ell \cdot e \neq 0\,, \tag{1.29}$$

and suppose that $\mathcal{A}_1 e$ is not in the range of $\mathcal{A}_0$. Then, look for the solution v as

$$v = u - \psi e. \tag{1.30}$$

The equation reads

$$\mathcal{A}u - \rho\psi\mathcal{A}_1 e = f\,. \tag{1.31}$$

Having added the scalar unknown ψ, we add a scalar additional equation:

$$\ell \cdot u + \rho a\psi = 0\,. \tag{1.32}$$

When $\rho \neq 0$, the equations are equivalent to

$$\mathcal{A}v = f\,, \quad \ell \cdot v = -(\ell \cdot e + \rho a)\psi\,.$$

As long as $\ell \cdot e + \rho a \neq 0$, the second equation determines ψ, thus (1.28) and (1.31) (1.32) are equivalent for $\rho > 0$.

The point is that one can expect that the problem (1.31) (1.32) is uniformly invertible for the unknowns $(u, \check{\psi})$ where $\check{\psi} = \rho\psi$, and it is certainly so in the finite dimensional case. Indeed, at $\rho = 0$, the kernel is given by the equations

$$\mathcal{A}_0 u - \check{\psi}\mathcal{A}_1 e = 0\,, \qquad \ell \cdot u + a\check{\psi} = 0\,.$$

Because $\mathcal{A}_1 e$ does not belong to the image of $\mathcal{A}_0$, the first equation implies that $\check{\psi} = 0$ and thus $u = \lambda e$. With the second equation, $\lambda = 0$.

Therefore, to solve (1.28), we can solve the nonsingular equation

$$\mathcal{A}u - \check{\psi}\mathcal{A}_1 e = f\,, \quad \ell \cdot u + a\check{\psi} = 0\,, \tag{1.33}$$

and recover $v = u - \rho^{-1}\check{\psi}e$ and in particular its polar part $-\rho^{-1}\check{\psi}e$.

Continuing the analogy, let us indicate in the case of finite dimensional equation (1.28) the difference between weak and maximal estimates. The weak estimate corresponds to the use of a lower bound for $\det \mathcal{A}$ and yields

$$\rho|v| \lesssim |f|\,. \tag{1.34}$$

On the other hand, the invertibility of (1.31) (1.32) implies

$$|u| + |\check{\psi}| \lesssim |f|\,. \tag{1.35}$$

With (1.30), this gives a precise description of the polar part of v. It implies that there is $\psi = \rho^{-1}\check{\psi}$ such that

$$|v + \psi e| + \rho|\psi| \lesssim |f|\,. \tag{1.36}$$

This is a noticeable improvement of the weak estimate. Of course, this estimate can be deduced from a spectral decomposition of $\mathcal{A}$, but it is precisely our point that the analysis sketched above does not use the detailed spectral properties of $\mathcal{A}$.

Let us indicate how the fully linearized equations plus the additional boundary condition enter the general procedure of desingularization sketched above. Indeed, the construction is rather going the other way. We have operators $\mathcal{A}(p, \zeta)$ which combine $\mathcal{L}(p, \zeta, \partial_z)$ and the transmission conditions. At $\zeta = 0$, we have a kernel of dimension one generated by $e := \partial_z w$. With $\rho = |\zeta|$, write $\mathcal{A}(p, \zeta) = \mathcal{A}_0 + \rho\mathcal{A}_1$. The fully linearized equations have the form

$$\mathcal{A}\dot{u} + \dot{\psi}\mathcal{B} = \dot{f} \tag{1.37}$$

and by Lemma 1.2 they are equivalent to

$$\mathcal{A}(\dot u - \dot\psi e) = \dot f\,.$$

Since $\mathcal{A}(p,0)e = 0$, this means that $\mathcal{A}(p,\zeta)e = \rho\mathcal{A}_1 e = -\mathcal{B}$ and therefore (1.37) is indeed the equation (1.31). After Fourier-Laplace transform and rescaling, the extra boundary condition (1.18) has the form (1.32).

In our application to the operators $\mathcal{L}(p,\zeta,\partial_z)$, the spectral decomposition of $\mathcal{A}$ would be related to the construction of Green's functions. It does not seem easy to use in space dimension larger than one. On the contrary, the symmetrizer techniques developed in [MZ1] are immediately available to get maximal estimates for the nonsingular modified problem $\mathcal{L}_m$.

However, let us make it clear that the analogy with finite dimensional problems sketched above cannot be pushed too far. Indeed, the linearized operators $\mathcal{L}(p,\zeta,\partial_z)$ have a continuous spectrum which contains the pole 0. Thus, the maximal estimates for the augmented nonsingular problem $\mathcal{L}_m$ which is analogous to (1.33) are not (1.36) but

$$(\gamma+\rho^2)\|\dot u\|_{L^2(\mathbb{R})} + (\gamma+\rho^2)^{1/2}|\check\psi| \lesssim |f|\,, \tag{1.38}$$

see [MZ1] (recall that $\zeta = (\tau - i\gamma,\eta)$ and $\rho = |\zeta|$ is small in this analysis). Similarly, the weak estimates proved in [GMWZ1], are not (1.34) but

$$(\gamma+\rho^2)^{1/2}\rho\|\dot v\|_{L^2(\mathbb{R})} \lesssim |f|. \tag{1.39}$$

This shows that the loss is more subtle than in (1.34). In accordance with our analysis above and in particular with Lemma 1.2, we see that if $(\dot u,\dot\psi)$ satisfies (1.38), then $\dot v = \dot u - \rho^{-1}\check\psi e$ satisfies (1.39). In addition, note that when $\gamma = 0$, which is used for the long time stability analysis discussed in [GMWZ1], the estimate (1.38) and (1.39) are equivalent.

On the other hand, for the small viscosity problem discussed in this paper or in [GMWZ2], scaling back to the original variables (see section 5 below), the maximal estimates (1.38) imply that the solutions $(\dot u,\dot\psi)$ of (1.17) (1.18) satisfy

$$\gamma\|e^{-\gamma t}\dot u\|_{L^2(\mathbb{R}^{1+d})} + \sqrt{\gamma}\|e^{-\gamma t}\nabla_{t,y}\dot\psi\|_{L^2(\mathbb{R}^d)} \lesssim \|e^{-\gamma t}\dot f\|_{L^2(\mathbb{R}^{1+d})}\,. \tag{1.40}$$

This implies (1.23) on any strip $[0,T]$. Next, we note that

$$\sqrt{\varepsilon}\|\dot\psi\partial_{\tilde x}e^{-\gamma t}u^\varepsilon_{app}\|_{L^2(\mathbb{R}^{1+d})} \lesssim \|e^{-\gamma t}\dot\psi\|_{L^2(\mathbb{R}^d} \lesssim \gamma^{-1}\|e^{-\gamma t}\nabla\dot\psi\|_{L^2}.$$

The first inequality comes from the form of $\partial_{\tilde{x}} u^\varepsilon_{app}$ given by (1.10). Therefore, $\dot{v} = \dot{u} - \dot{\psi}\partial_{\tilde{x}} u^\varepsilon_{app}$ satisfies

$$\gamma \min\{1, \sqrt{\varepsilon\gamma}\} \|e^{-\gamma t}\dot{v}\|_{L^2(\mathbb{R}^{1+d})} \lesssim \|e^{-\gamma t}\dot{f}\|_{L^2(\mathbb{R}^{1+d})}, \tag{1.41}$$

which explains where the loss of $\sqrt{\varepsilon}$ in (1.16) comes from. Clearly, (1.40) is a real improvement and the analysis of these estimates is the main goal of this paper.

In section 6 we construct high order approximate solutions $(u^a_\epsilon, \psi^a_\epsilon)$ to the parabolic system (1.3) which converge in an obvious sense to the given inviscid shock solution (u^0, ψ^0) of (1.1) on a time interval $[0, T_0]$. This solution exhibits the viscous boundary layers on each side of the shock and also determines the position of the viscous front to arbitrarily high order.

In section 7 we use the main linear estimate given by Theorem 5.5 to show that the approximate solutions are close to exact solutions $(u_\epsilon, \psi_\epsilon)$ of (1.3) on $[0, T_0]$. A precise statement of the relation between (u^a, ψ^a) and $(u_\epsilon, \psi_\epsilon)$ is given in Theorem 7.7. This theorem amounts to a demonstration of the stability of the viscous boundary layers on each side of the shock. As an immediate corollary of Theorem 7.7 we obtain for example

$$\begin{aligned} &\|u^0 - u_\epsilon\|_{L^2} = O(\sqrt{\epsilon}) \\ &\|u^0 - u_\epsilon\|_{L^\infty(\{|x-\psi^0(t,y)|>\kappa\})} = O(\epsilon),\ \kappa > 0 \end{aligned} \tag{1.42}$$

on the time interval $[0, T_0]$ (Corollary 7.8).

2 The stability conditions

In this section, we formulate the structural assumptions on the system and give the precise definition of the two Evans functions in play. Next we compare the different stability conditions.

2.1 Structural assumptions

Consider an $N \times N$ system of conservation laws (1.1) and the viscous perturbations (1.3). We denote by $A_j = f'_j$ the Jacobian matrix of f_j. We make the following assumptions:

Assumption 2.1.

(H0) *The f_j are C^∞ functions from $\mathcal{U}^* \subset \mathbb{R}^N$ to $\mathbb{R}^N$. The $B_{j,k}$ are $N \times N$ real matrices, C^∞ in $u \in \mathcal{U}^*$.*

(H1) *There is* $c > 0$ *such that for all* $u \in \mathcal{U}^*$ *and all* $\xi \in \mathbb{R}^d$ *the eigenvalues of* $\sum_{j,k=1}^d \xi_j \xi_k B_{j,k}(u)$ *satisfy* $\mathrm{Re}\mu \geq c|\xi|^2$.

(H2) *For* u *in the open subset* $\mathcal{U} \subset \mathcal{U}^*$ *and* $\xi \in \mathbb{R}^d \setminus \{0\}$*, the eigenvalues of* $\sum \xi_j A_j(u)$ *are real and semi-simple and have constant multiplicities.*

(H3) *There is* $c > 0$ *such that for all* $u \in \mathcal{U}$ *and* $\xi \in \mathbb{R}^d$ *the eigenvalues of* $i \sum_{j=1}^d \xi_j A_j(u) + \sum_{j,k=1}^d \xi_j \xi_k B_{j,k}(u)$ *satisfy* $\mathrm{Re}\mu \geq c|\xi|^2$.

We refer to [MZ1] or [Zu1] for comments on these standard and somewhat minimal assumptions.

2.2 Planar shocks and profiles

Consider a piecewise constant function

$$u = \begin{cases} u^-, & x < \sigma t + \theta y \\ u^+, & x > \sigma t + \theta y \end{cases}$$

where $(t, y, x) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}$ denote the variables. The front $x = \sigma t + \theta y$ is defined by $\sigma \in \mathbb{R}$ and $\theta \in \mathbb{R}^{d-1}$. We denote by $h = (\sigma, \theta) = (h_0, \dots, h_{d-1})$. With some abuse of notation, we refer to such a vector h as a *planar front*. The piecewise constant function u above is a weak solution of (1.1) if and only if the following Rankine-Hugoniot condition is satisfied:

$$f_\nu(u^+, h) - f_\nu(u^-, h) = 0\,, \tag{2.1}$$

with

$$f_\nu(u, h) := f_d(u) - \sum_{j=1}^{d-1} h_j f_j(u) - h_0 \mathrm{Id}\,. \tag{2.2}$$

Equivalently,

$$\tilde{u} = \begin{cases} u^-, & \tilde{x} < 0 \\ u^+, & \tilde{x} > 0 \end{cases}$$

and $\psi(t, y) = \sigma t + \theta y$ form a weak solution of (1.5).

Definition 2.2. *A planar shock is a point* $p = (u^-, u^+, h) \in \mathcal{U} \times \mathcal{U} \times \mathbb{R}^d$ *such that* $u^- \neq u^+$ *and which satisfies the Rankine-Hugoniot condition* (2.1)*.*

It is a Lax shock if the normal matrices $A_\nu(u^\pm, h) = \nabla_u f_\nu(u^\pm, h)$ *are invertible and if we let* N^+ *be the number of negative eigenvalues of* $A_\nu(u^+, h)$ *and* N^- *be the number of positive eigenvalues of* $A_\nu(u^-, h)$ *then*

$$N^+ + N^- = N + 1\,. \tag{2.3}$$

Note that by Assumption (H2), for $u \in \mathcal{U}$ and $h \in \mathbb{R}^d$, $A_\nu(u,h)$ has only real eigenvalues.

The travelling wave

$$w\left((x - \sigma t - \theta y)/\varepsilon\right) \tag{2.4}$$

is an exact solution of the viscous equation (1.3) if and only if w satisfies the profile equation

$$\mathcal{P}(w,h) := \partial_z\Big(B_\nu(w(z),h)\partial_z w\Big) - \partial_z\Big(f_\nu(w(z),h)\Big) = 0\,, \tag{2.5}$$

where B_ν is the normal viscosity matrix :

$$B_\nu(u,h) = \sum_{j,k=1}^{d} \nu_j \nu_k B_{j,k}(u)$$

with $\nu = (-h_1, \dots, -h_{d-1}, 1)$. Equivalently, $w(\tilde{x}/\varepsilon)$ and $\psi(t,y) = \sigma t + \theta y$ form a weak solution of (1.8)

Note that the equation is invariant by translation: if $w(z)$ is a solution of (2.5), then $w(z-a)$ is also a solution. Differentiating implies that $\partial_z w$ is a solution of the linearized equation

$$\mathcal{P}'_w \partial_z w = 0 \tag{2.6}$$

where

$$\begin{aligned}\mathcal{P}'_w \dot{w} := \partial_z\big(B_\nu(w,h)\partial_z \dot{w}\big) + \partial_z\big(\dot{w}\cdot\nabla_u B_\nu(w,h)\partial_z w\big) \\ - \partial_z\big(A_\nu(w,h)\dot{w}\big)\,.\end{aligned} \tag{2.7}$$

Definition 2.3. *A shock profile associated to the shock $p = (u^-, u^+, h)$ is a solution $w \in C^\infty(\mathbb{R};\mathcal{U}^*)$ of (2.5) such that*

$$\lim_{z\to-\infty} w(z) = u^-\,, \qquad \lim_{z\to+\infty} w(z) = u^+\,. \tag{2.8}$$

The profile w is transversal if the kernel of $\mathcal{P}'_w$ in $L^2(\mathbb{R})$ has dimension equal to one.

We recall a few known results about shock profiles. In order to avoid unnecessary repetition, we introduce the following definition.

Definition 2.4. *i) A profile W is a C^∞ mapping from $\mathbb{R}$ to $\mathcal{U}^*$, such that:*
a) for all $k > 0$, there are C and $\delta > 0$ such that

$$\forall z \geq 0\,, \qquad |\partial_z^k W(z)| + |\partial_z^k W(-z)| \leq C e^{-\delta z}\,, \tag{2.9}$$

b) the end states

$$W(\infty) = \int_0^\infty \partial_z W(z) dz + W(0)\,, \qquad W(-\infty) = W(0) - \int_{-\infty}^0 \partial_z W(z) dz$$

belong to $\mathcal{U}$.

ii) A set of profiles is bounded if the constants C and δ above can be chosen independent of the profile in the set; given parameters q in some smooth manifold, $\{W(\cdot, q)\}$ is a smooth family of profiles if the mapping $(z, q) \mapsto W(z, q)$ is C^∞ and for all $k > 0$ and α, there are C and $\delta > 0$ such that

$$\forall z \geq 0\,, \qquad |\partial_z^k \partial_q^\alpha W(z)| + |\partial_z^k \partial_q^\alpha W(-z)| \leq C e^{-\delta z}. \tag{2.10}$$

The exponential decay of the derivative implies that the integrals in the definition of $W(\pm\infty)$ converge and the end states are the limits of $W(z)$ as z tends to $\pm\infty$.

Proposition 2.5. *Suppose that w is a shock profile associated to a planar Lax shock $p = (u^-, u^+, h)$. Then w is a profile in the sense of Definition* 2.4, *with end states $u^\pm$. Moreover, $\partial_z w(z) \neq 0$ for all $z \in \mathbb{R}$.*

Proof. We can integrate (2.5) once, and it is equivalent to

$$B_\nu(w, h)\partial_z w = f_\nu(w(z), h) - k$$

where k is a constant. From this, we deduce that if $\partial_z w(z_0) = 0$, then $w(z_0)$ is a stationary point of the flux $f_\nu(u, h) - k$. Thus $w(z)$ is constant. If w is associated to p, this contradicts that

$$u^- = \lim_{z \to -\infty} w(z) \neq \lim_{z \to +\infty} w(z) = u^+\,.$$

If w is associated to p, then

$$\lim_{z \to +\infty} \partial_z w = \big(B_\nu(u^+, h)\big)^{-1} \big(f_\nu(u^+, h) - k\big)\,.$$

If the right hand side is not equal to zero, $\partial_z w$ has a non vanishing limit and w has no finite limit. This shows that $k = f_\nu(u^+, h)$. Moreover, Assumption 2.1 implies that $B_\nu^{-1} A_\nu(u^+, h)$ has no purely imaginary eigenvalues. Therefore w is in the stable manifold of the ode

$$B_\nu(w, h)\partial_z w = f_\nu(w, h) - f_\nu(u^+, h)$$

at $+\infty$. From the classical theory of ode, we deduce that $\partial_z w$ is exponentially decaying at infinity. $\square$

Proposition 2.6. *i) Suppose that $\underline{p}$ is a planar Lax shock. Then there is a neighborhood ω of $\underline{p}$ in $\mathcal{U} \times \mathcal{U} \times \mathbb{R}^d$ such that the set of shocks in ω form a smooth manifold $\mathcal{C}$ of dimension $N + d$ and all $p \in \mathcal{C}$ is a Lax shock.*

ii) Suppose in addition that $\underline{w}$ is a shock profile associated to $\underline{p}$ and that $\underline{w}$ is transversal. Then, shrinking ω if necessary, there is a C^∞ mapping W from $\mathbb{R} \times \mathcal{C}$ to $\mathcal{U}^* \subset \mathbb{R}^N$, such that $W(z, \underline{p}) = \underline{w}(z)$ and for all $p = (u^-, u^+, h) \in \mathcal{C}$, $W(\cdot, p)$ is a shock profile associated to p. This connection is unique, up to a translation in z by a smooth shift $k(p)$. Moreover, the W form a smooth family of profiles in the sense of Definition 2.4 above.

For the convenience of the reader, a proof of this proposition is recalled at the end of section 3.

2.3 The uniform stability condition

Consider a profile W and a planar front $h = (\sigma, \theta)$. We consider the linearized equations from (1.8) around

$$w^\varepsilon(t, y, x) = W(x/\varepsilon), \quad \psi(t, y) = \sigma t + \theta y .$$

For simplicity, we have changed the notation $\tilde{x}$ to x.

We first compute the partially linearized operator with respect to u. It has the form

$$L\dot{u} := -\varepsilon \partial_x \big(\widetilde{B}_\nu \partial_x \dot{u} \big) + \partial_x (A^\sharp \dot{u}) + \frac{1}{\varepsilon} M^\sharp \dot{u} \tag{2.11}$$

where

$$\begin{cases} A^\sharp v = \widetilde{A}_\nu v - \displaystyle\sum_{j=1}^{d-1} (\widetilde{B}_{j,\nu} + \widetilde{B}_{\nu,j}) \varepsilon \partial_j v - (\widetilde{\nabla_u B_\nu} \cdot v) \partial_z W , \\ M^\sharp v = \varepsilon \partial_t v + \displaystyle\sum_{j=1}^{d-1} A_j^\sharp \varepsilon \partial_j - \sum_{j=1,k}^{d-1} \widetilde{B}_{j,k} \varepsilon^2 \partial_j \partial_k , \end{cases}$$

with

$$A_j^\sharp v = \widetilde{A}_j - \big(\widetilde{\nabla_u B_{j,\nu}} \cdot v \big) \partial_z W + \big(\widetilde{\nabla_u B_{j,\nu}} \cdot \partial_z W \big) v ,$$

$$B_{j,\nu}(u, h) = \sum_{k=1}^{d} \nu_k B_{j,k}(u) , \qquad B_{\nu,j}(u, h) = \sum_{k=1}^{d} \nu_k B_{k,j}(u) ,$$

where $\nu = (-h_1, \dots, -h_{d-1}, 1)$ as before and $\widetilde{A}$ stands for the evaluation of the function $A(u,h)$ at $u = W(x/\varepsilon)$. Note that the coefficients are smooth functions of h and $z = x/\varepsilon$. Moreover, $A^\sharp$ and $M^\sharp$ are differential operators in $\varepsilon\partial_t$ and $\varepsilon\partial_y$.

Since the coefficients of L depend only on x, one can perform a Fourier-Laplace transform with respect to the tangential space-time variables (t,y). This leads to symbols $\mathcal{A}(z,\zeta)$ and $\mathcal{M}(z,\zeta)$, depending on $z \in \mathbb{R}$ and $\zeta = (\tau, \eta, \gamma) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}$, obtained by evaluating the coefficients at z and replacing in the definitions above ∂_j and ∂_t by $i\eta_j$, $j = 1, \dots, d-1$ and $\gamma + i\tau$ respectively. Denoting by $\hat{u}$ [resp. $\hat{f}$] the Fourier-Laplace transform of $\dot{u}$ [resp. $L\dot{u}$], one has:

$$\hat{f}(x,\hat{\zeta}) = -\varepsilon\partial_x(\widetilde{B}_\nu\partial_x\hat{u}) + \partial_x\left(\mathcal{A}\left(\frac{x}{\varepsilon}, \varepsilon\hat{\zeta}\right)\hat{u}\right) + \frac{1}{\varepsilon}\mathcal{M}\left(\frac{x}{\varepsilon}, \varepsilon\hat{\zeta}\right)\hat{u}. \tag{2.12}$$

Denote by $\hat{L}$ the operator in the right hand side acting on $\hat{u}$. It is then natural to rescale the variables. Setting

$$\zeta = \varepsilon\hat{\zeta}\,, \quad z = x/\varepsilon\,, \quad u^*(z,\zeta) = \hat{u}(x,\hat{\zeta})\,, \quad f^*(z,\zeta) = \varepsilon\hat{f}(x,\hat{\zeta})\,, \tag{2.13}$$

and

$$\mathcal{L}(z,\zeta,\partial_z)u^* := -\partial_z\left(\widetilde{B}_\nu(z)\partial_z u^*\right) + \partial_z\left(\mathcal{A}\left(z,\zeta\right)u^*\right) + \mathcal{M}\left(z,\zeta\right)u^*, \tag{2.14}$$

the equation (2.12) reads

$$f^* = \mathcal{L}(z,\zeta,\partial_z)u^*\,. \tag{2.15}$$

Dropping the stars, we now consider the well posedness of the equation

$$\mathcal{L}(z,\zeta,\partial_z)u = f. \tag{2.16}$$

This is a second order differential equation, and the equation is equivalent to the transmission problem where one looks for solutions u^+ and u^- on $\{z \geq 0\}$ and $\{z \leq 0\}$ separately, which satisfy the transmission conditions

$$u^-(0) = u^+(0)\,, \quad \partial_z u^-(0) = \partial_z u^+(0)\,.$$

All the constructions above depend on the initial choice of profile W and planar front h. When necessary we indicate this dependence in the notations and write $L_{W,h}$ and $\mathcal{L}_{W,h}(z,\zeta,\partial_z)$. Note that the coefficients are smooth functions of h and z.

Definition 2.7. *Given a profile W we denote by $\mathbb{E}^+_{W,h}(\zeta)$ [resp. $\mathbb{E}^-_{W,h}(\zeta)$] the set of initial data $(u(0), \partial_z u(0))$ such that the corresponding solution of $\mathcal{L}_{W,h}(z, \zeta, \partial_z)u = 0$ on $\{z \geq 0\}$ [resp. $\{z \leq 0\}$] is bounded as z tends to $+\infty$ [resp. $-\infty$].*

In the sequel, we denote by $\mathbb{R}^{d+1}_+$ the set of parameters $\zeta = (\tau, \eta, \gamma) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}$ such that $\gamma \geq 0$ and by $\mathbb{R}^{d+1}_+ \backslash \{0\}$ the set of $\zeta \neq 0$ with $\gamma \geq 0$. The proof of the next lemma is recalled in the next section.

Lemma 2.8. *Under Assumptions* (H0) *to* (H3), *$\mathbb{E}^+_{W,h}(\zeta)$ and $\mathbb{E}^-_{W,h}(\zeta)$ are smooth vector bundles of dimension N in $\mathbb{C}^{2N}$ over $\mathbb{R}^{d+1}_+ \backslash \{0\}$.*

There are nontrivial bounded solutions of $\mathcal{L}u = 0$ if and only if $\mathbb{E}^+ \cap \mathbb{E}^- \neq \{0\}$. The distance these two spaces can be measured via the *Evans' function*

$$D_{W,h}(\zeta) = \det \Big(\mathbb{E}^+_{W,h}(\zeta), \mathbb{E}^+_{W,h}(\zeta) \Big) \tag{2.17}$$

where the determinant is obtained by taking any orthonormal basis in the given spaces. Note that, by Lemma 2.8, the function D is smooth on $\mathbb{R}^{d+1}_+ \backslash \{0\}$.

There is an alternate way of computing the Evans function D. Considering the transmission problem, as a boundary problem, the natural space of initial data of bounded solutions is $\mathbb{E}^- \times \mathbb{E}^+ \subset \mathbb{C}^{2N} \times \mathbb{C}^{2N}$. Its dimension is $2N$. The boundary condition can be written $\Gamma(U^-, U^+) = 0$ where Γ is the mapping $(U^-, U^+) \mapsto U^+ - U^-$ from $\mathbb{C}^{2N} \times \mathbb{C}^{2N}$ to $\mathbb{C}^{2N}$. Thus $\dim \ker \Gamma = 2N$ and

$$D_{W,h}(\zeta) = \det \Big(\mathbb{E}^-_{W,h}(\zeta) \times \mathbb{E}^+_{W,h}(\zeta) \, , \, \ker \Gamma \Big) \; . \tag{2.18}$$

The weak stability condition requires that D does not vanish when $\zeta \neq 0$ and $\gamma \geq 0$. The uniform stability condition requires in addition an optimal control when ζ is small or large. It turns out that for large ζ the uniform condition follows from the Assumptions (H0) to (H5). For small ζ, we know from [ZS] that the determinant D is $O(|\zeta|)$. Following [ZS], [Zu1], the uniform stability condition reads:

Definition 2.9 (Stability conditions).

i) The shock profile W associated to a Lax shock $p = (u^-, u^+, h)$ is weakly stable if the Evans function $D_{W,h}$ does not vanish for $\zeta \in \mathbb{R}^{d+1} \backslash \{0\}$.

ii) It is uniformly stable if in addition there is a positive constant c such that for all $\zeta \in \mathbb{R}^{d+1} \backslash \{0\}$ with $|\zeta| \leq 1$,

$$|D_{W,h}(\zeta)| \geq c|\zeta|. \tag{2.19}$$

Proposition 2.10 ([ZS]). *Suppose that Assumption* 2.1 *is satisfied and that W is a shock profile associated to a planar Lax shock p.*

i) If W is uniformly stable, then W is transversal and the planar shock p is uniformly stable in the sense of Majda [Maj].

ii) Conversely, if W is transversal and the shock p is uniformly stable, then (2.19) *holds for $\zeta \in \mathbb{R}^{d+1} \backslash \{0\}$ small enough.*

The precise definition of Majda's uniform stability condition will be recalled in the next section, see Definition 3.9 below. We will also recall a proof of the proposition, as an introduction to the analysis of the modified Evans' function.

Corollary 2.11. *Under Assumptions* 2.1*, a profile W associated to a Lax shock p is uniformly stable if and only if:*

i) it is weakly stable ,

ii) W is transversal,

iii) p is uniformly stable in the sense of Majda.

In the general analysis of parabolic boundary value problems, the uniform stability condition for high frequencies (i.e. for $|\zeta|$ large) is described by a rescaled Evans function, see [MZ1]. With $\Lambda = (\tau^2 + \gamma^2 + |\eta|^4)^{1/4}$, introduce

$$\mathbb{E}^{\pm, rs}_{W,h}(\zeta) = \big\{ (\Lambda u(0), \partial_z u(0)) \, : \, (u(0), \partial_z u(0)) \in \mathbb{E}^{\pm}_{W,h} \big\} \tag{2.20}$$

and the scaled Evans function

$$D^{rs}_{W,h}(\zeta) = \det \big(\mathbb{E}^{-, rs}_{W,h}(\zeta) \, , \, \mathbb{E}^{+, rs}_{W,h}(\zeta) \big). \tag{2.21}$$

Of course, $D^{rs}(\zeta)$ vanishes if and only if $D(\zeta) = 0$. Following [MZ1], the next result means that the uniform stability condition is automatically satisfied for large frequencies :

Proposition 2.12. *For all profile W and planar front h, there are $\rho_1 > 0$ and $c > 0$ such that:*

$$\forall \zeta \in \mathbb{R}^{1+d}_+ , \quad |\zeta| \geq \rho_1 \, : \quad |D^{rs}_{W,h}(\zeta)| \geq c \, .$$

In particular, $D_{W,h}(\zeta) \neq 0$ for all $\zeta \in \mathbb{R}^{1+d}_+$ with $|\zeta| \geq \rho_1$.

2.4 The fully linearized equations

Consider a profile and a planar front $h = (\sigma, \theta)$. The fully linearized equation from (1.8) around $w^\varepsilon = W(x/\varepsilon)$, $\psi = \sigma t + \theta y$ reads

$$L\dot{u} + L^1\dot{\psi} = \dot{f}\,, \tag{2.22}$$

where L is given by (2.11) and

$$\begin{aligned} L^1\dot{\psi} =& -\partial_t\dot{\psi}\partial_x w^\varepsilon - \sum_{j=1}^{d-1}\partial_j\dot{\psi}\partial_x f_j(w^\varepsilon) \\ &+ \sum_{j=1}^{d-1}\varepsilon\partial_j\dot{\psi}\partial_x\Big((\widetilde{B}_{j,\nu} + \widetilde{B}_{\nu,j})\partial_x w^\varepsilon\Big) + \varepsilon\sum_{j,k=1}^{d-1}\partial_j\partial_k\dot{\psi}\widetilde{B}_{j,k}\partial_x w^\varepsilon \end{aligned} \tag{2.23}$$

with $\widetilde{B}_{j,k} = B_{j,k}(w^\varepsilon)$.

The main idea is to add an extra "boundary" condition to (2.22):

$$\partial_t\dot{\psi} - \varepsilon\Delta_y\dot{\psi} + \ell\cdot\dot{u}_{|x=0} = 0\,. \tag{2.24}$$

The special choice of the heat equation in the left hand side has no importance. It can be replaced by any parabolic operator of the same type, possibly depending on p. There is also a large freedom in the choice of ℓ. What we assume is that ℓ such that

$$\ell\cdot\partial_z W(0) > 0\,. \tag{2.25}$$

Such a choice is always possible since $\partial_z W(0) \neq 0$ by Proposition 2.5.

The coefficients of L^1 depend only on x. Again, we perform a Fourier-Laplace transform with respect to the tangential space-time variables (t, y). Denote by $\dot{f}^1$ the additional term $L^1\dot{\psi}$ and by $\hat{\psi}$ and $\hat{f}^1$ the Fourier Laplace transform of ψ and f^1 respectively. Parallel to (2.12), there holds

$$\hat{f}^1(x,\hat{\zeta}) = -\frac{1}{\varepsilon^2}\hat{\psi}(\hat{\zeta})\mathcal{L}^1(\frac{x}{\varepsilon},\varepsilon\hat{\zeta}) \tag{2.26}$$

where

$$\begin{aligned} \mathcal{L}^1(z,\zeta) =& (\gamma + i\tau)\partial_z W + \sum_{j=1}^{d-1} i\eta_j\partial_z f_j(W) \\ &- \sum_{j=1}^{d-1} i\eta_j\partial_z\Big((\widetilde{B}_{j,\nu} + \widetilde{B}_{\nu,j})\partial_z W\Big) + \sum_{j,k=1}^{d-1}\eta_j\eta_k\dot{\psi}\widetilde{B}_{j,k}\partial_z W \end{aligned} \tag{2.27}$$

and the coefficients $\widetilde{B}$ are now evaluated at $u = W(z)$. The natural rescaling for $\hat{f}^1$ and $\hat{\psi}$, which supplements (2.13), is:

$$(f^1)^*(z,\zeta) = \varepsilon \hat{f}^1(x,\hat{\zeta}), \quad \psi^*(\zeta) = \frac{1}{\varepsilon}\hat{\psi}(\hat{\zeta}), \tag{2.28}$$

so that

$$(f^1)^*(z,\zeta) = -\psi^*(\zeta)\mathcal{L}^1(z,\zeta).$$

Similarly, the Fourier-Laplace transform of the boundary condition (2.24) reads

$$(\hat{\gamma} + i\hat{\tau} + \varepsilon|\hat{\eta}|^2)\hat{\psi}(\hat{\zeta}) = \ell \cdot \hat{u}_{|x=0}(\hat{\zeta}).$$

Adding up, after Fourier-Laplace transform and rescaling, using (2.13) and (2.28), we see that the linearized equations read:

$$\begin{cases} \mathcal{L}(z,\zeta,\partial_z)u^* - \psi^*\mathcal{L}^1(z,\zeta) = f^* \\ c_0(\zeta)\psi^* + \ell \cdot u^*(0) = 0 \end{cases} \tag{2.29}$$

with $c_0(\zeta) = \gamma + i\tau + |\eta|^2$.

Lemma 2.13. *The following identity is satisfied:*

$$\mathcal{L}^1(z,\zeta) = \mathcal{L}(z,\zeta,\partial_z)\partial_z W(z) + \partial_z \mathcal{P}(W(z),h),$$

where $\mathcal{P}$ is defined in (2.5).

Proof. This is easily checked by direct computation; it can also be deduced from Lemma 1.2. □

For small ζ, it is natural to use polar coordinates:

$$\zeta = \rho\check{\zeta}, \quad \rho = |\zeta|, \quad |\check{\zeta}| = 1. \tag{2.30}$$

The definition (2.27) shows that $\mathcal{L}^1(z,0) = 0$ and therefore,

$$\mathcal{L}^1(z,\zeta) = \rho\check{\mathcal{L}}^1(z,\check{\zeta},\rho) \tag{2.31}$$

where $\check{\mathcal{L}}^1$ is smooth with respect to $z \in \mathbb{R}$, $\check{\zeta} \in S^d$ and $\rho \in [0,1]$. Similarly,

$$c_0(\zeta) = \rho\check{c}_0(\check{\zeta},\rho). \tag{2.32}$$

Thus, it is natural to introduce

$$\varphi = \rho\psi^* \tag{2.33}$$

so that the equation (2.29) reads

$$\begin{cases} \mathcal{L}(z,\rho\check{\zeta},\partial_z)u^* - \varphi\check{\mathcal{L}}^1(z,\check{\zeta},\rho) = f^*, \\ \check{c}_0(\check{\zeta},\rho)\varphi + \ell \cdot u^*(0) = 0. \end{cases} \tag{2.34}$$

Definition 2.14. *For $\rho > 0$, denote by $\widetilde{\mathbb{E}}_{W,h}(\check\zeta,\rho)$ the set of $(u_0^-, u_1^-, u_0^+, u_1^+, \varphi) \in \mathbb{C}^{4N+1}$ such that the solutions of*

$$\mathcal{L}(z,\rho\check\zeta,\partial_z)u^\pm - \varphi\check{\mathcal{L}}^1(z,\check\zeta,\rho) = 0\,, \qquad u^\pm(0) = u_0^\pm\,, \quad \partial_z u^\pm(0) = u_1^\pm$$

on $\{\pm z \geq 0\}$ are bounded as z tends to $\pm\infty$.

We denote by S_+^d the set of parameters $\check\zeta = (\check\tau,\check\eta,\check\gamma) \in \mathbb{R}\times\mathbb{R}^{d-1}\times\mathbb{R}$ such that $|\check\zeta| = 1$ and $\check\gamma \geq 0$.

Lemma 2.15. *Under Assumptions* (H0) *to* (H5)*, given a profile W and a planar front h, $\widetilde{\mathbb{E}}^+_{W,h}(\check\zeta,\rho)$ is a C^∞ vector bundle of dimension $2N+1$ in $\mathbb{C}^{4N+1}$ over $S_+^d\times]0,1]$ which has a continuous extension to $S_+^d\times[0,1]$.*

The proof is given in the next section. That $\widetilde{\mathbb{E}}^\pm$ is smooth for $\rho > 0$ follows from Lemma 2.8. The continuous extendability to $\rho = 0$ follows from [MZ2].

There is a nontrivial solution of (2.34) if and only if there is a nontrivial solution $(u_0^-, u_1^-, u_0^+, u_1^+, \varphi) \in \widetilde{\mathbb{E}}(\check\zeta,\rho)$ to

$$\widetilde\Gamma_\ell(u_0^-, u_1^-, u_0^+, u_1^+, \varphi) := \big(u_0^+ - u_0^-, u_1^+ - u_1^-, \check c_0\varphi + \ell\cdot u_0^+\big) = 0\,.$$

Note that $\ker\widetilde\Gamma_\ell$ is a smooth linear bundle of dimension $2N$ in $\mathbb{C}^{4N+1}$. Therefore, we can form the following determinant in $\mathbb{C}^{4N+1}$, which we call the *modified Evans' function*:

$$\widetilde{D}_{W,h,\ell}(\check\zeta,\rho) = \det\Big(\widetilde{\mathbb{E}}_{W,h}(\check\zeta,\rho)\,,\,\ker\widetilde\Gamma_\ell\Big)\,. \tag{2.35}$$

When W is a shock profile, Lemma 2.13 shows that for $\zeta \neq 0$ the equation (2.29) is equivalent to

$$u^* = v + \psi^*\partial_z W \quad \text{with} \quad \mathcal{L}v = f^*\,, \qquad \tilde c_0\psi^* + \ell\cdot v(0) = 0\,,$$

with $\tilde c_0(\zeta) = c_0(\zeta) + \ell\cdot\partial_z W(0)$. By Assumption (2.25), $\tilde c_0$ does not vanish for $\zeta \in \mathbb{R}_+^{1+d}$. Therefore, the equation (2.29) or equivalently (2.34) with $f^* = 0$ has a nontrivial solution if and only if there is a nontrivial solution of $\mathcal{L}v = 0$ on $\mathbb{R}$. Therefore, we have proved:

Proposition 2.16. *If W is a shock profile associated to a Lax shock $p = (u^-, u^+, h)$, then for all $\check\zeta \in S_+^d$ and $\rho > 0$, $\widetilde{D}_{W,h}(\check\zeta,\rho)$ vanishes if and only if $D_{W,h}(\rho\check\zeta) = 0$.*

However, as discussed in the introduction, the detailed behavior of D and $\widetilde{D}$ are quite different as ζ tend to zero. The first main result of this paper is to give an equivalent formulation of the uniform stability condition using the modified Evans function $\widetilde{D}$.

Theorem 2.17. *Under Assumptions* 2.1*, suppose that W is a shock profile associated to a Lax shock $p = (u^-, u^+, h)$ and ℓ satisfies* (2.25)*. Then W is uniformly stable if and only if:*

i) it is weakly stable,

ii) there is a constant $c > 0$ such that for all $(\check{\zeta}, \rho) \in S^d_+ \times]0, 1]$

$$|\widetilde{D}_{W,h,\ell}(\check{\zeta}, \rho)| \geq c\,. \tag{2.36}$$

The modified Evans function condition, is stable under perturbations. Suppose that the profile $\underline{W}$ associated to the Lax shock $\underline{p}$ is uniformly stable. Then it is transversal and by Proposition 2.6 there is a neighborhood ω of $\underline{p}$ in $\mathcal{U} \times \mathcal{U} \times \mathbb{R}^d$, such that the shocks in ω form a smooth manifold $\mathcal{C}$. Moreover, there is a smooth family of profiles $W(\cdot, p)$ extending $\underline{W}$ associated to $p \in \mathcal{C}$. We can also choose a smooth mapping ℓ from $\mathcal{C}$ to $\mathbb{R}^N$ such that for all $p \in \mathcal{C}$, $\ell(p) \cdot \partial_z W(0, p) \neq 0$. In this case, one can show that the modified function $\widetilde{D}_{W(p,\cdot),h,\ell(p)}$ extends continuously to $\mathcal{C} \times S^d_+ \times [0, 1]$ and therefore:

Theorem 2.18. *Suppose that $\underline{W}$ is a uniformly stable profile associated to a Lax shock $\underline{p}$. Then, with notations as above, there is a neighborhood ω of $\underline{p}$ such that all the profiles $W(\cdot, p)$ are uniformly stable when $p \in \mathcal{C} \cap \omega$ and there is $c > 0$ such that* (2.36) *is satisfied for all $p \in \mathcal{C}$ and $(\check{\zeta}, \rho) \in S^d_+ \times]0, 1]$.*

The proofs of these theorems are given in the next section.

3 Analysis of the Evans functions

This section is mainly devoted to the proof of Theorem 2.17 and the related results stated in section 2.

We assume that we are given a profile $\underline{W}$ associated to a Lax shock $\underline{p}$. To prepare the construction of symmetrizers, we need to consider neighboring values of $\underline{p}$ and we also need some extra parameters $p' \in \mathbb{R}^{N'}$. The precise conditions are summarized in the following assumption, which is supposed to hold throughout the section.

Assumption 3.1. *$\mathcal{Q}$ is a smooth manifold and $\{W(\cdot, q); q \in \mathcal{Q}\}$ is a smooth family of profiles in the sense of Definition* 2.4*. Ψ is a C^∞ mapping from*

$\mathcal{Q}$ to $\mathbb{R}^d$, bounded as well as its derivatives. Moreover, at $\underline{q} \in \mathcal{Q}$, $\underline{W}(z) = W(z, \underline{q})$ is a shock profile associated to a Lax shock $\underline{p} = (\underline{u}^-, \underline{u}^+, \underline{h})$ and $\Psi(\underline{q}) = \underline{h}$.

