L^2 well posed Cauchy Problems and Symmetrizability of First Order Systems (revised 2016)

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Abstract

The Cauchy problem for first order system $L(t, x, \partial_t, \partial_x)$ is known to be well posed in L^2 when a it admits a microlocal symmetrizer $S(t, x, \xi)$ which is smooth in ξ and Lipschitz continuous in (t, x). This paper contains three main results. First we show that a Lipshitz smoothness globally in (t, x, ξ) is sufficient. Second, we show that the existence of symmetrizers with a given smoothness is equivalent to the existence of *full symmetrizers* having the same smoothness. This notion was first introduced in [FrLa1]. This is the key point to prove the third result that the existence of microlocal symmetrizer is preserved if one changes the direction of time, implying local uniqueness and finite speed of propagation.

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1 Introduction

This paper is concerned with the well posedness in L^2 of the Cauchy problem for first order $N \times N$ systems

$$(1.1)\qquad \begin{cases} Lu := A_0(t,x)\partial_t u + \displaystyle\sum_{j=1}^{d} A_j(t,x)\partial_{x_j} u + B(t,x)u = f, \qquad t > 0, \\ u_{|t=0} = u_0. \end{cases}$$

The starting point is the well known theory of hyperbolic symmetric systems in the sense of Friedrichs ([Fr1, Fr2]): if the matrices A_j are Lipschitz continuous on $[0,T]\times\mathbb{R}^d$, hermitian symmetric, and if A_0 is definite positive with A_0^{-1} bounded, then for all $u_0 \in L^2(\mathbb{R}^d)$ and $f \in L^1([0,T]; L^2(\mathbb{R}^d))$, the equation (1.1) has a unique solution $u \in C^0([0,T]; L^2(\mathbb{R}^d))$ which satisfies

$$(1.2)\qquad \big\|u(t)\big\|_{L^2(\mathbb{R}^d)} \le C\big\|u_0\big\|_{L^2(\mathbb{R}^d)} + C\int_0^t \big\|Lu(s)\big\|_{L^2(\mathbb{R}^d)} ds,$$

for some constant C independent of u_0. Additional properties are local uniqueness and finite speed propagation. The question discussed in this paper is to know for which systems these properties remain true.

For scalar equations of order m, the analogue would be the well posedness in Sobolev spaces H^{m-1}, for which strict hyperbolicity is necessary ([IvPe]) and sufficient ([Gå2, Le]). This completely settles the question for scalar equations but for systems, the situation is much more complex.

A necessary condition has been given by V.Ivrii and V.Petkov ([IvPe]): they have shown that if the estimate (1.2) is valid for $u \in C_0^\infty(]0, T[\times \mathbb{R}^d)$, then there exists a bounded microlocal symmetrizer $S(t, x, \xi)$ for (1.1) (the precise definition is recalled below). This is equivalent to a strong form of hyperbolicity of the principal symbol, which we call *strong hyperbolicity of the symbol*, namely that $L+B_1$ is hyperbolic for all matrix $B_1(t, x)$. Of course it is stronger than hyperbolicity which is known to be a necessary condition for the Cauchy problem to be well posed in C^∞ (see [La1, Mi1, Gå1] and the review paper [Gå3]). In particular, (1.2) are the best estimates in terms of regularity that one can expect for the Cauchy problem.

On the side of sufficient conditions, except in the constant coefficient case, where the energy estimate (1.2) is easily obtained on the space-Fourier transform of the equation, the existence of a bounded symmetrizer does not imply in general that the problem is well posed, even in C^∞, A counterexample is given in [St] and another one is proposed in Section 3. Besides the case of symmetric systems recalled above for which the symmetrizer $S(t, x)$ is independent of ξ, the Cauchy problem is known to be well posed in L^2 when the microlocal symmetrizer is smooth in ξ and at least Lipschitz continuous in (t, x) (see [La2, Me] and Theorem 1.4 below for a precise statement). In this case, the energy estimates are proven using the usual pseudo-differential calculus or the para-differential calculus when the coefficient have limited smoothness. This covers the case of strictly hyperbolic systems and the more general case of hyperbolic systems with constant multiplicity (e.g. [Ca], [Ya]). This also applies to the case of "generic" double eigenvalues, still assuming the strong hyperbolicity of the symbol, see Theorem 3.6 below.

The first objective of this paper is to revisit these questions under the angle of the smoothness of the symmetrizer. We prove that the Lipschitz continuity in (t, x, ξ) for $\xi \neq 0$ of the symmetrizer S is sufficient to obtain the L^2 estimates and the L^2 well posedness. In addition, we give examples and counterexamples showing that the Lipschitz condition is sharp.

The second main result of this paper is to prove that the existence of microlocal symmetrizer is preserved by a change of time, as this is essential to obtain local uniqueness and the precise description of the propagation of the support of solutions (see [JMR1, Ra]). More surprisingly, we show that the existence of symmetrizers of a given smoothness is equivalent to the existence of *full symmetrizers* of the same smoothness, a notion introduced in [FrLa1]. This link is the key point in the proof of existence of microlocal symmetrizers in any direction of hyperbolicity.

We now briefly present the results. Note that (1.2) applied to $e^{\gamma t}u$, $u \in C_0^\infty(]0,T[\times\mathbb{R}^d)$ implies that

$$\forall \gamma \geq \gamma_0, : \quad \gamma \big\| u \big\|_{L^2(\mathbb{R}^{1+d})} \leq C \big\| (L + \gamma A_0) u \big\|_{L^2(\mathbb{R}^{1+d})}, \tag{1.3}$$

for some constants C and γ_0 independent of u. This estimate is elliptic like: with $\chi \in C_0^\infty(\mathbb{R}^{1+d})$ and $\underline{u} \in \mathbb{C}^N$, , applying it to

$$u(t,x) = e^{i\lambda(t\tau + x\xi)} \lambda^{-\alpha d/2} \chi(\lambda^{\frac{1}{2}}(t - t_0, x - x_0)) \underline{u},$$

and to $\gamma = \lambda\gamma_0$, and letting λ tend to $+\infty$ implies

Lemma 1.1. *Suppose that the coefficients of L are continuous and bounded on the open set Ω and there are constants γ_0 and C such that*

$$\gamma \big\| u \big\|_{L^2(\Omega)} \leq C \big\| \big(L + \gamma A_0) \big) u \big\|_{L^2(\Omega)}. \tag{1.4}$$

for all $\gamma \geq \gamma_0$ and $u \in C_0^\infty(\Omega)$. Then the principal symbol $L_1(t,x,\tau,\xi)$ of L satisfies for all $(t,x) \in \Omega$, all $\gamma \in \mathbb{R}$ and $u \in \mathbb{C}^N$:

$$|\gamma| \big| u \big| \leq C \big| (L_1(t,x,\tau,\xi) + i\gamma A_0(t,x)) \underline{u} \big|. \tag{1.5}$$

There is no sign condition on γ as seen by changing $\tilde{\xi}$ to $-\tilde{\xi}$. When it holds, we say that the *symbol is strongly hyperbolic* in the time direction. The condition (1.5) has several equivalent formulations, see Section 4. One of them is that the symbols admits a bounded symmetrizer.

Definition 1.2. *A microlocal symmetrizer for L_1 is a bounded matrix $S(t,x,\xi)$, homogeneous of degree 0 in $\xi \neq 0$, such that $S(t,x,\xi)A_0(t,x)$ is symmetric and uniformly definite positive, and $S(t,x,\xi)A(t,x,\xi)$ is symmmetric, where $A(t,x,\xi) = \sum A_j(t,x)\xi_j$.*

Combining Lemma 1.1 and Theorem 4.10 below, we recover the the necessary condition given in [IvPe]:

Proposition 1.3. *If L has continuous coefficient on the open set Ω and there are constants γ_0 and C such that* (1.4) *is satisfied, then the principal symbol L_1 must admit a bounded symmetrizer $S(t,x,\xi)$ on $\Omega \times \mathbb{R}^d \setminus 0$.*

In the constant coefficients case, the existence of a bounded symmetrizer is also sufficient, as immediately seen by Fourier synthesis. For variable coefficients, this condition is far from being sufficient for the well posedness of the Cauchy problem (1.1): in section 3 we give an example of a 3×3 systems in space dimension $d = 2$, whose symbol $L(x,\tau,\xi)$ is strongly hyperbolic uniformly in x, and such that the Cauchy problem (1.1) is ill posed, even locally and with C^∞ data.

On the side of sufficient conditions, let us first recall the following result:

Theorem 1.4. *Suppose that the coefficients $A_j \in W^{1,\infty}([0,T]\times\mathbb{R}^d)$ and there exists a microlocal symmetrizer S, homogenenous of degree* 0 *and C^∞ in $\xi \neq 0$ which satisfies $\partial^\beta_{t,x}\partial^\alpha_\xi S \in L^\infty([0,T]\times\mathbb{R}^d\times S^{d-1})$ for all $\alpha\in\mathbb{N}^d$ and all $|\beta|\leq 1$. Then, there are constants C and γ such that for all $u_0 \in L^2(\mathbb{R}^d)$ and $f \in L^1([0,T];L^2(\mathbb{R}^d))$, the Cauchy problem* (1.1) *has a unique solution $u \in C^0([0,T];L^2(\mathbb{R}^d))$ which satisfies*

$$\big\|u(t)\big\|_{L^2(\mathbb{R}^d)} \leq Ce^{\gamma_0 t}\big\|u_0\big\|_{L^2(\mathbb{R}^d)} + C\int_0^t e^{\gamma_0(t-s)}\big\|Lu(s)\big\|_{L^2(\mathbb{R}^d)}ds. \tag{1.6}$$

When the symmetrizer does not depend on ξ, this is Friedrichs theory, in which case the estimate (1.6) is easily obtained by forming the real part of the scalar product of SLu with u and performing integrations by parts. For microlocal symmetrizers, one replaces the multiplication by S by the action of the pseudodifferential operator $S(t,x,D_x)$ ([La2]) when the coefficients are also smooth in x, or by a paradifferential version when the coefficients are Lipschitz (see eg [Me]). This theorems applies to hyperbolic systems with constant multiplicities which admit smooth symmetrizers. Indeed, multiple eigenvalues of $A(t,x,\xi)$ with variable multiplicities are the main difficulty for the construction of smooth symmetrizers. However, we note in Proposition 3.6 that the theorem above applies to strongly hyperbolic systems which have only generic double eigenvalues.

The first main result of the paper extends this result to Lipshitz symmetrizers, using a Wick quantization of the symbols.

Theorem 1.5. *Suppose that the coefficients $A_j \in W^{2,\infty}(\mathbb{R}^{d+1})$ and there exists a microlocal symmetrizer, homogenenous of degree* 0 *in $\xi \neq 0$ and Lipschtiz continuous in (t,x,ξ) on $\mathbb{R}^{d+1}\times S^{d-1}$. Then, there are constants*

C and γ such that for all $u_0 \in L^2(\mathbb{R}^d)$ and $f \in L^1([0,T];L^2(\mathbb{R}^d))$, the Cauchy problem (1.1) *has a unique solution $u \in C^0([0,T];L^2(\mathbb{R}^d))$ which satisfies* (1.6).

This theorem is proved in Section 2. In Section 3 we discuss the existence of Lipschitz symmetrizers. In particular, we give examples of systems which admit a Lipschitz symmetrizer but no C^1 symmetrizer. We also prove that the Lipschitz condition is sharp, in the sense that for all $\mu < 1$, there are examples of systems admitting Hölder continuous symmetrizers of order $\mu < 1$, for which the Cauchy problem with C^∞ data is is locally ill posed.

Remark 1.6. In this theorem we assume that the coefficients are $W^{2,\infty}$ whereas $W^{1,\infty}$ was sufficient when the symmetrizers are smooth in ξ. This is due to the use of the Wick quantization. One one hand it helps to deal with symbols which are not smooth in ξ. On the other hand, the symbolic calculus is less precise, and the $W^{2,\infty}$ smoothness of the coefficient is used to prove that in $Op^{Wick}(S) \circ A(x,\partial_x) - Op^{Wick}(iSA)$ is bounded in L^2. At the present time, it is not known wether this additional smoothness which is crucial for the proof is necessary or not for the validity of the result.

The second part of the paper is concerned with the local theory of the Cauchy problem and the finite speed propagation property for the support of the solutions. A classical proof of this property relies on the invariance of the assumptions by changes of time variables, so that one can convexify the initial surface, The existence of a local symmetrizer is clearly invariant by change of time, as well as strict hyperbolicity or the property that the characteristic variety is smooth with constant multiplicities. In all these cases the local theory was well established. This invariance is not clear for the existence of smooth microlocal symmetrizers. However, when there are smooth symmetrizers, local uniqueness and finite speed of propagation are proved in [Ra] using another approach bases on finite difference approximation schemes and uniform estimates due to [La-Ni, Va].

The second main theorem of this paper asserts that the existence of a Lipschitz symmetrizer [resp. C^∞] is preserved by change of timelike directions. This is a key step for establishing a local theory, starting with local uniqueness, finite speed of propagation and ending with the sharp description of the propagation of support as stated in [JMR1, Ra].

Let $\tilde{x}$ denote the space-time variables (t,x) and set accordingly $\tilde{\xi} = (\tau, \xi)$ Assuming that $L_1(\tilde{x}, \tilde{\xi})$ is hyperbolic in the time direction $(1,0) \in \mathbb{R}^{1+d}$, denote by $\Gamma_{\tilde{x}}$ the cone of hyperbolic directions that is the component of $(1,0)$ in $\{\tilde{\xi} : \det L_1(\tilde{x}, \tilde{\xi}) \neq 0\}$.

Theorem 1.7. *Suppose that the coefficients A_j are Lipschitz continuous [resp C^∞] on $\mathbb{R}^{d+1}$ and that there exists a microlocal symmetrizer $S(t,x,\xi)$, homogenenous of degree 0 in $\xi \neq 0$ and Lipschtiz continuous in [resp C^∞] (t,x,ξ) on $\mathbb{R}^{d+1} \times S^{d-1}$. Then, for any time-like direction $\tilde\nu \in \Gamma_{t,x}$, the symbol $L(t,x,\tilde\nu)^{-1}A(t,x,\xi)$ admits a Lipschitz [resp C^∞] symmetrizer.*

Corollary 1.8. *Under the assumptions of Theorem 1.5, the Cauchy problem for L with initial data on any space like hyperplane is well posed in L^2.*

In Theorem 4.13, we prove that one can choose symmetrizers which also depend smoothly on $\tilde\nu$. As said above, with Theorems 1.4 and 1.5, this implies local uniqueness, and finite speed of propagation. Together with the Lipschitz dependence of the cone of propagation implied by Proposition 5.4, this allows to the results on the precise propagation of support stated in [JMR1, Ra] . We refer the reader to these papers for precise statements.

