TRANSPARENT NONLINEAR GEOMETRIC OPTICS AND MAXWELL-BLOCH EQUATIONS *

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1. Introduction

Many results have been obtained in the last decade about the justification of nonlinear geometric optics expansions (see references below and the survey papers [JMR1][JMR2]). All of them consider general equations and make no assumption on the structure of the nonlinear terms. There are cases where these general theorems are insufficiently precise. Typically, this happens when interaction coefficients vanish because of the special structure of the equations. This implies that the transport equations are linear so the leading order approximation does not reveal any nonlinear behavior. This is already a useful piece of information, showing that nonlinear phenomena do not assert their influence until anormaly large amplitudes or anormaly long interaction times are considered. This phenomenon is called *transparence*. To reach nonlinear regimes, one can consider waves of larger amplitude or, equivalently, of higher energy. The main goal of this paper is to start an analysis of this problem. We perform it within a class of equations which is interesting for three reasons. First, it contains several versions of Maxwell-Bloch equations which are of special interest in nonlinear optics. Second, it is sufficiently general to capture most of the expected phenomena. Third, it allows an almost complete analysis of the problem since the necessary and the sufficient conditions we state are very close. Two questions are raised. What are the conditions for the construction of BKW solutions? When they exist, what are the conditions for their stability, i.e. when are they close to exact solutions? The analysis relies on the study of resonant interaction of oscillations. Because of the large amplitudes, they may create strong instabilities in times $O(1)$, and they actually do so for general equations. The *compatibility* conditions simply mean that the interaction coefficients vanish at all the (possibly) unstable resonances. One important point is that there are many more interactions to control for the second question than for the first. In particular, unstable BKW solutions do exist. In fact, the Maxwell-Bloch equations lie at an extreme end of the class of equations which we consider. There exists a canonical change of unknowns which reduces them to the standard regime of nonlinear geometric optics, where the known

* Research partially supported by the U.S. National Science Foundation, U.S. Office of Naval Research, and the NSF-CNRS cooperation program under grants number NSF-DMS-9803296 and OD-G-N0014-92-J-1245 NSF-INT-9314095 respectively, and the CNRS through the Groupe de Recherche G1180 POAN.

results of [DR] apply. Our analysis explains how the existence of this change of unknows is deeply related to the very strong compatibility conditions which are satisfied by these equations. This is an extreme behaviour. The class of equations under consideration contains many other interesting examples such as coupled Klein-Gordon equations. We use them to illustrate the different levels of compatibility conditions.

To introduce the above mentionned class of equations, we start from the Maxwell-Bloch equations. They are widely used in nonlinear optics textbooks as a model for the description of the interaction between light and matter and the propagation of laser beams in nonlinear media, see e.g. [BW], [NM], [Bo] or [PP]. Because of their special interest, we will discuss them in detail in §12. Just recall here a model which comes from a two levels quantum system for the electrons. In suitable units and scales, the electromagnetic field (E, B), the polarization P of the medium and the difference N between the numbers of exited and nonexited atoms satisfy

$$\begin{cases} \partial_t B \;+\; \mathrm{curl} E = \;0\,, & \partial_t E \;-\; \mathrm{curl} B = \;-\partial_t P\,, \\ \varepsilon^2\, \partial_t^2 P \;+\; \Omega^2 P = \;\gamma_1\; N\, E\,, & \partial_t N = \;-\;\gamma_2 \partial_t P \cdot E\,, \end{cases} \tag{1.1}$$

where ε is a small parameter such that Ω/ε is the frequency associated to the electronic transition between the two levels. Introducing $Q = \varepsilon\partial_t P$ and $U = (B, E, P, Q, N - \underline{N})$ where $\underline{N}$ denotes the value of N at thermodynamical equilibrium, the equations (1.1) fall into the general framework of dispersive equations (see [DR])

$$\mathcal{L}(\varepsilon\partial_x)U \;=\; \mathcal{F}(U) \tag{1.2}$$

where $x = (t, y) \in \mathbb{R} \times \mathbb{R}^d$ denotes the space-time variables and $\mathcal{L}(\varepsilon\partial_x) = \varepsilon\partial_t + \sum \varepsilon\mathcal{A}_j\partial_j + \mathcal{L}_0$ is conservative. We study high frequency asymptotic solutions

$$U^\varepsilon(x) \;\sim\; \underline{U} + \varepsilon^p \sum_{n\geq 0} \varepsilon^{np}\, \mathbf{U}_n(x, \beta\cdot x/\varepsilon) \tag{1.3}$$

where $\underline{U}$ is a constant solution and the $\mathbf{U}_n(x, \theta)$ are periodic functions of θ. Note that the wavelength of the oscillation is exactly of order ε because the frequency of light is comparable to the frequency of the electronic transistion. This is an important feature of the problem which encodes the *dispersive* character of the propagation, see [Do], [DR].

The analysis of nonlinear geometric optics expansions for general systems (1.2) is made in [DR]. The first step is to determine the appropriate order ε^p for the solution. It must be sufficiently small so that the solutions exist on a domain independent of ε and it should be sufficiently large enough that nonlinear effects are captured in the leading term $\mathbf{U}_0$. The answer depends on the order of the nonlinearity f. When f is quadratic [resp. cubic], it is $p = 1$ [resp. $p = 1/2$]. This is the *standard* regime of semilinear geometric optics where exact solutions U^ε satisfying (1.3) are constructed in [DR] (see also [JR] for nondispersive equations). Recall that the wave number β satisfies the eikonal equation $\det L(i\beta) = 0$ and the Fourier coefficients of the principal term $\mathbf{U}_0 = \sum \mathbf{U}_0(\nu)e^{i\nu\theta}$ satisfy the polarization condition $\mathbf{U}_0(\nu) = P(\nu\beta)\mathbf{U}_0(\nu)$ where $P(\xi)$ is the orthogonal projector on $\ker L(i\xi)$. When f is quadratic, the nonlinear interaction term in the transport equation for $\mathbf{U}_0(\nu)$ is

$$P(\nu\beta) \sum_{\nu_1+\nu_2=\nu} q(\mathbf{U}_0(\nu_1), \mathbf{U}_0(\nu_2))\,, \tag{1.4}$$

where q is the symmetric bilinear form associated to f.

For the Maxwell-Bloch equations (1.1), f is quadratic and [DR] applies to solutions of amplitude $O(\varepsilon)$. This is insufficient for two reasons. First, for physically relevant choices of $\mathbf{U}_0$, the interactions terms (1.4) vanish. Thus the transport equations are linear showing that the BKW solutions are not affected by the nonlinearity of the medium. Second, Maxwell-Bloch equations are supposed to be a refinement of cubic models in nonlinear optics, such as the anharmonic oscillator model which is discussed in [Do] and [DR]. Both facts suggest that solutions of amplitude $O(\sqrt{\varepsilon})$ are natural. In addition, such BKW solutions of equation Maxwell-Bloch (1.1) are constructed in [Do]. They obey the following inhomogeneous scaling of the amplitudes :

$$(B, E, P, Q) = \sqrt{\varepsilon}\,(\widetilde{B}, \widetilde{E}, \widetilde{P}, \widetilde{Q}), \quad N - \underline{N} = \varepsilon \widetilde{N}\,. \tag{1.5}$$

This is a particular case of (1.3) with $p = 1/2$. However, one forces $N - \underline{N} = O(\varepsilon)$ because the terms in (1.4) vanish only when $\mathbf{U}_0 = (\widetilde{B}_0, \widetilde{E}_0, \widetilde{P}_0, \widetilde{Q}_0, 0)$. We refer to §12 for further motivations for introducing it. The Maxwell-Bloc equations (1.1) then read

$$\begin{cases} \varepsilon\, \partial_t \widetilde{B} + \varepsilon\, \mathrm{curl} \widetilde{E} = 0\,, & \varepsilon\, \partial_t \widetilde{E} - \varepsilon \mathrm{curl} \widetilde{B} = -\widetilde{Q}\,, \\ \varepsilon\, \partial_t \widetilde{P} - \widetilde{Q} = 0\,, & \varepsilon\, \partial_t \widetilde{Q} + \Omega^2 P = \gamma_1 \underline{N}\, \widetilde{E} + \varepsilon\, \gamma_1\, \widetilde{N}\, \widetilde{E}, \\ \varepsilon \partial_t \widetilde{N} = -\,\gamma_2\, \widetilde{Q} \cdot \widetilde{E}\,. & \end{cases} \tag{1.6}$$

The question is to study the existence of formal and exact solutions $\widetilde{U}$ of (1.6) with amplitude $O(1)$. A tricky argument gives the anwser. Consider the change of unknows

$$n = \widetilde{N} + \frac{\gamma_2}{2\gamma_1 \underline{N}}\,(\widetilde{Q}^2 + \Omega^2 \widetilde{P}^2)\,. \tag{1.7}$$

Then the last equation in (1.6) is transformed into

$$\varepsilon \partial_t n = \varepsilon \frac{\gamma_2}{\underline{N}}\, \widetilde{N}\, \widetilde{Q} \cdot \widetilde{E} = \varepsilon \left(c_1 n - c_2 (\widetilde{Q}^2 + \Omega^2 \widetilde{P}^2) \right) \widetilde{Q} \cdot \widetilde{E}\,. \tag{1.8}$$

The key point is that the bad $O(1)$ quadratic term in the equations for $\widetilde{N}$ has been eliminated. Introducing $U^\sharp := (\widetilde{B}, \widetilde{E}, \widetilde{P}, \widetilde{Q}, n)$, the system (1.6) is equivalent to an equation of the form

$$\mathcal{L}(\varepsilon \partial_x) U^\sharp = \varepsilon\, \mathcal{F}(U^\sharp) \tag{1.9}$$

where the key point is that the right hand side is $O(\varepsilon)$. For this equation, the *standard* regime of nonlinear geometric optics concerns $O(1)$ solutions and the results of [DR] apply. Changing back to the variables, we recover the BKW solutions of (1.1) constructed in [Do] and prove that they are stable, i.e. that there exist *exact* solutions of (1.1) which have the same asymptotic expansion. Because of their special interest, we will develop several examples of applications of this sort in §12.

This is the end of the story for the Maxwell-Bloch equations (1.1) but this is the starting point of this paper. Our goal is to understand what can be said for more general transparent systems and *why* there exists such a miraculous change of unknowns. In this paper we perform the analysis for systems of the following form which generalizes (1.6)

$$\begin{cases} L(\varepsilon \partial_x) u := \varepsilon \partial_t u + \sum \varepsilon A_j \partial_j u + L_0 u = \varepsilon f(u, v) \\ M(\varepsilon \partial_x) v := \varepsilon \partial_t v + \sum \varepsilon B_j \partial_j v + M_0 v = q(u) + \varepsilon g(u, v) \end{cases} \tag{1.10}$$

where L and M are symmetric hyperbolic, q is quadratic and f and g vanish at the origin. This is a particular case of equations (1.2). The triangular structure of the main quadratic interaction permits a very complete analysis. On the other hand, the problem we are now considering is more singular than the one sketched above for general equations (1.2) since, as in (1.6), we are looking for solutions $U = (u, v)$ of amplitude $O(1)$

$$U^\varepsilon(x) \sim \sum_{n\geq 0} \varepsilon^n \, \mathbf{U}_n(x, \varphi(x)/\varepsilon)\,, \tag{1.11}$$

where the $\mathbf{U}_n(x, \theta)$ are periodic functions of θ and $\varphi(x)$ is a given phase function or a finite set of phase functions if one considers interacting waves. For general systems (1.10) the *standard* regime of nonlinear geometric optics concerns solutions which are smaller by a factor ε. The analogue of (1.4) is

$$\sum_{\nu_1+\nu_2=\nu} Q(\nu d\varphi)\, q\big(\mathbf{u}_0(\nu_1), \mathbf{u}_0(\nu_2)\big)\,. \tag{1.12}$$

where the $\mathbf{u}_0(\nu)$ satisfy the polarization condition $\mathbf{u}_0(\nu) = P(\nu d\varphi)\mathbf{u}_0(\nu)$ and $P(\xi)$ [resp. $Q(\xi)$] is the orthogonal projector on $\ker L(i\xi)$ [resp. $M(i\xi)$]. It is *this* term which always vanishes for the Maxwell-Bloch equations.

In this paper, we discuss the following questions.

1. The existence of BKW solutions (1.10) of amplitude $O(1)$. Substituting (1.11) into (1.10) and ordering the terms yields a formal series $\sum \varepsilon^n \mathbf{F}_n$. A BKW solution is a formal series (1.11), such that after substitution, all the term $\mathbf{F}_n$ vanish. The equation $\mathbf{F}_0 = 0$ implies the polarisation conditions $\mathbf{u}_0(\nu) = P(\nu d\varphi)\mathbf{u}_0(\nu)$ and the necessary condition that the term in (1.12) vanishes. Thus, in order to construct solutions with arbitrary initial data for $P(\nu d\varphi)\mathbf{u}_0(\nu)$, it is necessary to assume that

$$Q((\nu_1+\nu_2)d\varphi)\, q\big(P(\nu_1 d\varphi)\,\cdot\,, P(\nu_2 d\varphi)\,\cdot\,\big) \;=\; 0\,, \quad \text{for all } \; \nu_1 \text{ and } \nu_2\,. \tag{1.13}$$

This is the *transparency* condition. Note that this assumption is not as strong as it may seem. Because the equations are dispersive, most of the $P(\nu d\varphi)$ and $Q(\nu d\varphi)$ vanish. Thus in practice, it reduces to a small number of cancellations and this is why this condition is not unrealistic.

When it is satisfied, the equations $\mathbf{F}_n = 0$ are equivalent to a triangular sequence of equations for $\mathbf{U}_0$, $\mathbf{U}_1$ etc. In general these equations are *quasilinear*. Moreover the transparency condition is not sufficient to imply that they have solutions. We give both sufficient conditions and also necessary conditions for their solvability. They are strictly stronger than (1.13).

2. The stability of BKW solutions. Using Borel's summation process, a BKW solution yields *approximate* solutions U^ε_{app}. They satisfy (1.11) and solve the equation (1.10) with infinite accuracy

$$\mathcal{L}(\varepsilon\partial_x)U^\varepsilon_{app} \,-\, \mathcal{F}(U^\varepsilon_{app}) \;=\; O(\varepsilon^\infty)\,. \tag{1.14}$$

The BKW solution is stable when there exists a family U^ε of *exact* solutions of (1.10) such that $U^\varepsilon - U^\varepsilon_{app} = o(1)$. This is not always true, and the main purpose of this paper is to study this question in detail. We give necessary and sufficient conditions for the stability. They are much stronger than the conditions which allow the construction of the BKW solution.

To see where the difficulty lies, consider $V^\varepsilon = \varepsilon^{-m}\big(U^\varepsilon - U^\varepsilon_{app}\big)$. Assuming that $\mathcal{F}(U) = \mathcal{Q}(U,U)$ is quadratic,

$$\widetilde{\mathcal{L}}^\varepsilon V^\varepsilon := \mathcal{L}(\varepsilon\partial_x)\, V^\varepsilon \;-\; 2\,\mathcal{Q}(U^\varepsilon_{app}, V^\varepsilon) \;=\; \varepsilon^m\, \mathcal{F}(V^\varepsilon) \;+\; O(\varepsilon^\infty)\,. \tag{1.15}$$

When $U^\varepsilon_{app} = O(\varepsilon)$ a standard energy estimate for $\mathcal{L}(\varepsilon\partial_x)$ and Gronwall's lemma imply that the solutions of $\widetilde{\mathcal{L}}^\varepsilon W^\varepsilon = 0$ satisfy on $[0,T]$

$$\|W^\varepsilon(t)\|_{L^2} \;\leq\; C_T \|W^\varepsilon(0)\|_{L^2} \tag{1.16}$$

where C_T is independent of ε. (Recall that the coefficient of ∂_t in L is ε). This kind of estimate is the starting point of the analysis in [DR] and [JR] leading to the linear and nonlinear stability results of standard nonlinear geometric optics. In sharp contrast, when $U^\varepsilon_{app} = O(1)$, the zero-th order term $\mathcal{Q}(U^\varepsilon_{app}, V^\varepsilon)$ is as strong as $\mathcal{L}(\varepsilon\partial_x)V^\varepsilon$ and cannot be neglected. The main task is to study the validity of a uniform estimate (1.16). Necessary and sufficient conditions are given. In short, they assert that

$$Q(\xi + \nu d\varphi)\, q\big(P(\nu d\varphi)\,\cdot\,, P(\xi)\,\cdot\,\big) \;=\; 0\,, \quad \text{for all } \nu \text{ and } \xi\,. \tag{1.17}$$

The idea is that the oscillations $\nu d\varphi$ in $\mathbf{u}_0$ interact with all the frequencies ξ of W^ε. The condition (1.17) ensures that there are no unbounded amplification in this mechanism. When (1.17) is not satisfied, we construct solutions which grow like $e^{\gamma t/\sqrt{\varepsilon}}$ showing that the uniform estimates (1.16) do not hold.

The last step is to prove that when (1.16) is satisfied, the BKW solution is actually stable.

3) The Maxwell-Bloch equations (1.6) and their multi-level extensions discussed in §12 satisfy the stronger property

$$Q(\xi + \xi')\, q\big(P(\xi)\,\cdot\,, P(\xi')\,\cdot\,\big) \;=\; 0\,, \quad \text{for all } \xi \text{ and } \xi'\,. \tag{1.18}$$

When this property is satisfied, we show that there is a bilinear mapping

$$(u, u') \mapsto J(u, u') \tag{1.19}$$

acting on functions of y, such that the change of variables

$$v^\sharp(t,\,.\,) = v(t,\,.\,) + \; J\big(u(t,\,.\,), u(t,\,.\,)\big) \tag{1.20}$$

transforms the second equation of (1.10) to

$$M(\varepsilon\partial_x)\, v^\sharp \;=\; 2\, J\big(u, L(\varepsilon\partial_x)u\big) \;+\; \varepsilon\, g(u,v) \;=\; \varepsilon\big(2\, J(u, f(u,v)) + g(u,v)\big)\,. \tag{1.21}$$

Therefore, $U^\sharp = (u, v^\sharp)$ satisfies an equation of the form (1.9). Thus, under the strong condition (1.18), the problem is reduced to the standard regime for equations (1.9). However, in general, the bilinear mapping J and thus the nonlinear $\mathcal{F}(U^\sharp)$ involve Fourier multipliers. The know results [DR], [JR], [JMR 3,4,5] should be adapted to cover this case to explain why $O(1)$ stable expansions are valid.

For the Maxwell-Bloch equations (1.6), it happens that the change of unknowns (1.20) is given by the polynomial substitution (1.7) and the known results of [DR] directly apply to the transformed equation (1.9). This is extended to more general versions of Maxwell-Bloch equations in §12. But in general, J does involve Fourier multipliers. We give examples of coupled Klein Gordon equations which illustrate this point.

As sketched above, the analysis only relies on the study of resonant interaction of oscillations which can cause strong instabilities because of the large amplitude of the leading term. The necessary and the sufficient compatibility conditions say that the interaction coefficients vanish at all the unstable resonances. The conditions (1.13) (1.17) and (1.18) are in increasing order of strength. Coupled Klein-Gordon equations give examples showing that thi orde is strict. Unstable BKW solutions exist. Stable BKW solutions may exist when (1.18) is not globally satisfied. The condition (1.17) can be satisfied for $d\varphi$ in an open subset of the characteritic variety while (1.18) does not hold.

2. Outline of the results

With variables $x = (t, y) \in \mathbb{R} \times \mathbb{R}^d$, consider a system

$$
\begin{cases} L(\varepsilon\, \partial_x)\, u \;+\; \varepsilon\, f(u, v) \;=\; 0\,, \\ M(\varepsilon \partial_x)\, v \;+\; q(u, u) \;+\; \varepsilon\, g(u, v) \;=\; 0\,, \end{cases} \tag{2.1}
$$

where f and g are smooth polynomial functions of their arguments and vanish at the origin, q is bilinear and

$$
\begin{aligned} L(\varepsilon\, \partial) \;&:=\; \varepsilon\, \partial_t \;+\; A(\varepsilon\, \partial_y) \;:= \varepsilon\, \partial_t \;+\; \sum \varepsilon\, A_j \partial_{y_j} \;+\; L_0 \;:=\; \varepsilon L_1(\partial_x) \;+\; L_0 \\ M(\varepsilon\, \partial) \;&:=\; \varepsilon\, \partial_t \;+\; B(\varepsilon\, \partial_y) \;:= \varepsilon\, \partial_t \;+\; \sum \varepsilon\, B_j \partial_{y_j} \;+\; M_0 \;:=\; \varepsilon M_1(\partial_x) \;+\; M_0 \end{aligned} \tag{2.2}
$$

are symmetric hyperbolic, meaning that the A_j and B_j are hermitian symmetric while L_0 and M_0 are skew adjoint. The main feature of this system is that the principal nonlinearity $q(u, u)$ appears only on the second equation and depends only on the first set of unknows u.

When one considers complex valued solutions, one should consider general quadratic interactions $q_1(u, u) + q_2(\overline{u}, u) + q_3(\overline{u}, \overline{u})$. Taking real and imaginary parts reduces to the case $q(u, u)$. However, even when we are interested only in real solutions of real systems, we want to use complex exponential and thus one has to extend q to the complex domain. For simplicity, we confine ourselves to the (complex) bilinear case. In Remark 2.12 below, we indicate briefly the necessary modifications to cover the general case.

We look for solutions satisfying

$$
u^\varepsilon(x) \;\sim\; \sum_{n\geq 0} \varepsilon^n\, \mathbf{u}_n(x, \beta\cdot x/\varepsilon)\,, \quad v^\varepsilon(x) \;\sim\; \sum_{n\geq 0} \varepsilon^n\, \mathbf{v}_n(x, \beta\cdot x/\varepsilon)\,. \tag{2.3}
$$

Here $\beta := (\beta_1, \ldots, \beta_m) \in (\mathbb{R}^{1+d})^m$ denotes a set of m space-time wave numbers. The profiles $\mathbf{u}_n(x, \theta)$ and $\mathbf{v}_n(x, \theta)$ are 2π-periodic in $\theta = (\theta_1, \ldots, \theta_m) \in \mathbb{T}^m := (\mathbb{R}/2\pi\mathbb{Z})^m$. Througout the paper we use the notations $\beta_j = (\omega_j, \kappa_j) \in \mathbb{R} \times \mathbb{R}^d$, $\omega := (\omega_1, \ldots, \omega_m)$ and $\kappa := (\kappa_1, \ldots, \kappa_m)$. Substituting the formal series (2.3) into (2.1) the left hand side of (2.1) has the formal expansion

$$
\sim \sum_{n\geq 0} \varepsilon^n\; \mathbf{\Phi}_n(x, \beta\cdot x/\varepsilon)\,.
$$

Definition. $\sum \varepsilon^n(\mathbf{u}_n, \mathbf{v}_n)$ *is a formal solution or a BKW solution when* $\mathbf{\Phi}_n = 0$ *for all* $n \geq 0$.

A) *The transparency condition.*

The first equation $\mathbf{\Phi}_0 = 0$ reads

$$L(\beta\partial_\theta)\mathbf{u}_0 \;=\; 0\,, \quad M(\beta\partial_\theta)\mathbf{v}_0 \;+\; q(\mathbf{u}_0, \mathbf{u}_0) \;=\; 0\,, \tag{2.4}$$

where $L(\beta\partial_\theta) := \sum_j L_1(\beta_j)\partial_{\theta_j} + L_0$ and $M(\beta\partial_\theta) := \sum_j M_1(\beta_j)\partial_{\theta_j} + M_0$. On Fourier series,

$$L(\beta\partial_\theta)\Big(\sum_{\nu\in\mathbb{Z}^m} a_\nu e^{i\nu\theta}\Big) \;=\; \sum_{\nu\in\mathbb{Z}^m} L(i\nu\beta)\, a_\nu\, e^{i\nu\theta}\,, \tag{2.5}$$

where $\nu\beta := \sum_j \nu_j\beta_j \in \mathbb{R}^{1+d}$ and $L(i\xi) = iL_1(\xi) + L_0$ denotes the symbol of L. We say that ξ is characteristic for L when $\det L(i\xi) = 0$. Introduce the projector $\mathbb{P}$ on the kernel of $L(\beta\partial_\theta)$

$$\mathbb{P}\Big(\sum_{\nu\in\mathbb{Z}^m} a_\nu e^{i\nu\theta}\Big) \;:=\; \sum_{\nu\in\mathbb{Z}^m} P(\nu\beta)\, a_\nu\, e^{i\nu\theta}\,, \tag{2.6}$$

where $P(\xi)$ denotes the orthogonal projector on $\ker L(i\xi)$. Similarly introduce $Q(\xi)$ the orthogonal projector on $\ker M(i\xi)$ and $\mathbb{Q}$ the projector on $\ker M(\beta\partial_\theta)$,

The first equation in (2.4) is equivalent to the polarisation condition $\mathbf{u}_0 = \mathbb{P}\mathbf{u}_0$. We are looking for solutions with $\mathbf{u}_0 \neq 0$ and thus at least one of the $P(\nu\beta)$ must be different of zero. In this work, we focus on dispersive equations, i.e. $L_0 \neq 0$, and for simplicity we assume throughout the paper that

$$\det L(i\nu\beta) \neq 0 \quad \text{and} \quad \det M(i\nu\beta) \neq 0 \quad \text{for } \nu \text{ large}\,. \tag{2.7}$$

Examples 1. To describe the propagation of a single wave, one considers one wave number $\beta := (\omega, \kappa) \in \mathbb{R}\times\mathbb{R}^d$ which satisfies the eikonal equation for L :

$$\det L(i\beta) \;=\; 0\,. \tag{2.8}$$

The condition (2.7) is satisfied for instance when $\det L_1(\beta) \neq 0$ and $\det M_1(\beta) \neq 0$. A typical example is that $\det L(i\nu\beta) = 0$ exactly for $\nu \in \{-1, 0, 1\}$ and $\det M(i\nu\beta) = 0$ only for $\nu = 0$.

2. To describe the interaction of waves one considers several wave numbers $(\beta_1, \ldots, \beta_m)$. Each of them satisfies the eikonal equation (2.8) and have at most a finite number of characteristic harmonics. The interaction is nonresonant when all the combinations $\sum \nu_j\beta_j$ with at least two nonvanishing coefficients which are not characteristic. In case of resonant interaction, there are often exactly two resonance relations $\sum \nu_j\beta_j = 0$ and $\sum(-\nu_j)\beta_j = 0$ and (2.7) is also satisfied.

The second equation in (2.4) requires that $\mathbb{Q}q(\mathbf{u}_0, \mathbf{u}_0) \;=\; 0$. The transparency condition states that this equation is a consequence of the polarisation $\mathbf{u}_0 = \mathbb{P}\mathbf{u}_0$.

Assumption 2.1. *For all ν and ν_1 in $\mathbb{Z}^m$ and for all vectors u and v, one has*

$$Q(\nu_1\beta)\, q\Big(P((\nu_1 - \nu)\beta)u\,,\, P(\nu\beta)v\Big) \;=\; 0\,. \tag{2.9}$$

Using Fourier series, (2.9) is equivalent to the condition that for all functions $\mathbf{u}$ and $\mathbf{u}'$ one has

$$\mathbb{Q}q\big(\mathbb{P}\mathbf{u}\,,\,\mathbb{P}\mathbf{u}'\big) \;=\; 0\,. \tag{2.10}$$

The assumption is trivially satisfied when all the $Q(\nu\beta)$ vanish, i.e. all the harmonics $\nu\beta$ are noncharacteristic for M. The interesting case occurs when there is at least one resonance, i.e. harmonics $\nu_1\beta$ and $\nu_2\beta$ which are characteristic for L and such that $(\nu_1+\nu_2)\beta$ is characteristic for M. Note, that (2.7) implies that there are only a finite number of such resonances.

B) *Triangulation of the equations for the formal equations.*

To find formal solutions, the first step is to put the system of equations $\{\mathbf{\Phi}_n = 0\}_{n\geq 0}$ in a triangular form in order to compute the $(\mathbf{u}_n, \mathbf{v}_n)$ inductively. In §3, we show that this can be done when Assumption 2.1 is satisfied leading to equations of the form

$$(2.11)_n \qquad \begin{cases} \mathbb{P}L_1(\partial_x)\mathbb{P}\mathbf{u}_n \;=\; \mathbf{r}_n\,, \\ \mathbb{Q}M_1(\partial_x)\mathbb{Q}\mathbf{v}_n \;+\; \mathbb{D}(\mathbf{u}_0, \partial_y)\mathbb{P}\mathbf{u}_n \;=\; \mathbf{r}_n \\ (\mathbb{I}-\mathbb{P})\,\mathbf{u}_n \;\;=\; \mathbf{r}_n\,, \\ (\mathbb{I}-\mathbb{Q})\,\mathbf{v}_n \;=\; \mathbf{r}_n\,. \end{cases}$$

The $\mathbf{r}_n$ denote different functionals which depends on $(\mathbf{u}_k, \mathbf{m}_k)_{k<n}$ and their derivatives and also on $(\mathbb{P}\mathbf{u}_n, \mathbb{Q}\mathbf{v}_n)$ but not on their derivatives. In particular, the firts two equations form a system for $(\mathbb{P}\mathbf{u}_n, \mathbb{Q}\mathbf{v}_n)$, which is quasilinear for $n = 0$ and linear for $n \geq 1$.

C) *Construction of formal solutions.*

Assumption 2.1 does not imply that the Cauchy problem for the first two equations in (2.11) is well posed. The principal part is

$$\begin{cases} \mathbb{P}L_1(\partial)\mathbb{P}\mathbf{u}\,, \\ \mathbb{Q}M_1(\partial_x)\mathbb{Q}\mathbf{v} \;+\; \mathbb{D}(\mathbb{P}\mathbf{u}_0, \partial_y)\mathbb{P}\mathbf{u}\,. \end{cases} \tag{2.12}$$

Introduce the space $\mathcal{P}$ [resp $\mathcal{Q}$] of trigonometric polynomials $\sum u_\nu e^{i\nu\theta}$ [resp. $\sum v_\nu e^{i\nu\theta}$] with coefficients $u_\nu \in \ker(Id - P(\nu\beta))$ [resp. $v_\nu \in \ker(Id - Q(\nu\beta))$]. By (2.7), $\mathcal{P}$ and $\mathcal{Q}$ are finite dimensional and (2.11) is a first order system for functions of variables $x = (t, y)$ with values in $\mathcal{P}\times\mathcal{Q}$. Its principal part (2.12) can be written

$$\partial_t \;+\; \mathcal{A}(\mathbf{u}_0(x), \partial_y) \;=\; \partial_t \;+\; \sum_{j=1}^{d} \mathcal{A}_j(\mathbf{u}_0)\partial_{y_j}\,. \tag{2.13}$$

To solve (2.11), and in particular $(2.11)_0$ which is quasilinear, we are led to impose hyperbolicity.

Assumption 2.2. *The system* (2.12) *is strongly hyperbolic, meaning that the matrices* $e^{i\mathcal{A}(\mathbf{a},\eta)}$ *are uniformly bounded for* $\eta \in \mathbb{R}^d$ *and* $\mathbf{a}$ *in bounded subsets of* $\mathcal{P}$. .

In §4 we give equivalent formulations using resonances. When it is satisfied, (2.13) is symmetrizable, but the symbol $\mathcal{S}(\mathbf{u}_0, \eta)$ of the symmetrizer is not necessarily smooth in η. However, in the present case, the lack of smoothness of the symmetrizer is not an obstacle for solving (2.11) and in §5 we prove the following result.

THEOREM 2.3 *Suppose that Assumptions* 2.1 *and* 2.2 *are satisfied. Given arbitrary initial data for* $\mathbb{P}\mathbf{u}_n$ *and* $\mathbb{Q}\mathbf{v}_n$ *in* $\mathbb{P}H^\infty(\mathbb{R}^d \times \mathbb{T}^m)$ *and* $\mathbb{Q}H^\infty(\mathbb{R}^d \times \mathbb{T}^m)$ *respectively, there is* $T > 0$ *and a sequence* $(\mathbf{u}_n, \mathbf{v}_n) \in C^0([0,T]; H^\infty(\mathbb{R}^d \times \mathbb{T}^m))$ *which satisfies the family of equations* (2.11)*.*

Here H^∞ denotes the intersection of the Sobolev spaces H^σ, for all σ.