3.1 Invariant spaces

We consider the rescaled linearized operators (2.14) associated to the profiles $W(\cdot, q)$ and normal front $\Psi(q)$:

$$
\begin{aligned}
\mathcal{L}(z, q, \zeta, \partial_z)u &= \mathcal{L}_{W(\cdot,q),\Psi(q)}(z, \zeta, \partial_z) \\
&= -\partial_z(\widetilde{B}_\nu(z, q)\partial_z u) + \partial_z\left(\mathcal{A}\left(z, q, \zeta\right) u\right) + \mathcal{M}\left(z, q, \zeta\right) u.
\end{aligned}
\tag{3.1}
$$

The coefficients are polynomial in ζ and smooth functions of W, $\partial_z W$, $\partial_z^2 W$ and $\Psi(q)$. Thus $\widetilde{B}_\nu$, $\mathcal{A}$ and $\mathcal{M}$ are smooth functions of (z, q, ζ) and converge at an exponential rate at infinity. It is convenient to write the equation

$$
\mathcal{L}(z, q, \zeta, \partial_z)u = f
\tag{3.2}
$$

as a first order system. Introducing $v = \widetilde{B}_\nu \partial_z u - \mathcal{A}u$, the equation reads

$$
\partial_z U = \mathcal{G}(z, q, \zeta)U + F,
\tag{3.3}
$$

with

$$
U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathcal{G} = \begin{pmatrix} \widetilde{B}_\nu^{-1}\mathcal{A} & \widetilde{B}_\nu^{-1} \\ \mathcal{M} & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix}.
$$

In the analysis of low and medium frequencies, the key remark is that (3.3) is conjugated to constant coefficients systems at infinity. The Assumption 3.1 implies that for $q \in \omega$ and ζ in bounded sets, there holds:

$$
\left|\partial_{z,q,\zeta}^\alpha \left(\mathcal{G}(z, q, \zeta) - \mathcal{G}^\pm(q, \zeta)\right)\right| \lesssim e^{-\delta|z|}, \qquad \pm z \geq 0,
\tag{3.4}
$$

with

$$
\mathcal{G}^\pm(q, \zeta) = \begin{pmatrix} (B_\nu^\pm)^{-1}(q)\mathcal{A}^\pm(q, \zeta) & (B_\nu^\pm)^{-1}(q) \\ \mathcal{M}^\pm(q, \zeta) & 0 \end{pmatrix},
$$

where

$$
\begin{cases}
\mathcal{A}^\pm = A_\nu^\pm - \displaystyle\sum_{j=1}^{d-1} i\eta_j (B_{j,\nu}^\pm + B_{\nu,j}^\pm), \\
\mathcal{M}^\pm = (\gamma + i\tau)\mathrm{Id} + \displaystyle\sum_{j=1}^{d-1} i\eta_j A_j^\pm + \sum_{j=1,k}^{d-1} \eta_j \eta_k B_{j,k}^\pm,
\end{cases}
$$

and the matrices $A_j^{\pm}$, $B_{j,k}^{\pm}$ are the corresponding matrices evaluated at the end states $W(\pm\infty, (q)$; similarly, the normal matrices A_ν, B_ν, $B_{j,\nu}$, $B_{\nu,k}$ are evaluated at the end states and normal front $\Psi(q)$.

In (3.4) and below, the notation $A(z,q,\zeta) \lesssim B(z,q,\zeta)$ means that there is a constant C such that $A(z,q,\zeta) \le CB(z,q,\zeta)$ for all values of (z,q,ζ) under consideration.

Therefore, Lemma 2.6 from [MZ1] implies

Lemma 3.2 ([MZ1]). *For all $q_0 \in \mathcal{Q}$ and $\zeta_0 \in \mathbb{R}_+^{1+d}$, there is a neighborhood Ω of (q_0, ζ_0) in $\mathcal{Q} \times \mathbb{R}^{1+d}$ and there are matrices $\mathcal{W}^{\pm}$ defined and C^∞ on $\{\pm z \ge 0\} \times \Omega$ and such that*

i) $\mathcal{W}^{\pm}$ and $(\mathcal{W}^{\pm})^{-1}$ are uniformly bounded and that for all $\alpha > 0$ there is $\delta_1 > 0$ such that for $(q,\zeta) \in \Omega$:

$$\left|\partial_{z,q,\zeta}^{\alpha}\big(\mathcal{W}^{\pm}(z,q,\zeta) - \mathrm{Id}\big)\right| \lesssim e^{-\delta_1|z|}, \quad \pm z \ge 0 .$$

ii) $\mathcal{W}^{+}$ and $\mathcal{W}^{-}$ satisfy on $\{z \ge 0\}$ and $\{z \le 0\}$ respectively:

$$\partial_z \mathcal{W}(z,q,\zeta) = \mathcal{G}(z,q,\zeta)\mathcal{W}(z,q,\zeta) - \mathcal{W}(z,q,\zeta)\mathcal{G}^{\pm}(q,\zeta) .$$

Recall that $\mathbb{R}_+^{1+d}$ denotes the set of $\zeta = (\tau, \eta, \gamma) \in \mathbb{R}^{1+d}$ such that $\gamma \ge 0$.

As a corollary, U is a solution of (3.3) on $\mathbb{R}^{\pm} := \{\pm z \ge 0\}$ if and only if $U_1 = (\mathcal{W}^{\pm})^{-1} U$ satisfies

$$\partial_z U_1 = \mathcal{G}^{\pm} U_1 + F_1 , \qquad F_1 = (\mathcal{W}^{\pm})^{-1} F . \tag{3.5}$$

Next we recall the spectral properties of the matrices $\mathcal{G}$ (cf [ZS], [Zu1], [MZ1] or [GMWZ2]). To deal with the high frequencies, we introduce the parabolic quasi-norm:

$$\langle\zeta\rangle = \big(1 + \tau^2 + \gamma^2 + |\eta|^4\big)^{\frac{1}{4}} . \tag{3.6}$$

Lemma 3.3.

i) For all $q_0 \in \mathcal{Q}$, there is a neighborhood ω_1 of q_0 and there are constants ρ_1 and $c_1 > 0$ such that for all $q \in \omega_1$, all $z \in \mathbb{R}$ and all $\zeta \in \mathbb{R}_+^{1+d}$ with $|\zeta| \ge \rho_1$, the matrix $\mathcal{G}(z,q,\zeta)$ has N eigenvalues counted with their multiplicities in $\{\mathrm{Re}\mu > 0\}$ and N eigenvalues in $\{\mathrm{Re}\mu < 0\}$. They all satisfy $|\mathrm{Re}\mu| \ge c_1\langle\zeta\rangle$.

ii) For $\zeta \in \mathbb{R}_+^{1+d}\backslash\{0\}$, the matrices $\mathcal{G}^{\pm}(q,\zeta)$ have N eigenvalues counted with their multiplicities in $\{\mathrm{Re}\mu > 0\}$ and N eigenvalues in $\{\mathrm{Re}\mu < 0\}$.

iii) For $\zeta = 0$, the matrices $\mathcal{G}^{\pm}(q,0)$ have 0 as a semi-simple eigenvalue of multiplicity N. The nonvanishing eigenvalues are those of $(B_\nu^{\pm})^{-1}A_\nu^{\pm}$. N^+ eigenvalues of $\mathcal{G}^+$ are in $\{\mathrm{Re}\mu < 0\}$ and N^- eigenvalues of $\mathcal{G}^-$ in $\{\mathrm{Re}\mu > 0\}$.

Consider $\zeta \in \mathbb{R}_+^{1+d}\backslash\{0\}$. The solutions of

$$\partial_z U_1 = \mathcal{G}^\pm U_1$$

which are bounded on $\mathbb{R}^\pm$, are exponentially decaying and given by:

$$U_1(z) = e^{z\mathcal{G}^\pm} U_1(0)\,, \qquad U_1(0) \in \mathbb{F}_1^\pm(q,\zeta)\,,$$

where $\mathbb{F}_1^\pm(q,\zeta)$ denotes the space generated by the eigenvectors associated to eigenvalues in $\{\pm\mathrm{Re}\mu < 0\}$. By the lemma above, $\mathbb{F}_1^\pm(q,\zeta))$ have dimension N and depend smoothly on (q,ζ).

Denote by $\mathbb{F}^\pm(q,\zeta)$ the space of initial data $U(0)$ such that the corresponding solution of

$$\partial_z U = \mathcal{G} U$$

is bounded as z tends to $\pm\infty$. There holds:

$$\mathbb{F}^\pm(q,\zeta) = \mathcal{W}(0,q,\zeta)\mathbb{F}_1^\pm(q,\zeta)\,. \tag{3.7}$$

Next, we note that the spaces $\mathbb{E}^\pm(q,\zeta)$ of initial data $(u(0), \partial_z u(0))$ such that the corresponding solution of (3.2) is bounded on $\mathbb{R}^\pm$ is directly linked to $\mathbb{F}^\pm(p,\zeta)$:

$$\mathbb{E}^\pm(q,\zeta) = \begin{pmatrix} \mathrm{Id} & 0 \\ \widetilde{B}_\nu^{-1}\mathcal{A} & \widetilde{B}_\nu^{-1} \end{pmatrix}_{\big| z=0} \mathbb{F}^\pm(q,\zeta). \tag{3.8}$$

In particular, the smooth dependence of $\mathbb{F}_1^\pm(q,\zeta)$ implies the following result which extends Lemma 2.8:

Lemma 3.4. *When $\underline{\zeta} \neq 0$, one can choose the neighborhood Ω in Lemma 3.2, such that $\mathbb{F}^\pm(q,\zeta)$ and $\mathbb{E}^\pm(q,\zeta)$ are smooth vector bundles of dimension N in $\mathbb{C}^{2N}$ over Ω.*

When $\underline{\zeta} = 0$, $\mathbb{F}^\pm(q,\zeta)$ and $\mathbb{E}^\pm(q,\zeta)$ are smooth vector bundles of dimension N in $\mathbb{C}^{2N}$ over $\Omega \cap \{\zeta \neq 0,\, \gamma \geq 0\}$.

According to the definition (2.17), the Evans function associated to the profile $W(\cdot, q)$ and the front $\Psi(q)$ is:

$$D(q,\zeta) = \det(\mathbb{E}^-(q,\zeta), \mathbb{E}^+(q,\zeta)\,. \tag{3.9}$$

3.2 The low frequency normal form

We now consider small frequencies ζ. Lemma 3.3 implies that there is a neighborhood Ω_0 of $(\underline{q}, 0)$ in $\mathcal{Q} \times \mathbb{R}^{1+d}$ and there are C^∞ invertible matrices $\mathcal{V}^\pm(q, \zeta)$ on Ω_0 such that

$$(\mathcal{V}^\pm)^{-1} \mathcal{G}^\pm \mathcal{V}^\pm = \begin{pmatrix} P^\pm & 0 \\ 0 & H^\pm \end{pmatrix} := \mathcal{G}_2^\pm(q, \zeta) \,,$$

with $H(q, 0) = 0$, $P^\pm(q, 0) = (B_\nu^\pm)^{-1} A_\nu^\pm$ and

$$\mathcal{V}^\pm(q, 0) = \begin{pmatrix} \mathrm{Id} & -(A_\nu^\pm)^{-1} \\ 0 & \mathrm{Id} \end{pmatrix} .$$

The eigenvalues of $P^\pm$ satisfy $|\mathrm{Re}\mu| \geq c > 0$, for some c independent of $(q, \zeta) \in \Omega_0$. Moreover (see e.g. [MZ1], Lemma 2.9)

$$H^\pm = -(A_\nu^\pm)^{-1} \Big((\gamma + i\tau)\mathrm{Id} + \sum_{j=1}^{d-1} i\eta_j A_j^\pm \Big) + O(|\zeta|^2) \,. \tag{3.10}$$

We now switch to polar coordinates $\zeta = \rho\check{\zeta}$, with $\rho = |\zeta|$ and $\check{\zeta} \in S_+^d$, the closed half sphere $\{|\check{\zeta}| = 1, \ \check{\gamma} \geq 0\}$. In particular, we use the notations

$$H^\pm(q, \zeta) = \rho \check{H}(q, \check{\zeta}, \rho) \,,$$

and note that $\check{H}(q, \check{\zeta}, 0)$, which is the main term in (3.10) evaluated at $\check{\zeta}$, is the symbol obtained by Laplace Fourier transform from the hyperbolic operator

$$\partial_x - (A_\nu^\pm)^{-1} \Big(\partial_t + \sum_{j=1}^{d-1} A_j^\pm \partial_j \Big) .$$

For $\rho > 0$, let us denote by $\mathbb{F}_P^+(q, \zeta)$ and $\mathbb{F}_H^+(q, \check{\zeta}, \rho)$ the negative spaces of the matrices $P^+(q, \zeta)$ and $\check{H}^+(q, \check{\zeta}, \rho)$ respectively, i.e. the spaces generated by generalized eigenvectors associated with eigenvalues in $\{\mathrm{Re}\mu < 0\}$. Symmetrically, we denote by $\mathbb{F}_P^-(q, \zeta)$ and $\mathbb{F}_H^-(q, \check{\zeta}, \rho)$ the positive spaces of $P^-(q, \zeta)$ and $\check{H}^-(q, \check{\zeta}, \rho)$. Thus the negative [resp. positive] space of $\mathcal{G}_2^+(q, \zeta)$ [resp. $\mathcal{G}_2^-(q, \zeta)$] are

$$\begin{aligned} \mathbb{F}_2^+(q, \zeta) &= \mathbb{F}_P^+(q, \zeta) \oplus \mathbb{F}_H^+(q, \check{\zeta}, \rho) \,, \\ \mathbb{F}_2^-(q, \zeta) &= \mathbb{F}_P^-(q, \zeta) \oplus \mathbb{F}_H^-(q, \check{\zeta}, \rho) \,. \end{aligned} \tag{3.11}$$

In addition:

$$\mathbb{F}_1^{\pm}(q,\zeta) = \mathcal{V}^{\pm}(p,\zeta)\mathbb{F}_2^{\pm}(q,\zeta)\,. \tag{3.12}$$

By Lemma 3.3, one has

$$\begin{aligned} \dim \mathbb{F}_P^+ = N^+\,, \quad \dim \mathbb{F}_H^+ = N - N^+\,,\\ \dim \mathbb{F}_P^- = N^-\,, \quad \dim \mathbb{F}_H^- = N - N^-\,. \end{aligned}$$

With (3.7), (3.12) and (3.11) we see that

$$\mathbb{F}^{\pm}(q,\zeta) = \mathcal{T}^{\pm}(0,q,\zeta)\big(\mathbb{F}_P^{\pm}(q,\zeta) \oplus \mathbb{F}_H^{\pm}(q,\check\zeta,\rho)\big) \tag{3.13}$$

where $\mathcal{T}^{\pm} = \mathcal{W}^{\pm}\mathcal{V}^{\pm}$. The vector bundles $\mathbb{F}_P^{\pm}(q,\zeta)$ are smooth for (q,ζ) near $(\underline{q},0)$. From [MZ2], we know that $\mathbb{F}_H^{\pm}(q,\check\zeta,\rho)$, which are smooth for $\rho > 0$ and $\check\zeta \in S^d_+$, have continuous extensions to $\rho = 0$. Thus:

Lemma 3.5. *The vector bundles $\mathbb{F}^{\pm}(q,\rho\check\zeta)$ are smooth for q close to $\underline{q}$, $\rho > 0$ small and $\check\zeta \in S^d_+$ and have continuous extensions to $\rho = 0$.*

Next, we consider the block decomposition of the matrix $\mathcal{T}^{\pm}$ into four $N \times N$ blocks:

$$\mathcal{T}^{\pm}(z,q,\zeta) := \mathcal{W}^{\pm}\mathcal{V}^{\pm}(z,q,\zeta) = \begin{pmatrix} \mathcal{T}^{\pm}_{1,1} & \mathcal{T}^{\pm}_{1,2} \\ \mathcal{T}^{\pm}_{2,1} & \mathcal{T}^{\pm}_{2,2} \end{pmatrix}. \tag{3.14}$$

On one side, the blocks correspond to the splitting of U into u and v and on the other side to the splitting of U_2 into its P and H components.

Lemma 3.6. *There holds*

$$\mathcal{T}^{\pm}_{2,2}(z,q,0) = \mathrm{Id}\,, \tag{3.15}$$

$$\mathcal{T}^{\pm}_{2,1}(z,q,0) = 0 \qquad \text{on } \mathbb{F}_P^{\pm}(q,0)\,. \tag{3.16}$$

Proof. At $\zeta = 0$, $U = (u,v)$ is a solution of

$$\partial_z u = \widetilde{B}_\nu^{-1}\mathcal{A}u + \widetilde{B}_\nu^{-1}v\,, \quad \partial_z v = 0\,,$$

if and only if

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \mathcal{T}^{\pm}_{1,1} & \mathcal{T}^{\pm}_{1,2} \\ \mathcal{T}^{\pm}_{2,1} & \mathcal{T}^{\pm}_{2,2} \end{pmatrix}_{|\zeta=0} \begin{pmatrix} u_P \\ u_H \end{pmatrix}$$

and

$$\partial_z u_P = P^{\pm}(q,0)u_P\,, \quad \partial_z u_H = 0\,.$$

Choosing $u_P = 0$ and u_H any constant vector in $\mathbb{C}^N$, one must have $\partial_z \mathcal{T}^{\pm}_{2,2} u_H = 0$, therefore $\mathcal{T}^{\pm}_{2,2}$ is constant, and since $\mathcal{T}^{\pm}$ tends to $\mathcal{V}^{\pm}$ at $\pm\infty$, this implies that at $\zeta = 0$, $\mathcal{T}^{\pm}_{2,2} = \mathrm{Id}$.

Similarly, choosing $u_P(z) = e^{zP^{\pm}(q,0)} u_P(0)$ and $u_H = 0$ implies that $\mathcal{T}^{\pm}_{2,1}(z, q, 0) = \mathcal{T}^{\pm}_{2,1}(0, q, 0) e^{-zP^{\pm}(q,0)}$. Since this matrix tends to zero at $\pm\infty$, it must vanish on the space $\mathbb{F}^{\pm}_P(p, 0)$ where $e^{-zP^{\pm}(q,0)}$ is exponentially growing. □

By (3.8), there are smooth nonvanishing function c and c' on Ω_0 such that

$$
\begin{aligned}
D(q,\zeta) &= c(q,\zeta) \det \big(\mathbb{F}^-(q,\zeta), \mathbb{F}^+(q,\zeta)\big) \\
&= c'(q,\zeta) \det \big(\mathbb{F}^-_P(q,\zeta) \oplus \mathbb{F}^-_H(q,\zeta), \mathbb{F}^+_P(q,\zeta) \oplus \mathbb{F}^+_H(q,\zeta)\big) .
\end{aligned} \tag{3.17}
$$

3.3 Low frequency analysis of the Evans function

In order to compute the Evans function, we now choose bases in $\mathbb{F}^{\pm}(q, \zeta)$. According to (3.13) we construct separately bases of $\mathcal{T}^{\pm}(0)\mathbb{F}^{\pm}_P$ and $\mathcal{T}^{\pm}(0)\mathbb{F}^{\pm}_H$.

When $\zeta = 0$, the linear operator $\mathcal{L}(z, \underline{q}, 0, \partial_z)$ is equal to the linearized operator $\mathcal{P}'$ (2.7) from the profile equation at $\underline{W}$:

$$
\mathcal{L}(z, \underline{q}, 0, \partial_z) = \mathcal{P}'_{\underline{W}}(z, \partial_z) . \tag{3.18}
$$

Since $\underline{W}$ is a shock profile, it is an exact solution of (2.5) and (2.6) implies that

$$
\mathcal{L}(z, \underline{q}, 0, \partial_z) \partial_z \underline{W} = 0 . \tag{3.19}
$$

Equivalently, this means that

$$
\underline{R}_1(z) = \begin{pmatrix} \partial_z \underline{W} \\ 0 \end{pmatrix}
$$

satisfies

$$
\partial_z \underline{R}_1 = \mathcal{G}(z, \underline{q}, 0) \underline{R}_1 . \tag{3.20}
$$

Lemma 3.7. *Shrinking the neighborhood Ω_0 of $(\underline{q}, 0)$ if necessary, there are functions $R^{\pm}_1(z, q, \zeta) = {}^t(r^{\pm}_1, s^{\pm}_1)$, C^∞ on $\mathbb{R}^{\pm} \times \Omega$, exponentially decaying in z and such that*

$$
\partial_z R^{\pm}_1 = \mathcal{G} R^{\pm}_1 \quad \text{on } \pm z \geq 0 , \qquad R^{\pm}_1(z, \underline{q}, 0) = \underline{R}_1(z) . \tag{3.21}
$$

Moreover, $s_1^\pm(z,q,0)=0$ *and in polar coordinates* $\zeta=\rho\check\zeta$ *there holds, uniformly for* $\check\zeta\in S_+^d$*:*

$$s_1^+(0,\underline{q},\zeta)-s_1^-(0,\underline{q},\zeta)=\rho\mathrm{m}(\underline{p},\check\zeta)+O(\rho^2)\,, \tag{3.22}$$

where

$$\mathrm{m}(\underline{p},\zeta)=(\gamma+i\tau)(\underline{u}^+-\underline{u}^-)+\sum_{j=1}^{d-1}i\eta_j\big(f_j(\underline{u}^+)-f_j(\underline{u}^-)\big)\,. \tag{3.23}$$

Proof. Because $\underline{R}_1$ is an exponentially decaying solution of (3.20), there are $\underline{c}_1^\pm\in\mathbb{F}_P^\pm(\underline{q},0)$ such that

$$\underline{R}_1(z)=\mathcal{T}(z,\underline{q},0)\begin{pmatrix}e^{zP^\pm(\underline{q},0)}\underline{c}_1^\pm\\0\end{pmatrix}.$$

Denoting by $\Pi^\pm(q,\zeta)$ the spectral projection on $\mathbb{F}^\pm(q,\zeta)$, we define

$$R_1(z,q,\zeta)=\mathcal{T}(z,q,\zeta)\begin{pmatrix}e^{zP^\pm(q,\zeta)}\Pi^\pm(q,\zeta)\underline{c}_1^\pm\\0\end{pmatrix}. \tag{3.24}$$

Then, R_1 satisfies (3.21). Moreover, Lemma 3.6 implies that $s_1=0$ when $\zeta=0$.

The equation (3.21) implies that

$$\partial_z s_1^\pm=\mathcal{M}r_1^\pm\,. \tag{3.25}$$

We note that

$$\mathcal{M}(z,q,\zeta)=(\gamma+i\tau)\mathrm{Id}+\sum_{j=1}^{d-1}i\eta_jf_j'(W)+O(\rho^2).$$

The definition of R_1 shows that $|r_1(z,\underline{q},\zeta)-r_1(z,\underline{q},0)|\lesssim\rho e^{-\delta|z|}$ for some $\delta>0$. Thus, $r_1(z,\underline{q},\zeta)=\partial_z\underline{W}+O(\rho e^{-\delta|z|})$ and therefore

$$(\mathcal{M}r_1^\pm)(z,\underline{q},\zeta)=\partial_z(f(\underline{W}(z),\zeta)+O(\rho^2e^{-\delta|z|})$$

where

$$f(u,\zeta)=(\gamma+i\tau)u+\sum_{j=1}^{d-1}i\eta_jf_j(u)\,.$$

Therefore, integrating (3.25) on the half line $\{\pm z\geq0\}$ yields

$$\begin{aligned}-s^+(0,\underline{q},\zeta)&=f(\underline{u}^+,\zeta)-f(\underline{W}(0),\zeta)+O(\rho^2)\,,\\ s^-(0,\underline{q},\zeta)&=f(\underline{W}(0),\zeta)-f(\underline{u}^-,\zeta)+O(\rho^2)\,.\end{aligned}$$

Adding up we get (3.22). $\square$

By (3.24), $R_1^\pm(0,q,\zeta)$ is a vector in $\mathcal{T}^\pm(0,q,\zeta)\mathbb{F}_P^\pm(q,\zeta)$. We take it as a first basis vector in this space and also construct bases in $\mathcal{T}^\pm(0,q,\zeta)\mathbb{F}_H^\pm(q,\zeta)$.

Lemma 3.8. *There are a neighborhood ω_0 of $\underline{q}$ in $\mathcal{Q}$ and $\rho_0 > 0$ such that :*

i) for $j \in \{2,\dots,N^\pm\}$, there are C^∞ functions $R_j^\pm(z,q,\zeta) = {}^t(r_j^\pm, s_j^\pm)$ on $\{\pm z \geq 0\} \times \omega_0 \times \{|\zeta| \leq \rho_0\}$, such that, together with $R_1^\pm(0,q,\zeta)$ given by Lemma 3.7, $\{R_j^\pm(0,q,\zeta)\}_{1\leq j\leq N^\pm}$ form a smooth basis of $\mathcal{T}^\pm(0,q,\zeta)\mathbb{F}_P^\pm(q,\zeta)$; moreover, for all $j \leq N^\pm$, $s_j^\pm(z,q,0) = 0$;

ii) for $j \in \{N^\pm+1,\dots,N\}$, there are C^∞ functions $\check{R}_j^\pm(z,q,\check{\zeta},\rho) = {}^t(\check{r}_j^\pm, \check{s}_j^\pm)$ on $\{\pm z \geq 0\} \times \omega_0 \times S^d_+ \times]0,\rho_0]$, which extend continuously to $\rho = 0$ and such that $\{\check{R}_j^\pm(0,q,\check{\zeta},\rho)\}_{N^\pm<j\leq N}$ form a continuous basis of $\mathcal{T}^\pm(0,q,\rho\check{\zeta})\mathbb{F}_H^\pm(q,\check{\zeta},\rho)$; moreover, $\{s_j^\pm(0,q,\check{\zeta},\rho)\}_{N^\pm<j\leq N}$ form a continuous basis of $\mathbb{F}_H^\pm(q,\check{\zeta},\rho)$.

Proof. Consider vectors $\underline{c}_j^\pm \in \mathbb{F}_P^\pm(\underline{q},0)$ for $j \in \{2,\dots,N^\pm\}$ such that $\{\underline{c}_j^\pm\}_{1\leq j\leq N^\pm}$ form a basis of $\mathbb{F}_P^\pm(\underline{q},0)$. Extend the definition (3.24) of R_1 to $j \in \{2,\dots,N^\pm\}$

$$R_j(z,q,\zeta) = \mathcal{T}(z,q,\zeta)\begin{pmatrix} e^{zP^\pm(q,\zeta)}\Pi^\pm(q,\zeta)\underline{c}_j^\pm \\ 0 \end{pmatrix}.$$

They satisfy the conditions i).

According to [MZ2], the spaces $\mathbb{F}_H^\pm(q,\check{\zeta},\rho)$ have continuous extensions to $\rho = 0$. Choose bases $\{\check{c}_j^\pm(q,\check{\zeta},\rho)\}_{N^\pm\leq j\leq N}$ of $\mathbb{F}_H^\pm(q,\check{\zeta},\rho)$ and define

$$\check{R}_j(z,q,\check{\zeta},\rho) = \mathcal{T}(z,q,\rho\check{\zeta})\begin{pmatrix} 0 \\ e^{zH^\pm(q,\rho\check{\zeta})}\check{c}_j^\pm(q,\check{\zeta},\rho) \end{pmatrix}.$$

for $N^\pm \leq j \leq N$. The properties of $s_j^\pm$ follow from Lemma 3.6. $\square$

We now introduce two important quantities. First, the *Lopatinski determinant* of the linearized inviscid shock problem: (cf [Maj])

$$\Delta(\underline{p},\check{\zeta}) = \det\left(\mathbb{C}\mathrm{m}(\underline{q},\check{\zeta}), \mathbb{F}_H^-(\underline{q},\check{\zeta},0), \mathbb{F}_H^+(\underline{q},\check{\zeta},0).\right) \tag{3.26}$$

Note that $\dim\mathbb{F}^- + \dim\mathbb{F}^+ = N-1$ so that the determinant above is $N\times N$. By Lemma 3.8, the $\underline{s}_j(\check{\zeta}) := \check{s}_j^\pm(0,\underline{q},\check{\zeta},0)$ for $j > N^\pm$ are continuous bases of $\mathbb{F}_H^\pm(\underline{p},\check{\zeta},0)$. Therefore, there is a nonvanishing function $c(\check{\zeta})$ on the closed half sphere S^d_+ such that

$$\begin{aligned}\Delta(\underline{p},\check{\zeta}) = c(\check{\zeta})& \\ \det\Big(\mathrm{m}(\underline{p},\check{\zeta}), \underline{s}^-_{N^-+1}(\check{\zeta}),\dots,\underline{s}^-_N(\check{\zeta}), \underline{s}^+_{N^++1}(\check{\zeta}),\dots,\underline{s}^+_N(\check{\zeta})\Big).&\end{aligned} \tag{3.27}$$

Definition 3.9 (Uniformly stable shocks, [Maj]). *The Lax shock $\underline{p}$ is uniformly stable if there is $c > 0$ such that for all $\check{\zeta} \in S^d_+$*

$$|\Delta(\underline{p}, \check{\zeta})| \geq c\,.$$

Next, we introduce the determinant

$$\beta(\underline{q}) = \det\left(\partial_z \underline{W}(0), \underline{r}^-_2, \ldots, \underline{r}^-_{N^-}, \underline{r}^+_2, \ldots, \underline{r}^+_{N^+}\right) \tag{3.28}$$

where $\underline{r}^\pm_j = r^\pm_j(0, \underline{q}, 0)$. Note that $\partial_z \underline{W}(0) = \underline{r}^-_1 = \underline{r}^+_1$. A more intrinsic definition can be given using the spaces $\mathbb{C}\partial_z \underline{W}(0)$ and orthogonal complements in the u projection of $\mathcal{T}^\pm(0, \underline{p}, 0)\mathbb{F}^\pm_P(\underline{p}, 0)$.

Lemma 3.10. *$\beta(\underline{q}) \neq 0$ if and only if the profile $\underline{W}$ is transversal in the sense of Definition* 2.3.

Proof. The identity (3.18) implies that $\dot{w}$ is a L^2 solution of the linearized equation $\mathcal{P}'_{\mathcal{W}}\dot{w} = 0$ if and only if

$$U = \begin{pmatrix} \dot{w} \\ 0 \end{pmatrix}$$

is an L^2 solution of $\partial_z U = \mathcal{G}(z, \underline{p}, 0)U$. Arguing on $\{\pm z \geq 0\}$ separately, this holds, if and only if $\dot{w}$ is exponentially decaying and $\dot{w}(0) \in \underline{\mathbb{P}}^- \cap \underline{\mathbb{P}}^+$, where $\underline{\mathbb{P}}^\pm$ is the space spanned by $\underline{r}^\pm_1, \ldots, \underline{r}^\pm_{N^\pm}$. By definition, the connection is transversal if and only if $\dim(\underline{\mathbb{P}}^- \cap \underline{\mathbb{P}}^+) = 1$. Since $\dim \underline{\mathbb{P}}^- + \dim \underline{\mathbb{P}}^+ = N + 1$ and $\underline{r}^-_1 = \underline{r}^+_1$, this condition holds if and only if the vectors $\underline{r}_1, \underline{r}^-_2, \ldots, \underline{r}^-_{N^-}, \underline{r}^+_2, \ldots, \underline{r}^+_{N^+}$ are independent. ☐

The next result implies Proposition 2.10, originally due to [ZS].

Proposition 3.11. *Shrinking the neigborhood ω_0 of $\underline{q}$ and $\rho_0 > 0$ if necessary, there is a continuous function $D_m(q, \check{\zeta}, \rho)$ up to $\rho = 0$ on $\omega \times S^d_+ \times [0, \rho_0]$, such that*

$$D_m(\underline{q}, \check{\zeta}, 0) = \beta(\underline{p})\Delta(\underline{p}, \check{\zeta}) \tag{3.29}$$

and the Evans function defined for $\zeta \in \mathbb{R}^{1+d}_+ \backslash \{0\}$ by (3.9) *satisfies for $q \in \omega_0$, $\check{\zeta} \in S^d_+$ and $\rho = |\zeta| \in]0, \rho_0]$:*

$$D(q, \rho\check{\zeta}) = \rho c(q, \check{\zeta}, \rho) D_m(q, \check{\zeta}, \rho)\,. \tag{3.30}$$

where $c(q, \check{\zeta}, \rho)$ is continuous up to $\rho = 0$ and does not vanish on $\omega \times S^d_+ \times [0, \rho_0]$.

Proof. By (3.17), there is a function $c(q,\check\zeta,\rho)$, continuous up to $\rho=0$, such that $D(q,\rho\check\zeta)=c(q,\check\zeta,\rho)D'(q,\check\zeta,\rho)$ where

$$D'=\det\begin{pmatrix}\{r_j^-\}_{j\le N^-}, & \{r_j^+\}_{2\le j\le N^+}, & r_1^+, & \{\check r_j^-\}_{j>N^-}, & \{\check r_j^+\}_{j>N^+}\\ \{s_j^-\}_{j\le N^-}, & \{s_j^+\}_{2\le j\le N^+}, & s_1^+, & \{\check s_j^-\}_{j>N^-}, & \{\check s_j^+\}_{j>N^+}\end{pmatrix}$$

and the functions are evaluated at $z=0$. The coefficients in the first $N^-+N^+=N+1$ columns are smooth functions of (q,ζ) on a neighborhood of $(\underline q,0)$ and the coefficients in the last $N-1$ columns are are smooth functions of $(q,\check\zeta,\rho)$ for q in a neighborhood of $\underline q$, $\check\zeta\in S^d_+$ and $\rho\in]0,\rho_0]$ which extends continuously to $\rho=0$.

All the coefficients in the $N\times(N+1)$ matrix $\left(\{s_j^-\}_{j\le N^-},\{s_j^+\}_{2\le j\le N^+},s_1^+\right)$ are smooth functions of (q,ζ) and vanish at $\zeta=0$. Thus one can factor out ρ, writing $s_j^\pm=\rho\check s_j^\pm$ with $\check s_j^\pm$ smooth on $\omega\times S^d_+\times[0,\rho_0]$. Developing the determinant, all terms must contain at least one coefficient from this $N\times(N+1)$ matrix. Thus ρ can be factored out in each term showing that the Evans function has the form (3.30) with $D_m(q,\check\zeta,\rho)$ continuous up to $\rho=0$.

Next, we compute the determinant at $q=\underline q$. We can subtract the first column from the $(N+1)$-th, that is replace R_1^+ by $R_1^+-R_1^-$. But we know that at $q=\underline q$ and $\zeta=0$, $R_1^+-R_1^-=0$. Thus, one can factor out ρ. Writing, at $q=\underline q$,

$$(r_1^+-r_1^-)(0,\underline q,\zeta)=\rho\mathrm{r}_1(\check\zeta,\rho)\,,\quad (s_1^+-s_1^-)(0,\underline q,\zeta)=\rho\mathrm{s}_1(\check\zeta,\rho),$$

we see that $D'(\underline q,\check\zeta,\rho)=\rho D''(\underline q,\check\zeta,\rho)$, with

$$D''=\det\begin{pmatrix}\{r_j^-\}_{j\le N^-}, & \{r_j^+\}_{2\le j\le N^+}, & \mathrm{r}_1, & \{\check r_j^-\}_{j>N^-}, & \{\check r_j^+\}_{j>N^+}\\ \{s_j^-\}_{j\le N^-}, & \{s_j^+\}_{2\le j\le N^+}, & \mathrm{s}_1, & \{\check s_j^-\}_{j>N^-}, & \{\check s_j^+\}_{j>N^+}\end{pmatrix}.$$

By Lemmas 3.7 and 3.8, we see that $D''(\underline q,\check\zeta,0)$ is the determinant

$$\begin{pmatrix}\{\underline r_j^-\}_{j\le N^-}, & \{\underline r_j^+\}_{2\le j\le N^+}, & \mathrm{r}_1, & \{\check r_j^-(0,\underline q,\check\zeta,0)\}_{j>N^-}, & \{\check r_j^+(0,\underline q,\check\zeta,0)\}_{j>N^+}\\ 0, & 0, & \mathrm{m}, & \{\check s_j^-(0,\underline q,\check\zeta,0)\}_{j>N^-}, & \{\check s_j^+(0,\underline q,\check\zeta,0)\}_{j>N^+}\end{pmatrix}$$

that is $\beta(\underline p)\Delta(\underline p,\check\zeta)$. □

3.4 Analysis of the modified Evans function

With W and Ψ given by Assumption 3.1, we now consider the fully linearized equations. After rescaling, as in section 2, they read

$$\mathcal L(z,q,\zeta,D_z)u-\psi\mathcal L^1(z,q,\zeta)=f\,, \tag{3.31}$$

with $\mathcal{L}^1(z, q, \zeta) = \mathcal{L}^1_{W(\cdot,q),\Psi(q)}(z, \zeta)$ given by (2.27). We consider this equation separately on $\{z \geq 0\}$ and $\{z \leq 0\}$, together with the transmission conditions, see (2.29):

$$u^-(0) = u^+(0) \quad \partial_z u^-(0) = \partial_z u^+(0)\,, \quad \ell(q) \cdot u(0) + c_0(\zeta)\psi = 0\,, \tag{3.32}$$

where $c_0(\zeta) = \gamma + i\tau + |\eta|^2$ and ℓ is a smooth mapping from $\mathcal{Q}$ to $\mathbb{R}^N$ such that

$$\min_{q\in\mathcal{Q}} \big\{\ell(q) \cdot \partial_z W(0, q)\big\} > 0\,. \tag{3.33}$$

In (3.32) and below, $u^\pm$ denotes the restriction of u to the half line $\{\pm z \geq 0\}$.

We summarize the useful properties of $\mathcal{L}^1$ in the next lemma.

Lemma 3.12. *i) $\mathcal{L}^1$ is a polynomial in ζ, vanishing at $\zeta = 0$, whose coefficients b_α satisfy : for all $k \geq 0$ and μ, there are C and $\delta > 0$ such that for $q \in \mathcal{Q}$, there holds*

$$|\partial_z^k \partial_q^\mu b_\alpha(z, q)| \leq Ce^{-\delta|z|}. \tag{3.34}$$

ii) At $q = \underline{q}$, there holds

$$\int_{-\infty}^{+\infty} \mathcal{L}^1(z, \underline{q}, \zeta)\, dz = \mathrm{m}(\underline{p}, \zeta) + O(|\eta|^2)\,, \tag{3.35}$$

where m *is defined in* (3.23). *Moreover,*

$$\mathcal{L}^1(z, \underline{q}, \zeta) = \mathcal{L}(z, \underline{q}, \zeta, \partial_z)\partial_z \underline{W}\,. \tag{3.36}$$

Proof. By (2.27), $\mathcal{L}^1$, is a polynomial in ζ, vanishing at $\zeta = 0$, with coefficients which all involve at least one derivative of W. Thus, they are exponentially decaying and (3.34) follows from the estimates (2.10) for W.

Integrating (2.27) on $\mathbb{R}$, the convergence at $\pm\infty$ of $\underline{W}$ and the exponential decay of $\partial_z W$ immediately imply (3.35).

Since $\underline{W}$ is a shock profile, (3.36) follows from Lemma 2.13. □

We note that $\mathcal{L}^1$ and c_0 vanish at $\zeta = 0$. In polar coordinates $\zeta = \rho\check{\zeta}$, we use the notations

$$\mathcal{L}^1(z, q, \zeta) = \rho\check{\mathcal{L}}^1(z, q, \check{\zeta}, \rho)\,, \qquad c_0(\zeta) = \rho\check{c}_0(\check{\zeta}, \rho). \tag{3.37}$$

Following (2.33), it is natural to introduce $\varphi = \rho\psi$ so that the transmission problem (3.31) (3.32) for (u, ψ) is equivalent to:

$$
(3.38) \qquad \begin{cases} \mathcal{L}(z, q, \zeta, D_z)u^{\pm} - \varphi\check{\mathcal{L}}^1(z, q, \check{\zeta}, \rho) = f^{\pm}, & \text{on } \{\pm z \geq 0\}, \\ u^-(0) = u^+(0), \quad \partial_z u^-(0) = \partial_z u^+(0), \\ \ell(q) \cdot u(0) + \check{c}_0(\check{\zeta}, \rho)\varphi = 0. \end{cases}
$$

Following Definition 2.14, $\widetilde{\mathbb{E}}(q, \check{\zeta}, \rho)$ denotes the space of triples $(U_0^-, U_0^+, \varphi) \in \mathbb{C}^{2N} \times \mathbb{C}^{2N} \times \mathbb{C}$ with $U_0^{\pm} = (u_0^{\pm}, v_0^{\pm})$, such that the solutions $u^{\pm}$ of

$$
\mathcal{L}u^{\pm} - \varphi\check{b} = 0, \quad \textit{on} \pm z \geq 0, \qquad u^{\pm}(0) = u_0^{\pm}, \quad \partial_z u^{\pm}(0) = v_0^{\pm}
$$

are bounded at infinity. Moreover, $\ker \widetilde{\Gamma}(q)$ denotes the set of $(U_0^-, U_0^+, \varphi) \in \mathbb{C}^{2N} \times \mathbb{C}^{2N} \times \mathbb{C}$ such that

$$
(3.39) \qquad U_0^- = U_0^+, \qquad \ell(q) \cdot u^-(0) + \check{c}_0(\check{\zeta}, \rho)\varphi = 0.
$$

Then, the modified Evans function is

$$
(3.40) \qquad \widetilde{D}(q, \check{\zeta}, \rho) = \det\Big(\widetilde{\mathbb{E}}(q, \check{\zeta}, \rho), \ker \widetilde{\Gamma}(q)\Big).
$$

The aim of the subsection is to prove the following result which is the analogue of Proposition 3.11. It implies Lemma 2.15. Moreover, together with Corollary 2.11 it also implies Theorem 2.17.

Proposition 3.13. *The vector bundle $\widetilde{\mathbb{E}}$ and the determinant $\widetilde{D}$ are C^∞ for q in a neighborhood of $\underline{q}$, $\check{\zeta}$ in the closed half sphere S^d_+ and $\rho > 0$ small enough. They have continuous extensions to $\rho = 0$.*

Moreover,

$$
\widetilde{D}(\underline{q}, \check{\zeta}, 0) = c(\check{\zeta})\beta(\underline{q})\Delta(\underline{q}, \check{\zeta})
$$

where c is continuous and does not vanish.

Before proving this result, we need some preparation. The definition (2.27) shows that

$$
(3.41) \qquad \mathcal{L}^1(q, \zeta) = (i\tau + \gamma)\mathcal{M}_0^1(q) + \sum_{j=1}^{d-1} i\eta_j \mathcal{M}_j^1(q, \eta).
$$

Similarly,

$$
c_0(\zeta) = (i\tau + \gamma)c_{0,0} + \sum_{j=1}^{d-1} i\eta_j c_{0,j}(\zeta).
$$

Lemma 3.14. *There are a neighborhood Ω of $(\underline{q}, 0)$ in $\mathcal{Q} \times \mathbb{R}^{1+d}$ and C^∞ functions $\mathcal{S}_j^+$ and $\mathcal{S}_j^-$ on $\Omega \times [0, +\infty[$, and $\Omega \times]-\infty, 0]$ respectively, such that*

$$\begin{cases} \mathcal{L}(z, q, \zeta, D_z)\mathcal{S}_j^\pm = \mathcal{M}_j^1(z, q, \zeta) & \text{on } \{\pm z \geq 0\}, \\ \ell(q) \cdot \mathcal{S}_j^\pm(0, q, \zeta) = -c_{0,j}(\zeta)\,, & \mathcal{S}_j^\pm(z, q, 0) = 0\,. \end{cases}$$

Moreover, the $\mathcal{S}_j^\pm$ and all their derivative are exponentially decaying as z tends to $\pm\infty$.