The proof of this theorem is based on an intrinsic characterization of the existence of Lipschitz symmetrizers which uses the notion of *full symmetrizers* introduced by K.O.Friedrichs and P.Lax [FrLa1]:

Definition 1.9. *A full symmetrizer for (1.1) is a bounded matrix $\widetilde S(t,x,\tau,\xi)$, homogeneous of degree 0 in $(\tau,\xi) \neq 0$, such that $\widetilde S(t,x,\tau,\xi)L(t,x,\tau,\xi)$ is symmmetric.*

$\widetilde S$ is said to be positive in the direction $\tilde\nu$, if $\mathrm{Re}\,\widetilde S(t,x,\tau,\xi)L(t,x,\tilde\nu)$ is definite positive on $\ker L(t,x,\tau,\xi)$ for all (t,x,τ,ξ).

Of course the condition is nontrivial only near characteristic points, but says nothing about hyperbolicity. Our third main theorem is the following;

Theorem 1.10. *Suppose that L is hyberpolic in the time direction. Then, L admits a continuous [resp. Lipschitz] microlocal symmetrizer $S(t,x,\xi)$, if and only if it admits a continuous [resp. Lipschitz] full symmetrizer $\widetilde S(t,x,\tau,\xi)$ which is positive in the time direction.*

In this case, $\widetilde S$ is positive in any direction of hyperbolicity $\tilde\nu$.

2 Lipschitz symmetrizability is sufficient for the L^2 well posedness

The goal of this section is to prove Theorems 1.5. We consider a system

$$Lu = \sum_{j=0}^{d} A_j(\tilde x)\partial_{\tilde x_j} u \tag{2.1}$$

with coefficients A_j which are at least $W^{1,\infty}([0;T] \times \mathbb{R}^d)$.

2.1 Wave packets and localization

For $u \in L^2(\mathbb{R}^s)$, $\lambda > 0$ and $B \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, let

$$W_{\lambda,B}u(x,\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{\lambda}{\pi}\right)^{\frac{d}{4}} \int e^{i(x-y)\xi - \frac{1}{2}\lambda|x-y|^2} B(x,y)u(y)dy \tag{2.2}$$

Lemma 2.1. *The operator $W_{\lambda,B}$ is bounded from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{2d})$ and*

$$\big\|W_{\lambda,B}u\big\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \leq \big\|B\big\|_{L^\infty}\big\|u\big\|_{L^2(\mathbb{R}^d)}. \tag{2.3}$$

Moreover, if $B(x,z) \equiv \mathrm{Id}$, $W_\lambda := W_{\lambda,\mathrm{Id}}$ is isometric from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d \times \mathbb{R}^d)$.

Proof. Let $\mathscr{F}$ denote the Fourier transform and $v_x(y) = B(x,y)e^{-\frac{1}{2}\lambda|x-y|^2}u(y)$. Then

$$W_{\lambda,B}u(x,\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{\lambda}{\pi}\right)^{\frac{d}{4}} e^{i\lambda x\xi}\mathscr{F}\big(v_x\big)(\xi).$$

Therefore

$$\begin{aligned}\int \big|W_B u(x,\xi)\big|^2 dxd\xi &= \left(\frac{\lambda}{\pi}\right)^{\frac{d}{2}} \int \big|v_x(y)\big|^2 dxdy \\ &= \left(\frac{\lambda}{\pi}\right)^{\frac{d}{2}} \int e^{-\lambda|x-y|^2}\big|B(x,y)u(y)\big|^2 dxdy \leq \big\|B\big\|^2_{L^\infty}\ \big\|u\big\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

When $B = 1$, the inequality is an equality. □

We will adapt the scale λ to the size of the frequency $|\xi|$. One has

$$W_\lambda u(x,\xi) = (2\pi)^{-d}\left(\frac{1}{\pi\lambda}\right)^{\frac{d}{4}} \int e^{ix\eta - \frac{1}{2\lambda}|\xi-\eta|^2}\hat{u}(\eta)d\eta.$$

This shows that for a fixed ξ, $W_\lambda u(\,\cdot\,,\xi)$ is the inverse Fourier transform of $w_\xi(\eta) = \big(\frac{1}{\pi\lambda}\big)^{\frac{d}{4}}\ e^{-\frac{1}{2\lambda}|\xi-\eta|^2}\hat{u}(\eta)$. Therefore

$$\begin{aligned}\int \big|W_\lambda u(x,\xi)\big|^2 dx &= (2\pi)^{-d}\int \big|w_\xi(\eta)\big|^2 d\eta \\ &= (2\pi)^{-d}\left(\frac{1}{\pi\lambda}\right)^{\frac{d}{2}} \int e^{-\frac{1}{\lambda}|\xi-\eta|^2}|\hat{u}(\eta)|^2 d\eta.\end{aligned} \tag{2.4}$$

Integrating in ξ, we recover the isometry of W_λ, but the important point is that we use (2.4) to localize in $|\xi|$.

Consider a dyadic partition of unity

$$1 = \varphi_0(\xi) + \sum_{j=1}^{\infty} \theta_j(\xi) \tag{2.5}$$

with $\varphi_0 \in C_0^\infty(\mathbb{R}^d)$, supported in $\{|\xi| \le 2\}$ and equal to one on $\{|\xi| \le 1\}$, $\theta_j(\xi) = \varphi_j(\xi) - \varphi_{j-1}(\xi)$ and $\varphi_j(\xi) = \varphi_0(2^j\xi)$ for $j \ge 1$. To unify notations, we set $\theta_0 = \varphi_0$ and for $j \ge 0$ we denote by define Θ_j the operator

$$\Theta_j u = \mathscr{F}^{-1}\big(\theta_j \hat{u}\big),$$

so that

$$u = \sum_{j=0}^{\infty} \Theta_j u \tag{2.6}$$

Proposition 2.2. *For all n, m and α, there is a constant C such that for all $j \ge 0$*

$$\big\| |\xi|^m (1 - \varphi_{j+2}) W_{2^j} \partial_y^\alpha \Theta_j u \big\|_{L^2(\mathbb{R}^{2d})} \le C 2^{-jn} \big\| \Theta_j u \big\|_{L^2(\mathbb{R}^d)}, \tag{2.7}$$

and for $j \ge 2$,

$$\big\| |\xi|^m \varphi_{j-2} W_{2^j} \partial_y^\alpha \Theta_j u \big\|_{L^2(\mathbb{R}^{2d})} \le C 2^{-jn} \big\| \Theta_j u \big\|_{L^2(\mathbb{R}^d)}. \tag{2.8}$$

Proof. By (2.4)

$$\big\| (1 - \varphi_{j+2}) W_{2^j} \partial_y^\alpha \Theta_j u \big\|^2_{L^2(\mathbb{R}^{2d})} = (2\pi)^{-d} \left(\frac{1}{\pi 2^j} \right)^{\frac{d}{2}}$$
$$\int e^{-2^{-j}|\xi-\eta|^2} (1 - \varphi_{j+2}(\xi))^2 |\eta^\alpha|^2 (\theta_j(\eta))^2 |\hat{u}(\eta)|^2 d\eta d\xi.$$

On the support of $(1 - \varphi_{j+2}(\xi))\theta_j(\eta)$, one has $|\xi| \ge 2^{j+2}$, $|\eta| \le 2^{j+1}$ so that $|\xi - \eta| \ge \frac{1}{2}|\xi|$ and therefore $2^{-j}|\xi - \eta|^2 \ge \frac{1}{2} 2^{-j} |\xi - \eta|^2 + \frac{1}{2}|\xi|$. Hence,

$$\big\| 2^{jn} |\xi|^m (1 - \varphi_{j+2}) W_{2^j} \partial_y^\alpha \Theta_j u \big\|^2_{L^2(\mathbb{R}^{2d})} \le (2\pi)^{-d} \left(\frac{1}{\pi 2^j} \right)^{\frac{d}{2}}$$
$$2^{2jn} 2^{2(j+1)|\alpha|} e^{-2^j} \int |\xi|^{2m} e^{-\frac{1}{2}|\xi|} e^{-\frac{1}{2} 2^{-j} |\xi-\eta|^2} |\theta_j(\eta) \hat{u}(\eta)|^2 d\eta d\xi$$
$$\le C \big\| \Theta_j u \big\|^2_{L^2(\mathbb{R}^d)}.$$

This proves (2.7). Similarly

$$\big\|(\varphi_{j-2})W_{2^j}\partial_y^\alpha\Theta_j u\big\|^2_{L^2(\mathbb{R}^{2d})} = (2\pi)^{-d}\left(\frac{1}{\pi 2^j}\right)^{\frac{d}{2}}$$
$$\int e^{-2^{-j}|\xi-\eta|^2}\varphi_{j-2}(\xi)^2|\eta^\alpha|^2(\theta_j(\eta))^2|\hat{u}(\eta)|^2 d\eta d\xi.$$

On the support of $\varphi_{j-2}(\xi)\theta_j(\eta)$, one has $|\xi| \le 2^{j-1}$ and $|\eta| \ge 2^j$ so that $|\xi-\eta| \ge \frac{1}{2}|\eta|$ and therefore $2^{-j}|\xi-\eta|^2 \ge \frac{1}{2}2^{-j}|\xi-\eta|^2 + 2^{j-3}$. Hence,

$$\big\|2^{jn}|\xi|^m\varphi_{j-2}W_{2^j}\partial_y^\alpha\Theta_j u\big\|^2_{L^2(\mathbb{R}^{2d})} \le (2\pi)^{-d}\left(\frac{1}{\pi 2^j}\right)^{\frac{d}{2}}$$
$$2^{2jn}2^{2(j+1)|\alpha|}2^{2jm}e^{-2^{j-3}}\int e^{-\frac{1}{2}2^{-j}|\xi-\eta|^2}|\theta_j(\eta)\hat{u}(\eta)|^2 d\eta d\xi$$
$$\le C\big\|\Theta_j u\big\|^2_{L^2(\mathbb{R}^d)},$$

and this proves (2.8). □

Corollary 2.3. *There is a constant C such that for all $j \ge 0$,*

$$\big\|(1-\varphi_{j+2})W_{2^j}\Theta_j\partial_{x_k}u\big\|_{L^2(\mathbb{R}^{2d})} \le C\big\|\Theta_j u\big\|_{L^2(\mathbb{R}^d)} \tag{2.9}$$

and for $j \ge 2$,

$$\big\|\varphi_{j-2}W_{2^j}\Theta_j\partial_{x_k}u\big\|_{L^2(\mathbb{R}^{2d})} \le C\big\|\Theta_j u\big\|_{L^2(\mathbb{R}^d)}. \tag{2.10}$$

2.2 The main estimate

Let

$$A(x,\partial_x) = \sum_{j=1}^d A_j(x)\partial_{x_j}, \qquad A(x,\xi) = \sum_{j=1}^d \xi_j A_j(x). \tag{2.11}$$

We assume that we are given a matrix $S(x,\xi)$, homogeneous of degree 0 in ξ such that $S(x,\xi)A(x,\xi)$ is hermitian symmetric.

Let $\psi \in C_0^\infty(\mathbb{R}^d)$, , vanishing on a neighborhood of the origin. For $\lambda \ge 1$ introduce $\psi_\lambda(\xi) = \psi(\lambda^{-1}\xi)$.

Proposition 2.4. *Suppose that the coefficients A_j belong to $W^{2,\infty}(\mathbb{R}^d)$ and that $S \in W^{1,\infty}(\mathbb{R}^d \times S^{d-1})$. Then, there is a constant C such that for all $\lambda \ge 1$ and u*

$$\Big|\mathrm{Re}\Big(\psi_\lambda S W_\lambda u, W_\lambda A(x,\partial_x)u\Big)_{L^2(\mathbb{R}^{2d})}\Big| \le C\big\|u\big\|^2_{L^2}. \tag{2.12}$$

Proof. Let $S_\lambda = \varphi_\lambda S$ and consider the self adjoint operator $\Sigma_\lambda = W_\lambda^* S_\lambda W_\lambda$:

$$\Sigma_\lambda u(x) = \kappa \lambda^{\frac{d}{2}} \int e^{\Phi_\lambda(x,y,z,\xi)} S_\lambda(z,\xi) u(y) dz d\xi dy$$

where κ is a normallization factor and

$$\Phi_\lambda(x,y,z,\xi) = i(x-y)\xi - \frac{1}{2}\lambda(|x-z|^2 + |y-z|^2).$$

The estimate to prove is

$$\Big| \mathrm{Re}\, \big(A(x,\partial_x)^* \Sigma_\lambda u, u \big)_{L^2} \Big| \le C \big\| u \big\|_{L^2}^2. \tag{2.13}$$

One can replace $A(x,\partial_x)^*$ by $\tilde{A}^* := \sum -A_j^* \partial_{x_j}$ since the difference is bounded in L^2 with norm $O(\sup_j \|A_j\|_{W^{1,\infty}})$. One has

$$A_j^* \partial_j \Sigma_\lambda u(x) = \kappa \int e^{\Phi_\lambda} A_j^*(x) \big(i\xi_j - \lambda(x_j - z_j) \big) S_\lambda(z,\xi) u(y) dz d\xi dy.$$

Therefore

$$\tilde{A}^*(x,\partial_x)\Sigma_\lambda = W_\lambda^*(-iA^* S_\lambda) W_\lambda + \sum_j (R_j^1 + R_j^2) \tag{2.14}$$

where

$$R_j^1 u(x) = i\kappa \int e^{\Phi_\lambda} \xi_j \big(A_j^*(z) - A_j^*(x)) \big) S_\lambda(z,\xi) u(y) dz d\xi dy,$$

$$R_j^2 u(x) = \kappa \int e^{\Phi_\lambda} \lambda (x_j - z_j) A_j^*(x) \big) S_\lambda(z,\xi) u(y) dz d\xi dy.$$