D) *Linear stability.*

Consider a formal solution on $[0,T] \times \mathbb{R}^d$, given by Theorem 2.3. For any k,

$$U^\varepsilon_{app}(x) = \mathbf{U}^\varepsilon_{app}(x, x \cdot \beta/\varepsilon)\,, \quad \mathbf{U}^\varepsilon_{app} := \sum_{n=0}^{k} \varepsilon^n (\mathbf{u}_n, \mathbf{v}_n) \tag{2.14}$$

is an approximate solution of (2.1), in the sense that the left hand side evaluated on U^ε_{app} is $O(\varepsilon^{k+1})$ in $L^\infty \cap L^2$ and its j-th derivatives are $O(\varepsilon^{k+1-j})$. In the discussion of the existence of an exact solution close to U^ε_{app}, the main step is to study the linear stability of the approximate solution, i.e. the well posedness of the linearized Cauchy problem :

$$\widetilde{L}^\varepsilon U \,+\, \varepsilon F'(U^\varepsilon_{app})U \;:=\; \begin{pmatrix} L_1(\varepsilon\partial_x)u \\ M(\varepsilon\partial_x)v \,+\, 2q(u^\varepsilon_0, u) \end{pmatrix} \,+\, \varepsilon F'(U^\varepsilon_{app})U \;=\; \varepsilon\, H\,, \tag{2.15}$$

where $U = (u, v)$ and $F(U) := (f(U), g(U))$. The Cauchy problem is stable when there is a constant C such that for all $\varepsilon \in]0,1]$, all smooth initial data $U(0)$ and right hand side H, the solution of the Cauchy problem for (2.15) satisfies for all $t \in [0,T]$:

$$\|U(t)\|_{L^2(\mathbb{R}^d)} \,\le\, C\,\|U(0)\|_{L^2(\mathbb{R}^d)} \,+\, C \int_0^t \|H(s)\|_{L^2(\mathbb{R}^d)}\, ds \tag{2.16}$$

In §6, we give the following necessary condition for the stability. Recall the notation $\beta = (\omega, \kappa)$.

PROPOSITION 2.4. *If the linearized Cauchy problem* (2.15) *is stable, there is a constant* C *such that for all* $\nu \in \mathbb{Z}^m$*, all* $\xi = (\tau, \eta)$ *in the characteristic variety of* L*, all* $\xi' = (\tau', \nu\kappa + \eta)$ *in the characteristic variety of* M*, all* $x \in [0,T] \times \mathbb{R}^d$ *and all vector* u

$$\Big| Q(\xi')q\big(\mathbf{u}_{0,\nu}(x)\,,\, P(\xi)u\big) \Big| \;\le\; C\,|\,\tau' - \tau - \nu\omega\,|\,|\,u\,|\,,$$

where $\mathbf{u}_{0,\nu}(x)$ *denotes the* ν*-th Fourier component of* $\mathbf{u}_0(x, \theta)$*.*

When Assumptions 2.1 and 2.2 are satisfied, approximate solutions are constructed with arbitrary initial data for $\mathbf{u}_{0,\nu}$ provided that they satisfy $\mathbf{u}_{0,\nu} = P(\nu\beta)\mathbf{u}_{0,\nu}$. Thus, we are led to the following condition

ASSUMPTION 2.5. *For all* $\nu \in \mathbb{Z}^m$*, there is a constant* C *such that for all* $\xi = (\tau, \eta)$ *in the characteristic variety of* L*, all* $\xi' = (\tau', \nu\kappa + \eta)$ *in the characteristic variety of* M *and all vectors* u *and* u'

$$\Big| Q(\xi')q\big(P(\nu\beta)u\,,\, P(\xi)u'\big) \Big| \;\le\; C\,|\,\tau' - \tau - \nu\omega\,|\,|\,u\,|\,|\,u'\,|\,. \tag{2.17}$$

In particular, Assumption 2.5 requires that

$$Q(\xi')q\big(P(\nu\beta)u\,,\,P(\xi)u'\big) = 0 \quad \text{at resonances, i.e. when} \quad \xi' = \nu\beta + \xi\,. \tag{2.18}$$

Conversely, near "regular" resonances, (2.18) implies that one can factor out the equation of resonance in $Q(\xi')q\big(P(\nu\beta)u\,,\,P(\xi)u'\big)$, and (2.17) follows. This is made precise in §7 where we also give other equivalent formulations of Assumption 2.5. An important remark is that it is strictly stronger than the previous Assumptions 2.1 and 2.2

PROPOSITION 2.6. *Assumption* 2.5 *implies Assumptions* 2.1 *and* 2.2.

When Assumptions 2.1 and 2.2 are satisfied, Proposition 2.4 asserts that Assumption 2.5 is necessary for the validity of the stability estimate (2.16) for all approximate solutions. Conversely, we prove in §8 that it is sufficient.

PROPOSITION 2.7. *When Assumption* 2.5 *is satisfied, then for the family of approximate solution* (2.14)*, the solutions of the linearized equation* (2.15) *satisfy the a priori estimate* (2.16) *with canstant C independent of $\varepsilon \in]0,1]$.*

E) *Nonlinear stability and exact solutions.*

In §8 we prove that Assumption 2.5 also implies the nonlinear stability of approximate solutions.

THEOREM 2.8. *Suppose that Assumption* 2.5 *is satisfied. Fix a positive integer $k \geq 3$ and consider a smooth approximate solution U^ε_{app}, as in* (2.14)*, defined on $[0, T_a] \times \mathbb{R}^d$. Then for all $T < T_a$, there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$ the Cauchy problem for* (2.1) *with initial data $U^\varepsilon_{app}(0, .)$ has a unique solution $U^\varepsilon = (u^\varepsilon, v^\varepsilon)$ on $[0, T] \times \mathbb{R}^d$. Moreover, there is a constant C such that for all $\varepsilon \in]0, \varepsilon_0]$ and all $t \in [0, T]$:*

$$\big\|\, U^\varepsilon(t) \,-\, U^\varepsilon_{app}(t)\,\big\|_{L^\infty(\mathbb{R}^d)} \;\leq\; C\,\varepsilon^k\,. \tag{2.19}$$

This theorem is a consequence of a more precise result which we now describe. We look for solutions (u, v) of (2.1) as functions

$$u(x) \;=\; \mathbf{u}(x, \beta\cdot x/\varepsilon)\,, \quad v(x) \;=\; \mathbf{v}(x, \beta\cdot x/\varepsilon) \tag{2.20}$$

with $\mathbf{u}(x, \theta)$ and $\mathbf{v}(x, \theta)$periodic in θ. For (u, v) to be solutions of (2.1), it is sufficient that

$$\begin{cases} L(\varepsilon\partial_x + \beta\partial_\theta)\mathbf{u} \,+\, \varepsilon\, f(\mathbf{u}, \mathbf{v}) \;=\; 0\,, \\ M(\varepsilon\partial_x + \beta\partial_\theta)\mathbf{v} \,+ q(\mathbf{u}, \mathbf{u}) \,+ \varepsilon\, g(\mathbf{u}, \mathbf{v}) \;=\; 0\,. \end{cases} \tag{2.21}$$

Introduce. $\mathbf{U} := (\mathbf{u}, \mathbf{v})$. With obviuous notations, write this system as

$$\mathbf{L}^\varepsilon \mathbf{U}^\varepsilon \;+\; \mathbf{F}^\varepsilon(\mathbf{U}^\varepsilon) \;=\; 0\,. \tag{2.22}$$

Consider a formal solution $\sum \varepsilon^n \mathbf{U}_n$ given by Theorem 2.3 on $[0, T_a] \times \mathbb{R}^d \times \mathbb{T}^m$. Fix a positive integer k and introduce $\mathbf{U}^\varepsilon_{app}$ as in (2.14). Proposition 3.1 below shows that $\mathbf{U}^\varepsilon_{app}$ is an approximate solution of (2.21), meaning that for all integer σ, there is a constant C such that for all $\varepsilon \in]0, 1]$ and all $t \in [0, T_a]$

$$\big\|\,\big(\mathbf{L}^\varepsilon \mathbf{U}^\varepsilon_{app} \,+\, \mathbf{F}^\varepsilon(\mathbf{U}^\varepsilon_{app})\big)(t,\,.\,)\,\big\|_{H^\sigma(\mathbb{R}^d\times\mathbb{T}^m)} \;\leq\; C\,\varepsilon^{k+1}\,, \tag{2.23}$$

Theorem 2.8 is a corollary of the following result which is proved in §8.

Theorem 2.9. *Suppose that Assumption 2.5 is satisfied. For $k \geq 2$ consider an approximate solution $\mathbf{U}^\varepsilon_{app}$ (2.14), defined on $[0, T_a] \times \mathbb{R}^d$. In addition, consider a bounded family $\mathbf{R}^\varepsilon_0$ in $H^\sigma(\mathbb{R}^d \times \mathbb{T}^m)$ with $\sigma > (d+m)/2$. Then for all $T < T_a$, there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$ the Cauchy problem for (2.32) with initial data $\mathbf{U}^\varepsilon_{app}(0, \,.\,) + \varepsilon^k \mathbf{R}^\varepsilon_0$ has a unique solution $\mathbf{U}^\varepsilon = (\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon)$ on $[0, T] \times \mathbb{R}^d \times \mathbb{T}^m$ and there is a constant C such that for all $\varepsilon \in]0, \varepsilon_0]$ and all $t \in [0, T]$*

$$\big\| \, \mathbf{U}^\varepsilon(t) \, - \, \mathbf{U}^\varepsilon_{app}(t) \, \big\|_{H^\sigma(\mathbb{R}^d \times \mathbb{T}^m)} \; \leq \; C \, \varepsilon^k \, . \tag{2.24}$$

F) *Compatible nonlinearities.*

The strongest condition in the stability analysis is to assume that all pump frequencies β give rise to stable oscillations.

Assumption 2.10. *There is a constant C such that for all $\xi = (\tau, \eta)$ and $\xi' = (\tau', \eta')$ in the characteristic variety of L, all $\xi'' = (\tau'', \eta + \eta')$ in the characteristic variety of M and all vectors u and u'*

$$\Big| Q(\xi'') q\big(P(\xi)u \, , \, P(\xi')u'\big) \Big| \; \leq \; C \, | \, \tau'' - \tau - \tau' \, | \, | \, u \, | \, | \, u' \, | \, . \tag{2.25}$$

This assumption is discussed and compared to Assumption 2.5 in § 9. There are interesting examples where not all the pump frequencies are stables. We also show that Assumption 2.10 implies that the system (2.1) is conjugated via a nonlinear change of unknowns to a similar system with $q = 0$.

Theorem 2.11 *Suppose that Assumption 2.10 is satisfied. Then, there exists a family of bilinear mappings J^ε, from $H^\infty(\mathbb{R}^d) \times H^\infty(\mathbb{R}^d)$ to $H^\infty(\mathbb{R}^d)$, such that for all $u \in C^1([0,T]; H^\infty(\mathbb{R}^d))$,*

$$q(u(t), u(t)) \; = \; M(\varepsilon\partial_x) J^\varepsilon\big(u(t), u(t)\big) \; - \; J^\varepsilon\big(L(\varepsilon\partial_x)u(t), u(t)\big) \; - \; J^\varepsilon\big(u(t), L(\varepsilon\partial_x)u(t)\big) \, . \tag{2.26}$$

The change of variables which eliminates q is

$$\tilde{v} := v + J^\varepsilon(u, u) \, ; \tag{2.27}$$

The general definition of J is of the form

$$J^\varepsilon(u, u')(y) \; = \; (2\pi)^{-d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{iy(\eta + \eta')} J(\varepsilon\eta, \varepsilon\eta'; \widehat{u}(\eta), \widehat{u'}(\eta')) \, d\eta d\eta' \tag{2.28}$$

where $J(\eta, \eta'; \cdot \, , \, \cdot)$ is a bounded family of quadratic forms on $\mathbb{C}^N \times \mathbb{C}^N$. The Maxwell-Bloch equations discussed in §12 satisfy Assumption 2.10, but in addition, the forms $J(\eta, \eta'; \cdot \, , \, \cdot)$ are independent of (η, η') implying that the change of variable (2.27) is algebraic. In §9, we also give examples of systems for which J is pseudodifferential.

G) *Examples of instability.*

When Assumption 2.5 is not satisfied, Proposition 2.4 implies that instabilities are expected in the linearized equation. In §10 we study a model problem for which a single resonant interaction is isolated. The strength of the instability depends on the lower order terms, that is on f. In the strongest case, the amplitudes $\widehat{U^\varepsilon}(t,\eta)$ of the exact solutions at frequencies η in a ball $|\eta - \eta_0/\varepsilon| \leq h/\sqrt{\varepsilon}$ are exponentially amplified by a factor $e^{\gamma t/\sqrt{\varepsilon}}$. This analysis is used in §11 to produce examples of systems (2.1) which satisfy Assumptions 2.1 and 2.2, together with very accurate approximate solutions U^ε_{app} which are uniformly bounded in $L^2 \cap L^\infty$ on $[0,T] \times \mathbb{R}^d$, but such that the exact solutions U^ε which have the same initial data satisfy

$$\| U^\varepsilon(t) \|_{L^2(K)} \ \geq \ c\, e^{\gamma t \sqrt{\varepsilon}} \tag{2.29}$$

where K is a ball, $c > 0$ and $\gamma > 0$. Thus, in time $t \approx \sqrt{\varepsilon}$, the exact solution has nothing to do with the approximate solution. Moreover, this gives examples of Cauchy problems which have infinely accurate and uniformly bounded approximate solutions, but which have no uniformly bounded excat solutions.

H) *Applications to the Maxwell-Bloch equations.*

In § 12, we come back to different versions of the Maxwell-Bloch equations. We compute the principal term $\mathbf{U}_0$ of the expansion in two different applications. The first concerns the propagation of one single beam in a two level isotropic medium. We recover equations similar to those found in [DR] for the anharmonic model and in [Do] for the model (1.1). The second application concerns the stimulated Raman scattering. This is a three waves mixing process. There we use expansions (2.3) with several phases, i.e. with $\beta \in (\mathbb{R}^{1+3})^3$. In § 12, we also outline another application of properties of the change of variables (1.7). It can be used to justify the long time diffractive expansions found in [Do] for equation (1.1), reducing the problem to a "'standard" regime treated in [Lan].

Remark 2.12. 1) The analysis of quadratic interaction sketched above for $\mathbb{C}$-bilinear q, relies on the rule that $q(ae^{ix\xi}, a'e^{ix\xi'}) = e^{ix(\xi+\xi')}q(a,a')$. When $q(u,u')$ is linear in u and antilinear in u' [resp. antilinear in u and linear in u'] [resp. bi-antilinear] one has $q(ae^{ix\xi}, a'e^{ix\xi'}) = e^{ix(\xi-\xi')}q(a,a')$ [resp. $q(ae^{ix\xi}, a'e^{ix\xi'}) = e^{ix(-\xi+\xi')}q(a,a')$] [resp. $q(ae^{ix\xi}, a'e^{ix\xi'}) = e^{-ix(\xi+\xi')}q(a,a')$] and the condition (2.9) (2.17) and (2.25) in Assumptions 2.1, 2.5 and 2.10 must be changed accordingly. For example, when q is linear-antilinear, (2.25) is to be replaced by

$$\Big| Q(\xi'')q\big(P(\xi)u\,,\,P(\xi')u'\big) \Big| \ \leq \ C\,|\,\tau'' - \tau + \tau'\,|\,|\,u\,|\,|\,u'\,|\,, \quad \text{for} \quad \xi" = (\tau", \eta - \eta')\,. \tag{2.30}$$

2) Suppose that L and M are real and suppose that q is a real quadratic form. Consider q_1 the $\mathbb{C}$-bilinear extension of q and $q_2(u,u') = q_1(\underline{u, \overline{u'})}$ [resp. $q_3(u,u') \underline{= q_1}(\overline{u}, \overline{u'})$] its sequilinear [resp. bi-antilinear] extension. Because $P(-\xi) = \overline{P(\xi)}$ and $Q(-\xi) = \overline{Q(\xi)}$, it is clear that the compatiblity conditions for q_2 [resp. q_3] just described are equivalent to the compatibility conditions for q_1.

Remark 2.13. In this paper we only consider planar phases $\beta \cdot x$. Part of the analysis can be extended to nonlinear phases $\varphi(x)$. As in [DR] or [JRM 3], one has to make *coherence* assumptions meaning that the rank of the matrices $L(i\nu d\varphi(x))$ is independent of x and that the projectors $P(\nu d\varphi(x))$ are smooth in x. However, technical difficulties arise. For example, in §8 we use a pseudodifferential calculus with non smooth symbols (of a special sort). To extend the proof to nonlinear phases, it seems reasonnable to make assumptions so that one gets smooth symbols. This would lead to assume that the characteristic varieties of L and M have constant multiplicity and that the resonances hold on smooth manifolds. Moreover, one should localize the analysis, to cover the case where the phases are defined only locally. We leave these extensions to the interested reader.

3. Equations for formal solutions

From now on, we fix $\beta \in (\mathbb{R}^{1+d})^m$ and write $(\omega, \kappa) \in \mathbb{R}^m \times (\mathbb{R}^d)^m$. We assume that (2.7) holds. We look for BKW solutions of equations (2.1), i.e. we look for formal series

$$\mathbf{u}^\varepsilon(x,\theta) \sim \sum_{n\geq 0} \varepsilon^n \, \mathbf{u}_n(x,\theta)\,, \qquad \mathbf{v}^\varepsilon(x,\theta) \sim \sum_{n\geq 0} \varepsilon^n \, \mathbf{v}_n(x,\theta) \tag{3.1}$$

which satisfy (2.21) in the sense of formal series. Formal substitution yields

$$\begin{aligned} &L(\varepsilon\partial_x + \beta\partial_\theta)\mathbf{u}^\varepsilon \sim \sum_{n\geq 0} \varepsilon^n \big(L(\beta\partial_\theta)\, \mathbf{u}_n + L_1(\partial_x)\mathbf{u}_{n-1} \big)\,, \\ &M(\varepsilon\partial_x + \beta\partial_\theta)\mathbf{v}^\varepsilon \sim \sum_{n\geq 0} \varepsilon^n \big(M(\beta\partial_\theta)\, \mathbf{v}_n + M_1(\partial_x)\mathbf{v}_{n-1} \big), \\ &f(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) \sim \sum_{n\geq 0} \varepsilon^n \, \mathbf{f}_n\,, \quad g(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) \sim \sum_{n\geq 0} \varepsilon^n \, \mathbf{g}_n\,, \quad q(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) \sim \sum_{n\geq 0} \varepsilon^n \, \mathbf{q}_n \end{aligned} \tag{3.2}$$

and the left hand side of (2.21) is

$$\sim \sum_{n\geq 0} \varepsilon^n \; \mathbf{\Phi}_n(x,\theta)\,.$$

Here and below we agree that all terms with negative index vanish.

We look for profiles $(\mathbf{u}_n, \mathbf{v}_n)$ which are trigonometric polynomials. Since f and g are polynomial functions of their arguments, we note that this implies that $\mathbf{f}_n$ and $\mathbf{g}_n$ are also trigonometric polynomials. The equation $\mathbf{\Phi}_n = 0$ reads

$$\begin{cases} L(\beta\partial_\theta)\, \mathbf{u}_n \; + \; L_1(\partial_x)\mathbf{u}_{n-1} \; + \; \mathbf{f}_{n-1} \; = \; 0\,, \\ M(\beta\partial_\theta)\mathbf{v}_n \; + \; M_1(\partial_x)\mathbf{v}_{n-1} \; + \; \mathbf{q}_n \; + \; \mathbf{g}_{n-1} \; = \; 0\,. \end{cases} \tag{3.3$_n$}$$

Introduce the operator

$$\mathbb{L}^{-1}\Big(\sum_{\nu\in\mathbb{Z}^m} a_\nu e^{i\nu\theta} \Big) \; := \; \sum_{\nu\in\mathbb{Z}^m} L^{(-1)}(i\nu\beta)\, a_\nu \, e^{i\nu\theta}\,. \tag{3.4}$$

where $L^{(-1)}(i\xi)$ denotes the partial inverse of $L(i\xi)$ definded by

$$L^{(-1)}(i\xi)\, L(i\xi) \; = \; L(i\xi)\, L^{(-1)}(i\xi) \; = \; Id - P(\xi)\,, \quad L^{(-1)}(\xi)\, P(\xi) \; = \; P(\xi)\, L^{(-1)}(\xi) \; = \; 0\,.$$

$\mathbb{L}^{-1}$ acts on formal Fourier series and on trigonometric polynomials. The definition of the partial inverse $\mathbb{M}^{-1}$ of $M(\beta\partial_\theta)$ is similar. One has

$$\begin{aligned} &\mathbb{P}L(\beta\partial_\theta) = L(\beta\partial_\theta)\mathbb{P} = 0\,, \\ &\mathbb{L}^{-1}L(\beta\partial_\theta) = L(\beta\partial_\theta)\mathbb{L}^{-1} = \mathbb{I} - \mathbb{P}\,, \qquad \mathbb{L}^{-1}\mathbb{P} = \mathbb{P}\mathbb{L}^{-1} = 0\,. \\ &\mathbb{Q}M(\beta\partial_\theta) = M(\beta\partial_\theta)\mathbb{Q} = 0\,, \\ &\mathbb{M}^{-1}M(\beta\partial_\theta) = M(\beta\partial_\theta)\mathbb{M}^{-1} = \mathbb{I} - \mathbb{Q}\,, \qquad \mathbb{M}^{-1}\mathbb{Q} = \mathbb{Q}\mathbb{M}^{-1} = 0\,. \end{aligned} \tag{3.5}$$

Thus $(3.3)_n$ is equivalent to

$$(3.6)_n \qquad \begin{cases} i) \quad (\mathbb{I}-\mathbb{P})\,\mathbf{u}_n \;=\; -\,\mathbb{L}^{-1}\,(L_1\mathbf{u}_{n-1}\,+\,\mathbf{f}_{n-1})\,, \\ ii) \;\; \mathbb{P}L_1\mathbf{u}_{n-1}\,+\,\mathbb{P}\mathbf{f}_{n-1}\;=\;0\,, \\ iii)\;(\mathbb{I}-\mathbb{Q})\mathbf{v}_n \;=\; -\,\mathbb{M}^{-1}(M_1\mathbf{v}_{n-1}\,+\,\mathbf{q}_n\,+\,\mathbf{g}_{n-1})\,, \\ iv)\;\mathbb{Q}M_1\mathbf{v}_{n-1}\,+\,\mathbb{Q}\mathbf{q}_n\,+\,\mathbb{Q}\mathbf{g}_{n-1}\;=\;0\,. \end{cases}$$

When $n=0$, the first equation reduces to $\mathbf{u}_0=\mathbb{P}\mathbf{u}_0$, the second is trivial and the fourth reads $\mathbb{Q}q(\mathbf{u}_0,\mathbf{u}_0)=0$. Therefefore, when the transparency Assumption 2.1 is satisfied, as we now assume, (2.10) implies that the fourth equation is a consequence of the first one. Thus $(3.6)_0$ reduces to

$$(3.6)_0 \qquad \begin{cases} i) \quad \mathbf{u}_0=\mathbb{P}\mathbf{u}_0, \\ iii)\;(\mathbb{I}-\mathbb{Q})\mathbf{v}_0\;=\;-\,\mathbb{M}^{-1}\mathbf{q}_0\,. \end{cases}$$

There is a similar analysis for $n>0$. Introduce the notation

$$(3.7) \qquad \mathbf{h}_{n-1}\;:=\;\sum_{k=1}^{n-1} q(\mathbf{u}_k,\mathbf{u}_{n-k})\;=\;\mathbf{q}_n\;-\;2q(\mathbf{u}_0,\mathbf{u}_n)\,.$$

The transparency assumption and $(3.6)_0$ imply that $\mathbb{Q}q(\mathbf{u}_0,\mathbb{P}\mathbf{u}_n)=0$. Using the first equation in $(3.6)_n$ to compute $\mathbb{Q}q(\mathbf{u}_0,(\mathbb{I}-\mathbb{P})\mathbf{u}_n)$, we see that if $(3.6)_0$ is satisfied, one can replace the fourth equation in $(3.6)_n$ by

$$(3.8)_n \qquad \mathbb{Q}M_1\mathbf{v}_{n-1}\;-\;2\mathbb{Q}q\big(\mathbf{u}_0\,,\,\mathbb{L}^{-1}(L_1\mathbf{u}_{n-1}+\mathbf{f}_{n-1})\big)\;+\;\mathbb{Q}(\mathbf{h}_{n-1}\,+\,\mathbf{g}_{n-1})\;=\;0\,.$$

For $n\ge 0$ consider the equations

$$(3.9)_n \qquad \begin{cases} i) \quad (3.6)_n \;\; i)\,, \\ ii) \;\; (3.6)_{n+1} \;\; ii)\,, \\ iii)\; (3.6)_n \;\; iii)\,, \\ iv)\; (3.8)_{n+1}\;\; . \end{cases}$$

The first and third equations give $(\mathbb{I}-\mathbb{P})\mathbf{u}_n$ and $(\mathbb{I}-\mathbb{Q})\mathbf{v}_n$ respectively. If we substitute their value in the second and fourth equation, we see that $(3.9)_n$ is equivalent to

$$(3.10)_n \quad \begin{cases} (\mathbb{I}-\mathbb{P})\,\mathbf{u}_n=-\mathbb{L}^{-1}(L_1\mathbf{u}_{n-1}+\mathbf{f}_{n-1})\,, \\ \mathbb{P}L_1\mathbb{P}\mathbf{u}_n+\mathbb{P}\mathbf{f}_n=\mathbb{P}L_1\mathbb{L}^{-1}(L_1\mathbf{u}_{n-1}+\mathbf{f}_{n-1})\,, \\ (\mathbb{I}-\mathbb{Q})\mathbf{v}_n=-\mathbb{M}^{-1}(M_1\mathbf{v}_{n-1}+\mathbf{q}_n+\mathbf{g}_{n-1})\,, \\ \mathbb{Q}M_1\mathbb{Q}\mathbf{v}_n-2\mathbb{Q}q\big(\mathbf{u}_0,\mathbb{L}^{-1}(L_1\mathbb{P}\mathbf{u}_n+\mathbf{f}_n)\big)-\mathbb{Q}M_1\mathbb{M}^{-1}\mathbf{q}_n+\mathbb{Q}(\mathbf{h}_{n-1}+\mathbf{g}_{n-1})=\mathbf{r}_{n-1}\;\;. \end{cases}$$

with

$$\mathbf{r}_{n-1}:=\mathbb{Q}M_1\mathbb{M}^{-1}(M_1\mathbf{v}_{n-1}+\mathbf{g}_{n-1})-2\mathbb{Q}q\big(\mathbf{u}_0,\mathbb{L}^{-1}L_1\mathbb{L}^{-1}(L_1\mathbf{u}_{n-1}+\mathbf{f}_{n-1})\big)\,.$$

Substituting $(\mathbb{I}-\mathbb{P})\mathbf{u}_n$ and $(\mathbb{I}-\mathbb{Q})\mathbf{v}_n$ in the nonlinear terms yields further simplifications. Denoting by $\mathbf{r}_{n-1}$ various expressions which depend only on $(\mathbf{u}_0,\ldots,\mathbf{u}_{n-1})$ and $(\mathbf{v}_0,\ldots,\mathbf{v}_{n-1})$, we get

$$\mathbf{q}_n=\mathbf{q}_n^*\,+\,\mathbf{r}_{n-1}\,,\quad \mathbf{f}_n=\mathbf{f}_n^*\,+\,\mathbf{r}_{n-1}\,,\quad \mathbf{g}_n=\mathbf{g}_n^*\,+\,\mathbf{r}_{n-1}\,.$$

with

$$\mathbf{q}_0^* := q(\mathbb{P}\mathbf{u}_0, \mathbb{P}\mathbf{u}_0)\,, \qquad \mathbf{q}_n^* := 2\,q(\mathbb{P}\mathbf{u}_0, \mathbb{P}\mathbf{u}_n) \quad \text{for} \quad n > 0\,, \tag{3.11}$$

$$\mathbf{f}_0^* := f(\mathbb{P}\mathbf{u}_0, \mathbb{Q}\mathbf{v}_0 - \mathbb{M}^{-1}\mathbf{q}_0^*)\,, \quad \mathbf{f}_n^* := \nabla f(\mathbf{u}_0, \mathbf{v}_0)(\mathbb{P}\mathbf{u}_n, \mathbb{Q}\mathbf{v}_n - \mathbb{M}^{-1}\mathbf{q}_n^*) \quad \text{for } n > 0 \tag{3.12}$$

and similar formulas for $\mathbf{g}_n^*$.

Next, by (3.7), one has $\mathbf{h}_0 = 0$, $\mathbf{h}_1 = q(\mathbf{u}_1, \mathbf{u}_1)$ and $\mathbf{h}_n = 2q(\mathbf{u}_1, \mathbf{u}_n) + \mathbf{r}_{n-1}$ for $n \geq 2$. We set $\mathbf{h}_0^* = 0$ and for $n \geq 1$, we see that the transparency assumption implies that

$$\mathbb{Q}\mathbf{h}_n = \mathbb{Q}\mathbf{h}_n^* + \mathbf{r}_{n-1}$$

where

$$\mathbf{h}_n^* := 2\,q\big((\mathbb{I} - \mathbb{P})\mathbf{u}_1, \mathbb{P}\mathbf{u}_n\big) = -2\,q\big(\mathbb{L}^{-1}(L_1\mathbf{u}_0 + f(\mathbf{u}_0, \mathbf{v}_0)), \mathbb{P}\mathbf{u}_n\big)\,. \tag{3.13}$$

Therefore, using the notation $\mathbf{r}_{n-1}$ introduced above, $(3.10)_n$ is equivalent to

$$(3.14)_n \qquad \begin{cases} \mathbb{P}L_1\mathbb{P}\mathbf{u}_n + \mathbb{P}\mathbf{f}_n = \mathbb{P}\mathbf{r}_{n-1} \\ \mathbb{Q}M_1\mathbb{Q}\mathbf{v}_n - 2\mathbb{Q}q(\mathbb{P}\mathbf{u}_0\,, \mathbb{L}^{-1}L_1\mathbb{P}\mathbf{u}_n) - \mathbb{Q}M_1\mathbb{M}^{-1}\mathbf{q}_n^* + \mathbb{Q}\mathbf{k}_n = \mathbb{Q}\mathbf{r}_{n-1} \\ (\mathbb{I} - \mathbb{P})\,\mathbf{u}_n = (\mathbb{I} - \mathbb{P})\mathbf{r}_{n-1}\,, \\ (\mathbb{I} - \mathbb{Q})\mathbf{v}_n = -\mathbb{M}^{-1}\mathbf{q}_n^* + (\mathbb{I} - \mathbb{Q})\mathbf{r}_{n-1}\,. \end{cases}$$

where

$$\mathbf{k}_n := -2\mathbb{Q}q(\mathbb{P}\mathbf{u}_0\,, \mathbb{L}^{-1}\mathbf{f}_n^*) + \mathbb{Q}(\mathbf{h}_n^* + \mathbf{g}_n^*) \tag{3.15}$$

is a linear function of $(\mathbb{P}\mathbf{u}_n, \mathbb{Q}\mathbf{v}_n)$ when $n > 0$.

The discussion above is summarized in the following statement.

PROPOSITION 3.1. *When Assumption* 2.1 *is satisfied, the equations* $(3.3)_0 \dots (3.3)_{n+1}$ *imply* $(3.14)_0 \dots (3.14)_n$ *which imply* $(3.3)_0 \dots (3.3)_n$.

The first two equations in (3.14) form a system for $(\mathbb{P}\mathbf{u}_n, \mathbb{Q}\mathbf{v}_n)$. When $n = 0$, it is quasilinear in the two terms $\mathbb{Q}q(\mathbb{P}\mathbf{u}_0, \mathbb{L}^{-1}L_1\mathbb{P}\mathbf{u}_0)$ and $\mathbb{Q}M_1\mathbb{M}^{-1}\mathbf{q}_0^* = \mathbb{Q}M_1\mathbb{M}^{-1}q(\mathbb{P}\mathbf{u}_0, \mathbb{P}\mathbf{u}_0)$. When $n > 0$, the definitions (3.11-12-13) show that it is linear. For $n = 1$, one could have expected a semilinear term coming from $q(\mathbf{u}_1, \mathbf{u}_1)$. This term is not present, because of the transparency Assumption which implies that $\mathbb{Q}\mathbf{h}_1 = \mathbb{Q}\mathbf{h}_1^*$

4. Hyperbolicity of formal equations

Assumption 2.1 does not say anything about the solvability of the equations (3.14). In this section we discuss the hyperbolicity of the first two equations in (3.14). For $n = 0$, they are

$$\begin{cases} \mathbb{P}L_1(\partial_x)\mathbb{P}\mathbf{u}_0 + \mathbb{P}\mathbf{f}_0 = 0 \\ \mathbb{Q}M_1(\partial_x)\mathbb{Q}\mathbf{v}_0 - \mathbb{D}(\mathbb{P}\mathbf{u}_0\,, \partial_y)\mathbb{P}\mathbf{u}_0 + \mathbb{Q}\mathbf{k}_0 = 0\,, \end{cases} \tag{4.1}$$

where the quasilinear term is

$$\mathbb{D}(\mathbb{P}\mathbf{a},\partial_y)\mathbb{P}\mathbf{u} \;:=\; \sum_{j=1}^{d} 2\,\mathbb{Q}\,q(\mathbb{P}\mathbf{a},\mathbb{L}^{-1}A_j\partial_{y_j}\mathbb{P}\mathbf{u}) \;+\; 2\,\mathbb{Q}B_j\mathbb{M}^{-1}q(\mathbb{P}\mathbf{a},\partial_{y_j}\mathbb{P}\mathbf{u})\,. \tag{4.2}$$

Note that there is no ∂_t in $\mathbb{D}$, since (3.5) implies that $\mathbb{L}^{-1}\partial_t\mathbb{P}=0$ and $\mathbb{Q}\partial_t\mathbb{M}^{-1}=0$.