Proof. The source terms $\mathcal{M}_j^1$ are exponentially decaying. It is sufficient to show that one when (q, ζ) remains in some small neighborhood of $(\underline{q}, 0)$ and $b(z, q, \eta)$ is exponentially decaying and $c(q, \zeta)$ is given, one can find exponentially decaying solutions of

$$\mathcal{L}(q, \zeta, \partial_z)r^\pm = b^\pm\,, \quad r^\pm(0, q, \zeta) = c(q, \zeta)\,.$$

We reduce this equation to a first order system (3.3) for $R = {}^t(r, s)$ with $s = \widetilde{B}_\nu \partial_z r - \mathcal{A}r$. Using the conjugation $\mathcal{T} = \mathcal{W}\mathcal{V}$ on a neighborhood of $(\underline{q}, 0)$, it is sufficient to solve for $R' = \mathcal{T}^{-1}R = {}^t(r', s')$ the constant coefficient equations

$$\begin{pmatrix} \partial_z r' - Pr' \\ \partial_z s' - Hs' \end{pmatrix} = \mathcal{T}^{-1}B\,.$$

with $B = {}^t(0, b)$ exponentially decaying. Since the spectrum of P is away form the imaginary axis, there are exponentially decaying solutions of $\partial_z r' - Pr' = O(e^{-\delta|z|})$ on each side $\{\pm z \geq 0\}$. Similarly, since $H = 0$ at $\zeta = 0$ and there are, for (q, ζ) is some neighborhood of $(\underline{q}, 0)$, exponentially decaying solutions of $\partial_z s' - Hs' = O(e^{-\delta|z|})$.

This shows that there are exponentially decaying $r^\pm$ which satisfy the equation $\mathcal{L}r^\pm = b_j^1$.

Let $R_1^\pm = {}^t(r_1^\pm, s_1^\pm)$ denote the exponentially decaying solutions of the homogeneous equation (3.21) constructed in Lemma 3.7. Thus $\mathcal{L}r_1^\pm = 0$ and at $q = \underline{q}$ and $\zeta = 0$, $r_1^\pm(z, \underline{q}, 0) = \partial_z \underline{W}$. Therefore, by (3.33) $\ell(q) \cdot r_1^\pm(0, q, \zeta) > 0$ for (q, ζ) close to $(\underline{q}, 0)$. Thus

$$a^\pm(q, \zeta) = \big(c(q, \zeta) + \ell(q) \cdot r^\pm(0, q, \zeta)\big)\big(\ell(q) \cdot r_1^\pm(0, q, \zeta)\big)^{-1}$$

are smooth on a neighborhood of $(\underline{q}, 0)$. Therefore,

$$\widetilde{r}^\pm(z, q, \zeta) = r^\pm(z, q, \zeta) - a(q, \zeta)r_1(z, q, \zeta)$$

satisfies the equation and the boundary condition. $\square$

Adding up, we see that

$$\mathcal{R}^{\pm}(q,\zeta) = (i\tau+\gamma)\mathcal{S}_0^{\pm}(q) + \sum_{j=1}^{d-1} i\eta_j \mathcal{S}_j^{\pm}(q,\eta) \tag{3.42}$$

is exponentially decaying, vanishes at $\zeta = 0$ and satisfies:

$$\begin{cases} \mathcal{L}(z,q,\zeta,D_z)\mathcal{R}^{\pm} = \mathcal{L}^1(z,q,\zeta) \quad \text{on } \{\pm z \geq 0\}, \\ \ell(q)\cdot\mathcal{R}^{\pm}(0,q,\zeta) = -c_0(\zeta)\,, \quad \mathcal{R}^{\pm}(z,q,0) = 0\,. \end{cases} \tag{3.43}$$

Proof of Proposition 3.13. **a)** With

$$U = {}^t(u, \widetilde{B}_\nu \partial_z u - \mathcal{A}u)\,,$$

the system (3.31)(3.32) is equivalent to

$$\begin{cases} \partial_z U^{\pm} - \mathcal{G}U^{\pm} - \varphi\check{\mathcal{G}}^1 = F^{\pm}\,, \qquad \mathit{on} \quad \pm z \geq 0\,, \\ U^-(0) = U^+(0)\,, \qquad \ell\cdot u^-(0) + \check{c}_0\varphi = 0\,, \end{cases} \tag{3.44}$$

with

$$\check{\mathcal{G}}^1 = \begin{pmatrix} 0 \\ \check{\mathcal{L}}^1 \end{pmatrix}, \quad F^{\pm} = \begin{pmatrix} 0 \\ f^{\pm} \end{pmatrix}.$$

The initial data $U^{\pm}(0)$ for (3.44) are linked to the initial data for (3.38) by the relation

$$U^{\pm}(0) = T\begin{pmatrix} u^{\pm}(0) \\ \partial_z^{\pm}(0) \end{pmatrix}, \qquad T = \begin{pmatrix} \mathrm{Id} & 0 \\ \widetilde{B}_\nu^{-1}\mathcal{A} & \widetilde{B}_\nu^{-1} \end{pmatrix}_{|z=0}.$$

Note that $T(q,\zeta)$ is common to both problems on $\{z \geq 0\}$.

Let $\widetilde{\mathbb{F}}$ denote the space of (U_0^-, U_0^+, φ) such that the solutions $U^{\pm}$ of $(\partial_z - \mathcal{G})U^{\pm} = \varphi\check{\mathcal{G}}^1$ on $\{z \geq 0\}$ with initial data $U_0^{\pm}$ are bounded. Thus $\widetilde{\mathbb{E}}$ is the image of $\widetilde{\mathbb{F}}$ by the mapping

$$\mathcal{T}' : \quad (V^-, V^+, \varphi) \mapsto (T^{-1}V^-, T^{-1}V^+, \varphi)\,.$$

Moreover, the space $\ker\widetilde{\Gamma}_\ell$ in (3.39) is invariant by $\mathcal{T}'$ and therefore it is sufficient to study the bundle $\widetilde{\mathbb{F}}(q,\zeta)$ and the determinant

$$\widetilde{D}'(z,\zeta) = \det(\widetilde{\mathbb{F}}(q,\zeta), \ker\widetilde{\Gamma}_\ell). \tag{3.45}$$

b) Next, we eliminate φ from the equations. Because $\mathcal{R}^{\pm}$ vanish at $\zeta = 0$, in polar coordinates there holds

$$\mathcal{R}^{\pm}(z,q,\rho\check{\zeta}) = \rho\check{r}^{\pm}(z,q,\check{\zeta},\rho) \tag{3.46}$$

with $\check r^\pm$ smooth for q close to $\underline{q}$, $\check\zeta \in S^d$ and $|\rho|$ small. Introducing

$$\check s^\pm(z,q,\zeta,\rho) = \widetilde B_\nu(z,q)\partial_z \check r^\pm(z,q,\zeta,\rho) - \mathcal{A}(z,q,\rho\check\zeta)\check r^\pm(z,q,\zeta,\rho)\,,$$

$\check R^\pm = {}^t(\check r^\pm, \check s^\pm)$ satisfies

$$(\partial_z - \mathcal{G})\check R^\pm = \check{\mathcal{G}}^1\,, \qquad \ell\cdot \check r^\pm{}_{|z=0} = -\check c_0\,.$$

Thus, $(U^\pm,\varphi)$ is a bounded solution of $(\partial_z - \mathcal{G})U^\pm = \varphi\check{\mathcal{G}}^1$ if and only if $V^\pm = U^\pm - \varphi\check R^\pm$ is a bounded solution of $(\partial_z - \mathcal{G})V^\pm = 0$.

Recall that $\mathbb{F}^\pm(q,\zeta)$ denotes the space of initial data such that the corresponding solution of $(\partial_z - \mathcal{G})V = 0$ on $\{\pm z \geq 0\}$ is bounded as z tends to $\pm\infty$. The analysis above implies that

$$\widetilde{\mathbb{F}} = \mathcal{T}_1\big(\mathbb{F}^- \times \mathbb{F}^+ \times \mathbb{C}\big)$$

where

$$\mathcal{T}_1(q,\check\zeta,\rho): \quad (V^-,V^+,\varphi) \mapsto (V^- + \varphi\check R^-(0),\, V^+ + \varphi\check R^+(0),\, \varphi)\,,$$

with $\check R^\pm(0) = \check R^\pm(0,q,\check\zeta,\rho)$. Moreover, with $\widetilde\ell$ denoting the vector $(\ell,0)$, there holds $\widetilde\ell\cdot\check R(0) = -\check c_0$ and therefore $\ker\widetilde\Gamma(q)$ is the image by $\mathcal{T}_1$ of

$$\mathbb{G}_1(q,\zeta) := \big\{(V^-,V^+,\varphi)\ :\ V^+ - V^- = \varphi(R^-(0) - R^+(0))\,,\ \widetilde\ell(q)\cdot V^+ = 0\big\}$$

Thus, it is sufficient to study $\widetilde{\mathbb{F}}_1 = \mathbb{F}^- \times \mathbb{F}^+ \times \mathbb{C}$ and the determinant

$$\widetilde D_1(q,\check\zeta,\rho) = \det(\widetilde{\mathbb{F}}_1,\mathbb{G}_1). \tag{3.47}$$

c) By Lemma 3.5, $\mathbb{F}^+$ and $\mathbb{F}^-$ are C^∞ vector bundles for q close to $\underline{q}$, $\check\zeta\in S^d_+$ and $\rho>0$ small and they have continuous extensions to $\rho=0$. Thus $\widetilde{\mathbb{F}}_1$ and $\widetilde{\mathbb{F}}$ have the same property. In particular, the determinant $\widetilde D_1(q,\check\zeta,\rho)$ is C^∞ for $\rho>0$ and continuous up to $\rho=0$.

We compute $D_1(\underline{q},\check\zeta,0)$, writing this determinant in suitable bases of $\widetilde{\mathbb{F}}_1(q,\zeta)$. By (3.13), Lemmas 3.7 and 3.8 provide us with bases $R_j^\pm(0)$ of $\mathbb{F}^\pm$. Elementary computations on the determinant, show that

$$\widetilde D_1(\underline{q},\check\zeta,0) = c(\check\zeta)D_2(\check\zeta)$$

where c does not vanish on the closed half sphere S^d_+ and $D_2(\check\zeta)$ is the $(1+2N)\times(1+2N)$ determinant

$$\det\begin{pmatrix} 0 & \{\ell\cdot r_j^+\}_{1\leq j\leq N^+} & 0 & \{\ell\cdot r_j^+\}_{j>N^+} & 0 \\ \{r_j^-\}_{j\leq N^-} & \{r_j^+\}_{1\leq j\leq N^+} & \{r_j^-\}_{j>N^-} & \{r_j^+\}_{j>N^+} & [\check r] \\ \{s_j^-\}_{j\leq N^-} & \{s_j^+\}_{1\leq j\leq N^+} & \{s_j^-\}_{j>N^-} & \{s_j^+\}_{j>N^+} & [\check s] \end{pmatrix}$$

with $[\check{r}] = r^+ - r^-$, $[\check{s}] = s^+ - s^-$ and the functions are evaluated at $z = 0$, $q = \underline{q}$ and $\rho = 0$. By Lemmas 3.7 and 3.8, the left lower hand entries $s_j^\pm$ for $j \le N^\pm$ are smooth functions of ζ and vanish at $\rho = 0$. Therefore the determinant is equal to the product of the right lower hand $N \times N$ determinant at $\rho = 0$,

$$\det(\{s_j^-\}_{j>N^-}, \{s_j^+\}_{j>N^+}, [\check{s}]), \tag{3.48}$$

and of the $(N+1) \times (N+1)$ left upper determinant

$$\det \begin{pmatrix} 0 & \{\ell \cdot r_j^+\}_{1\le j\le N^+} \\ \{r_j^-\}_{j\le N^-} & \{r_j^+\}_{1\le j\le N^+} \end{pmatrix}. \tag{3.49}$$

The equation for $R^\pm$ implies that

$$\partial_z s^\pm = \mathcal{M} r^\pm + \mathcal{L}^1 .$$

Since $\mathcal{M}$ and $r^\pm$ vanish at $\zeta = 0$, $\mathcal{M} r^\pm = O(|\zeta|^2)$. Integrating in z yields

$$-[s(q,\zeta)] = \int_{\mathbb{R}} \mathcal{L}^1(z,q,\zeta)dz + O(|\zeta|^2) .$$

Therefore, (3.35) implies that the jump of s at $z = 0$ satisfies

$$[s(\underline{q},\zeta)] := s^+(0,\underline{q},\zeta) - s^-(0,\underline{q},\zeta) = -\mathrm{m}(\underline{q},\zeta) + O(|\zeta|^2) \tag{3.50}$$

where m is defined at (3.23). Hence,

$$[\check{s}(\underline{q},\check{\zeta},0)] = \check{s}^+(0,\underline{q},\check{\zeta},0) - \check{s}^-(0,\underline{q},\check{\zeta},0) = -\mathrm{m}(\underline{q},\check{\zeta}),$$

implying that the determinant (3.48) is equal up to a sign to the determinant in (3.27), thus to $c(\check{\zeta})\Delta(\underline{q},\check{\zeta})$, where c does not vanish on S^d_+ and Δ is the Lopatinski determinant of the inviscid shock problem.

In the second determinant (3.49) we can subtract the first row from the last one, and since $r_1^- = r_1^+$ when $q = \underline{q}$ and $\zeta = 0$, this determinant is equal, up to the sign, to $\beta(\underline{q})(\ell \cdot r_+^1) = \beta(\underline{q})\partial_z \underline{W}(0)$. This finishes the proof of Proposition 3.13. $\square$

3.5 Elimination of the front

To prepare the proof of the energy estimates in the Appendix below, we make more explicit the argument of part b) of the proof of Proposition 3.13. The goal is to reduce the equations to problems where ψ appears only in the boundary conditions and to write the stability condition in terms of the corresponding Evans function.

Because $\mathcal{R}^\pm$ given by (3.42) vanishes at $\zeta = 0$, we factor out ρ and write $\mathcal{R}^\pm = \rho\check{\mathcal{R}}^\pm$. With this notation, the problem (3.38) is equivalent to

$$
\begin{cases}
u^\pm = v^\pm + \varphi\check{\mathcal{R}}^\pm \\
\mathcal{L}(z, q, \zeta, D_z) v^\pm = f^\pm , \quad \text{on } \{\pm z \geq 0\} , \\
[v(0)] + \varphi[\check{\mathcal{R}}(0)] = 0 \quad [\partial_z v(0)] + \varphi[\partial_z \check{\mathcal{R}}(0)] = 0, \\
\ell(q) \cdot v^-(0) = 0 .
\end{cases}
\tag{3.51}
$$

For functions $u^\pm$ on $\{\pm z \geq 0\}$, $[u(0)]$ denotes the jump $u^+(0) - u^-(0)$. By Lemma 3.14, $\ell \cdot [\check{\mathcal{R}}(0)] = 0$, and therefore the last condition $\ell(q) \cdot v^-(0) = 0$ can be equivalently replaced by $\ell(q) \cdot v^+(0) = 0$. Therefore,

$$
\widetilde{\mathbb{E}}(q, \check{\zeta}, \rho) = \mathcal{J}\big(\mathbb{E}^- \times \mathbb{E}^+ \times \mathbb{C}\big)
$$

where

$$
\mathcal{J}(q, \check{\zeta}, \rho) : \quad (V^-, V^+, \varphi) \mapsto (V^- + \varphi \mathrm{R}^-, V^+ + \varphi \mathrm{R}^+, \varphi)
$$

with

$$
\mathrm{R}^\pm(\check{\zeta}, \rho) = {}^t\big(\check{\mathcal{R}}^\pm(0), \partial_z \check{\mathcal{R}}^\pm(0)\big) .
$$

Moreover, $\ker \widetilde{\Gamma} = \mathcal{J}\mathbb{G}'$ with

$$
\mathbb{G}'(q, \check{\zeta}, \rho) = \big\{(V^-, V^+, \varphi) : V^+ - V^- = \varphi(\mathrm{R}^- - \mathrm{R}^+), \ \tilde{\ell} \cdot V^- = 0\big\}
$$

where $\tilde{\ell} = {}^t(\ell, 0)$. Therefore, the Evans function of the problem (3.51)

$$
D'(q, \check{\zeta}, \rho) = \det\Big(\mathbb{E}^- \times \mathbb{E}^+ \times \mathbb{C}, \mathbb{G}'\Big) .
\tag{3.52}
$$

satisfies:

$$
\frac{1}{C}|D(\underline{q}, \check{\zeta}, \rho)| \leq |D'(\underline{q}, \check{\zeta}, \rho)| \leq C|\widetilde{D}(\underline{q}, \rho\check{\zeta})|
\tag{3.53}
$$

As before, the vector bundles and determinants are smooth on $\omega \times S^d_+ \times]0, 1]$ for some neighborhood ω of $\underline{q}$, and they have continuous extensions to $\rho = 0$. Note that if $\mathrm{R}^+ - \mathrm{R}^- = 0$, then $(0, 0, 1) \in \mathbb{E}^- \times \mathbb{E}^+ \times \mathbb{C}$ also belongs to $\mathbb{G}'$. Therefore, Proposition 3.13 implies:

Proposition 3.15. *If the shock profile* $\underline{W} = W(\cdot, \underline{q})$ *is uniformly stable, then there is a neighborhood* ω *of* $\underline{q}$ *and there is* $c > 0$ *such that for all*

$$\forall (q, \check{\zeta}, \rho) \in \omega \times S^d_+ \times [0,1] \ : \quad |D'(q, \check{\zeta}, \rho)| \geq c\,.$$

In particular, the $\mathbb{C}^{2N}$ *valued function* $[\mathrm{R}] := \mathrm{R}^+ - \mathrm{R}^-$ *does not vanish on* $\omega \times S^d_+ \times [0,1]$.

We can push the analysis a little further. Since $[\mathrm{R}] \neq 0$, one at least locally, introduce a smooth $2N \times 2N$ matrix $\mathrm{K}(q, \check{\zeta}, \rho)$ such that

$$\ker \mathrm{K}(q, \check{\zeta}, \rho) = \mathbb{C}[\mathrm{R}(q, \check{\zeta}, \rho)]\,. \tag{3.54}$$

For instance, one can take K to be the orthogonal projector on $[\mathrm{R}]^\perp$. In this case, the boundary condition in (3.51) are equivalent to

$$\mathrm{K}[V(0)] = 0, \quad \ell \cdot v^-(0) = 0, \tag{3.55}$$

$$\varphi = -\mathrm{R} \cdot [V(0)]/|\mathrm{R}|^2, \tag{3.56}$$

with $V = (v, \partial_z v)$ and $[V(0)] = V^+(0) - V^-(0)$.

Denote by

$$\mathbb{G}''(q, \check{\zeta}, \rho) = \big\{ (V^-, V^+) : \mathrm{K}(V^+ - V^-) = 0,\ \tilde{\ell} \cdot V^- = 0 \big\}$$

and introduce the Evans function

$$D''(q, \check{\zeta}, \rho) = \det \big(\mathbb{E}^- \times \mathbb{E}^+, \mathbb{G}'' \big)\,. \tag{3.57}$$

of the problem

$$\begin{cases} \mathcal{L} v^\pm = f^\pm \quad \text{on } \{\pm z \geq 0\}, \\ \mathrm{K}([v(0)], [\partial_z v(0)]) = 0\,, \quad \ell \cdot v^-(0) = 0\,. \end{cases} \tag{3.58}$$

Proposition 3.16. *Suppose that the shock profile* $\underline{W} = W(\cdot, \underline{q})$ *is uniformly stable and that* K *satisfies* (3.54)*. Then there is a neighborhood* ω *of* $\underline{q}$ *and there is* $c > 0$ *such that for all*

$$\forall (q, \check{\zeta}, \rho) \in \omega \times S^d_+ \times [0,1] \ : \quad |D''(q, \check{\zeta}, \rho)| \geq c\,.$$

Remark 3.17. The Evans function D is associated to the equation (2.16), considered as a transmission problem:

$$\begin{cases} \mathcal{L} v^\pm = f^\pm \quad \text{on } \{\pm z \geq 0\}, \\ [v(0)] = [\partial_z v(0)] = 0\,. \end{cases} \tag{3.59}$$

The proposition above shows that the uniform stability condition for low frequencies is expressed through a natural "uniform lower bound condition" for a modified problem, for the same equation but with different transmission conditions.

3.6 The high frequency stability, proof of Proposition 2.12

We now prove that the transmission problem (3.59) satisfies the uniform stability condition for large frequencies.

For large ζ, we use the "parabolic polar coordinates"

$$\zeta = \Lambda\hat{\zeta}\,, \quad \Lambda = (\tau^2 + \gamma^2 + |\eta|^4)^{1/4}\,.$$

In this case, $\hat{\zeta}$ belongs to the "sphere" $\hat{S} := \{\tau^2 + \gamma^2 + |\eta|^4 = 1\}$. We also use the notations $\lambda = 1/\Lambda$, $\hat{S}_+ = \hat{S} \cap \{\hat{\gamma} \geq 0\}$.

With $V^\pm = {}^t(\Lambda v^\pm, \partial_z v^\pm)$, the homogeneous equation (3.59) is equivalent to the first order system

$$\partial_z V^\pm = \Lambda \mathcal{G}_2(z, q, \hat{\zeta}, \lambda) V^\pm\,, \quad V^+(0) = V^-(0) \tag{3.60}$$

where $\mathcal{G}_2$ is a matrix of the form

$$\mathcal{G}_2(z, q, \hat{\zeta}, \lambda) = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathcal{M}_2 & \mathcal{A}_2 \end{pmatrix}$$

with coefficients depending smoothly on $(z, q, \hat{\zeta}, \lambda)$ in $\mathbb{R} \times \mathcal{Q} \times \hat{S} \times [0,1]$. Note that we have factored out the large term Λ in (3.60), to ensure that $\mathcal{G}_2$ is bounded, and indeed continuous and smooth as λ tends to zero. The parabolicity Assumption (H1) implies that that for $\lambda \geq 0$ small and $\hat{\zeta} \in \hat{S}_+$, $\mathcal{G}_2$ has N eigenvalues in a compact set $\{\mathrm{Re}\mu > 0\}$ and N eigenvalues in a compact set $\{\mathrm{Re}\mu < 0\}$ (see Lemma 2.5 of [MZ1] or Lemma 3.3 above).

According to (2.20), one introduces the spaces $\mathbb{E}^{\pm, rs}(q, \zeta)$ of initial data $V_0^\pm \in \mathbb{C}^{2N}$ such that the solution of

$$(\partial_z - \Lambda\mathcal{G}_2)V^\pm = 0 \quad \text{for } \pm z \geq 0, \qquad V^\pm(0) = V_0^\pm$$

is bounded as z tends to infinity.

Introducing the scaling

$$\hat{V}(\hat{z}) = V(z),\ z = \lambda\hat{z}$$

(3.60) is transformed into:

$$\partial_{\hat{z}} \hat{V}^\pm = \hat{\mathcal{G}}(\hat{z}, q, \hat{\zeta}, \lambda)\hat{V}^\pm\,, \quad \hat{V}^+(0) = \hat{V}^-(0), \tag{3.61}$$

with $\hat{\mathcal{G}}(\hat{z}, q, \hat{\zeta}, \lambda) = \mathcal{G}_2(\lambda\hat{z}, q, \hat{\zeta}, \lambda)$. The limit equation as λ tends to zero is the constant coefficient system

$$\partial_{\hat{z}} \hat{V}^\pm = \hat{\mathcal{G}}_2(0, q, \hat{\zeta}, 0)\hat{V}^\pm\,, \quad \hat{V}^+(0) = \hat{V}^-(0)\,. \tag{3.62}$$

Using the uniform bounds $|\mathcal{G}_2(\lambda\hat{z}, q, \hat{\zeta}, \lambda) - \mathcal{G}_2(0, q, \hat{\zeta}, 0)| \le C\lambda(1 + |\hat{z}|)$ and the fact that $\mathcal{G}_2(0, q, \hat{\zeta}, 0)$ has no eigenvalues on the imaginary axis, one easily shows that the vector bundles $\hat{\mathbb{E}}^{\pm}(q, \hat{\zeta}, \lambda) = \mathbb{E}^{\pm}(q, \lambda^{-1}\hat{\zeta})$ are smooth in $(q, \hat{\zeta}, \lambda)$ for $q \in \mathcal{Q}$, $\hat{\zeta} \in \hat{S}_+$ and $\lambda \ge 0$ small enough. Moreover, at $\lambda = 0$, $\hat{\mathbb{E}}^{\pm}(q, \hat{\zeta}, 0)$ is the set of initial data $V_0^{\pm} \in \mathbb{C}^{2N}$ such that the solution of

$$(\partial_{\hat{z}} - \mathcal{G}_2(0, q, \hat{\zeta}, 0)\hat{V}^{\pm} = 0 \quad \text{for } \pm \hat{z} \ge 0, \qquad \hat{V}^{\pm}(0) = V_0^{\pm}$$

is bounded, and therefore exponentially decaying. Therefore, this shows that the rescaled Evans function (2.21), $D^{rs}(q, \zeta) = \det\big(\mathbb{E}^{-,rs}(q, \zeta), \mathbb{E}^{+,rs}(q, \zeta)\big)$, satisfies

$$D^{rs}(q, \zeta) = \hat{D}(q, \hat{\zeta}, \lambda) := \det\big(\hat{\mathbb{E}}^{-}(q, \hat{\zeta}, \lambda), \hat{\mathbb{E}}^{+}(q, \hat{\zeta}, \lambda)\big)$$

where $\hat{D}(q, \hat{\zeta}, \lambda)$ is smooth $q \in \mathcal{Q}$, $\hat{\zeta} \in \hat{S}_+$ and $\lambda \ge 0$ small enough.

Moreover, since the limit equation has constant coefficients, $\hat{\mathbb{E}}^{+}(q, \hat{\zeta}, 0)$ [resp. $\hat{\mathbb{E}}^{-}(q, \hat{\zeta}, 0)$] is the N dimensional space generated by the generalized eigenvectors of $\mathcal{G}_2(0, q, \hat{\zeta}, 0)$ associated to eigenvalues with negative [resp. positive] real part. Since $\mathcal{G}_2(0, q, \hat{\zeta}, 0)$ has no eigenvalues on the imaginary axis, there holds $\mathbb{C}^{2N} = \hat{\mathbb{E}}^{-}(q, \hat{\zeta}, 0) \oplus \hat{\mathbb{E}}^{+}(q, \hat{\zeta}, 0)$, thus

$$\hat{D}(q, \hat{\zeta}, 0) = \det\big(\hat{\mathbb{E}}^{-}(q, \hat{\zeta}, 0), \hat{\mathbb{E}}^{+}(q, \hat{\zeta}, 0)\big) \ne 0\,.$$

By continuity and compactness this implies Proposition 2.12. More precisely, we have proved:

Proposition 3.18. *For all compact subset $\mathcal{Q}_0 \subset \mathcal{Q}$ there there are $c > 0$ and $\lambda_0 > 0$ such that $|\hat{D}(q, \hat{\zeta}, \lambda)| \ge c$ when $q \in \mathcal{Q}_0$, $\hat{\zeta} \in \hat{S}_+$ and $\lambda \le \lambda_0$.*

3.7 Proof of Proposition 2.6

Proof. **a)** If $\underline{p}$ is a Lax shock, then the implicit function theorem implies that in a neighborhood ω of $\underline{p}$, the Rankine Hugoniot condition can be made explicit, giving u^+ as a smooth function of u^- and h. Thus, the set of shocks in ω is a smooth manifold $\mathcal{C}$ of dimension $N+d$. By continuity, shrinking ω if necessary, the eigenvalues of $A_\nu(u^{\pm}, h)$ do not vanish for $p = (u^-, u^+, h) \in \mathcal{C}$ and the numbers of positive and negative eigenvalues is constant. Thus all $p \in \mathcal{C}$ is a Lax shock.

b) Suppose that $\underline{w}$ is a profile associated to a Lax shock $\underline{p}$. For (u^+, h) close to $(\underline{u}^+, \underline{h})$ we consider the solutions of (2.2) on the half axis $\{z \ge 0\}$. The classical theory of stable manifolds implies the following

Lemma 3.19. *There are neighborhoods $\omega^+ \subset \mathcal{U} \times \mathbb{R}^d$ and $\omega_1^+ \subset \mathbb{R}^{N^+}$ of $(\underline{u}^+, \underline{h})$ and 0 respectively, and there is a smooth function W^+ from $\omega^+ \times \omega_1^+ \times [0, +\infty[$ to $\mathbb{R}^N$ and a constant $\delta > 0$, such that*

i) for all $(u^+, h) \in \omega^+$, all $a \in \omega_1^+$, $W^+(u^+, h, a, \cdot)$ is a solution of (2.2) *such that*

$$\left|W^+(u^+, h, a, z) - u^+\right| + \left|\nabla_a W^+(u^+, h, a, z)\right| \lesssim e^{-\delta z}\,,$$

ii) the matrix $\nabla_a W^+(\underline{u}^+, \underline{h}, 0, 0)$ has maximal rank equal to N^+.

In addition, all the solutions of (2.2) *close to $\underline{w}$ are of this form.*

We also apply this lemma to the equation

$$B_\nu(w, h)\partial_z w = f_\nu(w, h) - f_\nu(u^-, h) \tag{3.63}$$

on $]-\infty, 0]$. There are neighborhoods $\omega^- \subset \mathcal{U} \times \mathbb{R}^d$ and $\omega_1^- \subset \mathbb{R}^{N^-}$ of $(\underline{u}^-, \underline{h})$ and 0 respectively, and there is a smooth function W^- from $\omega^- \times \omega_1^+ \times [0, +\infty[$ to $\mathbb{R}^N$ and a constant $\delta > 0$, such that for all $(u^-, h) \in \omega^-$, all $a \in \omega_1^-$, $W^-(u^-, h, a, \cdot)$ is a solution of (3.63) on $\{z \leq 0\}$ such that

$$\left|W^-(u^-, h, a, z) - u^-\right| + \left|\nabla_a W^-(u^-, h, a, z)\right| \lesssim e^{\delta z}\,.$$

Moreover, the matrix $\nabla_a W^-(\underline{u}^-, \underline{h}, 0, 0)$ has rank equal to N^-.

To get a solution of (2.5) close to $\underline{w}$ and associated to $p = (u^-, u^+, h)$ with $(\underline{u}^-, \underline{h}) \in \omega^-$ and $(\underline{u}^-, \underline{h}) \in \omega^-$, it is necessary and sufficient that $f_\nu(u^-, h) = f_\nu(u^+, h)$, in which case the equations (2.2) and (3.63) are identical, and to glue together solutions on $\{\pm z \geq 0\}$. The first condition means that p satisfies the Rankine-Hugoniot conditions, that is $p \in \mathcal{C}$. The second condition is equivalent to to find $a^\pm \in \omega_1^\pm$ such that

$$F(p, a^-, a^+) := W^-(u^-, h, a^-, 0) - W^+(u^+, h, a^+, 0) = 0\,. \tag{3.64}$$

We denote by $\omega_1 = \omega_1^- \times \omega_1^+$ and by $a = (a^-, a^+)$ the variable in ω_1. We show that if $\underline{w}$ is transversal, then $\nabla_a F(\underline{p}, 0)$ has rank N. Thus, by the implicit function theorem, locally near $(\underline{p}, 0)$, the set of solutions $(p, a) \in \mathcal{C} \times \omega_1$ of (3.64) is a manifold of dimension $N + d + 1$ parametrized by p and some component $a_1 \in \mathbb{R}$ of a. Thus, locally, (3.64) is equivalent to $a = \alpha(p, a_1)$ for some smooth function α from $\mathcal{C}\times]-\delta, \delta[$ to ω_1.

Differentiating (2.2) and (3.63), we see that for all $j \leq N^+$ [resp. $k \leq N^-$], the functions $\dot{w}_j^+ = \partial_{a_j} W^+(\underline{u}^+, \underline{h}, 0, z)$ [resp. $\dot{w}_k^- = \partial_{a_k} W^-(\underline{u}^-, \underline{h}, 0, z)$] are solutions of the linearized equation (2.6)

$$\mathcal{P}'_{\underline{w}}\dot{w} = 0$$

on $\{z \geq 0\}$ [resp. $\{z \leq 0\}$], exponentially decaying at $+\infty$ [resp. $-\infty$]. By Lemma 3.19, the space $\mathbb{F}^+$ [resp. $\mathbb{F}^-$] generated by the $\dot{w}_j^+(0)$ [resp. $\dot{w}_k^-(0)$] is of dimension N^+ [resp. N^-]. If $\sum \dot{a}_j^+ \dot{w}_j^+(0) = \sum \dot{a}_k^- \dot{w}_k^-(0) \in \mathbb{F}^+ \cap \mathbb{F}^-$, then $\dot{w}^+ = \sum \dot{a}_j^+ \dot{w}_j^+$ and $\dot{w}^- = \sum \dot{a}_k^- \dot{w}_k^-$ are solutions of (2.6) on $\{z \geq 0\}$ and $\{z \leq 0\}$ respectively, exponentially decaying at infinity and piecing them together gives a solution $\dot{w}$ of (2.6) on the whole line. Therefore, the transversality assumption implies that $\dot{w}$ is proportional to $\partial_z \underline{w}$. Therefore, $\dim(\mathbb{F}^+ \cap \mathbb{F}^-) \leq 1$. Since $\dim \mathbb{F}^+ = N^+$ and $\dim \mathbb{F}^- = N^-$, the Lax shock condition $N^+ + N^- = N + 1$ implies that $\mathbb{F}^+ + \mathbb{F}^- = \mathbb{C}^N$. This means that $\nabla_a F(\underline{p}, 0)$ has rank N, as claimed, thus finishing the proof of Proposition 2.6. $\square$

4 Handbook of paradifferential calculus

For the convenience of the reader, we collect in this section the results about paradifferential calculus which are used to prove the linear stability estimates in section five. The proofs are omitted, they can be found in the Appendix of [MZ1].

4.1 The homogeneous calculus

We consider operators on $\mathbb{R}^d$. The variables are denoted $\tilde{y} = (t, y)$ and the frequency variables $\tilde{\eta} = (\tau, \eta)$. The symbols and operators also depend on a parameter γ which plays a distinguished role. We denote by $\mathbb{R}_+^{d+1}$ the set of frequencies $\zeta := (\tilde{\eta}, \gamma) \in \mathbb{R}^{d+1} \setminus \{0\}$ such that $\gamma \geq 0$ and by S_+^d the set of $(\tilde{\eta}, \gamma) \in \mathbb{R}_+^{d+1}$ such that $|\zeta| = 1$.

Definition 4.1 (Symbols). *Let $\mu \in \mathbb{R}$. i) Γ_0^μ denotes the space of locally L^∞ functions $a(\tilde{y}, \zeta)$ on $\mathbb{R}^d \times \mathbb{R}_+^{d+1}$ which are C^∞ with respect to ζ and such that for all $\alpha \in \mathbb{N}^d$ there is a constant C_α such that*

$$\forall (\tilde{y}, \zeta), \quad |\partial_{\tilde{\eta}}^\alpha a(\tilde{y}, \zeta)| \leq C_\alpha \, |\zeta|^{\mu - |\alpha|} . \tag{4.1}$$

ii) Γ_1^μ denotes the space of symbols $a \in \Gamma_0^\mu$ such that for all j, $\partial_{\tilde{y}_j} a \in \Gamma_0^\mu$.

For example, functions $a(\tilde{y}, \zeta)$ which are C^∞ and homogeneous of degree m in $(\tilde{\eta}, \gamma) \in \mathbb{R}_+^{d+1}$ and bounded on $\mathbb{R}^d \times S_+^d$, are symbols in Γ_0^m.

In the applications, we consider families of symbols $a^\varepsilon(x)$ in Γ_k^m, depending on parameters $\varepsilon \in [0, 1[$ and $x \in \mathbb{R}$. The key point is that they are bounded in Γ^m_k. Moreover, we want to study the action of the operators in conormal spaces. Consider the following set of vector fields on $\mathbb{R}^{d+1}$:

$$Z_0 = \partial_t\,, \quad Z_j = \partial_{y_j} \ \text{ for } 1 \le j \le d-1\,, \quad Z_d = \frac{x}{\sqrt{1+x^2}}\partial_x\,. \tag{4.2}$$

They commute, and for $\alpha \in \mathbb{Z}^{d+1}$ we use the notation $Z^\alpha = Z_0^{\alpha_0} \cdots Z_d^{\alpha_d}$.

Definition 4.2. *For $\mu \in \mathbb{R}$, $m \in \mathbb{N}$ and $k \in \{0,1\}$, $\Gamma^\mu_{k,m}$ is the set of functions $a(\tilde y, x, \zeta)$ such that for all $\alpha \in \mathbb{Z}^{d+1}$ with $|\alpha| \le m$, the set $\{(Z^\alpha a)(\cdot, x, \cdot) : x \in \mathbb{R}\}$ is bounded in Γ^μ_k.*

The spaces $\Gamma^\mu_{k,m}$ are equipped with semi-norms

$$\|a\|_{(\mu,k,m,N)} := \sup_{|\alpha|\le N}\ \sup_{|\beta|\le k}\ \sup_{|\sigma|\le m}\ \sup_{(x,\tilde y,\zeta)} \ |\zeta|^{|\alpha|-\mu} \left|Z^\sigma \partial_\zeta^\alpha \partial_{\tilde y}^\beta a(x,\tilde y,\zeta)\right|. \tag{4.3}$$

A family of symbols is bounded in $\Gamma^\mu_{k,m}$ when for all N, the semi norms are bounded. For $a \in \Gamma^\mu_{k,m}$ we denote by $Z^\alpha a(x)$ the symbol $(Z^\alpha a)(\cdot, x, \cdot) \in \Gamma^\mu_k$.

The para-differential calculus is a quantization of symbols in $a \in \Gamma^\mu_0$ to which are associated operators denoted by T^γ_a. This extends the Fourier multipliers calculus: when $a(\zeta)$ is a constant coefficient symbol, then

$$(T^\gamma_a u)(\tilde y) = \frac{1}{(2\pi)^d}\int e^{i\tilde y\tilde\eta} a(\tilde\eta,\gamma)\hat u(\tilde\eta)d\tilde\eta \tag{4.4}$$

where $\hat u$ denotes the Fourier transform of u. The T^γ_a act in the scale of Sobolev spaces $H^s(\mathbb{R}^d)$. These spaces are equipped with the family of norms

$$\left|u\right|_{0,s,\gamma} := \Big(\int_{\mathbb{R}^d} (\gamma^2 + |\tilde\eta|^2)^s\,|\hat u(\tilde\eta)|^2\,d\tilde\eta\Big)^{\frac12}. \tag{4.5}$$

Adding the normal variable x, we introduce the norms

$$\begin{aligned} \|u\|_{0,s,\gamma} &= \Big(\int \left|u(x,\cdot)\right|^2_{0,s,\gamma}\,dx\Big)^{\frac12}, \\ \|u\|_{m,s,\gamma} &= \sum_{|\alpha|\le m} \gamma^{m-|\alpha|}\,\|Z^\alpha u\|_{0,s,\gamma}\,, \end{aligned} \tag{4.6}$$

which are parameter dependent norms on spaces called $\mathcal{H}^{0,s}$ and $\mathcal{H}^{m,s}$ respectively.

When $a \in \Gamma^{\mu}_{0,m}$ the action of para-differential operators is extended to x-dependent functions:

$$(T^{\gamma}_a u)(x,\cdot) = T^{\gamma}_{a(x)} u(x,\cdot). \tag{4.7}$$

The paradifferential calculus in $\mathbb{R}^d$, was introduced by J.M.Bony [Bo] (see also [Mey] , [Hör], [Tay]) with γ fixed, say $\gamma = 1$). The parameter dependent version T^{γ}_a is introduced in [Mé1] [Mok] and applies in the scale of spaces $\mathcal{H}^{0,s}$. The extension to the scale $\mathcal{H}^{m,s}$ is immediate since one can construct the T^{γ} so that

$$Z T^{\gamma}_a u = T^{\gamma}_a Z u + T^{\gamma}_{Za} u\,. \tag{4.8}$$

Proposition 4.3 (Action). *i) When $a(\zeta)$ is a symbol independent of $\tilde y$, the operator T^{γ}_a is defined by* (4.4).

ii) For all $a \in \Gamma^{\mu}_{0,m}$, the family of operators $\{T^{\gamma}_a\}_{\gamma\geq 1}$ is of order $\leq \mu$, meaning that for $\gamma \geq 1$:

$$\| T^{\gamma}_a u\|_{m,s,\gamma} \leq C\ \| u\|_{m,s+\mu,\gamma}$$

where C is independent of $\gamma \geq 1$ and u.

Proposition 4.4 (Symbolic calculus). *Consider $a \in \Gamma^{\mu}_{1,m}$ and $b \in \Gamma^{\mu'}_{1,m}$. Then $ab \in \Gamma^{\mu+\mu'}_{1,m}$ and $\{T^{\gamma}_a \circ T^{\gamma}_b - T^{\gamma}_{ab}\}_{\gamma\geq 1}$ is of order $\leq \mu + \mu' - 1$, meaning that for $\gamma \geq 1$:*

$$\| (T^{\gamma}_a T^{\gamma}_b - T^{\gamma}_{ab}) u\|_{m,s,\gamma} \leq C\ \ \| u\|_{m,s+\mu+\mu'-1,\gamma}$$

where C is independent of $\gamma \geq 1$ and u.

If b is independent of $\tilde y$, then $T^{\gamma}_a \circ T^{\gamma}_b = T^{\gamma}_{ab}$.

These results extend to matrix valued symbols and operators.

Bounded functions of $\tilde y$ are particular examples of symbols in the class Γ^0_0, independent of the frequency variables ζ. In this case, T^{γ}_a is called a para-product in [Bo]. We introduce the spaces $\mathcal{W}^{m,k}$ of functions a on $\mathbb{R}^{d+1}$ which belong to $\Gamma^0_{k,m}$ when considered as symbols independent of ζ:

Definition 4.5. *$\mathcal{W}^{m,k}$ denotes the space of functions a on $\mathbb{R}^{d+1}$ such that $Z^{\alpha}\partial^{\beta}_{\tilde y} u \in L^{\infty}(\mathbb{R}^{d+1}_+)$ for all $|\alpha| \leq m$ and $|\beta| \leq k$, where $Z^{\alpha} := \Pi_j Z_j^{\alpha_j}$, with Z_j defined as in* (4.2).