Because S is a symmetrizer for $\sum A_j \xi_j$, the matrix $iA^*(x,\xi) S_\lambda$ is skew symmetric and thus the real part of the first term in (2.14) vanishes and it is sufficient to show that the remainders $R_j^{1,2}$ are bounded in L^2. Write

$$A_j^*(x) - A_j^*(z) = \sum_k (x_k - z_k) \tilde{A}_{j,k}(x,z), \tag{2.15}$$

with $\tilde{A}_{j,k} \in W^{1,\infty}$, and use that

$$2\lambda(x_k - z_k) = \partial_{z_k} \Phi_\lambda - i\lambda \partial_{\xi_k} \Phi_\lambda := Z_k \Phi_\lambda$$

Integrating by parts in (z,ξ) yields that $R_j^1 = \sum_k R_{j,k}^1$ with

$$R_{j,k}^1 u(x) = -\frac{i\kappa}{\lambda} \int e^{\Phi_\lambda} Z_k^* \big(\xi_j A_{j,k}^* S_\lambda \big) u(y) dz d\xi dy.$$

Note that $\frac{1}{\lambda} Z_k^*\big(\xi_j A_{j,k}^* S_\lambda\big)$ is a sum a terms of the form $B_l(x,z) S_l(z,\xi,\lambda)$ where the B_l and S_l are uniformly bounded. Therefore $R_{j,k}^1$ is of the sum of the operators $W_{\lambda,B_l}^* S_l W_\lambda$ where the definition of W_{λ,B_l} is given in 2.2. Lemma 2.1 implies that the R_j^1 are uniformly in λ,

The analysis of R_j^2 is similar. One has

$$R_j^2 u(x) = -\frac{i\kappa}{\lambda} \int e^{\Phi_\lambda} A_j^*(x) (Z_j^* S_\lambda) u(y) dz d\xi dy.$$

Hence $R_j^2 = -A_j^* W^*(Z_j S_\lambda) W$ is bounded in L^2 since $Z_j S_\lambda$ is bounded. $\square$

2.3 Proof of Theorem 1.5

Suppose that the operator L in (2.1) has $W^{2,\infty}$ coefficients. Without loss of generality, multiplying L on the left by A_0^{-1}, we assume that the coefficient of D_t is $A_0 = \mathrm{Id}$ so that $L = \partial_t + A(t,x,\partial_x)$. We are given a Lipschtiz symmetrizer $S(t,x,\xi)$ which is uniformly definite positive and such that

$$S, \ \partial_{t,x} S, \ |\xi| \partial_\xi S \in L^\infty. \tag{2.16}$$

Consider the energy

$$\mathcal{E}_t(u) = \sum_{j=0}^{\infty} \big(S(t) W_{2^j} \Theta_j u, W_{2^j} \Theta_j u \big)_{L^2(\mathbb{R}^{2d})}. \tag{2.17}$$

Lemma 2.5. *There are constants $C \geq c > 0$ such that*

$$c \big\| u \big\|_{L^2(\mathbb{R}^d)}^2 \leq \mathcal{E}(u) \leq C \big\| u \big\|_{L^2(\mathbb{R}^d)}^2.$$

Proof. Because $S(t,x,\xi)$ is definite positive bounded from above and from below,

$$\mathcal{E}_t(u) \approx \sum_{j=0}^{\infty} \big\| W_{2^j} \Theta_j u \big\|_{L^2(\mathbb{R}^{2d})}^2 = \sum_{j=0}^{\infty} \big\| \Theta_j u \big\|_{L^2(\mathbb{R}^{2d})}^2 \approx \big\| u \big\|_{L^2(\mathbb{R}^{2d})}^2.$$

$\square$

The Theorems follows from the energy estimate

Proposition 2.6. *If u satisfies $Lu = f$, then*

$$\frac{d}{dt} \mathcal{E}_t(u(t)) \leq C \Big(\big\| f(t) \big\|_{L^2(\mathbb{R}^{2d})} \big\| u(t) \big\|_{L^2(\mathbb{R}^{2d})} + \big\| u(t) \big\|_{L^2(\mathbb{R}^{2d})}^2 \Big). \tag{2.18}$$

Proof. One has

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_t(u(t)) &= (\partial_t\mathcal{E})(u(t)) + 2\mathrm{Re}\,\widetilde{\mathcal{E}}_t(u(t),\partial_t u(t)))\\ &= (\partial_t\mathcal{E})(u(t)) + 2\mathrm{Re}\,\widetilde{\mathcal{E}}_t(u(t),f(t))) - 2\mathrm{Re}\,\widetilde{\mathcal{E}}_t(u(t),Au(t)))\end{aligned}$$

where $\partial_t\mathcal{E}_t$ is the expression (2.17) with S replaced by $\partial_t S$ and $\widetilde{\mathcal{E}}$ is the bilinear version of $\mathcal{E}$. Because $\partial_t S \in L^\infty$, the first term is $O(\|u(t)\|^2_{L^2})$. Similarly, the second term is $O(\|u(t)\|_{L^2}\|f(t)\|_{L^2})$ and it remains to prove that

$$\big|\mathrm{Re}\,\widetilde{\mathcal{E}}_t\big(u(t),Au(t)\big)\big\| \le C\big\|u(t)\big\|^2_{L^2}. \tag{2.19}$$

For simplicity we drop the time from the notations, t being a parameter and all the estimate below being uniform in t.

The expression to consider is

$$\widetilde{\mathcal{E}}(u,Au) = \sum_{j=0}^{\infty}\big(SW_{2^j}\Theta_j u, W_{2^j}\Theta_j Au\big)_{L^2(\mathbb{R}^{2d})}. \tag{2.20}$$

For $j=0$ and $j=1$, the L^2 norms of $\Theta_j u$ and $\Theta_j Au$ are $O(\|u\|_{L^2})$ and therefore it is sufficient to consider the terms with $j\ge 2$ in the sum (2.21).

Corollary 2.3 implies that

$$\big\|(1-\varphi_{j+2})W_{2^j}\Theta_j(A_k\partial_{x_k}u)\big\|_{L^2(\mathbb{R}^{2d}} \lesssim \big\|\Theta_j(\partial_{x_k}A_k)u)\big\|_{L^2} + \big\|\Theta_j(A_k u)\big\|_{L^2},$$

and therefore

$$\begin{aligned}&\sum_{j\ge 2}\Big|\big((1-\varphi_{j+2})SW_{2^j}\Theta_j u, W_{2^j}\Theta_j Au\big)_{L^2(\mathbb{R}^{2d})}\Big|\\ &\qquad\lesssim \sum_k \big\|u\big\|_{L^2}\big(\big\|A_k u\big\|_{L^2} + \big\|(\partial_{x_k}A_k)u\big\|_{L^2}\big) \lesssim \big\|u\big\|^2_{L^2}.\end{aligned}$$

Similarly,

$$\sum_{j\ge 2}\Big|\big(\varphi_{j-2}SW_{2^j}\Theta_j u, W_{2^j}\Theta_j Au\big)_{L^2(\mathbb{R}^{2d})}\Big| \lesssim \big\|u\big\|^2_{L^2}.$$

Hence, it is sufficient to consider

$$\sum_{j=2}^{\infty}\big(\psi_j SW_{2^j}\Theta_j u, W_{2^j}\Theta_j Au\big)_{L^2(\mathbb{R}^{2d})} \tag{2.21}$$

where $\psi_j = \varphi_{j+2}-\varphi_{j-2}$. Next, we replace $\Theta_j(Au)$ by $A\Theta_j u$ using the following lemma

Lemma 2.7. *The $g_j = [A, \Theta_j]u$ satisfy*

$$\sum \|g_j(t)\|_{L^2}^2 \lesssim \|u(t)\|_{L^2}^2. \tag{2.22}$$

Since $S(x,\xi)$ is bounded and the W_λ are isometries, this lemma implies

$$\Big|\sum_{j=0}^{\infty} \big(\psi_j S W_{2^j}\Theta_j u, W_{2^j} g_j\big)_{L^2(\mathbb{R}^{2d})}\Big| \lesssim \sum_{j=0}^{\infty} \|\Theta_j u\|_{L^2}\|g_j\|_{L^2} \lesssim \|u\|_{L^2}^2.$$

Summing up, we have proved that

$$\widetilde{\mathcal{E}}(u, Au) = \sum_{j=0}^{\infty} \big(\psi_j S W_{2^j}\Theta_j u, W_{2^j} A\Theta_j u\big)_{L^2(\mathbb{R}^{2d})} + O(\|u\|_{L^2}^2).$$

We are now in position to apply Proposition 2.4 to each term of the sum above. It implies that the real part of this sum is $O(\sum \|\Theta_j u\|_{L^2}^2) = O(\|u\|_{L^2}^2)$ finishing the proof of 2.19 and of the proposition. □

Proof of Lemma 2.7. Using the para-differential calculus (see e.g. [Me]) , one has

$$A(x, \partial_x) = T_{iA} + R$$

where T_{iA} is a paradifferential operator of symbol $iA(x,\xi)$ and R is bounded from L^2 to L^2. Thus

$$g_j = [T_A, \Theta_j]u + R\Theta_j u - \Theta_j R u.$$

The last two terms satisfy (2.22). The symbolic calculus implies that the $[T_{iA}, \Theta_j]$ are uniformly bounded in L^2 since the coefficients of A belong to $W^{1,\infty}$. Moreover, the spectral properties of the paradifferential calculus shows that $[T_{iA}, \Theta_j] = [T_{iA}, \Theta_j]\widetilde{\Theta}_j$ where the cut-off function $\tilde\theta_j$ is supported in the annulus $\{2^{j-n} \le |\xi| \le 2^{j+n}\}$ for some fixed n if $j \ge 1$ and in the ball $\{|\xi| \le 2^n\}$ if $j = 0$. Therefore $g'_j = [T_{iA}, \Theta_j]u$ satisfies

$$\|g'_j\|_{L^2} \lesssim \|\widetilde{\Theta}_j u\|_{L^2}$$

and thus (2.22). □

3 Examples and counterexamples

In this section, we discuss the existence of symmetrizers of limited smoothness. The case of generic double eigenvalues is specific, as shown in the next subsection. But for eigenvalues of higher order, it is easy to construct examples of systems with symmetrizers which necessarily have no, or a limited, smoothness.

Second, we show on an example that Lipschitz smoothness is sharp, even for well posedness in C^∞.

3.1 Example of non smooth symmetrizers

In space dimension two consider, near the origin, a system of the form

$$L_0(x, \partial_t, \partial_x, \partial_y) = \mathcal{L}_0(\partial_t, \partial_x, x\partial_y) = \partial_t + A\partial_x + xB\partial_y, \tag{3.1}$$

with $\mathcal{L}_0(\tau, \xi, \eta)$ strictly hyperbolic. Consider next a perturbation

$$L_a(x, \partial_t, \partial_x, \partial_y) = L_0(x, \partial_t, \partial_x, \partial_y) + xa(x)C\partial_y = \mathcal{L}(a(x), \partial_t, \partial_x, x\partial_y) \tag{3.2}$$

We will give explicit examples below. For a small, $\mathcal{L}(a, \tau, \xi, \eta)$ is still strictly hyperbolic and therefore it has smooth symmetrizers $\mathcal{S}(a, \xi, \eta)$ for $(\xi, \eta) \neq (0, 0)$, providing bounded symmetrizers for $L_a(x, \tau, \xi, \eta)$

$$S(a, x, \xi, \eta) = \mathcal{S}(a, \xi, x\eta) \tag{3.3}$$

for $(x, \xi) \neq (0, 0)$. On the unit sphere $\xi^2 + \eta^2 = 1$, they are smooth when $(x, \xi) \neq (0, 0)$. The definition of S can be extended at $(x, \xi) = (0, 0)$, but in general they have a singularity there.

Lemma 3.1. *Suppose in addition that $\mathcal{L}_0$ is symmetric. Then, for a small, there is a symmetrizer $\mathcal{S}$ of the form*

$$\mathcal{S}(a, \xi, \eta) = \mathrm{Id} + a\mathcal{S}_1(a, \xi, \eta) \tag{3.4}$$

with $\mathcal{S}_1$ homogeneous of degree 0 in (ξ, η) and smooth in (a, ξ, η) for (ξ, η) in unit sphere $\xi^2 + \eta^2 = 1$.

Proof. The spectral projectors $\Pi_j(a, \xi, \eta)$ are smooth in (a, ξ, η) for (ξ, η) in unit sphere $\xi^2 + \eta^2 = 1$ and $\mathcal{S} = \sum \Pi_j^* \Pi_j$ is a symmetrizer. Since $\mathcal{L}_0$ is symmetric, the Π_j are symmetric when $a = 0$ and therefore $\mathcal{S}(0, \xi, \eta) = \mathrm{Id}$, implying (3.4). $\square$

Substituting in (3.3) implies the following

Corollary 3.2. *If $\mathcal{L}_0$ is symmetric and strictly hyperbolic and $a(x) = |x|^\alpha$ with $0 < \alpha < 1$, L_a admits Hölder continuous symmetrizers $S(a, x, \xi, \eta)$ of class C^α.*

If $a(x) = x$, it admits a Lipschtiz symmetrizer.

Example 1: consider

$$L_a = \partial_t + \begin{pmatrix} 0 & \partial_x + xa\partial_y & x\partial_y \\ \partial_x - xa\partial_y & 0 & 0 \\ x(1+a^2))\partial_y & 0 & 0 \end{pmatrix} \tag{3.5}$$

In this case, $\det \mathcal{L}(a, \tau, \xi, \eta) = \tau(\tau^2 - \xi^2 - \eta^2)$ is always strictly hyperbolic.