Introduce the spaces of trigonometric polynomials (with constant coefficients) $\mathcal{P}:=\ker(\mathbb{I}-\mathbb{P})$ and $\mathcal{Q}:=\ker(\mathbb{I}-\mathbb{Q})$. By (2.7), $\mathcal{P}\times\mathcal{Q}$ is finite dimensional and (4.1) is a quasilinear first order system for the function $\mathbf{U}_0(x)=(\mathbb{P}\mathbf{u}_0(x),\mathbb{P}\mathbf{v}_0(x))$ valued in $\mathcal{P}\times\mathcal{Q}$. It reads

$$(\partial_t \,+\, \mathcal{A}(\mathbb{P}\mathbf{u}_0,\partial_y))\mathbf{U}_0 \,+\, \mathcal{F}_0(\mathbf{U}_0) \;=\; 0\,. \tag{4.3}$$

Moreover, (3.12) and (3.15) show that $\mathcal{F}_0$ is a polynomial function on $\mathcal{P}\times\mathcal{Q}$.

For $n>0$, the analysis is similar. The first two equations in $(3.14)_n$ read

$$\begin{cases} \mathbb{P}L_1(\partial_x)\mathbb{P}\mathbf{u}_n \,+\, \mathbb{P}\mathbf{f}_n \;=\; \mathbb{P}\mathbf{r}_{n-1} \\ \mathbb{Q}M_1(\partial_x)\mathbb{Q}\mathbf{v}_n \,-\, \mathbb{D}(\mathbb{P}\mathbf{u}_0\,,\partial_y)\mathbb{P}\mathbf{u}_n \,-\, \mathbb{Q}\mathbf{k}'_n + \mathbb{Q}\mathbf{k}_n \;=\; \mathbb{Q}\mathbf{r}_{n-1} \end{cases} \tag{4.4}$$

where

$$\mathbf{k}'_n = 2\sum_j \mathbb{Q}B_j\mathbb{M}^{-1}q(\partial_j\mathbb{P}\mathbf{u}_0,\mathbb{P}\mathbf{u}_n)\,.$$

This is a linear first order system for the function $\mathbf{U}_n(x)=(\mathbb{P}\mathbf{u}_n(x),\mathbb{P}\mathbf{v}_n(x))$ valued in $\mathcal{P}\times\mathcal{Q}$. It reads

$$(\partial_t \,+\, \mathcal{A}(\mathbb{P}\mathbf{u}_0,\partial_y))\mathbf{U}_n \,+\, \mathcal{B}_0(\mathbf{u}_0,\partial_y\mathbf{u}_0)\,\mathbf{U}_n \;=\; \mathbf{R}_{n-1}\,. \tag{4.5}$$

$\mathbb{P}L_1(\partial_x)\mathbb{P}$ acts diagonaly on Fourier components. For $\mathbf{u}=\sum_\nu u_\nu e^{i\nu\theta}\in\mathcal{P}$,

$$\mathbb{P}L_1(\partial_x)\mathbb{P}\mathbf{u}=\sum_\nu L_{1,\nu}(\partial_x)u_\nu e^{i\nu\theta}\,,\qquad \text{with}\quad L_{1,\nu}(\partial_x):=P(\nu\beta)L_1(\partial_x)P(\nu\beta)$$

$L_{1,\nu}$ is symmetric hyperbolic on $\mathcal{P}_\nu:=\ker(Id-P(\nu\beta))$. Denote by $P_1(\nu,\xi)$ the orthogonal projector on $\ker L_{1,\nu}(\xi)\cap\mathcal{P}_\nu$. Similarly, introduce $M_{1,\nu}(\partial_x):=Q(\nu\beta)M_1(\partial_x)Q(\beta)$ and $Q_1(\nu,\xi)$ the orthogonal projector on $\ker M_{1,\nu}(\xi)\cap\ker(Id-Q(\nu\beta))$.

On the other hand, $\mathbb{D}(\mathbf{a},\eta)$ is not diagonal on Fourier series. For $\mathbf{a}=\sum a_\nu e^{i\nu\theta}\in\mathcal{P}$ and $\mathbf{u}=\sum u_\nu e^{i\nu\theta}\in\mathcal{P}$, one has

$$\mathbb{D}(\mathbf{a},\eta)\mathbf{u} \;=\; \sum_{\nu_1}\Big(\sum_\nu D_{\nu_1,\nu}(a_{\nu_1-\nu},\eta)u_\nu\Big)e^{i\nu_1\theta}\;\in\mathcal{Q} \tag{4.6}$$

where

$$\begin{aligned} D_{\nu_1,\nu}(a,\eta)u \;:=\; & 2\,Q(\nu_1\beta)\,q\Big(P((\nu_1-\nu)\beta)a\,,\,L^{(-1)}(\nu\beta)A_1(\eta)P(\nu\beta)u\Big) \\ & +\; 2\,Q(\nu_1\beta)B_1(\eta)M^{(-1)}(\nu_1\beta)\,q\Big(P((\nu_1-\nu)\beta)a\,,\,P(\nu\beta)u\Big)\,. \end{aligned} \tag{4.7}$$

and $A_1(\eta):=\sum\eta_jA_j$, $B_1(\eta):=\sum\eta_jB_j$.

Proposition 4.1. *The following properties are equivalent :*

i) The system (4.3) is strongly hyperbolic in the sense that for all $\mathbf{a} \in \mathcal{P}$ *the matrices* $e^{i\mathcal{A}(\mathbf{a},\eta)}$ *are uniformly bounded* $\eta \in \mathbb{R}^d$

ii) The system (4.3) is symmetrizable in the sense that for all $\mathbf{a} \in \mathcal{P}$ *there is a bounded family* $\{\mathcal{S}(\mathbf{a},\eta)\}_{\eta\in\mathbb{R}^d}$ *of uniformly symmetric definite positive matrices, such that* $\mathcal{S}(\mathbf{a},\eta)\,\mathcal{A}(\mathbf{a},\eta)$ *is symmetric.*

iii) The system (4.3) is conjugated to a symmetric system in the sense that for all $\mathbf{a} \in \mathcal{P}$ *there is a bounded family* $\{\mathcal{N}(\mathbf{a},\eta)\}_{\eta\in\mathbb{R}^d}$ *of invertible matrices, with uniformly bounded inverses, such that* $\mathcal{N}(\mathbf{a},\eta)^{-1}\,\mathcal{A}(\mathbf{a},\eta)\,\mathcal{N}(\mathbf{a},\eta)$ *is symmetric.*

iv) For all integers ν *and* ν_1*, there is a constant* C *such that for all* η*,* τ*,* τ' *and* $\mathbf{a} \in \mathcal{P}$

$$\Big|Q_1(\nu_1,\tau',\eta)\,D_{\nu_1,\nu}(a,\eta)\,P_1(\nu,\tau,\eta)\Big| \;\le\; C\,|\tau'-\tau|\,|a|\,. \tag{4.8}$$

v) There is a bounded family of bilinear mappings, $\{\mathbb{F}(\eta)\}_{\eta\in\mathbb{R}^d}$*, from* $\mathcal{P}\times\mathcal{P}$ *to* $\mathcal{Q}$*, such that for all* η*,* $\mathbf{a}\in\mathcal{P}$ *and* $\mathbf{u}\in\mathcal{P}$

$$\mathbb{D}(\mathbf{a},\eta)\,\mathbf{u} \;=\; \mathbb{F}(\eta)\big(\mathbf{a}\,,\,\mathbb{P}A_1(\eta)\mathbb{P}\mathbf{u}\big) \;-\; \mathbb{Q}B_1(\eta)\mathbb{Q}\,\mathbb{F}(\eta)\big(\mathbf{a}\,,\,\mathbf{u}\big)\,. \tag{4.9}$$

Proof. The implications $iii) \Rightarrow ii) \Rightarrow i)$ are always true.

a) $i) \Rightarrow iv)$. On Fourier components, $\mathcal{A}(\mathbf{a},\eta)$ has the following block structure

$$\mathcal{A}(\mathbf{a},\eta) \;=\; \begin{pmatrix} diag\{A_{1,\nu}(\eta)\}_\nu & 0 \\ \{D_{\nu_1,\nu}(a_{\nu_1-\nu},\eta)\}_{\nu_1,\nu} & diag\{B_{1,\nu_1}(\eta)\}_{\nu_1} \end{pmatrix}. \tag{4.10}$$

where $A_{1,\nu}(\eta) := P(\nu\beta)A_1(\eta)P(\nu\beta)$ and $B_{1,\nu_1}(\eta) := Q(\nu_1\beta)B_1(\eta)Q(\nu_1\beta)$. The exponential is explicitly computable. The diagonal terms $e^{itA_{1,\nu}(\eta)}$ and $e^{itB_{1,\nu_1}(\eta)}$ are unitary and the off-diagonal terms are

$$\int_0^t e^{i(t-s)B_{1,\nu_1}(\eta)}\,D_{\nu_1,\nu}(a_{\nu_1-\nu},\eta)\,e^{isA_{1,\nu}(\eta)}\,ds\,. \tag{4.11}$$

Introduce the eigenvalues $\lambda_k(\nu,\eta)$ [resp. $\mu_l(\nu_1,\eta)$] and the eigenprojectors $P_{1,k}(\nu,\eta)$ [resp. $Q_{1,l}(\nu_1,\eta)$] of $A_{1,\nu}(\eta)$ [resp. $B_{1,\nu_1}(\eta)$]. The integrals (4.11) are uniformly bounded if and only if the integrals

$$\int_0^t e^{is(\lambda_k(\nu,\eta)-\mu_l(\nu_1,\eta))}\,Q_{1,l}(\nu_1,\eta)D_{\nu_1,\nu}(a_{\nu_1-\nu},\eta)\,P_{1,k}(\nu,\eta)\,ds\,.$$

are uniformly bounded, hence if and only if there is $C(a_{\nu_1-\nu})$ such that

$$\forall\eta\,,\quad |Q_{1,l}(\nu_1,\eta)D_{\nu_1,\nu}(a_{\nu_1-\nu},\eta)\,P_{1,k}(\nu,\eta)| \;\le\; C(a_{\nu_1-\nu})\,|\lambda_k(\nu,\eta)-\mu_l(\nu_1,\eta)|\,. \tag{4.12}$$

The projectors $P_1(\nu,\tau,\eta)$ vanish except when $-\tau$ is equal to one of the eigenvalues of $A_{1,\nu}(\eta)$. Moreover, when $\tau = -\lambda_k(\nu,\eta)$, $P_1(\nu,\tau,\eta) = P_{1,k}(\nu,\eta)$. A similar argument holds for the Q's. Therefore, (4.12) is equivalent to

$$\forall\eta,\,\forall\tau\,,\forall\tau'\,,\quad \Big|Q_1(\nu_1,\tau',\eta)\,D_{\nu_1,\nu}(a_{\nu_1-\nu},\eta)\,P_1(\nu,\tau,\eta)\Big| \;\le\; C(a_{\nu_1-\nu})\,|\tau'-\tau|\,.$$

Because $D_{\nu_1,\nu}(a,\eta)$ is linear in a, this implies and thus is equivalent to the existence of a constant C such that (4.8) holds for all a, and all (η,τ,τ').

b) *iv)* $\Rightarrow$ *v)*. With notations as in a), introduce

$$F_{\nu_1,\nu}(a,\eta) := \sum_{k,l} \frac{1}{\lambda_k(\nu,\eta)-\mu_l(\nu_1,\eta)}\, Q_{1,l}(\nu_1,\eta) D_{\nu_1,\nu}(a,\eta)\, P_{1,k}(\nu,\eta)$$

with the convention that the summand vanishes when $\lambda_k(\nu,\eta)-\mu_l(\nu_1,\eta)=0$. If (4.8) is safisfied, one has

$$|F_{\nu_1,\nu}(a,\eta)| \leq C'\,|a| \tag{4.13}$$

and

$$D_{\nu_1,\nu}(a,\eta) = F_{\nu_1,\nu}(a,\eta)\, A_{1,\nu}(\eta) - B_{1,\nu_1}(\eta)\, F_{\nu_1,\nu}(a,\eta)\,. \tag{4.14}$$

Conversely, if (4.13) and (4.14) hold, multiplying (4.14) on the left by $Q_{1,l}(\nu_1,\eta)$ and on the right by $P_{1,k}(\nu,\eta)$, one obtains (4.12) and thus (4.8). Note that the $F_{\nu_1,\nu}$ depend linearly on a.

For $\mathbf{a}=\sum a_\nu e^{i\nu\theta}\in\mathcal{P}$ and $\mathbf{u}=\sum_\nu u_\nu e^{i\nu\theta}\in\mathcal{P}$, define

$$\mathbb{F}(\eta)(\mathbf{a},\mathbf{u}) = \sum_{\nu_1}\Big(\sum_{\nu} F_{\nu_1,\nu}(a_{\nu_1-\nu},\eta)u_\nu\Big)e^{i\nu_1\theta} \in \mathcal{Q} \tag{4.15}$$

This defines a bilinear mapping $\mathbb{F}(\eta):\mathcal{P}\times\mathcal{P}\mapsto\mathcal{Q}$. With this notation (4.14) is the componentwise expression of (4.9). Moreover, the bilinear mappings $\mathbb{F}(\eta)$ are uniformly bounded if and only if (4.13) holds.

c) *v)* $\Rightarrow$ *iii)*. Denote by $\mathbb{F}(\mathbf{a},\eta)$ the linear mapping $\mathbf{u}\mapsto\mathbb{F}(\eta)(\mathbf{a},\mathbf{u})$ and introduce

$$\mathcal{N}(\mathbf{a},\eta) := \begin{pmatrix} Id & 0 \\ \mathbb{F}(\mathbf{a},\eta) & Id\end{pmatrix} \tag{4.16}$$

where the blocks correspond to the components in $\mathcal{P}$ and $\mathcal{Q}$. The intertwinning relation (4.9) is equivalent to

$$\mathcal{N}(\mathbf{a},\eta)^{-1}\,\mathcal{A}(\mathbf{a},\eta)\,\mathcal{N}(\mathbf{a},\eta) = \begin{pmatrix}\mathbb{P}A_1(\eta)\mathbb{P} & 0 \\ 0 & \mathbb{Q}B_1(\eta)\mathbb{Q}\end{pmatrix}\,. \tag{4.17}$$

The proof of Proposition 4.1 is complete

Things are much simpler when $\nu\beta$ and $\nu_1\beta$ are regular points of the characteristic varieties.

Definition 4.2. *A point $\underline{\xi}$ in the characteristic variety of L [resp. M] is regular when, in a neighborhood of $\underline{\xi}$, the charactistic variety is a graph $\tau=-\lambda(\eta)$ and $i\lambda(\eta)$ is an eigenvalue of constant multiplicity of $A(i\eta)$, [resp. $B(i\eta)$].*

When $\nu\beta$ is a regular point of the characteristic variety of L corresponding to the eigenvalue $\lambda(\eta)$, $L_{1,\nu}$ is the vector field with symbol $\tau+\nabla_\eta\lambda(\nu\kappa)\cdot\eta$. Therefore, $P_1(\nu,\xi)=0$ when $\tau+\nabla_\eta\lambda(\nu\kappa)\cdot\eta\neq0$ and $P_1(\nu,\xi)=P(\nu\beta)$ when $\tau+\nabla_\eta\lambda(\nu\kappa)\cdot\eta=0$.

Proposition 4.3. *Suppose that $\nu\beta$ and $\nu_1\beta$ are regular points in the characteristic variety of L and M respectively with associated eigenvalues λ and μ respectively. Then the estimate* (4.8) *is satisfied if and only if for all η for all $\mathbf{a} \in \mathcal{P}$,*

$$D_{\nu_1,\nu}(a,\eta) = 0 \quad \text{when} \quad \nabla_\eta \lambda(\nu\kappa) \cdot \eta = \nabla_\eta \mu(\nu_1\kappa) \cdot \eta \,. \tag{4.18}$$

Moreover, if $\mathbb{Z}^m\beta$ intersects the characteristic varieties of L and M only at regular points, the symmetrizer $\mathcal{S}$ and the conjugation matrix $\mathcal{N}$ can be chosen independent of η.

Proof.

With the description of the projectors $P_1(\nu,\eta)$ and $Q_1(\nu_1,\eta)$ which follows Definition 4.2, it is clear that (4.18) follows from (4.8). Conversely, $D_{\nu_1,\nu}(a,\eta)$ is linear both in a and η. Thus,(4.18) means it vanishes on the hyperplane $\nabla_\eta \lambda(\nu\kappa) \cdot \eta = \nabla_\eta \mu(\nu_1\kappa) \cdot \eta$. This holds, if and only if there is a matrix $F_{\nu_1,\nu}(a)$, depending linearly on a, such that

$$\begin{aligned} D_{\nu_1,\nu}(a,\eta) = Q(\nu_1\beta)\, D_{\nu_1,\nu}(a,\eta)\, P(\nu\beta) &= \big(\nabla_\eta \mu(\nu_1\kappa) \cdot \eta \;-\; \nabla_\eta \lambda(\nu\kappa) \cdot \eta\big)\, F_{\nu_1,\nu}(a) \\ &= F_{\nu_1,\nu}(a)\, A_{1,\nu}(\eta) \;-\; B_{1,\nu_1}(\eta)\, F_{\nu_1,\nu}(a)\,. \end{aligned}$$

This is (4.14) with $F_{\nu_1,\nu}$ independent of η and the proposition follows.

5. Existence of formal solutions

In this section we prove Theorem 2.3. Suppose that the trigonometric polynomials $(\mathbf{U}_0, \dots \mathbf{U}_{n-1})$ and $(\mathbb{P}\mathbf{u}_n, \mathbb{Q}\mathbf{v}_n)$ are knowns. Then, since f and g are polynomials, the right hand sides in the third and fourth equation of (3.14) are trigonometric polynomials. Therefore, these two equations determine $(\mathbb{I}-\mathbb{P})\mathbf{u}_n$ and $(\mathbb{I}-\mathbb{Q})\mathbf{v}_n$ which are in their turn trigonometric polynomials. Therefore it is sufficient to solve the first two equations of (3.14), that is (4.3) when $n=0$ and (4.5) when $n>0$.

We suppose that Assumptions 2.1 and 2.2 are satisfied. The systems (4.3) (4.5) are hyperbolic and symmetrizable, thus the proof is quite standard, except for the fact that the symmetrizers have nonsmooth symbols since the bilinear $\mathbb{F}(\eta)$ are only L^∞. Thus we review the classical proof of existence and the only serious point to check is that the lack of smoothness of the symbols does not affect the a-priori estimates.

To solve (4.3) and (4.5), one uses Picard's iterations and thus one considers the Cauchy problem for

$$(\partial_t + \ \mathcal{A}(\mathbf{a}, \partial_y))\, \mathbf{U} \;=\; \mathbf{F}\,, \quad \mathbf{U}_{|t=0} = \mathbf{U}^0\,. \tag{5.1}$$

Proposition 5.1. *Suppose that*

$$\begin{aligned} &\mathbf{a} \in L^\infty([0,T]; H^\sigma(\mathbb{R}^d;\mathcal{P}))\,, \quad \partial_t\mathbf{a} \in L^\infty([0,T]; H^{\sigma-1}(\mathbb{R}^d;\mathcal{P}))\,, \\ &\mathbf{F} \in L^\infty([0,T]; H^\sigma(\mathbb{R}^d;\mathcal{P}\times\mathcal{Q}))\,, \quad \mathbf{U}^0 \in H^\sigma(\mathbb{R}^d;\mathcal{P}\times\mathcal{Q})\,, \end{aligned} \tag{5.2}$$

where $\sigma \in \mathbb{N}$ is strictly larger than $(d+1)/2$. Then, the Cauchy problem (5.3) *has a unique solution $\mathbf{U} \in C^0([0,T]; H^\sigma(\mathbb{R}^d;\mathcal{P}\times\mathcal{Q}))$ and it satifies the estimates* (5.11), (5.12) *and* (5.13) *below.*

Proof.

a) The specific definition of the spaces $\mathcal{P}$ and $\mathcal{Q}$ has no importance. The usefull properties are that $\mathcal{A}(\mathbf{a}, \partial_y)$ and the matrix $\mathcal{N}(\mathbf{a}, \eta)$ introduced at (4.16) have the following block structure

$$\mathcal{A}(\mathbf{a}, \partial_y) \;=\; \begin{pmatrix} \mathcal{A}_1(\partial_y) & 0 \\ \mathcal{D}(\mathbf{a}, \partial_y) & \mathcal{A}_2(\partial_y) \end{pmatrix}, \quad \mathcal{N}(\mathbf{a}, \eta) \;=\; \begin{pmatrix} Id & 0 \\ \mathcal{F}(\mathbf{a}, \eta) & Id \end{pmatrix}. \tag{5.3}$$

Moreover, $\mathcal{F}(\mathbf{a}, \eta)$ is linear in $\mathbf{a}$ and

$$\mathcal{F}(\mathbf{a}, \eta) \;=\; \sum_{l=1}^{\dim \mathcal{P}} a_l\, \mathcal{F}_l(\eta) \tag{5.4}$$

where the a_l denote the components of $\mathbf{a}$ in an arbitrary basis of $\mathcal{P}$ and the $\mathcal{F}_l(\eta)$ are uniformly bounded matrices for $\eta \in \mathbb{R}^d$. Finally, the intertwinnig relation (4.9) reads

$$\mathcal{D}(\mathbf{a}, \eta) \;=\; \mathcal{F}(\mathbf{a}, \eta)\, \mathcal{A}_1(\eta) \;-\; \mathcal{A}_2(\eta)\, \mathcal{F}(\mathbf{a}, \eta)\,. \tag{5.5}$$

For $\mathbf{a}$ valued in $\mathcal{P}$, introduce the operator

$$\big(\mathcal{F}(\mathbf{a}, D_y)\mathbf{u}\big)(y) \;:=\; (2\pi)^{-d} \int e^{iy\eta}\, \mathcal{F}(\mathbf{a}(y), \eta)\, \widehat{\mathbf{U}}(\eta)\,)\, d\eta \tag{5.6}$$

The operator $\mathcal{N}(\mathbf{a}, D_y)$ is defined similarly. The specific form (5.4) shows that

$$\big(\mathcal{F}(\mathbf{a}, D_y)\mathbf{u}\big)(y) = \sum_{l=1}^{\dim \mathcal{P}} a_l(y) \big(\mathcal{F}_l(D_y)\mathbf{u}\big)(y)\,.$$

Thus, when $\mathbf{a} \in L^\infty(\mathbb{R}^d; \mathcal{P})$, $\mathcal{F}(\mathbf{a}, D_y)$ is bounded from $L^2(\mathbb{R}^d; \mathcal{P})$ to $L^2(\mathbb{R}^d; \mathcal{Q})$. Moreover, the derivations ∂_t and ∂_y commute with Fourier multipliers $\mathcal{F}_l(D_y)$. Thus, because the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are differential, the identity (5.5) implies that for smooth functions

$$\mathcal{D}(\mathbf{a}, D_y) \;=\; \mathcal{F}(\mathbf{a}, D_y)\,(\partial_t + \mathcal{A}_1(\partial_y)) \;-\; (\partial_t + \mathcal{A}_2(\partial_y))\,\mathcal{F}(\mathbf{a}, D_y) \;+\; \mathcal{G}(\partial_x \mathbf{a}, D_y) \tag{5.7}$$

whith

$$\mathcal{G}(\partial_x \mathbf{a}, D_y) \;=\; \mathcal{F}(\partial_t \mathbf{a}, D_y) \;+ \sum_{j=1}^{d} \sum_{l=1}^{\dim \mathcal{P}} \mathcal{A}_{2,j}(\partial_j a_l)(y)\mathcal{F}_l(D_y)\,. \tag{5.8}$$

In this definition, $\mathcal{A}_{2,j}$ is the coefficient of ∂_j in $\mathcal{A}_2$. Note that $\mathcal{G}(\partial_x \mathbf{a}, D_y)$ is bounded in L^2 with norm dominated by $C\|\partial_x \mathbf{a}\|_{L^\infty}$.

Therefore, for smooth functions $\mathbf{U} = (\mathbf{u}, \mathbf{v})$ and $\mathbf{F} = (\mathbf{f}, \mathbf{g})$, (5.1) is equivalent to

$$\begin{cases} (\partial_t \,+ \mathcal{A}_1(\partial_y)\mathbf{u} \;=\; \mathbf{f}\,, \\ (\partial_t + \mathcal{A}_2(\partial_y))\big(\mathbf{v} \,+\, \mathcal{F}(\mathbf{a}, D_y)\mathbf{u}\big) \;=\; \mathbf{g} \,+\, \mathcal{F}(\mathbf{a}, D_y)\mathbf{f} \,+\, \mathcal{G}(\partial_x \mathbf{a}, D_y)\mathbf{u}\,. \end{cases} \tag{5.9}$$

b) The first equation in (5.9) is linear, has constant coefficient and is symmetric hyperbolic. Thus for all initial data $\mathbf{u}^0 \in H^\sigma(\mathbb{R}^d; \mathcal{P})$ and all $\mathbf{f} \in L^1([0, T]; H^\sigma(\mathbb{R}^d; \mathcal{P}))$, the Cauchy problem has a unique solution $\mathbf{u} \in C^0([0, T]; H^\sigma(\mathbb{R}^d; \mathcal{P}))$ and

$$\|\mathbf{u}(t)\|_{H^\sigma} \;\leq\; \|\mathbf{u}(0)\|_{H^\sigma} \;+\; \int_0^t \|\mathbf{f}(s)\|_{H^\sigma} ds\,. \tag{5.10}$$

Moreover, the equation implies that

$$\|\partial_t \mathbf{u}(t)\|_{H^{\sigma-1}} \;\le\; C\,\|\mathbf{u}(t)\|_{H^\sigma} \;+\|\mathbf{f}(t)\|_{H^{\sigma-1}}\,. \tag{5.11}$$

Knowing $\mathbf{u}$, the second equation in (5.1) determines $\mathbf{v}$. It reads

$$(\partial_t + \mathcal{A}_2(\partial_y))\mathbf{v} = \mathbf{g} - \mathcal{D}(a, \partial_y)\mathbf{u}\,.$$

Therefore, if the data satisfy (5.2), (5.1) has a unique solution in $C^0([0,T]; H^{\sigma-1}(\mathbb{R}^d))$.

c) We now prove the optimal a-priori estimate for $\mathbf{v}$. Assume first that the data satisfy (5.2) with $\sigma = \infty$. Thus the solution is valued in H^∞. Because $\mathcal{F}(\mathbf{a}, D_y)$ and $\mathcal{G}(\partial_x, D_y)$ are bounded in L^2, the usual L^2 energy estimates for (5.9) implies that the solution of (5.1) satisfies

$$\begin{aligned} \|\mathbf{v}(t)\|_{L^2} \;\le\; & \|\mathbf{v}(0)\|_{L^2} + C\|\mathbf{a}(0)\|_{L^\infty}\|\mathbf{u}(0)\|_{L^2} + C\|\mathbf{a}(t)\|_{L^\infty}\|\mathbf{u}(t)\|_{L^2} \\ & \int_0^t \Big(\|\mathbf{g}(s)\|_{L^2} + C\|\mathbf{a}(s)\|_{L^\infty}\|\mathbf{f}(s)\|_{L^2} + C\|\partial_x \mathbf{a}(s)\|_{L^\infty}\|\mathbf{u}(s)\|_{L^2}\Big)\, ds\,. \end{aligned} \tag{5.12}$$

To get the higher order estimates, the idea is to differentiate (5.1), and apply the L^2 estimate (5.12). For $|\alpha| \le \sigma$, the commutators

$$\gamma := \partial_y^\alpha \mathcal{D}(\mathbf{a}, \partial_y)\,\mathbf{u} \;-\; \mathcal{D}(\mathbf{a}, \partial_y)\partial_y^\alpha \mathbf{u}$$

are estimated using Gagliardo-Nirenberg's inequalities :

$$\|\,\gamma(t)\,\|_{L^2} \;\le\; C\,\big(\|\partial_y \mathbf{a}(t)\|_{L^\infty}\|\mathbf{u}(t)\|_{H^\sigma} \;+\; \|\mathbf{a}(t)\|_{H^\sigma}\|\partial_y \mathbf{u}(t)\|_{L^\infty}\big)$$

This yields

$$\begin{aligned} \|\mathbf{v}(t)\|_{H^\sigma} \;\le & \|\mathbf{v}(0)\|_{H^\sigma} \;+\; C\big(\|\mathbf{a}(0)\|_{L^\infty}\|\mathbf{u}(0)\|_{H^\sigma} + \|\mathbf{a}(t)\|_{L^\infty}\|\mathbf{u}(t)\|_{H^\sigma}\big) + \\ & \int_0^t \|\mathbf{g}(s)\|_{H^\sigma} ds \;+\; C\int_0^t \|\mathbf{a}(s)\|_{L^\infty}\|\mathbf{f}(s)\|_{H^\sigma}\, ds + \\ & C\int_0^t \big(\|\partial_x \mathbf{a}(s)\|_{L^\infty}\|\mathbf{u}(s)\|_{H^\sigma} \;+\; \|\mathbf{a}(s)\|_{H^\sigma}\|\partial_y \mathbf{u}(s)\|_{L^\infty}\big) ds\,. \end{aligned} \tag{5.13}$$

Mollifying the data and passing to the limit, one shows that for data satisfying (5.2) the unique solution $\mathbf{v}$ given by part b) belongs to $C^0([0,T]; H^\sigma)$ and satisfies (5.13). This finishes the proof of the proposition.

The estimates (5.10) (5.11) and (5.13) for the solutions of (5.1) are quite similar to the estimates available for quasilinear hyperbolic first order systems. Therefore, the standard Picard's iterations give the solutions of (4.3) and (4.5). We omit the details and give the results. Fix $\sigma_0 > (d+2) > 2$. For the quasilinear equation (4.3) with $n = 0$, there is $T > 0$ such that the Picard's iterates are bounded in $C^0([0,T]; H^{\sigma_0}(\mathbb{R}^d; \mathcal{P} \times \mathcal{Q}))$; they converge in $C^0([0,T]; H^{\sigma_0 - 1})$, and using the equation once more, the limit is shown to belong to $C^0([0,T]; H^{\sigma_0})$; the a-priori estimates also imply that the solution remains in H^σ, up to time T, if the Cauchy data belong to H^σ and $\sigma \ge \sigma_0$. The equation provides smoothness in t.

When $n \ge 1$, the equation (4.5) is linear, and knowing that the coefficient and the right hand side are C^∞ in $t \in [0,T]$ and H^∞ in y, the Picards iterates converge in $C^0([0,T]; H^\sigma)$ for all σ, and therefore the limit is a smooth solution.

6. Necesary conditions for linear stability.

In this section we prove Proposition 2.4 which gives necessary conditions for the stability of the linearized equations (2.15).

Consider $\mathbf{U}_0 = (\mathbf{u}_0, \mathbf{v}_0) \in H^\infty([0,T] \times \mathbb{R}^d \times \mathbb{T}^m)$ and $U^\varepsilon_{app} = (u^\varepsilon_{app}, v^\varepsilon_{app})$ such that

$$\sup_{x \in [0,T]\times\mathbb{R}^d} |\, U^\varepsilon_{app}(x) \;-\; \mathbf{U}_0(x, \beta\cdot x/\varepsilon)\,| \;=\; O(\varepsilon) \tag{6.1}$$

Consider the linearized equation (2.15). The estimate (6.1) and Gronwall's Lemma imply that the stability estimate (2.16) holds if and only if a similar estimate is satisfied by solutions of

$$\begin{cases} L(\varepsilon\partial_x)u \;=\; \varepsilon f\,, \\ M(\varepsilon\partial_x)v \;+\; 2q(u_0^\varepsilon, u) \;=\; \varepsilon g\,, \end{cases} \tag{6.2}$$

where $u_0^\varepsilon(x) \;=\; \mathbf{u}_0(x, \beta\cdot x/\varepsilon) \;=\; \sum \mathbf{u}_{0,\nu}(x)\, e^{i\nu\beta\cdot x/\varepsilon}$. Moreover, the estimate (2.16) for the Cauchy problem with initial data at time $t = 0$ implies similar estimates for data at all times. Therefore, we assume that there is a constant C such that for $\varepsilon \in]0,1]$, $t_0 \in [0,T[$, $F = (f,g)$ in $H^\infty([t_0,T]\times\mathbb{R}^d)$, the solution of (6.2) with initial data $U(t_0) \in H^\infty(\mathbb{R}^d)$ satisfies for $t \in [t_0, T]$

$$\|U(t)\|_{L^2} \;\le\; C\,\|U(t_0)\|_{L^2} \;+\; C\int_{t_0}^t \|F(s)\|_{L^2}\, ds\,. \tag{6.3}$$

We further assume that only a finite number of $\mathbf{u}_{0,\nu}$ do not vanish. Note that this condition is satisfied by approximate solutions. Therefore Proposition 2.4 is a corollary of the following result.