When $k = 0$, we simply denote by $\mathcal{W}^m$ the corresponding space.

These spaces are equipped with the norms

$$\|a\|_{\mathcal{W}^{m,k}} = \sum_{|\alpha|\leq m} \sum_{|\beta|\leq k} \|Z^\alpha \partial_{\tilde{y}}^\beta u\|_{L^\infty(\mathbb{R}^{d+1}_+)} . \tag{4.9}$$

A key point in the theory is the comparison between T_a^γ and the multiplication by a:

Proposition 4.6 (Para-products). *There is a constant C such that for all $a \in \mathcal{W}^{m,1}$ and all $u \in \mathcal{H}^{m,0}$*

$$\begin{aligned} \|au - T_a^\gamma u\|_{m,1,\gamma} &\leq C\, \|a\|_{\mathcal{W}^{m,1}} \|u\|_{m,0,\gamma} , \\ \gamma\|au - T_a^\gamma u\|_{m,0,\gamma} &\leq C\, \|a\|_{\mathcal{W}^{m,1}} \|u\|_{m,0,\gamma} , \\ \|a\partial_j u - T_a^\gamma \partial_j u\|_{m,0,\gamma} &\leq C\, \|a\|_{\mathcal{W}^{m,1}} \|u\|_{m,0,\gamma} . \end{aligned}$$

4.2 The semi-classical parabolic calculus

In the high frequency regime, the parabolic character of the equations prevails and we need a quasi-homogeneous calculus. In addition, the operators depend on the parameter ε and appear naturally as operators in $\varepsilon\partial_{\tilde{y}}$, leading to a semi-classical calculus. The parabolic quasi-homogeneity is associated to the dilations $\lambda\cdot(t,y) = (\lambda^2 t, \lambda y)$ and similarly $\lambda\cdot(\tau,\gamma,\eta) = (\lambda^2\tau, \lambda^2\gamma, \lambda\eta)$. The corresponding quasi-norm is

$$\langle\zeta\rangle = \Big(\gamma^2 + \tau^2 + |\eta|^4\Big)^{\frac{1}{4}}. \tag{4.10}$$

We also introduce the weight

$$\Lambda(\zeta) = (1 + \langle\zeta\rangle^4)^{\frac{1}{4}}. \tag{4.11}$$

Typical examples of symbols are smooth quasi-homogeneous functions of degree μ away from the origin. They satisfy

$$|\partial_\zeta^\alpha a(\zeta)| \leq C_\alpha \langle\zeta\rangle^{m-\langle\alpha\rangle}$$

where, for $\alpha = (\alpha_\tau, \alpha_\eta) \in \mathbb{N}\times\mathbb{N}^{d-1}$:

$$\langle(\alpha_\tau, \alpha_\eta)\rangle := 2|\alpha_\tau| + |\alpha_\eta|.$$

Definition 4.7 (Symbols). *Let $\mu \in \mathbb{R}$.*

i) $\mathrm{P}\Gamma_0^\mu$ *denotes the space of locally L^∞ functions $a(\tilde{y}, \zeta)$ on $\mathbb{R}^d \times \mathbb{R}_+^{d+1}$ which are C^∞ with respect to ζ and such that for all $\alpha \in \mathbb{N}^d$ there is a constant C_α such that*

$$\forall (\tilde{y}, \zeta)\,, \quad |\partial_{\tilde{\eta}}^\alpha a(\tilde{y}, \tilde{\eta}, \gamma)| \;\leq\; C_\alpha \,\Lambda(\zeta)^{\mu - \langle \alpha \rangle}\,. \tag{4.12}$$

ii) $\mathrm{P}\Gamma_1^\mu$ *denotes the space of symbols $a \in P\Gamma_0^\mu$ such that for all j, $\partial_{\tilde{y}_j} a \in P\Gamma_0^\mu$.*

iii) For $m \in \mathbb{N}$ and $k \in \{0, 1\}$, $\mathrm{P}\Gamma_{k,m}^\mu$ is the set of functions $a(\tilde{y}, x, \zeta)$ such that the set $\{(Z^\alpha a)(\cdot, x, \cdot) : |\alpha| \leq m,\ x \in \mathbb{R}\}$ is bounded in $\mathrm{P}\Gamma_k^\mu$.

The spaces $\mathrm{P}\Gamma_{k,m}^\mu$ are equipped with semi-norms

$$\|a\|_{(\mu,k,m,N)} := \sup_{\langle \alpha \rangle \leq N} \ \sup_{|\beta| \leq k} \ \sup_{|\sigma| \leq m} \ \sup_{(x, \tilde{y}, \zeta)} \ \Lambda^{\langle \alpha \rangle - \mu} |Z^\sigma \partial_\zeta^\alpha \partial_{\tilde{y}}^\beta a(x, \tilde{y}, \zeta)|\,. \tag{4.13}$$

Next we consider a semi-classical quantification of the symbols : when $a(\zeta)$ is independent of $\tilde{y}$, the associated operator is defined by the the Fourier multiplier $a(\varepsilon\tilde{\eta}, \varepsilon\gamma)$:

$$P_a^{\varepsilon,\gamma} u(\tilde{y}) = \frac{1}{(2\pi)^d} \int e^{i\tilde{y}\tilde{\eta}} a(\varepsilon\tilde{\eta}, \varepsilon\gamma) \hat{u}(\tilde{\eta}) d\tilde{\eta}. \tag{4.14}$$

Note that we use here the standard multiplication by ε, not the parabolic dilation $\varepsilon \cdot \tilde{\eta}$.

Similarly, the natural Sobolev spaces associated to the parabolic smoothness are the spaces PH^s of functions whose Fourier transform belong to the L^2 space with weight Λ^{2s}. Because we use a semi-classical analysis, this leads to introduce on PH^s the following family of norms

$$\big|u\big|_{0,s,\varepsilon,\gamma} := \Big(\int_{\mathbb{R}^d} \Lambda(\varepsilon\tilde{\eta}, \varepsilon\gamma)^{2s} \,|\hat{u}(\tilde{\eta})|^2 \, d\tilde{\eta} \Big)^{\frac{1}{2}}\,. \tag{4.15}$$

Adding the normal variable x, we introduce the norms

$$\begin{aligned} \|u\|_{0,s,\varepsilon,\gamma} &= \Big(\int \big|u(x, \cdot)\big|_{0,s,\varepsilon,\gamma}^2 \, dx \Big)^{\frac{1}{2}}\,, \\ \|u\|_{m,s,\varepsilon,\gamma} &= \sum_{|\alpha| \leq m} \gamma^{m - |\alpha|} \|Z^\alpha u\|_{0,s,\varepsilon,\gamma}\,, \end{aligned} \tag{4.16}$$

which are parameter dependent norms on spaces called $P\mathcal{H}^{0,s}$ and $P\mathcal{H}^{m,s}$ respectively.

The operators $P_a^{\varepsilon,\gamma}$ are first constructed for symbols $a \in \mathrm{P}\Gamma_0^\mu$. Next the action is extended to x dependent symbols:

$$(P_a^{\varepsilon,\gamma} u)(x,\cdot) = P_{a(x)}^{\varepsilon,\gamma} u(x,\cdot). \tag{4.17}$$

When $u \in P\mathcal{H}^{m,s}$ and $a \in \mathrm{P}\Gamma_{0,m}^\mu$, the following identity holds:

$$Z P_a^{\varepsilon,\gamma} u = P_a^{\varepsilon,\gamma} Z u + P_{Za}^{\varepsilon,\gamma} u\,. \tag{4.18}$$

Proposition 4.8 (Action). *i) When $a(\zeta)$ is a symbol independent of $\tilde y$, the operator $P_a^{\varepsilon,\gamma}$ is defined by (4.14).*

ii) For all $a \in \mathrm{P}\Gamma_{0,m}^\mu$ and $s \in \mathbb{R}$, there is C such that for $\varepsilon \in]0,1]$, $\gamma \geq 1$ and $u \in P\mathcal{H}^{m,s}$:

$$\| P_a^{\varepsilon,\gamma} u \|_{m,s-\mu,\varepsilon,\gamma} \leq C\, \| u \|_{m,s,\varepsilon,\gamma}\,.$$

The constant C is bounded when a remains in a bounded set of $\mathrm{P}\Gamma_{0,m}^\mu$.

iii) If $a \in \mathrm{P}\Gamma_0^\mu$ is supported in $\mathbb{R}^d \times \{\Lambda(\zeta) \leq R\}$, then, for all u, the spectrum of $P_a^{\varepsilon,\gamma} u$ is contained in $\{\Lambda(\varepsilon\zeta) \leq 2R\}$

iv) There is $\delta > 0$, such that If $a \in \mathrm{P}\Gamma_0^\mu$ is supported in $\mathbb{R}^d \times \{\Lambda(\zeta) \geq R\}$ then, for all u, the spectrum of $P_a^{\varepsilon,\gamma} u$ is contained in $\{\Lambda(\varepsilon\zeta) \geq \delta R\}$.

Proposition 4.9 (Symbolic calculus). *Consider $a \in \mathrm{P}\Gamma_{1,m}^\mu$ and $b \in \mathrm{P}\Gamma_{1,m}^{\mu'}$. Then $ab \in \mathrm{P}\Gamma_{1,m}^{\mu+\mu'}$ and there is C such that for $\varepsilon \in]0,1]$, $\gamma \geq 1$ and $u \in P\mathcal{H}^{m,s}$:*

$$\| (P_a^{\varepsilon,\gamma} \circ P_b^{\varepsilon,\gamma} - P_{ab}^{\varepsilon,\gamma}) u \|_{m,s-\mu-\mu'+1,\varepsilon,\gamma} \leq C\,\varepsilon\, \| u \|_{m,s,\varepsilon,\gamma}\,.$$

The constant C is bounded when a and b remain bounded in $\mathrm{P}\Gamma_{1,m}^\mu$ and $\mathrm{P}\Gamma_{1,m}^{\mu'}$ respectively.

Moreover, if b is independent of $\tilde y$, then $P_a^{\varepsilon,\gamma} \circ P_b^{\varepsilon,\gamma} = P_{ab}^{\varepsilon,\gamma}$.

Next we consider para-products, that is operators associated to symbols independent of ζ.

Proposition 4.10 (Para-products). *For all $a \in \mathcal{W}^{m,1}$, there is a constant C such that for all $u \in P\mathcal{H}^{m,1}$, $\varepsilon \in]0,1]$, and $\gamma \geq 1$:*

$$\begin{aligned} &\| au - P_a^{\varepsilon,\gamma} u \|_{m,1,\varepsilon,\gamma} + \sum_{|\alpha|=1} \varepsilon \| a\partial_y^\alpha u - P_a^{\varepsilon,\gamma} \partial_y^\alpha u \|_{m,0,\varepsilon,\gamma} \\ &\qquad\qquad \leq C\varepsilon \| u \|_{m,0,\varepsilon,\gamma}\,, \end{aligned} \tag{4.19}$$

$$\begin{aligned} &\gamma \| au - P_a^{\varepsilon,\gamma} u \|_{m,0,\varepsilon,\gamma} + \| a\partial_t u - P_a^{\varepsilon,\gamma} \partial_t u \|_{m,0,\varepsilon,\gamma} \\ &\qquad + \sum_{|\alpha|=2} \varepsilon \| a\partial_y^\alpha u - P_a^{\varepsilon,\gamma} \partial_y^\alpha u \|_{m,0,\varepsilon,\gamma} \leq C \| u \|_{m,1,\varepsilon,\gamma}\,. \end{aligned} \tag{4.20}$$

Corollary 4.11. *For all $a \in \mathcal{W}^{m,2}$, there is a constant C such that for all $u \in P\mathcal{H}^{m,2}$, $\varepsilon \in]0,1]$, and $\gamma \geq 1$:*

$$\|au - P_a^{\varepsilon,\gamma} u\|_{m,2,\varepsilon,\gamma} \leq C\varepsilon \|u\|_{m,1,\varepsilon,\gamma}\,,$$

4.3 A link between the two calculi

Remark first that for constant coefficient symbols a, T_a^γ and $P_a^{\varepsilon,\gamma}$ are Fourier multipliers by $a(\tilde\eta,\gamma)$ and $a(\varepsilon\tilde\eta,\varepsilon\gamma)$ respectively. Thus

$$P_a^{\varepsilon,\gamma} = T_{a^\varepsilon}^\gamma \quad \text{with} \quad a^\varepsilon(\zeta) = a(\varepsilon\zeta) \tag{4.21}$$

The next result extends this relation to symbols which also depend on the variables $\tilde y$.

Proposition 4.12. *Suppose that $b \in \mathrm{P}\Gamma^0_{1,0}$ has compact support in ζ. Then the family of symbols*

$$\mathrm{b}^\varepsilon(\tilde y, x, \zeta) = b(\tilde y, x, \varepsilon\zeta) \tag{4.22}$$

is bounded in $\Gamma^0_{1,0}$ and there is a constant C such that for all u, $\varepsilon \in]0,1]$ and $\gamma \geq 1$:

$$\gamma\|T^\gamma_{\mathrm{b}^\varepsilon} u - P_b^{\varepsilon,\gamma} u\|_{L^2} + \|T^\gamma_{\mathrm{b}^\varepsilon}\nabla_{\tilde y} u - P_b^{\varepsilon,\gamma}\nabla_{\tilde y} u\|_{L^2} \leq C\|u\|_{L^2}\,. \tag{4.23}$$

Moreover, in the scale of norms (4.6)*:*

$$\|T^\gamma_{\mathrm{b}^\varepsilon} u - P_b^{\varepsilon,\gamma} u\|_{0,0,\gamma} \leq C\|u\|_{0,-1,\gamma}, \tag{4.24}$$

$$\|T^\gamma_{\mathrm{b}^\varepsilon} u - P_b^{\varepsilon,\gamma} u\|_{0,1,\gamma} \leq C\|u\|_{0,0,\gamma}. \tag{4.25}$$

4.4 Calculi on the boundary and traces

We have stated the theorems above for symbols and functions depending on x, which indeed are extensions of results on $\mathbb{R}^d$, via the identities (4.7) and (4.17). We still denote by T^γ and $P^{\varepsilon,\gamma}$, the operators on $\mathbb{R}^d$. We do not make specific statements, they are in fact particular cases of the results above, provided that the set of vector fields $\{Z_j\}$ is restricted to the fields $\partial_{\tilde y_j}$.

However, we will use a specific result on the boundary. Proposition 4.10 and Corollary 4.11 imply that for $a \in W^{m+2,\infty}(\mathbb{R}^d)$ there holds

$$|au - P_a^{\varepsilon,\gamma} u|_{m,s+1,\varepsilon,\gamma} \leq C\varepsilon |u|_{m,s,\varepsilon,\gamma} \tag{4.26}$$

for $s = 0$ and $s = 1$. Interpolating implies that the estimate is satisfied for $s \in [0,1]$ and in particular for $s = 1/2$.

Taking traces, (4.7) and (4.17) imply that

$$\begin{cases} (T_a^\gamma u)_{|x=0} = T^\gamma_{a_{|x=0}} u_{|x=0}\,, \\ (P_a^{\varepsilon,\gamma} u)_{|x=0} = P^{\varepsilon,\gamma}_{a_{|x=0}} u_{|x=0}\,. \end{cases} \tag{4.27}$$

whenever the traces make sense. When u and a are smooth up to $x = 0$ on each side $\{\pm x > 0\}$, we denote by $u^\pm_{|x=0}$ the two traces and

$$\begin{cases} (T_a^\gamma u)^\pm_{|x=0} = T^\gamma_{a^\pm_{|x=0}} u^\pm_{|x=0}\,, \\ (P_a^{\varepsilon,\gamma} u)^\pm_{|x=0} = P^{\varepsilon,\gamma}_{a^\pm_{|x=0}} u^\pm_{|x=0}\,. \end{cases} \tag{4.28}$$

5 Linear stability

5.1 The main estimate

In this section we study the linear stability of approximate solutions of (1.8). We start with functions $u_0(t, y, x)$ on $\mathbb{R}^{1+d}$ and $\psi_0(t, y)$ on $\mathbb{R}^d$. The front ψ_0 and the restriction $u_0^\pm$ of u_0 to $\mathbb{R}^{1+d}_\pm = \{\pm x \geq 0\}$ are smooth enough (see Assumption 5.2 below). They are thought to be solutions of (1.5) but this is not needed in this section. What we require, is that they satisfy the Rankine-Hugoniot conditions (1.7) and that

$$p(t, y) = \big(u_0^-(t, y, 0), u_0^+(t, y, 0), d\psi_0(t, y)\big) \tag{5.1}$$

is a Lax shock. In addition, for all point $(t, y, 0)$ in the boundary, we assume that there is a profile associated to $p(t, y)$ which is uniformly stable in the sense of Definition 2.9. If $p(t, y)$ remains in a small neighborhood of $\underline{p} = p(\underline{t}, \underline{y})$, we can apply Proposition 2.6 and Theorem 2.18 to construct a smooth family of uniformly stable profiles $W(p, z)$. In the large, we have to *assume* that the different pieces can be glued together, and this yields to the following assumption.

Assumption 5.1. *We are given a smooth manifold $\mathcal{C} \subset \mathcal{U} \times \mathcal{U} \times \mathbb{R}^d$ of dimension $N + d$ and a smooth function W_0 from $\mathbb{R} \times \mathcal{C}$ to $\mathcal{U}^*$ such that:*

i) every $p = (u^-, u^+, h) \in \mathcal{C}$ is a Lax shock,

ii) for all $p \in \mathcal{C}$, the mapping $z \mapsto W_0(p, z)$ is a shock profile associated to p,

iii) $\{W_0(\cdot, p); p \in \mathcal{C}\}$ is a smooth family of profiles in the sense of Definition 2.4*,* *iv) for all $p \in \mathcal{C}$, the profile $W_0(\cdot, p)$ is uniformly stable in the sense of Definition* 2.9.

Moreover, we are given a smooth mapping ℓ_0 from $\mathcal{C}$ to $\mathbb{R}^N$ such that

$$\forall p \in \mathcal{C}, \qquad \ell(p) \cdot \partial_z W_0(0, p) > 0.$$

We consider the linearized equations around functions of the form

$$u_\varepsilon(t, y, x) = W_0\left(\frac{x}{\varepsilon}, p(t, y)\right) + u_\varepsilon^1(t, y, x) \tag{5.2}$$

and fronts

$$\psi_\varepsilon = \psi_0 + \varepsilon \psi_\varepsilon^1 , \tag{5.3}$$

with u_ε^1 small and having bounded derivatives near $x = 0$. (In particular, this includes the class of approximate solutions constructed in Section 6; see (6.40), Remark 6.10.) As in [MZ1], due to the rapid variations in the boundary layer, the natural smoothness for solutions is measured via conormal estimates. Recall the Definition 4.5 of spaces $\mathcal{W}^m$ and $\mathcal{W}^{m,k}$. In particular, recall the definition of the norm

$$\|a\|_{\mathcal{W}^m} = \sum_{|\alpha| \le m} \|Z^\alpha u\|_{L^\infty(\mathbb{R}_+^{d+1})} . \tag{5.4}$$

When necessary we specify $\mathcal{W}^m(\mathbb{R}_\pm^{1+d})$ to denote the space of functions defined on $\mathbb{R}_\pm^{1+d}$.

Given a function u, we denote by u^+ and u^- its restriction to $\mathbb{R}_+^{1+d} = \{x > 0\}$ and $\mathbb{R}_-^{1+d} = \{x < 0\}$ respectively. We make the following regularity assumptions:

Assumption 5.2. *m is a given non negative integer and*

i) the restrictions $u_0^\pm$ are C^{m+2} up to the boundary and belong to the Sobolev space $W^{m+2,\infty}(\mathbb{R}_\pm^{1+d})$ and $\psi_0 \in W^{m+3,\infty}(\mathbb{R}^d)$;

ii) p, defined at (5.1)*, takes values in a compact subset $\mathcal{C}_0$ of $\mathcal{C}$;*

iii) the families $\{u_\varepsilon^{1,\pm}\}$, $\{\nabla_{t,y,x} u_\varepsilon^{1,\pm}\}$, $\{\varepsilon \nabla_{t,y,x}^2 u_\varepsilon^{1,\pm}\}$ for $\varepsilon \in]0,1]$, are bounded in $\mathcal{W}^m(\mathbb{R}_\pm^{1+d})$;

iv) the traces $u^1_{\varepsilon|x=0}$ are $O(\varepsilon)$ in $L^\infty(\mathbb{R}^d)$ and $u_0^\pm(t, y, 0) + u_\varepsilon^{1,\pm}(t, y, x)$ take values in a compact subset of $\mathcal{U}$.

v) The family ψ_ε^1 is bounded in $W^{m+3,\infty}(\mathbb{R}^d)$.

To include the perturbations u^1_ε and ψ^1_ε, we extend the set of parameters and consider

$$q = (p, u^1, h^1) \in \mathcal{C} \times \mathbb{R}^N \times \mathbb{R}^d \tag{5.5}$$

where u^1 is a placeholder for u^1_ε and h^1 a placeholder for $\varepsilon d\psi^1_\varepsilon$: we denote by q_ε the mapping

$$q_\varepsilon(t, y, x) = \big(p(t, y), u^1_\varepsilon(t, y, x), \varepsilon d\psi^1_\varepsilon(t, y)\big) \ . \tag{5.6}$$

We denote by W the extended function

$$W(z, q) = W_0(z, p) + u^1 \tag{5.7}$$

so that the function (5.2) reads

$$u_\varepsilon(t, y, x) = W\left(\frac{x}{\varepsilon}, q_\varepsilon(t, y, x)\right) \ . \tag{5.8}$$

Moreover, introduce the function

$$q = (u^-, u^+, h, u^1, h^1) \quad \mapsto \quad \Psi(q) = h + h^1 \ , \tag{5.9}$$

so that

$$d\psi_\varepsilon = \Psi(q_\varepsilon) \ . \tag{5.10}$$

Similarly, we extend the function ℓ_0 to a function of q, setting for instance $\ell(q) = \ell_0(p)$, and we define

$$\ell_\varepsilon(t, y) = \ell(q_\varepsilon(t, y, 0)). \tag{5.11}$$

With notations as in section two, to the profile $W(\cdot, q)$ and planar front $\Psi(q)$ we associate the Evans function $D(q, \zeta)$, together with its rescaled form $D^{rs}(q, \zeta)$. Associated to W and to the extra boundary associated to $\ell(q)$, is the modified Evans function $\widetilde{D}(q, \check{\zeta}, \rho)$. Proposition 3.18 and Assumption 5.1 imply:

Lemma 5.3. *There is a relatively compact neighborhood $\mathcal{Q}_0$ of $\mathcal{C}_0 \times \{0\} \times \{0\}$ in $\mathcal{C} \times \mathbb{R}^N \times \mathbb{R}^d$ such that for all $(q, \zeta) \in \overline{\mathcal{Q}}_0 \times \mathbb{R}^{1+d}_+ \backslash \{0\}$, $D(q, \zeta) \neq 0$. Moreover, there is $c > 0$ such that*

$$\forall (q, \zeta) \in \mathcal{Q}_0 \times \mathbb{R}^{1+d}_+, \quad |\zeta| \geq 1 \ : \quad |D^{rs}(q, \zeta)| \geq c \, , \tag{5.12}$$

$$\forall (q, \check{\zeta}, \rho) \in \mathcal{Q}_0 \times S^d_+ \times]0, 1] \ : \quad |\widetilde{D}(q, \check{\zeta}, \rho)| \geq c \, . \tag{5.13}$$

The Assumption 5.2 immediately implies the following estimates:

Lemma 5.4. *i) The families* $\{q_\varepsilon^\pm\}$ *and* $\{u_\varepsilon^\pm\}$ *are bounded in* $\mathcal{W}^m(\mathbb{R}^{1+d}_\pm)$.

ii) The families $\{\nabla_{t,y,x} q_\varepsilon^\pm\}$, $\{\varepsilon\nabla^2_{t,y,x} q_\varepsilon^\pm\}$, $\{\nabla_{t,y} u_\varepsilon^\pm\}$, $\{\varepsilon\partial_x u_\varepsilon^\pm\}$, $\{\varepsilon\nabla_{t,y}\partial_x u_\varepsilon^\pm\}$ *and* $\{\varepsilon^2\partial_x^2 u_\varepsilon^\pm\}$ *are bounded in* $\mathcal{W}^m(\mathbb{R}^{1+d}_\pm)$.

iii) For all neighborhood $\mathcal{Q}$ *of* $\mathcal{C}_0 \times \{0\} \times \{0\}$ *in* $\mathcal{C} \times \mathbb{R}^N \times \mathbb{R}^d$, *there are* $\varepsilon_0 > 0$ *and* $\delta > 0$ *such that for all* $\varepsilon \in]0, \varepsilon_0]$ *and all* $(t, y, x) \in \mathbb{R}^{1+d}$ *with* $|x| \le 2\delta$, *there holds* $q_\varepsilon(t, y, x) \in \mathcal{Q}$.

iv) The family $\{\ell_\varepsilon\}$ *is bounded in* $W^{m+2,\infty}(\mathbb{R}^d)$.

Next, we consider the linearized equation from (1.8) around $(u_\varepsilon, \psi_\varepsilon)$. With $\partial_0 = \partial_t$, $D_j = \partial_j - \partial_j\psi\partial_x$ and $f_0(u) = u$, the equation reads

$$\mathcal{E}(u, \psi) := \sum_{j=0}^{d} D_j f_j(u) - \varepsilon \sum_{j,k=1}^{d} D_j\big(B_{j,k}(u) D_k u\big) = 0\,. \tag{5.14}$$

Thus the linearized equation at (u, ψ) are

$$\mathcal{E}'_u(u, \psi)\dot u + \mathcal{E}'_\psi(u, \psi)\dot\psi = \dot f \tag{5.15}$$

with

$$\begin{aligned}\mathcal{E}'_u(u, \psi)\dot u := \sum_{j=0}^{d} D_j\big(A_j(u)\dot u\big) &- \varepsilon \sum_{j,k=1}^{d} D_j\big(B_{j,k}(u) D_k \dot u\big)\\ &- \varepsilon \sum_{j,k=1}^{d} D_j\big(\dot u \cdot \nabla_u B_{j,k}(u) D_k u\big)\,,\end{aligned}$$

$$\begin{aligned}\mathcal{E}'_\psi(u, \psi)\dot\psi := -\sum_{j=0}^{d-1} \partial_j\dot\psi\, \partial_x f_j(u) &+ \varepsilon \sum_{j,k=1}^{d} \partial_j\dot\psi\partial_x\big(B_{j,k}(u) D_k u\big)\\ &+ \varepsilon \sum_{j,k=1}^{d} D_j\big(B_{j,k}(u)\partial_k\dot\psi\partial_x u\big)\,.\end{aligned}$$

where $A_j = f'_j$. The equation holds on $\mathbb{R}^{1+d}$, but we really think of it as a transmission problem, where the equations hold on both half space $\mathbb{R}^{1+d}_\pm$ and the restrictions $\dot u^\pm$ satisfy on $\{x = 0\}$ the transmission conditions

$$[\dot u(0)] = [\partial_x \dot u(0)] = 0\,. \tag{5.16}$$

To (5.15) (5.16), we add the "boundary" condition on $\{x = 0\}$:

$$\ell_\varepsilon \cdot \dot u^-(0) + (\partial_t - \varepsilon\Delta_y).\dot\psi = 0 \tag{5.17}$$

As in [MZ1] the maximal estimates involve weighted spaces. With $\zeta = (\tau, \eta, \gamma)$ introduce

$$\Lambda = \Lambda(\varepsilon\zeta) = \big(1 + (\varepsilon\tau)^2 + (\varepsilon\gamma)^2 + |\varepsilon\eta|^4\big)^{\frac{1}{4}} \tag{5.18}$$

and

$$\lambda = \lambda_\varepsilon(\zeta) = \begin{cases} \big(\gamma + \varepsilon|\zeta|^2\big)^{\frac{1}{2}} & \text{when} \quad |\varepsilon\zeta| \leq 1\,, \\ \sim \varepsilon^{-\frac{1}{2}} & \text{when} \quad 1 \leq |\varepsilon\zeta| \leq 2\,, \\ \frac{\Lambda}{\sqrt{\varepsilon}} \approx \big(\gamma + |\tau| + \varepsilon|\eta|^2\big)^{\frac{1}{2}} & \text{when} \quad |\varepsilon\zeta| > 1\,. \end{cases} \tag{5.19}$$

Note that the three terms are of the same order $\varepsilon^{-1/2}$ when $\varepsilon|\zeta| \approx 1$. Introduce next

$$\mu = \mu_\varepsilon(\zeta) = \begin{cases} |\zeta|\lambda_\varepsilon(\zeta) & \text{when} \quad |\varepsilon\zeta| \leq 1\,, \\ \sim \varepsilon^{-\frac{3}{2}} & \text{when} \quad 1 \leq |\varepsilon\zeta| \leq 2\,, \\ \big(\frac{\Lambda}{\varepsilon}\big)^{3/2} & \text{when} \quad |\varepsilon\zeta| > 1\,. \end{cases} \tag{5.20}$$

Given a weight function ϕ, we use the notation

$$|u|_{(\phi)} = \bigg(\int_{\mathbb{R}^d} \phi(\tau, \eta, \gamma)^2 |\hat{u}(\tau, \eta)|^2 d\tau d\eta\bigg)^{\frac{1}{2}}$$

where $\hat{u}$ is the Fourier transform of u on $\mathbb{R}^d$. When u also depends on the variable x we denote by $\|u\|_{(\phi)}$ the norm

$$\|u\|_{(\phi)} = \bigg(\int \ |u(\cdot, x)|^2_{(\phi)}\, dx\bigg)^{\frac{1}{2}}\ .$$

To be complete, the notation should include the interval where x varies which is $\mathbb{R}_\pm$ or $\mathbb{R}$. It will be clear from the context.

With the vector fields Z_j, for $m \in \mathbb{N}$, define

$$\|u\|_{m,(\phi)} = \sum_{|\alpha| \leq m} \gamma^{m-|\alpha|}\, \|Z^\alpha u\|_{(\phi)}\ , \quad |u|_{m,(\phi)} = \sum_{|\alpha| \leq m} \gamma^{m-|\alpha|}\, \big|\partial^\alpha_{t,y} u\big|_{(\phi)}\ . \tag{5.21}$$

They are norms, depending on the parameter γ, and on ε if the weight does as in (5.19), on spaces called $\mathcal{H}^m_\phi$. The L^2 norm corresponds to $m = 0$ and $\phi = 1$. We denote it by $\|u\|_0$. More generally, we denote by $\|u\|_m$ the norms $\|u\|_{m,(1)}$ associated to $\phi = 1$, which are also the norms defined in (4.6).

Because these norms only involve conormal and tangential derivatives, note that $u \in \mathcal{H}^m_\phi(\mathbb{R}^{1+d})$ if and only the restrictions $u^\pm$ to $\mathbb{R}^{1+d}_\pm$ belong to $\mathcal{H}^m_\phi(\mathbb{R}^{1+d}_\pm)$.

Theorem 5.5. *Under Assumptions* 2.1, 5.1 *and* 5.2 *there are constants* C, $\varepsilon_0 > 0$ *and* $\gamma_0 > 0$ *such that for all* $\varepsilon \in]0, \varepsilon_0]$, *all* $\dot u^\pm$ *and* $\dot f^\pm$, C^∞ *with compact support on* $\overline{\mathbb{R}}^{1+d}_\pm$, *all* $\dot\psi \in C_0^\infty(\mathbb{R}^d)$ *satisfying the linearized equation* (5.15) *for* $x \neq 0$ *and the boundary conditions* (5.16) (5.17), *there holds for all* $\gamma \geq \gamma_0$*:*

$$\begin{aligned}\left\|e^{-\gamma t}\dot u\right\|_{m,(\lambda^2)} + \sqrt{\varepsilon}\left\|e^{-\gamma t}\partial_x \dot u\right\|_{m,(\lambda)} + \left|e^{-\gamma t}\dot u(0)\right|_{m,(\lambda\sqrt{\Lambda})}\\ +\varepsilon\left|e^{-\gamma t}\partial_x \dot u(0)\right|_{m,(\lambda/\sqrt{\Lambda})} + \left|e^{-\gamma t}\dot\psi\right|_{m,(\mu\Lambda^2)} \leq C\left\|e^{-\gamma t}\dot f\right\|_m .\end{aligned} \tag{5.22}$$

Remark 5.6. We prove the estimates for $\dot u$ on both sides $\mathbb{R}^{1+d}_\pm$. But the transmission condition (5.16) implies that $\partial_x \dot u$ has no Dirac mass on the boundary and estimations for $\partial_x \dot u$ on both side are equivalent to estimations on the whole space.

Remark 5.7. In particular, (5.22) implies

$$\left\|e^{-\gamma t}\dot u\right\|_{m,(\lambda^2)} + \left|e^{-\gamma t}\dot u(0)\right|_{m,(\lambda\sqrt{\Lambda})} + \left|e^{-\gamma t}\dot\psi\right|_{m,(\mu\Lambda^2)} \leq C\left\|e^{-\gamma t}\dot f\right\|_m , \tag{5.23}$$

which may be viewed as a conormal version of Majda's estimate

$$\left\|e^{-\gamma t}\dot u\right\|_{H^{m,(\gamma)}} + \left|e^{-\gamma t}\dot u(0)\right|_{H^{m,(\sqrt{\gamma})}} + \left|e^{-\gamma t}\dot\psi\right|_{H^{m,(|\zeta|\sqrt{\gamma})}} \leq \quad C\left\|e^{-\gamma t}\dot f\right\|_{H^m} , \tag{5.24}$$

where the weighted Sobolev norms $|f|_{H^{m,(\phi)}}$, $\|f\|_{H^{m,(\phi)}}$ are defined exactly as were $|f|_{m,(\phi)}$, $\|f\|_{H^{m,(\phi)}}$, but with the conormal derivative $Z_d = \frac{x}{\sqrt{1+x^2}}\partial_x$ replaced by the usual partial derivative ∂_x in (5.21). Indeed, away from the shock layer $|x| \leq \varepsilon$, (5.23) implies (5.24); thus, we recover the bounds of Majda in the limit as $\varepsilon \to 0$.

To understand at a heuristic level the appearance of conormal derivatives in (5.23), consider solutions (u_0, ψ_0), $(u_0 + \dot u_0, \psi_0 + \dot\psi_0)$ of the inviscid equations with forcing terms f, $f + \dot f$ supported away from the shock boundary $\tilde x = 0$, and the associated zero-order viscous corrections (U_0, ψ_0), $(U_0 + \dot U_0, \psi_0 + \dot\psi_0)$ obtained by replacing the discontinuity from u^- to u^+ with viscous shock profile $w(x/\varepsilon, u^-(0), u^+(0))$. Then, the viscous corrections approximately satisfy the viscous equations (1.8)–(1.9) and $(\dot U_0, \dot\psi_0)$ approximately satisfy the linearized equations (1.17) but not the additional boundary condition (1.18); rather, $\dot\psi_0$ approximately satisfies the linearization of Rankine–Hugoniot equations (1.7). However, a calculation similar to that of (1.25), Remark 1.1.2 shows that $(\dot U_0, \dot\psi_0)$ is order ε close to the approximate solution

$$(\dot U, \dot\psi) := \Big((U_0 + \dot U_0)(x - \delta) - U_0, \dot\psi_0 + \delta\Big)$$

of (1.17)–(1.18) determined by the truncated version $\dot\psi_t = \ell \cdot \dot U_{|\tilde x=0}$ of (1.18) with $\delta(y,0) := 0$, which yields (after brief calculation)

$$\begin{aligned}\delta_t &\sim \mathcal{O}(|\dot u_0|_{|\tilde x=0}) - \delta\ell \cdot \partial_x w(x/\varepsilon, u^-(0), u^+(0)) \\ &\sim \mathcal{O}(|\dot u_0|_{|\tilde x=0}) - (c(y,t)/\varepsilon)\delta,\end{aligned} \tag{5.25}$$

$\delta(y,0) \equiv 0$ with $c(y,t) > 0$, and thereby $\delta(y,t) = \mathcal{O}(\varepsilon \sup_t |\dot u_0(y,t)|_{|\tilde x=0})$.

Observing that order $|\dot u_\pm|$ perturbations in the trace values $(u_0^\pm)$ lead to order $|\dot u_\pm|\varepsilon^{-k}e^{-\theta x/\varepsilon}$ perturbations in the k-th derivative of the profile, we obtain from the sharp bound (5.24) a sharp conormal bound

$$\left\|e^{-\gamma t}\nu(x)\dot u\right\|_{m,(\gamma)} + \left|e^{-\gamma t}\dot u(0)\right|_{m,(\sqrt\gamma)} + \left|e^{-\gamma t}\dot\psi\right|_{H^{m,(|\zeta|\sqrt\gamma)}} \leq \quad C\left\|e^{-\gamma t}\dot f\right\|_m, \tag{5.26}$$

on the approximate linearized solution $(\dot U, \dot\psi)$ of our construction, where $\nu(x) := \sqrt{\frac{1+|\tilde x|^2}{\gamma+|\tilde x|^2}}$ is order one away from the shock layer and order $\gamma^{-1/2}$ within. Apart from the difference between weights λ^2, $\lambda\sqrt\Lambda$, $\mu\Lambda^2$ and their $\varepsilon \to 0$ limits γ, $\sqrt\gamma$, $|\zeta|\sqrt\gamma$, reflecting the slight smoothing effects of viscosity in the medium and high frequency regimes, and the harmless $\nu(x)$ factor (γ is held fixed in the nonlinear arguments to follow), this is exactly estimate (5.23). Thus, (5.23) is a natural generalization of (5.24) to the singular-perturbation context. Note that computation (5.25) gives heuristic justification of our use of the inviscid shock location in the zero-order viscous approximation, rather than the location prescribed by (1.18) as done in higher-order approximations.

We set $u = e^{-\gamma t}\dot u$, $f = e^{-\gamma t}\dot f$, $\psi = e^{-\gamma t}\dot\psi$ and we prove the estimate

$$\begin{aligned}\|u\|_{m,(\lambda^2)} + \sqrt\varepsilon\,\|\partial_x u\|_{m,(\lambda)} &+ |u(0)|_{m,(\lambda\sqrt\Lambda)} \\ &+ \varepsilon\,|\partial_x u(0)|_{m,(\lambda/\sqrt\Lambda)} + |\psi|_{m,(\mu\Lambda^2)} \\ \lesssim \|f\|_m + \|u\|_{m,(\Lambda)} &+ \varepsilon\,\|\partial_x u\|_m + |u(0)|_{m,(\Lambda)} \\ &+ \varepsilon\,|\partial_x u(0)|_m + |\nabla_\gamma\psi|_{m,(\Lambda)}\end{aligned} \tag{5.27}$$

where $\lesssim$ means that the left hand side is estimated by constant times the right hand side, with a constant independent of ε, γ, u, f and ψ. In addition, $\nabla_\gamma\psi$ denotes $((\partial_t+\gamma)\psi, \partial_1\psi, \ldots, \partial_{d-1}\psi)$.

Indeed, (5.19) implies that $\lambda^2 \gtrsim \gamma$ and also that $\lambda^2 \gtrsim \gamma^{1/2}\Lambda$ for $\gamma \geq 1$.

Similarly, $\mu\Lambda \geq (\frac{\gamma}{\epsilon})^{\frac{1}{4}}|\zeta|$. Therefore

$$
\begin{aligned}
&\|u\|_{m,(\Lambda)} \lesssim \gamma^{-1/2}\|u\|_{m,(\lambda^2)}\,, &&\|\partial_x u\|_m \lesssim \gamma^{-1/2}\|\partial_x u\|_{m,(\lambda)}\,,\\
&|u(0)|_{m,(\Lambda)} \lesssim \gamma^{-1/4}|u(0)|_{m,(\lambda\sqrt{\Lambda})}\,, &&|\partial_x u(0)|_{m,(\Lambda)} \lesssim \gamma^{-1/4}|\partial_x u(0)|_{m,(\lambda\sqrt{\Lambda}}\,,\\
&|\nabla_\gamma\psi|_{m,(\Lambda)} \approx |\psi|_{m,(|\zeta|\Lambda)} \lesssim \gamma^{-1/4}|\psi|_{m,(\mu\Lambda^2)}\,.
\end{aligned}
$$

Hence, for γ large enough, (5.27) implies the estimate of Theorem 5.5.

The proof of this estimate is parallel to the analysis of [MZ1]. Away from the boundary, the hyperbolic-parabolic structure of the equation is sufficient to imply the estimate. So we concentrate on the analysis near the boundary. There, we first reduce ourselves to a first order system in ∂_x, neglecting some lower order terms in the equation. Next, we replace the differential equation by a para-differential one, as in [MZ1], [GMWZ2]. This costs only further admissible error terms. When this is achieved, we make two different choices of "good unknowns" in the high and low frequency regime respectively, to get paradifferential equations which have been analyzed in [MZ1].

5.2 Reductions

We compute the linearized operators. It is convenient to introduce the condensed notations:

$$
\begin{aligned}
&\widetilde{B}_{j,k}(u,h) = B_{j,k}(u) \quad \text{when } j,k<d\,,\\
&\widetilde{B}_{j,d}(u,h) = \sum_{k=1}^{d}\nu_k B_{j,k}(u)\,, \quad \widetilde{B}_{d,j}(u,h) = \sum_{k=1}^{d}\nu_k B_{k,j}(u) \quad \text{when } j<d\,,\\
&\widetilde{B}_{d,d}(u,h) = \sum_{j,k=1}^{d}\nu_k\nu_j B_{j,k}(u)\,,
\end{aligned}
$$

with $\nu = (-h_1,\ldots,-h_{d-1},1)$, and

$$
\begin{aligned}
&\widetilde{A}_j(u,h) = A_j(u) \quad \text{when } j<d\,,\\
&\widetilde{A}_d(u,h) = \sum_{j=1}^{d}\nu_k A_j(u) - h_0\mathrm{Id}\,.
\end{aligned}
$$

The matrices $\widetilde{A}_d$, $\widetilde{B}_{j,d}$ and $\widetilde{B}_{d,d}$ were called A_ν, $B_{j,\nu}$ and B_ν respectively, in sections one and two. Introduce next

$$
B^\sharp_{j,k}(z,q) = \widetilde{B}_{j,k}(W(q,z),\Psi(q))\,, \quad A^\sharp_j(z,q) = \widetilde{A}_j(W(q,z),\Psi(q))
$$

and

$$\begin{aligned} B^\varepsilon_{j,k}(t,y,x) &= B^\sharp_{j,k}(x/\varepsilon, q_\varepsilon) = \widetilde B_{j,k}(u_\varepsilon, d\psi_\varepsilon)\,,\\ A^\varepsilon_k(t,y,x) &= A^\sharp_j(x/\varepsilon, q_\varepsilon) = \widetilde A_j(u_\varepsilon, d\psi_\varepsilon)\,. \end{aligned}$$

Similarly, we consider the first and second derivatives of the coefficients. Calling $\nabla_u \widetilde A_j$ the derivative of $\widetilde A_j$ with respect to u, with some abuse of notation, $\nabla_u A^\varepsilon_j$ denotes the function $\nabla_u \widetilde A_j(u_\varepsilon, d\psi_\varepsilon)$. We use similar notation for the derivatives of the $B_{j,k}$.