Lemma 3.3. *If $a \neq 0$ is a constant, there are bounded symmetrizers $S(x, \xi, \eta)$ for L_a, but no continuous symmetrizers at $(x, \xi) = (0, 0)$ when $\eta = 1$.*

Proof. Fix $\eta = 1$. If $S(x, \xi)$ is a symmetrizer, then its complex conjugate is also a symmetrizer, so that $S + \overline{S}$ is a symmetrizer. Thus, it is sufficient to consider the case where S has real coefficients $s_{j,k}$. The symmetry condition reads

$$\begin{aligned} (\xi + ax)s_{11} &= (\xi - ax)s_{22} + (1 + a^2)xs_{23} \\ \eta s_{11} &= (\xi - ax)s_{23} + (1 + a^2)xs_{33} \\ \eta s_{12} &= (\xi + ax)s_{13}. \end{aligned}$$

The third condition is independent of the first two, it only involves s_{12} and s_{13}, and is trivially satisfied by $s_{12} = s_{13} = 0$.

There is no restriction in assuming that $s_{22} = 1$. Setting $s'_{11} = s_{11} - 1$, $s'_{33} = (1 + a^2)s_{33} - s_{11}$, one must have

$$\begin{aligned} (\xi + ax)s'_{11} &= -2ax + (1 + a^2)xs_{23} \\ xs'_{33} &= -(\xi - ax)s_{23}. \end{aligned} \tag{3.6}$$

Suppose that the coefficients are continuous at $(x, \xi) = (0, 0)$. Then taking $x = 0$ and $\xi \neq 0$ in the equations above, dividing by ξ and letting ξ tend to 0 implies that $s_{2,3}(0, 0) = s'_{11}(0, 0) = 0$. Taking $\xi = 0$ dividing by x and letting x tend to 0 implies that $s'_{11}(0, 0) = -2 + (1 + a^2)s'_{23}(0, 0)$ and $s'_{33}(0, 0) = as_{2,3}(0, 0)$. These conditions can be met only if $a = 0$. □

When $a(x) = x$, by Corollary 3.2, there is a Lipschitz symmetrizer, but it turns out that in this specific case, one can construct a C^∞ symmetrizer. The next example shows that this is not always the case.

Example 2 : Consider the 4×4 system with symbol

$$L_a = \partial_t + \begin{pmatrix} \Omega & aJ \\ 0 & 2\Omega \end{pmatrix}, \qquad \Omega = \begin{pmatrix} \xi & x\eta \\ x\eta & -\xi \end{pmatrix}, \qquad J = \begin{pmatrix} x\eta & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.7}$$

By Corollary 3.2, when $a(x) = x$, L_a has a Lipchitz symmetrizer but

Lemma 3.4. *When $a(x) = x$, there are no C^1 symmetrizers for L_a.*

Proof. Fix $\eta = 1$. Suppose that $S(\xi, x)$ is a C^1 symmetrizer near $(x, \xi) = (0, 0)$. We can assume that S has real coefficients. Using the block notation

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \tag{3.8}$$

the symmetry conditions imply

$$\Omega S_{12} - 2S_{12}\Omega = x^2 J_0 S_{11}, \qquad J_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.9}$$

This is a linear system in S_{12} and since Ω and 2Ω have no common eigenvalue it has a unique solution.

If S_{12} is C^1 near the origin, plugging its Taylor expansion $\Sigma_0 + x\Sigma_1 + \xi\Sigma_2$ in (3.9) and using the notation $\Omega = x\Omega_1 + \xi\Omega_2$, yields at first order

$$\Omega_1\Sigma_0 - 2\Sigma_0\Omega_1 = \Omega_2\Sigma_0 - 2\Sigma_0\Omega_2 = 0$$

which implies that $\Sigma_0 = 0$. The term in ξ^2 is

$$\Omega_2\Sigma_2 - 2\Sigma_2\Omega_2 = 0$$

showing that $\Sigma_2 = 0$. The term in $x\xi$ is then

$$\Omega_2\Sigma_1 - 2\Sigma_1\Omega_2 = 0$$

implying that $\Sigma_1 = 0$, which is incompatible with the equation given by the term in x^2:

$$\Omega_1\Sigma_1 - 2\Sigma_1\Omega_1 = -2J_0 S_{11}(0, 0) \neq 0$$

since $S_{11}(0, 0)$ must be definite positive. □

3.2 Existence of smooth symmetrizers for generic double eigenvalues

Consider a symbol $\tau \mathrm{Id} + A(a, \xi)$ which is strongly hyperbolic in the time direction, thus admitting a bounded symmetrizer $S(a, \xi)$. At $(\underline{a}, \underline{\xi})$, $\underline{\xi} \neq 0$, the characteristic polynomial $p(a, \tau, \xi) = \det(\tau \mathrm{Id} + A(a, \xi))$ has roots $\underline{\tau}_j$ of multiplicity m_j. Near this point, it can be smoothly factored

$$p(a, \tau, \xi) = \prod_j p_j(a, \tau, \xi) \tag{3.10}$$

with p_j of order m_j.

Assumption 3.5. *In a neighborhood of $(\underline{a}, \underline{\xi})$, $\underline{\xi} \neq 0$, the roots of p are either of constant multiplicity or of multiplicity at most two.*

In the second case, we assume that the multiplicity is two on a smooth manifold $\mathcal{M}$. Denoting by p_j the corresponding factor in (3.10)*, we further assume that either*

i) $\mathcal{M}$ has codimension one and the discriminant of p_j vanishes on $\mathcal{M}$ at finite order,

or

ii) $\mathcal{M}$ has codimension two and the discriminant of p_j vanishes on $\mathcal{M}$ exactly at order two.

Theorem 3.6. *Under these assumptions, there is a smooth symmetrizer $S(a, \xi)$ on a neighborhood of $(\underline{a}, \underline{\xi})$.*

Proof. The construction is local in $\rho = (a, \xi)$ and one can perform a block reduction of A near $\underline{\rho}$ and it is sufficient to construct a symmetrizer for each block. They are either diagonal and thus symmetric, or of dimension two. Eliminating the trace, it is therefore sufficient to consider matrices

$$A(\rho) = \begin{pmatrix} -a & b \\ c & a \end{pmatrix}. \tag{3.11}$$

The hyperbolicity condition is that the discriminant $\Delta = a^2 + bc$ is real and non negative. Strong hyperbolic, holds if and only if there is $\varepsilon > 0$ such that

$$\Delta = a^2 + bc \geq \varepsilon(|a|^2 + |b|^2 + |c|^2). \tag{3.12}$$

Our assumption is that Δ vanishes on a manifold $\mathcal{M}$, at finite order if $\operatorname{codim} \mathcal{M} = 1$ and at order two if $\operatorname{codim} \mathcal{M} = 2$.

a) If Δ vanishes at finite order on a manifold of codimension 1 of equation $\{\varphi = 0\}$, then (3.12) implies that for some integer k,

$$A = \varphi^k A_r \tag{3.13}$$

with $\det A_r \neq 0$ and still strongly hyperbolic. Thus A_r has distinct real eigenvalues and is therefore smoothly diagonalizable.

b) Suppose that Δ vanishes exactly at second order on $\mathcal{M}$ given by the equations $\{\varphi = \psi = 0\}$. This means that $\Delta \geq \varepsilon_1(\varphi^2 + \psi^2)$. Together with (3.12), this implies that A vanishes on $\mathcal{M}$ and that

$$A = \varphi A_1 + \psi A_2 \tag{3.14}$$

and A_1 and A_2 have distinct real eigenvalues at $\underline{\rho}$. We can smoothly conjugate A_1 to a real diagonal and traceless form and changing φ we are reduced to the case where

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -a_2 & b_2 \\ c_2 & a_2 \end{pmatrix}.$$

Moreover, changing φ to $\varphi - \operatorname{Re} a_2 \psi$, we can assume that $\operatorname{Re} a_2 = 0$. Since Δ is real

$$2\varphi \operatorname{Im} a_2 + \psi \operatorname{Im}(b_2 c_2) = 0 \tag{3.15}$$

Moroever, (3.12) implies for $\varphi = 0$

$$\operatorname{Re}(b_2 c_2) > (\operatorname{Im} a_2)^2 \tag{3.16}$$

and this remains true in a neighborhood of $\underline{\rho}$. In particular $b_2(\underline{\rho}) \neq 0$ and conjugating by a diagonal matrix with diagoanal entries $b_2/|b_2|$ and 1 changes b_2 into $|b_2|$, meaning that we can assume that b_2 is real. Having performed these reductions, one easily checks using (3.15) that

$$\begin{pmatrix} \operatorname{Re} c_2 & i \operatorname{Im} a_2 \\ -i \operatorname{Im} a_2 & b_2 \end{pmatrix} \tag{3.17}$$

is a smooth symetrizer for $\varphi A_1 + \psi A_2$, which is definite positive by (3.16). $\square$

3.3 Ill posedness for non Lipschitz symmetrizers

Consider the system (3.5) with $a = a(x) = |x|^\alpha$. For η large and β to be determined, we look for solutions of $L_a U = 0$ of the form

$$U(t,x,y) = e^{i\beta\sqrt{\eta}t + iy\eta} \begin{pmatrix} u(\sqrt{\eta}x) \\ v(\sqrt{\eta}x) \\ w(\sqrt{\eta}x) \end{pmatrix}. \tag{3.18}$$

With $\varepsilon = \eta^{-\alpha/2}$, the equation $LU = 0$ is equivalent to

$$v(x) = \frac{i}{\beta}(\partial_x - i\varepsilon x a)u(x), \qquad w = -\frac{1}{\beta}(1 + \varepsilon^2 a^2)u(x) \tag{3.19}$$

and the scalar equation for the first component is

$$\Big(\beta^2 + (\partial_x + i\varepsilon x a)(\partial_x - i\varepsilon x a(x)) - x^2(1 + \varepsilon^2 a^2)\Big)u = 0$$

that is, since $\partial_x(xa) = (\alpha + 1)a$,

$$\Big(\beta^2 + \partial_x^2 - x^2 - i\varepsilon(\alpha + 1)a\Big)u = 0. \tag{3.20}$$

The example has been cooked up precisely to get an eigenvalue problem for a perturbation of the harmonic oscillator.

When $\alpha = 0$, $\varepsilon = 1$ and $u(x) = e^{-\frac{1}{2}x^2}$ is a a solution when

$$\beta^2 = i - 1. \tag{3.21}$$

Choosing the root with negative imaginary part, this yieds exact solutions of $L_a U = 0$ of the form

$$U_\lambda(t,x,y) = \int e^{i\beta\sqrt{\eta}t + iy\eta} e^{-\frac{1}{2}\eta x^2}(U_0 + \sqrt{\eta}xU_1)\varphi(\eta/\lambda)d\eta \tag{3.22}$$

with constant vectors U_0 and U_1 not equal to 0 and $\varphi \in C_0^\infty(\mathbb{R})$ with support in the interval $[1,2]$. The exponential growth of $e^{it\sqrt{\eta}\beta}$ implies that there is no control of any H^{-s} norm at positive time by an $H^{s'}$ norm of the initial data. This can be localized in (x,y) and

Proposition 3.7. *When $a = 1$, L_a has bounded symmetrizers but the Cauchy problem for* (3.5) *is ill posed in L^2 but also in C^∞.*

Consider now the case $\alpha \in]0,1[$. By standard perturbation theory the eigenvalue problem (3.20) has a solution

$$u = e^{-\frac{1}{2}x^2} + \varepsilon u_1, \qquad \beta^2 = 1 + i\varepsilon\lambda_1 + O(\varepsilon^2) \tag{3.23}$$

with

$$\lambda_1 \int e^{-x^2} dx = (\alpha + 1) \int a(x) e^{-x^2} dx > 0 \tag{3.24}$$

so that $\lambda_1 > 0$. Therefore one can choose $\beta = -1 - \frac{i}{2}\varepsilon\lambda_1 + O(\varepsilon^2)$

$$\operatorname{Im}(\beta\sqrt{\eta}) \sim \frac{1}{2}\lambda_1 \eta^{\frac{1}{2}(1-\alpha)} < 0 \tag{3.25}$$

which is arbitrarily large if $\alpha < 1$. This provides solutions of $L_a U = 0$, with exponentially amplified L^2 norms implying the following proposition.

Proposition 3.8. *When $a = |x|^\alpha$ with $0 < \alpha < 1$, L_a has C^α symmetrizers but the Cauchy problem for* (3.5) *is ill posed in L^2.*

4 Strong hyperbolicity of first order symbols

In this section we introduce the notion of strong hyperbolicity and show that it is equivalent to the existence of symmetrizers. Next we discuss the existence of smooth symmetrizers. We show that these notions are preserved by a change of the time direction. For the convenience of the reader, we postpone to the appendix the proof of several independent results on matrices.

4.1 Basic properties

We denote by $\tilde{x} \in \mathbb{R}^{1+d}$ the time-space variables and by $\tilde{\xi}$ the dual variables. We consider $N \times N$ first order system systems $\sum_{j=0}^d A_j \partial_{\tilde{x}_j} + B$. Their characteristic determinant is $p(\tilde{\xi}) = \det\big(\sum_{j=0}^d i\tilde{\xi}_j A_j + B\big)$, the principal part of which is $p_N(\tilde{\xi}) = \det\big(\sum_{j=0}^d i\tilde{\xi}_j A_j\big)$

Definition 4.1. *i) $\sum_{j=0}^d A_j \partial_{\tilde{x}_j} + B$ is said to be hyperbolic in the direction $\nu \in \mathbb{R}^{1+d}$ if $p_N(\nu) \neq 0$ and there is γ_0 such that $p(i\tau\nu + \tilde{\xi}) \neq 0$ for all $\xi \in \mathbb{R}^{1+d}$ and all real τ such that $|\tau| > \gamma_0$.*

ii) $L = \sum_{j=0}^d A_j \partial_{\tilde{x}_j}$ is strongly hyperbolic in the direction ν if and only if for all matrix B, $L + B$ is hyperbolic in the direction ν.

The classical definition of hyperbolicity is that the roots of $p(i\tau\nu+\xi)\neq 0$ are located in $\tau<\gamma_0$. But, since hyperbolicity in the direction ν implies hyperbolicity in the direction $-\nu$, the definition above is equivalent to the usual one.