PROPOSITION 6.1. *If the estimate* (6.3) *is satisfied, there is a constant C such that for all $\nu \in \mathbb{Z}^m$, all $\xi = (\tau,\eta)$ in the characteristic variety of L, all $\xi' = (\tau', \nu\kappa + \eta)$ in the characteristic variety of M, all $x \in [0,T]\times\mathbb{R}^d$ and all vectors u*

$$\Big|\, Q(\xi')q\big(\mathbf{u}_{0,\nu}(x)\,,\, P(\xi)u\big)\Big| \;\le\; C\,|\,\tau' - \tau - \nu\omega\,|\,|\,u\,|\,. \tag{6.4}$$

Proof. **a)** Consider a characteristic covector ξ for L. Following [Lax], we construct oscillatory approximate solutions of (6.2). For $t_0 \in [0,T]$ and $\rho \in H^\infty(\mathbb{R}^d)$, independent of time, consider the solution σ of the symmetric hyperbolic Cauchy problem on $[t_0, T]$

$$P(\xi)\sigma(x) = \sigma(x)\,, \quad P(\xi)L_1(\partial_x)P(\xi)\,\sigma = P(\xi)\rho\,, \quad \sigma_{|t=t_0} = 0. \tag{6.5}$$

Introduce

$$\begin{cases} u^\varepsilon(x) \;:=\; \sigma^\varepsilon(x)\, e^{i\xi\cdot x/\varepsilon}\,, \quad \sigma^\varepsilon(x) \;=\sigma(x) \;-\; \varepsilon\, L^{(-1)}(i\xi)L_1(\partial_x)\sigma(x)\,, \\ f^\varepsilon(x) \;:=\; P(\xi)\rho(x)\, e^{i\xi\cdot x/\varepsilon}\,. \end{cases} \tag{6.6}$$

Then,

$$\sup_{t\in[t_0,T]} \|L(\varepsilon\partial_x)\, u^\varepsilon(t) - \varepsilon f^\varepsilon(t)\|_{L^2} \;=\; O(\varepsilon^2)\,, \quad u^\varepsilon_{|t=t_0} = 0\,. \tag{6.7}$$

The interaction term is

$$(6.8)\qquad 2q(u_0^\varepsilon, u^\varepsilon) \;=\; \sum_\nu b_\nu^\varepsilon(x)\, e^{i(\xi+\nu\beta)\cdot x/\varepsilon} \quad , \quad b_\nu^\varepsilon(x) := 2q\big(a_\nu(x)\,,\,\sigma^\varepsilon(x)\big)\,.$$

Next we compute an approximate solution of the second equation in (6.2) with source term $g = 0$, using standard linear geometric optics calculations (see [Lax]). We look for

$$(6.9)\qquad v^\varepsilon = \sum v_\nu^\varepsilon(x) e^{i(\xi+\nu\beta)x/\varepsilon}\,, \quad \text{with} \quad v_\nu^\varepsilon = \varepsilon^{-1} v_{-1,\nu} + v_{0,\nu} + \varepsilon\, v_{1,\nu}\,.$$

We determine the coefficients so that

$$(6.10)\qquad \begin{cases} \|v^\varepsilon(t_0)\|_{L^2} = O(\varepsilon)\,, \\ \sup\limits_{t\in[t_0,T]} \;\|M(\varepsilon\partial_x)\, v^\varepsilon(t) \,+\, 2\,q(u_0^\varepsilon(t), u^\varepsilon(t))\,\|_{L^2} \,=\, O(\varepsilon^2)\,. \end{cases}$$

Let Z denote the set of indices ν such that $\xi+\nu\beta$ is characteristic for M, and let Z' denote the complementary set. For $\nu \in Z$, we choose $v_{-1,\nu}$ satisfying $Q(\xi+\nu\beta)v_{-1,\nu} = v_{-1,\nu}$ and the symmetric hyperbolic equation

$$(6.11)\qquad Q(\xi+\nu\beta)\, M_1(\partial_x)\, Q(\xi+\nu\beta) v_{-1,\nu} \;=\; -2\, Q(\xi+\nu\beta)\, q(\mathbf{u}_{0,\nu}, \sigma)\,, \quad v_{-1,\nu\,|t=t_0} = 0\,.$$

Moreover,

$$(6.12)\qquad v_{0,\nu} \;=\; 2\, M^{(-1)}(i(\xi+\nu\beta))\, q(\mathbf{u}_{0,\nu}, \sigma) \;+\; w_\nu$$

where $w_\nu = Q(\xi+\nu\beta) w_\nu$ satisfies an equation similar to (6.11) and we choose the initial conditions for w_u to be equal to zero. In particular, we note that the initial condition in (6.4) implies that $v_{0,\nu}(t_0) = 0$. Recall that $M^{(-1)}(i(\xi+\nu\beta))$ is the partial inverse of $M(i(\xi+\nu\beta))$.

When $\nu \in Z'$, we choose $v_{-1,\nu} = 0$ and

$$(6.13)\qquad v_{0,\nu}(x) \;=\; -2\, M^{(-1)}(i(\xi+\nu\beta))\, q(\mathbf{u}_{0,\nu}, \sigma)\,.$$

and in particular $v_{0,\nu}(t_0) = 0$.

The explicit form of the second corrector $v_{1,\nu}$ has no importance.

With (6.7) and (6.10), the uniform estimate (6.3) implies that

$$(6.14)\qquad \forall t \in [t_0, T]\,, \quad \limsup_\varepsilon \;\|v^\varepsilon(t)\,\|_{L^2} \;\le\; C \limsup_\varepsilon \int_{t_0}^t \|f^\varepsilon(s)\,\|_{L^2}\, ds \;=\; C\,(t-t_0)\,\|\rho\|_{L^2}\,,$$

with the same constant C.

b) With notations similar to (6.9), we write $v^\varepsilon := \varepsilon^{-1} v_{-1}^\varepsilon + v_0^\varepsilon + \varepsilon\, v_1^\varepsilon$. The first consequence of (6.14) is that

$$\forall t \in [t_0, T]\,, \quad \limsup_\varepsilon \;\|v_{-1}^\varepsilon(t)\,\|_{L^2} \;=\; 0\,.$$

Because the $\xi+\nu\beta$ are pairwise distinct, this implies that

$$\lim_{\varepsilon\to 0} \|\, v_{-1}^\varepsilon\,\|^2_{L^2([t_0,t_1]\times\mathbb{R}^d)} \;=\; \sum_{\nu\in Z} \|v_{-1,\nu}\|^2_{L^2([t_0,t_1]\times\mathbb{R}^d)} \;=\; 0\,.$$

Thus all the $v_{-1,\nu}$ vanish. According to (6.11), this requires that for all ν, $Q(\xi+\nu\beta)q(\mathbf{u}_{0,\nu},\sigma)=0$. From (6.5), it follows that

$$\sigma(t)=(t-t_0)P(\xi)\rho+\ O\big((t-t_0)^2\big)\,. \tag{6.15}$$

Thus, we conclude that $Q(\xi+\nu\beta)q(\mathbf{u}_{0,\nu}(t_0),P(\xi)\rho)=0$. Since ρ and t_0 are arbitrary we have proved that a first necessary condition for the validity of (6.3) is that

$$Q(\xi+\nu\beta)\,q(\mathbf{u}_{0,\nu}(x),P(\xi)\rho)\ =\ 0 \tag{6.16}$$

for all x and all vectors ρ. In particular, this implies (6.4) when $\xi'=\xi+\nu\beta$.

c) Conversely, when (6.16) is satisfied, the right hand side of (6.11) vanishes, and the approximate solution (6.9) reducues to $v^\varepsilon=v_0^\varepsilon+\varepsilon\, v_1^\varepsilon$. Therefore, (6.14) implies that

$$\sum_\nu\ \|v_{0,\nu}\|^2_{L^2([t_0,t]\times\mathbb{R}^d)}\ =\ \lim_{\varepsilon\to0}\ \|\,v^\varepsilon\,\|^2_{L^2([t_0,t]\times\mathbb{R}^d)}\ \le\ C^2\,(t-t_0)^2\,\|\rho\|^2\,.$$

Thus each term in the sum is smaller than the right hand side and the definitions (6.12) and (6.13) show that for all ν,

$$\|M^{(-1)}(i(\xi+\nu\beta))\,q(\mathbf{u}_{0,\nu},\sigma)\|_{L^2([t_0,t]\times\mathbb{R}^d}\ \le\ C\,(t-t_0)\,\|\rho\|_{L^2} \tag{6.17}$$

Consider $\xi'=(\tau',\eta+\nu\kappa)$ in the characteristic variety of M. Remark that

$$M(i(\xi+\nu\beta))=i(\tau+\nu\omega)Id+B_1(\eta+\nu\,\kappa)=i(\tau+\nu\omega-\tau')Id+M(i\xi')\,.$$

Therefore

$$Q(\xi')\,M(i(\xi+\nu\beta))=i(\tau+\nu\omega-\tau')\,Q(\xi')$$

When $\xi+\nu\,\beta$ is not characteristic for M, this immediately implies that

$$Q(\xi')\,M^{(-1)}(i(\xi+\nu\beta))\ =\ \frac{1}{i(\tau+\nu\omega-\tau')}\,Q(\xi')\,. \tag{6.18}$$

When $\xi+\nu\,\beta$ is characteristic but $\tau'\neq\tau+\nu\omega$, τ' and $\tau+\nu\omega$ are distinct eigenvalues of $B_1(\eta+\nu\,\kappa)$ and therefore $Q(\xi')$ and $Q(\xi+\nu\,\beta)$ are projectors on two orthogonal eigenspaces and (6.18) satisfied in this case too. Hence, (6.18) is satsified for all $\xi'=(\tau',\eta+\nu\kappa)$ such that $\tau'\neq\tau+\nu\omega$.

Because $Q(\xi')$ is an orthogonal projector, (6.17) implies

$$\|Q(\xi')\,q(\mathbf{u}_{0,\nu},\sigma)\|_{L^2([t_0,t]\times\mathbb{R}^d)}\ \le\ C\,(t-t_0)\,|\tau+\nu\omega-\tau'|\,\|\rho\|_{L^2} \tag{6.19}$$

Using again (6.15), we conclude that

$$\|Q(\xi')\,q(\mathbf{u}_{0,\nu}(t_0),P(\xi)\rho)\|_{L^2(\mathbb{R}^d)}\ \le\ C\,|\tau+\nu\omega-\tau'|\,\|\rho\|_{L^2}$$

Because t_0 and ρ are arbitrary, this implies that

$$|Q(\xi')\,q(\mathbf{u}_{0,\nu}(x),P(\xi)\rho)\,|\ \le\ |\tau+\nu\omega-\tau'|\,|\rho|\,. \tag{6.20}$$

for all x and all vector ρ. In particular, this implies (6.4) when $\xi'\neq\xi+\nu\beta$ and the proof of Proposition 6.1 is complete.

7. The stability condition

In this section we discuss the link between Assumption 2.5 and Assumptions 2.1 and 2.2 and give several examples.

PROPOSITION 7.1. *Assumption 2.5 is satisfied if and only if for all ν in $\mathbb{Z}^m$ there is a bounded family of bilinear mappings $\{S_\nu(\eta)\}_{\eta\in\mathbb{R}^d}$ such that for all $\eta \in \mathbb{R}^d$, and all vectors a and u, one has*

$$(7.2)\quad q\Big(P(\nu\beta)\, a\ , u\Big) \;=\; S_\nu(\eta)\Big(P(\nu\beta)\, a\ ,\ A(i\eta)u\Big) \;-\; (i\nu\omega +\ B(i(\nu\kappa+\eta))\, S_\nu(\eta)\Big(P(\nu\beta)\, a\ ,\ u\Big)\,.$$

Proof.

For $\eta \in \mathbb{R}^d$, introduce the spectral decomposition $A(i\eta) = \sum i\lambda_k(\eta)P_k(\eta)$. The number of terms may depend on η and the P_k and λ_k may be nonsmooth. The important point is that the P_k are orthogonal projectors and therefore uniformly bounded. Then, $\xi = (\tau, \eta)$ is L-characteristic if and only if there is k such that $\tau = -\lambda_k(\eta)$. In this case, $P(\xi) = P_k(\eta)$. Introduce the similar decomposition $B(i\eta) = \sum \mu_l(\eta)Q_l(\eta)$. For fixed a and ν, introduce next the operator $G_\nu(a) : u \mapsto q(P(\nu\beta)a, u)$. Then Assumption 2.5 holds if and only if there is C such that for all η, k, l and a

$$(7.3)\qquad \big|\,Q_l(\eta + \nu\,\kappa)\,G_\nu(a)\,P_k(\eta)u\big| \;\le\; C\,|\mu_l(\eta) \;-\; \lambda_k(\eta) \;+\; \nu\omega|\,|a|\ .$$

Suppose that it is satisfied. Then define $S_\nu(\eta, a) = \sum S_{\nu,l,k}(\eta, a)$ where $S_{\nu,l,k}(\eta, a) = 0$ when $\mu_l(\eta) \;-\; \lambda_k(\eta) \;+\; \nu\omega = 0$ and

$$S_{\nu,l,k}(\eta, a) \;= i\big(\mu_l(\eta) \;-\; \lambda_k(\eta) \;+\; \nu\omega\big)^{-1}\; Q_k(\eta + \nu\,\kappa)\,G_\nu(a)\,P_k(\eta)u\big)$$

otherwise. Then (7.3) implies that

$$(7.4)\qquad |\,S_\nu(\eta, a)\,| \;\le\; C'\,|a|$$

and

$$(7.5)\qquad G_\nu(a) = -S_\nu(\eta, a)\,A(i\eta) + (i\nu\omega + B(\nu\kappa + \eta))S_\nu(\eta, a)\,.$$

Conversely, if (7.4) (7.5) hold, multiplying (7.5) on the left by $Q_l(\nu\kappa+\eta)$ and on the right by $P_k(\eta)$ implies (7.3). The S_ν are linear in a, thus setting $S_\nu(\eta)(a, u) = S_\nu(\eta, a)u$, the proposition follows.

PROPOSITION 7.2. *Assumption 2.5 implies both Assumptions 2.1 and 2.2. More precisely, for $\nu_1 \in \mathbb{Z}^m$ and $\nu \in \mathbb{Z}^m$, suppose that (2.18) or equivalently (7.2) is satisfied for $\nu_1 - \nu$ and η in a neighborhood of $\nu\kappa$. Then (2.9) is satisfied and (4.8) holds in a neighborhood of $\nu\kappa$.*

Proof. Fix, a, ν_1 and ν and assume that (2.18) holds with $\nu_1 - \nu$ in place of ν and for η in a neighborhood of $\nu\kappa$. Evaluating (2.18) at $\xi = \nu\beta$ and $\xi' = \nu_1\beta$ immediately yields (2.9). This proves the first part of the proposition.

To prove the other implication, introduce the notation $Gu := q(P((\nu_1 - \nu)\beta)a, u)$. Then (2.9) reads

$$(7.6)\qquad Q(\nu_1\beta)\,G\,P(\nu\beta) \;= 0\,.$$

Moreover, our assumption and Proposition 7.1 imply that there is a bounded family of matrices $S(\eta)$ defined for η close to $\nu\kappa$ and such that

$$G \;=\; S(\eta)\,(A(i\eta)+i\nu\omega) \;-\; (B(i\eta)+i\nu_1\omega)S(\eta)\,. \tag{7.7}$$

To prove (4.8) , the idea is to differentiate (7.7) at $\eta = \nu\kappa$. However, because we have made no assumption of smoothness of S we cannot take derivatives.

With notations as in (7.2), $S(\eta)u := S_{\nu_1-\nu}(P((\nu_1-\nu)\beta)a, u)$ and (7.7) is (7.5) applied to the the present situation. Thus, For, $\tilde\eta \in \mathbb{R}^d$, the matrix in (4.7) is

$$D(\tilde\eta) = 2\,Q(\nu_1\beta)\,G\,P'(\tilde\eta) + \;2\,Q'(\tilde\eta)\,G\,P(\nu\beta)\,. \tag{7.8}$$

with

$$P'(\tilde\eta) \;:=\; L^{(-1)}(i\nu\beta)\,A_1(\tilde\eta)\,P(\nu\beta)\,, \quad Q'(\tilde\eta) \;:=\; Q(\nu_1\beta)\,B_1(\tilde\eta)\,M^{(-1)}(i\nu_1\beta)\ . \tag{7.9}$$

For $s\in[0,1]$, we compute

$$\big(Q(\nu_1\beta) - isQ'(\tilde\eta)\big)\,G\,\big(P(\nu\beta) - isP'(\tilde\eta)\big)\,. \tag{7.10}$$

On one hand, by (7.6) it is equal to

$$\frac{1}{2i}\,s\,D(\tilde\eta) \;+\; O(s^2) \tag{7.11}$$

On the other hand, we use (7.7) with $\eta = \nu\kappa + s\tilde\eta$. Note that $A(i\eta)+i\nu\omega = L(i\nu\beta) + isA_1(\tilde\eta)$. Because $P(\nu\beta)L(i\nu\beta) = 0$ and $L(i\nu\beta)L^{(-1)}(i\nu\beta) = Id - P(\nu\beta)$, one has

$$(A(i\eta)+i\nu\omega)\,(P(\nu\beta) - sP'(\tilde\eta)) \;=\; i\,s\,A_{1,\nu}(\tilde\eta) \;+\; O(s^2)$$

where $A_{1,\nu}(\tilde\eta) := P(\nu\beta)\,A_1(\tilde\eta)\,P(\nu\beta)$. Similarly

$$(Q(\nu_1\beta) - isQ'(\tilde\eta))\,(B(i\eta)+i\nu_1\omega) \;=\; i\,sB_{1,\nu_1}(\tilde\eta) \;+\; O(s^2)$$

with $B_{1,\nu_1}(\tilde\eta) := Q\,(\nu_1\beta)\,B_1(\tilde\eta)\,Q(\nu_1\beta)$. Therefore, the term in (7.10) is equal to

$$i\,sQ(\nu_1\beta)\,S(\eta)\,A_{1,\nu}(\tilde\eta) \;-\; i\,s\,B_{1,\nu_1}(\tilde\eta)\,S(\eta)\,P(\nu\beta) \;+\; O(s^2)\,.$$

Comparing with (7.11) this shows that

$$\frac{1}{2}\,D(\tilde\eta) \;=\; -Q(\nu_1\beta)\,S(\eta)\,A_{1,\nu}(\tilde\eta) \;+\; B_{1,\nu_1}(\tilde\eta)\,S(\eta)\,P(\nu\beta) \;+\; O(s)\,. \tag{7.12}$$

When $S(\eta)$ is continuous at $\nu\beta$, the limit of (7.12) as s tends to 0 gives directly (4.14). In the general case, introduce the orthogonal projectors $P_1(\nu,\xi)$ and $Q_1(\nu,\xi)$ as in § 4. Then

$$A_{1,\nu}(\tilde\eta)P_1(\nu,\tau,\tilde\eta) \;=\; -\tau P_1(\nu,\tau,\tilde\eta) \qquad \text{and} \qquad B_{1,\nu_1}(\tilde\eta)Q_1(\nu_1,\tau',\tilde\eta) \;=\; -\tau' Q_1(\nu_1,\tau,\tilde\eta)\,,$$

Thus, (7.12) and the uniform estimate (7.4) imply that

$$|\,Q_1(\nu_1,\tau',\tilde\eta)\,D(\tilde\eta)\,P_1(\nu,\tau,\tilde\eta)\,| \;\le C\,|a|\,|\tau'-\tau| + O(s)\,. \tag{7.12}$$

Letting s tend to zero, this implies (4.8) and the proposition is proved.

Remark 7.3. Near regular points, the analysis can be pushed a little further. Suppose that $\underline{\xi} = (\underline{\tau}, \underline{\eta})$ and $\underline{\xi}' = \underline{\xi} + \nu\beta = (\underline{\tau}', \underline{\eta} + \nu\kappa)$ are regular points in the characteristic varieties of L and M respectively. Let λ and μ denote the smooth eigenvalues of L and M respectively, such that $\underline{\tau} = -\lambda(\underline{\eta})$ and $\underline{\tau}' = -\mu(\underline{\eta})$. We say that the *resonance is regular* when the group velocities $\nabla_\eta \mu(\underline{\eta} + \nu\kappa)$ and $\nabla_\eta \lambda(\underline{\eta})$ are diferent. In this case, the equation $\mu(\eta + \nu\kappa) = \lambda(\eta) - \nu\omega$, determines a smooth manifold $\mathcal{R}$ of codimension 1 in $\mathbb{R}^d$ near $\underline{\eta}$.

In this case, the estimate (2.18) in Assumption 2.5 is satisfied for ξ and ξ' in neighborhoods of $\underline{\xi}$ and $\underline{\xi}'$, if and only if for all $\xi = (\tau, \eta)$ near $\underline{\xi}$ in the characteristic variety of L, with $\eta \in \mathcal{R}$, one has

$$Q(\xi + \nu\beta)\, q(P(\nu\beta)u\,,\, P(\xi)v) \;=\; 0\,. \tag{7.13}$$

This is clearly necessary. Conversely, suppose that for $\xi = (\tau, \eta)$ and $\xi' = (\tau', \eta + \nu\kappa)$ belong to small neighborhoods of $\underline{\xi}$ and $\underline{\xi}'$ in the characteristic varieties of L and M respectively. This means that $\tau = -\lambda(\eta)$ and $\tau' = -\mu(\eta + \nu\kappa)$. The projectors $P(\xi)$ and $Q(\xi')$ are smooth functions on the characteristic varieties near regular points. Thus the function

$$Q(\xi')\, q(P(\nu\beta)u\,,\, P(\xi)v)$$

is a smooth function of η and (7.13) means that it vanishes on the resonant manifold $\mathcal{R}$. Hence, $\mu(\eta + \nu\kappa) - \lambda(\eta) + \nu\omega$ can be factored out in the left hand side of (7.13), implying (2.18).

In space dimension $d = 1$, if the resonance $(\underline{\xi}', \underline{\xi})$ is regular, $\mathcal{R} = \{\underline{\eta}\}$ and (7.13) reduces to the condition $Q(\underline{\xi} + \nu\beta)\, q(P(\nu\beta)u\,,\, P(\underline{\xi})v) \;=\; 0$. As a corollary, we can state.

Corollary 7.4 *In space dimension $d = 1$, assume that $\mathbb{Z}^m\beta$ intersects the characteristic varieties of L and M at finitely many regular points and that for all $\nu \in \mathbb{Z}^m$ all the resonances $(\xi, \underline{\xi}) \in \mathbb{Z}^m\beta \times \mathbb{Z}^m\beta$ are regular. Then, the transparency Assumption* 2.1 *implies that Assumption* 2.2 *is satisfied.*

Example 7.5. *Assumption* 2.5 *is strictly stronger than Assumptions* 2.1 *and* 2.2. In space dimension $d = 1$, consider

$$L(\varepsilon\partial_x) \;:=\; \begin{pmatrix} \varepsilon\partial_t - \varepsilon\partial_y & m \\ -m & \varepsilon\partial_t + \varepsilon\partial_y \end{pmatrix}\,, \quad M(\varepsilon\partial_x) := \varepsilon\partial_t\,. \tag{7.14}$$

The characteristic variety of L is $\mathcal{C} := \{\tau^2 = \eta^2 + m^2\}$. For $\xi = (\tau, \eta) \in \mathcal{C}$, the vector $e(\xi) := (m, i(\eta - \tau))$ is a basis of $\ker L(i\xi)$. Consider $\beta = (\omega, \kappa) \in \mathcal{C}$ with $\kappa \neq 0$. Then $\nu\beta \in \mathcal{C}$ if and only if $\nu = \pm 1$ and $\nu\beta$ is characteristic for M if and only if $\nu = 0$. Consider the quadratic form

$$q(u, u') \;:=\; u_1\, u_2' \;+\; u_1'\, u_2 \tag{7.15}$$

Then for all ξ one has $q(e(\xi), e(-\xi)) = 0$. Applied to $\xi = \pm\beta$, this shows that the transparency Assumption 2.1 is satisfied. With Corollary 7.4, this implies that Assumption 2.2 is also satisfied. On the other hand, consider $\beta' = (-\omega, \kappa) \in \mathcal{C}$. Then $\beta + \beta' = (0, 2\kappa)$ is characteristic for M and $q(e(\beta), e(\beta')) = 2\,i\,m\kappa \neq 0$ Thus (2.18) is not satified, showing that Assumption 2.5 does not hold.

This example will be used in §11 to produce an example of strongly unstable BKW solution.

8. Linear and nonlinear stability of approximate solutions

In this section we prove Theorem 2.9. Consider the system (2.21) and a formal solution $\sum \varepsilon^n \mathbf{U}_n$, with $\mathbf{U}_n = (\mathbf{u}_n, \mathbf{v}_n)$, given by Theorem 2.3. It is defined on $[0, T_a]$, with values in H^∞. Introduce the approximate solution

$$\mathbf{U}^\varepsilon_{app} := \sum_{n=0}^{k} \varepsilon^n \mathbf{U}_n \,. \tag{8.1}$$

It satisfies the estimate (2.23).

Neglecting $O(\varepsilon)$ zero-th order terms, the linearized operator is

$$\begin{cases} L(\varepsilon\partial_x + \beta\partial_\theta)\,\mathbf{u} \\ M(\varepsilon\partial_x + \beta\partial_\theta)\,\mathbf{v} \,+\, 2q(\mathbf{u}_0, \mathbf{u}) \end{cases} \tag{8.2}$$

We prove that, under Assumption 2.5, the operator (8.2) is conjugated to the free system, that is to the operator (8.2) with $\mathbf{u}_0 = 0$, modulo error terms which are $O(\varepsilon)$ in Sobolev spaces.

Theorem 8.1. *Suppose that Assumption 2.5 is satisfied. Then, there are families of operators* $\mathbf{S}^\varepsilon(t)$ *and* $\mathbf{T}^\varepsilon(t)$, *for* $\varepsilon \in]0, 1]$ *and* $t \in [0, T_a]$ *such that*

i) the mappings $t \mapsto \mathbf{S}^\varepsilon(t)$ *and* $t \mapsto \mathbf{T}^\varepsilon(t)$ *are* C^∞ *from* $[0, T_a]$ *to the space of bounded operators from* $H^\sigma(\mathbb{R}^d \times \mathbb{T}^m)$ *to itself for all* σ,

ii) for all σ *the operators* $\mathbf{S}^\varepsilon(t)$ *and* $\mathbf{T}^\varepsilon(t)$ *are uniformly bounded from* $H^\sigma(\mathbb{R}^d \times \mathbb{T}^m)$ *to itself,*

iii) one has the following relation

$$\mathbf{S}^\varepsilon L(\varepsilon\partial_x + \beta\partial_\theta) \,-\, M(\varepsilon\partial_x + \beta\partial_\theta)\,\mathbf{S}^\varepsilon \,=\, q(\mathbf{u}_0, \,\cdot\,) \,+\, \varepsilon\,\mathbf{T}^\varepsilon \,. \tag{8.3}$$

Proof.

a) Denote by Σ the space of functions on $[0, T_a] \times \mathbb{R}^d \times \mathbb{R}^d$ which are finite sums of products $a(x)\,p(\eta)$ with $a \in H^\infty([0, T_a] \times \mathbb{R}^d)$ and $p \in L^\infty(\mathbb{R}^d)$. For $S \in \Sigma$, $\varepsilon \in]0, 1]$, $t \in [0, T_a]$ and $\mu \in \mathbb{Z}^m$, the operator

$$u \;\mapsto\; S(t, y, \varepsilon D_y + \mu\kappa)\,u \;:=\; (2\pi)^{-d} \int e^{iy\eta} S(t, y, \varepsilon\eta + \mu\kappa)\,\widehat{u}(\eta)\,d\eta \tag{8.4}$$

maps $H^\sigma(\mathbb{R}^d)$ into itself, for all σ, with norm bounded independently of ε, t and μ, because this property is true both for operators of multiplication by functions H^∞ and for $p(\varepsilon D_y + \mu\kappa)$ with $p \in L^\infty$. Moreover, when $S \in \Sigma$, the times derivatives $\partial_t^j S$ also belong to Σ and the mapping $t \mapsto S(t, y, \varepsilon D_y + \mu\kappa))$ is C^∞ from $[0, T_a]$ to the space of bounded operators from $H^\sigma(\mathbb{R}^d \times \mathbb{T}^m)$ to itself for all σ,

Introduce next the space $\mathbf{\Sigma}$ of trigonometric polynomials $\mathbf{S}(x, \eta, \theta) \,=\, \sum S_\nu(x, \eta)\,e^{i\nu\theta}$ with coefficients in Σ. For $\mathbf{S} \in \mathbf{\Sigma}$, introduce the operators acting on Fourier series

$$\mathbf{S}^\varepsilon \mathbf{u} \,=\, \mathbf{S}^\varepsilon\Big(\sum_\mu e^{i\mu\theta} u_\mu\Big) := \sum_{\nu,\mu} e^{i(\mu+\nu)\theta} S_\nu(x, \varepsilon D_y + \mu\kappa)\,u_\mu \,. \tag{8.5}$$

Note that no time derivative acts on $\mathbf{u}$ in these formula and we denote by $\mathbf{S}^\varepsilon(t)$ and $\mathbf{T}^\varepsilon(t)$ the operators acting on functions of (y,θ) for the given value of time t.

Let $\sum e^{i\mu\theta} v_\mu$ be the Fourier series of $\mathbf{v} = \mathbf{S}^\varepsilon(t)\mathbf{u}$. Then, by definition

$$v_\mu = \sum_\nu S_\nu(x, \varepsilon D_y + (\mu-\nu)\kappa)\, u_{\mu-\nu}\,. \tag{8.6}$$

Note that the sum runs over a finite set of indices, say $|\nu| \le N$, since $\mathbf{S}$ is a trigonometric polynomial. Then, for all σ there is a constant C independent of t, ε, μ and $\mathbf{u}$ such that

$$\|v_\mu\|_{H^\sigma(\mathbb{R}^d)} \le C \sum_{|\nu|\le N} \|u_{\mu-\nu}\|_{H^\sigma(\mathbb{R}^d)}\,. \tag{8.7}$$

This implies that the $\mathbf{S}^\varepsilon(t)$ are uniformly bounded from $H^\sigma(\mathbb{R}^d \times \mathbb{T}^m)$ into itself for all σ.

b) When $\mathbf{S} \in \mathbf{\Sigma}$, the derivatives $\partial_x \mathbf{S}$ and $\partial_\theta \mathbf{S}$ also belong to $\mathbf{\Sigma}$. Moroever, the commutators $[\partial_{x_j}, S(x, \varepsilon D_y + \mu\kappa)]$ are equal to $(\partial_{x_j} S)(x, \varepsilon D_y + \mu\kappa)$. Therefore, acting on smooth functions, one has

$$\partial_x(\mathbf{S}^\varepsilon \mathbf{u}) \,=\, \mathbf{S}^\varepsilon(\partial_x \mathbf{u}) \,+\, (\partial_x \mathbf{S})^\varepsilon \mathbf{u}\,. \tag{8.8}$$

Similarly, one has

$$\partial_\theta\big(e^{i(\mu+\nu)\theta} S_\nu(x, \varepsilon D_y + \mu\kappa)\mathbf{u}_\mu\big) \,=\, e^{i(\mu+\nu)\theta}\, \big(S_\nu(x, \varepsilon D_y + \mu\kappa)\,(i\mu + i\nu) u_\mu\big)$$

and thus

$$\partial_\theta(\mathbf{S}^\varepsilon \mathbf{u}) \,=\, \mathbf{S}^\varepsilon(\partial_\theta \mathbf{u}) \,+\, (\partial_\theta \mathbf{S})^\varepsilon \mathbf{u}\,. \tag{8.9}$$

c) Introduce the bilinear mappings $S_\nu(\eta)$ given by Proposition 7.1. Introduce next the matrices

$$S_\nu(x,\eta)\, v := S_\nu(\eta)(P(\nu\beta) u_{0,\nu}(x)\,,\, v) \tag{8.10}$$

where $u_{0,\nu}(x) \in H^\infty([0,T_a]\times\mathbb{R}^d)$ denote the Fourier coefficients of $\mathbf{u}_0(x,\theta)$. Then, S_ν and $\partial_t S_\nu$ are matrices of symbols in the class Σ. Note that only finitely many S_ν do not vanish, because $P(\nu\beta) = 0$ when $|\nu|$ is large. Thus $\mathbf{S} = \sum S_\nu e^{i\nu\theta} \in \mathbf{\Sigma}$.

