

# The Mathematics of Nonlinear Optics

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# 1 Introduction

Nonlinear optics is a very active field in physics, of primary importance and with an extremely wide range of applications. For an introduction to the physical approach, we can refer the reader to classical text books such as [NeMo, Boy, Blo, BoWo, PaPu, Fey, LaLi].

Optics is about the propagation of electromagnetic waves. Nonlinear optics is more about the study of how high intensity light interacts with and propagates through matter. As long as the amplitude of the field is moderate, the linear theory is well adapted, and complicated fields can be described as the superposition of noninteracting simpler solutions (e.g. plane waves). Nonlinear phenomena such as double refraction or Raman effect, which are examples of wave interaction, have been known for a long time, but the major motivation for a nonlinear theory came from the discovery of the laser which made available highly coherent radiations with extremely high local intensities.

Clearly, *interaction* is a key word in nonlinear optics. Another fundamental feature is that many *different scales* are present: the wavelength, the length and the width of the beam, the duration of the pulse, the intensity of the fields, but also internal scales of the medium such as the frequencies of electronic transitions etc.

Concerning the mathematical set up, the primitive models are Maxwell's equations for the electric and magnetic intensity fields  $E$  and  $H$  and the electric and magnetic inductions  $D$  and  $B$ , coupled with constitutive relations between these fields and/or equations which model the interaction of the fields with matter. Different models are presented in Section 2. All these equations fall into the category of *nonlinear hyperbolic systems*. A very important step in modeling is to put these equations in a dimensionless form, by a suitable choice of units, and deciding which phenomena, at which scale, one wants to study (see e.g. [Don]). However, even in dimensionless form, the equations may contain one or several small/large parameters. For example, one encounters *singular systems* of the form

$$(1.1) \quad A_0(u)\partial_t u + \sum_{j=1}^d A_j(u)\partial_{x_j} u + \lambda L_0 u = F(u)$$

with  $\lambda \gg 1$ . In addition to the parameters contained in the equation, there are other length scales typical to the solutions under study. In linear and nonlinear optics, one is specially interested in *waves packets*

$$(1.2) \quad u(t, x) = e^{i(k \cdot x - \omega t)} U(t, x) + cc$$

where  $k$  and  $\omega$  are the central wave number and frequency respectively, and where the *envelop*  $U(t, x)$  has slow variations compared to the rapid oscillations of the exponential:

$$(1.3) \quad \partial_t^j U \ll \omega^j U, \quad \partial_x^j U \ll k^j U.$$

In dimensionless form, the wave length  $\varepsilon := \frac{2\pi}{k}$  is small ( $\ll 1$ ). An important property is that  $k$  and  $\omega$  must be linked by the *dispersion relation* (see Section 5). Interesting phenomena occur when  $\varepsilon \approx \lambda^{-1}$ . With this scaling there may be resonant interactions between the electromagnetic wave and the medium, accounting for the dependence of optical indices on the frequency and thus for diffraction of light (see e.g. [LaLi]).

More generally, several scales can be present in the envelop  $U$ , for instance

$$(1.4) \quad U = U\left(t, x, \frac{t}{\sqrt{\varepsilon}}, \frac{x}{\sqrt{\varepsilon}}\right) \quad (\text{speckles}),$$

$$(1.5) \quad U = U(t, x, \varepsilon t) \quad (\text{long time propagation})$$

etc. The typical size of the envelop  $U$  (the intensity of the beam) is another very important parameter.

Nonlinear systems is the appropriate framework to describe interaction of waves : wave packets with phases  $\varphi_j = k_j \cdot x - \omega_j t$  create by multiplication of the exponentials the new phase  $\varphi = \sum \varphi_j$  which may or may not satisfy the dispersion relation. In the first case, the oscillation with phase  $\varphi$  is propagated (or persists) as time evolves : this is the situation of *phase matching*. In the second case, the oscillation is not propagated and only creates a lower order perturbation of the fields. These heuristic arguments are the base of the formal derivation of envelop equations that can be found in physics books. It is part of the mathematical analysis to make them rigorous.

In particular, self interaction can create *harmonics*  $n\varphi$  of a phase  $\varphi$ . Reducing to a single small dimensionless parameter, this leads for instance to look for solutions of the form

$$(1.6) \quad u(y) = \varepsilon^p \mathcal{U}\left(\frac{\varphi}{\varepsilon}, y\right), \quad \varepsilon^p \mathcal{U}\left(\frac{\varphi}{\varepsilon}, \frac{y}{\sqrt{\varepsilon}}, y\right), \quad \varepsilon^p \mathcal{U}\left(\frac{\varphi}{\varepsilon}, y, \varepsilon y\right)$$

where  $y = (t, x)$ ,  $\varphi = (\varphi_1, \dots, \varphi_m)$  and  $\mathcal{U}$  is periodic in the first set of variables.

Summing up, we see that the mathematical setup concerns *high frequency* and *mutliscale* solutions of *nonlinear hyperbolic problem*. At this point we

merge with many other problems arising in physics or mechanics, in particular in fluid mechanics, where this type of multiscale analysis is also of fundamental importance. For instance, we mention the problems of low Mach number flows, or fast rotating fluids, which raise very similar questions.

These notes are intended to serve as an introduction to the field. We do not even try to give a complete overview of the existing results, that would be impossible within a single book, and probably useless. We will tackle only a few basic problems, with the aim of giving methods and landmarks in the theory. Neither do we give complete proofs, instead we will focus on the key arguments and the main ideas.

Different models arising from optics are presented in Section 2. It is important to understand the variety of problems and applications which can be covered by the theory. These models will serve as examples all along the exposition. Next, the mathematical analysis will present the following points:

- Basic results of the theory of multi-dimensional symmetric hyperbolic systems are recalled in Section 3. In particular we state the classical theorem of local existence and local stability of smooth solutions<sup>1</sup>. However, this theorem is of little use when applied directly to the primitive equations. For high frequency solutions, it provides existence and stability on very short intervals of time, because fixed bounds for the derivatives require small amplitudes. Therefore this theorem which is basic in the theory and provides useful method, does not apply directly to high frequency and high intensity problems.

One idea to circumvent this difficulty would be to use existence theorems of solutions in energy spaces (the minimal conditions that the physical solutions are expected to satisfy). But there is no such general existence theorem in dimension  $\geq 2$ . This is why existence theorems of energy solutions or weak solutions are important. We give examples of such theorems, noticing that, in these statements, the counterpart of global existence is a much weaker stability.

- In Section 4 we present two methods which can be used to build an existence and stability theory for high frequency / high intensity solutions. The first idea is to factor out the oscillations in the linear case or to consider directly equations for profiles  $\mathcal{U}$  (see (1.6)) in the nonlinear case, *introducing the fast variables* (the placeholder for  $\varphi/\varepsilon$  for instance) *as independent variables*. The resulting equations are singular, meaning that they have co-

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<sup>1</sup>From the point of view of applications and physics, stability is much more significant than existence

efficients of order  $\varepsilon^{-1}$ . The method of Section 3 does not apply in general to such system, but it can be adapted to classes of such singular equations. which satisfy symmetry and *good commutation properties*.

Another idea can be used if one knows an approximate solution. This is the goal of asymptotic methods presented below to construct such approximate solutions: they satisfy the equation up to an error term of size  $\varepsilon^m$ . The the exact solution is sought as the approximate solution plus a corrector, presumably of order  $\varepsilon^m$  or so. The equation for the corrector has the feature that the coefficients have two components: one is not small, highly oscillatory but known (coming from the approximate solution), the other one is not known but small (it depends on the corrector). The theory of Section 3 can be adapted to this context with a fairly general extent. However, there is a severe restriction, which is that the order of approximation  $m$  must be large enough. In practice, this does not mean that  $m$  must be necessarily very large, but this means that the knowledge of the principal term is not sufficient.

- *Geometric optics* is a high frequency approximation of solutions. It fits with the corpuscular description of light, giving a particle-like description of the propagation along *rays*. It concerns solutions of the form  $u(y) = \varepsilon^p \mathcal{U}(\frac{\varphi(y)}{\varepsilon}, y)$ . The *phase*  $\varphi$  satisfies the *eikonal equation* and the amplitude  $\mathcal{U}$  is transported along the rays. In Section 5, we give the elements of this description, using the *WKB method* (for Wentzel, Kramers and Brillouin) of asymptotic (formal) expansions. The scaling  $\varepsilon^p$  for the amplitude plays and important role in the discussion: if  $p$  is large, only linear effect are observed (at least in the leading term). There is a threshold value  $p_0$  where the nonlinear effects are launched in the equation of propagation (typically it becomes nonlinear). Its value depend on the structure of the equation and of the nonlinearity. For general quasilinear system,  $p_0 = 1$ . This is the standard regime of *weakly nonlinear geometric optics*. However, there are special cases where for  $p = 1$  the transport equation remain linear; this happens when some interaction coefficient vanishes. This phenomenon is called *transparency* and for symmetry reasons it occurs rather frequently in applications. An example is the case of waves associated to linearly degenerate modes. In this case it is natural to look for larger solution with  $p < 1$ , leading to *strongly nonlinear regimes*. We give two examples where formal large amplitude expansions can be computed.

The WKB construction, when it is possible, leads to approximate solutions,  $u_{app}^\varepsilon$ . They are functions which satisfy the equation up to an error term that is of order  $\varepsilon^m$  with  $m$  large. The main question is to study the

*stability of such approximate solutions.* In the weakly nonlinear regime, the second method evoked in the presentation above of Section 4 applies and WKB solutions are stable, implying that the formal solutions are actually asymptotic expansions of exact solutions. But for strongly nonlinear expansions, the answer is not simple. Indeed, strong instabilities can occur, similar to Rayleigh instabilities in fluid mechanics. These aspects will be briefly discussed.

Another very important phenomenon is *focusing* and *caustics*. It is a linear phenomenon, which leads to concentration and amplifications of amplitudes. In a nonlinear context, the large intensities can be over amplified, launching strongly nonlinear phenomena. Some results are presented at the end of the section. However a general analysis of nonlinear caustics is still a widely open problem.

- *Diffraction optics.* This is the usual regime of long time or long distance propagation. Except for very intense and very localized phenomena, the length of propagation of a laser beam is much larger than its width. To analyze the problem, one introduces an additional time scale (and possibly one can also introduce an additional scale in space). This is the *slow time*, typically  $T = \varepsilon t$ . The propagations are governed by equations in  $T$ , so that the description allows for times that are  $O(\varepsilon^{-1})$ . For such propagations, the classical linear phenomenon which is observed is *diffraction of light* in the direction transversal to the beam, along long distances. The canonical model which describes this phenomenon in a Schrödinger equation, which replaces the transport equation of geometric optics. This is the well known *paraxial approximation* which also applies to nonlinear equations. This material is presented in Section 6 with the goal of clarifying the nature and the universality of nonlinear Schrödinger equations as fundamental equations in nonlinear optics, and often presented as the basic equation in physics books.

- *Wave interaction.* This is really where the nonlinear nature of the equations is rich in applications but also of mathematical difficulties. The physical phenomenon which is central here is the resonant interaction of waves. They can be optical waves, for instance a pump wave interacting with scattered and back scattered waves; they can also be optical waves interacting with electronic transitions yielding Raman effects, etc. An important observation is that resonance or phase matching is a *rare* phenomenon : a linear combination of characteristic phases is very likely *not* characteristic. This does not mean that the resonance phenomenon is uninteresting, on the contrary it is of fundamental importance. But this suggests that, when it occurs, it should remain limited. This is more or less correct at the



level of formal or BKW expansions, if one retains only the principal terms and neglect all the alleged small residual terms. However, the mathematical analysis is much more delicate : the analysis of all the phases created by interaction, no only those in the principal term, can be terribly complicated, and possibly beyond the scope of a description by functions of finitely many variables; the focusing effects of these phases have to be taken into account; harmonic generation can cause small divisors problems... It turns out that the justification of the formal calculus requires strong assumptions, that we call *coherence*. Fortunately these assumptions are realistic in applications, and thus the theory applies to interesting examples such as the Raman interaction evoked above. Surprisingly, the coherence assumptions are very close to, if not identical with, the commutation requirements introduced in Section 4 for the equations with the fast variables. All these aspects, including examples, are presented in Section 7.

There are many other important questions which are not presented in these notes. We can mention:

- *Space propagation, transmission and boundary value problems.* In the exposition, we have adopted the point of view of describing the evolution in time, solving mainly Cauchy problems. For physicists, the propagation is more often thought in space : the beam propagates from one point to another, or the beam enters in a medium etc. In the geometric optics description, the two formulations are equivalent, as long as they are governed by the same transport equations. However, for the exact equations, this is a completely different point of view, with the main difficulty that the equations are *not* hyperbolic in space. The correct mathematical approach is to consider transmission problems or boundary value problems. In this framework, new questions are reflection and transmission of waves at a boundary. Another important question is the generation of harmonics or scattered waves at boundaries. In addition, the boundary or transmission conditions may reveal instabilities which may or may not be excited by the incident beam, but which in any case make the mathematical analysis harder. Several references in this direction are [ArMa, Wil1, Wil2, Che2, Chi, Les1, Les2, Don].

- *Short pulses.* We have only considered oscillating signals, that is of the form  $e^{i\omega t}a$  with  $a$  slowly varying, typically,  $\partial_t a \ll \omega a$ . For short pulses, and even more for ultra short pulses, which do exist in physics (femtosecond lasers), the duration of the pulse (the support of  $a$ ) is just a few periods. Therefore, the description with phase and amplitude is not any more adapted. Instead of periodic profiles, this leads to consider *confined profiles* in time and space, typically in the Schwartz class. On the one hand, the

problem is simpler because less nonlinear : the durations of interaction are much smaller, so the nonlinear effect require more intense fields. But on the other hand, the mathematical analysis is seriously complicated by passing from a discrete to a continuous Fourier analysis in the fast variables. Several references in this direction are [AR1, AR2, CaRa1, CaRa2, CaRa3, CaLa].

## 2 Examples of equations arising in nonlinear optics

The propagation of an electro-magnetic field is governed by Maxwell's equation. The nonlinear character of the propagation has several origin: it may come from self-interaction, or from the interaction with the medium in which its propagates. The general Maxwell's equations (in Minkowski space-time) read (see e.g. [Fey, NeMo, PaPu]) :

$$(2.1) \quad \begin{cases} \partial_t D - \text{curl} H = -j, \\ \partial_t B + \text{curl} E = 0, \\ \text{div} B = 0, \\ \text{div} D = q \end{cases}$$

where  $D$  is the electric displacement,  $E$  the electric field vector,  $H$  the magnetic field vector,  $B$  the magnetic induction,  $j$  the current density and  $q$  is the charge density;  $c$  is the velocity of light. They also imply the charge conservation law:

$$(2.2) \quad \partial_t q + \text{div} j = 0.$$

We mainly consider the case where there are no free charges and no current flows ( $j = 0$  and  $q = 0$ ).

All the physics of interaction of light with matter is contained in the relations between the fields. The *constitutive equations* read

$$(2.3) \quad D = \varepsilon E, \quad B = \mu H,$$

where  $\varepsilon$  is the dielectric tensor and  $\mu$  the tensor of magnetic permeability. When  $\varepsilon$  and  $\mu$  are known, this closes the system (2.1). In the opposite direction, (2.3) can be seen as the definition of  $\varepsilon$  and  $\mu$  and the links between the fields must be given through additional relations. In the description of interaction of light with matter, one uses the following constitutive relations (see [NeMo, PaPu]):

$$(2.4) \quad B = \mu H, \quad D = \varepsilon_0 E + P = \varepsilon E$$

where  $P$  is called the *polarization* and  $\varepsilon$  the dielectric constant. In vacuum,  $P = 0$ . When light propagates in a dielectric medium, the light interacts with the atomic structure, creates dipole moments and induces the polarization  $P$ .

In the standard regimes of optics, the magnetic properties of the medium are not prominent and  $\mu$  can be taken constant equal to  $\frac{c^2}{\varepsilon_0}$  with  $c$  the speed of light in vacuum and  $\varepsilon_0$  the dielectric constant in vacuum. We give below several models of increasing complexity which can be derived from (2.4), varying the relation between  $P$  and  $E$ .

- *Equations in vacuum.*

In vacuum,  $\varepsilon = \varepsilon_0$  and  $\mu = \frac{1}{\varepsilon_0 c^2}$  are scalar and constant. The constraint equations  $\operatorname{div} E = \operatorname{div} B = 0$  are propagated in time and the evolution is governed by the classical *wave equation*

$$(2.5) \quad \partial_t^2 E - c^2 \Delta E = 0.$$

- *Linear instantaneous polarization.*

For small or moderate values of the electric field amplitude,  $P$  depends linearly in  $E$ . In the simplest case when the medium is isotropic and responds instantaneously to the electric field,  $P$  is proportional to  $E$ :

$$(2.6) \quad P = \varepsilon_0 \chi E$$

$\chi$  is the *electric susceptibility*. In this case,  $E$  satisfies the wave equation

$$(2.7) \quad n^2 \partial_t^2 E - c^2 \Delta E = 0.$$

where  $n = \sqrt{1 + \chi} \geq 1$  is the *refractive index* of the medium. In this medium, light propagates at the speed  $\frac{c}{n} \leq c$ .

- *Crystal optics.*

In crystals, the isotropy is broken and  $D$  is not proportional to  $E$ . The simplest model is obtained by taking the dielectric tensor  $\varepsilon$  in (2.3) to be a positive definite symmetric matrix, while  $\mu = \frac{1}{\varepsilon_0 c^2}$  remains scalar. In this case the system reads:

$$(2.8) \quad \begin{cases} \partial_t(\varepsilon E) - \operatorname{curl} H = 0, \\ \partial_t(\mu H) + \operatorname{curl} E = 0, \end{cases}$$

plus the constraint equations  $\operatorname{div}(\varepsilon E) = \operatorname{div} B = 0$  which are again propagated from the initial conditions. That the matrix  $\varepsilon$  is not proportional to the identity reflects the *anisotropy* of the medium. For instance, for a bi-axial crystal  $\varepsilon$  has three distinct eigenvalues.

Moreover, in the equations above  $\varepsilon$  is constant or depends on the space variable modeling the *homogeneity or the inhomogeneity* of the medium.

- *Lorentz model*

Matter does not respond instantaneously to stimulation by light. The delay is captured by writing in place of (2.6)

$$(2.9) \quad P(x, t) = \varepsilon_0 \int_{-\infty}^t \chi(t - t') E(x, t') dt',$$

modeling a linear relation  $E \mapsto P$  satisfying the causality principle. On the frequency side, that is after Fourier transform in time, this relation reads

$$(2.10) \quad \widehat{P}(x, \tau) = \varepsilon_0 \widehat{\chi}(\tau) \widehat{E}(x, \tau).$$

It is a simple model where *the electric susceptibility  $\chi$  depends on the frequency  $\tau$* .

In a standard model due to Lorentz [Lor] of the linear dispersive behavior of electromagnetic waves,  $P$  is given by

$$(2.11) \quad \partial_t^2 P + \partial_t P / T_1 + \omega^2 P = \gamma E$$

with positive constants  $\omega$ ,  $\gamma$ , and  $T_1$  (see also [Fey], chap. I-31, and II-33). Resolving this equation yields an expression of the form (2.10). In particular

$$(2.12) \quad \widehat{\chi}(\tau) = \frac{\gamma_0}{\omega^2 - \tau^2 + i\tau/T_1}, \quad \gamma_0 = \frac{\gamma}{\varepsilon_0}.$$

The physical origin of equation 2.11 is a model of the electron as bound to the nucleus by a Hooke's law spring force with characteristic frequency  $\omega$ ;  $T_1$  is a damping time and  $\gamma$  a coupling constant.

For nonisotropic crystals, the equation reads

$$(2.13) \quad \partial_t^2 P + R \partial_t P + \Omega P = \Upsilon E$$

where  $R$ ,  $\Omega$  and  $\Upsilon$  are matrices which are diagonal in the the crystal frame.