Using these conventions and remarking that $\partial_x \psi = 0$, one checks that the left hand side of (1.8) reads

$$\mathcal{E}(u,\psi) = \partial_t u + \sum_{j=1}^d \widetilde A_j(u, d\psi)\partial_j u - \varepsilon \sum_{j,k}^d \partial_j\big(\widetilde B_{j,k}(u,d\psi)\partial_k u\big)\,. \tag{5.28}$$

Therefore

$$e^{-\gamma t}\mathcal{E}'_u(u_\varepsilon,\psi_\varepsilon)\big(e^{\gamma t}u\big) = L_{\varepsilon,\gamma,t}u$$

where

$$L_{\varepsilon,\gamma,t}u = (\partial_t + \gamma)u + \sum_{j=1}^d \mathrm{A}^{\varepsilon,t}_j \partial_j u - \varepsilon \sum_{j,k=1}^d B^\varepsilon_{j,k}\partial^2_{j,k}u + \frac{1}{\varepsilon}\mathrm{E}^{\varepsilon,t}u \tag{5.29}$$

with

$$\begin{aligned} \mathrm{A}^{\varepsilon,t}_j \dot v &= A^\varepsilon_j \dot v - \varepsilon\sum_k (\partial_k B^\varepsilon_{k,j})\dot v - \varepsilon \sum_k (\dot v\cdot \nabla_u B^\varepsilon_{j,k})\partial_k u_\varepsilon\,,\\ \mathrm{E}^{\varepsilon,t}\dot u =& \varepsilon \sum_j (\dot u\cdot\nabla_u A^\varepsilon_j)\partial_j u_\varepsilon\\ &- \varepsilon^2\sum_{j,k}(\dot u\cdot \nabla_u B^\varepsilon_{j,k})\partial^2_{j,k}u_\varepsilon - \varepsilon^2\sum_{j,k}\big((\dot u\cdot\partial_j(\nabla_u B^\varepsilon_{j,k})\big))\partial_k u_\varepsilon\,, \end{aligned}$$

The subscript t in $L_{\varepsilon,\gamma,t}$ or $\mathrm{A}^{\varepsilon,t}_j$ means “total”. We split the coefficients into their principal part and remainders : remainders are terms which are smaller by a factor ε, taking into account that the coefficients depend on $u_\varepsilon = W(q_\varepsilon, x/\varepsilon)$ and that

$$\partial_x u_\varepsilon = \varepsilon^{-1}\partial_z W(q_\varepsilon, x/\varepsilon) + \partial_x q_\varepsilon \cdot \nabla_q W(q_\varepsilon, x/\varepsilon)\,.$$

Accordingly, we have the splitting:

$$\mathrm{A}^{\varepsilon,t}_j = \mathrm{A}^\varepsilon_j + \varepsilon \mathrm{A}^{\varepsilon,r}_j\,,\quad \mathrm{E}^{\varepsilon,t} = \mathrm{E}^\varepsilon + \varepsilon\mathrm{E}^{\varepsilon,r} \tag{5.30}$$

with principal parts given by

$$
\begin{aligned}
&\mathrm{A}^\varepsilon_j \dot v = A^\varepsilon_j \dot v - (\partial_z W \cdot \nabla_u B^\varepsilon_{d,j})\dot v - (\dot v \cdot \nabla_u B^\varepsilon_{j,d})\partial_z W\,,\\
&\mathrm{E}^\varepsilon \dot u = (\dot u \cdot \nabla_u A^\varepsilon_d)\partial_z W - (\dot u \cdot \nabla_u B^\varepsilon_{d,d})\partial^2_z W - \nabla^2_{u,u} B^\varepsilon_{d,d}(\dot u, \partial_z W)\partial_z W\,.
\end{aligned}
$$

Here, $\partial_z W$ and $\partial^2_z W$ are evaluated at $q = q_\varepsilon(t,y,x)$ and $z = x/\varepsilon$. Moreover, $\nabla^2 B^\varepsilon_{d,d}(\cdot,\cdot)$ stands for the second derivative $\nabla^2 \widetilde B^\varepsilon_{d,d}(u,h)(\cdot,\cdot)$ evaluated at $u = u_\varepsilon, h = d\psi_\varepsilon$. With (5.8) (5.10), we note that the principal coefficients are functions of $(q_\varepsilon, x/\varepsilon)$:

$$
B^\varepsilon_{j,k} = B^\sharp_{j,k}(x/\varepsilon, q_\varepsilon)\,, \quad \mathrm{A}^\varepsilon_j = \mathrm{A}^\sharp_j(x/\varepsilon, q_\varepsilon)\,, \quad \mathrm{E}^\varepsilon = \mathrm{E}^\sharp(x/\varepsilon, q_\varepsilon)\,, \tag{5.31}
$$

with

$$
\begin{aligned}
&B^\sharp_{j,k}(z,q) = \widetilde B_{j,k}(W(q,z), \Psi(q))\\
&\mathrm{A}^\sharp_j(z,q)\dot v = \widetilde A_j \dot v - (\partial_z W \cdot \nabla_u \widetilde B_{d,j})\dot v - (\dot v \cdot \nabla_u \widetilde B_{j,d})\partial_z W\,,\\
&\mathrm{E}^\sharp(z,q)\dot u = (\dot u \cdot \nabla_u \widetilde A_d)\partial_z W - (\dot u \cdot \nabla_u \widetilde B_{d,d})\partial^2_z W - \nabla^2_{u,u} \widetilde B_{d,d}(\dot u, \partial_z W)\partial_z W\,.
\end{aligned}
$$

where the tilded functions are now evaluated at $u = W(q,z)$ and $h = \Psi(q)$, while $\partial_z W$ and $\partial^2_z W$ are functions of (q,z).

Thanks to Assumption 5.2, the remainders satisfy :

$$
\sup_\varepsilon \|\mathrm{A}^{\varepsilon,r}_j\|_{\mathcal W^m} < +\infty\,, \qquad \sup_\varepsilon \|\mathrm{E}^{\varepsilon,r}\|_{\mathcal W^m} < +\infty\,. \tag{5.32}
$$

Substituting the splitting (5.30) of the coefficients in the definition (5.29) of $L_{\varepsilon,\gamma,t}$ yields

$$
L_{\varepsilon,\gamma,t} = L_{\varepsilon,\gamma} + \varepsilon L_{\varepsilon,\gamma,r}\,. \tag{5.33}
$$

With (5.32), we obtain that

$$
\|\varepsilon L_{\varepsilon,\gamma,r} v\|_m \lesssim \|u\|_m + \|\varepsilon \nabla_{y,x} u\|_m\,. \tag{5.34}
$$

Similarly,

$$
e^{-\gamma t}\mathcal E'_\psi(u_\varepsilon, \psi_\varepsilon)\big(e^{\gamma t}\psi\big) = L^1_{\varepsilon,\gamma,t}\psi
$$

with

$$
L^1_{\varepsilon,t}\psi := -\frac{1}{\varepsilon}\left\{ \mathrm{C}^{\varepsilon,t}_0(\partial_t + \gamma)\psi + \sum_{j=1}^{d-1} \mathrm{C}^{\varepsilon,t}_j \partial_j \psi + \varepsilon \sum_{j,k=1}^{d-1} \mathrm{C}^{\varepsilon,t}_{j,k}\partial^2_{j,k}\psi \right\} \tag{5.35}
$$

and

$$\begin{aligned}
\mathrm{C}_0^{\varepsilon,t} &= \varepsilon\partial_x u_\varepsilon\,,\\
\mathrm{C}_j^{\varepsilon,t} &= \varepsilon\partial_x f_j(u_\varepsilon) - \sum_{k=1}^d \varepsilon^2\partial_x\big(B_{j,k}^\varepsilon\partial_k u_\varepsilon\big) - \sum_{k=1}^d \varepsilon^2\partial_k\big(B_{k,j}^\varepsilon\partial_x u_\varepsilon\big)\,,\\
\mathrm{C}_{j,k}^{\varepsilon,t} &= -\varepsilon B_{j,k}^\varepsilon\partial_x u_\varepsilon\,.
\end{aligned}$$

We have factored out ε^{-1} in the right hand side of (5.35) in order to get bounded coefficients. Again, we split the coefficients into their principal part plus remainders, getting

$$L^1_{\varepsilon,\gamma,t} = L^1_{\varepsilon,\gamma} + \varepsilon L^1_{\varepsilon,\gamma,r}\,. \tag{5.36}$$

The coefficients of $\varepsilon L^1_\varepsilon$ are

$$\mathrm{C}_j^\varepsilon = C_j^\sharp(x/\varepsilon, q_\varepsilon)\,,\quad \mathrm{C}_{j,k}^\varepsilon = C_{j,k}^\sharp(x/\varepsilon, q_\varepsilon) \tag{5.37}$$

with

$$\begin{aligned}
\mathrm{C}_0^\sharp(z,q) &= \partial_z W\,,\\
\mathrm{C}_j^\sharp(z,q) &= \partial_z f_j(W) - \partial_z\big(B_{j,d}(W,\Psi)\partial_z W\big) - \partial_z\big(B_{d,j}(W,\Psi)\partial_z W\big)\,,\\
\mathrm{C}_{j,k}^\sharp(z,q) &= -B_{j,k}(W)\partial_z W\,.
\end{aligned}$$

Using Lemma 5.4 and taking account of the decay in x of the coefficients $C_j^{\varepsilon,r}$ and $C_{j,k}^{\varepsilon,r}$ of $\varepsilon L^1_{\varepsilon,r}$, we have therefore

$$\|\varepsilon L^1_{\varepsilon,r}\dot\psi\|_m \lesssim |\nabla_{t,y}\dot\psi|_m + |\varepsilon\nabla_y^2\dot\psi|_m\,. \tag{5.38}$$

We remark that all the coefficients of $L^1_{\varepsilon,\gamma}$ involve at least one derivative in z of W thus are exponentially decaying in x/ε. Therefore, for all $\delta>0$ and $\kappa\in C_0^\infty(]-\delta,\delta[)$ equal to one for $|x|\le\delta/2$, the coefficients $\varepsilon^{-1}(1-\kappa(x))\mathrm{C}_*^\varepsilon$ decay in x. Hence, there holds:

$$\|(1-\kappa)L^1_\varepsilon\dot\psi\|_m \lesssim |\nabla_{t,y}\dot\psi|_m + |\varepsilon\nabla_{y,x}^2\dot\psi|_m\,. \tag{5.39}$$

With (5.34) (5.38) and (5.39), the main estimate (5.27) follows from the next proposition.

Proposition 5.8. *There are $\delta>0$ and $\kappa\in C_0^\infty(]-\delta,\delta[)$ equal to one for $|x|\le\delta/2$, such that the estimate* (5.27) *holds for all $\varepsilon\in]0,1]$, all $\gamma\ge1$, all smooth u, f and ψ with compact support satisfying*

$$L_{\varepsilon,\gamma}u + \kappa L^1_{\varepsilon,\gamma}\psi = f\,, \tag{5.40}$$

$$[u(0)] = 0\,,\qquad [\partial_x u(0)] = 0\,, \tag{5.41}$$

$$(\partial_t + \gamma - \varepsilon\Delta_y)\psi + \ell_\varepsilon\cdot u(0) = 0\,. \tag{5.42}$$

We first prove the estimate for functions u and f supported in the strip $\{|x| \leq 2\delta\}$. There, we reduce the equation to a first order system in ∂_x. According to sections 2 and 3, taking advantage of the conservative form of the equations, a natural additional unknown would be of the form $w = \varepsilon B_\nu \partial_x u - Au$. However, the estimates above show an improved smoothness in $v = \varepsilon \partial_x u$, since they give control of $v/\sqrt{\varepsilon}$, and no improvement for w. This leads to introduce explicitly v from the beginning and to forget about the conservative form of the equations.

Thus, with $U = {}^t(u, v)$, the equation (5.40) reads

$$\partial_x U - \frac{1}{\varepsilon} G_{\varepsilon,\gamma} = F - \frac{1}{\varepsilon} G^1_\varepsilon \nabla_\gamma \psi \tag{5.43}$$

with

$$G_{\varepsilon,\gamma} = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{M}_{\varepsilon,\gamma} & \mathrm{A}_\varepsilon \end{pmatrix}, \quad G^1_\varepsilon = \begin{pmatrix} 0 \\ \kappa \mathrm{M}^1_\varepsilon \end{pmatrix}, \qquad F = \begin{pmatrix} 0 \\ (B^\varepsilon_{d,d})^{-1} f \end{pmatrix},$$

and

$$\mathrm{A}_\varepsilon = (B^\varepsilon_{d,d})^{-1} \Big\{ \mathrm{A}^\varepsilon_d - \sum_{j<d} \big(B^\varepsilon_{j,d} + B^\varepsilon_{d,j} \big) \varepsilon \partial_j \Big\},$$

$$\mathrm{M}_{\varepsilon,\gamma,t} = (B^\varepsilon_{d,d})^{-1} \Big\{ \mathrm{E}^\varepsilon + \varepsilon(\partial_t + \gamma) + \sum_{j<d} \mathrm{A}^\varepsilon_j \varepsilon \partial_j - \sum_{j,k<d} B^\varepsilon_{j,k} \varepsilon^2 \partial_j \partial_k \Big\}.$$

Moreover, $\nabla_\gamma \psi = \big((\partial_t + \gamma)\psi, \partial_{y_1}\psi, \ldots, \partial_{d-1}\psi \big)$ and

$$\mathrm{M}^1_\varepsilon \nabla_\gamma \psi = \mathrm{M}^1_{\varepsilon,0} (\partial_t + \gamma)\psi + \sum_{j=1}^{d-1} \mathrm{M}^1_{\varepsilon,j} \partial_{y_1} \psi \tag{5.44}$$

with

$$\mathrm{M}^1_{\varepsilon,0} = (B^\varepsilon_{d,d})^{-1} \mathrm{C}^\varepsilon_0, \qquad \mathrm{M}^1_{\varepsilon,j} = (B^\varepsilon_{d,d})^{-1} \Big(\mathrm{C}^\varepsilon_j + \sum_{k=1}^{d-1} \mathrm{C}^\varepsilon_{j,k} \varepsilon \partial_k \Big) \quad \text{for } j \geq 1.$$

The boundary condition (5.41) is replaced by

$$[U(0)] = 0. \tag{5.45}$$

We have to prove that for (U, ψ, F) satisfying (5.43), (5.45), and (5.42), with U and F supported in $\{|x| \leq 2\delta\}$, there holds

$$\begin{aligned} &\|u\|_{m,(\lambda^2)} + \frac{1}{\sqrt{\varepsilon}} \|v\|_{m,(\lambda)} + |u(0)|_{m,(\lambda\sqrt{\Lambda})} \\ &\qquad\qquad + |v(0)|_{m,(\lambda/\sqrt{\Lambda})} + |\psi|_{m,(\mu\Lambda^2)} \lesssim RHS \end{aligned} \tag{5.46}$$

where RHS denotes the right hand side of (5.27).

5.3 Paralinearisation

Introduce the symbols of the operators defined after (5.43):

$$
\begin{aligned}
\mathrm{A}_\varepsilon &= a_\varepsilon(t,y,x,\varepsilon D_y)\,,\\
\mathrm{M}_\varepsilon &= m_\varepsilon(t,y,x,\varepsilon D_t,\varepsilon D_y,\varepsilon\gamma)\\
\mathrm{M}^1_{\varepsilon,j} &= m^1_{\varepsilon,j}(t,y,x,\varepsilon D_y)\,.
\end{aligned}
\tag{5.47}
$$

The symbols $a_\varepsilon(t,y,x,\zeta)$, $m_\varepsilon(t,y,x,\zeta)$ and $m^1_{\varepsilon,j}(t,y,x,\zeta)$ are polynomials of degree one, two and one respectively in $\zeta = (\tau,\eta,\gamma)$ and $D_y = -i\partial_y$, $D_t = -i\partial_t$. With (5.31) (5.37), they are functions of x/ε and q_ε:

$$
\begin{aligned}
a_\varepsilon(t,y,x,\zeta) &= \mathcal{A}\big(\frac{x}{\varepsilon}, q_\varepsilon(t,y,x),\zeta\big)\\
m_\varepsilon(t,y,x,\zeta) &= \mathcal{M}\big(\frac{x}{\varepsilon}, q_\varepsilon(t,y,x),\zeta\big)\\
m^1_{\varepsilon,j}(t,y,x,\zeta) &= \kappa(x)\mathcal{M}^1_j\big(\frac{x}{\varepsilon}, q_\varepsilon(t,y,x),\zeta\big)\,.
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{A}(z,q,\zeta) &= (B^\sharp_{d,d})^{-1}\Big(\mathrm{A}^\sharp_d - \sum_{j=1}^{d-1} i\eta_j(B^\sharp_{j,d}+B^\sharp_{d,j})\Big)\\
\mathcal{M}(z,q,\zeta) &= (B^\sharp_{d,d})^{-1}\Big(E^\sharp + (i\tau+\gamma)\mathrm{Id} + \sum_{j=1}^{d-1} i\eta_j\mathrm{A}^\sharp_j + \sum_{j,k=1}^{d-1}\eta_j\eta_k B^\sharp_{j,k}\Big)\\
\mathcal{M}^1_0(z,q) &= (B^\sharp_{d,d})^{-1}\mathrm{C}^\sharp_0\,,\\
\mathcal{M}^1_j(z,q,\zeta) &= (B^\sharp_{d,d})^{-1}\Big(\mathrm{C}^\sharp_j - \sum_{k=1}^{d-1} i\eta_k\mathrm{C}^\sharp_{j,k}\Big)\quad \text{for } j\ge 1\,.
\end{aligned}
$$

Remark 5.9. The reader should remark that these notations are slightly different from but parallel to the notations used in section three : the reduction to first order (5.43) is different from, but of course equivalent to the reduction (3.3). The choice made above is better adapted to the formulation of sharp energy estimates, while the reduction used in section three was better adapted to the computation of Evans functions.

The estimates of Lemma 5.4 implies that

Lemma 5.10. *The symbols a_ε and m^1_ε are uniformly bounded in $\mathrm{P}\Gamma^1_{1,m}$. The m_ε are uniformly bounded in $\mathrm{P}\Gamma^2_{1,m}$.*

Thus, by Proposition 4.10 , we can compare the differential operators A_ε, M_ε and M^1_ε to their paradifferential quantization $P^{\varepsilon,\gamma}_{a_\varepsilon}$, $P^{\varepsilon,\gamma}_{m_\varepsilon}$, $P^{\varepsilon,\gamma}_{m^1_\varepsilon}$. We note that the norms $\| \cdot \|_{m,(\Lambda^s)}$ coincide with the norms called $\| \cdot \|_{m,s,\varepsilon,\gamma}$ in (4.16). Therefore Proposition 4.10 implies:

$$\begin{aligned}
&\big\| A_\varepsilon v - P^{\varepsilon,\gamma}_{a_\varepsilon} v \big\|_m \lesssim \varepsilon \|v\|_m \,, \\
&\big\| M_\varepsilon u - P^{\varepsilon,\gamma}_{m_\varepsilon} v \big\|_m \lesssim \varepsilon \|u\|_{m,(\Lambda)} \,, \\
&\big\| \kappa M^1_\varepsilon \nabla_\gamma \psi - P^{\varepsilon,\gamma}_{m^1_\varepsilon} \nabla_\gamma \psi \big\|_m \lesssim \varepsilon |\nabla_\gamma \psi|_m \,.
\end{aligned}$$

Parallel to (5.44), we have used the notation

$$P^{\varepsilon,\gamma}_{m^1_\varepsilon} \nabla_\gamma \psi = P^{\varepsilon,\gamma}_{m^1_{\varepsilon,0}} (\partial_t + \gamma)\psi + \sum_{j=1}^{d-1} P^{\varepsilon,\gamma}_{m^1_{\varepsilon,j}} \partial_j \psi \,. \tag{5.48}$$

Therefore, if (U, F, ψ) satisfy (5.43), there holds

$$\partial_x U - \frac{1}{\varepsilon} P^{\varepsilon,\gamma}_{g_\varepsilon} U = F' - \frac{1}{\varepsilon} P^{\varepsilon,\gamma}_{g^1_\varepsilon} \nabla_\gamma \psi$$

with

$$g_\varepsilon = \begin{pmatrix} 0 & \mathrm{Id} \\ m_\varepsilon & a_\varepsilon \end{pmatrix} \,, \quad g^1_\varepsilon = \begin{pmatrix} 0 \\ m^1_\varepsilon \end{pmatrix} \,, \quad F' = \begin{pmatrix} 0 \\ f' \end{pmatrix}$$

and

$$\|f'\|_m \lesssim \|f\|_m + \|u\|_{m,(\Lambda)} + \|v\|_m + |\nabla_\gamma \psi|_{m,(\Lambda)} \tag{5.49}$$

Consider next a function $\chi \in C^\infty_0(\mathbb{R}^{d+1})$ supported in the ball $|\zeta| \le 2\rho_0$ and equal to one on the ball $|\zeta| \le \rho_0$. Then the commutators of $P^{\varepsilon,\gamma}_\chi$ with $P^{\varepsilon,\gamma}_{g_\varepsilon}$ and $P^{\varepsilon,\gamma}_{m^1_\varepsilon}$ are of order $O(\varepsilon)$ and therefore

$$\partial_x P^{\varepsilon,\gamma}_\chi U - \frac{1}{\varepsilon} P^{\varepsilon,\gamma}_{g_\varepsilon} P^{\varepsilon,\gamma}_\chi U = F' - \frac{1}{\varepsilon} P^{\varepsilon,\gamma}_{g^1_\varepsilon} \nabla_\gamma P^{\varepsilon,\gamma}_\chi \psi \tag{5.50}$$

with a new function $F' = {}^t(0, f')$ which still satisfies (5.49). Similarly,

$$\partial_x P^{\varepsilon,\gamma}_{1-\chi} U - \frac{1}{\varepsilon} P^{\varepsilon,\gamma}_{g_\varepsilon} P^{\varepsilon,\gamma}_{1-\chi} U = F' - \frac{1}{\varepsilon} P^{\varepsilon,\gamma}_{g^1_\varepsilon} \nabla_\gamma P^{\varepsilon,\gamma}_{1-\chi} \psi \tag{5.51}$$

with $F' = {}^t(0, f')$ which satisfies (5.49).

Next, we paralinearize the boundary conditions. Clearly, the jump conditions are preserved:

$$[P_\chi^{\varepsilon,\gamma} U] = 0\,, \qquad [P_{1-\chi}^{\varepsilon,\gamma} U] = 0\,. \tag{5.52}$$

We now investigate the extra boundary condition. By Assumption 5.2, ℓ_ε is bounded in $W^{m+2,\infty}$ and by (4.26) there holds

$$\big|\ell_\varepsilon \cdot u(0) - P_{\ell_\varepsilon}^{\varepsilon,\gamma} u(0)\big|_{m,(\Lambda^{3/2})} \lesssim \varepsilon \big|u(0)\big|_{m,(\Lambda^{1/2})}\,.$$

Thus, the extra boundary condition (5.42) implies that

$$(\partial_t + \gamma - \varepsilon\Delta_y)\psi + P_{\ell_\varepsilon}^{\varepsilon,\gamma} u(0) = e\,,$$

with

$$\big|e\big|_{m,(\Lambda^{3/2})} \lesssim \varepsilon\big|u(0)\big|_{m,(\Lambda^{1/2})}\,. \tag{5.53}$$

Commuting with $P_\chi^{\varepsilon,\gamma}$, implies that

$$(\partial_t + \gamma - \varepsilon\Delta_y) P_\chi^{\varepsilon,\gamma}\psi + P_{\ell_\varepsilon}^{\varepsilon,\gamma} P_\chi^{\varepsilon,\gamma} u(0) = e'\,, \tag{5.54}$$

$$(\partial_t + \gamma - \varepsilon\Delta_y) P_{1-\chi}^{\varepsilon,\gamma}\psi + P_{\ell_\varepsilon}^{\varepsilon,\gamma} P_{1-\chi}^{\varepsilon,\gamma} u(0) = e''\,. \tag{5.55}$$

where e' and e'' satisfy (5.53).

5.4 The high and medium frequencies analysis

In this subsection, we prove estimates for $P_{1-\chi}^{\varepsilon,\gamma} U$. The strategy is to use Lemma 1.2 to reduce the fully linearized equation to the partially linearized equation for $u - \psi\partial_x u_\varepsilon$, which is well posed for $\varepsilon|\zeta| \geq c > 0$. Next the front ψ is recovered from the extra boundary condition.

Consider

$$\mathcal{P}^\sharp(z,q) = \widetilde{A}_d(W,\Psi)\partial_z W - \partial_z\Big(\widetilde{B}_{d,d}(W,\Psi)\partial_z W\Big)\,.$$

Recall from Lemma 2.13 the following identity:

$$\begin{aligned} -\partial_z^3 W + \mathcal{A}(z,q,\zeta)\partial_z^2 W + &\mathcal{M}(z,q,\zeta)\partial_z W \\ &= \mathcal{L}^1(z,q,\zeta) + \big(\widetilde{B}_{d,d}(W,\Psi)\big)^{-1}\partial_z\mathcal{P}^\sharp \end{aligned} \tag{5.56}$$

with

$$\mathcal{L}^1(z,q,\zeta) = (i\tau+\gamma)\mathcal{M}_0^1(z,q) + \sum_{j=1}^{d-1} i\eta_j \mathcal{M}_j^1(z,q,\zeta)\,.$$

When $q = (p, 0, 0)$ and $p \in \mathcal{C}$, W is an exact solution of the profile equation $\mathcal{P}(z, q) = 0$ and therefore $\partial_z \mathcal{P}(z, q) = 0$. Thus:

$$\mathcal{E}^\sharp(z,q) := \big(\widetilde{B}_{d,d}(W,\Psi)\big)^{-1}\partial_z\mathcal{P}^\sharp(z,q) = u^1 W_1(z,q) + h^1 W_2(z,q) \tag{5.57}$$

where W_1 and W_2 are smooth functions of their argument with exponential decay in z. Introduce the symbols

$$\begin{aligned} r_\varepsilon &= \kappa(x)\partial_z W\big(\tfrac{x}{\varepsilon}, q_\varepsilon(t,y,x)\big)\,,\\ l^1_\varepsilon &= \kappa(x)\mathcal{L}^1\big(\tfrac{x}{\varepsilon}, q_\varepsilon(t,y,x),\zeta\big)\,,\\ e^1_\varepsilon &= \varepsilon^{-1}\kappa(x)\mathcal{E}^\sharp\big(\tfrac{x}{\varepsilon}, q_\varepsilon(t,y,x)\big)\,. \end{aligned} \tag{5.58}$$

Lemma 5.11. *r_ε, $\varepsilon\partial_x r_\varepsilon$ and e^1_ε are bounded families in $\mathrm{P\Gamma}^0_{1,m}$; the l^1_ε are bounded in $\mathrm{P\Gamma}^2_{1,m}$ and*

$$e^2_\varepsilon := -\varepsilon\partial_x^2 r_\varepsilon + a_\varepsilon\partial_x r_\varepsilon + \frac{1}{\varepsilon}m_\varepsilon r_\varepsilon - \frac{1}{\varepsilon}l^1_\varepsilon \tag{5.59}$$

is bounded in $\mathrm{P\Gamma}^1_{1,m}$. Moreover, r_ε is smooth across $\{x = 0\}$.

Proof. The statement for r_ε and l^1_ε are clear. When performing the substitution $q = q_\varepsilon$ in (5.57), h^1 is replaced by $\varepsilon d\psi^1_\varepsilon$. Thus the substitution in $\varepsilon^{-1}h^1W_2$ yields symbols $\kappa(x)d\psi^1_\varepsilon(t,y)W_2(x/\varepsilon, q_\varepsilon)$ which are bounded in $\mathcal{W}^m$ as well as their tangential derivatives.

Next, the placeholder u^1 is replaced by u^1_ε. Using Taylor expansion, there holds:

$$u^1_\varepsilon(t,y,x) = u^1_\varepsilon(t,y,0) + x\tilde{u}^1_\varepsilon(t,y,x)\,.$$

By Assumption 5.2, $\varepsilon^{-1}u^1_\varepsilon(t,y,0)$ is bounded and contributes to a bounded term in $\mathcal{W}^m$.

The $\tilde{u}^1_\varepsilon$ are bounded in $\mathcal{W}^m$ and the exponential decay of W_1 implies that $(x/\varepsilon)W_1(x/\varepsilon, q_\varepsilon)$ is bounded in $\mathcal{W}^m$. Therefore, the symbols $\kappa(x)\tilde{u}^1_\varepsilon(x/\varepsilon)W_1(x/\varepsilon, q_\varepsilon)$ are also bounded in $\mathcal{W}^m$ as well as their tangential derivatives.

The identity (5.56) implies that the left hand side of (5.59) is e^1_ε plus terms which involve derivatives of κ:

$$-\varepsilon(\partial_x\kappa)(\partial_x r_\varepsilon) - \varepsilon(\partial_x^2\kappa)r_\varepsilon + a_\varepsilon(\partial_x\kappa)r_\varepsilon\,.$$

This last term is of degree one and bounded in $\mathrm{P\Gamma}^1_{1,m}$.

Next we recall from (5.7) that the profile W is of the form $W(q,z) = W_0(p,z) + u^1$ when $q = (p, u^1, h^1)$ and W_0 is given by Assumption 5.1. Therefore, $\partial_z W(q,z) = \partial_z W_0(p,z)$ implying that

$$r_\varepsilon(t,y,x) = \kappa(x)\partial_z W_0(p(t,y), x/\varepsilon).$$

This shows that r_ε is C^∞ in x and the lemma is proved. $\square$

Next, we come to the microlocal version of the change of unknowns $u - \psi\partial_x u_\varepsilon$, introduced in section two. Transposed to the first order system, this would yield to consider $U - \psi\,{}^t(\partial_x u_\varepsilon, \varepsilon\partial_x^2 u_\varepsilon)$. However, we replace the multiplication by $(\partial_x u_\varepsilon, \varepsilon\partial_x^2 u_\varepsilon)$ by the corresponding the para-product and we keep only the main part of u_ε, that is $W(q_\varepsilon, x/\varepsilon)$. This leads to consider

$$(5.60)\quad U_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} := P^{\varepsilon,\gamma}_{1-\chi} U - P^{\varepsilon,\gamma}_{R_\varepsilon} P^{\varepsilon,\gamma}_{1-\chi}\theta\,, \quad \theta = \frac{1}{\varepsilon}\psi\,, \quad R_\varepsilon = \begin{pmatrix} r_\varepsilon \\ \varepsilon\partial_x r_\varepsilon \end{pmatrix}.$$

Introduce next the symbols

$$(5.61)\qquad c_\varepsilon(t,y,\zeta) = i\tau + \gamma + |\eta|^2 + \ell_\varepsilon \cdot \partial_z W(q_\varepsilon(t,y,0), 0)\,.$$

The family $\{c_\varepsilon\}$ is bounded in $\mathrm{P}\Gamma^2_{m,1}$.

Proposition 5.12. *U_1 and θ satisfy*

$$(5.62)\qquad \partial_x U_1 - \frac{1}{\varepsilon} P^{\varepsilon,\gamma}_{g_\varepsilon} U_1 = F_1'\,, \qquad [U_1(0)] = 0\,,$$

$$(5.63)\qquad P^{\varepsilon,\gamma}_{c_\varepsilon} P^{\varepsilon,\gamma}_{1-\chi}\theta + P^{\varepsilon,\gamma}_{\ell_\varepsilon} u_1(0) = e'$$

where $F_1' = {}^t(0, f_1')$ satisfies (5.49) *and*

$$(5.64)\qquad \big|e'\big|_{m,(\Lambda^{3/2})} \lesssim \varepsilon\big|u(0)\big|_{m,(\Lambda^{1/2})} + \varepsilon\big|\nabla_\gamma\psi\big|_{m,(\Lambda^{1/2})}$$

Proof. When φ is independent of x, the symbolic calculus implies that

$$\partial_x P^{\varepsilon,\gamma}_{r_\varepsilon}\varphi = P^{\varepsilon,\gamma}_{\partial_x r_\varepsilon}\varphi\,.$$

Moreover,

$$l^1_\varepsilon = m^1_{\varepsilon,0}(i\tau + \eta) + \sum_{j=1}^{d-1} m^1_{\varepsilon,j} i\eta_j\,.$$

Since $P^{\varepsilon,\gamma}$ is a semiclassical quantization, $P^{\varepsilon,\gamma}_{i\tau+\gamma} = \varepsilon(\partial_t + \gamma)$ and $P^{\varepsilon,\gamma}_{i\eta_j} = \varepsilon\partial_j$. Therefore,

$$(5.65)\qquad P^{\varepsilon,\gamma}_{m^1_\varepsilon}\nabla_\gamma\varphi = \frac{1}{\varepsilon} P^{\varepsilon,\gamma}_{l^1_\varepsilon}\varphi\,.$$

Substituting the definition of U_1 in equation (5.51) yields:

$$\partial_x U_1 - \frac{1}{\varepsilon} P^{\varepsilon,\gamma}_{g_\varepsilon} U_1 = F' + \begin{pmatrix} 0 \\ \mathbf{e}_\varepsilon \end{pmatrix} P^{\varepsilon,\gamma}_{1-\chi}\theta$$

with

$$\mathbf{e}_\varepsilon = -\varepsilon P^{\varepsilon,\gamma}_{\partial_x^2 r_\varepsilon} + P^{\varepsilon,\gamma}_{a_\varepsilon} P^{\varepsilon,\gamma}_{\partial_x r_\varepsilon} + \frac{1}{\varepsilon} P^{\varepsilon,\gamma}_{m_\varepsilon} P^{\varepsilon,\gamma}_{r_\varepsilon} - \frac{1}{\varepsilon} P^{\varepsilon,\gamma}_{l^1_\varepsilon}\,.$$

All the symbols are compactly supported in x. Hence, the symbolic calculus and (5.59) imply that

$$\mathbf{e}_\varepsilon = P^{\varepsilon,\gamma}_{e^2_\varepsilon} + \mathbf{e}'_\varepsilon$$

where the remainder $\mathbf{e}'_\varepsilon$ satisfies

$$\left\|\mathbf{e}'_\varepsilon(x)\varphi\right\|_m \lesssim |\varphi|_{m,(\Lambda)}\ .$$

Since $P^{\varepsilon,\gamma}_{e^2_\varepsilon}$ satisfies similar estimates, there holds

$$\left\|\mathbf{e}_\varepsilon(x) P^{\varepsilon,\gamma}_{1-\chi}\theta\right\|_m \lesssim \left|P^{\varepsilon,\gamma}_{1-\chi}\theta\right|_{m,(\Lambda)} \lesssim \left|\varepsilon P^{\varepsilon,\gamma}_{1-\chi}\nabla_\gamma\theta\right|_{m,(\Lambda)} = |\nabla_\gamma\psi|_{m,(\Lambda)}\ .$$

For the second inequality, we have used that $1-\chi$ is supported away from the origin, so that $|(1-\chi(\varepsilon\zeta)| \lesssim \varepsilon|\zeta|$. This shows that the right hand side F'_1 in (5.62) satisfies (5.49).

In addition, by (5.11), r_ε is continuous across $x=0$, as well as its derivative in x. Therefore, the jump of $P^{\varepsilon,\gamma}_{R_\varepsilon} P^{\varepsilon,\gamma}_{1-\chi}\theta$ is equal to zero. With (5.51) this implies that $[U_1(0)] = 0$.

Next we note that

$$P^{\varepsilon,\gamma}_{1-\chi} u = u_1 + P^{\varepsilon,\gamma}_{r_\varepsilon} P^{\varepsilon,\gamma}_{1-\chi}\theta \quad \text{and} \quad \big(\varepsilon(\partial_t+\gamma) - \varepsilon^2\Delta_y\psi\big) P^{\varepsilon,\gamma}_{1-\chi} = P^{\varepsilon,\gamma}_{i\tau+\gamma+|\eta|^2} P^{\varepsilon,\gamma}_{1-\chi}\theta\,.$$

The symbolic calculus implies that

$$e''_1 := \big(P^{\varepsilon,\gamma}_{\ell_\varepsilon} P^{\varepsilon,\gamma}_{r_\varepsilon(0)} - P^{\varepsilon,\gamma}_{\ell_\varepsilon r_\varepsilon(0)}\big) P^{\varepsilon,\gamma}_{1-\chi}\theta$$

satisfies

$$\left|e''_1\right|_{m,(\Lambda^{3/2})} \lesssim \varepsilon\left|P^{\varepsilon,\gamma}_{1-\chi}\theta\right|_{m,(\Lambda^{1/2})} = \left|P^{\varepsilon,\gamma}_{1-\chi}\psi\right|_{m,(\Lambda^{1/2})} \lesssim \left|\varepsilon\nabla_\gamma\psi\right|_{m,(\Lambda^{1/2})}\,.$$

Here, $r_\varepsilon(0)$ denotes the symbol r_ε evaluated at $x=0$. The boundary condition (5.55) and the definitions of r_ε and c_ε imply that

$$P^{\varepsilon,\gamma}_{c_\varepsilon} P^{\varepsilon,\gamma}_{1-\chi}\theta + P^{\varepsilon,\gamma}_{\ell_\varepsilon} u_1(0) = e'' + e''_1\,.$$

With the estimate (5.53) for e'', this implies (5.64). □

We now come to the main argument. In [MZ1] it was part of the argument to reduce the high and medium frequency regimes to problems similar to (5.62):

$$\partial_x P^{\varepsilon,\gamma}_{1-\chi} U - \frac{1}{\varepsilon} P^{\varepsilon,\gamma}_{g_\varepsilon} P^{\varepsilon,\gamma}_{1-\chi} U = F'$$

on $\{x \geq 0\}$, with Dirichlet boundary condition $P^{\varepsilon,\gamma}_{1-\chi} u(0) = 0$, assuming that the corresponding rescaled Evans function is bounded from below by a positive constant. The analysis clearly extends to general boundary condition, and to the transmission problem for (5.63) (see [GMWZ2]). By Assumption 5.2 and (5.12), the problem (5.62) satisfies the uniform stability condition for $\zeta \neq 0$, for parameters q in $\mathcal{Q}_0$. For $|x|$ and ε small enough, $q_\varepsilon(t, y, x)$ remains in $\mathcal{Q}_0$ and therefore, Propositions 4.6 and 4.10 of [MZ1] imply the following maximal estimates:

Theorem 5.13. *There are $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that if $\delta \in]0, \delta_0]$, then for all $\rho_0 > 0$, all cut off functions $\kappa(x)$ and $\chi(\zeta)$ as above, all $\varepsilon \in]0, \varepsilon_0]$, and all (u, f, ψ) satisfying* (5.40)(5.41) *with u and f supported in $|x| \leq 2\delta$, the function U_1 defined by* (5.60) *satisfies*

$$\frac{1}{\varepsilon} \|u_1\|_{m,(\Lambda^2)} + \frac{1}{\varepsilon} \|v_1\|_{m,(\Lambda)} + \frac{1}{\sqrt{\varepsilon}} |u_1(0)|_{m,(\Lambda^{3/2})} + \frac{1}{\sqrt{\varepsilon}} |v_1(0)|_{m,(\Lambda^{1/2})} \lesssim RHS \tag{5.66}$$

where RHS denotes the right hand side of (5.27).

Next, knowing U_1, we can estimate θ using the extra boundary condition (5.63).