For smooth trigonometric polynomials $\mathbf{u} = \sum u_\mu e^{i\mu\theta}$, (8.8) implies that

$$M_1(\partial_x)\, \mathbf{S}^\varepsilon \mathbf{u} \,=\, \mathbf{S}^\varepsilon(\partial_t \mathbf{u}) \,+\, \sum_j B_j \mathbf{S}^\varepsilon(\partial_j \mathbf{u}) \,-\, \mathbf{T}^\varepsilon \mathbf{u} \tag{8.11}$$

where $\mathbf{T} \in \mathbf{\Sigma}$. Therefore

$$\mathbf{S}^\varepsilon L(\varepsilon\partial_x + \beta\partial_\theta)\mathbf{u} \,-\, M(\varepsilon\partial_x + \beta\partial_\theta)\, \mathbf{S}^\varepsilon \mathbf{u} \,=\, \sum_{\mu,\nu} e^{i(\mu+\nu)\theta}\, v_{\mu,\nu} \,+\, \varepsilon\, \mathbf{T}^\varepsilon \mathbf{u} \tag{8.12}$$

with

$$v_{\mu,\nu}(x) \,=\, (2\pi)^{-d} \int e^{iy\eta}\, G_{\mu,\nu}(x,\eta)\, \widehat{u_\mu}(t,\eta)\, d\eta$$

$$\begin{aligned} G_{\mu,\nu}(x,\eta) \,:=\,& S_\nu(x, \varepsilon D_y + \mu\kappa)\, \big(i\mu\omega + A(i\varepsilon\eta + \mu\kappa)\big) \\ &- \big(\, i(\mu+\nu)\omega + B(i\varepsilon\eta + i(\mu+\nu)\kappa)\big) S_\nu(x, \varepsilon D_y + \mu\kappa)\,. \end{aligned}$$

The terms with $i\mu\omega$ cancel each other. Applying (8.2) to the frequency $\varepsilon\eta + \mu\kappa$ and vectors $a = \mathbf{u}_{0,\nu}(x)$, $u = \widehat{u_\mu}(\eta)$ shows that

$$G_{\mu,\nu}(x,\eta)\, \widehat{u_\mu}(\eta) \,=\, q\big(\mathbf{u}_{0,\nu}(x)\,,\, \widehat{u_\mu}(\eta)\big)\,.$$

Therefore, $v_{\mu,\nu} = q(u_{0,\nu}, u_\mu)$ and $\sum_{\mu,\nu} e^{i(\mu+\nu)\theta}\, v_{\mu,\nu} \,=\, q(\mathbf{u}_0, \mathbf{u})$ so (8.12) implies the theorem.

The main ingredient for solving the semilinear equation (2.22) is to prove Sobolev estimates for the solutions of the linearized equation

$$\mathbf{L}^\varepsilon \mathbf{U} \,+\, \nabla_{\mathbf{U}} \mathbf{F}^\varepsilon(\mathbf{U}^\varepsilon_{app})\mathbf{U} \tag{8.13}$$

Proposition 8.2. *Suppose that Assumption 2.5 is satisfied. Then for all $\sigma \in \mathbb{N}$, there is a constant C such that for all $T \in [0, T_a]$, $\mathbf{U} \in C^1([0,T]; H^{\sigma+1}(\mathbb{R}^d \times \mathbb{T}^m))$, $t \in [0,T]$ and $\varepsilon \in]0,1]$,*

$$\begin{aligned} \|\mathbf{U}(t)\|_{H^\sigma(\mathbb{R}^d\times\mathbb{T}^m)} \leq & C\,\|\mathbf{U}(0)\|_{H^\sigma(\mathbb{R}^d\times\mathbb{T}^m)} \\ & + C\,\varepsilon^{-1}\int_0^t \|\big(\mathbf{L} + \nabla_{\mathbf{U}}\mathbf{F}^\varepsilon\big(\mathbf{U}^\varepsilon_{app}\big)\big)\mathbf{U}(s)\|_{H^\sigma(\mathbb{R}^d\times\mathbb{T}^m)}\, ds\,. \end{aligned} \tag{8.14}$$

Recall that $\mathbf{L}$ starts with $\varepsilon\partial_t$. This is why there is a factor ε^{-1} in form of the integral. In the applications below the right hand side $\mathbf{L}\mathbf{U} + \nabla\mathbf{F}(\mathbf{U}^\varepsilon_{app})\mathbf{U}$ is $O(\varepsilon)$.

Proof. **a)** Let $(\mathbf{f},\mathbf{g}) := (\mathbf{L}^\varepsilon + \nabla_{\mathbf{U}}\mathbf{F}^\varepsilon(\mathbf{U}^\varepsilon_{app}))\mathbf{U}$, that is

$$\begin{cases} \mathbf{f} \,:=\, L(\varepsilon\partial_x + \beta\partial_\theta)\,\mathbf{u} \,+\, \varepsilon\nabla f(\mathbf{u}^\varepsilon_{app}, \mathbf{v}^\varepsilon_{app})(\mathbf{u},\mathbf{v}) \\ \mathbf{g} \,:=\, M(\varepsilon\partial_x + \beta\partial_\theta)\,\mathbf{v} \,+\, 2q(\mathbf{u}^\varepsilon_{app}, \mathbf{u}) \,+ \varepsilon\nabla g(\mathbf{u}^\varepsilon_{app}, \mathbf{v}^\varepsilon_{app})(\mathbf{u},\mathbf{v})\,. \end{cases} \tag{8.15}$$

The H^σ norm of the terms $\varepsilon\nabla f(\mathbf{u}^\varepsilon_{app}, \mathbf{v}^\varepsilon_{app})(\mathbf{u},\mathbf{v})$, $\varepsilon\nabla g(\mathbf{u}^\varepsilon_{app}, \mathbf{v}^\varepsilon_{app})(\mathbf{u},\mathbf{v})$ and $q(\mathbf{u}^\varepsilon_{app} - \mathbf{u}_0, \mathbf{u})$ are $O\big(\varepsilon\|\mathbf{U}(s)\|_{H^\sigma}\big)$. Therefore, Gronwall's lemma implies that it is sufficient to prove the estimate (8.14) when $f = 0$, $g = 0$ and $\mathbf{u}^\varepsilon_{app}$ is replaced by $\mathbf{u}_0$, that is for the system

$$\begin{cases} \mathbf{f} \,=\, L(\varepsilon\partial_x + \beta\partial_\theta)\,\mathbf{u}\,, \\ \mathbf{g} \,=\, M(\varepsilon\partial_x + \beta\partial_\theta)\,\mathbf{v} \,+\, 2q(\mathbf{u}_0, \mathbf{u})\,. \end{cases} \tag{8.16}$$

b) Introduce $\mathbf{w} := \mathbf{v} + 2\mathbf{S}^\varepsilon\mathbf{u}$. Theorem 8.1 implies that

$$M(\varepsilon\partial_x + \beta\partial_\theta)\,\mathbf{w} \,=\, \mathbf{h} \,:=\, \mathbf{g} \,+\, 2\,\mathbf{S}^\varepsilon\mathbf{f} \,+\, 2\,\varepsilon\,\mathbf{T}^\varepsilon\mathbf{u}\,. \tag{8.17}$$

Because $L(\varepsilon\partial_x + \beta\partial_\theta)$ and $M(\varepsilon\partial_x + \beta\partial_\theta)$ are symmetric hyperbolic, with coefficient of ∂_t equal to ε, one has

$$\begin{aligned} \|\mathbf{u}(t)\|_{H^\sigma} \,&\leq\, \|\mathbf{u}(0)\|_{H^\sigma} \,+\, \varepsilon^{-1}\int_0^t \|\mathbf{f}(s)\|_{H^\sigma}\, ds\,, \\ \|\mathbf{w}(t)\|_{H^\sigma} \,&\leq\, \|\mathbf{w}(0)\|_{H^\sigma} \,+\, \varepsilon^{-1}\int_0^t \|\mathbf{h}(s)\|_{H^\sigma}\, ds\,. \end{aligned} \tag{8.18}$$

Moreover, $\mathbf{S}^\varepsilon(t)$ and $\mathbf{T}^\varepsilon(t)$ are uniformly bounded in H^σ and

$$\begin{aligned} &\|\mathbf{h}(t)\|_{H^\sigma} \,\leq\, \|\mathbf{g}(t)\|_{H^\sigma} \,+ C\|\mathbf{f}(t)\|_{H^\sigma} \,+\, \varepsilon\, C\,\|\mathbf{u}(t)\|_{H^\sigma}. \\ &\|\mathbf{v}(t)\|_{H^\sigma} \,\leq\, \|\mathbf{w}(t)\|_{H^\sigma} \,+\, C\,\|\mathbf{u}(t)\|_{H^\sigma}\,, \quad \|\mathbf{w}(0)\|_{H^\sigma} \,\leq\, \|\mathbf{v}(0)\|_{H^\sigma} \,+\, C\,\|\mathbf{u}(0)\|_{H^\sigma}\,. \end{aligned} \tag{8.19}$$

When one substitutes the estimate of $\mathbf{h}$ in (8.18), the error term is $C\int_0^t \|u\|$ and therefore Gronwall's lemma implies the estimate (8.14) for the solutions of (8.16) finishing the proof of the proposition.

Proof of Theorem 2.8.

$\mathbf{U}^\varepsilon = \mathbf{U}^\varepsilon_{app} + \varepsilon^k \mathbf{V}^\varepsilon$ is a solution of (2.22) if and only if $\mathbf{V}^\varepsilon$ satisfies

$$\mathbf{L}^\varepsilon \mathbf{V}^\varepsilon \,+\, \nabla_{\mathbf{U}} \mathbf{F}^\varepsilon(\mathbf{U}^\varepsilon_{app})\mathbf{V}^\varepsilon \,=\, \varepsilon\, \mathbf{E}^\varepsilon \,+\, \varepsilon^k \mathbf{G}^\varepsilon(\mathbf{U}^\varepsilon_{app}, \mathbf{V}^\varepsilon)\,, \tag{8.20}$$

with

$$\mathbf{E}^\varepsilon := -\,\varepsilon^{-k-1}\big(\mathbf{L}^\varepsilon \mathbf{U}_{app} + \mathbf{F}(\mathbf{U}^\varepsilon_{app})\big)\,, \tag{8.21}$$

$$\mathbf{G}^\varepsilon(U,V) := \,-\,\varepsilon^{-2k}\big(\mathbf{F}^\varepsilon(U+\varepsilon^k V) \,-\, \varepsilon^k \nabla \mathbf{F}^\varepsilon(U)V\big) \,=\, O(V^2)\,. \tag{8.22}$$

The equation (8.20) is solved by Picard's iterations, noticing that by (2.23) $\mathbf{E}^\varepsilon$ is uniformly bounded in H^σ and that (8.22) defines a bounded family of smooth functions of the variables (U,V).

The linear operator in the left hand side is hyperbolic with smooth coefficients. Therefore the iterates are well defined in $C^0([0,T_a],H^\sigma)$ if the initial data belong to H^σ and $\sigma > (d+m)/2$ so that H^σ is an algebra. The iterates are estimated using Proposition 8.2. Note that the loss of ε^{-1} in (8.16) is compensated by the factors ε in front of $\mathbf{E}^\varepsilon$ and ε^k in front of $\mathbf{G}^\varepsilon(\mathbf{U}^\varepsilon_{app}, \mathbf{V}^\varepsilon)$. From there on, the proof is standard and we omit the details. When $k=1$, we obtain boundedness and convergence of the iterates on a uniform interval $[0,T]$. When $k>1$, the nonlinear term $\varepsilon^{k-1}\mathbf{G}^\varepsilon$ is arbitrarily small when ε is small, so that T can be chosen arbitrarily close to T_a for small ε.

9. Nonlinear conjugation

In this section we prove Theorem 2.11 and discuss the links between the Assumption 2.5 and 2.10. As a mater of fact, we consider the more general framework of equations (2.21) which includes the fast variables θ. In particular, Theorem 2.11 is a consequence of the following result applied to functions independent of θ.

THEOREM 9.1 *Suppose that Assumption* 2.10 *is satisfied. Then, there exists a family of symmetric bilinear mappings* $\mathbf{J}^\varepsilon$, *from* $H^\infty(\mathbb{R}^d\times\mathbb{T}^m)\times H^\infty(\mathbb{R}^d\times\mathbb{T}^m)$ *to* $H^\infty(\mathbb{R}^d\times\mathbb{T})$, *such that for all* $\mathbf{u}\in C^1([0,T];H^\infty(\mathbb{R}^d\times\mathbb{T}^m))$,

$$q\big(\mathbf{u}(t),\mathbf{u}(t)\big) \,=\, M(\varepsilon\partial_x + \beta\partial_\theta)\mathbf{J}^\varepsilon\big(\mathbf{u}(t),\mathbf{u}(t)\big) \,-\, 2\,\mathbf{J}^\varepsilon\big(L(\varepsilon\partial_x+\beta\partial_\theta)\mathbf{u}(t),\mathbf{u}(t)\big)\,. \tag{9.1}$$

Proof. Expanding functions into Fourier series we look for $\mathbf{J}^\varepsilon$ as

$$\mathbf{J}^\varepsilon\Big(\sum u_\nu e^{i\nu\theta}, \sum u_\nu e^{i\nu\theta}\Big) \,=\, \sum J^\varepsilon_{\nu,\nu'}(u_\nu,u'_\nu)\,e^{i(\nu+\nu')\theta}\,, \tag{9.2}$$

and, denoting by $\widehat{u}$ the Fourier transform of u on $\mathbb{R}^d$,

$$J^\varepsilon_{\nu,\nu'}(u,u')(y) \,=\, (2\pi)^{-d}\int_{\mathbb{R}^d\times\mathbb{R}^d} e^{iy(\eta+\eta')} J_{\nu,\nu'}(\varepsilon\eta,\varepsilon\eta';\widehat{u}(\eta),\widehat{u'}(\eta'))d\eta d\eta' \tag{9.3}$$

where $\{J_{\nu,\nu'}(\eta,\eta';\,\cdot\,,\,\cdot\,)\,;\,(\nu,\nu')\in\mathbb{Z}^m\times\mathbb{Z}^m\,,\,(\eta,\eta')\in\mathbb{R}^d\times\mathbb{R}^d\}$ is a bounded family of quadratic forms on $\mathbb{C}^N\times\mathbb{C}^N$. In this case, the relations above define a bounded family of continuous bilinear mappings $\{\mathbf{J}^\varepsilon\,;\,\varepsilon\in]0,1]\}$, from $H^\sigma(\mathbb{R}^d\times\mathbb{T}^m)\times H^\sigma(\mathbb{R}^d\times\mathbb{T}^m)$ to itself provided that $\sigma>(d+m)/2$. $\mathbf{J}^\varepsilon$ is symmetric if

$$J_{\nu,\nu'}(\eta,\eta';u,u')\;=\;J_{\nu',\nu}(\eta',\eta;u',u)\,. \tag{9.4}$$

One has

$$\varepsilon\partial_t\mathbf{J}^\varepsilon(\mathbf{u},\mathbf{u})\;=\;\mathbf{J}^\varepsilon(\varepsilon\partial_t\mathbf{u},\mathbf{u})\;+\;\mathbf{J}^\varepsilon(\mathbf{u},\varepsilon\partial_t\mathbf{u})$$

Therefore, to prove (9.1) it is sufficient to prove that

$$\begin{aligned} q\big(\mathbf{u},\mathbf{u}\big)\;=&\big(\omega\partial_\theta+\;B(\varepsilon\partial_y+\kappa\partial_\theta)\big)\mathbf{J}^\varepsilon\big(\mathbf{u},\mathbf{u}\big)\\ &-\;\mathbf{J}^\varepsilon\big((\omega\partial_\theta+A(\varepsilon\partial_y+\kappa\partial_\theta)\mathbf{u},\mathbf{u}\big)-\;\mathbf{J}^\varepsilon\big(\mathbf{u},(\omega\partial_\theta+A(\varepsilon\partial_y+\kappa)\partial_\theta)\mathbf{u}\big)\,. \end{aligned} \tag{9.5}$$

Taking Fourier expansions, this means that for all $(\nu,\nu',\eta,\eta',u,u')$ one has

$$\begin{aligned} q\big(u,u'\big)\;=\;&\Big(i(\nu+\nu')\omega+\;B\big(i(\eta+\eta'+\nu\kappa+\nu'\kappa)\big)\Big)J_{\nu,\nu'}(\eta,\eta';u,u')\\ &\qquad-J_{\nu,\nu'}\big(\eta,\eta';(i\nu\omega+A(i\eta+i\nu\kappa))u,u'\big)\\ &\qquad-J_{\nu,\nu'}\big(\eta,\eta';u,(i\nu'\omega+A(i\eta'+i\nu'\kappa))u'\big)\,. \end{aligned} \tag{9.6}$$

The terms with $i(\nu+\nu')\omega$, $i\nu\omega$ and $i\nu'\omega$ add to zero. Hence (9.6) is equivalent to

$$\begin{aligned} q\big(u,u'\big)\;=&B\big(i(\eta+\eta'+\nu\kappa+\nu'\kappa)\big)J_{\nu,\nu'}(\eta,\eta';u,u')\\ &-\;J_{\nu,\nu'}\big(\eta,\eta';A(i\eta+i\nu\kappa)u,u'\big)-\;J_{\nu,\nu'}\big(\eta,\eta';u,A(i\eta'+i\nu'\kappa)u'\big)\,. \end{aligned} \tag{9.7}$$

For fixed (ν,ν',η,η'), introduce $\tilde\eta=\eta+\nu\kappa$, $\tilde\eta'=\eta'+\nu'\kappa'$ and consider the spectral decompositions

$$\begin{aligned} &A(i\tilde\eta)=\sum i\lambda_j(\tilde\eta)P_j(\tilde\eta)\,,\quad A(i\tilde\eta')=\sum i\lambda_k(\tilde\eta')P_k(\tilde\eta')\\ &B(i(\tilde\eta+\tilde\eta'))=\sum i\mu_l(\tilde\eta+\tilde\eta')Q_l(\eta+\tilde\eta') \end{aligned}\,.$$

Then (9.7) is equivalent to the condition that for all (j,k,l)

$$\begin{aligned} &Q_l(\tilde\eta+\tilde\eta')q\big(P_j(\tilde\eta)u,P_k(\tilde\eta')u'\big)\;=\\ &\qquad\big(\mu_l(\tilde\eta+\tilde\eta')-\lambda_j(\tilde\eta)-\lambda_k(\tilde\eta')\big)\,Q_l(\tilde\eta+\tilde\eta')J_{\nu,\nu'}\big(\eta,\eta';(P_j(\tilde\eta)u,P_k(\tilde\eta')u'\big)\,. \end{aligned} \tag{9.8}$$

Assumption 2.10 implies that the left hand side is $O(\mu_l(\tilde\eta+\tilde\eta')-\lambda_j(\tilde\eta)-\lambda_k(\tilde\eta'))$. Therefore, this equation uniquely determines $J_{\nu,\nu'}(\eta,\eta')$. One has

$$J_{\nu,\nu'}(\eta,\eta')\;=\;J(\eta+\nu\kappa,\eta'+\nu'\kappa) \tag{9.9}$$

with

$$J\big(\eta,\eta';u,u'\big)\;:=\;\sum_{j,k,l}\big(\mu_l(\eta+\eta')-\lambda_j(\eta)-\lambda_k(\eta')\big)^{-1}Q_l(\eta+\eta')q\big(P_j(\eta)u,P_k(\eta')u'\big)\,. \tag{9.10}$$

Conversely, (9.9) and (9.10) define a bounded family of quadratic forms which satisfy (9.8) and the symmetry property (9.4). The theorem follows.

Corollary 9.2 *Consider the change of unknowns*

$$\tilde{\mathbf{v}} := \mathbf{v} + \mathbf{J}^\varepsilon(\mathbf{u},\mathbf{u})\,; \tag{9.11}$$

Then, for smooth solutions, the system (2.21) *is equivalent to*

$$\begin{cases} L(\varepsilon\partial_x + \beta\partial_\theta)\mathbf{u} \,+\, \varepsilon\, f(\mathbf{u}, \tilde{\mathbf{v}} - \mathbf{J}^\varepsilon(\mathbf{u},\mathbf{u})) \,=\, 0\,, \\ M(\varepsilon\partial_x + \beta\partial_\theta)\tilde{\mathbf{v}} \,+ \varepsilon\, \mathbf{J}^\varepsilon(\mathbf{u}, f(\mathbf{u}, \tilde{\mathbf{v}} - \mathbf{J}^\varepsilon(\mathbf{u},\mathbf{u}))) \,+ \varepsilon\, g(\mathbf{u}, \tilde{\mathbf{v}} - \mathbf{J}^\varepsilon(\mathbf{u},\mathbf{u})) \,=\, 0\,. \end{cases} \tag{9.12}$$

As mentionned in the introduction, for the Maxwell-Bloch equations, the bilinear operator $\mathbf{J}$ does not involve Fourier multipliers and has the simpler form

$$\mathbf{J}(\mathbf{u},\mathbf{u})(y,\theta) = \widetilde{J}(\mathbf{u}(y,\theta), \mathbf{u}(y,\theta)) \tag{9.13}$$

where $\widetilde{J}$ is a bilinear form on $\mathbb{C}^N \times \mathbb{C}^N$. We now give several other examples of systems which illustrate Assumption 2.10.

Example 9.3. *Fourier multipliers do occur.*

Consider

$$L(\varepsilon\partial_x) := \begin{pmatrix} \varepsilon\partial_t - \varepsilon\partial_y - i & 0 \\ 0 & \varepsilon\partial_t + \varepsilon\partial_y + i \end{pmatrix}, \quad M(\varepsilon\partial_x) := \begin{pmatrix} \varepsilon\partial_t - \varepsilon c\partial_y & m \\ -m & \varepsilon\partial_t + \varepsilon c\partial_y \end{pmatrix}. \tag{9.14}$$

The characterisitic varieties of L and M are $\mathcal{C}_L = \{\tau^2 = (\eta+1)^2\}$ and $\mathcal{C}_M = \{\tau^2 = c^2\eta^2 + m^2\}$ respectively. $\mathcal{C}_L$ is the union of two lines , $\mathcal{C}_\pm := \{\tau = \pm(\eta+1)\}$. The eigenvectors are $e_+ = (1,0)$ and $e_- = (0,1)$ respectively. Assume that

$$(m^2 - 4)\,(c^2 - 1) \,>\, 4\,. \tag{9.15}$$

This implies that when $\xi \in \mathcal{C}_L$ and $\xi' \in \mathcal{C}_L$ belong to the same line $\mathcal{C}_\pm$, then $\xi + \xi' \notin \mathcal{C}_M$. On the other hand, when ξ and ξ' belong to different lines, $\xi + \xi' \in \mathcal{C}_M$ if and only if

$$(\eta - \eta')^2 = c^2(\eta + \eta')^2 + m^2\,.$$

Denote by (u_1, u_2) the two components of u. If $q(u,u) = q_1 u_1^2 + q_2 u_2^2$, then $q(e_+, e_-) = 0$, showing that $q(P(\xi), P(\xi'))$ vanishes at resonances. Indeed, one can show that Assumption 2.10 is satisfied. We now compute explicitely the operator J^ε, when

$$q(u,u) \,=\, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} u_1^2\,. \tag{9.16}$$

In this case, J^ε depends only on u_1 :

$$J^\varepsilon(u,u) \,=\, \frac{1}{2\pi} \int e^{i(\eta+\eta')y} \begin{pmatrix} \rho(\eta,\eta') \\ \sigma(\eta,\eta') \end{pmatrix} \widehat{u_1}(\eta)\,\widehat{u_1}(\eta')\,d\eta\,d\eta'\,, \tag{9.17}$$

where ρ and σ satisfy

$$\begin{cases} i(\eta + \eta' + 2 -\, c(\eta+\eta'))\rho \,+\, m\,\sigma \,=\, b_1\,, \\ -\,m\rho \,+\, i(\eta+\eta'+2 +\, c(\eta+\eta'))\sigma \,=\, b_2\,. \end{cases} \tag{9.18}$$

The condition (9.15) implies that the determinant of this system is bounded from below by a positive constant and that $\rho(\eta,\eta')$ and $\sigma(\eta,\eta')$ are bounded symbols. Note that the definition of ρ and σ involves nontrivial rational fractions, implying that J^ε does involve Fourier multipliers.

Example 9.4. *The nonresonant case.*

Consider two coupled Klein-Gordon equations :

$$L(\varepsilon\partial_x) := \begin{pmatrix} \varepsilon\partial_t - \varepsilon\partial_y & m_1 \\ -m_1 & \varepsilon\partial_t + \varepsilon\partial_y \end{pmatrix}, \quad M(\varepsilon\partial_x) := \begin{pmatrix} \varepsilon\partial_t - \varepsilon\partial_y & m_2 \\ -m_2 & \varepsilon\partial_t + \varepsilon\partial_y \end{pmatrix}. \tag{9.19}$$

The characterisitec varieties of L and M are $\mathcal{C}_1 = \{\tau^2 = \eta^2 + m_1^2\}$ and $\mathcal{C}_2 = \{\tau^2 = \eta^2 + m_2^2\}$ respectively. When $m_2 < 2m_1$, the intersection $(\mathcal{C}_1 + \mathcal{C}_1) \cap \mathcal{C}_2$ is empty, which implies that there are no resonances. Moreover, there is a positive constant c such that

$$\Big| \pm \sqrt{(\eta + \eta')^2 + m_2^2} \pm \sqrt{\eta^2 + m_1^2} \pm \sqrt{\eta'^2 + m_1^2} \Big| \geq c\,. \tag{9.20}$$

This implies that for all quadratic form q, Assumption 2.10 is satisfied.

Example 9.5. *Resonant Klein Gordon equations.*

Consider the case where L itself is made of two Klein-Gordon operators :

$$L_1(\varepsilon\partial_x) := \begin{pmatrix} \varepsilon\partial_t - \varepsilon\partial_y & m_1 \\ -m_1 & \varepsilon\partial_t + \varepsilon\partial_y \end{pmatrix}, \quad L_2(\varepsilon\partial_x) := \begin{pmatrix} \varepsilon\partial_t - \varepsilon\partial_y & m_2 \\ -m_2 & \varepsilon\partial_t + \varepsilon\partial_y \end{pmatrix}. \tag{9.21}$$

We denote $u = (u_1, u_2)$ and L_j acts on u_j. In addition,

$$M(\varepsilon\partial_x) := \begin{pmatrix} \varepsilon\partial_t - \varepsilon\partial_y & m \\ -m & \varepsilon\partial_t + \varepsilon\partial_y \end{pmatrix}. \tag{9.22}$$

The characterisitec varieties are $\mathcal{C}_{L_1} = \{\tau^2 = \eta^2 + m_1^2\}$, $\mathcal{C}_{L_2} = \{\tau^2 = \eta^2 + m_2^2\}$ and $\mathcal{C}_M = \{\tau^2 = \eta^2 + m^2\}$. Assume that

$$m_2 < m_1\,, \quad \min(m_1 - m_2, 2m_2) < m < 2m_1\,. \tag{9.23}$$

This implies that

$$(\mathcal{C}_{L_1} + \mathcal{C}_{L_1}) \cap \mathcal{C}_M = \emptyset\,, \quad (\mathcal{C}_{L_1} + \mathcal{C}_{L_2}) \cap \mathcal{C}_M = \emptyset, \quad (\mathcal{C}_{L_2} + \mathcal{C}_{L_2}) \cap \mathcal{C}_M \neq \emptyset\,. \tag{9.24}$$

Thus, there are resonances, but only for L_2-characteristic frequencies. In particular, when

$$q(u,u) = q_1(u_1,u_1) + b(u_1,u_2)$$

does not involve quadratic terms in u_2, one has $Q(\xi+\xi')q(P(\xi)\,.\,, P(\xi')\,.\,) = 0$ at resonances. Moreover, estimates similar to (9.20) show that Assumption 2.10 is satisfied.

Example 9.6 *Assumption* 2.10 *is strictly stronger than Assumption* 2.5.

In this example, we show that the stability condition can be satisfied only for some pump frequencies β. Consider

$$L(\varepsilon\partial_x) := \begin{pmatrix} \varepsilon\partial_t - \varepsilon\partial_y & m_1 \\ -m_1 & \varepsilon\partial_t + \varepsilon\partial_y \end{pmatrix}, \quad M(\varepsilon\partial_x) := \begin{pmatrix} \varepsilon\partial_t - \varepsilon c\partial_y & m \\ -m & \varepsilon\partial_t + \varepsilon c\partial_y \end{pmatrix}. \tag{9.25}$$

The characterisitic varieties are $\mathcal{C}_L = \{\tau^2 = \eta^2 + m_1^2\}$ and $\mathcal{C}_M = \{\tau^2 = c^2\eta^2 + m^2\}$ respectively. Assume that

$$c > 1\,, \quad m > 2m_1\,. \tag{9.26}$$

Consider $\beta = (\omega, \kappa) \in \mathcal{C}_L$. One has $\mathbb{Z}\beta \cap \mathcal{C}_L = \{-\beta, \beta\}$. The resonance condition $(\mathcal{C}_L \pm \beta) \cap \mathcal{C}_M \neq \emptyset$ yields an equation of degree four for η. We show that this equation has no solution when κ is small, and that is has solutions when κ is large.

Consider first $\beta_0 = (m_1, 0) \in \mathcal{C}_L$. Then (9.26) clearly implies that $(\mathcal{C}_L \pm \beta) \cap \mathcal{C}_M = \emptyset$. This property remains true for β close to β_0. Consider next $\beta \in \mathcal{C}_L$ such that $\omega > m/2$. In this case, the functions $\varphi_1(\eta) := \omega + \sqrt{m_1^2 + (\eta - \kappa)^2}$ and $\varphi_2(\eta) := \sqrt{m^2 + c^2\eta^2}$ satisfy $\varphi_2(0) < \varphi_1(0)$ and $\varphi_2(\eta) > \varphi_1(\eta)$ when η is large. Therefore the graphs of φ_1 and φ_2 intersect each other and $(\mathcal{C}_L \pm \beta) \cap \mathcal{C}_M \neq \emptyset$.

Using Remark 7.3, this shows that there is $\delta > 0$, such that for all $\beta = (\omega, \kappa) \in \mathcal{C}_L$ with $|\kappa| < \delta$ and all quadratic form $q(u, u)$ the Assumption 2.5 is satisfied. On the other hand, when κ is large, there are resonances, and the condition

$$Q(\xi + \beta) q(P(\beta)\,.\,, P(\xi)\,.\,) = 0 \tag{9.27}$$

is certainly violated by an appropriate choice of q, in which case Assumption 2.10 is not satisfied.

Example 9.7. The example above is easily modified to include resonances for stable pump frequencies β. Consider $L = (L_1, L_2)$ as in (9.21) and M as in (9.25). Assume that

$$c > 1\,, \quad 2m_1 > m > m_1 + m_2\,. \tag{9.28}$$

For $\beta \in \mathcal{C}_{L_1}$ close to $\beta_0 = (m_1, 0)$, one has

$$(\mathcal{C}_{L_1} \pm \beta) \cap \mathcal{C}_M \neq \emptyset\,, \quad (\mathcal{C}_{L_2} \pm \beta) \cap \mathcal{C}_M = \emptyset\,. \tag{9.29}$$

The first condition means that there are resonances and the second that they do not involve the L_2 characteristic frequencies. Therefore, the condition (9.37) is satisfied for all

$$q(u, u) \;=\; q_2(u_2, u_2) \;+\; b(u_1, u_2) \tag{9.30}$$

which does not involve quadratic terms in u_1. Using Remark 7.3, this implies that Assumption 2.5 is satisfied for $\beta \in \mathcal{C}_{L_1}$ close to β_0.

For β large, one has $(\mathcal{C}_{L_2} \pm \beta) \cap \mathcal{C}_M \neq \emptyset$ and (9.27) is violated for appropriate choices of b in (9.30), showing that Assumption 2.10 is not satisfied.

10. Linearly unstable resonances

Proposition 2.4 asserts that when Assumption 2.5 is not satisfied, the linearized equations are not uniformly stable. What happens in this case depends on the lower order terms. In this section, we study a simplified model for the description of linear instabilities at resonances. In the next section, we apply this analysis to produce examples of systems (2.1) for which approximate solutions which are $O(1)$ and have residual $O(\varepsilon^\infty)$ differ from the exact solution with the same initial data by $O(1)$.