- *Phenomenological modeling of nonlinear interaction*

In a first attempt, nonlinear responses of the medium can be described by writing  $P$  as a power series in  $E$ :

$$(2.14) \quad P(x, t) = \varepsilon_0 \sum_{k=1}^{\infty} \int \chi^k(t - t_1, \dots, t - t_k) \bigotimes_{j=1}^k E(x, t_j) dt_1 \dots dt_k.$$

where  $\chi^k$  is a tensor of appropriate order. The symmetry properties of the susceptibilities  $\chi^k$  reflect the symmetry properties of the medium. For

instance, in a centrosymmetric and isotropic crystal, the quadratic susceptibility  $\chi^2$  vanishes.

The first term in the series represent the linear part and on often splits  $P$  into its linear (and main) part  $P_L$  and its nonlinear part  $P_{NL}$ .

$$(2.15) \quad P = P_L + P_{NL}.$$

Instantaneous responses correspond to susceptibilities  $\chi^k$  which are Dirac measures at  $t_j = t$ . One can also mix delayed and instantaneous responses.

- *Two examples*

In a centrosymmetric , homogeneous an isotropic medium (such as glasses or liquids), the fist nonlinear term is cubic. A model for  $P$  with a *Kerr nonlinearity* is  $P = P_L + P_{NL}$  with

$$(2.16) \quad \partial_t^2 P_L + \partial_t P_L / T_1 + \omega^2 P_L = \gamma E$$

$$(2.17) \quad P_{NL} = \gamma_{NL} |E|^2 E.$$

In a nonisotropic crystal (such as KDP), the nonlinearity is quadratic and model equations for  $P$  are

$$(2.18) \quad \partial_t^2 P_L + R \partial_t P_L + \Omega P_L = \Upsilon E$$

$$(2.19) \quad P_{NL} = \gamma_{NL} (E_2 E_3, E_1 E_2, E_1 E_2)^t.$$

- *The anharmonic model*

To explain nonlinear dispersive phenomena, a simple and natural model is to replace the linear restoring force (2.11) with a nonlinear law (see [Bl],[O])

$$(2.20) \quad \partial_t^2 P + \partial_t P / T_1 + (\nabla V)(P) = bE$$

For small amplitude solutions, the main nonlinear effect is governed by the Taylor expansion of  $V$  at the origin. in presence of symmetries, the first term is cubic, yielding the equation

$$(2.21) \quad \partial_t^2 P + \partial_t P / T_1 + \omega^2 P - \alpha |P|^2 P = \gamma E.$$

- *Maxwell-Bloch equations*

*Bloch's equation* are widely used in nonlinear optics textbooks as a theoretical background for the description of the interaction between light and matter and the propagation of laser beams in nonlinear media. They link  $P$  and the electronic state of the medium which is described through a simplified quantum model, see e.g. [BoWo, NeMo, Boy, PaPu]. The formalism

of density matrices is convenient to account for statistical averaging due for instance to the large number of atoms. The self-adjoint density matrix  $\rho$  satisfies

$$(2.22) \quad i\varepsilon \partial_t \rho = [\Omega, \rho] - [V(E, B), \rho],$$

where  $\Omega$  is the electronic Hamiltonian in absence of external field and  $V(E, B)$  is the potential induced by the external electromagnetic field. For weak fields,  $V$  is expanded into its Taylor's series (see e.g. [PP]). In the dipole approximation,

$$(2.23) \quad V(E, B) = E \cdot \Gamma, \quad P = \text{tr}(\Gamma \rho)$$

where  $-\Gamma$  is the dipole moment operator. An important simplification is that only a finite number of eigenstates of  $\Omega$  are retained. From the physical point of view, they are associated to the electronic levels which are actually in interaction with the electromagnetic field. In this case,  $\rho$  is a complex finite dimensional  $N \times N$  matrix and  $\Gamma$  is a  $N \times N$  matrix with entries in  $\mathbb{C}^3$ . It is hermitian symmetric in the sense that  $\Gamma_{k,j} = \bar{\Gamma}_{j,k}$  so that  $\text{tr}(\Gamma \rho)$  is real. In physics books, the reduction to finite dimensional systems (2.22) come with the introduction of phenomenological damping terms which would force the density matrix to relax towards a thermodynamical equilibrium in absence of the external field. For simplicity, we have omitted these damping terms in the equations above. The large ones only contribute to reduce the size of the effective system and the small ones contribute to perturbations which do not alter qualitatively the phenomena. Physics books also introduce “local field corrections” to improve the model and take into account the electromagnetic field created by the electrons. This mainly results in changing the values of several constants which is of no importance in our discussion.

The parameter  $\varepsilon$  in front of  $\partial_t$  in (2.22) plays a crucial role in the model. The quantities  $\omega_{j,k}/\varepsilon := (\omega_j - \omega_k)/\varepsilon$ , where the  $\omega_j$  are the eigenvalues of  $\Omega$ , have an important physical meaning. They are the characteristic frequencies of the electronic transitions from the level  $k$  to the level  $j$  and therefore related to the energies of these transitions. The interaction between light and matter is understood as a resonance phenomenon and the possibility of excitation of electrons by the field. This means that the energies of the electronic transitions are comparable to the energy of photons. Thus, if one chooses to normalize  $\Omega \approx 1$  as we now assume,  $\varepsilon$  is comparable to the pulsation of light. The Maxwell-Bloch model described above, is expected to be correct for weak fields and small perturbations of the ground state, in particular below ionization phenomena.

• *A two levels model* A simplified version of Bloch's equations for a two levels quantum system for the electrons links the polarization  $P$  of the medium and the difference  $N$  between the numbers of excited and nonexcited atoms:

$$(2.24) \quad \varepsilon^2 \partial_t P + \Omega^2 P = \gamma_1 N E ,$$

$$(2.25) \quad \partial_t N = -\gamma_2 \partial_t P \cdot E .$$

Here,  $\Omega/\varepsilon$  is the frequency associated to the electronic transition between the two levels.

• *Interaction Laser-Plasma*

We give here another example of systems that arise in nonlinear optics. It concerns the propagation of light in a plasma, that is a ionized medium. A classical model for the plasma is a bifluid description for ions and electrons. Then Maxwell equations are coupled to Euler equations for the fluids:

$$(2.26) \quad \partial_t B + c \operatorname{curl} \times E = 0 ,$$

$$(2.27) \quad \partial_t E - c \operatorname{curl} \times B = 4\pi e ((n_0 + n_e)v_e - (n_0 + n_i)v_i) ,$$

$$(2.28) \quad (n_0 + n_e) (\partial_t v_e + v_e \cdot \nabla v_e) = -\frac{\gamma_e T_e}{m_e} \nabla n_e - \frac{e(n_0 + n_e)}{m_e} (E + \frac{1}{c} v_e \times B) ,$$

$$(2.29) \quad (n_0 + n_i) (\partial_t v_i + v_i \cdot \nabla v_i) = -\frac{\gamma_i T_i}{m_i} \nabla n_i + \frac{e(n_0 + n_i)}{m_i} (E + \frac{1}{c} v_i \times B) ,$$

$$(2.30) \quad \partial_t n_e + \nabla \cdot ((n_0 + n_e)v_e) = 0 ,$$

$$(2.31) \quad \partial_t n_i + \nabla \cdot ((n_0 + n_i)v_i) = 0 ,$$

where

\*  $E$  and  $B$  are the electric and magnetic field respectively,

\*  $v_e$  and  $v_i$  denote the velocity of electrons and ions respectively,

\*  $n_0$  is the mean density of the plasma,

\*  $n_e$  and  $n_i$  are the variation of density with respect to the mean density  $n_0$  of electrons and ions respectively.

Moreover,

\*  $c$  is the velocity of light in the vacuum;  $e$  is the elementary electric charge,

\*  $m_e$  and  $m_i$  are the electron's and ion's mass respectively,

\*  $T_e$  and  $T_i$  are the electronic and ionic temperature respectively and  $\gamma_e$  and  $\gamma_i$  the thermodynamic coefficient.



For a precise description of this kind of model, we refer to classical textbooks such as [DeBe]. One of the main points is that the mass of the electrons is very small compared to the mass of the ions :  $m_e \ll m_i$ . Since the Lorentz force is the same for the ions and the electrons, the velocity of the ions will be negligible with respect to the velocity of the electrons. The consequence is that one can neglect the contribution of the ions in equation (2.27).

### 3 The framework of hyperbolic systems

The equations above fall into the general framework of *hyperbolic systems*. In this section we point out a few landmarks in this theory, concerning the local stability and existence theory, and some results of global existence.

#### 3.1 Equations

The general setting of *quasilinear first order systems* concerns equations of the form:

$$(3.1) \quad A_0(a, u)\partial_t u + \sum_{j=1}^d A_j(a, u)\partial_{x_j} u = F(a, u)$$

where  $a$  denote a set of parameters, which may depend and include the time-space variables  $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ ; the  $A_j$  are  $N \times N$  matrices and  $F$  is a function with values in  $\mathbb{R}^N$ ; they depend on the variables  $(a, u)$  varying in a subdomain of  $\mathbb{R}^M \times \mathbb{R}^N$  and we assume that  $F(0, 0) = 0$ . (Second order equations such as (2.11) or (2.21) are reduced to first order by introducing  $Q = \partial_t P$ ).

An important case is the case of *balance laws*

$$(3.2) \quad \partial_t f_0(u) + \sum_{j=1}^d \partial_{x_j} f_j(u) = F(u)$$

or *conservation laws* if  $F = 0$ . For smooth enough solutions, the chain rule can be applied and this system is equivalent to

$$(3.3) \quad A_0(u)\partial_t u + \sum_{j=1}^d A_j(u)\partial_{x_j} u = F(u)$$

with  $A_j(u) = \nabla_u f_j(u)$ . Examples of quasilinear systems are Maxwell's equations with the Kerr nonlinearity (2.17) or Euler-Maxwell equations.

The system is *semilinear* when the  $A_j$  do not depend on  $u$ :

$$(3.4) \quad A_0(a)\partial_t u + \sum_{j=1}^d A_j(a)\partial_{x_j} u = F(b, u)$$

Examples are the anharmonic model (2.21) or Maxwell-Bloch equations.

The system is *linear* when the  $A_j$  do not depend on  $u$  and  $F$  is affine in  $u$ , i.e. of the form  $F(b, u) = f + E(b)u$ . This is the case of systems such as (2.7), (2.8) or the Lorentz model.

Consider a solution  $u_0$  and the equation for small variations  $u = u_0 + \varepsilon v$ . Expanding in power series in  $\varepsilon$  yields at first order *the linearized equations*:

$$(3.5) \quad A_0(a, u_0)\partial_t v + \sum_{j=1}^d A_j(a, u_0)\partial_{x_j} v + E(t, x)v = 0$$

where

$$E(t, x)v = (v \cdot \nabla_u A_0)\partial_t u_0 + \sum_{j=1}^d (v \cdot \nabla_u A_j)\partial_{x_j} u_0 - v \cdot \nabla_u F$$

and the gradients  $\nabla_u A_j$  and  $\nabla_u F$  are evaluated at  $(a, u_0(t, x))$ .

In particular, the linearized equations from (3.2) or (3.3) near a constant solution  $u_0(t, x) = \underline{u}$  are the *constant coefficients equations*

$$(3.6) \quad A_0(\underline{u})\partial_t u + \sum_{j=1}^d A_j(\underline{u})\partial_{x_j} u = F'(\underline{u})u.$$

*The example of Maxwell's equations.*

Consider Maxwell's equation with no charge and no current:

$$(3.7) \quad \partial_t D - \operatorname{curl} H = -j, \quad \partial_t B + \operatorname{curl} E = 0,$$

$$(3.8) \quad \operatorname{div} B = 0, \quad \operatorname{div} D = 0,$$

together with constitutive relations between the fields as explained in Section 2. This system is not immediately of the form (3.1): it is *overdetermined* as it involves more equations than unknowns and as there is no  $\partial_t$  in the second set of equations. However, it satisfies *compatibility* conditions<sup>2</sup>: the first two equations (3.7) imply that

$$(3.9) \quad \partial_t \operatorname{div} B = 0, \quad \partial_t \operatorname{div} D = 0,$$

so that the *constraint conditions* (3.8) are satisfied for all time by solutions of (3.7) if and only if they are satisfied at time  $t = 0$ . As a consequence, one studies the evolution system (3.7) alone, which is of the form (3.1), and considers the constraints (3.8) as conditions on the initial data. With this modification, the framework of hyperbolic equations is well adapted to the various models involving Maxwell's equations.

For instance, the Lorentz model is the linearization both of the anharmonic model and of the Kerr Model at  $\underline{E} = 0, \underline{P} = 0$ .

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<sup>2</sup>This is a special case of a much more general phenomenon for fields equations, where the equations are linked through Bianchi's identities.

### 3.2 The dispersion relation & polarization conditions

Consider a *linear constant coefficient* system such as (3.6):

$$(3.10) \quad Lu := A_0 \partial_t u + \sum_{j=1}^d A_j \partial_{x_j} u + Eu = f$$

Particular solutions of the homogeneous equation  $Lu = 0$  are *plane waves*:

$$(3.11) \quad u(t, x) = e^{it\tau + ix \cdot \xi} a$$

where  $(\tau, \xi)$  satisfy the *dispersion relation* :

$$(3.12) \quad \det \left( i\tau A_0 + \sum_{j=1}^d i\xi_j A_j + E \right) = 0$$

and the constant vector  $a$  satisfies the *polarization condition*

$$(3.13) \quad a \in \ker \left( i\tau A_0 + \sum_{j=1}^d i\xi_j A_j + E \right).$$

The matrix  $i\tau A_0 + \sum_{j=1}^d i\xi_j A_j + E$  is called *the symbol* of  $L$ .

In many applications, the coefficients  $A_j$  and  $E$  are real and one is interested in real functions. In this case (3.11) is to be replaced by  $u = \text{Re}(e^{it\tau + ix \cdot \xi} a)$ .

When  $A_0$  is invertible, the equation (3.12) means that  $-\tau$  is an eigenvalue of  $\sum \xi_j A_0^{-1} A_j - iA_0^{-1} E$  and the polarization condition (3.13) means that  $a$  is an eigenvector.

We now illustrate these notions on four examples involving Maxwell's equations. Following the general strategy explained above, we forget the divergence equations (3.8). However, this has the effect of adding extra and unphysical eigenvalues  $\tau = 0$ , with eigenspace  $\{B \in \mathbb{R}\xi, D \in \mathbb{R}\xi\}$  incompatible with the divergence relations, which for plane waves require that  $\xi \cdot B = \xi \cdot D = 0$ . Therefore, these extra eigenvalues must be discarded in the physical interpretation of the problem.

- For the *Lorentz model*, the dispersion relation reads

$$(3.14) \quad \tau^2(\delta - \gamma_0)(\tau^2(\delta - \gamma_0) - c^2\delta|\xi|^2) = 0, \quad \delta = \tau^2 - i\tau/T_1 - \omega^2.$$

The root  $\tau = 0$  are unphysical as explained above. The roots of  $\delta - \gamma_0 = 0$  (that is  $\tau = \pm\sqrt{\omega^2 + \gamma}$  in the case  $\frac{1}{T_1} = 0$ ) do not correspond to optical

waves, since the corresponding waves propagate at speed 0 (see Section 5). The optical plane waves are associated to roots of the third factor. They satisfy

$$(3.15) \quad c^2|\xi|^2 = \tau^2(1 + \widehat{\chi}(\tau)), \quad \widehat{\chi}(\tau) = \frac{\gamma_0}{\omega^2 + i\tau/T_1 - \tau^2}.$$

For  $\xi \neq 0$ , they have multiplicity two and the polarization conditions are

$$(3.16) \quad E \in \xi^\perp, \quad P = \varepsilon_0 \widehat{\chi}(\tau) E, \quad B = -\frac{\xi}{\tau} \times E.$$

• Consider the *two level Maxwell-Bloch equations*. The linearized equations around  $\underline{E} = \underline{B} = \underline{P} = 0$  and  $\underline{N} = N_0 > 0$  read (in suitable units)

$$(3.17) \quad \begin{aligned} \partial_t B + \operatorname{curl} E &= 0, & \partial_t E - \operatorname{curl} B &= -\partial_t P \\ \varepsilon \partial_t^2 P + \Omega^2 P &= \gamma_1 N_0 E, & \partial_t N &= 0 \end{aligned}$$

This is the Lorentz model with coupling constant  $\gamma = \gamma_1 N_0$  augmented with the equation  $\partial_t N = 0$ . Thus we are back to the previous example.

• For *crystal optics*, in units where  $c = 1$ , the plane wave equations read

$$(3.18) \quad \begin{cases} \tau E - \varepsilon^{-1}(\xi \times B) = 0, \\ \tau B + \xi \times E = 0. \end{cases}$$

In coordinates where  $\varepsilon$  is diagonal with diagonal entries  $\alpha_1 > \alpha_2 > \alpha_3$ , the dispersion relation read

$$(3.19) \quad \tau^2(\tau^4 - \Psi(\xi)\tau^2 + |\xi|^2 \Phi(\xi))$$

with

$$\begin{cases} \Psi(\xi) = (\alpha_1 + \alpha_2)\xi_3^2 + (\alpha_2 + \alpha_3)\xi_1^2 + (\alpha_3 + \alpha_1)\xi_2^2 \\ \Phi(\xi) = \alpha_1\alpha_2\xi_3^2 + \alpha_2\alpha_3\xi_1^2 + \alpha_3\alpha_1\xi_2^2. \end{cases}$$

For  $\xi \neq 0$ ,  $\tau = 0$  is again a double eigenvalue. The nonvanishing eigenvalues are solutions of a second order equations in  $\tau^2$ , of which the discriminant is

$$\Psi^2(\xi) - 4|\xi|^2\Phi(\xi) = P^2 + Q^2$$

with

$$\begin{aligned} P &= (\alpha_1 - \alpha_2)\xi_3^2 + (\alpha_3 - \alpha_2)\xi_1^2 + (\alpha_3 - \alpha_1)\xi_2^2 \\ Q &= 2(\alpha_1 - \alpha_2)^{\frac{1}{2}}(\alpha_1 - \alpha_3)^{\frac{1}{2}}\xi_3\xi_2. \end{aligned}$$

For a bi-axial crystal,  $\varepsilon$  has three distinct eigenvalues. For general frequency  $\xi$ ,  $P^2 + Q^2 \neq 0$  and there are four simple eigenvalues,  $\pm \frac{1}{2} \left( \Psi \pm (P^2 + Q^2)^{\frac{1}{2}} \right)^{\frac{1}{2}}$ . There are double roots exactly when  $P^2 + Q^2 = 0$ , that is when

$$(3.20) \quad \xi_2 = 0, \quad \alpha_1 \xi_3^2 + \alpha_3 \xi_1^2 = \alpha_2 (\xi_1^2 + \xi_3^2) = \tau^2.$$

This is an example where the multiplicities of the eigenvalues change with  $\xi$ .

• Consider the *linearized Maxwell-Bloch equations* around  $\underline{E} = \underline{B} = 0$  and  $\underline{\rho}$  equal to the fundamental state i.e. the eigenprojector associated to the smallest eigenvalue  $\omega_1$  of  $\Omega$ . In appropriate units, they read

$$(3.21) \quad \begin{cases} \partial_t B + \text{curl} E = 0, \\ \partial_t E - \text{curl} B - i \text{tr}(\Gamma[\Omega, \rho]) + i \text{tr}(\Gamma(E \cdot G)) = 0, \\ \partial_t \rho + i[\Omega, \rho] - i E \cdot G = 0, \end{cases}$$

with  $G := [\Gamma, \rho]$ . The general expression of the dispersion relation is not simple, one reason for that is the lack of isotropy of the general model above. To simplify, we assume that the fundamental state is simple, that  $\Omega$  is diagonal with entries  $\alpha_j$  and we denote by  $\omega_j$  the *distinct* eigenvalues of  $\Omega$ , with  $\omega_1 = \alpha_1$ . Then  $G$  has the form

$$G = \begin{pmatrix} 0 & -\Gamma_{1,2} & \dots \\ \Gamma_{2,1} & 0 & \dots \\ \vdots & \vdots & 0 \end{pmatrix}$$

Assume that the models satisfies the following isotropy condition: for all  $\omega_j > \omega_1$  :

$$(3.22) \quad \sum_{\{k: \alpha_k = \omega_j\}} (E \cdot \Gamma_{1,k}) \Gamma_{k,1} = \gamma_j E$$

with  $\gamma_j \in \mathbb{C}$ . Then the optical frequencies satisfy

$$(3.23) \quad |\xi|^2 = \tau^2 (1 + \hat{\chi}(\tau)), \quad \hat{\chi}(\tau) = \sum 2 \frac{\text{Re} \gamma_j (\omega_j - \omega_1) + i \tau \text{Im} \gamma_j}{(\omega_j - \omega_1)^2 - \tau^2}$$

and the associated polarization conditions are

$$(3.24) \quad E \in \xi^\perp, \quad B = -\frac{\xi}{\tau} \times E, \quad \rho_{1,k} = \bar{\rho}_{k,1} = \frac{E \cdot \Gamma_{1,k}}{\alpha_k - \alpha_1 - \tau}$$

with the other entries  $\rho_{j,k}$  equal to 0.