Proposition 4.2. *$L=\sum_{j=0}^{d}A_j\partial_{\tilde x_j}$ is strongly hyperbolic in the direction ν if and only if there is a constant C such that*

i) for all $\tilde\xi\in\mathbb{R}^{1+d}$ and all matrix B, the roots of $\det\big(L(\tilde\xi+\lambda\nu)+B\big)=0$ *are located in the strip $|\mathrm{Im}\,\lambda|\leq C|B|$,*

With the same constant C, this condition is equivalent to

ii) for all $(\gamma,\tilde\xi,u)\in\mathbb{R}\times\mathbb{R}^{1+d}\times\mathbb{C}^N$:

$$\big|\gamma u\big|\leq C\big|L(\tilde\xi+i\gamma\nu)u\big|. \tag{4.1}$$

Other equivalent formulations can be deduced from Proposition 5.1 below.

Proof. **a)** By homogeneity, $ii)$ is equivalent to the condition

$$\big|\mathrm{Im}\,\lambda\big|\geq C \quad\Rightarrow\quad \big|L(\tilde\xi+\lambda\nu)^{-1}\big|\leq 1. \tag{4.2}$$

By Lemma 5.2 below, this is equivalent to the condition that for all matrices B such that $|B|<1$, $L(\tilde\xi+\lambda\nu)+B$ is invertible when $|\mathrm{Im}\,\lambda|\geq C$. This is equivalent to saying that the roots of $\det\big(L(\tilde\xi+\lambda\nu)+B\big)=0$ are contained in $\{|\mathrm{Im}\,\lambda|<C\}$. By homogeneity, this is equivalent to $i)$.

b) Note that (4.1) applied to $\tilde\xi=0$ implies that $L(\nu)$ is invertible. It is then clear that $i)$ implies strong hyperbolicity. Conversely, assume that L is strongly hyperbolic. Consider the matrix $B_{j,k}$ with all entries equal to zero, except the entry of indices (j,k) equal to one. Then

$$\det\Big(L(\tilde\xi)+B_{j,k}\Big)=\det L(\tilde\xi)+m_{jk}(\xi)$$

where $m_{j,k}$ is the cofactor of indices (j,k) in the matrix $L(\tilde\xi)$. Following Theorem 12.4.6 in [Hö1], the hyperbolicity condition implies that there is a constant C such that

$$\big|m_{j,k}(\tilde\xi+i\nu)\big|\leq C\big|\det L(\tilde\xi+i\nu)\big|.$$

Since $L(\tilde\xi+i\nu)^{-1}=(\det L(\tilde\xi+i\nu))^{-1}\widetilde{M}(\tilde\xi+i\nu)$ where $\widetilde{M}$ is the matrix with entries $(-1)^{j+k}m_{k,j}$, this implies that there is another constant C such that (4.1) is satisfied for $\gamma=1$. By homogeneity, it is also satisfied for all γ. $\square$

When p is hyperbolic in the direction ν, then the component of ν in the set $\{p_N \neq 0\}$ is a convex open cone, which we denote by $\Gamma(\nu)$ and p is hyperbolic in any direction $\nu' \in \Gamma(\nu)$. This property is also true for strong hyperbolicity. We give a quantitative version of this result, as we will need it later on.

Lemma 4.3. *Suppose that L is hyperbolic in the direction ν. For all $\nu' \in \Gamma(\nu)$, the ball centered at ν' of radius $\varepsilon := |p_N(\nu')|/K|\nu'|^{N-1}$ is contained in $\Gamma(\nu)$, where $K = \max_{|\tilde\xi|\leq 2} |\nabla_{\tilde\xi} \det L(\tilde\xi)|$.*

Proof. By homogeneity, one can assume that $|\nu'| = 1$. In this case, $|p_N(\nu'')| > |p_N(\nu')| - K|\nu' - \nu"|$ if $\nu" - \nu'| \leq 1$. Noticing that $|p_N(\nu')| = |\nu' \nabla_{\tilde\xi} p_N(\nu')| \leq K$, this implies that $|p_N(\nu'')| > 0$ if $|\nu" - \nu'| \leq \varepsilon$. □

Proposition 4.4. *Suppose that $L(\tilde\xi)$ satisfies* (4.1) *and let $\nu' \in \Gamma(\nu)$ such that $\big|\det L(\nu')\big| \geq c > 0$. Then*

$$\big|\gamma u\big| \leq C_1 \big|L(\tilde\xi + i\gamma\nu')u\big|. \tag{4.3}$$

with $C_1 = KC|\nu|/c|\nu'|$ and $K = \max_{|\tilde\xi|\leq 2} |\nabla_{\tilde\xi} \det L(\tilde\xi)|$

Proof. Let $p(\tilde\xi) = \det(L(\tilde\xi) + B)$ and $p_N = \det L(\xi)$ its principal part. Suppose that $|\nu| = |\nu'| = 1$. The general case follows immediately. By Proposition (4.2), one has

$$p(\tilde\xi + i\gamma\nu) \neq 0 \quad \text{for } |\gamma| > C|B|. \tag{4.4}$$

We choose $\nu'' = \nu' - \varepsilon\nu$ with $\varepsilon = c/K$. By Lemma 4.3 $\nu'' \in \Gamma(\nu)$ and following [Gå1, Hö1], (see e.g. [Hö1] vol 2, chap 12), one has

$$p(\tilde\xi + i\gamma\nu + i\sigma\nu'') \neq 0 \quad \text{for } \gamma > C,\ \sigma \geq 0. \tag{4.5}$$

and also for $\gamma < -C|B|$ and $\sigma \leq 0$. Indeed, all the roots of $p_N(t\nu + \nu'')$ are real and negative:

$$p_N(t\nu + \nu'') = 0 \quad \Rightarrow \quad t < 0. \tag{4.6}$$

By (4.4), $p(\tilde\xi + i\gamma\nu + z\nu'') = 0$ has no root on the real axis, so that the number of roots in $\{\operatorname{Im} z \geq 0\}$ is independent of $\tilde\xi$ and $\gamma > C|B|$. Taking $\tilde\xi = 0$ and letting γ tend to $+\infty$, (4.6) this implies that this number is equal to zero implying (4.5). The proof for $\gamma \leq -C|B|$ and $\sigma \leq 0$ is similar.

Substituting $\nu'' = \nu' - \varepsilon\nu$ in (4.5) and choosing $\gamma = \varepsilon'\sigma$ we conclude that

$$p(\tilde\xi + i\sigma\nu') \neq 0$$

if $\varepsilon|\sigma| > C|B|$. Applying again Proposition 4.2, (4.3) follows. □

In most applications, the coefficients of the system L and even the direction ν may depend on parameters, such as the space time variables, the unknown itself etc. The direction ν itself can be seen as a parameter. This leads to consider families of systems, $L(a, \partial_{\tilde{x}})$ and directions ν_a, depending on parameters $a \in \mathcal{A}$. Their symbol is

$$L(a, \tilde{\xi}) = \sum_{j=0}^{d} \tilde{\xi}_j A_j(a). \tag{4.7}$$

When considering such families, we always assume that the matrices $A_j(a)$ and the directions ν_a are uniformly bounded,

Definition 4.5. *We say that the family $L(a, \cdot\,)$ is uniformly strongly hyperbolic in the direction ν_a for $a \in \mathcal{A}$ if ,*

i) $c_{\mathcal{A}} := \inf_{a \in \mathcal{A}} |\det L(a, \nu_a)| > 0$,

ii) the equivalent conditions i) *and* ii) *of Proposition* 4.2 *are satisfied with a constant C independent of $a \in \mathcal{A}$.*

The next result is an immediate consequence of Proposition 4.4. It shows that one can enlarge the set of strongly hyperbolic direction, preserving uniformity: let Γ_a denote the component of ν_a in $\{\det L(a, \tilde{\xi}) \neq 0\}$; for $c \in]0, c_{\mathcal{A}}]$ and $C > 0$ introduce the set

$$\widetilde{\mathcal{A}} = \big\{(a, \nu); a \in \mathcal{A}, \nu \in \Gamma_a, |\nu| \leq C, |\det L(a, \nu)| \geq c\big\}. \tag{4.8}$$

Proposition 4.6. *Suppose that $L(a, \cdot\,)$ is uniformly strongly hyperbolic in the direction ν_a for $a \in \mathcal{A}$. Then $L(a, \cdot\,)$ is uniformly strongly hyperbolic in the direction ν, for $(a, \nu) \in \widetilde{\mathcal{A}}$.*

4.2 Symmetrizers

We start with the notion of *full symmetrizer* introduced in [FrLa1].

Definition 4.7. *A full symmetrizer for $L(\tilde{\xi}) = \sum \tilde{\xi}_j A_j$ is a bounded matrix $\mathbf{S}(\tilde{\xi})$, homogeneous of degree 0 on $\mathbb{R}^{1+d} \backslash \{0\}$, such that $\mathbf{S}(\tilde{\xi}) L(\tilde{\xi})$ is self adjoint. It is positive in the direction $\nu \neq 0$ if there is a constant $c > 0$ such that for all $\tilde{\xi} \neq 0$:*

$$u \in \ker L(\tilde{\xi}) \quad \Rightarrow \quad \operatorname{Re} \big(\mathbf{S}(\tilde{\xi}) L(\nu) u, u\big) \geq c |u|^2. \tag{4.9}$$

Given a family of systems $L(a, \cdot)$ and directions ν_a, a bounded family of full symmetrizers $\mathbf{S}(a, \cdot\,)$ for $a \in \mathcal{A}$ is said to be uniformly positive in the direction ν_a if the constant c above can be chosen independent of a.

In (4.9), (f,u) denotes the hermitian scalar product in $\mathbb{C}^N$. More intrinsically, it should be thought as the antiduality between covectors $f \in \mathbb{V}^*$ and vectors $u \in \mathbb{V}$, where $\mathbb{V}$ is a vector space of dimension N, so that the adjoint P^* of the operator P from $\mathbb{V}$ to $\mathbb{V}$ satisfies $(f, Pu) = (P^* f, u)$. In this spirit, the symbol $L(\tilde\xi)$ must be thought as linear mapping from a vector space $\mathbb{V}$ to another vector space $\mathbb{W}$ and the symmetrizer $S(\tilde\xi)$ maps $\mathbb{V}$ to $\mathbb{V}^*$ so that the antiduality (SLu, u) makes sense.

Note that outside a conical neighborhood of the the characteristic variety, the symmetrizer can be chosen arbitrarily and thus contains no information.

A different and more familiar notion of symmetrizer depends on the choice of a time direction ν. Choosing a space $\mathbb{E}$ such that $\mathbb{R}^{1+d} = \mathbb{E} \oplus \mathbb{R}\nu$ the symmetrizer is seen as a function of frequencies $\xi \in \mathbb{E}$. Since the open cone $\Gamma(\nu)$ is strictly convex, one can also require that $\Gamma(\nu) \cap \mathbb{E} = \emptyset$. In a more intrinsic definition, it can be seen as a symmetrizer invariant by translation in the direction ν, or defined on $\mathbb{R}^{1+d}/\mathbb{R}\nu$. To avoid technicalities, we choose the first option choosing a space $\mathbb{E}$. and when considering families $(L(a), \nu_a)$, we assume that we can choose $\mathbb{E}$ in such a way that there is a compact set K such that

$$\forall a \in \mathcal{A}, \nu_a \in K \qquad \text{and} \qquad K \cap \mathbb{E} = \emptyset. \tag{4.10}$$

This condition can always be met locally. In particular, uniformly in $a \in \mathcal{A}$:

$$c(|\xi| + |\tau|) \le |\xi + \tau\nu_a| \le C(|\xi| + |\tau|), \qquad \xi \in \mathbb{E},\ \tau \in \mathbb{R}. \tag{4.11}$$

Definition 4.8. *A symmetrizer for $L(\tilde\xi) = \sum \tilde\xi_j A_j$ in the direction ν is a bounded matrix $S(\xi)$, homogeneous of degree 0 in $\xi \in \mathbb{E}$ such that $S(\xi)L(\xi)$ and $S(\xi)L(\nu)$ are self adjoint for all ξ and there is $c > 0$ such that :*

$$\forall \xi \in \mathbb{E}\backslash\{0\},\ \forall u \in \mathbb{C}^N, \qquad \big(S(\xi)L(\nu)u, u\big) \ge c|u|^2. \tag{4.12}$$

Given a family of systems $L(a,\cdot)$ and directions ν_a satisfying (4.10), a uniform family of symmetrizers $S(a,\cdot)$ for $\{L(a,\cdot), \nu_a\}$, $a \in \mathcal{A}$, is a bounded family $S(a,\cdot)$ of symmetrizers for $L(a,\cdot)$ in the direction ν_a, such that the constant c can be chosen independent of a.

Remark 4.9. $\mathbf{S}(\xi + \tau\nu) = S(\xi)$, is almost a full symmetrizer, except that it not necessarily defined on the line $\mathbb{R}\nu$. But (4.12) implies that $L(\nu)$ is invertible and one can always choose $\mathbf{S}(\nu)$ so that $\mathbf{S}(\nu)L(\nu)$ is definite positive. This modification can be extended to a conical neighborhood of ν where $L(\tilde\xi))$ remains invertible. This construction obviously preserves positivity. This remains true for families and uniformity can be preserved.

The existence of symmetrizers is equivalent to strong hyperbolicity, in the following sense.

Theorem 4.10. *Consider a family* $\{L(a, \cdot), \nu_a, a \in \mathcal{A}\}$.

i) Assuming (4.10), $L(a,\cdot)$ *is strongly hyperbolic in the direction* ν_a *if and only if there exists a uniform family of symmetrizers* $S(a,\cdot)$.

ii) $L(a,\cdot)$ *is strongly hyperbolic in the direction* ν_a *if and only if*

a) it is hyperbolic in the direction ν_a *and* $\inf_{a\in\mathcal{A}} |\det L(a,\nu_a)| > 0$,

b) there is a bounded family of full symmetrizer $\mathbf{S}(a,\cdot)$ *which is uniformly positive in the direction* ν_a.

Proof. **i)** If $L(a,\cdot)$ is uniformly strongly hyperbolic in the direction ν_a, then $L(a,\nu_a)$ and $L(a,\nu_a)^{-1}$ are uniformly bounded, Similarly, (4.12) implies that $L(a,\nu_a)^{-1}$ is bounded.