Consider the linearized version of (2.1)

$$
\begin{cases} L(\varepsilon\partial_x)u \,+\, \varepsilon f(\overline{u}_0^\varepsilon, v) \,=\, 0\,, \\ M(\varepsilon\partial_x)v \,+\, q(u_0^\varepsilon, u) \,=\, 0\,, \end{cases} \tag{10.1}
$$

where L and M are two symmetric hyperbolic systems of the form (2.2) and f and q are bilinear forms. The coefficient u_0^ε comes from $\mathbf{u}_0(x, \beta\cdot x/\varepsilon)$. The main simplification is to assume here that $\mathbf{u}_0$ is a monochromatic plane wave

$$
u_0^\varepsilon(x) \,=\, a\, e^{i\beta\cdot x/\varepsilon}\,, \tag{10.2}
$$

where $\beta = (\omega, \kappa) \in \mathbb{R}\times\mathbb{R}^d$ and the amplitude a is constant. The second simplification is due to the special placement of complex conjugate in the interaction. On the Fourier side, $f(\overline{u}_0^\varepsilon, v)$ translates the frequencies of v by $-\beta$, while $q(u_0^\varepsilon, u)$ translates back the frequencies of u by β. Therefore, it makes sense to look for monochromatic solutions

$$
u(x) \,:=\, \tilde{u}(x)\, e^{i\alpha\cdot x/\varepsilon}\,, \quad v(x) \,:=\, \tilde{v}(x)\, e^{i(\alpha+\beta)\cdot x/\varepsilon}\,,. \tag{10.3}
$$

with α characteristic for L and $\alpha+\beta$ characteristic for M. This means that the resonance $(\alpha,\beta)\mapsto(\alpha+\beta)$ has been singled out. Without restriction, we assume that $\alpha = 0$.

For (u, v) given by (10.3), the equations (10.1) are equivalent to the constant coefficient system

$$
(\varepsilon\partial_t \,+ \mathcal{A}(\varepsilon, \varepsilon\partial_y)) \begin{pmatrix} u \\ \tilde{v} \end{pmatrix} \,=\, 0\,, \quad \mathcal{A}(\varepsilon, i\eta) := \begin{pmatrix} A(i\eta) & \varepsilon F \\ G & \omega I + B(i\eta + i\kappa) \end{pmatrix}\,. \tag{10.4}
$$

where F [resp. G] is the matrix such that $f(\overline{a}, v) = Fv$ [resp. $q(a, u) = Gu$].

In this case, the uniform stability estimate (2.16) holds if and only if the matrices $e^{-t\mathcal{A}(\varepsilon,i\eta)/\varepsilon}$ are uniformly bounded for $t \in [0,T]$. By Gronwall's lemma, this is equivalent to the uniform boundedness of $e^{-t\mathcal{A}(0,i\eta)}$ for all time. As in the proof of Proposition 4.1, this is true if and only if there is a constant C such that for all $\xi = (\tau, \eta)$ and all $\xi' = (\tau', \eta + \kappa)$, one has

$$
\big|Q(\xi')GP(\xi)\big| \,\le\, C\,|\tau' - \tau - \omega|\,.
$$

This is Assumption 2.5. In the opposite direction, we consider the following situation.

Assumption 10.1 *Suppose that $\xi_0 = (\tau_0, \eta_0)$ and $\xi'_0 = \xi_0 + \beta = (\tau'_0, \eta'_0)$ are regular points in the characteristic variety of L and M respectively and*

$$
Q(\xi'_0)\, G\, P(\xi_0) \neq 0 \tag{10.5}
$$

Moreover, $-i\tau_0$ is the unique common eigenvalue of $A(i\eta_0)$ and $B(i\eta'_0) + i\omega$.

Our goal is to investigate the growth propeties of $e^{-t\mathcal{A}(\varepsilon,i\eta)/\varepsilon}$ for η in a small neighborhood of η_0.

a) In a neighborhood of η_0, [resp. $\eta'_0 = \eta_0 + \kappa$] there is a smooth eigenvalue of constant multiplicity $i\lambda(\eta)$ [resp. $i\mu(\eta')$] of $A(i\eta)$ [resp. $B(i\eta')$] such that $\tau_0 = -\lambda(\eta_0)$ [resp. $\tau'_0 = \tau_0 + \omega = -\mu(\eta'_0)$]. With some abuse of notation, we denote by $P(\eta)$ [resp. $Q(\eta')$] the corresponding eigenprojector.

There are smooth unitary matrices $S(\eta)$ and $T(\eta')$ such that

$$S(\eta)^{-1}A(i\eta)S(\eta) \;=\; \begin{pmatrix} A^\flat(i\eta) & 0 \\ 0 & i\lambda(\eta) \end{pmatrix}, \quad T(\eta')^{-1}B(i\eta')T(\eta') \;=\; \begin{pmatrix} i\mu(\eta') & 0 \\ 0 & B^\flat(i\eta') \end{pmatrix}. \tag{10.6}$$

Therefore

$$S(\eta)^{-1}P(\eta)S(\eta) \;=\; \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad T(\eta')^{-1}Q(\eta')T(\eta') \;=\; \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

With the notation $\eta' = \eta + \kappa$ which we use systematically in this section, $A^\flat(i\eta)$ and $B^\flat(i\eta') + i\omega$ have no common eigenvalue, $i\mu(\eta')$ is not an eigenvalue of $A^\flat(i\eta)$ and $i\lambda(\eta)$ is not an eigenvalue of $B^\flat(i\eta') + i\omega\}$. Therefore, there is a matrix $K(\eta)$ which depends smoothly on η,

$$K(\eta) = T(\eta') \begin{pmatrix} K_{11}(\eta) & 0 \\ K_{21}(\eta) & K_{22}(\eta) \end{pmatrix} S^{-1}(\eta),$$

such that

$$\begin{cases} K(\eta)A(i\eta) \;-\; (B(i\eta') + i\omega)K(\eta) \;=\; G \;-\; Q(\eta')\, G\, P(\eta), \\ Q(\eta')K(\eta)P(\eta) \;=\; 0, \end{cases} \tag{10.7}$$

Introduce

$$\mathcal{P}(\eta) \;:=\; \begin{pmatrix} S(\eta) & 0 \\ K(\eta)S(\eta) & T(\eta') \end{pmatrix}. \tag{10.8}$$

Then $T(\eta')^{-1}Q(\eta')GP(\eta)S(\eta)$ is of the form

$$T(\eta')^{-1}Q(\eta')GP(\eta)S(\eta) \;=\; \begin{pmatrix} 0 & \rho(\eta) \\ 0 & 0 \end{pmatrix} \tag{10.9}$$

and therefore, (10.7) implies that

$$\mathcal{A}_1(0,\eta) := \mathcal{P}(\eta)^{-1}\mathcal{A}(0,\eta)\mathcal{P}(\eta) \;=\; \begin{pmatrix} A^\flat(i\eta) & 0 & 0 & 0 \\ 0 & i\lambda(\eta) & 0 & 0 \\ 0 & \rho(\eta) & i\mu(\eta') + i\omega & 0 \\ 0 & 0 & 0 & B^\flat(\eta') + i\omega \end{pmatrix}. \tag{10.10}$$

Using (10 4), we see that

$$\mathcal{A}_1(\varepsilon,\eta) \;:=\; \mathcal{P}(\eta)^{-1}\,\mathcal{A}(\varepsilon,\eta)\,\mathcal{P}(\eta) \;=\; \mathcal{A}_1(0,\eta) \;+\; \varepsilon\mathcal{F}(\eta) \tag{10.11}$$

where

$$\mathcal{F}(\eta) = \mathcal{P}(\eta)^{-1} \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \mathcal{P}(\eta)$$

We compute only one element matrix of $\mathcal{F}$. In the basis where (10.10) holds, the second line of the third column is the matrix $\sigma(\eta)$ such that

$$S(\eta)^{-1}P(\eta)FQ(\eta')T(\eta') \;=\; \begin{pmatrix} 0 & 0 \\ \sigma(\eta) & 0 \end{pmatrix} \tag{10.12}$$

The matrix $\mathcal{A}_1(0,\eta)$ is block diagonal and the blocks have no common eigenvalue. Therefore, there is a smooth family of matrices $\mathcal{Q}(\varepsilon,\eta) = Id + O(\varepsilon)$ defined for ε small enough and η in a small neighborhood of η_0, such that

$$\begin{aligned} \mathcal{A}_2(\varepsilon,\eta) &:= \mathcal{Q}(\varepsilon,\eta)^{-1}\mathcal{A}_1(\varepsilon,\eta)\mathcal{Q}(\varepsilon,\eta) \\ &= \begin{pmatrix} A^\flat(\varepsilon,i\eta) & 0 & 0 & 0 \\ 0 & i\lambda(\varepsilon,\eta) & \varepsilon\sigma(\varepsilon,\eta) & 0 \\ 0 & \rho(\varepsilon,\eta) & i\mu(\varepsilon,\eta) + i\omega & 0 \\ 0 & 0 & 0 & B^\flat(\varepsilon,i\eta') + i\omega \end{pmatrix}. \end{aligned} \tag{10.13}$$

where $A^\flat(\varepsilon,i\eta)$, $\lambda(\varepsilon,\eta)$... are smooth extensions of $A^\flat(i\eta)$, $\lambda(\eta)$... .

b) We study the exponentials $e^{-t\mathcal{A}_2/\varepsilon}$. Because $A^\flat(\varepsilon, i\eta) = A^\flat(i\eta) + O(\varepsilon)$ and $A^\flat(i\eta)$ is anti-adjoint, Gronwall's Lemma implies that the matrices $e^{-tA^\flat(\varepsilon,i\eta)/\varepsilon}$ are uniformly bounded for $t \in [0,T]$, $\varepsilon \in]0, \varepsilon_0]$ and η in a small neighborhood of η_0. Similarly, $e^{-t(B^\flat(\varepsilon,i\eta)+i\omega)/\varepsilon}$ is uniformly bounded in the same range of parameters.

c) Next we evaluate the exponential of the second block in $\mathcal{A}_2$:

$$\mathcal{B}(\varepsilon,\eta) := \begin{pmatrix} i\lambda(\varepsilon,\eta) & \varepsilon\sigma(\varepsilon,\eta) \\ \rho(\varepsilon,\eta) & i\mu(\varepsilon,\eta') + i\omega \end{pmatrix} = i\lambda(\varepsilon,\eta)\, Id + \begin{pmatrix} 0 & \varepsilon\sigma(\varepsilon,\eta) \\ \rho(\varepsilon,\eta) & i\tilde\mu(\varepsilon,\eta) \end{pmatrix} := i\lambda(\varepsilon,\eta)\, Id + \mathcal{B}_1(\varepsilon,\eta)\,, \tag{10.14}$$

where $\tilde\mu(\varepsilon,\eta) = \mu(\varepsilon,\eta') + \omega - \lambda(\varepsilon,\eta)$. One has $e^{-t\mathcal{B}/\varepsilon} = e^{-it\lambda(\varepsilon,\eta)/\varepsilon} e^{-t\mathcal{B}_1/\varepsilon}$. Since $\lambda(\varepsilon,\eta) = \lambda(\eta) + O(\varepsilon)$ and $\lambda(\eta)$ is real, $e^{-it\lambda(\varepsilon,\eta)/\varepsilon}$ is uniformly bounded for $t \in [0,T]$. Thus it remains to study $e^{-t\mathcal{B}_1/\varepsilon}$.

The resonance condition $\tau_0' = \tau_0 + \omega$ implies that $\tilde\mu(0,\eta_0) = 0$ and therefore

$$\tilde\mu(\varepsilon,\eta) = O(|\eta - \eta_0)| + \varepsilon)\,. \tag{10.15}$$

One has

$$\begin{pmatrix} 1/\sqrt{\varepsilon} & 0 \\ 0 & Id \end{pmatrix} \mathcal{B}_1(\varepsilon,\eta) \begin{pmatrix} \sqrt{\varepsilon} & 0 \\ 0 & Id \end{pmatrix} = \sqrt{\varepsilon} \begin{pmatrix} 0 & \sigma(\varepsilon,\eta) \\ \rho(\varepsilon,\eta) & i\tilde\mu(\varepsilon,\eta)/\sqrt{\varepsilon} \end{pmatrix} := \sqrt{\varepsilon}\, \mathcal{B}_2(\varepsilon,\eta)\,. \tag{10.16}$$

Because of (10.15), $\tilde\mu(\varepsilon,\eta)/\sqrt{\varepsilon}$ is small when ε is small and η is restricted to a small neighborhood of η_0, i.e. when

$$|\eta - \eta_0| \le h\sqrt{\varepsilon} \tag{10.17}$$

and h is small. In this case, $\mathcal{B}_2(\varepsilon,\eta)$ is a perturbation of

$$\mathcal{B}_0 := \begin{pmatrix} 0 & \sigma(\eta_0) \\ \rho(\eta_0) & 0 \end{pmatrix} \tag{10.18}$$

and the behaviour of $e^{-t\mathcal{B}_1/\varepsilon}$ depends on the sprectrum of $\mathcal{B}_0$. Note that $\varphi \ne 0$ is an eigenvalue of $\mathcal{B}_0$ if and only if φ^2 is an eigenvalue of $\sigma(\eta_0)\rho(\eta_0)$ or equivalently of $\rho(\eta_0)\sigma(\eta_0)$, which means here that φ^2 is a nonvanishing eigenvalue of $P(\eta_0)\, FQ(\eta_0')GP(\eta_0)$ and of $Q(\eta_0')GP(\eta_0)FQ(\eta_0')$. If $\mathcal{B}_0$ has an eigenvalue φ_0 with negative real part, then for ε and h are small enough, and for η satisfying (10.17), $\mathcal{B}_2(\varepsilon,\eta)$ also has an eigenvalue φ with negative real part. In this case, $\sqrt{\varepsilon}\,\varphi$ is an eigenvalue of $\mathcal{A}(\varepsilon,\eta)$.

On the other hand, the imaginary part of $\tilde\mu(\varepsilon,\eta)$ is $O(\varepsilon)$ so that the real part of $\mathcal{B}_2$ is $O(1)$. Thus $e^{-t\mathcal{B}_2} = O(e^{Ct})$ implying that $e^{-t\mathcal{A}} = O(\varepsilon^{-1/2} e^{C\sqrt{\varepsilon}t})$.

Therefore, we have proved :

Proposition 10.2. *i) There is a constant C and a neighborhood $\mathcal{O}$ of η_0, such that for all $\varepsilon \in]0,1]$, $\eta \in \mathcal{O}$, $t \in [0,T]$*

$$\left| e^{-t\mathcal{A}(\varepsilon,\eta)/\varepsilon} \right| \le \frac{C}{\sqrt{\varepsilon}}\, e^{Ct/\sqrt{\varepsilon}}\,. \tag{10.19}$$

ii) If $Q(\eta_0')\, GP(\eta_0)FQ(\eta_0')$ has a nonreal eigenvalue or a real positive eigenvalue, then there are constants $\gamma > 0$, $\varepsilon_0 > 0$ and $h > 0$, such that for all $\varepsilon \in]0, \varepsilon_0]$, all η satisfying (10.17) *and all $t \in [0,T]$*

$$\left| e^{-t\mathcal{A}(\varepsilon,\eta)/\varepsilon} \right| \ge \gamma\, e^{\gamma t/\sqrt{\varepsilon}}\,. \tag{10.20}$$

We now give a more precise estimate to be used in the next section. Consider the following situation

Assumption 10.3. *$Q(\eta_0')GP(\eta_0)FQ(\eta_0')$ has a simple eigenvalue $\delta_0 = \varphi_0^2 \notin]-\infty, 0]$, such that the real part of the square roots of the other eigenvalues is strictly smaller than $|\mathrm{Re}\varphi_0|$.*

Examples are given in the next section. Choose the square root φ_0 of δ_0 such that its real part is positive. Assumption 10.3 means that $\pm\varphi_0$ are simple eigenvalues of $\mathcal{B}_0$ and the real part of the other eigenvalues belong to the open interval $]-\mathrm{Re}\varphi_0, \mathrm{Re}\varphi_0[$.

When Assumption 10.3 is satisfied, we denote by Π_0 the spectral projection on the eigenspace associated to δ_0.

Proposition 10.4. *Suppose that Assumptions 10.1 and 10.3 are satisfied. For all $C \geq 0$ there are positive constants ε_0, h, γ, r and c such that for all $\varepsilon \in]0, \varepsilon_0]$, all η satisfying (10.17) all $t \in [r\sqrt{\varepsilon}, T]$ and all vectors $U = (u, v)$ such that*

$$|u| \leq C\varepsilon\,|v|\,, \quad |v| \leq C\,|\Pi_0 v|\,, \tag{10.21}$$

one has

$$|e^{-t\mathcal{A}(\varepsilon,\eta)/\varepsilon}\,U\,| \geq c\,e^{\gamma t/\sqrt{\varepsilon}}\,|v|. \tag{10.22}$$

Proof **a)** Introduce $U_1 = \mathcal{Q}^{-1}(\varepsilon, \eta)\mathcal{P}^{-1}(\eta)U$. In the basis where $\mathcal{A}_2$ has the block decomposition (10.13), write $U_1 = {}^t(u^\flat, u_1, v_1, v^\flat)$. Note that (10.21) implies that for ε small enough, one has

$$u^\flat = O(\varepsilon|v|)\,, \quad u_1 = O(\varepsilon|v|)\,, \quad |v^\flat| \leq C'|v_1| \approx |v| \leq C''|\pi_0 v_1| \tag{10.23}$$

where π_0 denotes the eigenprojector of $\rho(\eta_0)\sigma(\eta_0)$ associated to the eigenvalue $-\varphi_0$.

Introduce $V := {}^t(u_1, v_1)$. For $t \in [0, T]$, one has

$$|e^{-t\mathcal{A}(\varepsilon,\eta)/\varepsilon}U| \approx |e^{-t\mathcal{A}_2(\varepsilon,\eta)/\varepsilon}U_1| \geq |e^{-t\mathcal{B}(\varepsilon,\eta)/\varepsilon}V| \approx |e^{-t\mathcal{B}_1(\varepsilon,\eta)/\varepsilon}V|\,. \tag{10.24}$$

Thus it is sufficient to give a lower bound for the last term.

b) Assumption 10.3 implies that $-\varphi_0$ is a simple eigenvalue of $\mathcal{B}_0$. The eigenprojector is

$$\Sigma_0 = \frac{1}{2\varphi_0}\begin{pmatrix} \varphi_0\pi_1 & -\sigma(\eta_0)\pi_0 \\ -\rho(\eta_0)\pi_1 & \varphi_0\pi_0 \end{pmatrix} \tag{10.25}$$

where π_0 and π_1 are the eigenprojectors of $\rho(\eta_0)\sigma(\eta_0)$ and $\sigma(\eta_0)\rho(\eta_0)$ respectively, associated to the eigenvalue φ_0^2. They satisfy $\sigma(\eta_0)\pi_0 = \pi_1\sigma(\eta_0)$ and $\pi_0\rho(\eta_0) = \rho(\eta_0)\pi_1$ so that the matrix in (10.25) defines a projector, which commutes with $\mathcal{B}_0$.

For ε and h small enough and for η satisfying (10.17), $\mathcal{B}_2(\varepsilon, \eta)$ has a simple eigenvalue, $-\varphi(\varepsilon, \eta) = -\varphi_0 + O(h/\sqrt{\varepsilon})$ and the eigenprojector $\Sigma(\varepsilon, \eta)$ satisfies $\Sigma(\varepsilon, \eta) = \Sigma_0 + O(h + \sqrt{\varepsilon})$. Thus

$$\mathcal{B}_2(\varepsilon, \eta) = -\varphi(\varepsilon, \eta)\,\Sigma(\varepsilon, \eta) + \mathcal{B}_2^\flat(\varepsilon, \eta) \tag{10.26}$$

where $\mathcal{B}_2^\flat(\varepsilon, \eta) = \mathcal{B}_0^\flat + O(h + \sqrt{\varepsilon})$. Assumption 10.3 implies that all the eigenvalues of $\mathcal{B}_0^\flat$ have real part strictly smaller than $\mathrm{Re}\varphi_0$. Therefore, if ε_0 and h are small enough, there are $\gamma > \gamma' \geq 0$ and C such that for η satisfying (10.17), one has

$$\begin{cases} \mathrm{Re}\,\varphi(\varepsilon, \eta) \geq \gamma\,, \\ \forall t \geq 0\,, \quad |\,e^{-t\mathcal{B}_2^\flat(\varepsilon,\eta)}| \leq C\,e^{t\gamma'}\,. \end{cases} \tag{10.27}$$

c) Next, introduce

$$\widetilde{V} = \begin{pmatrix} \tilde{u}_1 \\ v_1 \end{pmatrix} := \begin{pmatrix} 1/\sqrt{\varepsilon} & 0 \\ 0 & Id \end{pmatrix} V = \begin{pmatrix} u_1/\sqrt{\varepsilon} \\ v_1 \end{pmatrix} . \tag{10.28}$$

The identity (10.16) and the spectral decomposition (10.26) imply that

$$\begin{aligned} e^{-t\mathcal{B}_1(\varepsilon,\eta)/\varepsilon} V &= \begin{pmatrix} 1/\sqrt{\varepsilon} & 0 \\ 0 & Id \end{pmatrix} e^{-t\mathcal{B}_2(\varepsilon,\eta)/\sqrt{\varepsilon}}\, \widetilde{V} , \\ e^{-t\mathcal{B}_2(\varepsilon,\eta)/\sqrt{\varepsilon}}\, \widetilde{V} &= e^{t\varphi(\varepsilon,\eta)/\sqrt{\varepsilon}}\, \Sigma(\varepsilon,\eta)\, \widetilde{V} + e^{-t\mathcal{B}_2^\flat(\varepsilon,\eta)/\sqrt{\varepsilon}}\, \widetilde{V} . \end{aligned} \tag{10.29}$$

Using (10.23) and (10.25), we see that if h and ε are small enough,

$$\Sigma(\varepsilon,\eta)\, \widetilde{V} = \frac{1}{2\varphi_0} \begin{pmatrix} -\sigma(\eta_0)\pi_0 v_1 \\ \varphi_0 \pi_0 v_1 \end{pmatrix} + O\big((h+\sqrt{\varepsilon})|v|\big) .$$

With (10.27) it follows that

$$|e^{-t\mathcal{B}_1(\varepsilon,\eta)/\varepsilon} V| \ge \frac{1}{2} e^{\gamma t \sqrt{\varepsilon}} \left(|\pi_0 v_1| - O\big((h+\sqrt{\varepsilon})|v|\big) \right) - O\big(e^{\gamma' t/2\sqrt{\varepsilon}}|v|\big) . \tag{10.30}$$

With (10.24) and recalling that $\gamma > \gamma' \ge 0$, the proposition follows.

Remarks 10.5 1) The second assumption in (10.21) means that the component of v in the crucial direction does not vanish and dominates the length of v.

2) The proof uses a weaker version of (10.17). What is needed is that

$$|\lambda(\eta) - \mu(\eta+\kappa) - \omega| \le h\sqrt{\varepsilon} . \tag{10.31}$$

In particular, when the resonance is regular, i.e. when $\nabla\lambda(\eta_0) \ne \nabla\mu(\eta_0')$ the set of resonances $\{\eta \,|\, \lambda(\eta) = \mu(\eta+\kappa) + \omega\}$ is a smooth manifold near η_0 and (10.31) is the set of points whose distance to the resonance manifold is less than $h\sqrt{\varepsilon}$. Thus, all the frequencies in this $\sqrt{\varepsilon}$-neighborhood of the resonance manifold are amplified.

11. An example of instability

Consider a system of the form

$$\begin{cases} L_1(\varepsilon\partial_x)\, u_1 = 0 , \\ L_2(\varepsilon\partial_x) u_2 + \varepsilon f(\overline{u_1}, v) = 0 , \\ M(\varepsilon\partial_x) v + q(u_1,u_2) = 0 , \end{cases} \tag{11.1}$$

where L_1, L_2 and M are symmetric hyperbolic systems of the form (2.2) and f and g are bilinear forms. Taking real an imaginary parts of the unknows yields a real system. With notations as in (2.1), one has

$$q(u,u') = \frac{1}{2} g(u_1,u_2') + \frac{1}{2} g(u_1',u_2) \quad \text{for} \quad u = (u_1,u_2)\,, u' = (u_1',u_2'). \tag{11.2}$$

For simplicity, we further assume that M is homogeneous, i.e. $M(\varepsilon\partial_x) = \varepsilon M(\partial_x)$.

We suppose that the wave number $\beta = (\omega,\kappa) \in \mathbb{R}\times\mathbb{R}^d$ is so chosen that

$$\det L_1(i\beta) = 0 , \quad \det L_2(-i\beta) \ne 0 , \quad \det M(i\beta) \ne 0 . \tag{11.3}$$

Because M is homogeneous, all the projectors $Q(\nu\beta)$ vanish, except when $\nu = 0$ and $Q(0) = Id$.

EXAMPLE 11.1. In space dimension $d = 1$ consider the 3×3 system

$$\begin{cases} (\partial_t - \partial_y)\, u_1 \;=\; 0\,, \\ (\partial_t - 2\partial_y) u_2 \;+\; \delta\, \overline{u_1}\, v \;=\; 0\,, \\ \varepsilon \partial_t v \;+\; u_1\, u_2 \;=\; 0\,, \end{cases} \tag{11.4}$$

and choose $\beta = (1,1)$. In the second equation, δ is a parameter.

EXAMPLE 11.2. In space dimension $d = 1$, consider $\beta = (\sqrt{2}, 1)$ and

$$\begin{aligned} &L_1(\varepsilon\partial_x) = \begin{pmatrix} \varepsilon(\partial_t - \partial_y) & 1 \\ -1 & \varepsilon(\partial_t + \partial_y) \end{pmatrix}, \quad L_2(\varepsilon\partial_x) = \begin{pmatrix} \varepsilon(\partial_t - \partial_y) & 1/2 \\ -1/2 & \varepsilon(\partial_t + \partial_y) \end{pmatrix}, \\ &M(\varepsilon\partial_x) = \varepsilon\partial_t\,. \end{aligned} \tag{11.5}$$

The conditions (2.7) are satisfied and

$$\det L_1(i\nu\beta) = 0 \;\Leftrightarrow\; \nu = \pm 1 \quad \text{and} \quad \forall \nu \in \mathbb{Z}\,,\ \det L_2(i\nu\beta) \neq 0\,. \tag{11.6}$$

Moreover,

$$\det M(i\nu\beta) \;=\; 0 \;\Leftrightarrow\; \nu = 0\,. \tag{11.7}$$

Near $\pm\beta$, the characteristic variety of (L_1, L_2) coincides with the characteristic variety of L_1 and there, the spectral projector $P(\xi)$ projects onto a subspace of $\{u_2 = 0\}$. With (11.2) this implies that $q(P(-\nu\beta)u\,,\, P(\xi)u') = 0$. Thus (2.18) is satisfied for ξ close to $\pm\beta$ and Proposition 7.2 and Remark 7.3 imply that Assumptions 2.1 and 2.2 are satisfied.

Consider a family $\{u_1^\varepsilon\}_{\varepsilon\in]0,1]}$ of exact solutions of the linear equation $L_1(\varepsilon\partial_x)u_1^\varepsilon = 0$, such that

$$u_1^\varepsilon(x) \;=\; a(x,\varepsilon)\, e^{i\beta\cdot x/\varepsilon}\,, \quad a(x,\varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n a_n(x) \tag{11.8}$$

and $a^\varepsilon \in H^\infty([0,T] \times \mathbb{R}^d \times [0,1])$ (see e.g.[Lax]). When the initial data of u_2 and v vanish, the solution of (11.1) is $(u_1^\varepsilon, 0, 0)$. We now discuss the stability of these solutions. We consider (small) perturbations of the initial data and show :

1) there are uniformly bounded families of formal and approximate solutions,

2) the exact solutions wih the same Cauchy data may diverge exponentially, like $e^{\gamma t/\sqrt{\varepsilon}}$, from the approximate solution.

Formal solutions

The system (11.1) reduces to a linear system (10.1) for (u_2, v). Following the general theory in §4, one can construct formal solutions of (11.1). But, thanks to the special form of the interaction, one can look for solutions

$$u_2^\varepsilon(x) \sim \Big(\sum_{n=1}^{\infty} \varepsilon^n b_n(x)\Big) e^{-i\beta\cdot x/\varepsilon}\,, \quad v^\varepsilon(x) \sim \sum_{n=0}^{\infty} \varepsilon^n v_n(x)\,. \tag{11.9}$$

The b_n are defined for $n \geq 1$ by

$$b_{n+1} \;=\; -\,L_2(-i\beta)^{-1}\Big(\sum_{k=0}^{n} f(\overline{a_k}, v_{n-k}) \;+\; L_{2,1}(\partial_x) b_n\Big) \tag{11.10}$$

where $L_{2,1}$ denotes the first order part of L_2. By induction this implies that

$$b_{n+1} \;=\; -\,L_2(-i\beta)^{-1} f(\overline{a_0}, v_n) \;+\; \varphi_{n-1} \tag{11.11}$$

where φ_{n-1} depends only on $(v_0, \ldots, v_{n-1})$ and their derivatives. The v_n satisfy

$$M(\partial_x) v_n \;+\; \sum_{k=0}^{n} g(a_k, b_{n+1-k}) \;=\; 0\,. \tag{11.12}$$

Substituting (11.11) in (11.12) yields a symmetric hyperbolic linear equation for v_n with source term depending only on $(v_0, \ldots, v_{n-1})$. Therefore

Proposition 11.3. *For any sequence v_n^0 in $H^\infty(\mathbb{R}^d)$, the system* (11.1) *has a unique formal solution $(u_1^\varepsilon, u_2^\varepsilon, v^\varepsilon)$, $(u_2^\varepsilon, v^\varepsilon)$ given by* (11.9), *with coefficients b_n and v_n in $H^\infty([0,T] \times \mathbb{R}^d)$ and such that $v_{n|t=0} = v_n^0$.*

Note that $v_n = 0$ for $n < p$ and $b_n = 0$ for $n \leq p$ when the Cauchy data v_n^0 vanish for $n < p$. Given a formal solution, consider

$$u_{2,app}^\varepsilon = \sum_{n=1}^{k+1} \varepsilon^k b_n \, e^{-i\beta\cdot x/\varepsilon}, \quad v_{app}^\varepsilon = \sum_{n=0}^{k} \varepsilon^k v_n \;. \tag{11.13}$$

They are approximate solutions of (11.1), meaning that

$$e^{i\beta\cdot x/\varepsilon}\Big(L_2(\varepsilon\partial_x) u_{2,app}^\varepsilon \;+\; \varepsilon f(\overline{u}_1^\varepsilon, v_{app}^\varepsilon)\Big) \quad \text{and} \quad \varepsilon M(\partial_x) v_{app}^\varepsilon \;+\; g(u_1^\varepsilon, u_{2,app}^\varepsilon)$$

are $O(\varepsilon^{k+2})$ in $H^\infty([0,T] \times \mathbb{R}^d)$.

Exact solutions

Given an approximate solution (11.13) we now consider the family of exact solutions of (11.1) which have the same Cauchy data. The existence is clear, since the problem is linear in $(u_2^\varepsilon, v^\varepsilon)$. The question is to know how long the exact solution remains close to the approximate solution when Assumption 2.5 is violated.

To simplify the analysis, we make several choices for u_1^ε and the data v_n^0. Consider a bounded open set $\Omega^0 \subset \mathbb{R}^d$, and let Ω be the domain of influence of Ω^0 in $[0,T] \times \mathbb{R}^d$ for the system $(L_2(\varepsilon\partial_x), M(\varepsilon\partial_x))$. Consider a constant vector $\underline{a}$ such that $P_1(\beta)\underline{a} = \underline{a}$, where $P_1(\beta)$ denotes the orthogonal projector on $\ker L_1(i\beta)$. Note that $\underline{a} e^{i\beta\cdot x/\varepsilon}$ is an exact plane wave solution of $L_1(\varepsilon\partial_x)u = 0$. The classical theory ([Lax]) shows that one can construct the family u_1^ε in (11.8) so that

$$u_1^\varepsilon = \underline{a} e^{i\beta\cdot x/\varepsilon} \quad \text{on } \Omega \tag{11.14}$$

Next, we choose initial data $v_0^0 \in C_0^\infty(\Omega^0)$ and $v_n^0 = 0$ when $n > 0$. The equations (11.11) (11.12) show that there are differential operators $\mathcal{D}_n(\partial_y)$ of order n such that

$$b_{n+1}(0, .) = \mathcal{D}_n(\partial_y)\, v_0^0 \tag{11.15}$$

Note that $\mathcal{D}$ has constant coefficients, since a is constant on a neighborhood of $\{0\} \times \Omega^0$. In particular, the initial values of b_n are supported in Ω^0. Therefore the initial values of u_2^ε and v^ε which are equal to the initial values of $u_{2,app}^\varepsilon$ and v_{app}^ε are supported in Ω^0. Thus, the exact solutions u_2^ε and v^ε are supported in Ω and therefore they satisfy

$$\begin{cases} L_2(\varepsilon\partial_x)u_2^\varepsilon + \varepsilon f(\overline{u_0^\varepsilon}, v^\varepsilon) = 0\,, \\ M(\varepsilon\partial_x)v^\varepsilon + q(u_0^\varepsilon, u_2^\varepsilon) = 0\,, \end{cases} \qquad u_0^\varepsilon = \underline{a}e^{i\beta\cdot x/\varepsilon}\,. \tag{11.16}$$

Therefore, we are in position to apply the results of §10. Recall the following notation : F [resp. G] is the matrix such that $f(\underline{\overline{a}}, v) = Fv$ [resp. $g(\underline{a}, u) = Gu$]. We suppose that Assumptions 10.1 and 10.3 are satisfied at $\xi_0 = (\tau_0, \eta_0)$ for the system (L_2, M). We denote by Π_0 the spectral projection introduced after Assumption 10.3.