### 3.3 Existence and stability

The equations presented in Section 2 fall into the category of *symmetric hyperbolic* systems. More precisely they satisfy the following condition:

**Definition 3.1** (Symmetry). *A system (3.1) is said to be symmetric hyperbolic in the sense of Friedrichs if there is a matrix  $S(a, u)$*

- *it is a  $C^\infty$  function of its arguments ;*
- *for all  $j$ ,  $a$  and  $u$ , the matrices  $S(a, u)A_j(a, u)$  are self-adjoint and in addition  $S(a, u)A_0(a, u)$  is positive definite.*

The Cauchy problem consists in solving the equation (3.1) together with initial conditions

$$(3.25) \quad u|_{t=0} = h.$$

The first basic result of the theory is the local existence of smooth solutions:

**Theorem 3.2** (Local Existence). *Suppose that the system (3.1) is symmetric hyperbolic. Then for  $s > \frac{d}{2} + 1$ ,  $h \in H^s(\mathbb{R}^d)$  and  $a \in C^0([0; T]; H^s(\mathbb{R}^d))$  such that  $\partial_t a \in C^0([0; T]; H^{s-1}(\mathbb{R}^d))$ , there is  $T' > 0$ ,  $T' \leq T$ , which depends only on the norms of  $a$ ,  $\partial_t a$  and  $h$ , such that the Cauchy problem has a unique solution  $u \in C^0([0; T']; H^s(\mathbb{R}^d))$ .*

In the semilinear case, that is when the matrices  $A_j$  do not depend on  $u$ , the limiting lower value for the local existence is  $s > \frac{d}{2}$ :

**Theorem 3.3.** *Consider the semilinear system (3.4) assumed to be symmetric hyperbolic. Suppose that  $a \in C^0([0; T]; H^\sigma(\mathbb{R}^d))$  is such that  $\partial_t a \in C^0([0; T]; H^{\sigma-1}(\mathbb{R}^d))$  where  $\sigma > \frac{d}{2} + 1$ . Then, for  $\frac{d}{2} < s \leq \sigma$ ,  $h \in H^s(\mathbb{R}^d)$  and  $b \in C^0([0; T]; H^s(\mathbb{R}^d))$ , there is  $T' > 0$ ,  $T' \leq T$  such that the Cauchy problem with initial data  $h$  has a unique solution  $u \in C^0([0; T']; H^s(\mathbb{R}^d))$ .*

As it is important in the understanding of the remaining part of these notes we will give the main steps in the proof of this important result. The analysis of linear symmetric hyperbolic problems goes back to [Fr1, Fr2]. For the nonlinear version we refer to [Miz, Maj, Hö3, Tay].

*Scheme of the proof.* Solutions can be constructed through an iterative scheme

$$(3.26) \quad \begin{cases} A_0(a, u_n)\partial_t u_{n+1} + \sum_{j=1}^d A_j(a, u_n)\partial_{x_j} u_{n+1} = F(a, u_n), \\ u_{n+1}|_{t=0} = h. \end{cases}$$

There are four steps:

1 - [Definition of the scheme.] Prove that if  $u_n \in C^0([0; T]; H^s(\mathbb{R}^d))$  and  $\partial_t u_n \in C^0([0; T]; H^{s-1}(\mathbb{R}^d))$ , the system has a solution  $u_{n+1}$  with the same smoothness;

2 - [Boundedness in high norm.] Prove that there is  $T' > 0$  such that the sequence is bounded in  $C^0([0; T']; H^s(\mathbb{R}^d))$  and  $C^1([0; T']; H^{s-1}(\mathbb{R}^d))$ .

3 - [Convergence in low norm.] Prove that the sequence converges in  $C^0([0; T']; L^2(\mathbb{R}^d))$ . Together with the uniform bounds, this implies that the convergence holds in  $C^0([0; T']; H^{s'}(\mathbb{R}^d))$  and in  $C^1([0; T']; H^{s'-1}(\mathbb{R}^d))$  for all  $s' < s$ . Since  $s > \frac{d}{2} + 1$ , the convergence holds in  $C^1([0, T'] \times \mathbb{R}^d)$  and the limit  $u$  is a solution of the cauchy problem (3.1) (3.25). The convergence also holds in  $C^0([0; T']; H_w^s(\mathbb{R}^d))$  where  $H_w^s$  denotes the space  $H^s$  equipped with the weak topology and  $u \in L^\infty([0, T'], H^s(\mathbb{R}^d)) \cap C^0([0; T']; H_w^s(\mathbb{R}^d))$ .

4 - [Strong continuity.] Use the equation to prove that  $u$  is actually continuous in time with values in  $H^s(\mathbb{R}^d)$  equipped with the strong topology.

This analysis relies on the study of linear problems

$$(3.27) \quad \begin{cases} L(a, \partial)u := A_0(a)\partial_t u + \sum_{j=1}^d A_j(a)\partial_{x_j} u = f \\ u|_{t=0} = h. \end{cases}$$

The main step is to prove *a priori estimates* for the solutions of such systems.

**Proposition 3.4.** *If the system is symmetric hyperbolic, then for  $u$  smooth enough*

$$(3.28) \quad \begin{aligned} \|u(t)\|_{H^s} &\leq C_0 e^{(K_0 + K_s)t} \|u(0)\|_{H^s} \\ &+ C_0 \int_t^{t'} e^{(K_0 + K_s)(t-t')} \|L(a, \partial)u\|_{H^s} dt' \end{aligned}$$

where  $C_0$  [resp.  $K_0$ ] [resp.  $K_s$ ] depends only on the  $L^\infty$  norm [resp.  $W^{1,\infty}$  norm] [resp.  $L^\infty(H^s)$  norm] of  $a$  on  $[0, T] \times \mathbb{R}^d$ .

They are used first to prove existence and uniqueness of solutions and next to control the solutions. In particular, they serve to prove points 1 and 2 in the scheme above. The convergence in low norm is also a consequence of energy estimates in  $L^2$  ( $s = 0$ ) applied to differences  $u_{n+1} - u_n$ . The additional smoothness consists in proving that if  $a \in L^\infty([0, T']; H^s)$  and  $\partial_t a \in L^\infty([0, T']; H^{s-1})$ , then the solution  $u$  actually belongs to  $C^0([0, T']; H^s)$ .  $\square$



*Notes on the proof of Proposition 3.4.*

When  $s = 0$ , the estimate (with  $K_s = 0$ ) follows easily by multiplying the equation by  $S(a)u$  and integration by parts, using the symmetry properties of  $SA_j$  and the positivity of  $SA_0$ .

When  $s$  is a positive integer,  $s > \frac{d}{2} + 1$ , the estimates of the derivatives are deduced from the  $L^2$  estimates writing

$$(3.29) \quad L(a, \partial)\partial_x^\alpha u = A_0\partial_x^\alpha(A_0^{-1}L(a, \partial)u) - A_0[\partial_x^\alpha, A_0^{-1}L(a, \partial)]u$$

and commutation estimates for  $|\alpha| \leq s$ :

$$(3.30) \quad \|[\partial_x^\alpha, A_0^{-1}L(a, \partial)]u(t)\|_{L^2} \leq K_s \|u(t)\|_{H^s}$$

where  $K_s$  depends only on the  $H^s$  norm of  $a(t)$ .

The bound (3.30) follow from two classical nonlinear estimates which are recalled in the lemma below.

**Lemma 3.5.** *For  $\sigma > \frac{d}{2}$ ,*

*i)  $H^\sigma(\mathbb{R}^d)$  is a Banach algebra embedded in  $L^\infty(\mathbb{R}^d)$ .*

*ii) for  $u \in H^{\sigma-l}(\mathbb{R}^d)$  and  $v \in H^{\sigma-m}(\mathbb{R}^d)$  with  $l \geq 0$ ,  $m \geq 0$  and  $l + m \leq \sigma$ , the product  $w$  belongs to  $H^{\sigma-l-m}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ .*

*iii) For  $s > \frac{d}{2}$  and  $F$  a smooth function such that  $F(0) = 0$ , the mapping  $u \mapsto F(u)$  is continuous from  $H^s(\mathbb{R}^d)$  into itself and maps bounded sets to bounded sets.*

Indeed, the commutator  $[\partial_x^\alpha, A_0^{-1}L(a, \partial)]u$  is a linear combination of terms

$$(3.31) \quad \partial_x^\gamma \partial_{x_k} A_j(a) \partial_x^\beta \partial_{x_j} u, \quad |\beta| + |\gamma| \leq |\alpha| - 1.$$

Noticing that  $A_j(a(t)) \in H^s(\mathbb{R}^d)$ , and applying *ii)* of the lemma with  $\sigma = s - 1$  yields the estimate (3.30).  $\square$

**Remark 3.6.** The use of  $L^2$  based Sobolev spaces is well adapted to the framework of symmetric systems, but it is also dictated by the consideration of multidimensional problems (see [Ra1]).

Besides the existence statement, which is interesting from a mathematical point of view, the proof of Theorem 3.2 contains an important *stability* result:

**Theorem 3.7** (Local stability). *Under assumptions of Theorem 3.2, if  $u_0 \in C^0([0, T], H^s(\mathbb{R}^d))$  is a solution of (3.1), then there is  $\delta > 0$  such that for all initial data  $h$  such that  $\|h - u_0(0)\|_{H^s} \leq \delta$ , the Cauchy problem for with*

initial data  $h$  has a unique solution  $u \in C^0([0, T], H^s(\mathbb{R}^d))$ . Moreover, for all  $t \in [0, T]$ , the mapping  $h \mapsto u(t)$  defined in this way, is Lipschitz continuous in the  $H^{s-1}$  norm and continuous in the  $H^{s'}$  norm for all  $s' < s$ , uniformly in  $t$ .

### 3.4 Continuation of solutions

Uniqueness in Theorem 3.2 allows one to define *the maximal interval of existence of a smooth solution* : under the assumptions of this theorem, let  $T^*$  be the supremum of the  $T' \in [0, T]$  such that the Cauchy problem as a solution in  $C^0([0, T']; H^s(\mathbb{R}^d))$ . By uniqueness, this defines a (unique) solution  $u \in C^0([0, T^*]; H^s(\mathbb{R}^d))$ . With this definition, Theorem 3.2 immediately implies the following

**Lemma 3.8.** *It  $T^* < T$ , then*

$$\sup_{t \in [0, T^*[} \|u(t)\|_{H^s} = +\infty.$$

We mention here the more precise result (see e.g. [Ma.j]):

**Theorem 3.9.** *It  $T^* < T$ , then*

$$\sup_{t \in [0, T^*[} \|u(t)\|_{L^\infty} + \|\nabla_x u(t)\|_{L^\infty} = +\infty.$$

We also deduce from Theorem 3.2 two continuation arguments based on *a priori* estimates :

**Lemma 3.10.** *Suppose that there is a constant  $C$  such that if  $T' \in ]0, T]$  and  $u \in C^0([0, T']; H^s)$  is solution of the Cauchy problem for (3.1) with initial data  $h \in H^s$ , then it satisfies for  $t \in [0, T']$ :*

$$(3.32) \quad \|u(t)\|_{H^s} \leq C.$$

*Then, the maximal solution corresponding to the initial data  $h$  is defined on  $[0, T]$  and satisfies (3.32) on  $[0, T]$ .*

**Lemma 3.11.** *Suppose that there are constants  $C$  and  $C' > C \geq \|h\|_{H^s}$  such that if  $u \in C^0([0, T']; H^s)$  is solution of the Cauchy problem for (3.1) with initial data  $h \in H^s$  the following implication holds:*

$$(3.33) \quad \sup_{t \in [0, T']} \|u(t)\|_{H^s} \leq C' \quad \Rightarrow \quad \sup_{t \in [0, T']} \|u(t)\|_{H^s} \leq C.$$

*Then, the maximal solution is defined on  $[0, T]$  and satisfies (3.32) on  $[0, T]$ .*

### 3.5 Global existence

As mentioned in the introduction, the general theorem of local existence is of little use for high frequency initial data, since the time of existence depends on high regularity norms and thus may be very small. In the theory of hyperbolic equations, there is a huge literature on global existence and stability results. We do not mention here the results which concern *small data* (see e.g. [Hö3] and the references therein), since the smallness is again measured in high order Sobolev spaces and thus is difficult to apply to high frequency solutions.

There is another class of classical global existence theorem of *weak or energy* solutions for hyperbolic *maximal dissipative* equations, using only the conservation or dissipation of energy and a weak compactness argument (see e.g. [LiSt] and Section 7.6.4 below.)

In this section, we illustrate on an example another approach which is more adapted to our context. Indeed, for most of the physical examples, there are conserved (or dissipated) quantities, such as energies. This provides *a priori* estimates that are valid for all time, independently of the size of the data. The question is to use this particular additional information to improve the general analysis and eventually conclude to global existence.

The case of two levels Maxwell-Bloch equations has been studied in [DoRa2]. Their results have been extended to general Maxwell Bloch equations (2.22) in [Dum3] and to the anharmonic model (2.20) in [JMR5] (see also [JMR11] for Maxwell's equations in a ferromagnetic medium). In the remaining part of this section, we briefly present the example of levels Maxwell-Bloch equations. Recall the equations from Section 2:

$$(3.34) \quad \begin{cases} \partial_t B + \operatorname{curl} E = 0, \\ \partial_t(E + P) - \operatorname{curl} B = 0, \\ \partial_t P + \Omega^2 P = \gamma_1 N E, \\ \partial_t N = -\gamma_2 \partial_t P \cdot E, \end{cases}$$

together with the constraints

$$(3.35) \quad \operatorname{div}(E + P) = \operatorname{div} B = 0.$$

Recall that  $N$  is the difference between the number of electrons in the excited state and in the ground state per unit of volume.  $N_0$  is the equilibrium value of  $N$ . This system can be written as a first order semilinear symmetric hyperbolic system for

$$(3.36) \quad U = (B, E, P, \partial_t P, N - N_0).$$

Since the system is semilinear, with matrices  $A_j$  that are constant, the local existence theorem proves that the Cauchy problem is locally well posed in  $H^s(\mathbb{R}^3)$  for  $s > \frac{3}{2}$ , see Theorem 3.3. The special form of the system implies that the maximal time of existence is  $T^* = +\infty$ :

**Theorem 3.12.** *If  $s \geq 2$  and the initial data  $U(0) \in H^s(\mathbb{R}^3)$  satisfies (3.35), then the Cauchy problem for (3.34) has a unique solution  $U \in C^0([0, +\infty[; H^s(\mathbb{R}^3))$ , which satisfies (3.35) for all time.*

*Notes on the proof.* (see [DoRa2]).

The total energy

$$\mathcal{E} = N_0 \|B\|_{L^2}^2 + N_0 \|E\|_{L^2}^2 + \frac{\Omega^2}{\gamma_1} \|P\|_{L^2}^2 + \frac{1}{\gamma_1} \|\partial_t P\|_{L^2}^2 + \frac{1}{\gamma_2} \|N - N_0\|_{L^2}^2$$

is conserved, proving that  $U$  remains bounded in  $L^2$  for all time.

There is also a pointwise conservation of

$$\frac{1}{\gamma_1} |\partial_t P|^2 + \frac{\Omega^2}{\gamma_1} |P|^2 + \frac{1}{\gamma_2} |N|^2$$

proving that  $P$ ,  $\partial_t P$  and  $N$  remain bounded in  $L^\infty$  for all time.

The  $H^1$  estimates are obtained by differentiating the equations with respect to  $x$ :

$$\begin{cases} \partial_t \partial B + \text{curl} \partial E = 0, \\ \partial_t \partial E - \text{curl} \partial B = -\partial_t \partial P, \\ \partial_t \partial P + \Omega^2 \partial P = \gamma_1 \partial(N E), \\ \partial_t \partial N = -\gamma_2 \partial(\partial_t P \cdot E), \end{cases}$$

Then

$$\mathcal{E}_1 = \|\partial B\|_{L^2}^2 + \|\partial E\|_{L^2}^2 + \frac{\Omega^2}{\gamma_1} \|\partial P\|_{L^2}^2 + \frac{1}{\gamma_1} \|\partial_t \partial P\|_{L^2}^2 + \frac{1}{\gamma_2} \|\partial N\|_{L^2}^2$$

satisfies

$$\partial_t \mathcal{E}_1 = 2 \int \Phi dx$$

with

$$\begin{aligned} \Phi &= -(\partial Q) \partial E + \partial(N E) \partial Q - \partial(Q E) \partial N \\ &= -(\partial Q) \partial E + N \partial E \partial Q - \partial N Q \partial E, \end{aligned}$$

with  $Q = \partial_t P$ . Thus using the known  $L^\infty$  bounds for  $N$  and  $Q$  implies that

$$\partial_e \mathcal{E}_1 \leq C \mathcal{E}_1$$

implying that  $\mathcal{E}_1(t) \leq e^{Ct} \mathcal{E}_1(0)$  for all time.

The estimates of the second derivatives is more subtle, but follows the same ideas: use the known  $L^2$ ,  $L^\infty$  and  $H^1$  bounds to obtain  $H^2$  estimates valid for all time. For the details we refer to [DoRa2].  $\square$

Using the a priori  $H^1$  bounds, it is not difficult to prove the global existence of global  $H^1$  solutions (see [DoRa2]):

**Theorem 3.13.** *For arbitrary  $U(0) \in H^1(\mathbb{R}^3)$  satisfying (3.35) and such that  $(P(0), \partial_t P(0), N(0)) \in L^\infty(\mathbb{R}^3)$ , there is a unique global solution  $U$  such that for all  $T > 0$   $U \in L^\infty([0, T]; H^1(\mathbb{R}^3))$  and  $(P, \partial_t P, N) \in L^\infty([0, +\infty[ \times \mathbb{R}^3)$ .*

### 3.6 Local results

A fundamental property of hyperbolic systems is that they reproduce the physical idea that *waves propagate at a finite speed*. Consider solutions of a symmetric hyperbolic linear equation

$$(3.37) \quad L(a, \partial)u := A_0(a)\partial_t u + \sum_{j=1}^d A_j(a)\partial_{x_j} u + E(a)u = f$$

on domains of the form:

$$(3.38) \quad \Omega = \{(t, x) : t \geq 0, |x| + t\lambda_* \leq R\},$$

Let  $\omega = \{x : |x| \leq R\}$ . One has the following result

**Proposition 3.14 (Local uniqueness).** *There is a real valued function  $\lambda_*(M)$ , which depends only the matrices  $A_j$  in the principal part such that if  $\lambda_* \geq \lambda_*(M)$ ,  $a$  is Lipschitz continuous on  $\bar{\Omega}$  with  $|a(t, x)| \leq M$  on  $\Omega$ ,  $u \in H^1(\Omega)$  satisfies (3.37) on  $\Omega$  with  $f = 0$  and  $u|_{t=0} = 0$  on  $\omega$ , then  $u = 0$  on  $\Omega$ .*

*Proof.* By Green's formula

$$\begin{aligned} 0 = 2\text{Re} \int_{\Omega} e^{-\gamma t} L u \cdot \bar{u} dt dx &= \int_{\Omega} e^{-\gamma t} (\gamma A_0 u - K u) \cdot \bar{u} dt dx \\ &\quad + \int_{\Sigma} e^{-\gamma t} L(a, \nu) u \cdot u d\Sigma \end{aligned}$$

where  $K = \partial_t A_0(a) + \sum \partial_{x_j} A_j(a)$ ,  $\Sigma = \{\lambda_* t + |x| = R\}$  and  $L(a, \nu) = \sum \nu_j A_j$  is the symbol of  $L$  in co-direction  $\nu = (\nu_0, \dots, \nu_d)$  which is the exterior normal to  $\Omega$ . Because  $\nu$  is proportional to  $(\lambda_*, x_1/|x|, \dots, x_d/|x|)$ , if  $\lambda_*$

is large enough, the matrix  $L(a, \nu)$  is nonnegative. More precisely, this condition is satisfied if

$$(3.39) \quad \lambda_* \geq \lambda_*(M) = \sup_{|a| \leq M} \sup_{|\xi|=1} \sup_p |\lambda_p(a, \xi)|$$

where the  $\lambda_p(a, \xi)$  denote the eigenvalues of  $A_0(a)^{-1} \sum_{j=1}^d \xi_j A_j(a)$ .