Proposition 5.14. *With assumptions as in Theorem* 5.13*, for ε is small enough, there holds*

$$\varepsilon^{-3/2} \big| P^{\varepsilon,\gamma}_{1-\chi} \psi \big|_{m,(\Lambda^{7/2})} = \varepsilon^{-1/2} \big| P^{\varepsilon,\gamma}_{1-\chi} \theta \big|_{m,(\Lambda^{7/2})} \lesssim RHS\,. \tag{5.67}$$

Proof. By Assumptions 5.2, there holds

$$\|\ell_\varepsilon \cdot \partial_z W(q_\varepsilon, 0) - \ell(p(\cdot)) \cdot \partial_z (p(\cdot), 0)\|_{L^\infty} \to 0 \quad \text{as} \quad \varepsilon \to 0\,.$$

Since the range of $p(\cdot)$ is contained in a compact subset of $\mathcal{C}$, Assumption 5.2 implies that for ε small enough,

$$1 \lesssim \ell_\varepsilon \cdot \partial_z W(q_\varepsilon, 0) \tag{5.68}$$

and therefore

$$1+|\tau|+\gamma+|\eta|^2 \lesssim |c_\varepsilon|\,.$$

Thus c_ε is elliptic and $1/c_\varepsilon \in \mathrm{P}\Gamma^{-2}_{1,m}$. We use standard ellipticity arguments for the equation $P^{\varepsilon,\gamma}_{c_\varepsilon}\varphi = g$: multiplying the equation by $P^{\varepsilon,\gamma}_{1/c_\varepsilon}$ and using the symbolic calculus implies that

$$\big|P^{\varepsilon,\gamma}_{1-\chi}\theta\big|_{m,(\Lambda^{7/2})} \lesssim \big|P^{\varepsilon,\gamma}_{c_\varepsilon}P^{\varepsilon,\gamma}_{1-\chi}\theta\big|_{m,(\Lambda^{3/2})} + \varepsilon\big|P^{\varepsilon,\gamma}_{1-\chi}\theta\big|_{m,(\Lambda^{3/2})}\,.$$

Together with equation (5.63), this implies

$$\big|P^{\varepsilon,\gamma}_{1-\chi}\theta\big|_{m,(\Lambda^{7/2})} \lesssim \big|P^{\varepsilon,\gamma}_{1-\chi}u_1(0)\big|_{m,(\Lambda^{3/2})} + \big|e'\big|_{m,(\Lambda^{3/2})} + \varepsilon\big|P^{\varepsilon,\gamma}_{1-\chi}\theta\big|_{m,(\Lambda^{3/2})}\,.$$

By (5.66) and by (5.64) the first two terms in the right hand side are dominated by $\sqrt{\varepsilon}RHS$ and εRHS respectively. Moreover,

$$\big|P^{\varepsilon,\gamma}_{1-\chi}\theta\big|_{m,(\Lambda^{3/2})} = \varepsilon^{-1}\big|P^{\varepsilon,\gamma}_{1-\chi}\psi\big|_{m,(\Lambda^{3/2})} \lesssim \big|\nabla_\gamma\psi\big|_{m,(\Lambda^{1/2})} \lesssim RHS\,.$$

Therefore $\big|P^{\varepsilon,\gamma}_{1-\chi}\theta\big|_{m,(\Lambda^{7/2})} \lesssim \sqrt{\varepsilon}RHS$ and the proposition is proved. $\square$

Knowing bounds U_1 and ψ, we can estimate U:

Theorem 5.15. *With notations as above, if δ and ε are small enough, then*

$$\frac{1}{\varepsilon}\left\|P^{\varepsilon,\gamma}_{1-\chi}u\right\|_{m,(\Lambda^2)} + \frac{1}{\varepsilon}\left\|P^{\varepsilon,\gamma}_{1-\chi}v\right\|_{m,(\Lambda)} + \frac{1}{\sqrt{\varepsilon}}\left|P^{\varepsilon,\gamma}_{1-\chi}u(0)\right|_{m,(\Lambda^{3/2})}$$
$$+\frac{1}{\sqrt{\varepsilon}}\left|P^{\varepsilon,\gamma}_{1-\chi}v(0)\right|_{m,(\Lambda^{1/2})} + \frac{1}{\varepsilon^{3/2}}\left|P^{\varepsilon,\gamma}_{1-\chi}\psi\right|_{m,(\Lambda^{7/2})} \lesssim RHS\,.$$

Proof. Adding up the estimates (5.66) and (5.67), one obtains:

$$\frac{1}{\varepsilon}\|u_1\|_{m,(\Lambda^2)} + \frac{1}{\varepsilon}\|v_1\|_{m,(\Lambda)} + \frac{1}{\sqrt{\varepsilon}}|u_1(0)|_{m,(\Lambda^{3/2})}$$
$$+\frac{1}{\sqrt{\varepsilon}}|v_1(0)|_{m,(\Lambda^{1/2})} + \frac{1}{\varepsilon^{3/2}}\big|P^{\varepsilon,\gamma}_{1-\chi}\psi\big|_{m,(\Lambda^{7/2})} \lesssim RHS\,.$$

Next, we switch back to $P^{\varepsilon,\gamma}_{1-\chi}U = U_1 + \frac{1}{\varepsilon}P^{\varepsilon,\gamma}_{R_\varepsilon}P^{\varepsilon,\gamma}_{1-\chi}\psi$. The estimate of Theorem 5.15 follows from the estimate above and

$$\frac{1}{\varepsilon^2}\big\|P^{\varepsilon,\gamma}_{R_\varepsilon}P^{\varepsilon,\gamma}_{1-\chi}\psi\big\|_{m,(\Lambda^2)} \lesssim \frac{1}{\varepsilon^{3/2}}\big|P^{\varepsilon,\gamma}_{1-\chi}\psi\big|_{m,(\Lambda^{7/2})}\,. \tag{5.69}$$

Indeed, we note that the symbol R_ε has the special form

$$R_\varepsilon(t,y,x,\zeta) = \widetilde{R}_\varepsilon(\frac{x}{\varepsilon},t,y,x,\zeta)$$

with $\widetilde{R}_\varepsilon$ exponentially decaying in z, as well as all its derivatives. Recall that in the definition (4.17) of $P^{\varepsilon,\gamma}$, x is a parameter. Therefore, for all x:

$$\big|(P^{\varepsilon,\gamma}_{R_\varepsilon}\varphi)(x)\big|_{0,(\Lambda^2)} \lesssim e^{-\delta|x|/\varepsilon}\big|\varphi\big|_{0,(\Lambda^2)}\,.$$

Taking the L^2 norm in x inplies

$$\big\|P^{\varepsilon,\gamma}_{R_\varepsilon}\varphi\big\|_{0,(\Lambda^2)} \lesssim \sqrt{\varepsilon}\big|\varphi\big|_{0,(\Lambda^2)}\,.$$

Commuting with Z-derivatives, one obtains:

$$\big\|P^{\varepsilon,\gamma}_{R_\varepsilon}\varphi\big\|_{m,(\Lambda^2)} \lesssim \sqrt{\varepsilon}\big|\varphi\big|_{m,(\Lambda^2)}\,. \tag{5.70}$$

This imples (5.69) and the theorem is proved. □

5.5 The low frequency analysis

In this subsection, we prove estimates for $P^{\varepsilon,\gamma}_{\chi}U$. Following the analysis of the modified Evans function in section three, the idea is to use for ζ small, a new symbol $\tilde r_\varepsilon(t,y,x,\zeta)$, vanishing at $\zeta=0$ but still satisfying (5.59), that is, such that:

$$\tilde e_\varepsilon := -\varepsilon\partial_x^2\tilde r_\varepsilon + a_\varepsilon\partial_x\tilde r_\varepsilon + \varepsilon^{-1}m_\varepsilon\tilde r_\varepsilon - \varepsilon^{-1}l^1_\varepsilon \in \mathrm{P}\Gamma^1_{0,m}\,.$$

The choice $r_\varepsilon = \partial_z W(q_\varepsilon, z)$ made in the previous subsection satisfied for ε small (see (5.68)):

$$[r_\varepsilon(0)] := r^+_{\varepsilon\,|x=0} - r^-_{\varepsilon\,|x=0} = 0\,,\qquad \ell_\varepsilon\cdot r_\varepsilon(0) > 0\,. \tag{5.71}$$

As in section 3.3, we now require that the symbol $\tilde r_\varepsilon$ vanishes at $\zeta=0$, which forces to relax the jump condition.

Recall that l^1_ε given by (5.58) satisfies:

$$l^1_\varepsilon = (i\tau+\gamma)m^1_{\varepsilon,0} + \sum_{j=1}^{d-1} i\eta_j m^1_{\varepsilon,j}\,. \tag{5.72}$$

It depends upon the choice of a cut-off function κ supported in $\{|x|\leq 2\delta\}$.

Lemma 5.16. *There are $\varepsilon_0>0$, $\rho_0>0$, $\delta>0$ such that for κ supported in $\{|x|\leq 2\delta\}$ as above, there are bounded families of symbols $s_{\varepsilon,j}\in\mathrm{P}\Gamma^0_{1,m}$, for $\varepsilon\in]0,\varepsilon_0]$, defined for $|\zeta|\leq 2\rho_0$ and supported in $\{|x|\leq 2\delta\}$, such that*

$$\tilde e^{\pm}_{\varepsilon,j} := -\varepsilon\partial_x^2 s_{\varepsilon,j} + a_\varepsilon\partial_x s_{\varepsilon,j} + \varepsilon^{-1}m_\varepsilon s_{\varepsilon,j} - \varepsilon^{-1}m^1_{\varepsilon,j}$$

is bounded in $P\Gamma^0_{m,0}$ *on both side* $\pm x \ge 0$ *and*

$$\ell_\varepsilon \cdot s^{\pm}_{\varepsilon,j\,|x=0} = -c_{0,j}\,,$$

with $c_{0,0} = 1$ *and* $c_{0,j} = -i\eta_j$ *for* $j \in \{1, \dots, d-1\}$.

Proof. Lemma 3.14 implies that for all $\underline{q} = (\underline{p}, 0, 0)$ with $\underline{p} \in \mathcal{C}$, there is $\rho_0 > 0$ such that for $|\zeta| \le 2\rho_0$ there are functions $\mathcal{S}^{\pm}_j(z, q, \zeta)$, defined and smooth for $\{\pm z \ge 0\}$, q in a neighborhood of $\underline{q} \in \overline{\mathcal{Q}}_0$, exponentially decaying at infinity as well as their derivative, and such that

$$\begin{cases} -\partial_z^2 \mathcal{S}^{\pm}_j + \mathcal{A}\partial_z \mathcal{S}^{\pm}_j + \mathcal{M}\mathcal{S}^{\pm}_j = \mathcal{M}^1_j(z, q, \zeta) \quad \text{on } \pm z \ge 0 \\ \ell(q) \cdot \mathcal{S}^{\pm}_j(0, q, \zeta) = -c_{0,j}(\zeta), \qquad \mathcal{S}_j(z, q, 0) = 0\,. \end{cases}$$

These identities are linear in $\mathcal{S}^{\pm}_j$. Thus, using a partition of unity and compactness of $\mathcal{C}_0$, we can assume that the $\mathcal{S}^{\pm}_j$ are defined for $q \in \overline{\mathcal{Q}}_0$ and $\rho \le \rho_0$, with ρ_0 small enough.

For $\delta > 0$ small and $|x| \le 2\delta$, $q_\varepsilon(t, y, x)$ remains in $\mathcal{Q}_0$. With $\kappa(x)$ supported in $\{|x| \le 2\delta\}$ consider for $|\zeta| \le 2\rho_0$

$$s^{\pm}_{\varepsilon,j} = \kappa(x)\mathcal{S}^{\pm}_j\big(\frac{x}{\varepsilon}, q_\varepsilon(t, y, x), \zeta\big)\,. \tag{5.73}$$

The equations for $\mathcal{S}^{\pm}_j$ imply that the error term $\widetilde{e}_{\varepsilon,j}$ is bounded in $P\Gamma^0_{m,0}$ (note that the order has no significance since we consider bounded frequencies ζ). $\square$

Consider $\chi \in C^\infty_0(\mathbb{R}^{d+1})$ supported in $|\zeta| < 2\rho_0$ and such that $\chi = 1$ for $|\zeta| \le 1$. Introduce next $\chi_1 \in C^\infty_0(\mathbb{R}^{d+1})$ supported in $|\zeta| < 2\rho_0$ and such that $\chi_1\chi = \chi$. Consider the matrices

$$S^{\pm}_{\varepsilon,j} = \begin{pmatrix} s^{\pm}_{\varepsilon,j}\chi_1 \\ \varepsilon\partial_x s^{\pm}_{\varepsilon,j}\chi_1 \end{pmatrix} \tag{5.74}$$

They are bounded families of symbols in $P\Gamma^0_{1,m}$, compactly supported in ζ (so that the order 0 is unessential). Next, we introduce the jumps

$$[S_{\varepsilon,j}](t, y, \zeta) = S^{+}_{\varepsilon,j}(t, y, 0, \zeta) - S^{-}_{\varepsilon,j}(t, y, 0, \zeta)\,. \tag{5.75}$$

They are bounded families of symbols in $P\Gamma^0_{1,m}$ on the boundary, compactly supported in ζ.

Consider again (u, f, ψ) satisfying the equations (5.40), (5.41), and (5.42), with u and f supported in $\{|x| \leq 2\delta\}$. Introduce $U = {}^t(u, \varepsilon\partial_x u)$ and $F = {}^t(0, f)$ as in (5.43) and

$$U_1^\pm = \begin{pmatrix} u_1^\pm \\ v_1^\pm \end{pmatrix} = P_\chi^{\varepsilon,\gamma} U^\pm - P_{S_\varepsilon^\pm}^{\varepsilon,\gamma} P_\chi^{\varepsilon,\gamma} \nabla_\gamma \psi \,, \tag{5.76}$$

where

$$P_{S_\varepsilon^\pm}^{\varepsilon,\gamma} P_\chi^{\varepsilon,\gamma} \nabla_\gamma \psi = P_{S_{\varepsilon,0}^\pm}^{\varepsilon,\gamma} P_\chi^{\varepsilon,\gamma} (\partial_t + \gamma)\psi + \sum_{j=1}^{d-1} P_{S_{\varepsilon,j}^\pm}^{\varepsilon,\gamma} P_\chi^{\varepsilon,\gamma} \partial_{y_j} \psi \,.$$

Proposition 5.17. *U_1 satisfies*

$$\partial_x U_1^\pm - \frac{1}{\varepsilon} P_{g_\varepsilon}^{\varepsilon,\gamma} U_1^\pm = F'^\pm \quad on \quad \{\pm x > 0\}\,, \tag{5.77}$$

where $F' = {}^t(0, f')$ satisfies (5.49)*. Moreover, on $\{x = 0\}$, there holds*

$$[U_1(0)] + P_{[S_\varepsilon]}^{\varepsilon,\gamma} P_\chi^{\varepsilon,\gamma} \nabla_\gamma \psi = 0\,, \quad P_{\ell_\varepsilon}^{\varepsilon,\gamma} u_1^-(0) = e'' \tag{5.78}$$

where

$$\left|e''\right|_{m,(\lambda)} \lesssim \left|U(0)\right|_m + \left|\nabla_\gamma \psi\right|_m . \tag{5.79}$$

Proof. The symbolic calculus and Lemma 5.16 imply that

$$(\partial_x - \frac{1}{\varepsilon} P_{g_\varepsilon}^{\varepsilon,\gamma}) P_{S_\varepsilon}^{\varepsilon,\gamma} P_\chi^{\varepsilon,\gamma} \nabla_\gamma \psi + \frac{1}{\varepsilon} P_{g_\varepsilon^1}^{\varepsilon,\gamma} P_\chi^{\varepsilon,\gamma} \nabla_\gamma \psi = \begin{pmatrix} 0 \\ f'' \end{pmatrix}$$

with

$$\|f''\|_m \lesssim \left|\nabla_\gamma \psi\right|_m .$$

With (5.50), this implies (5.77).

The first boundary conditions immediately follows from (5.52). Introducing the matrix $C_0 = (c_{0,0}, \ldots, c_{0,d-1})$, there holds

$$(\partial_t + \gamma - \varepsilon\Delta_y) P_\chi^{\varepsilon,\gamma} = P_{C_0}^{\varepsilon,\gamma} P_\chi^{\varepsilon,\gamma} \nabla_\gamma \psi \,.$$

The boundary conditions for $s_{\varepsilon,j}$ and the symbolic calculus imply that

$$P_{\ell_\varepsilon}^{\varepsilon,\gamma} P_{S_\varepsilon^- \,|x=0}^{\varepsilon,\gamma} P_\chi^{\varepsilon,\gamma} \nabla_\gamma \psi = -(\partial_t + \gamma - \varepsilon\Delta_y) P_\chi^{\varepsilon,\gamma} \psi + e_1'$$

where $|e_1'|_m \leq |\nabla_\gamma \psi|_m$. Therefore, $P_{\ell_\varepsilon}^{\varepsilon,\gamma} u_1^-(0) = e' + e_1'$ where e' is the right hand side of the boundary condition (5.54). Next we remark that both e'

and e_1' are spectrally supported in a domain where $\varepsilon\zeta$ and hence $\sqrt{\varepsilon}\lambda$, are bounded. Therefore,

$$|e' + e_1'|_{m,(\lambda)} \lesssim \frac{1}{\sqrt{\varepsilon}} |e' + e_1'|_m$$

and the estimate (5.79) follows from (5.53). $\square$

The Evans function for the problem (5.77) with boundary conditions (5.78) is the determinant called D' defined at (3.52). Lemma 5.3 and Proposition 3.15 imply that this determinant is bounded from below by a positive constant for $\zeta \in \mathbb{R}_+^{1+d}$ with $|\zeta| \le \rho_0$. In [MZ1] we proves maximal estimates for the equation (5.77) on $\{x \ge 0\}$ with Dirichlet boundary conditions, $u_1^+(0) = 0$. In the Appendix we show that the analysis extends to the present case. The analogue of Propositions 4.14 [MZ1] is:

Proposition 5.18. *Suppose that δ and $\rho_0 > 0$ are small enough. Consider cut off functions κ , χ and χ_1 as above and χ' such that $\chi\chi' = \chi$ and $\chi'\chi_1 = \chi'$. Then, there holds*

$$\begin{aligned} \Big\| P_{\chi'}^{\varepsilon,\gamma} u_1 \Big\|_{m,(\lambda^2)} &+ \frac{1}{\sqrt{\varepsilon}} \Big\| P_{\chi'}^{\varepsilon,\gamma} v_1 \Big\|_{m,(\lambda)} + \Big| P_{\chi'}^{\varepsilon,\gamma} u_1(0) \Big|_{m,(\lambda)} \\ &+ \Big| P_{\chi'}^{\varepsilon,\gamma} v_1(0) \Big|_{m,(\lambda)} + \big| P_{\chi}^{\varepsilon,\gamma} \nabla_\gamma \psi \big|_{m,(\lambda)} \lesssim RHS \end{aligned} \tag{5.80}$$

where RHS denotes the right hand side of (5.27)*.*

Here, the traces $U_1(0)$ stand for the pair $\big(U_1^-(0), U_1^+(0)\big)$ of the traces from both side $\pm x > 0$.

Knowing bounds for U_1 and $\nabla_\gamma \psi$ immediately provides estimates for U.

Theorem 5.19. *With notations as above, if δ and ε are small enough, then*

$$\begin{aligned} \big\| P_{\chi}^{\varepsilon,\gamma} u \big\|_{m,(\lambda^2)} &+ \frac{1}{\sqrt{\varepsilon}} \big\| P_{\chi}^{\varepsilon,\gamma} v \big\|_{m,(\lambda)} + \big| P_{\chi}^{\varepsilon,\gamma} u(0) \big|_{m,(\lambda)} \\ &+ \big| P_{\chi}^{\varepsilon,\gamma} v(0) \big|_{m,(\lambda)} + \big| P_{\chi}^{\varepsilon,\gamma} \nabla_\gamma \psi \big|_{m,(\lambda)} \lesssim RHS\,. \end{aligned}$$

Proof. By (5.76),

$$P_{\chi}^{\varepsilon,\gamma} U = P_{\chi'}^{\varepsilon,\gamma} U_1 + P_{\chi'}^{\varepsilon,\gamma} P_{S_\varepsilon \chi_1}^{\varepsilon,\gamma} \theta \quad \text{with} \quad \theta := P_{\chi}^{\varepsilon,\gamma} \nabla_\gamma \psi.$$

Therefore, it is sufficient to check that

$$\big\|P^{\varepsilon,\gamma}_{s_\varepsilon\chi_1}\theta\big\|_{m,(\lambda^2)} + \frac{1}{\sqrt{\varepsilon}}\big\|P^{\varepsilon,\gamma}_{\varepsilon\partial_x s_\varepsilon\chi_1}\theta\big\|_{m,(\lambda)} \lesssim \big|\theta\big|_{m,(\lambda)},$$

The spectrum of θ, and thus of $P^{\varepsilon,\gamma}_{s_\varepsilon\chi_1}\theta$ is contained in a ball $\{|\varepsilon\zeta| \le \rho_2\}$ where $\sqrt{\varepsilon}\lambda \le \varepsilon^{-1/2}$ is bounded. Thus

$$\big\|P^{\varepsilon,\gamma}_{S_\varepsilon\chi_1}\theta\big\|_{m,(\lambda^2)} \lesssim \frac{1}{\sqrt{\varepsilon}}\big\|P^{\varepsilon,\gamma}_{S_\varepsilon\chi_1}\theta\psi\big\|_{m,(\lambda)}.$$

It remains to prove that

$$\big\|P^{\varepsilon,\gamma}_{r_\varepsilon}\theta\big\|_{m,(\lambda)} \lesssim \sqrt{\varepsilon}\big|\theta\big|_{m,(\lambda)}.$$

when r_ε is either $s_\varepsilon\chi_1$ or $\varepsilon\partial_x s_\varepsilon\chi_1$. The proof of this estimate is similar to the proof of (5.70), using that r_ε is exponentially decaying in $|x|/\varepsilon$. □

Proof of Theorem 5.5. We prove Proposition 5.8, as it implies the theorem. Combining Theorems 5.15 and 5.19, we see that the estimate (5.27) is satisfied when u is supported in a small strip $|x| \le 2\delta$. Next, for u supported in $|x| \ge \delta$ satisfying

$$L_\varepsilon(\partial_t + \gamma, \partial_y, \partial_x)u = f$$

Proposition 5.5 of [MZ1] implies that the estimate (5.27) is also satisfied.

□

6 Approximate solutions

In this section we adapt the construction of high order approximate solutions in [GW] for the case of Laplacian viscosity to the more general viscosities considered in this paper. A precise statement of the properties of the approximate solutions is given in Proposition 6.8. We continue to denote coordinates by $(t, x_1, \ldots, x_d) = (t, y, x)$.

We seek an approximate solution $(u^a_\epsilon, \psi^a_\epsilon)$ to the $N \times N$ system

$$u_t + \sum_{j=1}^{d} \partial_j f_j(u) - \epsilon \sum_{j,k=1}^{d} \partial_j(B_{j,k}(u)\partial_k u) = 0, \tag{6.1}$$

given a shock solution (u^0, ψ^0) to the associated hyperbolic system.

As before we introduce the unknown front $x = \psi_\epsilon(t,y)$, change variables $\tilde{x} = x - \psi_\epsilon(t,y)$, drop tildes and epsilons, and rewrite (6.1) in the notation of (5.28)

$$(6.2) \quad u_t + \sum_{j=1}^{d-1} A_j(u)\partial_j u + \tilde{A}_d(u, d\psi)\partial_d u - \epsilon \sum_{j,k=1}^{d} \partial_j(\tilde{B}_{j,k}(u, d\psi)\partial_k u) = 0.$$

We are also given a leading profile $\mathcal{U}^0(t,y,x,z)$, which in terms of our earlier notation is given by

$$(6.3) \qquad \mathcal{U}^0(t,y,x,z) = W_0(\frac{x}{\epsilon}, p(t,y)) + (u^0(t,y,x) - u^0(t,y,0)).$$

Recall that we view (6.2) as representing two problems for (u,ψ), one on $x \geq 0$ and one on $x \leq 0$ with transmission boundary conditions

$$(6.4) \qquad [u] = 0, \ [\partial_x u] = 0 \text{ on } x = 0.$$

We add the extra boundary condition on $\{x = 0\}$:

$$(6.5) \quad \partial_t\psi - \epsilon\triangle_y\psi + \ell(t,y)\cdot u_{|x=0} = \partial_t\psi^0 - \epsilon\triangle_y\psi^0 + \ell(t,y)\cdot\mathcal{U}^0(t,y,0,0)$$

where $\ell(t,y)$ has been chosen so that

$$(6.6) \qquad \ell(t,y)\cdot\partial_z\mathcal{U}^0(t,y,0,0) > 0.$$

We seek an approximate solution $(u^a_\epsilon, \psi^a_\epsilon)$ of the form (dropping epsilons)

$$(6.7) \qquad \psi^a = \psi^0(t,y) + \epsilon\psi^1(t,y) + \cdots + \epsilon^M\psi^M(t,y),$$

$$(6.8) \quad u^a = \big(\mathcal{U}^0(t,y,x,z) + \epsilon\mathcal{U}^1(t,y,x,z) + \cdots + \epsilon^M\mathcal{U}^M(t,y,x,z)\big)\,|_{z=\frac{x}{\epsilon}},$$

where

$$\mathcal{U}^j(t,y,x,z) = U^j(t,y,x) + V^j(t,y,z),$$

$U^0_\pm(t,y,x)$ is the original shock (in the new variables), and the $V^j_\pm(t,y,z)$ are boundary layer profiles exponentially decreasing to 0 as $z \to \pm\infty$.

Remark 6.1. *More precisely, if $(u^0(t,y,x), x = \psi^0(t,y))$ is the inviscid shock in the original coordinates, then $U^0(t,y,x)$ in the above expansion is $u^0(t,y,x+\psi^0(t,y))$.*

6.1 Interior profile equations

We substitute (6.8) into (6.2) and write the result as

$$\sum_{-1}^{M} \epsilon^j \mathcal{F}^j(t,y,x,z)|_{z=\frac{x}{\epsilon}} + \epsilon^M R^{\epsilon,M}(t,y,x), \tag{6.9}$$

where

$$\mathcal{F}^j(x,z) = F^j(x) + G^j(x',z), \tag{6.10}$$

and the G^j decrease exponentially to 0 as $z \to \pm\infty$.

In writing out the F^j, G^j we use the following notation.

Notation 6.2.

1. $\tilde{f}_d(u,d\phi) \equiv f_d(u) - \sum_0^{d-1} f_j(u)\partial_j\phi$, *where* $f_0(u) = u$.

2. $H(U^0,d\psi^0)\partial \equiv \sum_0^{d-1} A_j(U^0)\partial_j + \tilde{A}_d(U^0,d\psi^0)\partial_d$.

3. $d_u\tilde{A}_d(\mathcal{U}^0,d\psi^0)(v,w) \equiv \sum_1^N v_i\partial_{u_i}\tilde{A}_d(\mathcal{U}^0,d\psi^0)w = \partial_u\tilde{A}_d(\mathcal{U}^0,d\psi^0)(w,v)$, *by symmetry of hessians.*

4. $B(u)d\phi \equiv -\sum_0^{d-1} A_j(u)\partial_j\phi$.

5. $\mathbb{B}(u)d\phi \equiv -\sum_0^{d-1} f_j(u)\partial_j\phi$.

6. $[h(u)] \equiv h(u_+) - h(u_-)$ *on* $x = 0$, *where* $u_\pm$ *denote the limits from the right/left at* $x_N = 0$*..*

7. ψ_k^j *means* $\partial_k\psi^j$.

Next we recall our notation for viscosity matrices:

Notation 6.3.

1. Let $\nu = (-\psi_1,\dots,-\psi_{d-1},1)$, $\nu^0 = (-\psi_1^0,\dots,-\psi_{d-1}^0,1)$, *and* $\nu^1 = (-\psi_1^1,\dots,-\psi_{d-1}^1,0)$

2. $\tilde{B}_{j,k}(u) = B_{j,k}(u)$, *if* $j < d$, $k < d$.

3. $\tilde{B}_{j,d}(u) = \sum_{k=1}^d B_{j,k}(u)\nu_k$ *if* $j < d$; $\tilde{B}_{d,k}(u) = \sum_{j=1}^d B_{j,k}(u)\nu_j$ *if* $k < d$.

4. $\tilde{B}_{d,d}(u) = \sum_{j,k=1}^d B_{j,k}(u)\nu_j\nu_k$.

5. $\tilde{B}_{j,k}^0$ *is defined just like* $\tilde{B}_{j,k}$, *except that* $(\mathcal{U}^0,\nu^0)$ *is substituted for* (u,ν). *Similarly,* $B_{j,k}^0 = B_{j,k}(\mathcal{U}^0)$.

6. $d\tilde{B}_{d,d}^0(v,w) = \sum_1^N v_i\partial_{u_i}\tilde{B}_{d,d}^0 w$.

Observe that in (6.2) we wrote $\tilde{B}_{j,k}(u,d\psi)$ instead of $\tilde{B}_{j,k}(u)$.

The interior profile equations are obtained by setting the F^j, G^j equal to zero. In the following expressions for $G^j(t,y,z)$, the functions $U^j(t,y,x)$ and their derivatives are evaluated at $(t,y,0)$. We have

$$
\begin{aligned}
&F^{-1}(t,y,x) = 0 \\
&G^{-1}(t,y,z) = -\partial_z(\tilde{B}^0_{d,d}\partial_z\mathcal{U}^0) + \partial_z f_\nu(\mathcal{U}^0, d\psi^0),
\end{aligned}
\tag{6.11}
$$

(6.12)

$$
\begin{aligned}
&F^0(t,y,x) = H(U^0, d\psi^0)\partial U^0, \\
&G^0(t,y,z) = -\partial_z(\tilde{B}^0_{d,d}\partial_z V^1) + \\
&\qquad \partial_z\Big(\tilde{A}_d(\mathcal{U}^0, d\psi^0)(U^1+V^1) - d\tilde{B}^0_{d,d}(U^1+V^1, \partial_z V^0) + \mathbb{B}(\mathcal{U}^0)d\psi^1\Big) + \\
&\qquad Q^0(U^0, V^0, d\psi^0, d\psi^1),
\end{aligned}
$$

where $Q^0 = Q^0(t,y,z)$ (for short) is exponentially decaying in z. In fact

(6.13)

$$
\begin{aligned}
&Q^0 = \sum_0^{d-1} A_j(\mathcal{U}^0)\partial_j V^0 + \\
&\sum_1^{d-1}(A_j(\mathcal{U}^0) - A_j(U^0))\partial_j U^0 + (\tilde{A}_d(\mathcal{U}^0, d\psi^0) - \tilde{A}_d(U^0, d\psi^0))\partial_d U^0 - \\
&\{\sum_{j=1}^{d-1}\partial_j(\tilde{B}^0_{j,d}\partial_z V^0) + \sum_{k=1}^{d-1}\partial_z(\tilde{B}^0_{d,k}\partial_k\mathcal{U}^0) + \\
&\sum_{j,k=1}^{d}\nu^0_j\nu^0_k\partial_u B^0_{j,k}(\partial_z V^0, \partial_d U^0) + \sum_{j,k=1}^{d}\nu^0_j\nu^0_k\partial_u B^0_{j,k}(\partial_d U^0, \partial_z V^0) + \\
&\sum_{j,k=1}^{d} B^0_{j,k}(\nu^0_j\nu^1_k + \nu^0_k\nu^1_j)\partial_z^2 V^0 + \sum_{j,k=1}^{d}(\nu^0_j\nu^1_k + \nu^0_k\nu^1_j)\partial_u B^0_{j,k}(\partial_z V^0, \partial_z V^0)\}.
\end{aligned}
$$

For $j \geq 1$

(6.14)

$$
\begin{aligned}
&F^j(t,y,x) = H(U^0, d\psi^0)\partial U^j - P^{j-1}(x) \\
&G^j(t,y,z) = -\partial_z(\tilde{B}^0_{d,d}\partial_z V^{j+1}) + \\
&\qquad \partial_z\Big(\tilde{A}_d(\mathcal{U}^0, d\psi^0)(U^{j+1}+V^{j+1}) - d\tilde{B}^0_{d,d}(U^{j+1}+V^{j+1}, \partial_z V^0) + \mathbb{B}(\mathcal{U}^0)d\psi^{j+1}\Big) + \\
&\qquad Q^j(t,y,z),
\end{aligned}
$$

where P^j, Q^j depend only on $(\mathcal{U}^k, d\psi^k)$, $(\mathcal{U}^k, d\psi^k, d\psi^{k+1})$ respectively, and their derivatives, for $k \leq j$.

Remark 6.4. *1. Recall that a term like* $(A_j(\mathcal{U}^0) - A_j(U^0))\partial_j U^0$ *in* (6.13) *is evaluated at* $(t, y, x, z) = (t, y, 0, z)$*. This introduces a fast decaying error which can be incorporated into* $G^1(t, y, z)$ *in view of the fact that* $x = \epsilon \frac{x}{\epsilon}$*. This kind of observation is applied to all such errors.*

2. Define $\mathbb{Q}^0(t, y, z)$ *for* $z \geq 0$ *by* $\int_{+\infty}^z Q^0(t, y, s)ds$ *and for* $z \leq 0$ *by* $\int_{-\infty}^z Q^0(t, y, s)ds$*. As we'll see shortly, it is essential that the terms involving* ψ^1 *do not contribute to the jump of* $\mathbb{Q}^0$ *at* $z = 0$*. These terms come from the last line in* (6.13)*, which can be expressed as*

$$\partial_z\left(\sum_{j,k=1}^{d} (\nu_j^0 \nu_k^1 + \nu_k^0 \nu_j^1) B_{j,k}^0 \partial_z V^0\right) \equiv h(t, y, z).$$

Since this derivative is smooth at $z = 0$ *and* $\int_{-\infty}^{+\infty} h(t, y, z)dz = 0$*, the desired conclusion follows. The same remark applies to the terms involving* ψ^{j+1} *in the jump of* $\mathbb{Q}^j$ *at* $z = 0$*.*

6.2 Boundary profile equations

In the boundary profile equations (t, y, x, z) is evaluated at $(t, y, 0, 0)$. These equations are obtained by substituting the expansions into (6.4) and (6.5) and setting coefficients of the different powers of epsilon equal to 0. Here $U_\pm^0$ or $V_\pm^0$ denote limits as x (resp. z) approaches $0^\pm$.

From (6.4) and (6.5) we obtain the conditions:

$$\begin{aligned}
&(a)\epsilon^0: \quad U_+^0 + V_+^0 = U_-^0 + V_-^0 \\
&(b)\epsilon^{-1}: \ \partial_z V_+^0 = \partial_z V_-^0, \\
&(c)\epsilon^0: \quad \partial_t \psi^0 - l(t, y) \cdot \mathcal{U}^0 = \partial_t \psi^0 - l(t, y) \cdot \mathcal{U}^0,
\end{aligned} \tag{6.15}$$

$$\begin{aligned}
&(a)\epsilon^1: \ U_+^1 + V_+^1 = U_-^1 + V_-^1 \\
&(b)\epsilon^0: \ \partial_x U_+^0 + \partial_z V_+^1 = \partial_x U_-^0 + \partial_z V_-^1, \\
&(c)\epsilon^1: \ \partial_t \psi^1 - \triangle_y \psi^0 + l \cdot \mathcal{U}^1 = -\triangle_y \psi^0.
\end{aligned} \tag{6.16}$$

and for $j \geq 2$,

$$\begin{aligned}
&(a)\epsilon^j: \ U_+^j + V_+^j = U_-^j + V_-^j \\
&(b)\epsilon^{j-1}: \ \partial_x U_+^{j-1} + \partial_z V_+^j = \partial_x U_-^{j-1} + \partial_z V_-^j, \\
&(c)\epsilon^j: \ \partial_t \psi^j - \triangle_y \psi^{j-1} + l \cdot \mathcal{U}^j = 0.
\end{aligned} \tag{6.17}$$

6.3 Solution of the profile equations.

We'll postpone a careful discussion of regularity of solutions until Proposition 6.8. Here we note simply that we need to assume

$$U^0 \in H^{s_0}([-T_0,T_0]\times\overline{\mathbb{R}}^d_\pm),\ \ \psi^0 \in H^{s_0+1}([-T_0,T_0]\times\mathbb{R}^d) \tag{6.18}$$

for some large enough s_0 depending on M.

1. Note that $F^0=0$ already by our assumption that $(U^0,d\psi^0)$ is a shock.

2. Solve for V^0. Recall that $G^{-1}=0$ represents two equations. Define

$$\mathbb{G}^{-j}(t,y,z)=\begin{cases}\int_{+\infty}^z G^{-j}(t,y,s)ds \text{ for } z\geq 0\\ \int_{-\infty}^z G^{-j}(t,y,s)ds \text{ for } z\leq 0\end{cases}.$$

The equations $\mathbb{G}^{-1}=0$ can be written

$$\tilde{B}^0_{d,d}\partial_z\mathcal{U}^0=\tilde{f}_d(\mathcal{U}^0,d\psi^0)-\tilde{f}_d(U^0,d\psi^0), \tag{6.19}$$

where we have used the fact that our given profile $\mathcal{U}^0$ satisfies

$$\lim_{z\to\pm\infty}\mathcal{U}^0(t,y,0,z)=U^0_\pm(t,y,0),$$

so that $U^0(t,y,0)$ in (6.19) means $U^0_+(t,y,0)$ in one equation and $U^0_-(t,y,0)$ in the other. The Rankine-Hugoniot condition implies the two equations (6.19) piece together to give one equation on $\mathbb{R}_z$, and our given profile $\mathcal{U}^0(t,y,0,z)$ is a smooth solution, exponentially decaying to its endstates. Define $V^0(t,y,z)=\mathcal{U}^0(t,y,0,z)-U^0(t,y,0)$. Clearly, the boundary conditions (6.15) are satisfied.

3. Compatibility condition for V^1**.** The equations $\mathbb{G}^0=0$ can be written

$$\begin{aligned}\tilde{B}^0_{d,d}\partial_z V^1 =&\tilde{A}_d(\mathcal{U}^0,d\psi^0)(U^1+V^1)-d\tilde{B}^0_{d,d}(U^1+V^1,\partial_z V^0)\\ &+\mathbb{B}(\mathcal{U}^0)d\psi^1-(\tilde{A}_d(U^0,d\psi^0)U^1+\mathbb{B}(U^0)d\psi^1)+\mathbb{Q}^0.\end{aligned} \tag{6.20}$$

In step 6 we seek an exponentially decaying solution to (6.20) which satisfies (6.16). Suppose for a moment V^1 satisfies (6.20) and that (6.16)(a) holds. Then, since $\tilde{B}^0_{d,d}$ is invertible, (6.16)(b) holds if and only if on $x=0$, $z=0$ we have

$$[\tilde{B}^0_{d,d}\partial_z V^1]=-[\tilde{B}^0_{d,d}\partial_x U^0]. \tag{6.21}$$

Using (6.20) we see this is equivalent to

$$[\tilde{A}_d(U^0,d\psi^0)U^1]+[\mathbb{B}(U^0)d\psi^1]=[\tilde{B}^0_{d,d}\partial_x U^0]+[\mathbb{Q}^0]. \tag{6.22}$$

This equation provides the boundary condition in step 4.

Remark 6.5. (6.16)*(b) implies in general that*

$$\partial_z V_+^1 \neq \partial_z V_-^1$$

at $z = 0$*, so it is not possible to solve* (6.20) *with the boundary conditions* (6.16) *by solving a single ODE on* $\mathbb{R}_z$ *(as was the case for* V^0*).*

4. Solve for (U^1, ψ^1)**.** These are determined by solving

$$\begin{aligned} &H(U^0, d\psi^0)\partial U^1 = P^0(x) \\ &[\mathbb{B}(U^0)d\psi^1] + [\tilde{A}_d(U^0, d\psi^0)U^1] = [\tilde{B}^0_{d,d}\partial_x U^0] + [\mathbb{Q}^0], \text{ on } x = z = 0. \end{aligned} \tag{6.23}$$

The right sides in the boundary and interior equations of (6.23) are initially defined for $t \in [-T_0, T_0]$. We can modify them to be zero in $t \leq -T_0 + \delta$, say. We thereby obtain a problem for $(U^1, d\psi^1)$ that is forward well-posed in the sense of Majda [Maj], since $(U^0, d\psi^0)$ is uniformly stable and ψ^1 does not appear on the right side of the boundary equation (Remark 6.4). Thus, we obtain a solution to (6.23) on $[-\frac{T_0}{2}, T_0]$.

5. Stable and unstable manifolds Let $W_0^s(t,y)$ and $W_0^u(t,y)$ denote the stable and unstable manifolds of (6.19) for the rest points $U^0_\pm(t,y,0)$. Our assumptions (Lax shock, Evans condition) imply they intersect transversally in a smooth curve containing $\mathcal{U}^0(t,y,0,0)$. The tangent spaces to $W_0^s(t,y)$ and $W_0^u(t,y)$ at $\mathcal{U}^0(t,y,0,0)$, denoted $\mathbb{W}_0^s(t,y)$ and $\mathbb{W}_0^u(t,y)$, are the stable and unstable subspaces for the equations

$$\tilde{B}^0_{d,d}\partial_z V^1 = \tilde{A}_d(\mathcal{U}^0, d\psi^0)V^1 - d\tilde{B}^0_{d,d}(V^1, \partial_z V^0). \tag{6.24}$$

Observe that (6.20) has the form

$$\tilde{B}^0_{d,d}\partial_z V^1 = \tilde{A}_d(\mathcal{U}^0, d\psi^0)V^1 - d\tilde{B}^0_{d,d}(V^1, \partial_z V^0) + \mathcal{F}(t,y,z), \tag{6.25}$$

where $\mathcal{F}$ is exponentially decreasing to 0 as $z \to \pm\infty$. Let $W_1^s(t,y)$ and $W_1^u(t,y)$ be the linear submanifolds of $\mathbb{R}^m$ consisting of initial data at $z = 0$ of solutions to (6.25) that decay as $z \to \pm\infty$. Standard ODE facts [Co] imply that $W_1^s(t,y)$ and $W_1^u(t,y)$ are translates of $\mathbb{W}_0^s(t,y)$ and $\mathbb{W}_0^u(t,y)$. The sum of the dimensions of $W_1^s(t,y)$ and $W_1^u(t,y)$ is $N+1$ and they intersect transversally, so their intersection is a line in $\mathbb{R}^N$ with direction $\partial_z\mathcal{U}^0(t,y,0,0)$.

6. Solve for V^1**.** Since the compatibility condition (6.22) holds, to obtain exponentially decaying solutions $V^1_\pm$ to (6.25) satisfying both (6.16)(a) and (b), we choose initial data

$$\begin{aligned} &(V_+^1(t,y,0), V_-^1(t,y,0)) \in (\mathbb{W}_1^s(t,y) \times \mathbb{W}_1^u(t,y)) \cap \\ &\qquad \{(v_1, v_2) \in \mathbb{R}^{2N} : v_1 - v_2 = U_-^1(t,y,0) - U_+^1(t,y,0)\}, \end{aligned} \tag{6.26}$$

The discussion in step 5 implies this is a transversal intersection of linear submanifolds of $\mathbb{R}^{2N}$ of dimensions $N+1$ and N respectively. Call this intersection (which is necessarily nonempty)

$$\mathcal{L}^1(t,y), \text{ the } \textit{line of connection initial data} \text{ for } V^1_\pm(t,y,z). \tag{6.27}$$

For a given (t,y), any point on this line gives a choice of initial data for (6.20) corresponding to a decaying solution that satisfies (6.16)(a) and (b). To arrange (6.16)(c) as well, note that $\mathcal{L}^1$ has direction $\mathbb{U}^0(t,y,0) \equiv (\partial_z\mathcal{U}^0(t,y,0,0), \partial_z\mathcal{U}^0(t,y,0,0))$. So

$$\mathcal{L}^1(t,y) = \{\mathbb{K}(t,y) + s\mathbb{U}^0(t,y,0), s \in \mathbb{R}\}, \tag{6.28}$$

for some initial point $\mathbb{K}(t,y)$. The boundary condition (6.16)(c) holds provided

$$\partial_t\psi^0(t,y) + \ell(t,y)\cdot(U^1_+(t,y,0) + V^1_+(t,y,0)) = 0. \tag{6.29}$$

Since $\ell(t,y)\cdot\partial_z\mathcal{U}^0(t,y,0,0) \neq 0$, there is a unique smooth choice of $s(t,y)$ that gives V^1_+ satisfying (6.29). We now have exponentially decaying $V^1_\pm$ satisfying (6.20) and (6.16).

7. (Continue) The solution of the remaining profile equations follows the same pattern:

$$(U^1,\psi^1) \to V^1 \to (U^2,\psi^2) \to V^2 ... \tag{6.30}$$

The boundary condition for the problem satisfied by (U^j,ψ^j) is always the compatibility condition for V^j. In view of Remark 6.4 the boundary problems for the (U^j,ψ^j) are all Majda well-posed linearized shock problems. The line $\mathcal{L}^j(t,y)$ of connection initial data for $V^j_\pm$ always has direction $\mathbb{U}^0(t,y,0)$.