In both case, $A(a,\xi) = L(a,\nu_a)^{-1}L(a,\xi)$ for $|\xi| = 1$ is bounded, and strong hyperbolicity is equivalent to the existence of a constant C such that for all λ, a, $\xi \in \mathbb{E}$ and u:

$$|\operatorname{Im}\lambda|\big|u\big| \le C\big|A(a,\xi)u - \lambda u\big|. \tag{4.13}$$

By Proposition 5.1 this is equivalent to the existence of a symmetric matrix $S_A(a,\xi)$, bounded and uniformly definite positive, such that $S_A A$ is symmetric. This is equivalent to the condition that $S(a,\xi) = S_A(a,\xi)L(a,\nu_a)^{-1}$ is a symmetrizer for $L(a,\cdot)$ bounded and uniformly positive in the direction ν_a.

ii) Strong hyperbolicity implies the existence of a symmetrizer, thus of a full positive symmetrizer by Remark 4.9. Hence it only remains to prove the converse part of *ii*).

Let $\mathbf{S}(a,\cdot)$ be a full symmetrizer for $L(a,\cdot)$, positive in the direction ν_a. Suppose in addition that $L(a,\cdot)$ is hyperbolic in this direction, so that $L(a,\nu)$ is invertible. Then, Proposition 6.1 implies that when $\ker L(a,\tilde{\xi}) \ne \{0\}$, 0 is a semi simple eigenvalue of $A(a,\tilde{\xi}) = L(a,\nu_a)^{-1}La,\tilde{\xi})$. Moreover, the spectral projectors, that is the projectors on $\ker A = \ker L$ parallel to the range of A , are uniformly bounded. Applied to $\tilde{\xi} + \tau\nu$, this implies that all the real eigenvalues of $A(a,\tilde{\xi})$ are semi-simple and all the corresponding spectral projectors are uniformly bounded. Since L is hyperbolic, all the eigenvalues are real and with Proposition 5.1 this implies that (4.13) is satisfied. □

4.3 Smooth symmetrizers

We now consider a family $\{L(a,\cdot), \nu_a, a \in \mathcal{A}\}$ where $\mathcal{A}$ is an open set of some space $\mathbb{R}^m$. We assume that the condition (4.10) is satisfied.

Theorem 4.11. *Suppose that the coefficients of $L(a,\cdot)$ are continuous, [resp. $W^{1,\infty}$] [resp. C^∞] on $\mathcal{A}$ and that the mapping $a \mapsto \nu_a$ is continuous, [resp. $W^{1,\infty}$] [resp. C^∞]. Then, there exists a full symmetrizer $\mathbf{S}(a,\tilde{\xi})$ which is continuous, [resp. $W^{1,\infty}$] [resp. C^∞] on $\mathcal{A}\times S^d$ and uniformly positive in the direction ν_a, if and only if there is a symmetrizer $S(a,\xi)$ which is continuous, [resp. $W^{1,\infty}$] [resp. C^∞] on $\mathcal{A}\times S^{d-1}$ and uniformly positive in the direction ν_a,*

Proof. By Remark 4.9 passing from a symmerizer to a full symmetrizer is immediate. The converse statement follows from a more general result given in Theorem 4.13 where the construction is extended to other directions $\nu \in \Gamma(\nu_a)$. □

4.4 Invariance by change of time

Proposition 4.6 shows that strong hyperbolicity, thus the existence of bounded symmetrizers or of full symmetrizers, extends from ν_a to all directions in the cone of hyperbolicity Γ_a, preserving uniformity in sets such as (4.8). We now prove that this is also true for smooth symmetrizers. The key point, is to prove that for a continuous full symmetrizer, positivity extends from ν_a to $\Gamma(\nu_a)$.

Proposition 4.12. *Consider a family $\{L(a,\cdot), \nu_a, a \in \mathcal{A}\}$ and assume that $L(a,\cdot)$ is uniformly strongly hyperbolic in the direction ν_a. For $c > 0$ and C given, define $\widetilde{\mathcal{A}}$ as in (4.8).*

Suppose that $\mathbf{S}(a,\cdot)$ is a full symmetrizer of $L(a,\cdot)$ which depends continuously on $\tilde{\xi} \in \mathbb{R}^{1+d}\backslash\{0\}$, such that $\mathbf{S}(a,\cdot)$ is uniformly positive in the direction ν_a. Then, $\mathbf{S}(a,\cdot)$ is uniformly positive in the direction ν for $(a,\nu) \in \widetilde{\mathcal{A}}$.

Proof. Since $\mathbf{S}(a,\tilde{\xi}+s\eta)L(a,\tilde{\xi}+s\tilde{\eta})$ is symmetric, for u and v in $\ker L(a,\tilde{\xi})$ one has

$$\big(\mathbf{S}(a,\tilde{\xi}+s\eta)L(a,\tilde{\eta})u, v\big) = \big(u, \mathbf{S}(a,\tilde{\xi}+s\eta)L(a,\tilde{\eta})v\big).$$

Letting s tend to 0, shows that for all η, the matrices $\mathbf{S}(a,\tilde{\xi})L(a,\tilde{\eta})$ are symmetric on $\ker L(a,\tilde{\xi})$.

By Lemma 4.3, there is ε such that for all $(a,\nu) \in \mathcal{P}$, the ball centered at ν and radius ε is contained in $\Gamma_a(\nu_a)$. Therefore, there is $t_0 \in]0,1[$ such that for all $(a,\nu) \in \mathcal{P}$, there is $\nu' \in \Gamma_a(\nu_a)$ on the line joining ν_a and ν such that $\nu = t\nu_a + (1-t)\nu'$ with $t \in [t_0, 1[$. By Proposition 4.4, $L(a,\dot{})$ is strongly hyperbolic in the direction ν', implying that $L(a,\nu')^{-1}L(a,\tilde{\xi})$ has only real an semi-simple eigenvalues. Therefore, the result follows from Proposition 6.2 applied to $J_t = (1-t)L(a,\nu_a) + tL(a,\nu)$ □

We are now ready to prove that the existence of a regular full symmetrizer implies the existence of symmetrizer, having the same smoothness, in all directions $\nu \in \Gamma(\nu_a)$. In particular, this finishes the proof of Theorem 4.11. Consider a strongly hyperbolic family $\{L(a,\cdot),\nu_a, a \in \mathcal{A}\}$. Assume that the condition (4.10) holds. For $c > 0$ and $C > 0$ and $\mathcal{O}$ an open neighborhood of K such that $\overline{\mathcal{O}} \cap \mathbb{E} = \emptyset$ let

$$\widetilde{\mathcal{A}}_0 = \big\{(a,\nu); a \in \mathcal{A}, \nu \in \Gamma_a \cap \mathcal{O}, |\nu| \le C, |\det L(a,\nu)| \ge c\big\}. \tag{4.14}$$

Theorem 4.13. *Suppose that the coefficients of $L(a,\cdot)$ are continuous, [resp. $W^{1,\infty}$] [resp. C^∞] on $\mathcal{A}$ and that the mapping $a \mapsto \nu_a$ is continuous, [resp. $W^{1,\infty}$] [resp. C^∞]. Suppose that $S(a,\xi)$ is a uniform bounded family of symmetrizers for $\{L(a,\cdot)$ in the directions ν_a for $a \in \mathcal{A}$, which is continuous, [resp. $W^{1,\infty}$] [resp. C^∞] on $\mathcal{A} \times \mathbb{E}$. Then, there exist a continuous, [resp. $W^{1,\infty}$] [resp. C^∞] uniform family of symmetrizers $S(a,\nu,\xi)$ for $L(a,\cdot)$ in the direction ν, which is continuous, [resp. $W^{1,\infty}$] [resp. C^∞] on $\widetilde{\mathcal{A}}_0 \times \mathbb{E}$.*

Proof. For $\tilde{a} = (a,\nu) \in \widetilde{\mathcal{A}}_0$, consider

$$\tilde{L}(\tilde{a},\tau,\xi) = \tau L(a,\nu) + L(a,\xi) = L(a,\xi+\tau\nu).$$

Then $\tilde{\mathbf{S}}(\tilde{a},\tau,\xi) = \mathbf{S}(a,\xi+\tau\nu)$ symmetrizes $\tilde{L}(\tilde{a},\tau,\xi)$. By (4.11), $\tilde{L}$ and $\tilde{\mathbf{S}}$ are continuous, [resp. $W^{1,\infty}$] [resp. C^∞] functions on $\widetilde{\mathcal{A}}_0 \times \mathbb{R} \times S^{d-1}$. The positivity condition

$$\operatorname{Re}\big(\tilde{\mathbf{S}}(\tilde{a},\tau,\xi)L(a,\nu)u,u\big) \ge c|u|^2 \tag{4.15}$$

on $\ker \tilde{L}(\tilde{a},\tau,\xi) = \ker L(a,\xi+\tau\nu)$ follows from Proposition 4.12 and and the construction of a symmetrizer $S(a,\nu,\xi)$, with the same smoothness as $\tilde{\mathbf{S}}$, is given by Theorem 6.5. □

5 Appendix A : Strongly hyperbolic matrices

We collect here the various technical results on matrices which have been used in the previous section. Changing slightly the notations, for instance including ξ or ν among the parameters, we consider a family of $N \times N$ matrices, $A(a)$ depending on parameters $a \in \Omega$ where Ω is an open subset of $\mathbb{R}^n$. We denote by $\Sigma(a)$ the spectrum of $A(a)$.

5.1 Definition and properties

Proposition 5.1. *The following properties are equivalent*

i) There is a real C_1 such that

$$\forall t \in \mathbb{R}, \forall a \in \Omega \ : \qquad \big| e^{itA(a)} \big| \leq C_1. \tag{5.1}$$

ii) All the the eigenvalues λ of $A(a)$ are real and semi-simple and there is a real C_2 such that all the eigen-projectors $\Pi_\lambda(a)$ satisfy

$$\forall a \in \Omega \ : \qquad \big| \Pi_\lambda(a) \big| \leq C_2. \tag{5.2}$$

iii) $A(a) - \lambda \mathrm{Id}$ is invertible when $\operatorname{Im} \lambda \neq 0$ and there is a real C_3 such that

$$\forall \lambda \notin \mathbb{R} \ \forall a \in \Omega \ : \qquad \big| \big(A(a) - \lambda \mathrm{Id} \big)^{-1} \big| \leq C_3 \big| \operatorname{Im} \lambda \big|^{-1}. \tag{5.3}$$

iv) There are definite positive matrices $S(a)$ and there are constants C_4 and $c_4 > 0$ such that for all $a \in \Omega$, $S(a)A(a)$ is symmetric, and

$$\big| S(a) \big| \leq C_4, \qquad S(a) \geq c_4 \mathrm{Id}. \tag{5.4}$$

v) There is a real C_5 such that for all matrix B, all $a \in \Omega$ and all $\rho \in \mathbb{R}$, the eigenvalues of $\rho A(a) + B$ are located in $\{ |\operatorname{Im} \lambda| < C_5 |B| \}$.

Proof. **a)** $ii)$ implies that $A(a)$ has the spectral decomposition $A = \sum \lambda_j \Pi_j$ with real λ_j's. Thus (5.2) implies that $\big| e^{itA)} \big| = \big| \sum e^{it\lambda_j} \Pi_j \big| \leq N C_2$.

Conversely, $i)$ implies that the eigenvalues λ_j of $A(a)$ are real, and semi-simple and thus that $A(a) = \sum \lambda_j \Pi_j$. Moreover,

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{it(A(a) - \lambda_j \mathrm{Id})} dt = \sum_k \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{it(\lambda_k - \lambda_j \mathrm{Id})} \Pi_k dt = \Pi_j.$$

Thus, $|\Pi_j| \leq C_1$ if (5.1) is true.

b) Suppose that $ii)$ is satisfied so that $A = \sum \lambda_j \Pi_j$ and $\mathrm{Id} = \sum \Pi_j$. Then

$$S(a) = \sum \Pi_j^* \Pi_j \tag{5.5}$$

is definite positive, satisfies $S \geq N^{-1} \mathrm{Id}$, $|S| \leq N C_2^2$, and $SA = \sum \lambda_j \Pi_j^* \Pi_j$ is self adjoint.

If $iv)$ holds then, with $\epsilon = \mathrm{sign}(\gamma)$,

$$c_4|\gamma||u|^2 \leq \mathrm{Re}\,\epsilon\big(S(-iA+\gamma\mathrm{Id})u,u\big) \leq C_4\big|(A+i\gamma)u\big|\ \big|u\big|$$

implying $iii)$ with $C_3 = C_4/c_4$.

If $iii)$ is satisfied, then the eigenvalues of $A(a)$ are real, and semi-simple, for if there were a nondiagonal block in the Jordan's decomposition of $A - \lambda_j\mathrm{Id}$, the norm of $(A-(\lambda_j - i\gamma)\mathrm{Id})^{-1}$ would be at least of order γ^{-2} when $\gamma \to 0$. Thus $A = \sum \lambda_j \Pi_j$ and

$$\lim_{\gamma\to 0} i\gamma\big(A-(\lambda_j - i\gamma)\mathrm{Id}\big)^{-1} = \sum_k \lim_{\gamma\to 0}\frac{i\gamma}{(\lambda_k - \lambda_j + i\gamma}\Pi_k = \Pi_j,$$

hence $|\Pi_j| \leq \ C_3$.

c) By homogeneity, $iii)$ is equivalent to the condition

$$\forall a \in \Omega, \forall \rho \in \mathbb{R}, \qquad \big|\mathrm{Im}\,\lambda\big| \geq C_3 \quad \Rightarrow \quad \big|\big(\rho A(a) - \lambda\mathrm{Id}\big)^{-1}\big| \leq 1.$$

By Lemma 5.2 below, this is equivalent to the condition that for all matrix B such that $|B| < 1$, $\rho A - \lambda\mathrm{Id} + B$ is invertible when $|\mathrm{Im}\,\lambda| \geq C_3$, meaning that the spectrum of $\rho A + B$ is contained in $\{|\mathrm{Im}\,\lambda| < C_3$. By homogeneity, this is equivalent to $v)$ with $C_5 = C_3$.

The proof of the proposition is now complete. □

Lemma 5.2. *The matrix A is invertible with $|A^{-1}| \leq \kappa$ if and only if $A+B$ is invertible for all B such that $|B| < \kappa^{-1}$.*

Proof. If $|A^{-1}| \leq \kappa$, then $A + B = A^{-1}(\mathrm{Id} + A^{-1}B)$ is invertible for all B such that $|A^{-1}B| \leq \kappa|B| < 1$.