If  $\gamma$  is large enough, the matrix  $\gamma A_0 - K$  is definite positive, and the energy identity above implies that  $u = 0$  on  $\Omega$ .  $\square$

This result implies that the solution  $u$  of (3.37) is uniquely determined in  $\Omega$  by the values of the source term  $f$  on  $\Omega$  and the values of the initial data on  $\omega$ . One says that  $\Omega$  is contained in the domain of determinacy of  $\omega$ .

The proposition can be improved, giving the optimal domain of determinacy  $\Omega$  associated to an initial domain  $\omega$ , see [Lax2, Ler, JMR12] and the references therein.

On domains  $\Omega$ , one uses the following spaces:

**Definition 3.15.** *We say that  $u$  defined on  $\Omega$  is continuous in time with values in  $L^2$  if its extension by 0 outside  $\Omega$  belongs to  $C^0([0, T_0]; L^2(\mathbb{R}^d))$ ; for  $s \in \mathbb{N}$ , we say that  $u$  is continuous with values in  $H^s$  if the derivatives  $\partial_x^\alpha u$  for  $|\alpha| \leq s$  are continuous in time with values in  $L^2$ . We denote these spaces by  $C^0 H^s(\Omega)$ .*

Proposition (3.14) extends to semilinear equations, as the domain  $\Omega$  does not depend on the source term  $f(u)$ . The energy estimates can be localized on  $\Omega$ , using integrations by parts on  $\Omega$ , and Theorem 3.3 can be extended as follows:

**Theorem 3.16.** *Consider the semilinear system (3.4) assumed to be symmetric hyperbolic. Suppose that  $a \in C^0 H^\sigma(\Omega)$  is such that  $\partial_t a \in C^0 H^{\sigma-1}(\Omega)$  where  $\sigma > \frac{d}{2} + 1$  and  $\|a\|_{L^\infty(\Omega)} \leq M$ . Then, for  $\frac{d}{2} < s \leq \sigma$ ,  $h \in H^s(\omega)$  and  $b \in C^0 H^s(\Omega)$ , there is  $T > 0$ , such that the Cauchy problem with initial data  $h$  has a unique solution  $u \in C^0 H^s(\Omega \cap \{t \leq T\})$ .*

For quasilinear systems (3.1), the situation is more intricate since then the eigenvalues now depend on the solution itself, so that  $\lambda_*(M)$  in (3.39) must be replaced by a function  $\lambda_*(M, M')$  which dominates the eigenvalues of  $A_0(a, u)^{-1} \sum \xi_j A_j(a, u)$  when  $|a| \leq M$  and  $|u| \leq M'$ . Note that  $\lambda_*(M, M')$  is a continuous increasing function of  $M'$ , so that if  $\lambda_* > \lambda_*(M, M')$  then  $\lambda_* \geq \lambda_*(M, M'')$  for a  $M'' > M'$ .

**Theorem 3.17.** *Suppose that the system (3.1) is symmetric hyperbolic. Fix  $M$ ,  $M'$  and  $s > \frac{d}{2} + 1$ . Let  $\Omega$  denote the set (3.38) with  $\lambda_* > \lambda(M, M')$ . Let  $h \in H^s(\mathbb{R}^d)$  and  $a \in C^0H^s(\Omega)$  be such that  $\|h\|_{L^\infty(\omega)} \leq M'$ ,  $\partial_t a \in C^0H^{s-1}(\Omega)$  and  $\|a\|_{L^\infty(\Omega)} \leq M$ . Then there is  $T > 0$ , such that the Cauchy problem has a unique solution  $u \in C^0H^s(\Omega \cap \{t \leq T\})$ .*

## 4 Equations with parameters

A general feature of problems in optics is that very different scales are present: for instance, the wavelength of the light beam is much smaller than the length of propagation, the length of the beam is much larger than its width. Many models (Lorentz, anharmonic, Maxwell-Bloch, Euler-Maxwell) contain many parameters, which may be large or small. In applications to optics, we are facing two opposite requirements:

- optics concern high frequency regimes, that is functions Fourier transform localized around large values of the wave number  $\xi$ ;
- we want to consider waves with large enough amplitude so that non-linear effects are present in the propagation of the main amplitude.

Obviously, large frequencies and not too small amplitudes are incompatible with uniform  $H^s$  bounds for large  $s$ . Therefore, a direct application of Theorem 3.2 for highly oscillatory and not small data, yields existence and stability for  $t \in [0, T]$  with  $T$  very small, and often much smaller than any relevant physical time in the problem. This is why, one has to keep track of the parameters in the equations and to look for existence or stability results which are independent of these parameters. In this section, we give two examples of such results, which will be used for solving envelope equations and proving the stability of approximate solutions respectively.

Note that all the results given in this section have local analogues on domains of determinacy, in the spirit of Section 3.6.

### 4.1 Singular equations

We start with two examples which will serve as a motivation:

- The Lorentz model (2.11) (see also (2.21) and (2.24)) contain a large parameter  $\omega$  in front of the zero-th order term. Similarly, Bloch's equations (2.22) contain a small parameter  $\varepsilon$  in front of the derivative  $\partial_t$ .

- To take into account the multiscale character of the phenomena, one can introduce explicitly the *fast scales* and look for solutions of (3.1) of the form

$$(4.1) \quad u(t, x) = U(t, x, \varphi(t, x)/\varepsilon)$$

where  $\varphi$  is valued in  $\mathbb{R}^m$  and  $U$  is a function of  $(t, x)$  and additional *independent* variables  $y = (y_1, \dots, y_m)$ .

Both cases lead to equations of the form

$$(4.2) \quad A_0(a, u)\partial_t u + \sum_{j=1}^d A_j(a, u)\partial_{x_j} u + \frac{1}{\varepsilon}\mathcal{L}(a, u, \partial_x)u = F(a, u)$$



with possibly an augmented number of variables  $x_j$  and an augmented number of parameters  $a$ . In (4.2)

$$(4.3) \quad \mathcal{L}(a, u, \partial_x) = \sum_{j=1}^d \mathcal{L}_j(a, u) \partial_{x_j} + \mathcal{L}_0(a, u).$$

We again assume that  $F(0, 0) = 0$ . This setting occurs in many other fields, in particular in fluid mechanics, in the study of low Mach number flows (see e.g. [KM1, KM2, Sch1]) or in the analysis of rotating fluids.

Multiplying by a symmetrizer  $S(a, u)$  if necessary, we always assume that the following condition is satisfied:

**Assumption 4.1** (Symmetry). *For  $j \in \{0, \dots, d\}$ , the matrices  $A_j(a, u)$  are self-adjoint and in addition  $A_0(a, u)$  is positive definite.*

*For all  $j \in \{1, \dots, m\}$  the matrices  $\mathcal{L}_j(a, u)$  are self-adjoint and  $\mathcal{L}_0(a, u)$  is skew adjoint.*

Theorem 3.2 implies that the Cauchy problem is locally well posed for systems (4.2), but the time of existence given by this theorem in general shrinks to 0 as  $\varepsilon$  tends to 0. To have a uniform interval of existence, additional conditions are required. We first give two examples, before giving hints for a more general discussion.

#### 4.1.1 The weakly non linear case

Consider a system (4.2) where all the coefficients  $A_j$  and  $\mathcal{L}_k$  are functions of  $(\varepsilon a, \varepsilon u)$ . Expanding  $\mathcal{L}_k(\varepsilon a, \varepsilon u) = \underline{\mathcal{L}}_k + \varepsilon \tilde{A}_k(\varepsilon, a, u)$  yields to systems with the following structure:

$$(4.4) \quad A_0(\varepsilon a, \varepsilon u) \partial_t u + \sum_{j=1}^d A_j(a, u) \partial_{x_j} u + \frac{1}{\varepsilon} \underline{\mathcal{L}}(\partial_x) u = F(a, u)$$

where  $\underline{\mathcal{L}}$  has the form (4.3) with *constant coefficients*  $\underline{\mathcal{L}}_j$ . We still assume that the symmetry Assumption 4.1 is satisfied and  $F(0, 0) = 0$ . The matrices  $A_j$  and  $F$  could also depend smoothly on  $\varepsilon$ , but for simplicity we forget this easy extension.

**Theorem 4.2** (Uniform local Existence). *Suppose that  $h \in H^s(\mathbb{R}^d)$  and  $a \in C^0([0; T]; H^s(\mathbb{R}^d)) \cap C^1([0; T]; H^{s-1}(\mathbb{R}^d))$ , where  $s > \frac{d}{2} + 1$ . Then, there is  $T' > 0$  such that for all  $\varepsilon \in ]0, 1]$  the Cauchy problem for (4.4) with initial data  $h$  has a unique solution  $u \in C^0([0; T']; H^s(\mathbb{R}^d))$ .*

*Sketch of proof.* Consider the linear version of the equation

$$(4.5) \quad L_\varepsilon(a, \partial)u := A_0(\varepsilon a)\partial_t u + \sum_{j=1}^d A_j(a)\partial_{x_j} u + \frac{1}{\varepsilon} \underline{\mathcal{L}}(\partial_x)u = f.$$

Thanks to the symmetry, the  $L^2$  estimate is immediate. There holds

$$(4.6) \quad \begin{aligned} \|u(t)\|_{L^2} &\leq C_0 e^{K_0 t} \|u(0)\|_{L^2} \\ &+ C_0 \int_t^{t'} e^{K_0(t-t')} \|L_\varepsilon(a, \partial)u\|_{H^s} dt' \end{aligned}$$

with  $C_0$  [resp.  $K_0$ ] depending only on the  $L^\infty$  norm [resp.  $W^{1,\infty}$  norm] of  $\varepsilon a$ .

Next, one commutes the  $A_0^{-1}L_\varepsilon$  with the derivatives  $\partial_x^\alpha$  as in (3.29). The key new observation is that the derivatives  $\partial_x^\alpha(\varepsilon^{-1}A_0^{-1}(\varepsilon a)\underline{\mathcal{L}}_j)$  are bounded with respect to  $\varepsilon$  as well as the derivatives  $\partial_x^\alpha(A_0^{-1}(\varepsilon a)A_j(a))$ . The precise estimate is that for  $s > \frac{d}{2} + 1$ , there is  $K_s$  which depends only on the  $H^s$  norm of  $a(t)$  such that for  $|\alpha| \leq s$ , there holds

$$(4.7) \quad \|[\partial_x^\alpha, A_0^{-1}L_\varepsilon(a, \partial)]u(t)\|_{L^2} \leq K_s \|u(t)\|_{H^s}.$$

From here, the proof is as in the nonsingular case.  $\square$

#### 4.1.2 The case of prepared data

We relax here the weakly nonlinear dependence of the coefficient  $A_0$  and consider a system:

$$(4.8) \quad A_0(a, u)\partial_t u + \sum_{j=1}^d A_j(a, u)\partial_{x_j} u + \frac{1}{\varepsilon} \underline{\mathcal{L}}(\partial_x)u = F(a, u)$$

and its linear version

$$(4.9) \quad L_\varepsilon(a, \partial)u := A_0(a)\partial_t u + \sum_{j=1}^d A_j(a)\partial_{x_j} u + \frac{1}{\varepsilon} \underline{\mathcal{L}}(\partial_x)u = f.$$

where  $\underline{\mathcal{L}}$  has *constant coefficients*  $\underline{\mathcal{L}}_j$ . We assume that the symmetry Assumption 4.1 is satisfied and  $F(0, 0) = 0$ .

The commutators  $[\partial_x^\alpha, \varepsilon^{-1}A_0^{-1}(a)L_\varepsilon]$  are of order  $\varepsilon^{-1}$ , so that the method of proof of Theorem 4.2 cannot be used any more. Instead, one can use the

following path: the commutators  $[\partial_x, \varepsilon^{-1}\underline{\mathcal{L}}]$  are excellent, since they vanish identically. Thus one can try to commute  $\partial_x^\alpha$  and  $L_\varepsilon(a)$  directly. However, this commutator contains terms  $\partial_x^{\alpha-\beta}A_0(a)\partial_t\partial_x^\beta u$  and hence mixed time-space derivative  $\partial_t\partial_x^\beta u$ . One cannot use the equation to replace  $\partial_t u$  by spatial derivative, since this would reintroduce singular terms. Therefore, to close the estimate one is led to estimate all the derivatives  $\partial_t^{\alpha_0}\partial_x^\alpha u$ . Then, the commutator argument closes, yielding an existence result on a uniform interval of time, *provided that* the initial values of *all* the derivative  $\partial_t^{\alpha_0}\partial_x^\alpha u$  are uniformly bounded. Let us proceed to the details.

For  $s \in \mathbb{N}$ , introduce the space  $CH^s([0, T] \times \mathbb{R}^d)$  of functions  $u \in C^0([0, T]; H^s(\mathbb{R}^d))$  such that for all  $k \leq s$ ,  $\partial_t^k u \in C^0([0, T]; H^{s-k}(\mathbb{R}^d))$ . It is equipped with the norm

$$(4.10) \quad \|u\|_{CH^s} = \sup_{t \in [0, T]} \|u(t)\|_s, \quad \|u(t)\|_s = \sum_{k=0}^s \|\partial_t^k u(t)\|_{H^{s-k}(\mathbb{R}^d)}$$

For  $s > \frac{d}{2}$ , the space  $CH^s$  is embedded in  $L^\infty$ . Therefore, the chain rule and Lemma 3.5 imply that for  $F$  smooth such that  $F(0) = 0$ , the mapping  $u \mapsto F(u)$  is continuous from  $CH^s$  into itself and maps bounded sets to bounded sets.

Moreover, the commutator  $[\partial_{t,x}^\alpha, L_\varepsilon(a, \partial)]u$  is a linear combination of terms

$$(4.11) \quad \partial_{t,x}^\gamma A_j(a) \partial_{t,x}^\beta u,$$

with

$$0 \leq |\gamma| - 1 \leq s - 1, \quad 0 \leq |\beta| - 1, \quad |\gamma| - 1 + |\beta| - 1 \leq s - 1.$$

Since  $A_j(a(t)) \in CH^s$ , *ii*) of Lemma 3.5 applied with  $\sigma = s - 1$  implies for  $s > \frac{d}{2} + 1$  the following commutator estimates

$$(4.12) \quad \|[\partial_{t,x}^\alpha, L_\varepsilon(a, \partial_x)]u(t)\|_{L^2} \leq C \|u(t)\|_s$$

where  $C$  depends on the norm  $\|a(t)\|_s$ .

For systems (4.9), the  $L^2$  energy estimates are again straightforward from the symmetry assumption. Together with the commutator estimates above, they provide bounds for the norms  $\|u(t)\|_s$ , uniform in  $t$  and  $\varepsilon$ , *provided that their initial values  $\|u(0)\|_s$  are bounded.*

The initial values  $\partial_t^k u(0)$  are computed by induction using the equation: for instance

$$(4.13) \quad \partial_t u|_{t=0} = -A_0^{-1}(a_0, u_0) \left( \sum_{j=1}^d A_j(a_0, u_0) \partial_{x_j} u_0 + \frac{1}{\varepsilon} \underline{\mathcal{L}}(\partial_x) u_0 - F(a_0, u_0) \right)$$

where  $a_0 = a|_{t=0}$  and  $u_0 = u|_{t=0}$ . In particular, for a fixed initial data  $u|_{t=0} = h$ , the term  $\partial_t u|_{t=0}$  is bounded independently of  $\varepsilon$  if and only if

$$(4.14) \quad \underline{\mathcal{L}}(\partial_x) h = 0.$$

The analysis of higher order derivatives is similar. To simplify notations, let  $\partial^{(k)} u$  denote a product of derivatives of  $u$  of total order  $k$ :

$$(4.15) \quad \partial^{\alpha_1} u \dots \partial^{\alpha_p} u \quad \text{with} \quad \alpha_j > 0 \quad \text{and} \quad |\alpha_1| + \dots + |\alpha_p| = k.$$

**Lemma 4.3.** *For  $s > (d+1)/2$  and  $k \in \{0, \dots, s\}$ , there are nonlinear functionals  $\mathcal{F}_k^\varepsilon(a, u)$ , which are finite sums of terms*

$$(4.16) \quad \varepsilon^{-l} \Phi(a, h) (\partial_{t,x}^{(p)} a) (\partial_x^{(q)} u), \quad l \leq k, \quad p+q \leq k$$

with  $\Phi$  smooth, such that for  $a \in CH^s$  and  $h \in H^s$  and all  $\varepsilon > 0$  the local solution of the Cauchy problem for (4.8) with initial data  $h$  belongs to  $CH^s$  and

$$(4.17) \quad \partial_t^k u = \mathcal{F}_k^\varepsilon(a, h).$$

*Proof.* For  $C^\infty$  functions, (4.17) is immediate by induction on  $k$ . Lemma 3.5 implies that the identities extend to coefficients  $a \in CH^s$  and  $u \in C^0([0, T'], H^s)$ , proving by induction that  $\partial_t^k u \in C^0([0, T'], H^{s-k})$ .  $\square$

In particular, if  $u$  is a solution of (4.8) with initial data  $h$ , there holds

$$(4.18) \quad (\partial_t^k u)|_{t=0} = \mathcal{H}_k^\varepsilon(a, h) := \mathcal{F}_k^\varepsilon(a, h)|_{t=0}$$

Note that  $\mathcal{H}_k^\varepsilon$  is singular as  $\varepsilon \rightarrow 0$ , since in general it is of order  $\varepsilon^{-k}$ .

**Theorem 4.4.** *Suppose that  $s > \frac{d}{2} + 1$  and  $a \in CH^s([0, T] \times \mathbb{R}^d)$  and  $h \in H^s(\mathbb{R}^d)$  are such that the families  $\mathcal{H}_k^\varepsilon(a, h)$  are bounded for  $\varepsilon \in ]0, 1]$ .*

*Then, there is  $T' > 0$  such that for all  $\varepsilon \in ]0, 1]$  the Cauchy problem for (4.8) with initial data  $h$  has a unique solution  $u \in CH^s([0, T'] \times \mathbb{R}^d)$ .*

*Sketch of the proof.* (See e.g. [BrKr]).