6.4 Summary

Let $\mathcal{E}(u,\psi)$ be the operator in the left side of (6.2). Our approximate solution (u^a,ψ^a) as in (6.7), (6.8) satisfies

$$\begin{aligned}
&\mathcal{E}(u^a,\psi^a) = \epsilon^M R^{\epsilon,M}(t,y,x) \text{ on } [-\frac{T_0}{2}, T_0]\times\overline{\mathbb{R}}^d_\pm \\
&[u^a] = 0;\ \ [\partial_x u^a] = \epsilon^M r^M(t,y) \text{ on } x=0 \\
&\partial_t\psi^a - \epsilon\triangle_y\psi^a + \ell(t,y)\cdot u^a \\
&\qquad\qquad = \partial_t\psi^0 - \epsilon\triangle_y\psi^0 + \ell(t,y)\cdot\mathcal{U}^0(t,y,0,0) \text{ on } x=0,
\end{aligned} \tag{6.31}$$

with remainders $\epsilon^M R^M$ and $\epsilon^M r^M$ as described in the next step. We can make $[\partial_x u^a] = 0$ without changing the other conditions in (6.31) by adding $-x\rho(x)\epsilon^M r^M(t,y)$ to u^a_+, where ρ is a smooth cutoff equal to one near $x = 0$.

Remark 6.6. *1. The construction does not require the full strength of the uniform stability assumption on the profile $W_0(z, p(t,y))$. We need only the properties that follow from this assumption by the Zumbrun-Serre theorem in the low frequency limit; namely, transversality of the connection and uniform stability of the inviscid shock (U^0, ψ^0).*

2. Observe that with the extra boundary condition, the higher profiles are uniquely determined by this construction once the leading profile $\mathcal{U}^0(t,y,0,z)$ and inviscid shock $(U^0(t,y,x), \psi^0(t,y))$ are fixed.

In the next Proposition we use the following spaces:

Definition 6.7. *1. Let H^s be the set of functions $U(t,y,x)$ on $[-T_0,T_0]\times\mathbb{R}^d$ such that the restrictions $U_\pm$ belong to $H^s([-T_0,T_0]\times\overline{\mathbb{R}}^d_\pm)$.*

2. Let $\tilde{H}^s$ be the set of functions $V(t,y,z)$ on $[-T_0,T_0]\times\mathbb{R}^{d-1}\times\mathbb{R}$ such that the restrictions $V_\pm$ belong to $C^\infty(\overline{\mathbb{R}}_\pm, H^s(t,y))$ and satisfy

$$|\partial_z^k V(t,y,z)|_{H^s(t,y)} \leq C_{k,s} e^{-\delta|z|} \text{ for all } k \tag{6.32}$$

for some $\delta > 0$.

Proposition 6.8 (Approximate solutions). *For given integers $m \geq 0$ and $M \geq 1$ let*

$$s_0 > m + \frac{7}{2} + 2M + \frac{d+1}{2}. \tag{6.33}$$

Suppose the given inviscid shock (U^0, ψ^0) is uniformly stable in the sense of Majda and satisfies $U^0 \in H^{s_0}$, $U^0_\pm(t,y,0) \in H^{s_0}(t,y)$, and $\psi^0(t,y) \in H^{s_0+1}(t,y)$. Suppose also that the connection given by $W_0(z,p(t,y))$ is transversal. Then one can construct (u^a, ψ^a) as above,

$$\psi^a = \psi^0(t,y) + \epsilon\psi^1(t,y) + \cdots + \epsilon^M\psi^M(t,y), \tag{6.34}$$

$$u^a = \left(\mathcal{U}^0(t,y,x,z) + \epsilon\mathcal{U}^1(t,y,x,z) + \cdots + \epsilon^M\mathcal{U}^M(t,y,x,z)\right)|_{z=\frac{x}{\epsilon}}, \tag{6.35}$$

where now $U_+^M(t,y,x)$ *is replaced by* $U_+^M(t,y,x) - x\rho(x)r^M(t,y)$ *for* r^M *as in* (6.31)*. The approximate solution* (u^a, ψ^a) *satisfies*

$$\begin{aligned}
&\mathcal{E}(u^a,\psi^a) = \epsilon^M R^M(t,y,x) \text{ on } [-\frac{T_0}{2}, T_0] \times \overline{\mathbb{R}}^d_\pm \\
&[u^a] = 0; \ \ [\partial_x u^a] = 0 \text{ on } x = 0 \\
&\partial_t \psi^a - \epsilon \triangle_y \psi^a + \ell(t,y)\cdot u^a \\
&\qquad = \partial_t \psi^0 - \epsilon\triangle_y \psi^0 + \ell(t,y)\cdot \mathcal{U}^0(t,y,0,0) \text{ on } x = 0.
\end{aligned} \tag{6.36}$$

We have

$$\begin{aligned}
&U^j(t,y,x) \in H^{s_0-2j}, \ \psi^j(t,y) \in H^{s_0-2j+1}(t,y) \\
&V^j(t,y,z) \in \tilde{H}^{s_0-2j} \\
&r^M(t,y) \in H^{s_0-2M-\frac{3}{2}}(t,y),
\end{aligned} \tag{6.37}$$

and $R^M(t,y,x)$ *satisfies for* $Z = (Z_0, \dots, Z_d)$ *as in* (4.2)

$$\begin{aligned}
&(a)\ |(Z,\epsilon\partial_x)^\alpha R^M|_{L^2(t,y,x)} \le C_\alpha \text{ for } |\alpha| \le m + \frac{d+1}{2} \\
&(b)\ |(Z,\epsilon\partial_x)^\alpha R^M|_{L^\infty(t,y,x)} \le C_\alpha \text{ for } |\alpha| \le m.
\end{aligned} \tag{6.38}$$

Definition 6.9. *We'll refer to* (u^a, ψ^a) *as in Proposition 6.8 as an* approximate solution of order M.

Proof of Proposition 6.8. It just remains to check (6.37) and (6.38). (U^0, ψ^0) has the given regularity by assumption and V^0 by construction since $U^0{}_{|x=0}$ belongs to H^{s_0}.

In the linearized shock problem (6.23) satisfied by (U^1, ψ^1), the interior forcing term $P^0(t,y,x)$ involves terms in which U^0 is differentiated twice, and so belongs to H^{s_0-2}. Similarly, the boundary data lies in $H^{s_0-2}(t,y)$. Thus, Majda's estimates for (6.23) imply $U^1 \in H^{s_0-2}$, $U^1{}_{|x=0} \in H^{s_0-2}$, and $\psi^1 \in H^{s_0-1}$.

$V^1(t,y,z)$ satisfies an ODE in z, (6.20), in which the coefficients and boundary data at $z = 0$ depend on (U^1, ψ^1); so $V^1 \in \tilde{H}^{s_0-2}$. Following this pattern establishes the stated regularity of (U^j, ψ^j) and V^j for any j.

From the boundary profile equation (6.17) we obtain

$$r^M(t,y) = \partial_x U_+^M - \partial_x U_-^M. \tag{6.39}$$

Since $U^M(t,y,x) \in H^{s_0-2M}$, we have $r^M \in H^{s_0-2M-\frac{3}{2}}(t,y)$. This finishes (6.37).

Finally, since $x\rho(x)r^M(t,y) \in H^{s_0-2M-\frac{3}{2}}$ and the least regular terms in R^M involve two derivatives of $x\rho(x)r^M(t,y)$, we obtain (6.38). Observe that we do not deduce (6.38)(b) from (6.38)(a). (6.38)(b) is verified separately using (6.37) and the Sobolev embedding theorem.

$\square$

Remark 6.10. *1. Let m and M be given nonnegative integers, and set*

$$\begin{aligned} &u_0^\pm = U_\pm^0(t,y,x) \\ &u_\epsilon^{1,\pm} = (U_\pm^0(t,y,x) - U_\pm^0(t,y,0)) + \\ &\qquad\qquad \epsilon\mathcal{U}_\pm^1(t,y,x,\frac{x}{\epsilon}) + \cdots + \epsilon^M\mathcal{U}_\pm^M(t,y,x,\frac{x}{\epsilon}) \\ &\psi_\epsilon^1 = \epsilon\psi^1 + \cdots + \epsilon^M\psi^M, \end{aligned} \tag{6.40}$$

where the terms on the right in (6.40) *are as in Proposition 6.8. It is now easy to check, using* (6.37) *and the Sobolev embedding theorem, that $u^\pm$, ψ^0, $u_\epsilon^{1,\pm}$, and ψ_ϵ^1 have the regularity stated in Assumption 5.2.*

7 Nonlinear stability

In this section we show that the approximate solutions constructed in the previous section are close for ϵ small to exact solutions of the parabolic transmission problem

$$\begin{aligned} &\mathcal{E}(u,\psi) = 0 \text{ on } [0,T_0] \times \overline{\mathbb{R}}^d_\pm \\ &[u] = 0; \;\; [\partial_x u] = 0 \text{ on } x = 0 \\ &\partial_t\psi - \epsilon\triangle_y\psi + \ell(t,y)\cdot u \\ &\qquad = \partial_t\psi^0 - \epsilon\triangle_y\psi^0 + \ell(t,y)\cdot\mathcal{U}^0(t,y,0,0) \text{ on } x = 0. \end{aligned} \tag{7.1}$$

As an immediate corollary we will obtain a precise statement of the sense in which the original inviscid shock u_0 satisfying (1.1) is the limit as $\epsilon \to 0$ of solutions u_ϵ to the associated viscous perturbation problem (1.3).

7.1 The error problem

With (u^a, ψ^a) as in Proposition 6.8 we look for exact solutions to (7.1) of the form

$$u = u^a + \epsilon^M v, \;\; \psi = \psi^a + \epsilon^M\phi. \tag{7.2}$$

Expanding around (u^a, ψ^a) we find

$$
\begin{aligned}
&0 = \mathcal{E}(u^a + \epsilon^M v, \psi^a + \epsilon^M \phi) = \\
&\qquad \mathcal{E}(u^a, \psi^a) + \mathcal{E}'_u(u^a, \psi^a)\epsilon^M v + \mathcal{E}'_\psi(u^a, \psi^a)\epsilon^M \phi + \epsilon^M \mathcal{Q}_\epsilon(v, \phi),
\end{aligned} \tag{7.3}
$$

where $\mathcal{Q}_\epsilon$ is a quadratic error that we'll describe carefully later. Thus, (v, ϕ) must satisfy the error transmission problem

$$
\begin{aligned}
&\mathcal{E}'_u(u^a, \psi^a)v + \mathcal{E}'_\psi(u^a, \psi^a)\phi = -\mathcal{Q}_\epsilon(v, \phi) - R^M \text{ on } [0, T_0] \times \overline{\mathbb{R}}^d_\pm, \\
&[v] = 0,\ [\partial_x v] = 0 \text{ on } x = 0 \\
&\partial_t \phi - \epsilon \triangle_y \phi + \ell(t, y) \cdot v = 0 \text{ on } x = 0.
\end{aligned} \tag{7.4}
$$

This transmission problem needs some initial conditions in order to be well-posed. Recall that (u^a, ψ^a) satisfies (6.36) on the time interval $[-\frac{T_0}{2}, T_0]$. Introduce a smooth cutoff function $\theta(t)$ such that

$$
\theta(t) = \begin{cases} 1, \text{ for } t \geq \frac{-T_0}{4} \\ 0 \text{ for } t \leq \frac{-T_0}{3} \end{cases}. \tag{7.5}
$$

We will solve (7.4) by solving the following forward problem:

$$
\begin{aligned}
&\mathcal{E}'_u(u^a, \psi^a)v + \mathcal{E}'_\psi(u^a, \psi^a)\phi \\
&\qquad\qquad = -\theta(t)\{\mathcal{Q}_\epsilon(v, \phi) + R^M\} \text{ on } [-T_0, T_0] \times \overline{\mathbb{R}}^d_\pm, \\
&[v] = 0,\ [\partial_x v] = 0 \text{ on } x = 0, \\
&\partial_t \phi - \epsilon \triangle_y \phi + \ell(t, y) \cdot v = 0 \text{ on } x = 0, \\
&v = 0,\ \phi = 0 \text{ in } t < \frac{-T_0}{3}.
\end{aligned} \tag{7.6}
$$

7.2 Linear estimates

Definition 7.1. *1. For a nonnegative integer m define the conormal spaces*

$$
\begin{aligned}
&\mathcal{H}^m = \{u(t, y, x) \in L^2([-T_0, T_0] \times \mathbb{R}^d) : \\
&\qquad \textit{the restrictions } u_\pm \textit{ satisfy } \sup_{|\alpha| \leq m} |Z^\alpha U|_{L^2(t,y,x)} < \infty\}
\end{aligned} \tag{7.7}
$$

and

$$
\begin{aligned}
&\mathcal{W}^m = \{u(t, y, x) \in L^\infty([-T_0, T_0] \times \mathbb{R}^d) : \\
&\qquad \textit{the restrictions } u_\pm \textit{ satisfy } \sup_{|\alpha| \leq m} |Z^\alpha u|_{L^\infty(t,y,x)} < \infty\}.
\end{aligned} \tag{7.8}
$$

These spaces are equipped with the obvious norms denoted $\|\cdot\|_{\mathcal{H}^m}$ *and* $\|\cdot\|_{\mathcal{W}^m}$*, respectively.*

2. $H^m = \{u(t,y) \in L^2([-T_0,T_0]\times\mathbb{R}^{d-1}) : \sup_{|\alpha|\leq m} |\partial^\alpha_{t,y}u|_{L^2(t,y)} < \infty\}$ *and is equipped with the norm* $|\cdot|_{H^m}$*.*

3. $W^m = \{u(t,y) \in L^\infty([-T_0,T_0]\times\mathbb{R}^{d-1}) : \sup_{|\alpha|\leq m} |\partial^\alpha_{t,y}u|_{L^\infty(t,y)} < \infty\}$ *and is given the usual norm* $\|\cdot\|_{W^m}$*.*

Our main tool for studying nonlinear stability is the estimate for the fully linearized transmission problem given by Theorem 5.5. First, we need to extract from that theorem convenient $\|\cdot\|_{\mathcal{H}^m}$ and $\|\cdot\|_{\mathcal{W}^m}$ estimates for the forward linear problem:

$$\begin{aligned}
&\mathcal{E}'_u(u^a,\psi^a)v + \mathcal{E}'_\psi(u^a,\psi^a)\phi = f \text{ on } [-T_0,T_0]\times\overline{\mathbb{R}}^d_\pm\\
&[v]=0,\ [\partial_x v]=0 \text{ on } x=0\\
&\partial_t\phi - \epsilon\triangle_y\phi + \ell(t,y)\cdot v = 0 \text{ on } x=0\\
&v=0,\ \phi=0,\ f=0 \text{ in } t<\frac{-T_0}{3}.
\end{aligned} \tag{7.9}$$

Theorem 7.2 ($\mathcal{H}^m$ estimate). *Let m be a nonnegative integer and let (u^a,ψ^a) be as in Proposition 6.8. There are $C>0$ and ϵ_0 such that for $\epsilon\in(0,\epsilon_0]$ and all $f\in\mathcal{H}^m$ vanishing in $t<\frac{-T_0}{3}$, the solution (v,ϕ) of (7.9) is unique and satisfies*

$$\begin{aligned}
\|v\|_{\mathcal{H}^m}+&\sqrt{\epsilon}\|\partial_{x,y}v\|_{\mathcal{H}^m} + \epsilon^{\frac{3}{2}}\|\partial^2_{x,y}v\|_{\mathcal{H}^m}\\
&+|v(0)|_{H^m}+|\phi|_{H^{m+1}}+\sqrt{\epsilon}|\partial_y d\phi|_{H^m}\leq C\|f\|_{\mathcal{H}^m},
\end{aligned} \tag{7.10}$$

where $d\phi = \nabla_{t,y}\phi$*.*

Proof. We'll deduce the estimate (7.10) from the weighted norm estimate for the linearized problem given in Theorem 5.5. We need to examine the size of the weights λ^2, λ, and $\mu\Lambda^2$ appearing there and defined in (5.19) and (5.20).

The weight function λ satisfies

$$\lambda^2 \geq C(\gamma+\epsilon|\eta|^2+\min(\epsilon\tau^2,|\tau|)), \tag{7.11}$$

so

$$\begin{aligned}
&\lambda^2 \geq C(\gamma+\sqrt{\epsilon\gamma}|\eta|+\epsilon|\eta|^2+\sqrt{\epsilon}|\tau|),\\
&\lambda \geq C(\sqrt{\gamma}+\sqrt{\epsilon}|\eta|).
\end{aligned} \tag{7.12}$$

Next consider $\mu\Lambda^2$. In $|\epsilon\zeta| \le 1$ we have

$$\mu\Lambda^2 \ge \mu = |\zeta|\lambda \sim |\zeta|(\sqrt{\gamma} + \sqrt{\epsilon}|\tau| + \sqrt{\epsilon}\gamma + \sqrt{\epsilon}|\eta|), \tag{7.13}$$

so

$$\mu\Lambda^2 \ge \sqrt{\gamma}|\zeta| \text{ and } \mu\Lambda^2 \ge \sqrt{\epsilon}|\zeta|^2 \text{ in } |\epsilon\zeta| \le 1. \tag{7.14}$$

In $|\epsilon\zeta| \ge 1$ we have

$$\mu\Lambda^2 = \frac{\Lambda^{\frac{7}{2}}}{\epsilon^{\frac{3}{2}}} \sim \epsilon^{-\frac{3}{2}} + \epsilon^{\frac{1}{4}}|\tau|^{\frac{7}{4}} + \epsilon^{\frac{1}{4}}\gamma^{\frac{7}{4}} + \epsilon^2|\eta|^{\frac{7}{2}} \ge |\zeta|^{\frac{3}{2}}. \tag{7.15}$$

Apply the inequality

$$a^2 + b^2 \ge a^{\frac{6}{7}}b^{\frac{8}{7}}, \text{ for } a > 0, b > 0 \tag{7.16}$$

to five pairs of terms in (7.15) to obtain

$$\begin{aligned} \mu\Lambda^2 &\ge \epsilon|\tau||\eta|^{\frac{3}{2}} + \epsilon\gamma|\eta|^{\frac{3}{2}} + \sqrt{\epsilon}|\eta|^2 + \frac{|\tau|}{\sqrt{\epsilon}} + \frac{\gamma}{\sqrt{\epsilon}} \\ &\ge \sqrt{\epsilon}|\eta||\tau| + \sqrt{\epsilon}|\eta|\gamma + \sqrt{\epsilon}|\eta|^2 \sim \sqrt{\epsilon}|\eta||\zeta|, \end{aligned} \tag{7.17}$$

where the second inequality follows by considering separately the cases $|\eta| \ge \frac{1}{\epsilon}$ and $|\eta| < \frac{1}{\epsilon}$.

Summarizing we have shown that for arbitrary $|\epsilon\zeta|$,

$$\mu\Lambda^2 \ge \sqrt{\gamma}|\zeta| \text{ and } \mu\Lambda^2 \ge \sqrt{\epsilon}|\eta||\zeta|. \tag{7.18}$$

For a fixed $\gamma \ge \gamma_0$ we have

$$e^{-\gamma t} \sim 1, \text{ for } t \in [-T_0, T_0], \tag{7.19}$$

so it follows directly from Theorem 5.5, (7.12), (7.18), and (7.19) that all terms in the left side of the estimate (7.10), with the exception of the $\epsilon^{\frac{3}{2}}\partial_x^2 v$ term, are dominated by $C\|f\|_{\mathcal{H}^m}$. The estimate for the remaining term is obtained by using the equation (7.9) to express $\epsilon^{\frac{3}{2}}\partial_x^2 v$ in terms of previously estimated quantities. Observe that the term $\epsilon^{-\frac{1}{2}}E^\epsilon v$ can be estimated using

$$\|e^{-\theta x/\epsilon}w\|_{\mathcal{H}^m} \le C(\epsilon\|\partial_x w\|_{\mathcal{H}^m} + \sqrt{\epsilon}|w(0)|_{H^m}). \tag{7.20}$$

□

Remark 7.3. *For each fixed ϵ we can use standard parabolic theory to solve the (nonstandard) linear problem (7.9) by the following iteration scheme. Let $(v_0, \phi_0) = 0$ and for given (v_n, ϕ_n) define (v_{n+1}, ϕ_{n+1}) by solving the parabolic problems:*

$$\begin{aligned}&\mathcal{E}'_u(u^a, \psi^a)v_{n+1} = f - \mathcal{E}'_\psi(u^a, \psi^a)\phi_n \text{ on } [-T_0, T_0] \times \overline{\mathbb{R}}^d_\pm \\ &[v_{n+1}] = 0,\ [\partial_x v_{n+1}] = 0 \text{ on } x = 0 \\ &v_{n+1} = 0 \text{ in } t < \frac{-T_0}{3},\end{aligned} \tag{7.21}$$

$$\begin{aligned}&\partial_t\phi_{n+1} - \epsilon\triangle_y\phi_{n+1} = -\ell(t, y) \cdot v_n = 0 \text{ on } x = 0 \\ &\phi_{n+1} = 0 \text{ in } t < \frac{-T_0}{3}.\end{aligned} \tag{7.22}$$

Theorem 7.4 ($\mathcal{W}^m$ estimate). *Suppose $m > 2 + \frac{d+1}{2}$ and let (u^a, ψ^a) be as in Proposition 6.8. There are $C > 0$ and ϵ_0 such that for $\epsilon \in (0, \epsilon_0]$ and all $f \in \mathcal{H}^m$ vanishing in $t < \frac{-T_0}{3}$, the solution (v, ϕ) of (7.9) is unique and satisfies*

$$\|v\|_{\mathcal{W}^2} + \epsilon\|\partial_{x,y}v\|_{\mathcal{W}^1} + |\phi|_{W^3} \le C\|f\|_{\mathcal{H}^m}. \tag{7.23}$$

If in addition $f \in L^\infty$, then

$$\epsilon^2\|\partial^2_{x,y}v\|_{L^\infty} \le C(\|f\|_{\mathcal{H}^m} + \epsilon\|f\|_{L^\infty}). \tag{7.24}$$

Proof. The estimate for the terms involving v follows from Theorem 7.2 by almost exactly the same argument as that used to prove Theorem 5.8 of [MZ1], with the one difference that the boundary term that appears on the right in applications of (7.20) does not vanish in our setting.

Since $m > 2 + \frac{d+1}{2}$, the estimate for ϕ is a consequence of

$$|\phi|_{W^3} \le C|\phi|_{H^{m+1}}. \tag{7.25}$$

□

7.3 Nonlinear estimates

In the proof of the next Proposition we'll need to estimate norms of products of derivatives of $v(t, y, x)$ with derivatives of $d\phi(t, y)$. The fact that ϕ is independent of x means that Moser inequalities can't be applied directly to estimate the $\mathcal{H}^m$ norm of $\mathcal{Q}_\epsilon(v, \phi)$ as in (7.6). This minor difficulty is easily circumvented by introducing a C^∞ cutoff in x. Fix $K>0$ arbitrarily large and let

$$\chi_K(x) = \begin{cases} 1, & |x| \le K \\ 0, & |x| \ge K+1 \end{cases} . \tag{7.26}$$

Instead of solving (7.6), we will now solve

$$\begin{aligned} &\mathcal{E}'_u(u^a,\psi^a)v + \mathcal{E}'_\psi(u^a,\psi^a)\phi \\ &\qquad\qquad = -\theta(t)\chi_K(x)\{\mathcal{Q}_\epsilon(v,\phi) + R^M\} \text{ on } [-T_0,T_0]\times\overline{\mathbb{R}}^d_\pm, \\ &[v]=0,\ [\partial_x v]=0 \text{ on } x=0 \\ &\partial_t\phi - \epsilon\triangle_y\phi + \ell(t,y)\cdot v = 0 \text{ on } x=0 \\ &v=0,\ \phi=0 \text{ in } t<\frac{-T_0}{3}. \end{aligned} \tag{7.27}$$

When estimating $\chi_K(x)\mathcal{Q}_\epsilon(v,\phi)$, we are now free to replace $\phi(t,y)$ by $\chi(x)\phi(t,y)$ for any C^∞ cutoff $\chi(x)$ satisfying

$$\chi\chi_K = \chi_K. \tag{7.28}$$

In writing out the form of $\mathcal{Q}_\epsilon$ it will be convenient to set

$$w = (v, d\phi),\ \text{where as always } d\phi = \partial_{t,y}\phi, \tag{7.29}$$

and to let $\Phi = \Phi(u^a, d\psi^a, \partial_y d\psi^a, \epsilon w)$ denote a C^∞ function of the given arguments which may change from line to line. In what (7.30) and (7.31) ∂ will always denote some *spatial* derivative.

An examination of the expansions shows that $Q_\epsilon(v,\phi)$ is a sum of terms of the form

$$\begin{aligned} \mathcal{Q}_1 &= \epsilon^M \Phi w\partial w \\ \mathcal{Q}_2 &= \epsilon^M \Phi ww\partial u^a \\ \mathcal{Q}_3 &= \epsilon^{M+1}\partial(\Phi w\partial w) \\ \mathcal{Q}_4 &= \epsilon^{M+1}\partial(\Phi ww\partial u^a) \end{aligned} \tag{7.30}$$

Here $\mathcal{Q}_1$ and $\mathcal{Q}_3$ are bilinear in w and ∂w, while $\mathcal{Q}_2$ and $\mathcal{Q}_4$ are bilinear in w and linear in ∂u^a.

The terms $\mathcal{Q}_3$ and $\mathcal{Q}_4$ involve

$$\begin{aligned}
\mathcal{Q}_{3,1} &= \epsilon^{M+1}\Phi w\partial^2 w\\
\mathcal{Q}_{3,2} &= \epsilon^{M+1}\Phi \partial w\partial w\\
\mathcal{Q}_{3,3} &= \epsilon^{M+1}\Phi w\partial w\partial u^a\\
\mathcal{Q}_{4,1} &= \epsilon^{M+1}\Phi w\partial w\partial u^a\\
\mathcal{Q}_{4,2} &= \epsilon^{M+1}\Phi ww\partial u^a\partial u^a\\
\mathcal{Q}_{4,3} &= \epsilon^{M+1}\Phi ww\partial^2 u^a
\end{aligned} \tag{7.31}$$

It is important to note that *there are no terms where $\partial^2 d\phi$ appears*, although $\mathcal{Q}_{3,1}$ would seem to allow such terms.

Introduce the norms

$$\|f\|_{\mathcal{Y}^m} := \|f\|_{\mathcal{H}^m} + \epsilon\|f\|_{L^\infty}, \tag{7.32}$$

$$\begin{aligned}
\|(v,\phi)\|_{\mathcal{X}^m} := \|v\|_{\mathcal{H}^m} + \sqrt{\epsilon}\|\partial_{x,y}v\|_{\mathcal{H}^m} + \epsilon^{\frac{3}{2}}\|\partial^2_{x,y}v\|_{\mathcal{H}^m} + |v(0)|_{H^m} +\\
|\phi|_{H^{m+1}} + \sqrt{\epsilon}|\partial_y d\phi|_{H^m} +\\
\|v\|_{\mathcal{W}^2} + \epsilon\|\partial_{x,y}v\|_{\mathcal{W}^1} + \epsilon^2\|\partial^2_{x,y}v\|_{L^\infty} + |\phi|_{W^3},
\end{aligned} \tag{7.33}$$

where we've suppressed ϵ dependence in the $\mathcal{X}^m$, $\mathcal{Y}^m$ notation. Observe that the $\mathcal{Y}^m$ and $\mathcal{X}^m$ norms are obtained by adding the right (respectively, left) sides of the estimates (7.10), (7.23), and (7.24). We denote by $\mathcal{Y}^m$ and $\mathcal{X}^m$ the natural spaces (independent of ϵ) associated to these norms. Denote by $\mathcal{Y}^m_0$ (resp., $\mathcal{X}^m_0$) the subspace of $(v,\phi) \in \mathcal{Y}^m$ (resp., $\mathcal{X}^m$) which vanish for $t < \frac{-T_0}{3}$ (resp., vanish for $t < \frac{-T_0}{3}$ and satisfy the boundary conditions in (7.27)).

Let $\mathcal{P}_\epsilon$ denote the fully linearized operator on the left side of (7.27). Theorems 7.2 and 7.4 imply there is a constant C_0 such that for all $\epsilon \in (0,\epsilon_0]$ and $f \in \mathcal{Y}^m_0$ the problem

$$\mathcal{P}_\epsilon(v,\phi) = f,\ (v,\phi) \in \mathcal{X}^m_0 \tag{7.34}$$

has a unique solution which satisfies

$$\|(v,\phi)\|_{\mathcal{X}^m} \le C_0\|f\|_{\mathcal{Y}^m}. \tag{7.35}$$

We are now in a position to solve (7.27) by a fixed point argument that is essentially identical to the nonlinear stability argument in [MZ1], section 6. The main step is to prove the following estimates for the quadratic terms. The cutoff $\chi_K(x)$ was introduced in (7.26).

Proposition 7.5. *For all $\mathcal{M} \geq 0$ there is a constant $C(\mathcal{M})$ such that for all $\epsilon \in (0, \epsilon_0]$ and all $(v_i, \phi_i) \in \mathcal{X}_0^m$, $i = 1, 2$, $\chi_K(x)\mathcal{Q}_\epsilon(v_i, \phi_i)$ belong to $\mathcal{Y}_0^m$ and*

$$\begin{aligned} &\|\chi_K(x)\mathcal{Q}_\epsilon(v_1, \phi_1)\|_{\mathcal{Y}^m} \leq \epsilon^{\frac{1}{4}} C(\mathcal{M}), \\ &\|\chi_K(x)(\mathcal{Q}_\epsilon(v_1, \phi_1) - \mathcal{Q}_\epsilon(v_1, \phi_1))\|_{\mathcal{Y}^m} \\ &\qquad\qquad \leq \epsilon^{\frac{1}{4}} C(\mathcal{M}) \|(v_1, \phi_1) - (v_2, \phi_2)\|_{\mathcal{X}^m}, \end{aligned} \tag{7.36}$$

provided that

$$\epsilon \|(v_i, d\phi_i)\|_{L^\infty} \leq 1, \; i = 1, 2 \tag{7.37}$$

and

$$\|(v_i, \phi_i)\|_{\mathcal{X}^m} \leq \mathcal{M}, \; i = 1, 2. \tag{7.38}$$

Proof. When estimating $\chi_K \mathcal{Q}_\epsilon$, we are free to replace ϕ by $\chi(x)\phi$ for any smooth cutoff χ such that $\chi\chi_K = \chi_K$, so fix such a cutoff function. Since

$$\begin{aligned} &\|\chi d\phi\|_{\mathcal{H}^m} + |\chi(0) d\phi|_{H^m} \leq C|\phi|_{H^{m+1}}, \\ &\sqrt{\epsilon}\|\partial_{x,y}(\chi d\phi)\|_{\mathcal{H}^m} \leq C\sqrt{\epsilon}|\partial_y d\phi|_{H^m}, \end{aligned} \tag{7.39}$$

and

$$\|\chi d\phi\|_{\mathcal{W}^2} + \epsilon\|\partial_{x,y}(\chi d\phi)\|_{\mathcal{W}^1} + \epsilon^2 \|\partial^2_{x,y}(\chi d\phi)\|_{L^\infty} \leq C|\phi|_{W^3}, \tag{7.40}$$

for each norm of v or its first derivatives appearing in the definition of $\mathcal{X}^m$, our linear estimates give control over the same norm of $\chi d\phi$ or its first derivatives. We do not have control over $\epsilon^{\frac{3}{2}} \|\partial^2_{x,y}(\chi d\phi)\|_{\mathcal{H}^m}$, but recall that no derivatives of the form $\partial^2(d\phi)$ appear in $\mathcal{Q}_\epsilon$. Thus, we don't need control over second derivatives of $d\phi$. This means that in the proof of Proposition 7.5, terms involving $d\phi$ or its first derivatives can be handled exactly like terms involving v or its first derivatives. Thus, we reduce to the proof of the analogous proposition in [MZ1], Proposition 6.4, whose proof can be repeated word for word to establish Proposition 7.5.

$\square$

Remark 7.6. *The proof in [MZ1] is designed to handle the case of a first order approximate solution, $M = 1$, which is more delicate than the case $M > 1$. The argument of [MZ1] applies to any M, but the presence of extra factors of ϵ allows a much simpler argument to be given in the case $M > 1$.*

Theorem 7.7. *Let $m > 2 + \frac{d+1}{2}$ and $M \geq 1$ be integers and suppose*

$$s_0 > m + \frac{7}{2} + 2M + \frac{d+1}{2}. \tag{7.41}$$

Suppose the given inviscid shock (U^0, ψ^0) satisfies $U^0 \in H^{s_0}$, $U^0_\pm(t,y,0) \in H^{s_0}(t,y)$, and $\psi^0(t,y) \in H^{s_0+1}(t,y)$. Under Assumptions 2.1 and 5.1 an approximate solution of order M, (u^a, ψ^a), can be constructed as in Proposition 6.8. In addition there is a unique exact solution (v, ϕ) to the nonlinear forward error problem (7.27) *such that*

$$\|(v,\phi)\|_{\mathcal{X}^m} < \infty. \tag{7.42}$$

With $K > 0$ arbitrarily large as in (7.27)*, let*

$$\Omega_K = [0, T_0] \times \mathbb{R}^{d-1} \times [-K, K]. \tag{7.43}$$

The nonlinear transmission problem (7.1) *has an exact solution on Ω_K*

$$u = u^a + \epsilon v, \quad \psi = \psi^a + \epsilon^M \phi \tag{7.44}$$

such that

$$\|(u,\psi) - (u^a, \psi^a)\|_{\mathcal{X}^m(\Omega_K)} = O(\epsilon^M). \tag{7.45}$$

Thus, in particular,

$$\begin{aligned} &\|u - u^a\|_{\mathcal{H}^m(\Omega_K)} + \|u - u^a\|_{L^\infty(\Omega_K)} = O(\epsilon^M), \\ &\|\psi - \psi^a\|_{H^{m+1}([0,T_0]\times\mathbb{R}^{d-1})} + \|\psi - \psi^a\|_{W^3([0,T_0]\times\mathbb{R}^{d-1})} = O(\epsilon^M). \end{aligned} \tag{7.46}$$

Proof. We just need to prove that the forward error problem (7.27) has a unique solution (v,ϕ) satisfying (7.42). The rest is immediate from the definitions of the cutoffs and norms.

Theorems 7.2 and 7.4 imply that $\mathcal{P}^\epsilon$ as in (7.34) is an isomorphism from $\mathcal{X}^m_0$ onto $\mathcal{Y}^m_0$. Thus, the problem (7.27) is equivalent to

$$(v,\phi) = \mathcal{P}^{-1}\left(-\theta(t)\chi_K(x)\{\mathcal{Q}_\epsilon(v,\phi) + R^M\}\right), \ (v,\phi) \in \mathcal{X}^m_0. \tag{7.47}$$

The estimates in Theorems 7.2 and 7.4 and Proposition 6.8 imply there is a constant C_1 such that for all $\epsilon \in (0, \epsilon_0]$

$$\|\mathcal{P}^{-1}(\theta(t)\chi_K(x)R^M)\|_{\mathcal{X}^m} \leq C_1. \tag{7.48}$$

For all $\mathcal{M} > 0$ introduce

$$\mathcal{X}_0^m(\mathcal{M}, \epsilon) = \{(v, \phi) \in \mathcal{X}_0^m \ : \epsilon\|(v, d\phi)\|_{L^\infty} \leq 1 \text{ and } \|(v, \phi)\|_{\mathcal{X}^m} \leq \mathcal{M}\}. \tag{7.49}$$

Apply Theorems 7.2 and 7.4 and Proposition 7.5 to deduce that for all $\mathcal{M} > 0$ and $\epsilon \in (0, \epsilon_0]$:

$$\begin{aligned} &\|\mathcal{P}^{-1}\left(\theta(t)\chi_K(x)\mathcal{Q}_\epsilon(v, \phi)\right)\|_{\mathcal{X}^m} \leq \epsilon^{\frac{1}{4}} C(\mathcal{M}), \\ &\|\mathcal{P}^{-1}\big(\theta(t)\chi_K(x)(\mathcal{Q}_\epsilon(v_1, \phi_1) - \mathcal{Q}_\epsilon(v_1, \phi_1))\big)\|_{\mathcal{X}^m} \\ &\qquad\qquad \leq \epsilon^{\frac{1}{4}} C(\mathcal{M})\|(v_1, \phi_1) - (v_2, \phi_2)\|_{\mathcal{X}^m}. \end{aligned} \tag{7.50}$$

Choose $\mathcal{M} > C_1$ and apply the above estimates to see that, after shrinking ϵ_0 if necessary, the equation (7.47) has a unique solution (v, ϕ) in $\mathcal{X}^m(\mathcal{M}, \epsilon)$ for all $\epsilon \in (0, \epsilon_0]$. In particular,

$$\|(v, \phi)\|_{\mathcal{X}^m} \leq \mathcal{M}. \tag{7.51}$$

□

We have constructed exact solutions $(u^\epsilon, \psi^\epsilon)$ to the parabolic problem (1.8) in the new coordinates $(t, y, \tilde{x})$, where $\tilde{x} = x - \psi^\epsilon(t, y)$. Returning to the original coordinates we set

$$u^\epsilon_{or}(t, y, x) = u^\epsilon(t, y, x - \psi^\epsilon(t, y)). \tag{7.52}$$

The following corollary is an immediate consequence of Theorem 7.7.

Corollary 7.8. *Suppose that the assumptions of Theorem 7.7 hold, and for K arbitrarily large let Ω_K be as in* (7.43)*, but defined in the original coordinates. Let $(u^0(t, y, x), \psi^0(t, y))$ be the inviscid shock solution to the system of hyperbolic conservation laws* (1.1) *in the original coordinates. There are smooth exact solutions u^ϵ_{or} to the parabolic system* (1.3) *on Ω_K such that*

$$\begin{aligned} &(a)\|u^0 - u^\epsilon_{or}\|_{L^2(\Omega_K)} = O(\sqrt{\epsilon}), \\ &(b)\|u^0 - u^\epsilon_{or}\|_{L^\infty(\Omega_K \cap \{(t,y,x): |x - \psi^0(t,y)| \geq \kappa\})} = O(\epsilon), \\ &(c)\|u^0 - u^\epsilon_{or}\|_{L^\infty(\Omega_K \cap \{(t,y,x): |x - \psi^0(t,y)| \geq -\epsilon \log \epsilon\})} = O(\epsilon^{\min(\delta, 1)}), \end{aligned} \tag{7.53}$$

where $\kappa > 0$ is arbitrary and $\delta > 0$ satisfies

$$|V^0(t, y, z)| \leq Ce^{-\delta|z|}. \tag{7.54}$$

A Appendix : Proof of Proposition 5.18

Proposition 4.14 in [MZ1] gives estimates similar to (5.80) for solutions of equations (5.77) on $\{x \geq 0]$ with Dirichlet boundary conditions. The extension to transmission problems is immediate, but the introduction of boundary conditions like (5.78) requires some care. In this appendix, we go over the analysis of [MZ1], pointing out the modifications which are necessary to cover the case of equations (5.77), (5.78).

A.1 The symbolic analysis

We are given matrices

$$\mathcal{G}(z, q, \zeta) = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathcal{M} & \mathcal{A} \end{pmatrix} \tag{A.1}$$

on $\mathbb{R} \times \mathcal{Q}_0 \times \mathbb{R}^{1+d}$. They converge at an exponential rate to $\mathcal{G}^{\pm}\infty(q, \zeta)$ when $z \to \pm\infty$. Moreover, we are given matrices $\mathcal{S}_j^{\pm}(z, q, \zeta)$, and consider the boundary matrix

$$\Gamma(q, \zeta)(U^-, U^+, \psi) = \begin{pmatrix} [u] + \psi[\mathcal{R}(0, q, \zeta)] \\ [v] + \psi[\partial_z \mathcal{R}(0, q, \zeta)] \\ \ell(q) \cdot u^- \end{pmatrix} \tag{A.2}$$

with $U^{\pm} = {}^t(u^{\pm}, v^{\pm})$, $[U] = U^+ - U^-$ and

$$\mathcal{R}^{\pm} = (i\tau + \gamma)\mathcal{S}_0^{\pm} + \sum i\eta_j \mathcal{S}_j^{\pm}. \tag{A.3}$$

The spaces of initial data $U(0)$ such that the associated solution of $\partial_z U - \mathcal{G}U = 0$ is bounded when $z \to \pm\infty$ are denoted by $\mathbb{E}^{\pm}(q, \zeta)$. They are well defined for $\zeta \neq 0$ with $\gamma \geq 0$. According to Proposition 3.15, a version of the modified Evans' function reads

$$\widetilde{D}(q, \zeta) = \det\big(\mathbb{E}^- \times \mathbb{E}^+ \times \mathbb{C}, \ker \Gamma\big). \tag{A.4}$$

By Lemma 2.6 of [MZ1], there are smooth matrices $\mathcal{W}^{\pm}$ for $z \in \mathbb{R}_{\pm} = \{\pm z \geq 0\}$, converging at an exponential rate at infinity to the identity matrix and such that

$$\partial_z \mathcal{W}^{\pm} - \mathcal{G}\mathcal{W}^{\pm} = \mathcal{W}^{\pm}\mathcal{G}^{\pm}. \tag{A.5}$$

The $\mathcal{W}^\pm(z, q, \zeta)$ are defined and smooth for $z \in \mathbb{R}_\pm$, $q \in \mathcal{Q}_0$ and ζ in a neighborhood $\mathcal{Z}$ of 0 in $\mathbb{R}^{1+d}$. Next, shrinking $\mathcal{Z}$ if necessary, there are matrices $\mathcal{V}^\pm(q, \zeta)$ such that

$$\mathcal{G}^\pm \mathcal{V}^\pm = \mathcal{V}^\pm \mathcal{G}_2^\pm, \qquad \mathcal{G}_2^\pm = \begin{pmatrix} H^\pm & 0 \\ 0 & P^\pm \end{pmatrix}. \tag{A.6}$$

Moreover, at $\zeta = 0$, $\mathcal{V}$ has the form

$$\mathcal{V}(q, 0) = \begin{pmatrix} \mathrm{Id} & V_{HP}^\pm \\ 0 & \mathrm{Id} \end{pmatrix}. \tag{A.7}$$

We consider $\mathcal{T}^\pm = \mathcal{W}^\pm \mathcal{V}^\pm$, so that the positive [resp. negative] spaces for $\mathcal{G}_2^-$ [resp. $\mathcal{G}_2^+$] are $\mathbb{E}_2^- = (\mathcal{T}^-)^{-1}\mathbb{E}^-$ [resp. $\mathbb{E}_2^+ = (\mathcal{T}^+)^{-1}\mathbb{E}^+$]. Similarly, the boundary conditions are transformed to

$$\Gamma_2(U_2^-, U_2^+, \psi) = \Gamma(\mathcal{T}^-(0)U_2^-, \mathcal{T}^+(0)U_2^+, \psi). \tag{A.8}$$

The Evans' function (A.4) is equivalent to

$$D_2(q, \zeta) = \det(\mathbb{E}_2^- \times \mathbb{E}_2^+, \times\mathbb{C}, \ker \Gamma_2). \tag{A.9}$$

The diagonal form of $\mathcal{G}_2$ implies that

$$\mathbb{E}_2^\pm(q, \zeta) = \mathbb{E}_H^\pm(q, \zeta) \oplus \mathbb{E}_P^\pm(q, \zeta) \tag{A.10}$$

where $\mathbb{E}_H^\pm(q, \zeta)$ and $\mathbb{E}_P^\pm(q, \zeta)$ are the negative space for the $+$ sign and the positive space for the $-$ sign of $H^\pm$ and $P^\pm$ respectively.