In addition to the previous choices for the initial data, assume that the Fourier transform of v_0^0 satisfies

$$\exists C\,,\ \forall\eta\,, \quad |\widehat{v_0^0}(\eta)| \leq C\,|\Pi_0 F\widehat{v_0^0}(\eta)|\,. \tag{11.17}$$

$$\exists s < 1/2\,,\ \exists c_1 > 0\,,\ \forall\eta\,, \quad |\widehat{v_0^0}(\eta)| \geq \gamma_1\, e^{-\gamma_2|\eta|^s}\,. \tag{11.18}$$

Recall that Π_0 is the projector introduced afer Assumption 10.3. Hence, the first condition is a polarization condition, ensuring that the unstable mode is activated. The second condition, is a "nonGevrey 2" condition. It is related to the rate of growth $e^{\gamma/\sqrt{\varepsilon}}$ of unstable frequencies of size $\approx 1/\varepsilon$.

Exemples are given after the proof of the next theorem.

Theorem 11.4. *Suppose that Assumptions* 10.1 *and* 10.3 *are satisfied and the initial data are chosen as indicated above. Then there are* $c > 0$, $\gamma > 0$ *and* $C > 0$ *such that*

i) the approximate solutions (11.13) *are uniformly bounded on* $[0, T] \times \mathbb{R}^d$ *and compactly supported,*

ii) for ε *small enough, the exact solutions with the same initial data satisfy*

$$\|U^\varepsilon(t)\|_{L^2} \geq c\, e^{\gamma t/2\sqrt{\varepsilon}}\,, \quad \text{when} \quad t \in [C\,\varepsilon^{\frac{1}{2}-s}, T]\,. \tag{11.19}$$

Proof.

a) Recall that the coefficients b_n of the formal solution are given by (11.15). In particular,

$$|\widehat{b_{n+1}}(0, \eta)\,| \leq C_n(1 + |\eta|)^n |\widehat{v_0^0}(\eta)|\,.$$

Therefore, (11.13) implies that

$$|\widehat{u_2^\varepsilon}(0,\eta)| \;=\; |\widehat{u_{2,app}^\varepsilon}(0,\eta)| \;\le\; C\,\varepsilon\,(1+\varepsilon|\eta+\kappa/\varepsilon|)^k\,|\widehat{v_0^0}(\eta+\kappa)|\,. \tag{11.20}$$

As in (10.3) introduce $\tilde v^\varepsilon := v^\varepsilon e^{-i\beta\cdot x/\varepsilon}$ and its Fourier transform $\widehat{\tilde v^\varepsilon}(t,\eta) = e^{-it\omega/\varepsilon}\widehat{v^\varepsilon}(t,\eta+\kappa/\varepsilon)$. Let $\mathcal{A}(\varepsilon,\varepsilon\partial_y)$ denote the matrix (10.4) associated to the system (11.16) satisfied by $(u_2^\varepsilon, v^\varepsilon)$. Then

$$\begin{pmatrix} \widehat{u_2^\varepsilon}(t,\eta) \\ e^{-it\omega/\varepsilon}\widehat{v^\varepsilon}(t,\eta+\kappa/\varepsilon)\end{pmatrix} \;=\; e^{-it\mathcal{A}(\varepsilon,\varepsilon\eta)/\varepsilon}\begin{pmatrix} \widehat{u_2^\varepsilon}(0,\eta) \\ \widehat{v^\varepsilon}(0,\eta+\kappa/\varepsilon)\end{pmatrix} \tag{11.21}$$

In addition, note that $v^\varepsilon(0,\,.\,) = v_{app}^\varepsilon(0,\,.\,) = v_0^0$.

b) Let ε_0, h, γ, c and r denote the constants given by Proposition 10.4. For $\varepsilon\le\varepsilon_0$ and η in the ball

$$|\varepsilon\eta-\eta_0| \;\le\; h\sqrt{\varepsilon} \tag{11.22}$$

one has $|\eta| = O(1/\varepsilon)$. Thus (11.20) and (11.17) imply that the assumption (10.21) in Proposition 10.4 is satisfied. Therefore, for $t\ge r\sqrt{\varepsilon}$ and η satisfying (11.22), the estimate (10.22) in Proposition 10.4 applies to the frequency $\varepsilon\eta$. With (11.21) this implies that

$$|\widehat{U}(t,\eta)| \;\ge\; c_1\, e^{\gamma t\sqrt{\varepsilon}}\,|\widehat{v_0^0}(\eta+\kappa/\varepsilon)|^2\, d\eta\,. \tag{11.23}$$

Integrating over the ball (11.22), and using the assumption (11.18), yields

$$\|U^\varepsilon(t)\|_{L^2} \;\ge\; c_2\, e^{\gamma t/\sqrt{\varepsilon}-\gamma'/\varepsilon^s}\,, \tag{11.24}$$

and the Theorem follows.

Remarks 11.5. Let $P_2(\xi_0)$ denote the orthogonal projector on $\ker L_2(\xi_0)$.

1) When v is one dimensional, $Q(\xi_0')GP_2(\xi_0)FQ(\xi_0')$ is the multiplication by a scalar δ. In this case, Assumption 10.3 reduces to the condition $\delta\notin]-\infty,0]$. Morever, the polarization condition (11.17) is trivially satisfied.

2) When the eigenvalue $\lambda(\eta)$ [resp $\mu(\eta')$] is simple, $P_2(\xi_0)$ [resp. $Q(\xi_0')$] is a rank one projector. Thus there is $\delta\in\mathbb{C}$ such that $P_2(\xi_0)FQ(\xi_0')GP_2(\xi_0) = \delta P_2(\xi_0)$ [resp $Q(\xi_0')G_2P(\xi_0)FQ(\xi_0') = \delta Q(\xi_0')$]. In this case also, Assumption 10.3 reduces to the condition $\delta\notin]-\infty,0]$.

3) The condition (11.17) is satisfied when v_0^0 takes its values in a space which does not intersect the kernel of the operator $Q(\xi_0')GP_2(\xi_0)FQ(\xi_0')$.

Examples 11.6 1) Consider the system (11.4) of Example 11.1. In this case, $\beta=(1,1)$, and there is a unique $\xi_0=(-1,-1/2)$ which is characteristic for $\partial_t-2\partial_y$ and such that $\xi_0+\beta$ is characteristic for ∂_t. The matrix $\mathcal{A}$ is the 2×2 matrix

$$\mathcal{A}(\varepsilon,\eta) = \begin{pmatrix} -2i\eta & \varepsilon\delta\overline{a} \\ a & i\end{pmatrix}.$$

G and F are 1×1 matrices, equal to a and $\overline{a}$ respectively. Moreover $Q(\xi_0')GP(\xi_0)FQ(\xi_0') = \delta|a|^2$. Thus Assumption 10.3 is satisfied if and only if $\delta\notin]-\infty,0]$.

The exponential of $\mathcal{A}$ can be computed explicitely. For example, when δ is a positive real number, the eigenvalues are purely imaginary when $|\eta+1/2| > \sqrt{\varepsilon}\sqrt{\delta}|a|$ but there are real positive and negative eigenvalues, $\sqrt{\varepsilon}\varphi_\pm(\varepsilon,\eta)$, when $|\eta+1/2| < \sqrt{\varepsilon}\sqrt{\delta}|a|$. The analysis in §10 is a simple extension of this particular case.

2) Consider the operators L_1, L_2 and M (11.5) of Example 11.2 and $\beta = (\sqrt{2}, 1)$. There are two resonances $(-\sqrt{2}, \pm\sqrt{7}/2)$. The eigenvalues of L_1 and L_2 are simple and Assumption 10.1 is satisfied. Moreover, we are in the situation of Remark 10.5 2, $Q(\xi_0') = Id$ and $P(\xi_0)$ is a rank one projector. Let ℓ be a nonvanishing vector in the kernel of $L_2(\xi_0)$. There is $\delta \in \mathbb{C}$ such that

$$P(\xi_0)\, f(\overline{a}, g(a, \ell)) \;=\; \delta \ell.$$

Assumption 10.3 is satisfied exactly when $\delta \notin]-\infty, 0]$. This condition is met by a suitable choice of f and g.

Remark 11.7. In space dimension one, squaring and integrating (11.23) yields

$$\| U^\varepsilon(t) \|_{L^2}^2 \;\geq\; c_1\, e^{t\gamma/\sqrt{\varepsilon}} \int_{\eta_0'/\varepsilon - h/\sqrt{\varepsilon}}^{\eta_0'/\varepsilon + h/\sqrt{\varepsilon}} |\, \widehat{v_0^0}(\eta)\, |^2\, d\eta \tag{11.25}$$

with $\eta_0' = \eta_0 + \kappa$.

a) If $\eta_0' = 0$ and $v_0^0 \neq 0$, this implies that for ε small enough and $t \geq r\sqrt{\varepsilon}$ the L^2 norm of the exact solution U^ε is larger than $ce^{\gamma t/\sqrt{\varepsilon}}$.

b) If $\eta_0' \neq 0$, (11.25) implies that if the family $U^\varepsilon(t)$ is uniformly bounded in L^2 for $t \in [0, T]$ and $\varepsilon \leq \varepsilon_0$, then the integral

$$\int_{-\infty}^{+\infty} e^{\gamma' \sqrt{|\eta|}}\;\; |\widehat{v_0^0}(\eta)|^2\, d\eta \;<\; \infty$$

for some $\gamma' > 0$. This means that v_0^0 belongs to the Gevrey class G^2. Conversely, using the the upper bound (10.19) in Proposition 10.2, one can show that this condition implies that U^ε is uniformly bounded for small times.

Remark 11.8. For the sake of completeness, we check that the set of functions $v_0^0 \in C_0^\infty(\mathbb{R})$ which statisfy (11.18) is not empty. Introduce $\chi(y)$ the inverse Fourier transform of $e^{-(1+|\eta|^2)^{s/2}}$. This function belongs to the Gevrey class $G^{1/s}$ and therefore is C^∞. Consider next $\chi_1 \neq 0$ a real and even C^∞ function with compact support. Its Fourier transform $\widehat{\chi_1}$ is real and even. Thus $\chi_2 = \chi_1 * \chi_1 \in C_0^\infty$ and $\widehat{\chi_2}$ is real and nonnegative. Consider $v = \chi\chi_2$. It is C^∞ and Gevrey $G^{1/s}$ if we choose χ_1 in a Gevrey class G^a with $1 < a \leq 1/s$. The support of v is compact, and

$$\widehat{v}(\eta) \;\geq\; \int_{-1}^{1} e^{-|\eta - \zeta|^s}\;\; \widehat{\chi_2}(\zeta)\, d\zeta$$

Because $\widehat{\chi_2}$ is real analytic, nonnegative and does not vanish identically, its integral over $[-1, 1]$ is positive. This implies that $\widehat{v}$ satisfies (11.18).

12. Maxwell-Bloch equations

The Maxwell-Bloch equations present a theoretical background for the description of the interaction between light and matter, see e.g. [BW], [NM], [Bo] or [PP]. The electromagnetic field satisfies

$$\begin{cases} \partial_t B \;+\; \mathrm{curl} E = \; 0\,, & \mathrm{div} B \;=\; 0, \\ \partial_t E \;-\; \mathrm{curl} B = \; -\partial_t P\,, & \mathrm{div}(E + P) \;=\; 0, \end{cases} \tag{12.1}$$

where P is the polarization of the medium. The divergence equations are propagated in time from the initial conditions, and we forget them competely in the discussion below. Bloch's equations link P and the electronic state of the medium which is described through a simplified quantum model. The formalism of density matrices is convenient to account for statistical averagings due for instance to the large number of atoms. The density matrix ρ satisifies

$$i\varepsilon\,\partial_t\rho \;=\; [\Omega,\rho] \;-\; [V(E,B),\rho]\,, \tag{12.2}$$

where Ω is the electronic Hamiltonian in absence of external field and $V(E,B)$ is the potential induced by the external electromagnetic field. For weak fields, V is expanded into its Taylors's series (see e.g. [PP]). In the dipole approximation,

$$V(E,B) \;=\; E\cdot\Gamma\,, \quad P \;= \mathrm{tr}(\Gamma\rho) \tag{12.3}$$

where $-\Gamma$ is the dipole moment operator. An important simplification is that only a finite number of eigenstates of Ω are retained. From the physical point of view, they are associated to the electronic levels which are excited by the electronomagnetic field. In this case, ρ is a complex finite dimensional $N\times N$ matrix and Γ is a hermitian symmetric $N\times N$ matrix with entries in $\mathbb{C}^3$. In physics books, this reduction is captured by introducing phenomenological damping terms which would force the density matrix to relax towards a thermodynamical equilibrium in absence of the external field. For simplicity, we do not consider here the damping terms. The large ones only contribute to reduce the size of the effective system and the small ones contribute to perturbations which do not alter qualitatively the phenomena. Physics books also introduce "local field corrections" to improve the model and take into account the electromagnetic field created by the electrons. This mainly results in changing the values of several constants which is of no importance in our discussion.

The parameter ε in front of ∂_t in (12.2) plays a crucial role in the statement of the problem. It intervenes at three different places. First it makes our problem fall into the category of "dispersive" equations (1.2) (see[Do], [DR])

$$\mathcal{L}(\varepsilon\partial_x)\,U \;=\; F(U)\,. \tag{12.4}$$

Next, the quantities $\omega_{j,k}/\varepsilon := (\omega_j-\omega_k)/\varepsilon$, where the ω_j are the eigenvalues of Ω, have an important physical meaning. They are the characterisitic frequencies of the electronic transitions from the level k to the level j and therefore related to the energies of these transitions. The interaction between light and matter is understood as a resonance phenomenon and the possibility of excitation of electrons by the field. This means that the energies of the electronic transitions are comparable to the energy of photons. Thus, if one chooses to normalize $\Omega\approx 1$ as we now assume, ε is comparable to the pulsation of light. The choice of units which is tacitely used in (12.1) fixes the speed of light in vacuum to be equal to one. Thus, in these units, ε is also comparable to the wavelength of light. Therefore, this means that in (12.4), we are interested in oscillatory solutions of wavelength of order ε. At last, the Maxwell-Bloch model described above, is expected to be correct for weak fields and small perturbations of a steady state, which can be for instance either the ground state or a thermodynamical equilibrium. This means that in (12.4) we consider small perturbations of a constant solution $\underline{U}$. How small they are with respect to ε is a very important part of the discussion. This is the third occurence of ε in the statement of the problem. Summing up, we look for solutions of the form (1.3)

$$U^\varepsilon(x) \;\sim\; \underline{U} \,+\, \varepsilon^p\sum_{n\geq 0}\varepsilon^{np}\,\mathbf{U}_n(x,\beta\cdot x/\varepsilon) \tag{12.5}$$

where the $\mathbf{U}_n(x,\theta)$ are periodic functions of θ and β is a given space-time wave number, or a finite set of wave numbers if one considers interacting waves.

In applications, an important point is the notion of parity for the eigenstates of Ω. For instance, consider the Hamiltonian associated to one electron. It acts in a space $L^2(\mathbb{R}^3)$ and the eigenvectors of Ω are functions φ_j of variables $\tilde{y} \in \mathbb{R}^3$. The dipole moment operator is $-e\,\tilde{y}$ where e is the electric charge of the electron and thus the entries of Γ in the basis $\{\varphi_j\}$ are

$$\Gamma_{j,k} \;=\; \int_{\mathbb{R}^3} \tilde{y}\,\varphi_j(\tilde{y})\overline{\varphi_k(\tilde{y})}\,d\tilde{y}\,. \tag{12.6}$$

(see [PP]). For physical reasons, Ω may have symmetries which may imply that some of the coefficients $\Gamma_{j,k}$ vanish. For instance, Ω is often invariant under the change $\tilde{y} \mapsto -\tilde{y}$, and the eigenstates φ_j can often been choosen to have a definite parity, i.e. to be either odd or even. In this case, all the coefficients $\Gamma_{j,k}$ associated to states φ_j and φ_k of the same parity vanish.

Next, we note that equation (12.2) has the important consequence that the spectrum of $\rho(t,y)$ is time independent. This implies the existence of many conserved quantities, such as the trace and the determinant of ρ. This means that among the entries of the density matrix, the actual number of unknowns is much less than it could seem. This remark is crucial to understand the special properties of the Maxwell-Bloch equations.

Examples.

Example 12.1. *The one space dimension equations.*

They describe a one dimensional propagation, say along the first axis. E is assumed to have a constant direction, orthogonal to the direction of propagation. B is perpendicular both to E and the axis of propagation. P is parallel to E. This leads to the following set of equations

$$\begin{cases} \partial_t b + \partial_y e \;=\; 0\,, \\ \partial_t e + \partial_y b \;=\; -\partial_t \mathrm{tr}(\Gamma\rho)\,, \\ i\varepsilon\partial_t \rho \;=\; [\Omega,\rho] \;-\; e\,[\Gamma,\rho]\,, \end{cases} \tag{12.7}$$

where e and b take their value in $\mathbb{R}$ and Γ is a Hermitian symmetric matrix, with entries in $\mathbb{C}$. In this case, note that $\mathrm{tr}(\Gamma[\Gamma,\rho]) = 0$ and

$$\varepsilon\partial_t \mathrm{tr}(\Gamma\rho) \;=\; \frac{1}{i}\,\mathrm{tr}(\Gamma[\Omega,\rho])\,. \tag{12.8}$$

Example 12.2. *The two levels equations in space dimension one.*

In this case, ρ and Γ are 2×2 hermitian symmetric matrices. We chose a basis such that Ω is diagonal whith entries $\omega_1 < \omega_2$. It is convenient to introduce

$$p \;:=\; \gamma_{1,2}\rho_{2,1} \;+\gamma_{2,1}\rho_{1,2}\,, \quad q \;:=\; \frac{1}{i}\,(\gamma_{1,2}\rho_{2,1} \;-\gamma_{2,1}\rho_{1,2})\,, \quad n \;:=\; \rho_{1,1} \;-\; \rho_{2,2}\,. \tag{12.9}$$

Using (12.8), (12.7) implies

$$\begin{cases} \varepsilon\partial_t b + \varepsilon\partial_y e \;=\; 0\,, \\ \varepsilon\partial_t e + \varepsilon\partial_y b \;=\; -\,\omega_{2,1} q\,, \\ \varepsilon\partial_t p \;=\; \omega_{2,1} q \;+\; (\gamma_{1,1} - \gamma_{2,2})\,e\,q\,, \\ \varepsilon\partial_t q \;=\; -\omega_{2,1} p \;+\; (\gamma_{2,2} - \gamma_{1,1})\,p\,e \;+\; 2|\gamma_{2.1}|^2\,n\,e\,, \\ \varepsilon\partial_t n \;=\; -2\,e\,q\,. \end{cases} \tag{12.10}$$

Conversely, (12.9) together with the condition $\rho_{1,1}+\rho_{2,2}=1$, which is natural thanks to the general conservation $\partial_t \mathrm{tr}\rho = 0$, allow to recover ρ from (p,q,n), and (12.10) implies (12.7).

When the states 1 and 2 have definite parities, then $\gamma_{1,1}=\gamma_{2,2}=0$. When the parities are different, $\gamma_{2,1}$ is not equal to zero in general. When the states 1 and 2 have definite parities, or more generally, when $\gamma_{1,1}-\gamma_{2,2}=0$, the equations take the simpler form :

$$
\begin{cases} \partial_t b + \partial_y e \;=\; 0\,, & \partial_t e + \partial_y b \;=\; -\partial_t p \\ \varepsilon^2\,\partial_t^2 p \;+\; (\omega_{2,1})^2 p = \;2\omega_{2,1}|\gamma_{2,1}|^2\; n\,e\,, & \partial_t n = \;-\;\dfrac{2}{\omega_{2,1}}\partial_t p e\,. \end{cases} \tag{12.11}
$$

Note that the equations (1.1) are the rotation invariant extension of equations (12.11) to $\mathbb{R}^3$.

Example 12.3 *Other isotropic two levels models.*

Consider 4×4 matrices Ω and Γ such that

$$
\Omega = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 Id_{3\times 3} \end{pmatrix}\,, \quad E\cdot\Gamma = \begin{pmatrix} 0 & \gamma^t E \\ \overline{\gamma} E & i\,\delta E\times\cdot \end{pmatrix}\,, \tag{12.12}
$$

where the blocks correspond to a splitting $\mathbb{C}\times\mathbb{C}^3$ of the space. The Hamiltonian Ω has two eigenvalues and we suppose that $\omega_1<\omega_2$. This means that the ground state is simple and that the excited level has multiplicity 3. The group of rotations $SO(3)$ acts on $\mathbb{R}^3$ and $\mathbb{C}^3$, and thus on the electric and magnetic fields E and B. For $R\in SO(3)$, define

$$
\widetilde{R} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}\,.
$$

Then the matrices Ω and Γ satisfy

$$
\Omega\widetilde{R} \;=\; \widetilde{R}\Omega\,, \quad (RE)\cdot\Gamma\,\widetilde{R} \;=\; \widetilde{R}(E\cdot\Gamma), \quad \mathrm{tr}(\Gamma\widetilde{R}\rho\widetilde{R}^{-1}) \;=\; R\mathrm{tr}(\Gamma\rho)\,.
$$

Therefore, the equations (12.1) (12.2) are invariant under the change of unknows $(E,B,\rho)\mapsto (RE,RB,\widetilde{R}\rho\widetilde{R}^{-1})$, which means that they are invariant by rotation and thus isotropic.

Example 12.4. *A model for Raman scattering.*

We now give a classical model which is used to describe Raman scattering in one space dimension (see e.g. [Bo], [NM], [PP]). This is a particular case of equations (12.7) in which Ω has three simple eigenvalues, $\omega_1<\omega_2<\omega_3$. Moreover, the states 1 and 2 have the same parity, while the state 3 has the opposite parity. This yields

$$
\Omega = \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix}\,, \quad \Gamma = \begin{pmatrix} 0 & 0 & \gamma_{1,3} \\ 0 & 0 & \gamma_{2,3} \\ \gamma_{3,1} & \gamma_{3,2} & 0 \end{pmatrix}\,. \tag{12.13}
$$

The linearized equations.

We look for solutions (12.5) with $\underline{U}=(0,0,\underline{\rho})$ where $\underline{\rho}$ is a constant hermitian matrix such that $[\Omega,\underline{\rho}]=0$. Let $\omega_1<\omega_2<\ldots$ be the distinct eigenvalues of Ω and $\Pi_1,\Pi_2\ldots$, the corresponding orthogonal spectral projectors. We denote by $\rho_{j,k}$ the block decomposition of the matrix ρ in the eigenspaces of Ω i.e. $\rho=\sum\Pi_j\rho_{j,k}\Pi_k$, and $\rho_{j,k}=\Pi_j\rho\Pi_k$.

Assumption 12.5. $\underline{\rho}=\sum_j r_j\Pi_j$, *with* $r_1\geq r_2\geq\ldots\geq 0$.

EXAMPLES 12.6. 1) The ground state corresponds to $\underline{\rho} = \Pi_1$.

2) The thermodynamical equilibrium corresponds to $\underline{\rho} = \sum r_j \Pi_j$ with (see [PP])

$$r_j = \frac{e^{-\hbar\tilde{\omega}_j/\kappa T}}{\sum_k e^{-\hbar\tilde{\omega}_k/\kappa T}}\,.$$

where $\tilde{\omega}_j$ is the physical frequency of the state j expressed in s^{-1}. For applications one has to compare r_j/r_1 to ε. If $r_j/r_1 = O(\varepsilon)$ for $j > 1$, the distinction between the ground state and the thermodynamical equilibrium makes no sense in our analysis

DEFINITION 12.7. *Introduce $I := \{(j,k) \,:\, r_j \neq r_k\}$ and II the complementary set. Given a matrix ρ set*

$$\rho^I := \sum_{(j,k)\in I} \Pi_j \rho \Pi_k\,, \quad \rho^{II} := \sum_{(j,k)\in II} \Pi_j \rho \Pi_k\,. \tag{12.14}$$

Introduce $u := (E, B, \rho^I)$ and $v = \rho^{II} - \underline{\rho}$. Note that $\underline{\rho}^I = 0$. The Maxwell-Bloch equations

$$\begin{cases} \varepsilon\partial_t B \;+\; \varepsilon\,\mathrm{curl} E = \;0\,, \\ \varepsilon\partial_t E \;-\; \varepsilon\,\mathrm{curl} B = \; i\,\mathrm{tr}\Gamma[\Omega,\rho] \;-\; i\mathrm{tr}\Gamma[E\cdot\Gamma,\rho]\,, \\ i\varepsilon\,\partial_t\rho \;=\; [\Omega,\rho] \;-\; [E\cdot\Gamma,\rho] \end{cases} \tag{12.15}$$

take the form

$$\begin{cases} L(\varepsilon\partial)u \;+\; Kv \;+\; q_1(u) \;+\; f_1(u,v) \;=\; 0\,, \\ M(\varepsilon\partial)\,v \;+\; q_2(u) \;+\; f_2(u,v) \;=\; 0\,. \end{cases} \tag{12.16}$$

where

$$L(\varepsilon\partial_x)u := \begin{cases} \varepsilon\partial_t B \;+\; \varepsilon\,\mathrm{curl} E \\ \varepsilon\partial_t E \;-\; \varepsilon\,\mathrm{curl} B \;-\; i\,\mathrm{tr}\big(\Gamma^I[\Omega,\rho^I]\big) \;+\; i\,\mathrm{tr}\big(\Gamma^I(E\cdot G)\big) \\ \varepsilon\partial_t\,\rho^I \;+ i\,[\Omega,\rho^I] \;-i\,E\cdot G \end{cases} \tag{12.17}$$

with $G := [\Gamma, \underline{\rho}]$

$$M(\varepsilon\partial_x)v := \varepsilon\partial_t + i[\Omega, v]\,, \tag{12.18}$$

$$Kv := \begin{cases} 0 \\ -\,i\,\mathrm{tr}(\Gamma^{II}[\Omega,v]) \\ 0 \end{cases}\,, \tag{12.19}$$

$$q_1(u) := \begin{cases} 0 \\ i\,\mathrm{tr}\big(\Gamma[E\cdot\Gamma,\rho^I]\big) \\ -\,i\,\big([E\cdot\Gamma,\rho^I]\big)^I \end{cases}\,, \quad f_1(u,v) := \begin{cases} 0 \\ i\mathrm{tr}\big(\Gamma[E\cdot\Gamma,v]\big) \\ -\;i\,\big([E\cdot\Gamma,v]\big)^I \end{cases}\,, \tag{12.20}$$

$$q_2(u) := -i\,\big([E\cdot\Gamma,\rho^I]\big)^{II}\,, \quad f_2(u,v) \;:=\; -\,i\,\big([E\cdot\Gamma,v]\big)^{II}\,. \tag{12.21}$$

Note that q_1 and q_2 are quadratic in u and f_1 and f_2 are bilinear in (u,v). The definition 12.7 is motivated by the remark that $G := [\Gamma, \underline{\rho}] = G^I$. It implies the triangular form of the linear part of the equations (12.16).

Proposition 12.8. *The systems $L(\varepsilon\partial_x)$ and $M(\varepsilon\partial_x)$ are conservative, i.e. they are symmetric hyperbolic in the sense given after* (2.2).

Proof.

a) The conserved quantity for M is

$$\mathrm{tr}(vv^*) = \sum_{(j,k)\in II} \mathrm{tr}(v_{j,k}\, v_{j,k}^*)$$

as a consequence of the identity $\mathrm{Im}\,\mathrm{tr}([\Omega, v]v^*) = 0$.

b) For solutions of $L(\varepsilon\partial_x)u = 0$, the following quantity is conserved

$$|E|^2 + |B|^2 + \sum_{(j,k)\in I} \frac{\omega_k - \omega_j}{r_j - r_k}\, \mathrm{tr}(\rho_{j,k}\, \rho_{j,k}^*)$$

Note that the denominator $r_j - r_k$ does not vanish precisely when $(j,k) \in I$. Moreover, Assumption 12.5 implies that the coefficients $(\omega_k - \omega_j)/(r_j - r_k)$ are positive, showing that the quadratic form above is definite positive. Therefore $L(\varepsilon\partial_x)$ is symmetric hyperbolic.

The linear part of (12.16) is

$$\begin{pmatrix} L(\varepsilon\partial_x) & K \\ 0 & M(\varepsilon\partial_x) \end{pmatrix} \tag{12.23}$$

When $K \neq 0$, in general, it is not conservative. At intersection points of the characteristic manifolds of L and M, the coupling term Kv introduces a nondiagonal Jordan factor. Hence, there are no L^2 energy estimates for (12.23) independent of ε.

When $K = 0$ (12.23) is hyperbolic symmetric. Analogously, introducing $u = (E, B, P, Q)$ with $Q := \varepsilon\partial_t P$) and $v = N - \underline{N}$, the equations (1.1) take the form (12.19) with $K = 0$:

$$\left\{ \begin{array}{l} \begin{cases} \varepsilon\,\partial_t B + \varepsilon\,\mathrm{curl} E = 0\,, & \varepsilon\,\partial_t E - \varepsilon \mathrm{curl} B + Q = 0\,, \\ \varepsilon\,\partial_t P - Q = 0\,, & \varepsilon\,\partial_t Q + \Omega^2 P = \gamma_1 \underline{N} E + \gamma_1\, vE\,, \end{cases} \\ \varepsilon\partial_t v = -\gamma_2\, Q \cdot E\,. \end{array} \right. \tag{12.24}$$

Compatibility and change of unknowns.

Proposition 12.9. *The operators $L(\varepsilon\partial_x)$, $M(\varepsilon\partial_x)$ and the quadratic form q_2 satisfy Assumption* 2.10. *Moreover the associated bilinear mapping J given by Theorem* 2.11 *is*

$$J_{j,k}(\rho, \rho') := \sum_{l\in I_{j,k}} \frac{1}{r_j - r_l} \rho_{j,l}\rho'_{l,k} \qquad (j,k) \in II\,, \tag{12.25}$$

where $I_{j,k} := \{l \,:\, (j,l) \in I\}$.

$J(\rho, \rho)$ is a matrix of type II. Note that for $(j,k) \in II$, the conditions $(j,l) \in I$ and $(k,l) \in I$ are equivalent and $r_l - r_j = r_l - r_k \neq 0$.

Proof.

We leave to the reader to check that Assumption 2.10 is satisfied. This is a consequence of the identity (2.26) which we now check. Note that

$$\begin{aligned}
&\partial_t J(\rho^I,\rho^I) = J(\partial_t\rho^I,\rho^I) + J(\rho^I,\partial_t\rho^I)\,,\\
&[\Omega, J(\rho^I,\rho^I)] = J([\Omega,\rho^I],\rho^I) + J(\rho^I,[\Omega,\rho^I])\\
&[\rho^I, E\cdot\Gamma]^{II} = J(E\cdot G,\rho^I) + J(\rho^I, E\cdot G)\,.
\end{aligned}$$

These identities follow directly from the definition (12.25). They imply that

$$q_2(u) = M(\varepsilon\partial_x)J(\rho^I,\rho^I) - J\Big(\varepsilon\partial_t\rho^I + i[\Omega,\rho^I] - iE\cdot G,\rho^I\Big) - J\Big(\rho^I,\varepsilon\partial_t\rho^I + i[\Omega,\rho^I] - iE\cdot G\Big)$$

The proposition follows.

The change of unknows (2.27) reads

$$\sigma := v + J(\rho^I,\rho^I) \tag{12.26}$$

and the second equation in (12.16) is equivalent to

$$M(\varepsilon\partial_x)\sigma + g(U) + f_2(u,v) \;=\; 0 \tag{12.27}$$

where $g(U)$ is cubic in U

$$g(u) := J([\rho, E\cdot\Gamma]^I,\rho^I) + J(\rho^I,[\rho,E\cdot\Gamma]^I)\,. \tag{12.28}$$

The case where $\underline{\rho}$ is the ground state.