The assumption means that the initial norms  $\|u(0)\|_s$  are bounded, providing uniform estimates of  $\|u(t)\|_s$  for  $t \leq T'$ , for some  $T' > 0$  independent of  $\varepsilon$ . This implies that the local solution can be continued up to time  $T'$ .  $\square$

The data which satisfy the condition that the  $\mathcal{H}_k^\varepsilon(a, h)$  are often called *prepared data*. The first condition (4.14) is quite explicit, but the higher order conditions are less explicit, and the construction of prepared data is a nontrivial independent problem. However, there is an interesting application of Theorem 4.4 when the wave is created not by an initial data but by a forcing term which vanishes in the past: consider the problem

$$(4.19) \quad A_0(a, u)\partial_t u + \sum_{j=1}^d A_j(a, u)\partial_{x_j} u + \frac{1}{\varepsilon} \underline{\mathcal{L}}(\partial_x)u = F(a, u) + f$$

with  $F(a, 0) = 0$ . (In the notations of (4.8), this means that  $f = F(a, 0)$ ). We consider  $f$  as one of the parameter entering in the equation. We assume that  $f$  is given in  $CH^s([0, T] \times \mathbb{R}^d)$  and vanishes at order  $s$  on  $t = 0$ :

$$(4.20) \quad \partial_t^k f|_{t=0} = 0, \quad k \in \{0, \dots, s-1\}.$$

Then, one can check by induction that for a vanishing initial data  $h = 0$  the traces of the solution vanishes:

$$(4.21) \quad \partial_t^k u|_{t=0} = 0, \quad k \in \{0, \dots, s\}.$$

Therefore:

**Theorem 4.5.** *Suppose that  $s > \frac{d}{2} + 1$ ,  $a \in CH^s([0, T] \times \mathbb{R}^d)$  and  $f \in CH^s([0, T] \times \mathbb{R}^d)$  satisfies (4.20).*

*Then, there is  $T' > 0$  such that for all  $\varepsilon \in ]0, 1]$  the Cauchy problem for (4.19) with vanishing initial data has a unique solution  $u \in CH^s([0, T'] \times \mathbb{R}^d)$  and  $u$  satisfies (4.21)*

### 4.1.3 Remarks on the commutation method

The proofs of Theorems 3.2, 4.2 and 4.4 are based on commutation properties which strongly depend on the structure of the equation. In particular, it was crucial in (4.4) and (4.8) that the singular part  $\mathcal{L}(\partial_x)$  had *constant* coefficients and that  $A_0$  depended on  $(\varepsilon a, \varepsilon u)$  in the former case. Without pretending to maximal generality, we investigate here to what extent the commutation method can be extended to more general equations (4.2) where  $\mathcal{L}$  and  $A_0$  could also depend on  $a$ .

1. *The principle of the method.* Consider a linear equation  $L(t, x, \partial)u = f$  and assume that there is a good  $L^2$  energy estimate. We suppose that the coefficients are smooth functions of  $a$  and that  $a$  is smooth enough. Consider a set of vector fields  $\mathcal{Z} = \{Z_1, \dots, Z_m\}$ , which we want to commute with the equation. The commutation properties we used were of the form:

$$L(t, x, \partial)Z_j = Z_j L(t, x, D) + \sum R_{j,k} Z_k$$

with bounded matrices  $R_{j,k}$ . The energy estimate is then applied to  $Z_j u$ . However, one must keep in mind that one could perform a change of variables  $u = Vv$  or pre-multiply the equation by a matrix before commuting with the  $Z_j$ . This is equivalent to adding zero order terms to the  $Z_j$  and looking for relations of the form

$$(4.22) \quad L(t, x, \partial)(Z_j + B_j) = (Z_j + C_j)L(t, x, D) + \sum R_{j,k} Z_k + R_{j,0}$$

with matrices  $B_j$  and  $C_j$  to be found, depending on  $a$  and its derivatives. Together with the  $L^2$  estimate, this clearly implies estimates of derivatives  $Z_{j_1} \dots Z_{j_m} u$ , knowing similar estimates for the right hand side  $f$  and the initial data.

2. *Analysis of the commutation conditions.* The operator  $L$  is split into a good part, linear combination of the  $Z_j$  and a bad or singular part:

$$(4.23) \quad L(t, x, \partial) = \sum A_j(t, x)Z_j + \sum G_k(t, x)T_k$$

where the  $T_k$  are some other vector fields or singular terms which commute with the  $Z_j$ . Only the commutation with terms  $G_k(t, x)T_k$  may cause problem, and the commutation conditions read

$$(4.24) \quad Z_j(G_k) = G_k B_j - C_j G_k,$$

assuming that

$$(4.25) \quad \text{the } [T_k, B_j] \text{ are bounded.}$$

We now analyze the geometric implications of (4.24).

**Lemma 4.6.** *Suppose that the  $G_k$  are smooth matrices and  $\mathcal{Z}$  is an integrable system of vector fields. Then, locally, there are smooth matrices  $B_j$  et  $C_j$  satisfying (4.24), if and only if there are smooth invertible matrices  $W$  et  $V$ , such that*

$$(4.26) \quad \forall j, \forall k, \quad [Z_j, W G_k V] = 0.$$

*In this case, for all  $\eta \in \mathbb{R}^m$ , the rank of  $\sum \eta_k G_k$  is constant along the integral leaves of  $\mathcal{Z}$ .*

*Proof.* If  $\tilde{G}_k = W G_k V$ , then

$$Z_j(\tilde{G}_k) = W \left( W^{-1}(Z_j W) G_k + Z(G_k) + G_k Z_j(V) V^{-1} \right) V.$$

Thus, if  $Z(\tilde{G}_k) = 0$ , then (4.24) holds with  $C_j = W^{-1} Z_j(W)$  and  $B_j = -Z_j(V) V^{-1}$ .

Conversely, one can assume locally that  $\mathcal{Z} = \{\partial_{x_1}, \dots, \partial_{x_m}\}$  and prove the result by induction on  $m$ . One determines locally  $W_1$  and  $V_1$  such that

$$\partial_{x_1} W_1 = W_1 C_1, \quad \partial_{x_1} V_1 = -B_1 V_1$$

and (4.24) implies  $\partial_{x_1}(W_1 G_1 V_1) = 0$ . This finishes the proof when  $m = 1$ . When  $m > 1$ , let  $\tilde{G}_k = W_1 G_k V_1$ . The commutation properties (4.24) are stable under such a transform: the  $\tilde{G}_k$  satisfy

$$(4.27) \quad \partial_{x_j} \tilde{G}_k = \tilde{G}_k \tilde{B}_j - \tilde{C}_j \tilde{G}_k,$$

with new matrices  $\tilde{B}_j$  and  $\tilde{C}_j$ , which vanish when  $j = 1$ . In particular, the  $\tilde{G}_k$  do not depend on the first variable  $x_1$ . Freezing the variable  $x_1$  at  $\underline{x}_1$ , we see that the commutation relation (4.27) also holds with matrices  $\tilde{B}_j$  and  $\tilde{C}_j$  that are independent of  $x_1$ . From here, one can apply the induction hypothesis for the matrices  $\tilde{G}_k$  and the vector fields  $\{\partial_{x_j}, j \geq 2\}$  and find matrices  $W'$  and  $V'$ , independent of  $x_1$ , such that  $\partial_{x_j}(W' \tilde{G}_k V') = 0$  for  $j \geq 2$ . The property (4.26) follows with  $W = W' W_1$  and  $V = V_1 V'$ .

The condition (4.26) imply that  $W(\sum \eta_k G_k)V$  is independent of the variables  $(x_1, \dots, x_m)$ . In particular, its rank is constant when the other variables are fixed.  $\square$

**Remark 4.7.** Suppose that  $l \geq 2$  and that  $G_1$  is invertible. Saying that the rank of  $\sum \eta_k G_k$  is independent of  $x$ , is equivalent to saying that the eigenvalues of  $\sum_{k \geq 2} \eta_k G_1^{-1} G_k$  are independent of  $x$  as well as their multiplicity.

The next lemma shows that the commutation properties imply that the equation can be transformed to another equivalent equation with constant coefficients in the singular part.

**Lemma 4.8.** *i) Suppose that  $G(a)$  is a smooth matrix of rank independent of  $a$ . Then, locally near any point  $\underline{a}$ , there are smooth matrices  $W(a)$  and  $V(a)$  such that  $W(a)G(a)V(a)$  is constant.*

*ii) Suppose that  $G_0(a)$  and  $G(a)$  are a smooth self adjoint matrices with  $G_0$  definite positive, such that the eigenvalues of  $G_0^{-1}G(a)$  are independent of  $a$ . Then locally, there is a smooth matrix  $V(a)$  such that, with  $W = V^{-1}G_0^{-1}$ ,  $W(a)G_0(a)V(a)$  and  $W(a)G(a)V(a)$  constant.*

*In both cases, if  $a(t, x)$  is such that the  $Z_j a$  are bounded, the commutation relations (4.24) are satisfied with bounded matrices  $B_j$  and  $C_j$ .*

*Proof.* i) If the rank of  $G$  is constant equal to  $\mu$ , in a neighborhood of  $\underline{a}$ , there are smooth invertible matrices  $W$  and  $V$  such that

$$W(a)G(a)V(a) = \begin{pmatrix} \text{Id}_\mu & 0 \\ 0 & 0 \end{pmatrix}.$$

ii) Since  $G_0$  is definite positive and  $G$  is self adjoint,  $G_0^{-1}G$  is diagonalizable. Since the eigenvalues are constant, say equal to  $\underline{\lambda}_j$ , there is a smooth invertible matrix  $V$  such that

$$V^{-1}(a)G_0^{-1}(a)G(a)V(a) = \text{diag}(\underline{\lambda}_j).$$

□

**Conclusion 4.9.** *Lemmas 4.6 and 4.8 give key indications for the validity of the commutation method, applied to equations (4.23).*

#### 4.1.4 Application 1

Consider a system with coefficients depending smoothly on  $a$

$$(4.28) \quad A_0(a)\partial_t u + \sum_{j=1}^d A_j(a)\partial_{x_j} u + \frac{1}{\varepsilon} B(a) = f.$$

We assume that the matrices  $A_j$  are self adjoint with  $A_0$  definite positive and  $B(a)$  skew adjoint.

- For  $\mathcal{Z} = \{\partial_t, \partial_{x_1}, \dots, \partial_{x_d}\}$ , the commutation condition reads

$$(4.29) \quad \text{the rank of } B(a(t, x)) \text{ is independent of } (t, x).$$



Locally in  $a$ , this is equivalent to

$$(4.30) \quad \begin{aligned} &\text{there are smooth matrices } W(t, x) \text{ and } V(t, x) \text{ such that} \\ &W(t, x)B(a(t, x))V(t, x) \text{ is constant.} \end{aligned}$$

**Theorem 4.10.** *Suppose that  $a$  is a smooth function of  $(t, x)$ . Under Assumption (4.30), there are energy estimates in the spaces  $CH^s([0, T] \times \mathbb{R}^d)$  for the solutions of (4.28), of the form*

$$(4.31) \quad \|u(t)\|_s \leq Ce^{Kt} \|u(0)\|_s + C \int_0^t e^{K(t-t')} \|f(t')\|_s dt'$$

• For  $\mathcal{Z} = \{\partial_{x_1}, \dots, \partial_{x_d}\}$ , then  $\partial_t$  becomes a “bad” vector field to be included in the second sum in (4.23). Following Remark 4.7, the condition reads

$$(4.32) \quad \begin{aligned} &\text{the eigenvalues of } A_0^{-1}(a(t, x))B(a(t, x)) \text{ have constant} \\ &\text{multiplicity.} \end{aligned}$$

Locally in  $a$ , this is equivalent to

$$(4.33) \quad \begin{aligned} &\text{there is a smooth matrix } V(t, x) \text{ such that} \\ &V^{-1}(t, x)A_0^{-1}(a(t, x))B(a(t, x))V(t, x) \text{ is constant.} \end{aligned}$$

**Theorem 4.11.** *Suppose that  $a$  is a smooth function of  $(t, x)$ . Under Assumption (4.33), there are energy estimates in the spaces  $C^0([0, T]; H^s(\mathbb{R}^d))$  for the solutions of (4.28), of the form*

$$(4.34) \quad \|u(t)\|_s \leq Ce^{Kt} \|u(0)\|_s + C \int_0^t e^{K(t-t')} \|f(t')\|_s dt'$$

#### 4.1.5 Application 2

The following systems arise when one introduces fast scales (see Section 7.4):

$$(4.35) \quad A_0(a)\partial_t u + \sum_{j=1}^d A_j(a)\partial_{x_j} u + \frac{1}{\varepsilon} \left( \sum_{k=1}^m B_k(a)\partial_{\theta_k} u + E(a)u \right) = f.$$

We assume that the matrices  $A_j$  and  $B_k$  are self adjoint with  $A_0$  is definite positive and  $E$  skew symmetric. The additional variables  $\theta = (\theta_1, \dots, \theta_m)$  correspond to the fast variables  $\varphi/\varepsilon$  in (4.1). In this framework functions are periodic in  $\theta$ , and up to a normalization of periods, this means that  $\theta \in \mathbb{T}^m$ ,

$\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ . There are other situations where systems of the form (4.35) occur with no periodicity assumption in the variables  $\theta_j$ , for instance in the low mach limit analysis Euler equation. The analysis can be adapted to some extent to this case, but we do not investigate this question in details here.

An important assumption is that

$$(4.36) \quad \partial_{\theta_j} a = 0, \quad j \in \{1, \dots, m\}.$$

In this case one can expand  $u$  and  $f$  in Fourier series in  $\theta$ , and the Fourier coefficients  $\hat{u}_\alpha$ ,  $\alpha \in \mathbb{Z}^m$  are solutions of

$$(4.37) \quad A_0(a) \partial_t \hat{u}_\alpha + \sum_{j=1}^d A_j(a) \partial_{x_j} \hat{u}_\alpha + \frac{1}{\varepsilon} \mathcal{B}(a, \alpha) \hat{u}_\alpha = \hat{f}_\alpha.$$

where

$$(4.38) \quad \mathcal{B}(a, \eta) = \sum_{j=1}^m i \eta_j B_j(a) + E(a).$$

They are systems of the form (4.28). The conditions (4.29) (4.32) read

$$(4.39) \quad \text{for all } \alpha \in \mathbb{Z}^m, \text{ the rank of } \mathcal{B}(a(t, x), \alpha) \text{ is independent of } (t, x).$$

$$(4.40) \quad \text{for all } \alpha \in \mathbb{Z}^m, \text{ the eigenvalues of } A_0^{-1}(a(t, x)) \mathcal{B}(a(t, x), \alpha) \text{ have constant multiplicity.}$$

respectively. To get estimates uniform in  $\alpha$ , we reinforce their equivalent formulation (4.29) (4.32) as follows:

**Assumption 4.12.** *There are matrices  $\mathcal{W}(t, x, \alpha)$  and  $\mathcal{V}(t, x, \alpha)$  for  $\alpha \in \mathbb{Z}^m$ , which are uniformly bounded as well as their derivatives with respect to  $(t, x)$ , and a matrix  $\mathcal{B}^\sharp(\alpha)$  independent of  $(t, x)$  such that*

$$(4.41) \quad \mathcal{W}(t, x, \alpha) \mathcal{B}(a(t, x), \alpha) \mathcal{V}(t, x, \alpha) = \mathcal{B}^\sharp(\alpha).$$

**Assumption 4.13.** *There are matrices  $\mathcal{V}(a(t, x), \alpha)$  for  $\alpha \in \mathbb{Z}^m$ , which are uniformly bounded as well as their derivatives with respect to  $(t, x)$ , and a matrix  $\mathcal{B}^\sharp(\alpha)$  independent of  $(t, x)$  such that*

$$(4.42) \quad \mathcal{V}^{-1}(t, x, \alpha) A_0^{-1}(a(t, x)) \mathcal{B}(a(t, x), \alpha) \mathcal{V}(t, x, \alpha) = \mathcal{B}^\sharp(\alpha).$$

Note that no smoothness in  $\alpha$  is required in these assumptions.

**Theorem 4.14.** *Suppose that  $a$  is a smooth function of  $(t, x)$ .*

*i) Under Assumption 4.12 there are energy estimates in the spaces  $CH^s([0, T] \times \mathbb{R}^d \times \mathbb{T}^m)$  for the solutions of (4.35)*

$$(4.43) \quad \left\| \|u(t)\| \right\|_s \leq C e^{Kt} \left\| \|u(0)\| \right\|_s + C \int_0^t e^{K(t-t')} \left\| \|f(t')\| \right\|_s dt'$$

*ii) Under Assumption 4.13, there are energy estimates in the spaces  $C^0([0, T]; H^s(\mathbb{R}^d \times \mathbb{T}^m))$ :*

$$(4.44) \quad \|u(t)\|_{H^s} \leq C e^{Kt} \|u(0)\|_{H^s} + C \int_0^t e^{K(t-t')} \|f(t')\|_{H^s} dt'$$

*Proof.* The symmetry conditions immediately imply an  $L^2$  energy estimate for (4.37). The commutation properties are satisfied by  $\varepsilon^{-1}\mathcal{B}(a, \alpha)$  in the first case and by  $\{A_0(a)\partial_t, \varepsilon^{-1}\mathcal{B}(a, \alpha)$  in the second case, providing estimates which are uniform in  $\alpha$ . Moreover, the equation commutes with  $|\alpha|$ , allowing for estimates of the  $\partial_\theta$  derivatives.  $\square$

**Remark 4.15.** These estimates extend to equations

$$(4.45) \quad \begin{aligned} A_0(a, \varepsilon u)\partial_t u + \sum_{j=1}^d A_j(a, \varepsilon u)\partial_{x_j} u \\ + \frac{1}{\varepsilon} \left( \sum_{k=1}^m B_j(a, \varepsilon u)\partial_{\theta_j} u + E(a, \varepsilon u)u \right) = f. \end{aligned}$$

since all the additional commutators which involve  $\varepsilon u$  are nonsingular. We stress that the conditions (4.41) or (4.42) are unchanged and bear on the coefficients  $A_j(a, v)$  and  $B_j(a, v)$  at  $v = 0$ . This yields existence theorems, which extend of Theorem 4.4 for prepared data under Assumption 4.12 and of Theorem 4.2 for general data under Assumption 4.13. We refer to [JMR4] for details.

Next we give explicit conditions which ensure that the Assumptions are satisfied. The first remark is trivial, but useful.

**Remark 4.16.** *If  $a(t, x) = \underline{a}$  is constant, the Assumptions 4.12 and 4.13 are satisfied.*

Next we consider the case where  $E = 0$ . Though the next proposition is does not apply to geometric optics, it is natural and useful for other applications.

**Proposition 4.17.** *Suppose that  $E = 0$  and there is  $\delta > 0$  such that for all  $a$  and  $\eta$  in the sphere  $S^{m-1}$ ,  $0$  is the unique eigenvalue of  $\mathcal{B}(a, \eta)$  in the disk centered at  $0$  and of radius  $\delta$ , and that its multiplicity is constant.*

*Then, the Assumption 4.12 is satisfied.*

In Section 7 we will consider systems of the form above such that the symmetric system  $\mathcal{B}(a(t, x), \eta)$  is hyperbolic in a direction  $\underline{\eta}$ , which we can choose to be  $\underline{\eta} = (1, 0, \dots, 0)$ . We use the notations

$$(4.46) \quad \eta = (\eta_1, \eta'), \quad \mathcal{B}(a, \eta) = \eta_1 B_1(a) + \mathcal{B}'(a, \eta')$$

and  $B_1$  is definite positive. Then the kernel of  $\mathcal{B}(a(t, x), \eta)$  is nontrivial if and only if  $-\eta_1$  is an eigenvalue of  $B_1^{-1} \mathcal{B}'(a(t, x), \eta')$ . In this case, a natural choice for a projector  $P(t, x, \eta)$  on  $\ker \mathcal{B}$  is to consider the spectral projector, which is orthogonal for the scalar product defined by  $B_1(a(t, x))$ . If  $-\eta_1$  is not an eigenvalue, define  $P(t, x, \eta) = 0$ .