Conversely, if A is not invertible or if $|A^{-1}| > \kappa$, there is $\underline{u}$ such that $|\underline{u}| = 1$ and $|A\underline{u}| < \kappa^{-1}$. Pick a linear form ℓ such that $\ell(\underline{u}) = 1$ and $|\ell| = 1$. Then the matrix B defined by $Bu = \ell(u)A\underline{u}$ satisfies $|B| = |A\underline{u}| < \kappa^{-1}$ but $A - B$ is not invertible since $\underline{u}$ is in its kernel. □

5.2 Lipschitz dependence of the eigenvalues

Assumption 5.3. *The family $\{A(a), a \in \Omega\}$ of $N\times N$ matrices, is uniformly strongly hyperbolic in the sense that the equivalent properties of Proposition* 5.1 *are satisfied.*

Proposition 5.4. *Suppose that $A(\cdot) \in W^{1,\infty}(\Omega)$ satisfies Assumption* 5.3. *Denote by $\lambda_j(a)$, $1 \le j \le N$, the eigenvalues of $A(a)$, labelled in the increasing order and repeated accordingly to their multiplicity. Then, the functions λ_j belong to $W^{1,\infty}(\Omega)$*

Proof. The continuity of the roots of a polynomial with respect to the coefficients is well known. The Lipschitz smoothness with respect to parameters of the roots of hyperbolic polynomials is true in general, provided that the coefficients are smooth enough (see [Br]). The proposition says that when the polynomial is the characteristic determinant of a strongly hyperbolic system, the Lipschitz smoothness of the coefficients is sufficient.

Fix $\underline{a} \in \Omega$ and an eigenvalue $\underline{\lambda} = \lambda_p(\underline{a}) = \lambda_{p+m}(\underline{a})$ of $A(\underline{a})$ of multiplicity $m+1$. Let $\delta > 0$ denote the distance of $\underline{\lambda}$ to the remainder part of the spectrum of $A(\underline{a})$. By Assumption 5.3, there is C which depends only on an upper bound of the norms of the spectral projectors, thus independent of $\underline{a}$, such that

$$\forall z \in \mathbb{C},\ |z - \underline{\lambda}| \le \delta/2, \qquad \big|\big(A(\underline{a}) - z\mathrm{Id}\big)^{-1}\big| \le C|z - \underline{\lambda}|^{-1}$$

Therefore, $A - z\mathrm{Id}$ is invertible when $|A - A(\underline{a})| < |z - \underline{\lambda}| \le \delta/2C$.

Let $\Omega_1 \subset \Omega$ denote a convex open neighborhhood of $\underline{a}$. Because $A \in W^{1,\infty}(\Omega)$ for a and $a' \in \Omega_1$ there holds

$$\big|A(a) - A(a')\big| \le K|a - a'| \tag{5.6}$$

with $K = \big\|\nabla_a A\big\|_{L^\infty(\Omega)}$. Therefore, $A(a) - z\mathrm{Id}$ is invertible if $a \in \Omega_1$ and $CK|a - \underline{a}| \le |z - \underline{\lambda}| \le \delta/2$. By Rouché's theorem, this implies that $A(a)$ has $m+1$ eigenvalues (counted with their multiplicity) in the disk $\{|\lambda - \underline{\lambda}| \le KC|a - \underline{a}|\}$. They must be real by assumption, and by continuity they are $\{\lambda_p(a), \ldots, \lambda_{p+m}(a)\}$. Hence, $|\lambda_j(a) - \underline{\lambda}| \le KC|a - \underline{a}|$ for $p \le j \le p+m$, provided that $a \in \Omega_1$ and $KC|a - \underline{a}| < \delta/2$.

Gluing these estimates together, we have proved that there are constants C and K such that : for all $\underline{a} \in \Omega$, there is a convex neighborhood ω of $\underline{a}$, such that for $a \in \omega$ and all j,

$$|\lambda_j(a) - \lambda_j(\underline{a})| \le CK|a - \underline{a}|.$$

The following independent lemma implies that $\big|\nabla_a \lambda_j\big|_{L^\infty(\Omega)} \le CK$ and the proposition follows. □

Lemma 5.5. *Suppose that K is a positive real number and f is a function defined on the open set $\Omega \subset \mathbb{R}^n$ such that for all $\underline{a} \in \Omega$, there is a neighborhood ω of $\underline{a}$, such that*

$$\forall a \in \omega, \qquad |f(a) - f(\underline{a})| \leq K|a - \underline{a}|. \tag{5.7}$$

Then $\big\|\nabla_a f\big\|_{L^\infty(\Omega)} \leq K$.

Proof. Note that (5.7) implies that f is continuous at $\underline{a}$, thus f is continuous.

We first shown that for all convex open set $\Omega_1 \subset \Omega$ the inequality

$$|f(b) - f(a)| \leq K|b - a|. \tag{5.8}$$

is satisfied for all a and b in Ω_1. Indeed, let $\mathbf{T}$ denote the set of real numbers $t \in [0, 1]$ such that

$$\forall s \in [0, t], \qquad |f(a + s(b - a)) - f(a)| \leq Ks|b - a|. \tag{5.9}$$

By assumption, the property (5.8) is satisfied on a neighborhood of a, implying that $\mathbf{T}$ is not empty. By definition $\mathbf{T}$ is an interval, and by continuity of f it is closed. Using the assumption (5.7) near $a + t(b - a)$, implies that $\mathbf{T}$ is open so that $T = 1$ and (5.8) is proved.

This implies that f is Lipschitz continuous on Ω_1 and that $\big\|\nabla_a f\big\|_{L^\infty(\Omega_1)} \leq K$. Since this is true for all ball $\Omega_1 \subset \Omega$, the lemma follows. □

5.3 Lipschitz dependence of the eigenprojectors

Proposition 5.6. *Suppose that $A(\cdot) \in W^{1,\infty}(\Omega)$ satisfies Assumption 5.3. Let $\underline{a} \in \Omega$ and consider $\underline{\Lambda} := \{\lambda_j(\underline{a}), j \in J\}$ a subset of the spectrum of $A(\underline{a})$. Let $\underline{\Lambda}' = \Sigma(\underline{a})\backslash\underline{\Lambda} = \{\lambda_j(\underline{a}), j \in J'\}$ and define*

$$\delta = \operatorname{dist}(\underline{\Lambda}, \underline{\Lambda}') = \min_{(j,j') \in J, \times J'} \big|\lambda_j(\underline{a}) - \lambda_{j'}(\underline{a})\big| > 0 \tag{5.10}$$

Let ω be a neighborhood of $\underline{a}$ such that $|\lambda_j(a) - \lambda_j(\underline{a})| \leq \delta/4$ for all $j \in \{1, \ldots, N\}$ and $a \in \omega$. With $\Lambda(a) = \{\lambda_j(a), j \in J\}$, consider the spectral projector

$$\Pi_\Lambda(a) = \sum_{\lambda \in \Lambda(a)} \Pi_\lambda(a). \tag{5.11}$$

Then, $\Pi_\Lambda(a)$ is continuous on ω and

$$\big|\Pi_\Lambda(a) - \Pi_\Lambda(a')\big| \leq CK\delta^{-1}\big|a - a'\big|, \tag{5.12}$$

where C depends only on an upper bound of the norms of the spectral projectors of $A(\cdot)$ and $K = \big\|\nabla_a A\big\|_{L^\infty(\Omega)}$.

Proof. There are finitely many Jordan curves Γ_k in the complex domain, of total length less than $C\delta$, such that $|z - \lambda_j(\underline{a})| \geq \delta/2$ for all $z \in \cup\Gamma_k$ and all $j \in J$, surrounding Λ so that

$$\Pi_\Lambda(\underline{a}) = \sum_k \frac{1}{2i\pi} \int_{\Gamma_k} (z\mathrm{Id} - A(\underline{a}))^{-1} dz$$

This formula extends to $a \in \omega$. Moreover, using the estimate

$$\begin{aligned} &\left|(z\mathrm{Id} - A(a))^{-1} - (z\mathrm{Id} - A(a'))^{-1}\right| \\ &\quad \leq \left|(z\mathrm{Id} - A(a))^{-1}\right| \left|A(a) - A(a')\right| \left|(z\mathrm{Id} - A(a'))^{-1}\right| \\ &\qquad\qquad \leq CK\delta^{-2}\left|a - a'\right| \end{aligned}$$

for $z \in \cup\Gamma_k$, implies (5.12). □

5.4 A piece of functional calculus

We study the smoothness of $f(A(a))$, given the smoothness of f and A. We extend the analysis to vector or matrix valued functions $S(\lambda, a)$ using the following definition

$$S_A(a) = \sum_{\lambda \in \Sigma(a)} S(\lambda, a)\Pi_\lambda(a). \tag{5.13}$$

Theorem 5.7. *Suppose that $A(\cdot)$ is continuous [resp. $W^{1,\infty}$] [resp. C^∞] on Ω and satisfies Assumption* 5.3. *Suppose that $S(\lambda, a)$ is continuous [resp. $W^{1,\infty}$] [resp. C^∞] on $\mathbb{R} \times \Omega$. Then $S_A(a)$ is continuous [resp. $W^{1,\infty}$] [resp. C^∞] on Ω.*

1) *The C^∞ case.* If S were holomorphic in λ one would have

$$S_A(a) = \frac{1}{2i\pi} \int_{\partial D} S(z, a)(z\mathrm{Id} - A(a))^{-1} dz$$

where D is a rectangle $[-R, R] + i[-\delta, \delta]$ containing $\Sigma(a)$ in its interior, implying the result since $(z\mathrm{Id} - A(a))^{-1}$ is smooth in a for $z \in \partial D$. In the C^∞ case, we modify this proof considering an almost holomorphic extension of S in the variable λ. It is a C^∞ function in $(z, a) \in \mathbb{C} \times \Omega$ such that, for z in bounded sets,

$$\partial_{\overline{z}} S(z, a) = O(|\mathrm{Im}\, z|^\infty) \tag{5.14}$$

Since the result is local, we can assume that Ω is bounded and fix R such that for $a \in \Omega$, the spectrum of $A(a)$ is contained in $\{|z| \leq R\}$. Let D denote the disc of radius $R+1$ in $\mathbb{C}$. Then,

$$\begin{aligned} S_A(a) = \frac{1}{2i\pi} \int_{\partial D} S(z,a)(z\mathrm{Id} - A(a))^{-1} dz \\ + \frac{1}{2i\pi} \int_{D} \partial_{\overline{z}} S(z,a)(z\mathrm{Id} - A(a))^{-1} dz d\overline{z} \end{aligned}$$

The first integral is C^∞ in a as explained above. In the second, we note that for all m, $\partial_{\overline{z}} S(z,a) = (\mathrm{Im}\, z)^m R_m(z,a)$ with R_m smooth and $(\mathrm{Im}\, z)^m (z\mathrm{Id} - A(a))^{-1}$ has $m-1$ uniformly bounded derivatives in a, for $z \in D \backslash \mathbb{R}$, implying that the second integral is C^{m-1} with respect to $a \in \Omega$. □

2) *Continuity.* Fix $\underline{a} \in \Omega$ and denote by $\mu_j, 1 \leq j \leq m$, the *distinct* eigenvalues of $A(\underline{a})$ and introduce $\delta = \min_{j \neq k} |\mu_j - \mu_k|$. For a in a neighborhood ω of $\underline{a}$, the spectrum of $A(a)$ is contained in nonoverlapping intervals $I_j =]\mu_j - \delta/4, \mu_j + \delta/4[$. Write

$$\begin{aligned} S_A(a) = \sum_j S(\mu_j, \underline{a}) \Pi_{I_j}(a) \\ + \sum_j \sum_{\lambda \in I_j \cap \Sigma(a)} \Big(S(\lambda, a) - S(\mu_j, \underline{a}) \Big) \Pi_\lambda(a) \end{aligned} \tag{5.15}$$

with

$$\Pi_j(a) = \sum_{\lambda \in I_j \cap \Sigma(a)} \Pi_\lambda(a). \tag{5.16}$$

Using a uniform bound for the $\Pi_\lambda(a)$ and the continuity of the eigenvalues at $\underline{a}$ one concludes that the second sum in (5.15) tends to 0 as a tends to $\underline{a}$. By Proposition 5.6, Π_j is continuous and hence S_A is continuous at $\underline{a}$. □

3) *Lipschitz continuity.* By Lemma 5.5 it is sufficient to prove that there is a positive constant K such that for all $\underline{a} \in \Omega$, there is a neighborhood ω of $\underline{a}$, such that

$$\forall a \in \omega, \qquad |S_A(a) - S_A(\underline{a})| \leq K |a - \underline{a}|. \tag{5.17}$$

The proof starts as in 2) with the decomposition (5.15). Shrinking ω if necessary, the Lipschitz continuity of S and of the eigenvalues implies that for $a \in \omega$ and $\lambda \in I_j \cap \Sigma(a)$

$$\Big| S(\lambda, a) - S(\mu_j, \underline{a}) \Big| \leq C |a - \underline{a}|$$

where C depends only $\|\nabla_a A\|_{L^\infty(\Omega)}$ and $\|\nabla_{\lambda,a} S\|_{L^\infty(\mathbb{R}\times\Omega)}$. Since the Π_λ are uniformly bounded, this implies that the second sum in (5.15) is uniformly $O(|a-\underline{a}|)$ so that it remains to prove that, with $S_j = S(\mu_j, \underline{a})$ and $p_j(a) = \Pi_j(a) - \Pi_j(\underline{a})\big)$, one has

$$\Big|\sum_j S_j p_j(a)\Big| \le K|a-\underline{a}|. \tag{5.18}$$

with K independent of $\underline{a}$ and ω. Since $S \in W^{1,\infty}$, the S_j satisfy

$$\big|S_j\big| \le K_1, \qquad \big|S_j - S_k\big| \le K_1|\mu_j - \mu_k|. \tag{5.19}$$

Moreover, Proposition 5.6 implies that for all $J \subset \{1,\dots,m\}$ with $J \ne \emptyset$ and $J \ne \{1,\dots,m\}$,

$$P_J = \sum_{j\in J} p_j(a). \tag{5.20}$$

satisfies, with $\varepsilon = |a-\underline{a}|$:

$$\big|P_J\big| \le K_2\varepsilon \left(\min_{j\in J, k\notin J} |\mu_j - \mu_k|\right)^{-1}. \tag{5.21}$$

Moreover, when $J = \{1,\dots,m\}$,

$$P_{\{1,\dots,m\}} = 0. \tag{5.22}$$

The next lemma implies (5.18), finishing the proof of the proposition. □

Lemma 5.8. *There is a constant C_m which depends only on m, such that for all S_j and p_j, $1 \le j \le m$ satisfying* (5.19) (5.21)*and* (5.22)*, the sum $S = \sum S_j p_j$ satisfies*

$$\big|S\big| \le C_m K_1 K_2 \varepsilon. \tag{5.23}$$

Proof. By homogeneity, we can assume that $K_1 = K_2 = 1$. The proof is by induction on m. When $m = 1$, the condition (5.19) reduces to $|S_1| \le 1$, the condition (5.21) is void and $p_1 = 0$.