When $\underline{\rho}$ is the ground state one has

$$\rho^I = \begin{pmatrix} 0 & \rho_{1,2} & \dots & \rho_{1,m}\\ \rho_{2.1} & 0 & \dots & 0\\ \vdots & \vdots & & \vdots\\ \rho_{m1} & 0 & \dots & 0\end{pmatrix},\quad \rho^{II} = \begin{pmatrix} \rho_{1,1} & 0 & \dots & 0\\ 0 & \rho_{2.2} & \dots & \rho_{2,m}\\ \vdots & \vdots & & \vdots\\ 0 & \rho_{m2} & \dots & \rho_{m,m}\end{pmatrix}, \tag{12.29}$$

$$J(\rho^I,\rho^I) = \rho^I J\rho^I\,,\quad \text{with}\quad J = \begin{pmatrix} -1 & 0\\ 0 & Id\end{pmatrix}, \tag{12.30}$$

$$q_1(u) := \begin{cases} 0\\ i\,\mathrm{tr}\big(\Gamma^{II}[E\cdot\Gamma^I,\rho^I]\big) + i\,\mathrm{tr}\big(\Gamma^I[E\cdot\Gamma^{II},\rho^I]\big)\\ -\,i\,\big([E\cdot\Gamma^{II},\rho^I]\big)^I\end{cases}, \tag{12.31}$$

$$q_2(u) := -i\,\big([E\cdot\Gamma^I,\rho^I]\big)^{II},\quad f_2(u,v) \;:=\; -\,i\,\big([E\cdot\Gamma^{II},v]\big)^{II}\,. \tag{12.32}$$

After the change of variables (12.26), further simplifications occur in the equation (12.27). It reads

$$M(\varepsilon\partial_x)\sigma - i\,[E\cdot\Gamma^{II},\sigma] - i\,\big[[E\cdot\Gamma^{I},\sigma-\rho^I J\rho^I]\,,J\,\rho^I\big] = 0\,. \tag{12.33}$$

The similar change of variables for the system (12.24) was given in (1.7)

$$n = v + \frac{\gamma_2}{2\gamma_1\underline{N}}\,(Q^2+\Omega^2P^2)\,. \tag{12.34}$$

Then the last equation in (12.24) is transformed into

$$\varepsilon\partial_t n = \varepsilon\,\big(c_1 n - c_2(Q^2+\Omega^2P^2)\big)\,Q\cdot E\,. \tag{12.35}$$

Scaling the amplitudes.

We discuss the scale ε^p of asymptotic solutions (12.5) of equations (12.16).

A) The case $K\neq 0$. This case occurs when Γ^{II} is not diagonal.

Because the linear system (12.13) is not conservative, the term Kv is a source term in the equation for u. Suppose that $u=O(\varepsilon^p)$ and $v=O(\varepsilon^{p'})$. The source term for u is then of order $O(\varepsilon^\alpha)$ with $\alpha=\min(p',2p,p+p')$. Because the propagation is in $\varepsilon\partial_t$, the cumulative effects of the source terms for times $O(1)$ is $O(\varepsilon^{\alpha-1})$. Thus, one is lead to require that $p\leq\alpha-1$. Similarly, the equation for v yields the condition $p'\leq\min(2p,p+p')$. These conditions are satisfied when $p\geq 2, q\in[p+1,2p-1]$. Thus, for general equations of the form (12.16), the largest amplitudes are reached for $p=2$, $p'=3$.

We now take advantage of the special form of the nonlinear terms, using Proposition 12.9. We replace the second equation in (12.16) by (12.27) and use (12.26) to obtain a system for (u,σ). Because g is cubic, the conditions for (p,p') are now $p\leq\min(p',2p,p+p')-1$ and $p'\leq\min(3p,p+p')-1$. The optimal solution is $p=1$, $p'=2$. This yields to introduce new unknowns $(\tilde u,\tilde\sigma)$ such that

$$u=\varepsilon\tilde u\,,\quad \sigma=\varepsilon^2\tilde\sigma\,. \tag{12.36}$$

In the original variables this means

$$E=\varepsilon\,\widetilde E\,,\quad B=\varepsilon\,\widetilde B\,,\quad \rho^I=\varepsilon\,\tilde\rho^I\,,\quad \rho^{II}=\underline{\rho}+\varepsilon^2\,\tilde\rho^{II}\,. \tag{12.37}$$

Note that the change of variables (12.26) is compatible with this scaling. The equations for $(\tilde u,\tilde\sigma)$ are of the form

$$\begin{cases} L(\varepsilon\partial)\tilde u + \varepsilon K\tilde v + \varepsilon F_1(\tilde u,\tilde\sigma) = 0\,,\\ M(\varepsilon\partial)\,\tilde\sigma + \varepsilon F_2(\tilde u,\tilde\sigma) = 0\,.\end{cases} \tag{12.38}$$

The difference of scales for u and σ provides the factor ε in front of K which becomes an admissible source term. On the other hand, because g is cubic, the nonlinearity has also a factor ε. The standard results of geometric optics apply to (12.38) ([DR], [JR], [JMR 3,4,5])

B) The case $K = 0$.

B1) When $K = 0$, the sandard results apply to solutions (12.5) with $p = 1$.

B2) When $\Gamma^{II} = 0$ and $\underline{\rho}$ is the ground state, one can construct larger solutions with $p = 1/2$. In this case, (12.31) and (12.32) imply that $K = 0$, $q_1 = 0$, $f_2 = 0$. Moreover, (12.33) simplifies and the equations read

$$\begin{cases} L(\varepsilon\partial_x)u & + & f_1(u, \sigma - \rho^I J\rho^I) & = & 0\,, \\ M(\varepsilon\partial_x)\sigma & - & i\,\big[[E\cdot\Gamma^I, \sigma - \rho^I J\rho^I]\,, J\,\rho^I\big] & = & 0\,. \end{cases} \tag{12.39}$$

Introduce the new scaling which is well adapted to the nonlinear terms

$$u = \sqrt{\varepsilon}\tilde{u}\,, \quad \sigma = \varepsilon\tilde{\sigma}\,. \tag{12.40}$$

Then (12.39) in transformed to

$$\begin{cases} L(\varepsilon\partial_x)\tilde{u} & + & \varepsilon f_1(\tilde{u}, \tilde{\sigma} - \tilde{\rho}^I J\tilde{\rho}^I) & = & 0\,, \\ M(\varepsilon\partial_x)\tilde{\sigma} & - & i\,\varepsilon\,\big[[\tilde{E}\cdot\Gamma^I, \tilde{\sigma} - \tilde{\rho}^I J\tilde{\rho}^I]\,, J\tilde{\rho}^I\big] & = & 0\,. \end{cases} \tag{12.41}$$

The standard results of [DR] apply to solutions In the original variables this means that

$$E = \sqrt{\varepsilon}\,\widetilde{E}\,, \quad B = \sqrt{\varepsilon}\,\widetilde{B}\,, \quad \rho^I = \sqrt{\varepsilon}\,\tilde{\rho}^I\,, \quad \rho^{II} = \underline{\rho} + \varepsilon\,\tilde{\rho}^{II}\,. \tag{12.42}$$

Applications.

The idea is to apply known results to the equations satisfied by $\tilde{u} = (\widetilde{B}, \widetilde{E}, \tilde{\rho}^I)$ and $\tilde{\sigma}$. We give three examples where we use results and ideas from [DR], [JRM 3], [Lan]. For the first two applications, the results also follow from Therems 2.3, 2.8 and 2.9 above. From now on, we assume that $\underline{\rho}$ is the ground state.

A) $O(\sqrt{\varepsilon})$ fields for an isotropic two level model.

To illustrate the case $B2$), we compute the principal term $\mathbf{U}_0$ for the isotropic two level system (12.12) when $\delta = 0$. Introduce the scaling (12.42) and perform the change of unknonws

$$\tilde{\rho}^{II} = \tilde{\sigma} + \tilde{\rho}^I \begin{pmatrix} -1 & 0 \\ 0 & Id_{3\times 3} \end{pmatrix} \tilde{\rho}^I$$

Introduce next the notations

$$\tilde{\rho}^I = \begin{pmatrix} 0 & {}^t\varphi \\ \psi & 0 \end{pmatrix}\,, \quad \sigma = \begin{pmatrix} n & 0 \\ 0 & N \end{pmatrix}\,.$$

where φ and ψ are vectors in $\mathbb{C}^3$, n is a complex number and N is a 3×3 matrix. Using the identities

$$[\widetilde{E}\cdot\Gamma^I, \sigma] = \begin{pmatrix} 0 & \gamma^t E\,(N - nId) \\ -\overline{\gamma}(N - nId)E & 0 \end{pmatrix}\,, \quad \mathrm{tr}\Gamma^I \begin{pmatrix} 0 & \gamma^t h \\ -\overline{\gamma}k & 0 \end{pmatrix} = |\gamma|^2(h - k)\,,$$

we obtain that the equations (12.41) read

$$
\begin{cases}
\varepsilon\partial_t\widetilde B \,+\,\varepsilon\,\mathrm{curl}\widetilde E \,=\,0\,,\\
\varepsilon\partial_t\widetilde E \,-\,\varepsilon\,\mathrm{curl}\widetilde B \,-\,i\,(\gamma\omega_{1,2}\psi+\overline{\gamma}\omega_{2,1}\varphi)\,-\,i\varepsilon|\gamma|^2(A-{}^tA)\widetilde E\,=0\,,\\
\varepsilon\partial_t\,\varphi\,+\,i\,\omega_{1,2}\,\varphi\,+\,i\gamma\widetilde E-\,i\varepsilon\,\gamma\,{}^tA\widetilde E\,=\,0\,,\\
\varepsilon\partial_t\,\psi\,+\,i\,\omega_{2,1}\,\psi\,-\,i\overline{\gamma}\widetilde E+\,i\varepsilon\,\overline{\gamma}\,A\widetilde E\,=\,0\,,\\
\varepsilon\partial_t\,n\,+i\,\varepsilon\,\big(\gamma\,{}^t\widetilde E A\psi-\overline{\gamma}\,{}^t\varphi A\widetilde E\big)\,=\,0\,,\\
\varepsilon\partial_t\,N\,+i\,\varepsilon\,\big(\overline{\gamma}\,A\widetilde E^t\varphi-\gamma\,\psi^t\widetilde E A\big)\,=\,0\,.
\end{cases}
\tag{12.43}
$$

with $A := N-\psi^t\varphi-(n+{}^t\psi\varphi)Id$.

The characteristic variety of the operator L in (12.17), which now acts on the components $(\tilde B,\tilde E,\varphi,\psi)$ is the union of $\{\tau=0\}$, $\{\tau=\pm\sqrt{(\omega_{2,1})^2+2|\gamma|^2\omega_{2,1}}\}$ and the variety $\mathcal C$ of equation

$$
|\eta|^2\,=\,\tau^2(1+\chi(\tau))\,,\quad \chi(\tau)\,:=\,\frac{2\omega_{2,1}|\gamma|^2}{(\omega_{2,1})^2-\tau^2}\,.
\tag{12.43}
$$

Optical wave numbers $\xi=(\tau,\eta)$ satisfy (12.43). When $\xi\in\mathcal C$ and $\eta\neq 0$, the kernel of $L(i\xi)$ is parametrized by $E\in\eta^\perp$ and the ohter components are

$$
B=-\frac{1}{\tau}\,\eta\times E,\quad \varphi=\frac{\gamma}{\omega_{2,1}-\tau}E\,,\quad \psi=\frac{\overline{\gamma}}{\omega_{2,1}+\tau}E\,.
\tag{12.44}
$$

Moreover, $f=(f_B,f_E,f_\varphi,f_\psi)$ is in the image of $L(i\xi)$ if and only if

$$
-\eta\times f_B\,+\,\tau\big(f_E+\frac{\omega_{2,1}\overline{\gamma}}{\omega_{2,1}-\tau}f_\varphi+\frac{\omega_{2,1}\gamma}{\omega_{2,1}+\tau}f_\psi\big)\,\in\mathbb C\eta\,.
\tag{12.45}
$$

Consider $\beta=(\omega,\kappa)\in\mathcal C$ with $\kappa\neq 0$ and assume that $\nu\beta$ is not characteristic when $\nu\notin\{-1,0,1\}$. The only frequency $\nu\beta$ which is characteristic for $\varepsilon\partial_t$ is $\nu=0$. The polarisation condition $\mathbb P\mathbf U_0=\mathbf U_0$ of the general theory of [DR] or in §2 shows that the principal terms $\mathbf u_0$ and $\mathbf v_0$ satisfy

$$
\mathbf u_0(x,\theta)=\sum_{\nu=-1}^{1}u_\nu(x)\,e^{i\nu\theta}\,,\quad P(\nu\beta)u_\nu=u_\nu\,,\quad \mathbf v_0(x,\theta)\,=\,v_0(x)
\tag{12.46}
$$

and transport equations of the form

$$
\begin{cases}
P(\nu\beta)\,L_1(\partial_x)\,u_\nu\,+\,P(\nu\beta)\,\Phi(\mathbf u_0,\mathbf v_0)\,=\,0\\
\partial_t\,v_0\,+\,\big\langle\Psi(\mathbf u_0,\mathbf v_0)\big\rangle\,=\,0\,,
\end{cases}
\tag{12.47}
$$

where $P(\nu\beta)$ is the spectral projector on $\ker L(i\nu\beta)$ and $\langle\mathbf v\rangle$ denotes the mean value of the periodic function $\mathbf v$. In general, these equations couple $u_{\pm1}$ and (u_0,m_0). However, (u_0,v_0) remain equal to zero when their initial data of vanish. This means that instead of (12.46) one has

$$
\mathbf u_0(x,\theta)=\sum_{\nu\in\{-1,1\}}u_\nu(x)\,e^{i\nu\theta}\,,\quad P(\nu\beta)u_\nu=u_\nu\,,\quad \mathbf v_0(x,\theta)\,=\,0\,.
\tag{12.48}
$$

This follows from the fact that if $\mathbf{u}_0$ satisfies (12.46), then

$$P(0)\,\Phi(\mathbf{u}_0,0) \;=\; 0\,, \quad \big\langle \Psi(\mathbf{u}_0,0)\big\rangle \;=\; 0\,. \tag{12.49}$$

The first equality follows from the fact that $\Phi(u,0)$ is cubic in u and therefore the average of $\Phi(\mathbf{u}_0,0)$ vanishes when $\mathbf{u}_0$ has only the harmonics $+1$ and -1. To prove the second, note that

$$\Psi(u,0) \;=\; \begin{pmatrix} -\gamma({}^t\widetilde{E}\psi)({}^t\varphi\psi) - \gamma({}^t\varphi\psi))({}^t\widetilde{E}\psi) + \overline{\gamma}({}^t\varphi\psi)({}^t\varphi\widetilde{E}) + \overline{\gamma}({}^t\varphi\widetilde{E})({}^t\varphi\psi) \\ -\overline{\gamma}({}^t\varphi\widetilde{E})\psi\otimes{}^t\varphi - \overline{\gamma}({}^t\varphi\psi)\widetilde{E}\otimes{}^t\varphi + \gamma\,({}^t\widetilde{E}\psi)\psi\otimes{}^t\varphi + \gamma({}^t\varphi\psi)\psi\otimes{}^t\widetilde{E} \end{pmatrix}.$$

Thus, when $\mathbf{u}_0$ is given by (12.46), the polarisation conditions (12.44) imply

$$\Psi(\mathbf{u}_0,0) \;=\; \sum_{\nu_1,\nu_2,\nu_3,\nu_4} e^{i(\nu_1+\nu_2+\nu_3+\nu_4)\theta}(\nu_1+\nu_2+\nu_3+\nu_4)\omega \begin{pmatrix} ({}^t\varphi_{\nu_1}\psi_{\nu_2})({}^t\varphi_{\nu_3}\psi_{\nu_4}) \\ -({}^t\varphi_{\nu_1}\psi_{\nu_2})\psi_{\nu_3}\otimes{}^t\varphi_{\nu_4} \end{pmatrix}$$

Thus the average vanishes and the second identity of (12.49) is proved. When the principal profile satisfies (12.48) the transport equation (12.47) reduces to

$$P(\nu\beta)\,L_1(\partial_x)\,u_\nu + P(\nu\beta)\,f(\mathbf{u}_0,0) \;=\; 0\;, \quad \nu\in\{-1,1\}\,.$$

For $\nu=\pm1$, $P(\nu\beta)L_1P(\nu\beta)$ is the transport field $\partial_t + v_g\cdot\partial_y$ where the v_g is group velocity deduced form the dispersion relation (12.43)

$$v_g := \frac{\partial(-\tau)}{\partial\eta}(\kappa) \;=\; \frac{-\kappa}{\omega(1+\chi(\omega)+\omega\chi'(\omega)/2}\,.$$

Thanks to (12.31), the Fourier coefficients $u_{\pm1}$ are determined by $\widetilde{E}_{\pm1}$. Thus the equations (12.37) are equations for $\widetilde{E}_\nu$. For real solutions, $\widetilde{E}_{-1}=\overline{\widetilde{E}_1}$, and using the characterization (12.32) of the image of $L(i\beta)$, we end up with the following equation for $\widetilde{E}_1$:

$$(\partial_t + v_g\cdot\partial_y)\,\widetilde{E}_1 \;+\; i\,c_1|\widetilde{E}_1|^2\widetilde{E}_1 \;+\; ic_2\,({}^t\widetilde{E}_1\cdot\widetilde{E}_1)\overline{\widetilde{E}_1} \;=\; 0\,, \tag{12.50}$$

where c_1 and c_2 are real constants which we do not compute explicitely. This is the familiar form of the equations found for differents models in nonlinear optics ([Bo], [NM], [Do] and [DR]).

B) The case $\Gamma^{II}\neq 0$. Application to stimulated Raman scattering.

This is the general case of equation (12.15) and we use the scaling (12.37). The explicit form of the equations for $(\widetilde{B},\widetilde{E},\tilde{\rho}^I)$ and $\tilde{\sigma}$ is

$$\begin{cases} \varepsilon\partial_t\widetilde{B} \,+\, \varepsilon\,\mathrm{curl}\widetilde{E} \;=\; 0\,, \\ \varepsilon\partial_t\widetilde{E} \,-\, \varepsilon\,\mathrm{curl}\widetilde{B} \,-\, i\,\mathrm{tr}(\Gamma^I[\Omega,\rho^I]) \,+\, i\mathrm{tr}(\Gamma^I(\widetilde{E}\cdot G)) \\ \qquad\quad +\, i\varepsilon\mathrm{tr}\big(\Gamma[\widetilde{E}\cdot\Gamma,\tilde{\rho}^I]\big) \,-\, i\,\varepsilon\mathrm{tr}(\Gamma^{II}[\Omega,\tilde{\sigma}-\tilde{\rho}J\tilde{\rho}]) \;=\varepsilon^2F_1\,, \\ \varepsilon\partial_t\,\tilde{\rho}^I \,+\, i\,[\Omega,\tilde{\rho}^I] \,-\, i\,\widetilde{E}\cdot G \,-\, i\,\varepsilon[\widetilde{E}\cdot\Gamma^{II},\tilde{\rho}^I] \;=\; \varepsilon^2F_2\,, \\ \varepsilon\partial_t\,\tilde{\sigma} \,+ i\,[\Omega,\tilde{\sigma}] \,-\, i\,\varepsilon\,[\widetilde{E}\cdot\Gamma^{II},\tilde{\sigma}] \;=\; \varepsilon^2F_3 \end{cases} \tag{12.51}$$

where we do not specify F_1, F_2 and F_3 because they do not affect the principal terms of the expansions.

The stimulated Raman scattering is modelled through a resonant three phases geometric optics expansion for equations (12.7) with Ω and Γ given by (12.13). We jump directly to the corresponding system (12.51). Because we are interested in the computation of the principal term of the asymptotic expansion, we can assume that $F_1 = F_2 = F_3 = 0$. Moreover, assuming that the initial data of $\tilde\sigma$ vanish, we find that $\tilde\sigma = 0$ (or is $O(\varepsilon)$) and therefore can be neglected.

The equations for $u = (\tilde b, \tilde e, \tilde\rho^I)$ read

$$\tag{12.52} \begin{cases} \varepsilon\partial_t \tilde b + \varepsilon\partial_y \tilde e \;=\; 0\,, \\ \varepsilon\partial_t \tilde e + \varepsilon\partial_y \tilde b \;-i\,\omega_{3,1}(\gamma_{1,3}\tilde\rho_{3,1} - \gamma_{3,1}\tilde\rho_{1,3}) \;+ i\,\varepsilon\omega_{3,2}(\gamma_{2,3}\tilde\rho_{3,1}\tilde\rho_{1,2} - \gamma_{3,2}\tilde\rho_{2,1}\tilde\rho_{1,3}) \;=\; 0\,, \\ \varepsilon\partial_t \tilde\rho_{1,3} - i\omega_{3,1}\tilde\rho_{1,3} + i\tilde e\gamma_{1,3} \;+\; i\,\varepsilon\,\tilde e\,\rho_{1,2}\gamma_{2,3} \;=\; 0\,, \\ \varepsilon\partial_t \tilde\rho_{3,1} + i\omega_{3,1}\tilde\rho_{3,1} - i\tilde e\gamma_{3,1} \;-\; i\,\varepsilon\,\tilde e\,\gamma_{3,2}\rho_{2,1} \;=\; 0\,, \\ \varepsilon\partial_t \rho_{1,2} - i\omega_{2,1}\rho_{1,2} + \; i\,\varepsilon\,\tilde e\,\rho_{1,3}\gamma_{3,2} \;=\; 0\;, \\ \varepsilon\partial_t \rho_{2,1} + i\omega_{2,1}\rho_{2,1} - \; i\,\varepsilon\,\tilde e\,\gamma_{2,3}\rho_{3,1} \;=\; 0\,. \end{cases}$$

This is of the form

$$\tag{12.53} L(\varepsilon\partial_x)u \;+\; \varepsilon f(u) \;=\; 0$$

where f is quadratic. As a consequence of the equality $\gamma_{1,2} = 0$, the linear operator $L(\varepsilon\partial_x)$ splits into two independent systems :

$$\tag{12.54} L_1(\varepsilon\partial_x)\begin{pmatrix} b \\ e \\ \rho_{1,3} \\ \rho_{3,1}\end{pmatrix} := \begin{cases} \varepsilon\partial_t b + \varepsilon\partial_y e, , \\ \varepsilon\partial_t e + \varepsilon\partial_y b - i\,\omega_{3,1}(\gamma_{1,3}\tilde\rho_{3,1} - \gamma_{3,1}\tilde\rho_{1,3})\,, \\ \varepsilon\partial_t \rho_{1,3} - i\omega_{3,1}\rho_{1,3} + ie\gamma_{1,3}\;, \\ \varepsilon\partial_t \rho_{3,1} + i\omega_{3,1}\rho_{3,1} - ie\gamma_{3,1}\;, \end{cases}$$

and

$$\tag{12.55} L_2(\varepsilon\partial_x)\begin{pmatrix} \rho_{1,2} \\ \rho_{2,1}\end{pmatrix} := \begin{cases} \varepsilon\partial_t \rho_{1,2} - i\omega_{2,1}\rho_{1,2}\;, \\ \varepsilon\partial_t \rho_{2,1} + i\omega_{2,1}\rho_{2,1}\;. \end{cases}$$

The characteristic varieties of L_1 and L_2 are

$$\tag{12.56} \begin{cases} \mathcal{C}_{L_1} \;=\; \{\xi = (\tau,\eta) \in \mathbb{R}^2\,;\, \eta^2 \;=\; \tau^2(1+\chi(\tau)\}\,, \quad \chi(\tau) \;:=\; \dfrac{2\omega_{3,1}|\gamma_{1,3}|^2}{(\omega_{3,1})^2 - \tau^2}\,, \\ \mathcal{C}_{L_2} \;=\; \{\xi = (\tau,\eta) \in \mathbb{R}^2\,;\, \tau = \pm\omega_{2,1}\}\,. \end{cases}$$

Raman interaction occurs when a laser beam of wave number $\beta_L = (\omega_L, \kappa_L) \in \mathcal{C}_{L_1}$ interacts with an electronic exitation $\beta_E = (\omega_{2,1}, \kappa_E) \in \mathcal{C}_{L_2}$ to produce a scattered wave $\beta_S = (\omega_S, \kappa_S) \in \mathcal{C}_{L_1}$ via the resonance relation

$$\tag{12.57} \beta_L = \beta_E + \beta_S\,.$$

One further assumes that $\beta_L \notin \mathcal{C}_{L_2}$, $\beta_E \notin \mathcal{C}_{L_1}$ and $\beta_S \notin \mathcal{C}_{L_2}$.

To fit the general framework of expansions (12.5), introduce $\beta := (\beta_L, \beta_S)$. Then, the principal profile $\mathbf{u}_0(x,\theta)$ has a two dimensional fast variable $\theta \in \mathbb{T}^2$ and its Fourier coefficients satisfy

$$\tag{12.58} P(\nu\beta)\,u_\nu = u_\nu\,, \quad P(\nu\beta)L_1(\partial)P(\nu\beta)u_\nu \;+\; P(\nu\beta)f(\mathbf{u}_0) \;=\; 0.$$

Because f is quadratic, the conditions on $(\beta_L, \beta_S, \beta_E)$ show that (12.58) has solutions with spectrum contained in $\{\pm(1,0), \pm(0,1), \pm(1,-1)\}$. We now restrict our attention to these solutions. It is convenient to label the Fourier coefficients $u_{\pm L}$, $u_{\pm S}$ and $u_{\pm E}$ so that

$$\mathbf{u}_0(x, \beta\cdot x/\varepsilon) \;=\; u_L(x)e^{i\beta_L x/\varepsilon} \;+\; u_S(x)e^{i\beta_S x/\varepsilon} \;+\; u_E(x)e^{i\beta_E x/\varepsilon} \;+\; u_{-L}(x)e^{-i\beta_L x/\varepsilon} \;+\dots \tag{12.59}$$

In addition, introduce the notations $u_L = (b_L, e_L, \rho_{j,k,L})$ etc. The polarization conditions imply that $b_{\pm E} = e_{\pm E} = \rho_{1,3,\pm E} = \rho_{3,1,\pm E} = 0$ and $\rho_{1,2,\pm L} = \rho_{1,2,\pm S} = \rho_{2,1,\pm L} = \rho_{2,1,\pm S} = 0$. Moreover, $\rho_{1,2,-E} = \rho_{2,1,E} = 0$,

$$b_L = -\frac{1}{\omega_L}\,\kappa_L \times E, \quad \rho_{1,3,L} = \frac{\gamma_{1,3}}{\omega_{3,1}-\omega_L} e_L\,, \quad \rho_{3,1,L} = \frac{\gamma_{3,1}}{\omega_{3,1}+\omega_L} e_L\,,$$

and similar formula hold for the subscripts $-L$ and $\pm S$. Furthermore, for real fields and hermitian density matrices, one has the relations

$$e_{-L} = \overline{e_L}\,, \quad e_{-S} = \overline{e_S}\,, \quad \rho_{2,1,-E} = \overline{\rho_{1,2,E}}\,.$$

Therefore, the profile equations (12.58) reduce to the following system for (e_L, e_S) and $\sigma_E := \rho_{1,2,E}$, which is the familar equations of three waves mixing ([Bo], [NM], [PP]).

$$\begin{cases} (\partial_t + v_L\partial_y)e_L \;+ i\,c_1\,e_S\,\sigma_E \;=\; 0\,, \\ (\partial_t + v_S\partial_y)e_S \;+\; ic_2\; e_S\,\overline{\sigma_E} \;=\; 0\,, \\ \partial_t\sigma_E \;+\; i\,c_3\; e_L\,\overline{e_S} \;=\; 0\,. \end{cases} \tag{12.60}$$

Here v_L and v_S are the group velocities associated to the frequencies β_L and β_S respectively. We refer for example to [Bo], [NM] for an explicit calculation of the constants c_k and the discussion of the amplification properties of this system.

C) Long time diffraction.

Consider the system (1.1). Consider the new scaling

$$(B,E,P,Q) = \varepsilon(\widetilde{B}, \widetilde{E}, \widetilde{P}, \widetilde{Q})\,, \quad N \;=\; \underline{N} + \varepsilon^2 \widetilde{N} \tag{12.61}$$

and introduce the new variable n as in (1.7). Then (1.1) is equivalent to

$$\begin{cases} \varepsilon\,\partial_t\widetilde{B} \;+\; \varepsilon\,\mathrm{curl}\widetilde{E} = \;0\,, \\ \varepsilon\,\partial_t\widetilde{E} \;-\; \varepsilon\mathrm{curl}\widetilde{B} = \;-\widetilde{Q}\,, \\ \varepsilon\,\partial_t\widetilde{P} \;-\; \widetilde{Q} \;=\; 0\,, \\ \varepsilon\,\partial_t\widetilde{Q} \;+\; \Omega^2\,P \;=\; \gamma_1\;\underline{N}\,\widetilde{E} \;+\; \varepsilon^2\,\gamma_1\,\big(n \;-\; c\,(\widetilde{Q}^2 + \Omega^2\widetilde{P}^2)\big)\,\widetilde{E}\,, \\ \varepsilon\partial_t n \;=\; \varepsilon^2\,\dfrac{\gamma_2}{\underline{N}}\big(n \;-\; c\,(\widetilde{Q}^2 + \Omega^2\widetilde{P}^2)\big)\,\widetilde{Q}\cdot\widetilde{E}\,. \end{cases} \tag{12.62}$$

This equation is of the form

$$L(\varepsilon\partial_x)\widetilde{U} \;+\; \varepsilon^2 f(\widetilde{U}) \;=\; 0\,, \tag{12.63}$$

and long time diffractive geometric optics expansions are available in [Lan] (see also [DJMR] and [JMR 7] for nondispersive equations). Formal solutions of (1.1) of the form

$$U(x) \sim \varepsilon \sum \varepsilon^n U_n(\varepsilon t, x - v_g t, \beta \cdot x/\varepsilon) \tag{12.64}$$

were computed in [Do]. They correspond to formal solutions of (12.62) of the form

$$\tilde{U}(x) \sim \sum \varepsilon^n U_n(\varepsilon t, x - v_g t, \beta \cdot x/\varepsilon) \tag{12.65}$$

The existence of exact solutions of (12.63) which satisfy (12.64) uniformly for times $t \le T_*/\varepsilon$ follows from [Lan]. In particular, this justifies the stability of the formal solutions found in [Do].

This discussion also applies to the general form of Maxwell-Bloch equations discussed above when $\Gamma^{II} = 0$.

References

[Bl] N. Bloembergen, *Nonlinear Optics*, W.A. Benjamin Inc., New York, 1965

[BW] M.Born and E.Wolf, *Principles of Optics*, Pergamon Press, 1959.

[Bo] R. Boyd, *Nonlinear Optics*, Academic Press, 1992

[Do] P. Donnat, *Quelques contributions mathématiques en optique non linéaire*, Thèse, École Polytechnique, 1994.

[DJMR] P.Donnat, J.L.Joly, G.Métivier and J.Rauch, *Diffractive Nonlinear Optics*, Séminaire E.D.P., École Polytechnique 1995-96.

[DR] P. Donnat and J. Rauch, *Dispersive Nonlinear Geometric Optics*, J.Math.Physics, 38 (1997) pp 1484-1523.

[JMR 1] J.L.Joly, G.Métivier and J.Rauch, *Several Recent Results in Nonlinear Geometric Optics*, Birkhäuser

[JMR 2] J.L.Joly, G.Métivier and J.Rauch, *Recent Results in Nonlinear Geometric Optics*, Zürich

[JMR 3] J.L.Joly, G.Métivier and J.Rauch, *Coherent and focusing multidimensional nonlinear geometric optics*, Annales de L'École Normale Supérieure de Paris, 28 (1995), PP 51-113.

[JMR 4] J.L.Joly, G.Métivier and J.Rauch, *Coherent nonlinear waves and the Wiener algebra*, Ann. Inst. Fourier, 44, 1994, pp 167-196.

[JMR 5] J.L.Joly, G.Métivier and J.Rauch, *Generic rigorous asymptotic expansions for weakly nonlinear geometric optics*, Duke. Math. J., 70, 1993, pp 373-404.

[JMR 6] J.L.Joly, G.Métivier and J.Rauch, *Diffractive nonlinear geometric optics with rectification*, Indiana J. Math., to appear.

[JR] J.L.Joly and J.Rauch, *Justification of multidimensional single phase semilinear geometric optics*, Trans. Amer. Math. Soc. 330 (1992), 599-625.

[Lan] D.Lannes, *Dispersive effects for nonlinear geometrical optics with rectification,* Asymptotic Anal., 18 (1998), 111-146.

[Lax] P. Lax, *Asymptotic solutions of oscillatory initial value problems*, Duke Math. Journal, 24 (1957) pp627-645.

[NM] A. Newell and J. Moloney *Nonlinear Optics* Addison-Wesley, Reading Mass., 1992.

[PP] R.Pantell and H.Puthoff, *Fundamentals of Quantum Electronics*, J.Wiley&Sons Inc., New York and *Electronique quantitique en vue des applications*, Dunod, 1973.