**Proposition 4.18.** *With  $E = 0$ , assume that the symmetric symbol  $\mathcal{B}$  is hyperbolic with timelike codirection  $\underline{\eta} \in \mathbb{R}^m \setminus \{0\}$  and use the notations above. Then, locally in  $(t, x)$ , the Assumption 4.12 is satisfied if*

- i) for all  $\eta \in \mathbb{R}^m$  the rank of  $\mathcal{B}(a(t, x), \eta)$  is independent of  $(t, x)$ ,*
- ii) for all  $\eta \in \mathbb{R}^m$  the projectors  $P(t, x, \eta)$  are smooth in  $(t, x)$ , and the set  $\{P(\cdot, \cdot, \eta)\}_{\eta \in \mathbb{R}^m}$  is bounded in  $C^\infty$ .*

*In particular, the second condition is satisfied when the eigenvalues of  $B_1^{-1} \mathcal{B}'(a(t, x), \eta')$  have their multiplicity independent of  $(t, x, \eta')$ , for  $\eta'$  in the unit sphere.*

*Proof.* Condition *i*) implies that for all  $\eta'$ , the eigenvalues of  $B_1^{-1} \mathcal{B}'(a(t, x), \eta')$ , which are all real by the hyperbolicity assumption, are independent of  $(t, x)$ . By *ii*), locally, there are smooth families of eigenvectors, which are orthonormal for the scalar product  $A_0(a(t, x))$ , and uniformly bounded in  $C^\infty$  independently of  $\eta'$ . This yields matrices  $\mathcal{V}(t, x, \eta')$  such that  $\mathcal{V}(\cdot, \eta')$  and  $\mathcal{V}^{-1}(\cdot, \eta')$  are uniformly bounded in  $C^\infty$  and

$$\mathcal{V}^{-1}(t, x, \eta') B_1^{-1} a(t, x) \mathcal{B}'(a(t, x), \eta') \mathcal{V}(t, x, \eta') = \text{diag}(\lambda_j(\eta')) := \mathcal{B}^b(\eta').$$

Thus, with  $\mathcal{W} = \mathcal{V}^{-1} B_1(a(t, x), \eta')$  there holds

$$\mathcal{W}(t, x, \eta') \mathcal{B}(a(t, x), \eta) \mathcal{V}(t, x, \eta') = i\eta_1 \text{Id} + \mathcal{B}^b(\eta') := \mathcal{B}^b(\eta).$$

If the eigenvalues of  $B_1^{-1}\mathcal{B}'(a(t, x), \eta')$  have constant multiplicity in  $(t, x)$  and  $\eta' \neq 0$ , the spectral projectors are smooth in  $(t, x, \eta')$  and homogeneous of degree 0 in  $\eta'$ , yielding *ii*).  $\square$

Concerning Assumption 4.13, the same proof applied to  $i\tau A_0 + \mathcal{B}(a, \eta)$  implies the following:

**Proposition 4.19.** *With  $E = 0$ , the Assumption 4.12 is satisfied locally in  $(t, x)$  if:*

*i) for all  $\eta \in \mathbb{R}^m$ , the eigenvalues of  $A_0^{-1}(a(t, x))\mathcal{B}(a(t, x), \eta)$  are independent of  $(t, x)$ ,*

*ii) the spectral projectors of  $A_0^{-1}(a(t, x))\mathcal{B}(a(t, x), \eta)$  are smooth in  $(t, x)$ , and belong to a bounded set in  $C^\infty$  when  $\eta \in \mathbb{R}^m$ .*

*In particular the second condition is satisfied if the multiplicities of the eigenvalues of  $A_0^{-1}(a(t, x))\mathcal{B}(a(t, x), \eta)$  are constant in  $(t, x, \eta)$  for  $\eta \neq 0$ .*

In Section 7, we will use the following extensions:

**Proposition 4.20.** *Suppose that*

$$(4.47) \quad \tilde{\mathcal{B}}(a, \tilde{\eta}) = \sum_{j=1}^{\tilde{m}} i\tilde{\eta}_j \tilde{B}_j(a)$$

*satisfies the assumption of Proposition 4.18 [resp. Proposition 4.19], and that there is a  $\tilde{m} \times m$  matrix  $M$  such that*

$$(4.48) \quad \mathcal{B}(a, \eta) = \tilde{\mathcal{B}}(a, M\eta).$$

*Then,  $\mathcal{B}$  satisfies the Assumption 4.12 [resp. Assumption 4.13].*

There are analogous results when  $E \neq 0$ , but we omit them here for the sake of brevity.

## 4.2 Equations with rapidly varying coefficients

In this subsection we consider systems

$$(4.49) \quad A_0(a, u)\partial_t u + \sum_{j=1}^d A_j(a, u)\partial_{x_j} u + \frac{1}{\varepsilon}E(a)u = F(a, u)$$

with the idea that  $a$  and  $u$  have rapid oscillations. More precisely, we assume that  $a \in L^\infty([0, T] \times \mathbb{R}^d)$  and that its derivative satisfy for  $0 < |\alpha| \leq s$ :

$$(4.50) \quad \varepsilon^{|\alpha|-1} \|\partial_{t,x}^\alpha a\|_{L^\infty([0, T] \times \mathbb{R}^d)} \leq C_1,$$

Following the general existence theory, it is assumed that the integer  $s$  satisfies  $s > \frac{d}{2} + 1$ . We suppose that  $u = 0$  is almost a solution, that is that there is a real number  $M > 0$  such that  $f = F(a, 0)$  satisfies for  $|\alpha| \leq s$

$$(4.51) \quad \varepsilon^{|\alpha|} \|\partial_x^\alpha f^\varepsilon(t)\|_{L^2(\mathbb{R}^d)} \leq \varepsilon^M C_2,$$

How large  $M$  must be chosen is part of the analysis. Similarly, we consider initial data which satisfy for  $|\alpha| \leq s$

$$(4.52) \quad \varepsilon^{|\alpha|} \|\partial_x^\alpha h\|_{L^2(\mathbb{R}^d)} \leq \varepsilon^M C_3.$$

Note that these assumptions allow for families of data  $a$ , which have amplitude  $O(\varepsilon)$  at frequencies  $|\xi| \approx \varepsilon^{-1}$ . In applications,  $a$  will be an approximate solution of the original equation. The equation (4.49) with coefficients  $A_j(a, 0)$  is the linearized equation near this approximate solutions. Its well posedness accounts for the stability of the approximate solution and (4.49) has to be understood as an equation for a corrector.

We always assume that the equations is symmetric hyperbolic, that is that the matrices  $A_j$  are self adjoint with  $A_0$  positive definite and that  $E$  is skew symmetric.

**Theorem 4.21.** *Under the assumptions above, if  $M > 1 + d/2$ , there are  $\varepsilon_1 > 0$  and  $C_4$ , depending only on the constants  $C_1, C_2, C_3$  and the coefficients  $A_j$  and  $F$ , such that for  $\varepsilon \in ]0, \varepsilon_1]$ , the Cauchy problem for (4.49) with initial data  $h$ , has a unique solution  $u \in C^0([0, T]; H^s(\mathbb{R}^d))$  which satisfies*

$$(4.53) \quad \varepsilon^{|\alpha|} \|\partial_x^\alpha u(t)\|_{L^2(\mathbb{R})} \leq \varepsilon^M C_4,$$

*Proof.* (See [Gu2]). Write  $u = \varepsilon^M v$ ,  $f = \varepsilon^M g$  and  $F(a, u) = f + \varepsilon^M G(a, u)v$ . The equation for  $v$  reads

$$(4.54) \quad L(a, u, \partial)v := A_0(a, u)\partial_t v + \sum_{j=1}^d A_j(a, u)\partial_{x_j} v + \frac{1}{\varepsilon} E(a)v - G(a, u)v = g.$$

Introduce the norms

$$(4.55) \quad \|u\|_{H_\varepsilon^s} = \sup_{|\alpha| \leq s} \varepsilon^{|\alpha|} \|\partial_x^\alpha u\|_{L^2(\mathbb{R}^d)}.$$

The main new ingredient is the weighted Sobolev inequality:

$$(4.56) \quad \|\varepsilon^M v\|_{L^\infty(\mathbb{R}^d)} \leq K \varepsilon^{M - \frac{d}{2}} \|v\|_{H_\varepsilon^{s-1}},$$

$$(4.57) \quad \|\varepsilon^M \partial_x v\|_{L^\infty(\mathbb{R}^d)} \leq K \varepsilon^{M - \frac{d}{2} - 1} \|v\|_{H_\varepsilon^s}.$$

Since  $M > \frac{d}{2} + 1$ , this implies that, if the  $H_\varepsilon^s$ -norm of  $v$  is bounded by some constant  $C_4$ , then the Lipschitz norms of the coefficients  $A_j(a, \varepsilon^M v)$  are bounded by some constant independent of  $C_4$  if  $\varepsilon$  is small enough.

Therefore the symmetry implies the following  $L^2$  estimate: there are constants  $C$  and  $K$ , which depend only on  $C_1, C_2, C_3$ , and  $\varepsilon_0 > 0$  which depends in addition on  $C_4$ , such that for  $\varepsilon \leq \varepsilon_0$  and  $u$  satisfying (4.53) on  $[0, T']$ , there holds for  $t \in [0, T']$ :

$$(4.58) \quad \|v(t)\|_{L^2} \leq C e^{Kt} \|v(0)\|_{L^2} + C \int_0^t e^{K(t-t')} \|L(a, u, \partial)v(t')\|_{L^2} dt'.$$

Next one commutes the equation (pre-multipied by  $A_0^{-1}$ ) with the weighted derivatives  $\varepsilon \partial_{x_j}$ . One proves that for  $|\alpha| \leq s$ ,

$$(4.59) \quad \|\varepsilon^{|\alpha|} \partial_x^\alpha [A_0^{-1} L(a, u, \partial)]v(t)\|_{L^2} \leq K \|v(t)\|_{H_\varepsilon^s}$$

with  $K$  depending only on  $C_1, C_2, C_3$ , provided that  $u$  satisfies (4.53) and  $\varepsilon \leq \varepsilon_0$  where  $\varepsilon_0 > 0$  depends on  $C_1, C_2, C_3$  and  $C_4$ . Indeed, by homogeneity, Lemma 3.5 implies that for  $\sigma > \frac{d}{2}$  and  $l \geq 0, m \geq 0$  with  $l + m \leq \sigma$ , there holds

$$(4.60) \quad \|uv\|_{H_\varepsilon^{\sigma-l-m}} \leq C \varepsilon^{-\frac{d}{2}} \|u\|_{H_\varepsilon^{\sigma-l}} \|v\|_{H_\varepsilon^{\sigma-m}}.$$

The commutator  $[\varepsilon^{|\alpha|} \partial_x^\alpha, A_0^{-1} A_j \partial_j]v$  is linear combination of terms

$$\varepsilon^{|\alpha|} \partial_x^{\beta^1} a \dots \partial_x^{\beta^q} a \partial_x^{\alpha^1} u \dots \partial_x^{\alpha^p} u \partial_x^\gamma v$$

with  $|\beta^j| \leq s, |\alpha^k| \leq s, |\gamma| \leq s$  and  $\sum |\beta^j| + \sum |\alpha^k| + |\gamma| \leq |\alpha| + 1 \leq s + 1$ . These terms belong to  $L^2$ , as explained in the proof of Proposition 3.4. More precisely, using the assumptions (4.52) and 4.53 and the product rule (4.60), one obtains that the  $L^2$  norm of such terms is bounded by

$$\varepsilon^\mu C C_1^q C_4^p \|v\|_{H_\varepsilon^s}.$$

for some numerical constant  $C$  and

$$\mu = |\alpha| + pM - \sum (|\beta^j| - 1)_+ - \sum |\alpha^k| - |\gamma| - p\frac{d}{2}.$$

If  $\sum |\beta^j| > 0$ , then  $\sum (|\beta^j| - 1)_+ \leq (\sum |\beta^j|) - 1$  and  $\mu = 0$  when  $p = 0$  and  $\mu > 0$  when  $p > 0$  since  $M > \frac{d}{2}$ . If  $\sum |\beta^j| = 0$ , then  $p > 0$  and  $\mu > 0$  since  $M > \frac{d}{2} + 1$ . This implies that this term satisfies (4.59).

The commutator  $[\varepsilon^{|\alpha|}\partial_x^\alpha, \varepsilon^{-1}A_0^{-1}E]v$  is linear combination of terms

$$\varepsilon^{|\alpha|-1}\partial_x^{\beta^1}a\dots\partial_x^{\beta^q}a\partial_x^\gamma v$$

with  $\sum|\beta^j|+|\gamma|\leq|\alpha|\leq s$  and  $\sum|\beta^j|>0$ . With (4.50), this term is also dominated as in (4.59).

Hence, there are constants  $C$  and  $K$ , which depend only on  $C_1, C_2, C_3$ , and  $\varepsilon_0 > 0$  which depends in addition on  $C_4$ , such that for  $\varepsilon \leq \varepsilon_0$  and  $u$  satisfying (4.53) on  $[0, T']$ , there holds for  $t \in [0, T']$ :

$$(4.61) \quad \|v(t)\|_{H_\varepsilon^s} \leq Ce^{Kt}\|v(0)\|_{H_\varepsilon^s} + C \int_0^t e^{K(t-t')}\|L(a, u, \partial)v(t')\|_{H_\varepsilon^s} dt'.$$

Choose  $C_4 \geq 2Ce^{KT}(C_3+C_2)$  and  $\varepsilon_0$  accordingly and assume that  $\varepsilon \leq \varepsilon_0$ . Therefore, (4.61) shows that if  $u$  satisfies  $\|u(t)\|_{H_\varepsilon^s} \leq C_4$  on  $[0, T']$ , then it also satisfies  $\|u(t)\|_{H_\varepsilon^s} \leq \frac{1}{2}C_4$  on this interval. By continuation, this implies that the local solution  $u$  of (4.49) can be continued on  $[0, T]$  and satisfies (4.53).  $\square$



## 5 Geometrical Optics

In this section we present the WKB method for two scale asymptotic expansions, applied to the construction of high frequency wave packet solutions. This method rapidly leads to the geometric optics equations, of which the main features are the eikonal equations, the polarization conditions and the transport equation most often along rays. The formal asymptotic solutions can be converted into approximate solutions by truncating the expansion. The next main step is to study their stability, with the aim of constructing exact solutions close to them. We first review the linear case and next we consider the general regime of *weakly nonlinear geometric optics* where general results of existence and stability of oscillating solutions are available. The transport equations are in general nonlinear. In particular Burger's equation appears as the generic transport equations for quasilinear non dispersive systems. However, for some equations, because of their special structure, the general results do not allow one to reach nonlinear regimes. In these cases one is led to increase the intensity of the amplitudes. There are cases of such large solutions where the construction of WKB solution can be carried out. However, the stability analysis is much more delicate and strong instabilities exist. We end this section with several remarks about caustics and the focusing of rays, which is a fundamental feature of multi-dimensional geometric optics.

### 5.1 Linear geometric optics

We sketch below the main outcomes of Lax' analysis (see P.Lax [Lax1]), applied to linear equations of the form

$$(5.1) \quad L(a, \partial)u := A_0(a)\partial_t u + \sum_{j=1}^d A_j(a)\partial_{x_j} u + \frac{1}{\varepsilon}E(a)u = 0$$

where  $a = a(t, x)$  given, with values belonging to some domain  $\mathcal{O} \subset \mathbb{R}^M$ .

**Assumption 5.1.** (Symmetry). *The matrices  $A_j$  and  $E$  are smooth functions on  $\mathcal{O}$ . The  $A_j$  are self-adjoint, with  $A_0$  positive definite and  $E$  skew-adjoint.*

#### 5.1.1 An example using Fourier synthesis

Consider a constant coefficient system (5.1) and assume that the eigenvalues of  $A_0^{-1}(\sum \xi_j A_j + E)$ , which are real by Assumption 5.1, have constant

multiplicity. We denote them by  $\lambda_p(\xi)$  and call  $\Pi_p(\xi)$  the corresponding eigenprojectors. Then the solution of (5.1) with initial data  $h$  is given by

$$(5.2) \quad u(t, x) = \frac{1}{(2\pi)^d} \sum_p \int e^{i(\varepsilon\xi x - t\lambda_p(\varepsilon\xi))/\varepsilon} \Pi_p(\varepsilon\xi) \hat{h}(\xi) d\xi.$$

For oscillating initial data

$$(5.3) \quad u^\varepsilon|_{t=0} = h(x) e^{ikx/\varepsilon}$$

the solution is

$$(5.4) \quad \begin{aligned} u^\varepsilon(t, x) &= \frac{1}{(2\pi)^{d/2}} \sum_p \int e^{i\{x(k+\varepsilon\xi) - t\lambda_p(k+\varepsilon\xi)\}/\varepsilon} \Pi_p(k + \varepsilon\xi) \hat{h}(\xi) d\xi \\ &= \sum_p e^{i(kx - t\omega_p)/\varepsilon} a_p^\varepsilon(t, x) \end{aligned}$$

with an obvious definition of  $a_p^\varepsilon$  and  $\omega_p = \lambda_p(k)$ . Expanding the phases to first order in  $\varepsilon$  yields

$$\lambda_p(k + \varepsilon\xi) = \lambda_p(k) + \varepsilon\xi \cdot \mathbf{v}_p + O(\varepsilon^2|\xi|^2), \quad \mathbf{v}_p = \nabla_\xi \lambda_p(k)$$

and  $\Pi_p(k + \varepsilon\xi) = \Pi_p(k) + O(\varepsilon|\xi|)$  so that

$$(5.5) \quad a_p^\varepsilon(t, x) = a_{p,0}(t, x) + O(\varepsilon t) + O(\varepsilon)$$

with

$$\begin{aligned} a_{p,0}(t, x) &= \frac{1}{(2\pi)^{d/2}} \int e^{i(x\xi - t\xi \mathbf{v}_k)} \Pi_p(k) \hat{h}(\xi) d\xi \\ &= \Pi_p(k) h(x - t\mathbf{v}_p). \end{aligned}$$

Indeed, using the estimates  $e^{itO(\varepsilon|\xi|^2)} - 1 = tO(\varepsilon|\xi|^2)$ ,  $\Pi_p(k + \varepsilon\xi) - \Pi_p(k) = O(\varepsilon|\xi|)$  implies the precise estimate

$$(5.6) \quad \|a^\varepsilon(t, \cdot) - a_{p,0}(t, \cdot)\|_{H^s(\mathbb{R}^d)} \leq C \varepsilon(1+t) \|h\|_{H^{s+2}(\mathbb{R}^d)}$$

Together with (5.4), this gives an asymptotic description of  $u^\varepsilon$  with error  $O(\varepsilon t)$ . The amplitudes  $a_{p,0}$  satisfy the *polarization condition*:

$$(5.7) \quad a_{p,0} = \Pi_p(k) a_{p,0}$$

and the simple *transport equation*:

$$(5.8) \quad (\partial_t + \mathbf{v}_p \cdot \nabla_x) a_{p,0} = 0.$$

The analysis just performed can be carried out without fundamental change for initial oscillations with nonlinear phase  $\psi(x)$  (see the beginning of the paragraph about caustics below) and also for variable coefficient operators (see e.g. [Lax1, Ra3, Hö2]). The typical results of linear geometric optics are contained in the description (5.4) (5.5), with the basic properties of polarization (5.7) and transport (5.8).

This method based on an explicit writing of the solutions using Fourier synthesis (or more generally Fourier Integral Operators) is limited to linear problems, because no such representation is available for nonlinear problems. On the contrary, the BKW method which is presented below is extendable to nonlinear problem and we now concentrate on this approach. Of course, in the linear case we will recover the same properties as those presented in the example above.

### 5.1.2 The BKW method and formal solutions

In the BKW method one looks *a priori* for solutions which have a *phase-amplitude* representation:

$$(5.9) \quad u^\varepsilon(t, x) = e^{i\varphi(t, x)/\varepsilon} \sigma^\varepsilon(t, x), \quad \sigma^\varepsilon(t, x) \sim \sum_{n \geq 0} \varepsilon^n \sigma_n(t, x),$$

We will consider here only *real* phase functions  $\varphi$ . That the parameter in front of  $L_0$  has the same order as the inverse of the wave length has already been discussed in Section 2: in this scaling, the zero-th order term  $L_0$  comes in the definition of the dispersion relation and in all the aspects of the propagation, as shown in the computations below. Examples are various versions of Maxwell's equations coupled with Lorentz model, or anharmonic oscillators, or with Bloch's equations.