We assume that the lemma is proved up to order $m-1 \ge 1$ and we prove it at the order m. Let

$$\delta := \min_{j\ne k} |\mu_j - \mu_k| > 0. \tag{5.24}$$

Permuting the indices we can assume that the infimum is attained for $(j,k) = (m-1,m)$, which means that

$$\forall j \neq k : \quad |\mu_{m-1} - \mu_m| \leq |\mu_j - \mu_k|. \tag{5.25}$$

We spilt S in two terms:

$$S = \widetilde{S} \ + \ (S_m - S_{m-1})p_m \tag{5.26}$$

with

$$\widetilde{S} = \sum_{j=1}^{m-1} S_j \tilde{p}_j \tag{5.27}$$

where $\tilde{p}_j = p_j$ when $j \leq m-2$ and $\tilde{p}_{m-1} = p_{m-1} + p_m$.

The condition (5.21) applied to $J = \{m\}$ and (5.25) imply that $|p_m| \leq \varepsilon\delta^{-1}$ while (5.19) and (5.25) imply that $|S_m - S_{m-1}| \leq \delta$. This shows that the second term in (5.26) satisfies $|(S_m - S_{m-1})p_m| \leq \varepsilon$.

We now check that the induction hypothesis can be applied to $\widetilde{S}$. The condition (5.19) is clear, so we only have to show that the conditions (5.21) and (5.22) are satisfied for the $\tilde{p}_j$.

Consider a non empty subset $\widetilde{J} \subset \{1,\ldots,m-1\}$. Then $\widetilde{P}_{\widetilde{J}} = P_J$ with

1) $J = \widetilde{J} \cup \{m\}$ if $m-1 \in \widetilde{J}$,
2) $J = \widetilde{J}$ if $m-1 \notin \widetilde{J}$,

In particular, $\widetilde{P}_{\{1,\ldots,m-1\}} = P_{\{1,\ldots,m\}} = 0$ so that (5.22) for $\widetilde{S}$ is satisfied.

Suppose that $\widetilde{J} \neq \{1,\ldots,m-1\}$ and let J be as above. Introduce also $\widetilde{K} = \{1,\ldots,m-1\}\backslash\widetilde{J}$ and $K = \{1,\ldots,m\}\backslash J$. One has $K = \widetilde{K}$ in case 1) and $K = \widetilde{K} \cup \{m\}$ in case 2). By assumption, we know that

$$\big|\widetilde{P}_{\tilde{J}}\big| = \big|P_J\big| \leq \varepsilon \left(\min_{j\in J, k\in K} |\mu_j - \mu_k| \right)^{-1}.$$

We claim that

$$\min_{j\in\tilde{J}, k\in\tilde{K}} |\mu_j - \mu_k| \leq 2 \min_{j\in J, k\in K} |\mu_j - \mu_k|. \tag{5.28}$$

Indeed, if it is true, it implies that

$$\big|\widetilde{P}_{\tilde{J}}\big| \leq 2\varepsilon \left(\min_{j\in\tilde{J}, k\in\tilde{K}} |\mu_j - \mu_k| \right)^{-1} \tag{5.29}$$

so that the induction hypothesis is satisfied for $\widetilde{S}$ with ε replaced by 2ε. Thus $\big|\widetilde{S}\big| \leq 2C_{m-1}\varepsilon$ and (5.23) follows with $C_m = 1 + 2C_{m-1}$.

Therefore, to complete the proof, it remains to prove the claim (5.28). In this estimate, $\tilde{J}$ and $\tilde{K}$ play symmetric roles, and therefore we can assume that $m \in J$, that is that $m-1 \in \tilde{J}$. Comparing the sets $\tilde{J} \times \tilde{K}$ and $J \times K$, we see that the only nontrivial case concerns $|\mu_m - \mu_k|$ when $k \in K = \tilde{K}$. In this case, since $m-1 \in \tilde{J}$, the claim follows from the inequality

$$|\mu_{m-1} - \mu_k| \leq |\mu_m - \mu_k| + |\mu_{m-1} - \mu_m| \leq 2|\mu_m - \mu_k|$$

where we have used (5.25). The proof of the lemma is now complete. □

6 Appendix B : Symmetrizable matrices

6.1 Positivity of symmetrizers and bounds

Proposition 6.1. *Suppose that L and $\mathbf{S}$ are matrices such that $\mathbf{S}L$ is hermitian symmetric. Suppose that J is an invertible matrix such that*

$$\forall u \in \ker L, \qquad \mathrm{Re}\,\big(\mathbf{S}Ju, u\big) \geq c|u|^2. \tag{6.1}$$

Then, 0 is a semi-simple eigenvalue of $J^{-1}L$ and the associated eigenprojector Π satisfies

$$\Pi^*\mathbf{S}J\Pi = \Pi^*\mathbf{S}J \tag{6.2}$$

and

$$\big|\Pi\big| \leq |\Sigma J|\ /c. \tag{6.3}$$

Moreover,

$$\mathbf{S}f \in (\ker L)^\perp \quad \Leftrightarrow \quad f \in \mathrm{range}(L). \tag{6.4}$$

Proof. Let $\mathbf{K}$ and $\mathbf{R}$ denote respectively the kernel and the range of L. The identity $(\mathbf{S}Lu, v) = (u, \mathbf{S}Lv)$ implies that $\mathbf{S}\mathbf{R} \subset \mathbf{K}^\perp$ and hence $\mathbf{R} \subset \mathbf{S}^{-1}(\mathbf{K}^\perp)$. Next we note that (6.1) implies that if $u \in \mathbf{K}$ and $\mathbf{S}Ju \in \mathbf{K}^\perp$, then $u = 0$, so that $J\mathbf{K} \cap \mathbf{S}^{-1}(\mathbf{K}^\perp) = \{0\}$:

$$\mathbf{R} \subset \mathbf{S}^{-1}(\mathbf{K}^\perp), \qquad J\mathbf{K} \cap \mathbf{S}^{-1}(\mathbf{K}^\perp) = \{0\}. \tag{6.5}$$

In particular $J\mathbf{K} \cap \mathbf{R} = \{0\}$ and $\mathbf{K} \cap J^{-1}\mathbf{R} = \{0\}$. This means that the kernel $\mathbf{K}$ of $A_J - \tau\mathrm{Id}$ has a trivial intersection with the range $J^{-1}\mathbf{R}$ of

$A_J - \tau\mathrm{Id}$, that is that τ is a semisimple eigenvalue of A_J. Moreover, since $\dim \mathbf{R} + \dim J\mathbf{K} = N$, (6.5) implies that $\mathbf{R} = \mathbf{S}^{-1}(\mathbf{K}^\perp)$, that is (6.4).

In the splitting $u = \Pi u + (\mathrm{Id} - \Pi)u$, $\Pi u \in \mathbf{K}$ and there is v such that $(\mathrm{Id}-\Pi)u = J^{-1}Lv$. Therefore $(\mathbf{S}Lv, \Pi u) = 0$ and $\big(\mathbf{S}J\Pi u, \Pi u\big) = \big(\mathbf{S}Ju, \Pi u\big)$. Hence,

$$c\big|\Pi u\big|^2 \leq \mathrm{Re}\,\big(\mathbf{S}J\Pi u, \Pi u\big) = \mathrm{Re}\,\big(\mathbf{S}Ju, \Pi u\big) \leq \big|\mathbf{S}Ju\big|\;\big|\Pi u\big|$$

and (6.3) follows.

For $f \in \mathbf{K}^\perp$ and $u \in \mathbb{C}^N$ one has $(\Pi^* f, u) = (f, \Pi u) = 0$. Thus $\mathbf{K}^\perp \subset \ker \Pi^*$ and indeed $\mathbf{K}^\perp = \ker \Pi^*$ since the two spaces have the same dimension. For all u, $(\mathrm{Id} - \Pi)u \in J^{-1}\mathbf{R}$, hence $\mathbf{S}J(\mathrm{Id} - \Pi)u \in \mathbf{K}^\perp$ and therefore $\Pi^*\Sigma J(\mathrm{Id} - \Pi)u = 0$ that is (6.2). $\square$

Proposition 6.2. *Suppose that L and $\mathbf{S}$ ares matrices such that $\mathbf{S}L$ is hermitian symmetric. We now assume that we are given two matrices J_0 and J_1 such that for all $t \in [0,1]$ $J_t = (1-t)J_0 + tJ_1$ is invertible, and for all u and v in* ker*,*

$$\big(\mathbf{S}J_t u, v\big) = \big(u, \mathbf{S}J_t v\big). \tag{6.6}$$

Suppose that J_0 satisfies

$$\forall u \in \mathbf{K}_a, \qquad \big(\mathbf{S}J_0 u, u\big) \geq c|u|^2. \tag{6.7}$$

and suppose that for $t \in [0,1[$, 0 is a semi simple eigenvalue of $J_t^{-1}L$. Then for all $t \in [0,1]$,

$$\forall u \in \ker L, \qquad \big(\mathbf{S}J_t u, u\big) \geq (1-t)c|u|^2. \tag{6.8}$$

Proof. The assumption (6.6) means that the restriction of $\mathbf{S}J_t$ to $\ker L$ is symmetric. By (6.7), it is positive definite for $t = 0$. By continuity, it remains positive as long as it remains definite. It is indefinite when there is a $u \in \ker L$, $u \neq 0$ such that $\mathbf{S}J_t u(\in \ker L)^\perp$. By (6.4), this would imply that $u \neq 0$ would belong both to the kernel and to the range of $J_t^{-1}L$, contradicting the assumption that 0 is a semisimple eigenvalue. By continuity, $\mathbf{S}J_1$ is nonnegative and (6.8) follows. $\square$

6.2 From full symmetrizers to symmetrizers

We consider here $N \times N$ matrices

$$L(\tau, a) = \tau J(a) - A(a) \tag{6.9}$$

which depend on parameters a in an open set Ω and $\tau \in \mathbb{R}$. We always assume that $J(a)$ is invertible. We link the spectral properties of $A_J(a) := J(a)^{-1}A(a)$ to the existence of symmetrizers and full symmetrizers of $L(\tau, a)$.

Assumption 6.3. *We assume that the matrices $J(a)$, $J(a)^{-1}$ are uniformly bounded and that there are uniformly bounded matrices $\mathbf{S}(\tau, a)$ is such that for all τ and a, $\mathbf{S}(\tau, a)L(\tau, a)$ is hermitian symmetric and*

$$\forall u \in \ker L(\tau, a), \qquad \operatorname{Re}\big(\mathbf{S}(\tau, a)J(a)u, u\big) \geq c|u|^2. \tag{6.10}$$

where c is independent of a and τ.

We further assume that all the complex roots in τ of $\det L(\tau, a) = 0$ are real.

Proposition 6.1 implies that the eigenvalues τ of $J(a)^{-1}A(a)$ are real and semi simple and that the corresponding eigenprojectors are uniformly bounded:

Corollary 6.4. *Under Assumption 6.3, the family $A_J(a)$ is uniformly strongly hyperbolic in the sense of Assumption* 5.3*.*

Theorem 6.5. *In addition to Assumption* 6.3*, suppose that J, A and $\mathbf{S}$ are continuous [resp. Lipschitz continuous] [resp. C^∞] in $a \in \Omega$ and $\tau \in \mathbb{R}$. Then there is a bounded and continuous [resp. Lipschitz continuous] [resp. C^∞] matrix $S(\cdot)$ on Ω such that*

$$S(a)J(a) = \big(S(a)J(a)\big)^* \geq c_1 \mathrm{Id}, \qquad S(a)A(a) = \big(S(a)A(a)\big)^*, \tag{6.11}$$

with $c_1 > 0$ independent of a.

Proof. Multiplying L by J^{-1} reduces to the case $J = \mathrm{Id}$. A symmetrizer is

$$S(a) = \sum_{\tau \in \Sigma(a)} \Pi(\tau, a)^* \mathbf{S}(\tau, a)\Pi(\tau, a) \tag{6.12}$$

where $\Sigma(a) \subset \mathbb{R}$ denotes the spectrum of $A(a)$.

For u and v in $\ker L(\tau, a)$, one has $L(\tau + \sigma s, a)u = \sigma J(a)u$ and a similar expression for v. The symmetry implies

$$\big(\mathbf{S}(\tau + \sigma\nu)J(a)u, v\big) = \big(u, \mathbf{S}(\tau + \sigma u)J(a)v\big)$$

and letting σ tend to zero implies

$$\big(\mathbf{S}(\tau, a)J(a)u, v\big) = \big(u, \mathbf{S}(\tau, a)J(a)v\big). \tag{6.13}$$

This shows that each term of the sum (6.12) is symmetric and S is symmetric. Moroever, by (6.10) and Corollary 6.4, it is uniformly bounded and uniformly positive. By construction, $S(a)A(a)$ is symmetric and by (6.2) and symmetry, one has

$$S(a) = \sum_{\tau \in \Sigma(a)} \Pi(\tau, a)^* \mathbf{S}(\tau, a) = \sum_{\tau \in \Sigma(a)} \mathbf{S}^*(\tau, a) \Pi(\tau, a). \tag{6.14}$$

Theorem 5.7 implies that S is continuous [resp. Lipschitz continuous] [resp. C^∞] in a, finishing the proof of the theorem. □

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