Next, we pass to the construction of symmetrizers. The eigenvalues of $P^\pm(q, \zeta)$ stay away from the imaginary axis. Thus, for all $\kappa > 0$ large, there are smooth symmetric matrices $\Sigma_2^\pm(q, \zeta)$ smooth matrices $\Sigma_P^\pm(q, \zeta)$ such that:

$$\pm\Sigma_P^\pm(q, \zeta) \geq \kappa(\Pi_P^\pm)^*\Pi_P^\pm - (\mathrm{Id} - \Pi_P^\pm)^*(\mathrm{Id} - \Pi_P^\pm) \tag{A.11}$$

where $\mathrm{Id} - \Pi_P^\pm(q, \zeta)$ is the spectral projection on $\mathbb{E}_P^\pm(q, \zeta)$. Moreover, there is $c_\kappa > 0$ such that for all $(q, \zeta) \in \mathcal{Q}_0 \times \mathcal{Z}$:

$$\mathrm{Re}(\Sigma_P^\pm P^\pm) \geq c_\kappa. \tag{A.12}$$

We refer to [Kr], [Ch-P], [MZ1], [MZ2] for a detailed construction of Σ_P^+. On the side $\{z \leq 0\}$, that is for the minus sign, $-\Sigma_P^-$ is the symmetrizer for $-P^-$ given by this construction.

The analysis of the hyperbolic-like component H is more delicate. Passing to polar coordinates $\zeta = \rho\check{\zeta}$, with $\rho = |\zeta|$, there holds

$$H(q,\zeta) = \rho \check{H}(q,\check{\zeta},\rho). \tag{A.13}$$

Following [MZ1] or [MZ2], for all $\kappa > 0$ large, there are symmetric matrices $\check{\Sigma}_H^{\pm}(q,\check{\zeta},\rho)$ defined for $q \in \mathcal{Q}_0$, $\check{\zeta} \in \overline{S}_+^d := \{|\check{\zeta}| = 1, \check{\gamma} \geq 0\}$ and ρ small enough, such that

$$\pm\check{\Sigma}_H^{\pm}(q,\check{\zeta},\rho) \geq \kappa(\check{\Pi}_H^{\pm})^*\check{\Pi}_H^{\pm} - (\mathrm{Id} - \check{\Pi}_H^{\pm})^*(\mathrm{Id} - \check{\Pi}_H^{\pm}) \tag{A.14}$$

where $\mathrm{Id} - \check{\Pi}_P^{\pm}(q,\check{\zeta},\rho)$ is a projection on $\check{\mathbb{E}}_H^{\pm}(q,\check{\zeta},\rho)$ which is the continuous extension to $\rho = 0$ of the fiber bundle $\mathbb{E}_H^{\pm}(q,\rho\check{\zeta})$. We warn the reader that $\mathrm{Id} - \check{\Pi}_P^{\pm}(q,\check{\zeta},\rho)$ is *not* a spectral projection near glancing mode. Moreover, $\mathrm{Re}(\Sigma_H^{\pm}\check{H}^{\pm})$ has the special form indicated in Lemma 2.13 of [MZ2]. For the convenience of the reader, we give here the weak form of this statement: it implies that there is $c_\kappa > 0$ such that for all $(q,\check{\zeta},\rho) \in \mathcal{Q}_0 \times \overline{S}_+^d \times [0,\rho_0]$:

$$\mathrm{Re}(\Sigma_H^{\pm}\check{H}^{\pm}) \geq c_\kappa(\check{\gamma} + \rho). \tag{A.15}$$

Introduce the matrices

$$\check{\Sigma}^{\pm}(q,\check{\zeta},\rho) = \begin{pmatrix} \check{\Sigma}_H^{\pm}(q,\check{\zeta},\rho) & 0 \\ 0 & \check{\Sigma}_P^{\pm}(q,\check{\zeta},\rho) \end{pmatrix}, \tag{A.16}$$

where $\check{\Sigma}_P^{\pm}(q,\check{\zeta},\rho) = \Sigma_P^{\pm}(q,\rho,\check{\zeta})$. As used above, the vector bundles $\mathbb{E}_H^{\pm}(q,\rho\check{\zeta})$ and hence $\mathbb{E}_2(q,\rho\check{\zeta})$ have continuous extensions to $\rho = 0$. We denote the later by $\check{\mathbb{E}}_2(q,\check{\zeta},\rho)$. By (A.10), (A.11) and (A.14), there holds

$$\pm\check{\Sigma}^{\pm}(q,\check{\zeta},\rho) \geq \kappa(\check{\Pi}^{\pm})^*\check{\Pi}^{\pm} - (\mathrm{Id} - \check{\Pi}^{\pm})^*(\mathrm{Id} - \check{\Pi}^{\pm}) \tag{A.17}$$

where $\mathrm{Id} - \check{\Pi}^{\pm}(q,\check{\zeta},\rho)$ is a projection on $\check{\mathbb{E}}_2^{\pm}(q,\check{\zeta},\rho)$.

Next, we rewrite the boundary conditions:

$$\check{\Gamma}(q,\check{\zeta},\rho)(U^-,U^+,\varphi) = \begin{pmatrix} [u] + \varphi[\check{\mathcal{R}}(0,q,\check{\zeta},\rho)] \\ [v] + \varphi[\partial_z\check{\mathcal{R}}(0,q,\check{\zeta},\rho)] \\ \ell(q)\cdot u^- \end{pmatrix} \tag{A.18}$$

where we have used the the notation

$$\mathcal{R}^{\pm}(z,q,\zeta) = \rho\check{\mathcal{R}}^{\pm}(z,q,\check{\zeta},\rho). \tag{A.19}$$

Transformed by $\mathcal{T}(0)$, they become

$$\check{\Gamma}_2(U_2^-,U_2^+,\varphi) = \check{\Gamma}(\mathcal{T}^-(0)U_2^-,\mathcal{T}^+(0)U_2^+,\varphi).$$

By Proposition 3.15, we know that the vector ${}^t([\check{\mathcal{R}}(0,q,\check{\zeta},\rho)],[\partial_z\check{\mathcal{R}}(0,q,\check{\zeta},\rho)])$ does not vanish and that the Evans' function $D_2(q,\rho\check{\zeta})$ extends continuously as

$$\check{D}_2(q,\check{\zeta},\rho) = \det\big(\check{\mathbb{E}}_2^- \times \check{\mathbb{E}}_2^+ \times \mathbb{C}, \ker \check{\Gamma}_2\big).$$

The uniform stability condition implies that this function is bounded from below by a positive constant when $q \in \mathcal{Q}_0$, $\check{\zeta} \in \overline{S}^d_+$ and $\rho \in [0,\rho_0]$ for some $\rho_0 > 0$. In particular,

$$\Big(\check{\mathbb{E}}_2^-(q,\check{\zeta},\rho) \times \check{\mathbb{E}}_2^+(q,\check{\zeta},\rho) \times \mathbb{C}\Big) \cap \ker \check{\Gamma}_2(q,\check{\zeta},\rho) = \{0\},$$

for all $(q,\check{\zeta},\rho)$ in the compact set $\overline{\mathcal{Q}}_0 \times \overline{S}^d_+ \times [0,\rho_0]$. Therefore, if κ is chosen large enough, there are constants C and $c > 0$ such that for all $(q,\check{\zeta},\rho) \in \overline{\mathcal{Q}}_0 \times \overline{S}^d_+ \times [0,\rho_0]$ and all $(U^-,U^+,\varphi) \in \mathbb{C}^{2N} \times \mathbb{C}^{2N} \times \mathbb{C}$:

$$\begin{aligned}(\check{\Sigma}^+U^+,U^+) - (\check{\Sigma}^-U^-,U^-) + &C|\check{\Gamma}_2(U^-,U^+,\varphi)|^2 \\ &\geq c\big(|U^-|^2 + |U^+|^2 + |\varphi|^2\big).\end{aligned} \tag{A.20}$$

On the left hand side, $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{C}^{2N}$.

Indeed, the construction of $\check{\Sigma}^\pm_H$ is made locally near points $(\underline{q},\underline{\check{\zeta}},0)$, because the choice of projectors $\check{\Pi}^\pm_H$ is local. Then, by compactness, one can can find $\rho_0 > 0$ small enough, κ large, constants C and $c > 0$, a finite covering $\cap\Omega_k$ of $\overline{\mathcal{Q}}_0 \times \overline{S}^d_+ \times [0,\rho_0]$ and symmetrizers $\check{\Sigma}^\pm_{H,k}$ such that (A.20) is satisfied by $\check{\Sigma}^\pm_k$ on Ω_k. Using a partition of unity, yields $\check{\Sigma}^\pm_H = \sum \chi_k \check{\Sigma}^\pm_{H,k}$ which satisfy (A.15) and $\check{\Sigma}^\pm$ defined by (A.16) satisfies (A.20) globally.

A.2 Symmetrizers and L^2 estimates

We fix $\delta > 0$ and $\varepsilon_0 > 0$ such that for $|x| \leq 2\delta$ and $(t,y) \in \mathbb{R}^{1+d}$, $q_\varepsilon(t,y,x) \in \mathcal{Q}_0$. Next, we fix $\rho_0 > 0$ such that the symbols introduced in the previous subsection are defined for $|\zeta| \leq 2\rho_0$ with $\gamma \geq 0$. As indicated in section 5, we fix cut off functions $\kappa(x)$ supported in $\{|x| \leq 2\delta\}$ and equal to one for $|x| \leq \delta$, $\chi(\zeta)$ and $\chi_1(\zeta)$ supported in $|\zeta| \leq 2\rho_0$, with $\chi_1\chi = \chi$. We consider additional cut-off functions κ_a, with $\kappa_a\kappa_{a'} = \kappa_a$ when $a' > a > 0$, supported in $\{|x| \leq 2\delta\}$ and equal to one on the support of κ, and χ_a supported in $|\zeta| \leq 2\rho_0$, equal to one on the support of χ and such that $\chi_a\chi_{a'} = \chi_a$ when $0 < a < a'$.

With $\mathcal{G}_2(q,\zeta)$ and $\mathcal{T}(z,q,\zeta) = \mathcal{W}\mathcal{V}$ as in the previous subsection, define:

$$\begin{aligned} g^\pm_{2,\varepsilon}(t,y,x,\zeta) &= \kappa_3(x)\chi_1(\zeta)\mathcal{G}^\pm_2(q_\varepsilon(t,x,y),\zeta), \\ w^\pm_\varepsilon(t,y,x,\zeta) &= \kappa_2(x)\chi_1(\zeta)(\mathcal{T}^\pm)^{-1}(\frac{x}{\varepsilon},q_\varepsilon(t,x,y,\zeta) \end{aligned}$$

By (A.6), $g_{2,\varepsilon}$ is block diagonal and we use the notation

$$g^{\pm}_{2,\varepsilon} = \begin{pmatrix} h^{\pm}_{\varepsilon} & 0 \\ 0 & \pi^{\pm}_{\varepsilon} \end{pmatrix}$$

They are bounded families of symbols in the class $P\Gamma^{0}_{1,m}$, compactly supported in ζ. The intertwining relations (A.5) (A.6) imply that

$$\varepsilon \partial_x w_\varepsilon + w_\varepsilon g_\varepsilon = g_{2,\varepsilon} w_\varepsilon + \varepsilon r_\varepsilon \tag{A.21}$$

with r_ε bounded in $P\Gamma^{0}_{0,m}$.

With $U^{\pm}_1$ defined by (5.76), let

$$U^{\pm}_2 = P^{\varepsilon,\gamma}_{\chi_{1/2}} P^{\varepsilon,\gamma}_{w^{\pm}_{\varepsilon}} U^{\pm}_1 . \tag{A.22}$$

The equation (5.77), the identity (A.21) and the symbolic calculus imply that the components (u_2, v_2) of U_2 satisfy:

$$\partial_x u^{\pm}_2 = \frac{1}{\varepsilon} P^{\varepsilon,\gamma}_{h^{\pm}_{\varepsilon}} u^{\pm}_2 + f^{\pm}_2 , \tag{A.23}$$

$$\partial_x v^{\pm}_2 = \frac{1}{\varepsilon} P^{\varepsilon,\gamma}_{\pi^{\pm}_{\varepsilon}} v^{\pm}_2 + g^{\pm}_2 , \tag{A.24}$$

where f_2 and g_2 satisfy (5.49). We refer to [MZ1], section 4, for details.

To the symbols Σ_P, we associate the bounded family in $P\Gamma^{0}_{1,m}$:

$$\sigma^{\pm}_{P,\varepsilon}(t,y,x,\zeta) = \kappa_4(x)\chi_2(\zeta)\Sigma^{\pm}_P(q_\varepsilon(t,y,x),\zeta)$$

and the symmetrizers:

$$\mathfrak{S}^{\pm}_P = \gamma \mathrm{Re} P^{\varepsilon,\gamma}_{\sigma^{\pm}_{P,\varepsilon}} - \varepsilon \sum_{j=0}^{d-1} \partial_j \mathrm{Re} P^{\varepsilon,\gamma}_{\sigma^{\pm}_{P,\varepsilon}} \partial_j .$$

In this definition, ∂_0 stands for ∂_t. We have the identities

$$\begin{aligned} -\big(\mathfrak{S}^{-}_P(0) v^{-}_2(0), v^{-}(0)\big) + \mathrm{Re}\big(\mathcal{S}^{-}_P P^{\varepsilon,\gamma}_{\pi^{-}_{\varepsilon}} v^{-}_2 , v^{-}_2\big) \\ = -\big([\partial_x, \mathfrak{S}^{-}_P] v^{-}_2 , v^{-}\big) - 2\mathrm{Re}\big(\mathcal{S}^{-}_P g^{-}_2 , v^{-}_2\big) \end{aligned}$$

$$\begin{aligned} \big(\mathfrak{S}^{+}_P(0) v^{+}_2(0), v^{+}(0)\big) + \mathrm{Re}\big(\mathcal{S}^{+}_P P^{\varepsilon,\gamma}_{\pi^{+}_{\varepsilon}} v^{+}_2 , v^{+}_2\big) \\ = -\big([\partial_x, \mathfrak{S}^{+}_P] v^{+}_2 , v^{+}\big) - 2\mathrm{Re}\big(\mathcal{S}^{+}_P g^{+}_2 , v^{+}_2\big) \end{aligned}$$

where $(\cdot,\cdot)$ denotes the L^2 scalar product, on $\{x \leq 0\}$, $\{x \geq 0\}$ or $\{x = 0\}$, the domain being clear from the context. Using the symbolic calculus and (A.12) as in [MZ1], one derives the estimates

$$\frac{1}{\varepsilon}\|v_2\|^2_{0,(\lambda)} + I \lesssim \varepsilon\|g_2\|^2_{0,(\lambda)} + \|v_2\|^2_0, \tag{A.25}$$

where

$$I := \big(\mathfrak{S}^+_P(0)v_2^+(0), v_2^+(0)\big) - \big(\mathfrak{S}^-_P(0)v_2^-(0), v_2^-(0)\big). \tag{A.26}$$

In (A.25), the norms of v_2 are the sum of the norms of v_+^- and v_2^+ on $\{x \geq 0\}$ and $\{x \leq 0\}$ respectively. We use a similar notation for g_2.

We proceed in a similar way for the hyperbolic component u_2. However, because the symbolic analysis is made in polar coordinates, we are led to switch to the homogeneous calculus. Introduce

$$\mathrm{h}^\pm_\varepsilon(t,y,x,\zeta) := \frac{1}{\varepsilon}h^\pm_\varepsilon(t,y,x,\varepsilon\zeta) = \kappa_3(x)\chi_1(\varepsilon\zeta)|\zeta|\check{H}\big(q_\varepsilon(t,y,x), \frac{\zeta}{|\zeta|}, \varepsilon|\zeta|\big)$$

where the last equality follows from (A.13). The $\mathrm{h}^\pm_\varepsilon$ are bounded families of symbols in $\Gamma^1_{1,m}$. Using Proposition 4.12 and arguing as in [MZ1], one can show that

$$\|\frac{1}{\varepsilon}P^{\varepsilon,\gamma}_{h^\pm_\varepsilon}u_2^\pm - T^\gamma_{\mathrm{h}^\pm_\varepsilon}u_2^\pm\|_0 \lesssim \|u_2^\pm\|_0.$$

Therefore, we can replace (A.23) by

$$\partial_x u_2^\pm - T^\gamma_{\mathrm{h}^\pm_\varepsilon}u_2^\pm = \tilde{f}_2^\pm$$

where $\|f_2'\|_0 \lesssim \|f_2\| + \|u_2\|_0$. Given the symmetrizers $\check{\Sigma}_H(q,\check{\zeta},\rho)$, introduce the symbols:

$$\check{\sigma}^\pm_{H,\varepsilon}(t,y,x,\zeta) = \kappa_4(x)\chi_2(\varepsilon\zeta)\check{\Sigma}^\pm_H\big(q_\varepsilon(t,y,x), \frac{\zeta}{|\zeta|}, \varepsilon|\zeta|\big).$$

They are bounded in $\Gamma^0_{1,m}$. Introduce next the symmetrizers

$$\mathfrak{S}^\pm_H = \gamma\mathrm{Re}T^\gamma_{\check{\sigma}^\pm_{H,\varepsilon}} - \varepsilon\sum_{j=0}^{d-1}\partial_j\mathrm{Re}T^\gamma_{\check{\sigma}^\pm_{H,\varepsilon}}\partial_j.$$

It yields to the following estimates:

$$\|u_2\|^2_{0,(\lambda^2)} + II \lesssim \|f_2\|^2_0 + \|u_2\|^2_{0,(\lambda)}, \tag{A.27}$$

where

$$II := \big(\mathfrak{S}^+_H(0)u^+_2(0), u^+_2(0)\big) - \big(\mathfrak{S}^-_H(0)u^-_2(0), u^-_2(0)\big). \tag{A.28}$$

For a detailed proof we refer again to the proof of Proposition 4.12 of [MZ1]. Using that $\lambda \le \alpha\lambda^2 + 1/4\alpha$, we can replace $\|u_2\|^2_{0,(\lambda)}$ by $\|u_2\|^2_0$ in the right hand side of (A.27).

We now come to the new part of the proof, which is the analysis of the boundary terms $I + II$. With obvious notations, they split into $I^\pm$ and $II^\pm$, and

$$II^\pm = \gamma \mathrm{Re}\big(T^\gamma_{\check\sigma^\pm_{H,\varepsilon}} u^\pm_2(0), u^\pm_2(0)\big) + \sum_{j=0}^{d-1} \varepsilon \mathrm{Re}\big(T^\gamma_{\check\sigma^\pm_{H,\varepsilon}} \partial_j u^\pm_2(0), \partial_j u^\pm_2(0)\big). \tag{A.29}$$

To compare the terms I and II, we use the same quantization T^γ for I. On $x = 0$, introduce the symbols

$$\check\sigma^\pm_{P,\varepsilon}(t, y, \zeta) = \sigma^\pm_{P,\varepsilon}(t, y, 0, \varepsilon\zeta)$$

which are bounded in $\Gamma^0_{1,m}$. Replacing $P^{\varepsilon,\gamma}_{\sigma_{P,\varepsilon}}$ by $T^\gamma_{\check\sigma_{P,\varepsilon}}$ in the definition of $\mathfrak{S}_P$, and using Proposition 4.12, we see that

$$I^\pm = \gamma \mathrm{Re}\big(T^\gamma_{\check\sigma^\pm_{P,\varepsilon}} v^\pm_2(0), v^\pm_2(0)\big) + \sum_{j=0}^{d-1} \varepsilon \mathrm{Re}\big(T^\gamma_{\check\sigma^\pm_{P,\varepsilon}} \partial_j v^\pm_2(0), \partial_j v^\pm_2(0)\big) + e. \tag{A.30}$$

where

$$|e| \lesssim |v_2(0)|_0 |v_2(0)|_{0,(1+\varepsilon|\zeta|)} \lesssim |v_2(0)|_0 |v_2(0)|_{0,(\lambda)}.$$

The last estimate is the consequence of the inequality $1 + \varepsilon|\zeta| \lesssim \lambda$ valid on support of the Fourier transform of $v_2(0)$.

Next, we rewrite the boundary conditions (5.78) for U_2 and ψ. Introduce the symbols

$$\check w^\pm_\varepsilon(t, y, \zeta) = \chi_2(\zeta)\mathcal{T}^\pm(0, q_\varepsilon(t, y, 0), \zeta).$$

Thus $\check w^\pm_\varepsilon \chi_{1/2} w^\pm_{\varepsilon\,|x=0} = \chi_{1/2}(\zeta)$ and Proposition 4.9 implies that

$$P^{\varepsilon,\gamma}_{\check w^\pm_\varepsilon} U^\pm_2(0) = P^{\varepsilon,\gamma}_{\chi_{1/2}} U^\pm_1(0) + e^\pm_1 \tag{A.31}$$

where

$$|e^\pm_1|_{m,(\lambda)} \lesssim |e^\pm_1|_{m,(\Lambda/\sqrt{\varepsilon})} \lesssim \sqrt{\varepsilon}|U^\pm_1(0)|_m. \tag{A.32}$$

We switch to the homogeneous quantization and introduce

$$\mathrm{w}_\varepsilon^\pm(t,y,\zeta) = \breve{w}_\varepsilon^\pm(t,y,\varepsilon\zeta).$$

They are bounded families of symbols in $\Gamma^0_{1,m}$ on the boundary. By Proposition 4.12,

$$e_2^\pm := P^{\varepsilon,\gamma}_{\breve{w}_\varepsilon^\pm} U_2^\pm(0) - T^\gamma_{\mathrm{w}_\varepsilon^\pm} U_2^\pm(0)$$

satisfies

$$|e_2^\pm|_{m,(\lambda)} \lesssim |e_2^\pm|_{m,(|\zeta|)} \lesssim |U_2^\pm(0)|_m. \tag{A.33}$$

Similarly, we replace the $P^{\varepsilon,\gamma}_{[S_{j,\varepsilon}]\chi}$ which act on $\nabla_\gamma\psi$ by $T^\gamma_{\mathrm{S}_{j,\varepsilon}\chi(\varepsilon\zeta)}$ where

$$\mathrm{S}_j(t,y,\zeta) = [S_{j,\varepsilon}](t,y,\varepsilon\zeta),$$

to the price of an error e_3 which satisfies

$$|e_3|_{0,(\lambda)} \lesssim |\nabla_\gamma\psi|_0.$$

Introduce next the bounded family in $\Gamma^0_{1,m}$:

$$\mathrm{R}_\varepsilon = \frac{i\tau+\gamma}{|\zeta|}\mathrm{S}_{0,\varepsilon} + \sum_{j=1}^{d-1}\frac{i\eta_j}{|\zeta|}\mathrm{S}_{j,\varepsilon}.$$

With

$$\varphi = T^\gamma_{|\zeta|\chi(\varepsilon\zeta)}\psi, \tag{A.34}$$

there holds

$$T^\gamma_{\mathrm{S}_\varepsilon\chi(\varepsilon\zeta)}\nabla_\gamma\psi = T^\gamma_{\mathrm{R}_\varepsilon}\varphi.$$

The last term in (5.78) is $P^{\varepsilon,\gamma}_{\ell_\varepsilon} u_1^-(0)$. Since ℓ_ε is independent of ζ, we replace $P^{\varepsilon,\gamma}_{\ell_\varepsilon}$ by $T^\gamma_{\ell_\varepsilon}$ and $u_1^-(0)$ by $T^\gamma_{\mathrm{w}_\varepsilon^{-,1}} U_2^-(0)$ where $\mathrm{w}_\varepsilon^{-,1}$ denotes the first N rows of the $2N\times 2N$ matrix w_ε^-. Summing up, introduce the $(2N+1)\times(4N+1)$ matrix of zero-th order symbols

$$\Upsilon_\varepsilon = \begin{pmatrix} -\mathrm{w}_\varepsilon^- & \mathrm{w}_\varepsilon^+ & \mathrm{R}_\varepsilon \\ \ell_\varepsilon\cdot\mathrm{w}_\varepsilon^{-,1} & 0 & 0 \end{pmatrix}. \tag{A.35}$$

Applying $P^{\varepsilon,\gamma}_{\chi_{1/2}} = T^\gamma_{\chi_{1/2}(\varepsilon\zeta)}$ to the boundary conditions (5.78) implies that

$$T^\gamma_{\Upsilon_\varepsilon}\Phi = \theta \tag{A.36}$$

(A.37) $$|\theta|_{m,(\lambda)} \lesssim |U(0)|_m + |\nabla_\gamma \psi|_m.$$

where

(A.38) $$\Phi = \begin{pmatrix} U_2^-(0) \\ U_2^+(0) \\ \varphi \end{pmatrix}.$$

Similarly, introduce the self adjoint matrix of symbols

$$J_\varepsilon = \begin{pmatrix} -\check\sigma_\varepsilon^- & 0 & 0 \\ 0 & \check\sigma_\varepsilon^+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{with} \quad \check\sigma_\varepsilon^\pm = \begin{pmatrix} \check\sigma_{H,\varepsilon}^\pm & 0 \\ 0 & \check\sigma_{P,\varepsilon}^\pm \end{pmatrix}.$$

The condition (A.20) implies that $J_\varepsilon + C\Upsilon_\varepsilon\Upsilon_\varepsilon$ is definite positive for $\varepsilon\zeta$ in the support of $\chi_{1/2}$. Now, we are again in a situation studied in [MZ1], proof of Proposition 4.12. Since U_2 and φ are spectrally supported in the support of $\chi_{1/2}(\varepsilon\zeta)$, Gårding's inequality implies that

$$\big|T^\gamma_{\chi_{1/2}(\varepsilon\zeta)}\Phi\big|_0^2 \lesssim \big(T^\gamma_{J_\varepsilon}\Phi, \Phi\big) + C\big|T^\gamma_{\Upsilon_\varepsilon}\Phi\big|_0^2 + |\Phi|^2_{0,(|\zeta|^{-1})}.$$

Commuting with $\varepsilon\partial_j$, one obtains similar estimates for the derivatives. Since $\lambda^2 \approx \gamma + \varepsilon(\tau^2 + |\eta|^2)$ on the support of $\chi_{1/2}(\varepsilon\zeta)$, this implies the following estimate:

$$\begin{aligned}\big|T^\gamma_{\chi_{1/2}(\varepsilon\zeta)}\Phi\big|^2_{0,(\lambda)} \lesssim \gamma\big(T^\gamma_{J_\varepsilon}\Phi, \Phi\big) &+ \sum_{j=0}^{d-1}\big(T^\gamma_{J_\varepsilon}\partial_j\Phi, \partial_j\Phi\big) \\ &+ \big|T^\gamma_{\Upsilon_\varepsilon}\Phi\big|^2_{0,(\lambda)} + |\Phi|_0^2.\end{aligned}$$

Adding up, with (A.25), (A.27), (A.33), (A.30), one proves the following estimates:

Proposition A.1. *The solution* (U_2, φ) *of* (A.23), (A.24), (A.36) *satisfies*

$$\begin{aligned}\|u_2\|_{0,(\lambda^2)} + \frac{1}{\sqrt{\varepsilon}}\|v_2\|_{0,(\lambda)} + |U_2(0)|_{0,(\lambda)} + |\varphi|_{0,(\lambda)} \lesssim \|f_2\|_0 \\ + \sqrt{\varepsilon}\|g_2\|_{0,(\lambda)} + \|u_2\|_{0,(\lambda)} + \|v_2\|_0 + |U_2(0)|_0 + |\varphi|_0.\end{aligned}$$

The norms $\|u\|$ stand for the sum of the norms $\|u^+\|$ and $\|u^-$, taken on the half space $\{x \geq 0\}$ and $\{x \leq 0\}$ respectively. Similarly, the trace $U_2(0)$ denotes the couple $(U_2^-(0), U_2^+(0))$.

A.3 End of proof

The proposition above is the exact analogue of Proposition 4.12 of [MZ1]. Commuting the equations (A.23), (A.24), (A.36) with the vector fields Z_j yields:

Proposition A.2. *The solution* (U_2, φ) *of* (A.23), (A.24), (A.36) *satisfies*

$$\|u_2\|_{m,(\lambda^2)} + \frac{1}{\sqrt{\varepsilon}} \|v_2\|_{m,(\lambda^2)} + |U_2(0)|_{m,(\lambda)} + |\varphi|_{m,(\lambda)} \lesssim \|f_2\|_m + \sqrt{\varepsilon}\|g_2\|_{m,(\lambda)} + \|u_2\|_{m,(\lambda)} + \|v_2\|_m + |U_2(0)|_m + |\varphi|_m.$$

The proof, which is by induction on m, is identical to the proof of Proposition 4.13 of [MZ1], and we do not repeat the details. Using that $\lambda \leq \alpha\lambda^2 + 1/(4\alpha)$ with α small enough, the error term $\|u_2\|_{m,(\lambda)}$ in the right hand side can be changed to $\|u_2\|_m$.

Next, we come back to U_1 and ψ. First, we note that (A.34) immediately implies that

$$|P^{\varepsilon,\gamma}_{\chi} \nabla_\gamma \psi|_{m,(\lambda)} \lesssim |\varphi|_{m,(\lambda)}.$$

Next, we extend the previous definition of the symbols $\breve{w}_\varepsilon$ to $x \neq 0$, setting:

$$\breve{w}^{\pm}_{\varepsilon}(t, y, x, \zeta) = \kappa_3(x)\chi_2(\zeta)\mathcal{T}^{\pm}(\frac{x}{\varepsilon}, q_\varepsilon(t, y, x), \zeta).$$

Thus $\breve{w}^{\pm}_{\varepsilon}\chi_{1/2} w^{\pm}_{\varepsilon\,|x=0} = \chi_{1/2}(\zeta)$ and Proposition 4.9 implies that

$$P^{\varepsilon,\gamma}_{\breve{w}^{\pm}_{\varepsilon}} U_2^{\pm} = P^{\varepsilon,\gamma}_{\chi_{1/2}} U_1^{\pm} + \varepsilon e^{\pm}$$

where

$$\|\varepsilon e^{\pm}\|_{m,(\lambda^2)} \lesssim |e^{\pm}|_m \lesssim \sqrt{\varepsilon}|U_1^{\pm}|_m.$$

(For the first inequality, we have used that the U_1 and U_2 are spectrally supported in a domain where $\varepsilon\zeta$ is bounded). Repeating the proof of Proposition 4.14 of [MZ1], which takes into account the exponential decay of $\mathcal{W} - \mathrm{Id}$ at infinity and the special form of the matrices $\mathcal{V}(q, 0)$, yields the estimates:

$$\|P^{\varepsilon,\gamma}_{\chi_{1/2}} u_1\|_{0,(\lambda^2)} + \frac{1}{\sqrt{\varepsilon}} \|P^{\varepsilon,\gamma}_{\chi_{1/2}} v_1\|_{0,(\lambda^2)} + |P^{\varepsilon,\gamma}_{\chi_{1/2}} U_1(0)|_{0,(\lambda)} + |P^{\varepsilon,\gamma}_{\chi} \nabla_\gamma \psi|_{0,(\lambda)} \lesssim \|F'\|_m + \|u\|_m + \|v\|_m + |U_1(0)|_m + |\nabla_\gamma \psi|_m,$$

where F' is the right hand side of the equation (5.77) for U_1. With the estimate (5.49) of F', this implies (5.80), finishing the proof of Proposition 5.18.

References

[Ag] S. Agmon, *Problèmes mixtes pour les équations hyperboliques d'ordre supérieur, Les Équations aux Dérivées Partielles.* Éditions du Centre National de la Recherche Scientifique, Paris (1962) 13–18.

[AGJ] J. Alexander-R. Gardner-C.K.R.T. Jones, *A topological invariant arising in the analysis of traveling waves.* J. Reine Angew. Math. 410 (1990) 167–212.

[Al] S.Alinhac, *Existence d'ondes de raréfaction pour des sytèmes quasi-linéaires hyperboliques multidimensionnels*, Comm. Partial Diff.Equ., 14 (1989), 173-230.

[AMPZ] A. Azevedo-D. Marchesin-B. Plohr-K. Zumbrun, *Nonuniqueness of solutions of Riemann problems.* Z. Angew. Math. Phys. 47 (1996), 977–998.

[BiB] S. Bianchini and A. Bressan, *The vanishing viscosity limit for a class of viscous hyperbolic systems.* preprint (2002).

[BBB] C.Bardos-D.Brezis-H.Brezis, *Perturbations singulières et prolongement maximaux d'opérateurs positifs*, Arch.Rational Mech. Anal., 53 (1973, 69–100.

[Ba-Ra] C.Bardos-J.Rauch *Maximal positive boundary value problems as limits of singular perturbation problems*, Trans. Amer.Math.Soc., 270 (1982), 377–408.

[Bo] J.M.Bony, . *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann.Sc.E.N.S. Paris, 14 (1981) pp 209-246.

[Ch-P] J. Chazarain-A. Piriou, *Introduction to the theory of linear partial differential equations,* Translated from the French. Studies in Mathematics and its Applications, 14. North-Holland Publishing Co., Amsterdam-New York, 1982. xiv+559 pp. ISBN: 0-444-86452-0.

[Co-Me] R.Coifmann-Y.Meyer. *Au delà des opérateurs pseudo-différentiels*, Astérisque 57, (1978).

[Co] W. A. Coppel, *Stability and asymptotic behavior of differential equations*, D.C. Heath, Boston.

[E1] J.W. Evans, *Nerve axon equations: I. Linear approximations*. Ind. Univ. Math. J. 21 (1972) 877–885.

[E2] J.W. Evans, *Nerve axon equations: II. Stability at rest*. Ind. Univ. Math. J. 22 (1972) 75–90.

[E3] J.W. Evans, *Nerve axon equations: III. Stability of the nerve impulse*. Ind. Univ. Math. J. 22 (1972) 577–593.

[E4] J.W. Evans, *Nerve axon equations: IV. The stable and the unstable impulse*. Ind. Univ. Math. J. 24 (1975) 1169–1190.

[FS] Freistühler, H. and Szmolyan,P., *Spectral stability of small shock waves*, to appear in Arch. Rat. Mech. Analysis. Preprint, 2002.

[GZ] R. Gardner-K. Zumbrun, *The Gap Lemma and geometric criteria for instability of viscous shock profiles*. Comm. Pure Appl. Math. 51 (1998), 797–855.

[Gi-Se] M. Gisclon-D. Serre, *Conditions aux limites pour un système strictement hyperbolique fournies par le schéma de Godunov*. RAIRO Modél. Math. Anal. Numér. 31 (1997), 359–380.

[Go1] J. Goodman, *Remarks on the stability of viscous shock waves*, in: Viscous profiles and numerical methods for shock waves. (Raleigh, NC, 1990), 66–72, SIAM, Philadelphia, PA, (1991).

[Go2] J. Goodman, *Stability of viscous scalar shock fronts in several dimensions*, TAMS 311. 1989, 683-695.

[GM] J. Goodman and J. R. Miller, *Long-time behavior of scalar viscous shock fronts in two dimensions*, J. Dynam. Diff. Eqns. 11 (1999), 255-277.

[GX] J. Goodman and Z. Xin, *Viscous limits for piecewise smooth solutions to systems of conservation laws*, Arch. Rational Mech. Analysis 121. 1992, 235-265.

[Gr-Gu] E. Grenier and O. Guès, *Boundary layers for viscous perturbations of noncharacteristic quasilinear hyperbolic problems*. J. Differential Equations 143 (1998), 110–146.

[Gr-Ro] E. Grenier-F. Rousset, *Stability of one-dimensional boundary layers by using Green's functions*. Comm. Pure Appl. Math. 54 (2001), 1343–1385.

[Gu1] O.Guès, *Perturbations visqueuses de problèmes mixtes hyperboliques et couches limites*, Ann.Inst.Fourier, 45 (1995),973–1006.

[Gu2] O.Guès, *Problème mixte hyperbolique quasilinéaire caractéristique*, Comm. in Part.Diff.Equ., 15 (1990), 595–645.

[GMWZ1] O.Guès-G.Métivier-M.Williams-K.Zumbrun *Multi-dimensional viscous shocks I: Degenerate symmetrizers and long time stability* , submitted, available at http://www.math.unc.edu/Faculty/williams/

[GMWZ2] O.Guès-G.Métivier-M.Williams-K.Zumbrun *Multidimensional viscous shocks II: The small viscosity limit* , to appear in Comm. Pure Appl. Math., 2004, available at http://www.math.unc.edu/Faculty/williams/

[GMWZ3] O.Guès-G.Métivier-M.Williams-K.Zumbrun *Navier–Stokes regularization of multidimensional Euler shocks*, in preparation.

[GW] O.Guès and M. Williams, *Curved shocks as viscous limits: a boundary problem approach*, Indiana Univ. Math. J. 51. 2002, 421-450.

[HoZ] D. Hoff-K. Zumbrun, *Multi-dimensional diffusion waves for the Navier-Stokes equations of compressible flow.* Indiana Univ. Math. J. 44 (1995), 603–676.

[Hör] L.Hörmander. *Lectures on Nonlinear Hyprbolic Differential Equations*, Mathématiques et Applications 26, Sringer Verlag, 1997.

[J] C.K.R.T. Jones, *Stability of the travelling wave solution of the FitzHugh–Nagumo system.* Trans. Amer. Math. Soc. 286 (1984), 431–469.

[K] T. Kapitula, *On the stability of travelling waves in weighted L^∞ spaces.* J. Diff. Eqs. 112 (1994), 179–215.

[Kr] H.O. Kreiss, *Initial boundary value problems for hyperbolic systems,* Comm. Pure Appl. Math. 23 (1970) 277-298.

[Lio] J.L.Lions, *Perturbations singulières dans les problèmes aux limites et en contrôle optimal*, Lectures Notes in Math., 323, Sringer Verlag, 1973.

[Maj] A. Majda, *The stability of multi-dimensional shock fronts – a new problem for linear hyperbolic equations.* Mem. Amer. Math. Soc. 275 (1983).

[Mé1] G. Métivier, *Interaction de deux chocs pour un système de deux lois de conservation, en dimension deux d'espace.* Trans. Amer. Math. Soc. 296 (1986) 431–479.

[Mé2] G. Métivier, *Stability of multidimensional shocks.* Advances in the theory of shock waves, 25–103, Progr. Nonlinear Differential Equations Appl., 47, Birkhäuser Boston, Boston, MA, 2001.

[Mé3] G.Métivier. *The Block Structure Condition for Symmetric Hyperbolic Problems*, Bull. London Math.Soc., 32 (2000), 689–702

[MZ1] G.Métivier-K.Zumbrun, *Viscous Boundary Layers for Noncharacteristic Nonlinear Hyperbolic Problems*, preprint.

[MZ2] G.Métivier-K.Zumbrun, *Symmetrizers and continuity of stable subspaces for parabolic-hyperbolic boundary value problems*, to appear in Disc. Cont. Dyn. Syst.

[Mey] Y.Meyer,. *Remarques sur un théorème de J.M.Bony*, Supplemento al Rendiconti der Circolo Matematico di Palermo, Serie II, No1, 1981.

[Mok] A.Mokrane *Problèmes mixtes hyperboliques non linéaires*, Thesis, Université de Rennes 1, 1987.

[PW] R. L. Pego-M.I. Weinstein, *Eigenvalues, and instabilities of solitary waves.* Philos. Trans. Roy. Soc. London Ser. A 340 (1992), 47–94.

[PZ] Plaza, R. and Zumbrun, K., *An Evans function approach to spectral stability of small-amplitude shock profiles*, Preprint, 2002.

[Ral] J.V. Ralston, *Note on a paper of Kreiss,* Comm. Pure Appl. Math. 24 (1971) 759–762.

[Ra1] J.Rauch, *BV estimates fail for most quasilinear hyperbolic systems in dimensions greater than one*, Comm. Math. Phys., 106 (1986), 481–484.

[Ra2] J.Rauch, *Symmetric positive systems with boundary characteristic of constant multiplicity*, Trans. Amer.Math.Soc, 291 (1985), 167–185.

[Ra-Ma] J. Rauch-F. Massey, *Differentiability of solutions to hyperbolic intitial boundary value problems.* Trans. Amer. Math. Soc. 189 (1974) 303-318.

[Ro1] F. Rousset, *Inviscid boundary conditions and stability of viscous boundary layers.* Asymptot. Anal. 26 (2001), no. 3-4, 285–306.

[Ro2] F. Rousset, *Viscous approximation of strong shocks of systems of conservation laws.* SIAM J. Math. Anal. 35 (2003), no. 2, 492–519.

[S] D. Serre, *Sur la stabilité des couches limites de viscosité.* Ann. Inst. Fourier (Grenoble) 51 (2001), 109–130.

[St] E.Stein,. *Singular Integrals and Differentiability Propseties of Functions*, Princeton Univ. Press, Princeton, 1970.

[Tay] M.Taylor. *Partial Differential Equations*III, Applied Mathematical Sciences 117, Springer, 1996.

[Yu] S.-H. Yu. *Zero-dissipation limit of solutions with shocks for systems of hyperbolic conservation laws.* Arch. Ration. Mech. Anal. 146 (1999), no. 4, 275–370.

[ZS] K. Zumbrun-D.Serre, *Viscous and inviscid stability of multidimensional planar shock fronts.* Indiana Univ. Math. J. 48 (1999), 937–992.

[ZH] K. Zumbrun-P. Howard, *Pointwise semigroup methods and stability of viscous shock waves.* Indiana Mathematics Journal V47 (1998), 741–871.

[Zu1] K. Zumbrun, *Multidimensional stability of planar viscous shock waves.* Advances in the theory of shock waves, 307–516, Progr. Nonlinear Differential Equations Appl., 47, Birkhäuser Boston, Boston, MA, 2001.

[Zu2] K. Zumbrun, *Refined Wave–tracking and Nonlinear Stability of Viscous Lax Shocks.* Methods Appl. Anal. 7 (2000) 747–768.