Introduce the symbol of the equation:

$$(5.10) \quad \mathcal{L}(a, \tau, \xi) = \tau A_0(a) + \sum_{j=1}^d \xi_j A_j(a) - iE(a) = A_0(a)(\tau \text{Id} + \mathcal{G}(a, \xi))$$

Plugging the expansion (5.9) into the equation, and ordering in powers of  $\varepsilon$  yields the following cascade of equations:

$$(5.11) \quad \mathcal{L}(a, d_{t,x}\varphi)\sigma_0 = 0$$

$$(5.12) \quad i\mathcal{L}(a, d_{t,x}\varphi)\sigma_{n+1} + L_1(a, \partial)\sigma_n = 0, \quad n \geq 0.$$

with  $L_1(a, \partial) = A_0\partial_t + \sum A_j\partial_{x_j}$ .

**Definition 5.2.** A formal solution is a formal series (5.9) which satisfies the equation in the sense of formal series, that is satisfies the equations (5.11) and (5.12) for all  $n$ .

### 5.1.3 The dispersion relation and phases

The first equation (5.11) has a nontrivial solution  $\sigma_0 \neq 0$  if and only if  
-  $\varphi$  solves the *eikonal equation*

$$(5.13) \quad \det \left( \mathcal{L}(a(t, x), d\varphi(t, x)) \right) = 0$$

-  $\sigma_0$  satisfies the *polarization condition*

$$(5.14) \quad \sigma_0(t, x) \in \ker \left( \mathcal{L}(a(t, x), d\varphi(t, x)) \right).$$

**Definitions 5.3.** *i)* The equation  $\det \mathcal{L}(a, \tau, \xi) = 0$  is called the dispersion relation. We denote by  $\mathcal{C}$  the set of its solutions  $(a, \tau, \xi)$ .

*ii)* A point  $(\underline{a}, \underline{\tau}, \underline{\xi}) \in \mathcal{C}$  is called regular if on a neighborhood of this point,  $\mathcal{C}$  is given by an equation  $\tau + \lambda(a, \xi) = 0$ , where  $\lambda$  is a smooth function near  $(\underline{a}, \underline{\xi})$ . We denote by  $\mathcal{C}_{reg}$  the manifold of regular points.

*iii)* Given a  $C^k$  function  $a(t, x)$ ,  $k \geq 2$ ,  $\varphi$  is called a characteristic phase if it satisfies the eikonal equation (5.13).

*iv)* A characteristic phase is said to be of constant multiplicity  $m$  if the dimension of  $\ker \mathcal{L}(a(t, x), d\varphi(t, x))$  is equal to  $m$  for all points  $(t, x)$ .

*v)* A characteristic phase is said to be regular if  $(a(t, x), d\varphi(t, x)) \in \mathcal{C}_{reg}$  for all  $(t, x)$ .

**Remarks 5.4.** 1) A point  $(a, \tau, \xi)$  belongs to  $\mathcal{C}$  if and only if  $-\tau$  is an eigenvalue of  $\mathcal{G}(a, \xi)$ .

2) Because of the symmetry Assumption 5.1,  $\mathcal{G}(a, \xi)$  has only real and semi-simple eigenvalues and the dispersion relation has real coefficients. In particular the kernel and image of  $\tau \text{Id} + \mathcal{G}(a, \xi)$  satisfy:

$$(5.15) \quad \ker(\tau \text{Id} + \mathcal{G}(a, \xi)) \cap \text{im}(\tau \text{Id} + \mathcal{G}(a, \xi)) = \{0\}.$$

3) If  $(\underline{a}, \underline{\tau}, \underline{\xi}) \in \mathcal{C}_{reg}$ , then for  $(a, \xi)$  in a neighborhood of  $(\underline{a}, \underline{\xi})$ ,  $\lambda(a, \xi)$  is the only eigenvalue of  $\mathcal{G}(a, \xi)$  close to  $-\underline{\tau}$ , proving that  $\lambda(a, \xi)$  has constant multiplicity. In particular, a regular phase has constant multiplicity.

4) If  $\varphi$  is characteristic phase of constant multiplicity, then the kernel  $\ker \mathcal{L}(a(t, x), d\varphi(t, x))$  and the image  $\text{im} \mathcal{L}(a(t, x), d\varphi(t, x))$  are vector bundles of constant dimension, as smooth as  $d\varphi$  with respect to  $(t, x)$ .

**Examples 5.5. 1) (Planar phases.)** If  $L$  has constant coefficients, for instance if  $a = \underline{a}$  is fixed, then any solution of the dispersion relation, that is any eigenvalue  $-\tau$  of  $\mathcal{G}(\underline{a}, \xi)$ , yields a phase

$$(5.16) \quad \varphi(t, x) = \tau t + \xi \cdot x.$$

This phase has constant multiplicity since  $\mathcal{L}(\underline{a}, d\varphi)$  is independent of  $(t, x)$ .

**2) (Regular phases.)** If  $\mathcal{C} = \{\tau + \lambda(a, \xi) = 0\}$  near a regular point, then one can solve, locally, the Hamilton-Jacobi equation

$$(5.17) \quad \partial_t \varphi + \lambda(a(t, x), \partial_y \varphi) = 0$$

using the method of characteristics (see e.g. [Dui, Hö2]). The graph of  $d\varphi$  is a union of integral curves, called bicharacteristic curves, of the *Hamiltonien field*

$$(5.18) \quad H = \partial_t + \sum_{j=1}^d \partial_{\xi_j} \lambda(a(t, x), \xi) \partial_{x_j} - \sum_{k=0}^d \partial_{x_k} \lambda(a(t, x), \xi) \partial_{\xi_k}.$$

Starting from an initial phase,  $\varphi_0(x)$ , one determines the initial manifold  $\Lambda_0 = \{(0, x, -\lambda(0, x, d_x \varphi_0(x)), d_x \varphi_0(x))\}$  and next, the manifold  $\Lambda$  which is the union of the bicharacteristic curves launched from  $\Lambda_0$ . As long as the projection  $(t, x, \tau, \xi) \mapsto (t, x)$  from  $\Lambda$  to  $\mathbb{R}^{1+d}$  remains invertible,  $\Lambda$  is the graph of the differential of a function  $\varphi$ , that is  $\Lambda = \{(t, x, \partial_t \varphi, \partial_x \varphi)\}$ , and  $\varphi$  is a solution of (5.17).

The projections on the  $(t, x)$  space of the bicharacteristic curves drawn in  $\Lambda$  are called the *rays*. They are integral curves of the field

$$(5.19) \quad X = \partial_t + \sum_{j=1}^d \partial_{\xi_j} \lambda(a(t, x), d_x \varphi(t, x)) \partial_{x_j}.$$

#### 5.1.4 The propagator of amplitudes

The equation (5.12) for  $n = 1$  reads

$$i\mathcal{L}(a, d_{t,x}\varphi)\sigma_1 + L_1(a, \partial)\sigma_0 = 0.$$

A necessary and sufficient condition for the existence of solutions is that  $L_1(a, \partial)\sigma_0$  belongs to the range of  $\mathcal{L}(a, d\varphi)$ . Therefore,  $\sigma_0$  must satisfy:

$$(5.20) \quad \sigma_0 \in \ker \mathcal{L}(a, d\varphi), \quad L_1(a, \partial)\sigma_0 \in \text{im} \mathcal{L}(a, d\varphi).$$

Suppose that  $a$  is smooth and  $\varphi$  is a smooth characteristic phase of constant multiplicity  $m$ . In this case  $\mathcal{N}(t, x) = \ker \mathcal{L}(a(t, x), d\varphi(t, x))$  and  $\mathcal{I}(t, x) = \text{im} \mathcal{L}(a(t, x), d\varphi(t, x))$  define smooth vector bundles  $\mathcal{N}$  and  $\mathcal{I}$  with fiber of dimension  $m$  and  $N - m$  respectively. Introduce the quotient bundle  $\mathcal{N}' := \mathbb{C}^N / \mathcal{I}$  and the natural projection  $\pi : \mathbb{C}^N \mapsto \mathbb{C}^N / \mathcal{I}$ . The profile equations (5.20) read

$$(5.21) \quad \sigma_0 \in C_{\mathcal{N}}^\infty, \quad L_\varphi \sigma_0 = 0$$

where  $C_{\mathcal{N}}^\infty$  denotes the  $C^\infty$  sections of the bundle  $\mathcal{N}$  and

$$(5.22) \quad L_\varphi = \pi L_1(a, \partial)$$

is a first order system from  $C_{\mathcal{N}}^\infty$  to  $C_{\mathcal{N}'}^\infty$ . We will prove below that the Cauchy problem for (5.21) is well posed and thus determines  $\sigma_0$ . The operator  $L_\varphi$  acting from sections of  $\mathcal{N}$  to sections of  $\mathcal{N}'$  is the *intrinsic formulation of the propagation operator*.

In order to make computations and proofs, it is convenient to use more practical forms of  $L_\varphi$ . We give two of them.

- *Formulation using projectors.* We still assume that  $\varphi$  is a smooth characteristic phase of constant multiplicity  $m$ . Thus, there are smooth projectors  $P(t, x)$  and  $Q(t, x)$  on  $\ker \mathcal{L}(a(t, x), d\varphi(t, x))$  and  $\text{im} \mathcal{L}(a(t, x), d\varphi(t, x))$  respectively (see Remark 5.4 4). They satisfy at each point  $(t, x)$ :

$$(5.23) \quad (I - Q)\mathcal{L}(a, d\varphi) = 0, \quad \mathcal{L}(a, d\varphi)P = 0$$

With these notations, the conditions (5.20) are equivalent to

$$(5.24) \quad \sigma_0(t, x) = P(t, x)\sigma_0(t, x),$$

$$(5.25) \quad (I - Q)L_1(a, \partial)\sigma_0 = 0.$$

In this setup, the transport operator reads

$$(5.26) \quad L_\varphi \sigma := (I - Q)L_1(a, \partial)(P\sigma) = A_{0,\varphi}\partial_t + \sum_{j=1}^d A_{j,\varphi}\partial_{x_j} + E_\varphi$$

where  $A_{j,\varphi} = (\text{Id} - Q)A_jP$  and  $E_\varphi = (I - Q)L_1(a, \partial)(P)$ .

**Remark 5.6 (About the choice of projectors).** The choice of the projectors  $P$  and  $Q$  is completely free. This does not mean a lack of uniqueness in the determination of  $\sigma_0$  since all the formulations are equivalent to (5.20).

Moreover, if  $P'$  and  $Q'$  are other projectors on  $\ker \mathcal{L}$  and  $\text{im} \mathcal{L}$  respectively, there are matrices  $\alpha$  and  $\beta$  such that  $P' = P\alpha$  and  $(\text{Id} - Q') = \beta(\text{Id} - Q)$  since  $P$  and  $P'$  have the same image and  $(\text{Id} - Q)$  and  $(\text{Id} - Q')$  have the same kernel. Therefore the operators  $L'_\varphi = (\text{Id} - Q')A_j P'$  and  $L_\varphi$  are obviously conjugated and thus share the same properties.

Of course, the symmetry assumption suggests a **natural choice**: one can take  $P(t, x)$  to be the spectral projector on  $\ker(\partial_t \varphi + \mathcal{G}(a, d_x \varphi))$ , that is the projector on the kernel along the image of  $\partial_t \varphi + \mathcal{G}(a, d_x \varphi)$ . By the symmetry assumption, it is an orthogonal projector for the scalar product induced by  $A_0(a(t, x))$  so that  $A_0 P = P^* A_0$ , and the natural associated projector on the image is  $Q = A_0(\text{Id} - P)A_0^{-1} = \text{Id} - P^*$ . This choice is natural and sufficient for a mathematical analysis and the reader can assume that this choice is made all along these notes.

• *Formulation using parametrization of the bundles.* Alternately, one can parametrize (at least locally) the kernel as

$$(5.27) \quad \ker \mathcal{L}(a(t, x), d\varphi(t, x)) = \rho(t, x) \mathbb{R}^m$$

where  $\rho(t, x)$  is a smooth and injective  $N \times m$  matrix. For instance, in the case of Maxwell's equations, the kernel can be parametrized by the  $E$  component (see Example 5.11 below). Similarly, one can parametrize the co-kernel of  $\mathcal{L}(a, d\varphi)$  and introduce a smooth  $m \times N$  matrix  $\ell$  of rank  $m$  such that

$$(5.28) \quad \ell(t, x) \text{im} \mathcal{L}(a(t, x), d\varphi(t, x)) = \{0\}.$$

With these notations, the conditions (5.20) are equivalent to

$$(5.29) \quad \sigma_0 = \rho \sigma^b$$

$$(5.30) \quad \ell L_1(a, \partial) \rho \sigma^b = 0.$$

For instance,  $\mathcal{L}(a, d\varphi)$  being self adjoint, one can choose  $\ell = \rho^*$ . In this setup, the transport operator reads

$$(5.31) \quad L^b \sigma^b := \ell L_1(a, \partial)(\rho \sigma^b) = A_0^b \partial_t + \sum_{j=1}^d A_j^b \partial_{x_j} + E^b$$

where  $A_j^b = \ell A_j \rho$  and  $E^b = \ell L_1(a, \partial)(\rho)$ . It is clear that changing  $\rho$  [resp.  $\ell$ ] is just a linear change of unknowns [resp. change of variables in the target space].

For instance, let  $(r_1, \dots, r_m)$  be a smooth basis of  $\ker \mathcal{L}(a, d\varphi)$  (the columns of the matrix  $\rho$ ) and let a smooth dual basis  $(\ell_1, \dots, \ell_m)$  of left null vectors of  $\mathcal{L}(a, d\varphi)$  (the rows of  $\ell$ ). They can be normalized so that

$$(5.32) \quad \ell_p(t, x)A_0(a(t, x)r_q(t, x)) = \delta_{p,q}, \quad 1 \leq p, q, \leq m.$$

(One can choose the  $\{r_p\}$  forming an orthonormal basis for the hermitian scalar product induced by  $A_0$  and thus  $\ell_p = r_p^*$ ). The polarization condition (5.29) reads

$$(5.33) \quad \sigma(t, x) = \sum_{p=1}^m \sigma_p(t, x)r_p(t, x),$$

with scalar functions  $\sigma_p$ . The equation (5.30) is

$$(5.34) \quad \partial_t \sigma_p(t, x) + \sum_{j=1}^d \sum_{q=1}^m \ell_p(t, x) \partial_{x_j} (A_j(a(t, x)) \sigma_q(t, x) r_q(t, x)) = \ell_p f.$$

With  $\vec{\sigma}$  denoting the column  $m$ -vector with entries  $\sigma_p$ , the operator in the left hand side reads

$$(5.35) \quad L^b \vec{\sigma} := \partial_t \vec{\sigma} + \sum_{j=1}^d A_j^b \partial_{x_j} \vec{\sigma} + E^b \vec{\sigma}$$

where  $A_j^b$  is the  $m \times m$  matrix with entries  $\ell_p A_j r_q$ .

**Lemma 5.7 (Hyperbolicity of the propagation operator).** *Suppose that  $\varphi$  is a smooth characteristic phase of constant multiplicity. Then the first order system  $L_\varphi$  on the smooth fiber bundle  $\mathcal{N} = \ker \mathcal{L}(a, d\varphi)$  is symmetric hyperbolic, in the sense that*

*i) for any choice of projectors  $P$  and  $Q$  as above, there is a smooth matrix  $S_\varphi(t, x)$  such that the matrices  $S_\varphi A_{j,\varphi}$  occurring in (5.26) are self-adjoint and  $S_\varphi A_{0,\varphi}$  is definite positive on  $\mathcal{N}$ ,*

*or equivalently,*

*ii) for any choice of matrices  $\rho$  and  $\ell$ , the  $m \times m$  system  $L^b$  (5.31) is symmetric hyperbolic.*

*Proof.* Using the Remark 5.6, it is sufficient to make the proof when  $P$  is the spectral projector and  $(\text{Id} - Q) = P^*$ , in which case the result is immediate with  $S = \text{Id}$ .



For a general direct proof, note that  $f \in \ker(\text{Id} - Q)$  when  $f \in \text{im}\mathcal{L}(a, d\varphi)$ . Because of the symmetry,  $\text{im}\mathcal{L}(a, d\varphi) = \ker \mathcal{L}(a, d\varphi)^\perp$ . Therefore, this space is equal to  $(\text{im}P)^\perp = \ker P^*$ . This shows that  $\ker(\text{Id} - Q) = \ker P^*$ , implying that there is a smooth matrix  $S$  such that

$$S(\text{Id} - Q) = P^*.$$

Therefore  $SA_{j,\varphi} = P^*A_jP$  is symmetric and  $SA_{0,\varphi}P = P^*A_0P$  is definite positive on  $\mathcal{N}$ .

The proof for the  $L^b$  representation is quite similar.  $\square$

The classical existence theory for symmetric hyperbolic systems can be transported to vector bundles, for instance using the existence theory for  $L^b$ . It can also be localized on domains of determinacy as sketched in section 3.6. In the remaining part of these notes we generally use the formulation (5.26) of the equations.

**Theorem 5.8.** *Suppose that  $a \in C^\infty$  and that  $\varphi$  is a  $C^\infty$  characteristic phase of constant multiplicity near  $\Omega$  of  $(0, \underline{x})$ . Given a neighborhood  $\omega$  of  $\underline{x}$  in  $\mathbb{R}^d$ , then, shrinking  $\Omega$  if necessary, for all  $C^\infty$  section  $h$  over  $\omega$  of  $\ker \mathcal{L}(a(0, x), d\varphi(0, x))$  there is a unique solution  $\sigma \in C^\infty(\Omega)$  of*

$$(5.36) \quad \sigma \in \ker \mathcal{L}(a, d\varphi), \quad L_1(a, \partial)\sigma \in \text{im}\mathcal{L}(a, d\varphi), \quad \sigma|_{t=0} = h.$$

Equivalently, the equations can be written

$$(5.37) \quad \sigma = P\sigma, \quad (\text{Id} - Q)L_1(a, \partial)P\sigma = 0 \quad P\sigma|_{t=0} = h.$$

for any set of projectors  $P$  and  $Q$ .

**Lemma 5.9 (Transport equations for regular phases).** *If  $\varphi$  is a regular phase associated with the eigenvalue  $\lambda(a, \xi)$ , then the principal part of the operator  $L_\varphi$  is the transport operator*

$$(5.38) \quad A_{0,\varphi} X(t, x, \partial_{t,x}),$$

where  $A_{0,\varphi} = (I - Q)A_0P$  and  $X = \partial_t + \mathbf{v}_g \cdot \partial_c$  is the ray propagator (5.19).

**Definition 5.10 (Group velocity).** *Under the assumptions of the previous lemma,  $\mathbf{v}_g = \nabla_\xi \lambda(a, \partial_x \varphi)$  is called the group velocity. It is constant when  $a = \underline{a}$  is constant and  $a$  is a planar phase.*

*Proof.* Near a regular point  $(\underline{a}, \underline{\tau}, \underline{\xi})$  there are smooth projectors  $\mathcal{P}(a, \xi)$  and  $\mathcal{Q}(a, \xi)$  on  $\ker \mathcal{L}(a, -\lambda(a, \xi), \xi)$  and  $\text{im} \mathcal{L}(a, -\lambda(a, \xi), \xi)$  respectively. Differentiating the identity

$$(5.39) \quad \left( -\lambda(a, \xi)A_0(a) + \sum \xi_j A_j(a) - iE(a) \right) \mathcal{P}(a, \xi) = 0$$

with respect to  $\xi$  and multiplying on the left by  $(I - \mathcal{Q})$  implies that

$$(5.40) \quad -\partial_{\xi_j} \lambda (I - \mathcal{Q}) A_0 \mathcal{P} + (I - \mathcal{Q}) A_j \mathcal{P} = 0.$$

Evaluating at  $\xi = d_x \varphi$  implies that

$$(5.41) \quad (I - Q) A_j P = -\partial_{\xi_j} \lambda(a, d_x \varphi) (I - Q) A_0 P,$$

that is (5.38). □

**Example 5.11 (Maxwell-Lorentz equations).** Consider the system

$$(5.42) \quad \partial_t B + \text{curl} E = 0, \quad \partial_t E - \text{curl} B = -\partial_t P, \quad \varepsilon^2 \partial_t^2 P + P = \gamma E$$

and an optical planar phase  $\varphi = \omega t + kx$  which satisfies the dispersion relation

$$(5.43) \quad |k|^2 = \omega^2 \left( 1 + \frac{\gamma}{1 - \omega^2} \right) := \mu^2(\omega).$$

(see (3.15)). The polarization conditions are

$$(5.44) \quad E \in k^\perp, \quad B = -\frac{1}{\omega} k \times E, \quad P = \frac{\gamma E}{1 - \omega^2}$$

(see (3.16)). Thus the kernel  $\ker \mathcal{L}$  is parametrized by  $E \in k^\perp$  and the transport equation reads

$$(5.45) \quad \mu'(\omega) \partial_t E - \frac{k}{|k|} \cdot \partial_x E = 0.$$

In the general case, the characteristic determinant of  $\mathcal{L}_\varphi$  can be related to the Taylor expansion of the dispersion relation:

**Proposition 5.12.** *Suppose that  $\varphi$  is a characteristic phase with constant multiplicity  $m$ . Then the polynomial in  $(\tau, \xi)$*

$$(5.46) \quad \det \mathcal{L}(a(t, x), \partial_t \varphi(t, x) + \tau, \partial_x \varphi(t, x) + \xi)$$

*vanishes at order  $m$  at the origin, and its homogeneous part of degree  $m$  is proportional to characteristic determinant of  $L_\varphi$ .*

*Proof.* Fix  $(t, x)$ . For  $\xi$  small, there is a smooth decomposition of  $\mathbb{C}^N = \mathbb{E}_0(\xi) + \mathbb{E}_1(\xi)$ , into invariant spaces  $\mathbb{E}_l$  of  $\mathcal{G}(a, \partial_x \varphi + \xi)$ , such that  $\mathbb{E}_0(0) = \ker(\partial_t \varphi + \mathcal{G}(a, \partial_x \varphi))$ . In bases  $\{r_p(\xi)\}_{1 \leq p \leq m}$  for  $\mathbb{E}_0$  and  $\{r_p(\xi)\}_{m+1 \leq p \leq N}$  for  $\mathbb{E}_1$ ,  $\mathcal{G}(a, \partial_x \varphi + \xi)$  has a block diagonal form :

$$\partial_t \varphi I + \mathcal{G}(a, \partial_x \varphi + \xi) = \begin{pmatrix} \mathcal{G}_0 & 0 \\ 0 & \mathcal{G}_1 \end{pmatrix}.$$

with  $\mathcal{G}_0 = 0$  and  $\mathcal{G}_1$  invertible at  $\xi = 0$ . Therefore

$$\det \mathcal{L}(a, \partial_{t,x} \varphi + (\tau, \xi)) = c(\tau, \xi) \det(\tau I + \mathcal{G}_0(\xi))$$

where  $c(0, 0) \neq 0$ . Let  $\{\psi_p\}$  denote the dual basis of  $\{r_p\}$ . The entries of  $\mathcal{G}_0$  are

$$\begin{aligned} \psi_p(\xi)(\partial_t \varphi I + \mathcal{G}(a, \partial_x \varphi + \xi))r_q(\xi) &= \psi_p(0)\mathcal{G}(a, \xi)r_q(0) + O(|\xi|^2) \\ &= \sum_j \xi_j \psi_p(0)A_j(a)r_q(0) + O(|\xi|^2). \end{aligned}$$

for  $1 \leq p, q \leq m$ . At  $\xi = 0$ ,  $\{r_1, \dots, r_m\}$  is a basis of  $\ker \mathcal{L}(a, d\varphi)$  and  $\{\ell_p\} = \{\psi_p(A_0)^{-1}\}$  is a basis of left null vectors of  $\ker \mathcal{L}(a, d\varphi)$  which satisfies (5.32). Therefore the  $\psi_p(0)A_j r_q(0)$  are the entries of the matrix  $A_j^b$  introduced in (5.35) and hence

$$\det(\tau I + \mathcal{G}_0(\xi)) = \det(\tau I + \sum \xi_j A_j^b) + O((|\tau| + |\xi|)^{m+1}).$$

The proposition follows.  $\square$

**Remark 5.13.** When the phase is regular of multiplicity  $m$ , Lemma 5.9 asserts that the first order part of the propagator is  $L^b = XI$ , whose characteristic determinant is

$$\det L^b(\tau, \xi) = \left( \tau + \sum \xi_j \partial_{\xi_j} \lambda(a, \partial_x \varphi) \right)^m.$$

Therefore, we recover that

$$\begin{aligned} \det \mathcal{L}(a, \partial_{t,x} \varphi + (\tau, \xi)) &= c(\tau, \xi) (\tau + \partial_t \varphi + \lambda(a, \partial_x \varphi + \xi))^m \\ &= c \det L^b(\tau, \xi) + O((|\tau| + |\xi|)^{m+1}). \end{aligned}$$

From a geometrical point of view, this means that the characteristic manifold of  $L^b$  is the tangent space of  $\mathcal{C}_{reg}$  at  $(a, d\varphi)$ . In general, the geometrical meaning of Proposition 5.12 is that the characteristic manifold of  $L^b$  is the tangent cone of  $\mathcal{C}$  at  $(a, d\varphi)$ .

**Example 5.14 (Conical refraction).** Consider Maxwell's equations in a bi-axial crystal. With notations as in see (3.19), the characteristic determinant satisfies

$$\det \mathcal{L}(\tau, \xi) = \frac{1}{4} \tau^2 \left( (2\tau^2 - \Psi)^2 - (P^2 + Q^2) \right)$$

There are exceptional points in the characteristic variety  $\mathcal{C}$  which are *not* regular : this happens when  $P^2 + Q = 0$  and  $\tau = \pm \frac{1}{2} \Psi$ , whose explicit solution is given in (3.20). Consider such a point  $(\omega, k) \in \mathcal{C} \setminus \mathcal{C}_{reg}$  and the planar phase  $\varphi = \omega t + kx$ , which has constant multiplicity  $m = 2$ , see Example 5.5 1). Near this point,

$$\det \mathcal{L}(\partial_{t,x} \varphi + (\tau, \xi)) = c \left( (4\tau - \xi \cdot \Psi'(k))^2 - (\xi \cdot P'(k))^2 - (\xi \cdot Q'(k))^2 \right) + h.o.t.$$

Thus the relation dispersion of  $L_\varphi$  is

$$(4\tau - \xi \cdot \Psi'(k))^2 - (\xi \cdot P'(k))^2 - (\xi \cdot Q'(k))^2 = 0.$$

Because  $P'(k)$  and  $Q'(k)$  are linearly independent, this cannot be factored and this is the dispersion relation of a *wave* equation, not of a transport equation. We refer the reader for instance to [Lud2] [JMR4] for more details.

### 5.1.5 Construction of WKB solutions

**Theorem 5.15 (WKB solutions).** *Suppose that  $a \in C^\infty$  and that  $\varphi$  is a  $C^\infty$  characteristic phase of constant multiplicity  $m$  on a neighborhood  $\Omega$  of  $(0, \underline{x})$ . Given a neighborhood  $\omega$  of  $\underline{x}$ , shrinking  $\Omega$  is necessary, for all sequence of functions  $h_n \in C^\infty(\omega)$  satisfying  $P(0, x)h_n = h_n$ , there is a unique sequence of functions  $\sigma_n \in C^\infty(\Omega; \mathbb{C}^N)$  which satisfies (5.11) (5.12) and the initial conditions:*

$$(5.47) \quad P\sigma_n|_{t=0} = h_n.$$

*In addition,  $\sigma_0 = P\sigma_0$  is polarized.*

*Sketch of proof.* The equation (5.12) is of the form

$$(5.48) \quad \mathcal{L}(a(t, x), d\varphi(t, x))\sigma = f.$$

A necessary condition for the existence of  $\sigma$  is that  $f \in \text{im} \mathcal{L}(a, d\varphi)$ , equivalently that  $(\text{Id} - Q)f = 0$  where  $Q$  is a smooth projector on the range of  $\mathcal{L}(a, d\varphi)$ . When this condition is satisfied, this equation determines  $\sigma$  up to

an element of the kernel. More precisely, this the class  $\tilde{\sigma}$  of  $\sigma$  in the quotient space  $\mathbb{C}^N / \ker \mathcal{L}(a, d\varphi)$  is

$$(5.49) \quad \tilde{\sigma} = (\tilde{\mathcal{L}}(a, d\varphi))^{-1} f$$

where  $\tilde{\mathcal{L}}$  is the natural isomorphism from  $\mathbb{C}^N / \ker \mathcal{L}(a, d\varphi)$  to  $\text{im} \mathcal{L}(a, d\varphi)$  induced by  $\mathcal{L}(a, d\varphi)$ .

This relation can be lifted to  $\mathbb{C}^N$  introducing a partial inverse of  $\mathcal{L}(a, d\varphi)$ . Given projectors  $P$  and  $Q$  on  $\ker \mathcal{L}(a, d\varphi)$  and  $\text{im} \mathcal{L}(a, d\varphi)$  respectively, there is a unique partial inverse  $R(t, x)$  such that for all  $(t, x)$ :

$$(5.50) \quad R\mathcal{L}(a, d\varphi) = I - P, \quad PR = 0, \quad R(I - Q) = 0.$$

In particular, (5.48) is equivalent to

$$(5.51) \quad (\text{Id} - Q)f = 0, \quad \sigma = Rf + P\sigma.$$

Using these notations the cascade of equations (5.11) (5.12) is equivalent to

$$(5.52) \quad P\sigma_0 = \sigma_0$$

$$(5.53) \quad (I - Q)L_1(a, \partial)\sigma_n = 0, \quad n \geq 0,$$

$$(5.54) \quad \sigma_{n+1} = iRL_1(a, \partial)\sigma_n + P\sigma_{n+1} \quad n \geq 0.$$

thus to

$$(5.55) \quad P\sigma_0 = \sigma_0, \quad (I - Q)L_1(a, \partial)P\sigma_0 = 0,$$

and for  $n \geq 1$ :

$$(5.56) \quad (I - Q)L_1(a, \partial)P\sigma_n = -i(I - Q)L_1(a, D)RL_1(a, \partial)\sigma_{n-1},$$

$$(5.57) \quad (I - P)\sigma_n = iRL_1(a, \partial)\sigma_{n-1}$$

By (5.59), The Cauchy problems (5.55) and (5.56) with unknowns  $P\sigma_n$  can be solved on a neighborhood of  $(0, \underline{x})$  using Theorem 5.8.  $\square$

The initial conditions (5.47) can be replaced by conditions of the form

$$(5.58) \quad H\sigma_n|_{t=0} = h_n$$

where  $H$  a  $m \times N$  matrix, which depends smoothly on  $x$ , and such that

$$(5.59) \quad \ker H(x) \cap \ker \mathcal{L}(a(0, x), d\varphi(0, x)) = \{0\}.$$

**Theorem 5.16.** *Suppose that  $a \in C^\infty$  and that  $\varphi$  is a  $C^\infty$  characteristic phase of constant multiplicity  $m$  on a neighborhood  $\Omega$  of  $(0, \underline{x})$ . Given a neighborhood  $\omega$  of  $\underline{x}$ , shrinking  $\Omega$  is necessary, for all sequence of functions  $h_n \in C^\infty(\omega; \mathbb{C}^m)$ , there is a unique sequence of solutions  $\sigma_n \in C^\infty(\Omega; \mathbb{C}^N)$  of (5.11) (5.12) and the initial conditions (5.58).*

Indeed, the  $n$ -th initial condition reads

$$HP\sigma_n|_{t=0} = h_n - H(\text{Id} - P)\sigma_n|_{t=0} = h_n - iHRL_1\sigma_{n-1}|_{t=0}.$$

Knowing  $\sigma_{n-1}$ , the assumption (5.59) implies that this equation can be solved and give the desired initial conditions for the  $P\sigma_n|_{t=0}$ .

**Remark 5.17 (About the choice of projectors).** The projector  $P$  intervene in the formulation of the Cauchy condition (5.47). We point out here that the coefficients  $\sigma_n$  do not depend on the splitting (5.53) (5.54) of the equations, where one could use as well other projectors  $P'$  and  $Q'$ : the equations are always equivalent to (5.11) (5.12) and the theorem asserts uniqueness. Moreover, the set of asymptotic solutions obtained by this construction does not depend on the projector  $P$  used for the initial condition

**Remark 5.18.** Imposing initial conditions for the  $P\sigma_n$  is natural from the proof. We follow this approach in the remaining part of these notes. However, in applications, the formulation (5.58) may be more adapted to physical considerations. For instance, when considering Maxwell's equations as in (5.42), it makes sense to impose initial conditions on the electric field, or on the electric induction (see Example 5.11).

### 5.1.6 Approximate solutions

Given  $(\sigma_0, \dots, \sigma_n) \in C^\infty$  solutions of (5.11) (5.12) for  $0 \leq k \leq n$ , the family of functions

$$(5.60) \quad u_{app,n}^\varepsilon(t, x) = \sum_{k=0}^n \varepsilon^k \sigma_k(t, x) e^{i\varphi(t,x)/\varepsilon}$$

satisfies the equation (5.1) with an error term of order  $O(\varepsilon^n)$ , that is

$$(5.61) \quad L(a, \partial)u_{app,n}^\varepsilon = \varepsilon^n f_n e^{i\varphi/\varepsilon}$$

where  $f_n = QL_1(a, \partial)\sigma_n$  is smooth.

If  $\{\sigma_n\}_{n \in \mathbb{N}}$  is a family of solutions of (5.11) (5.12), one can use Borel's Theorem to construct asymptotic solutions

$$(5.62) \quad u_{app}^\varepsilon(t, x) = \sigma^\varepsilon(t, x) e^{i\varphi(t, x)/\varepsilon}, \quad \sigma^\varepsilon(t, x) \sim \sum_{k \geq 0} \varepsilon^k \sigma_k(t, x)$$

where the symbol  $\sim$  means that for all  $n$  and all  $m$  :

$$(5.63) \quad \sigma^\varepsilon - \sum_{k \leq n} \varepsilon^k \sigma_k = O(\varepsilon^{n+1}) \quad \text{in } C^m.$$

In this case

$$(5.64) \quad L(a, \partial) u_{app}^\varepsilon = r^\varepsilon e^{i\varphi/\varepsilon}$$

where  $r^\varepsilon = O(\varepsilon^\infty)$  in  $C^\infty$ , meaning that for all  $n$  and all  $m$ ,

$$(5.65) \quad r^\varepsilon = O(\varepsilon^n) \quad \text{in } C^m.$$

Note that this also implies that

$$(5.66) \quad r^\varepsilon e^{i\varphi/\varepsilon} = O(\varepsilon^\infty) \quad \text{in } C^\infty.$$

### 5.1.7 Exact solutions

The construction of exact solutions  $u^\varepsilon$  of  $L(a, \partial)u^\varepsilon = 0$  close to approximate solutions  $u_{app}^\varepsilon$  amounts to solving the equation for the difference

$$(5.67) \quad L(a, \partial)(u^\varepsilon - u_{app}^\varepsilon) = -r^\varepsilon e^{i\varphi/\varepsilon}.$$

Because this system is symmetric hyperbolic and linear, the Cauchy problem is locally well posed. The question is to obtain estimates for the difference  $u^\varepsilon - u_{app}^\varepsilon$ , which are uniform in  $\varepsilon$ . The symmetry immediately implies uniform  $L^2$  estimates. The proof of uniform Sobolev estimates is more delicate when the term  $\varepsilon^{-1}E(a)$  is present in the equation (5.1), as discussed in Section 4. In all cases, the method of weighted estimates using  $H_\varepsilon^s$  norms can be applied. Combining Theorem 5.15 and Theorem 4.21, localized on domains of determinacy as in Section 3.6, implies the following.

**Theorem 5.19.** *Suppose that  $a \in C^\infty$  and that  $\varphi$  is a  $C^\infty$  characteristic phase of constant multiplicity near  $(0, \underline{x})$ . Suppose that  $\{h_n\}_{n \in \mathbb{N}}$  is a sequence of  $C^\infty$  functions on a fixed neighborhood of  $\underline{x}$ , such that  $P(0, x)h_n(x) =$*

$h_n(x)$ . Then there are  $C^\infty$  solutions  $u^\varepsilon$  of  $L(a, \partial)u^\varepsilon = 0$  on a neighborhood of  $(0, \underline{x})$  independent of  $\varepsilon$  such that

$$u^\varepsilon = \sigma^\varepsilon e^{i\varphi/\varepsilon}, \quad \sigma^\varepsilon \sim \sum_{n \geq 0} \varepsilon^n \sigma_n$$

where the  $\{\sigma_n\}_{n \in \mathbb{N}}$  are the unique solutions of (5.11) (5.12) such that  $P\sigma_n|_{t=0} = h_n$ . In particular

$$Pu^\varepsilon|_{t=0} = h^\varepsilon e^{i\varphi(0, \cdot)/\varepsilon}, \quad h^\varepsilon \sim \sum_{n \geq 0} \varepsilon^n h_n.$$

## 5.2 Weakly nonlinear geometric optics

To fix the notations consider here the first order system

$$(5.68) \quad L(a, u, \partial)u := (\varepsilon A_0(a, u)\partial_t + \sum_{j=1}^d \varepsilon A_j(a, u)\partial_{x_j} + E(a))u = f(a, u),$$

where  $f$  is a smooth function with  $f(a, 0) = 0, \nabla_u f(a, 0) = 0$ . The  $A_j$  are smooth and symmetric matrices and  $E$  is skew symmetric. To model high frequency oscillatory nonlinear waves, one looks for solutions of the form

$$(5.69) \quad u^\varepsilon(t, x) \sim \varepsilon^p \sum_{n \geq 0} \varepsilon^n U_n(t, x, \varphi(t, x)/\varepsilon),$$

There are two differences with (5.9) :

- because of the nonlinearity, *harmonics*  $e^{ik\varphi/\varepsilon}$  are expected, yielding to general periodic functions of  $\varphi/\varepsilon$  and thus profiles  $U_n(t, x, \theta)$  that are periodic in  $\theta$ , see Section 7 for an elementary example.

- a prefactor  $\varepsilon^p$  which measures the amplitude of the oscillations. When  $p$  is large, the wave is small and driven by the linear part  $L(a, 0, \partial)$ , the nonlinear effects appearing only as perturbations of the principal term. Decreasing  $p$ , the *weakly nonlinear regime* is reached when the nonlinear effects are present in the propagation of the principal term  $U_0$ .

For general quasilinear equations (5.68) and quadratic nonlinearities, this corresponds to  $p = 1$ . When the nonlinearities are cubic, then the natural scaling is  $p = \frac{1}{2}$  and so forth (see the general discussion in [DoRa1]). In the remaining part of this section, we mainly concentrate on the most general case with  $p = 1$  (see however the third example in Remarks 5.26).

Other regimes of *strongly nonlinear geometric optics* will be briefly discussed in the subsequent sections.



**Remark 5.20 (Other frameworks).** In the framework presented here,  $u_0 = 0$  is a solution of (5.68) and we study perturbations of this particular solution. The analysis applies as well to slightly different equations. For instance, one can replace the condition  $f = O(|u|^2)$  by  $f = \varepsilon \tilde{f}$  and consider perturbations of nonconstant solutions. Examples are systems

$$(5.70) \quad L(u, \partial)u := \left( A_0(u)\partial_t + \sum_{j=1}^d A_j(u)\partial_{x_j} + \frac{1}{\varepsilon}E \right)u = f(u).$$

Note that may change the prefactor  $\varepsilon^p$  in (5.69). We leave to the reader to write down the corresponding modifications. Some of the examples below concern this class of equations.

### 5.2.1 Asymptotic equations

With  $p = 1$ , plug the expansion (5.69) into the equation, expand in power series of  $\varepsilon$  and equate to zero the coefficients. We obtain a cascade of equations :

$$(5.71) \quad \mathcal{L}_0(a, d\varphi, \partial_\theta)U_0 = 0$$

$$(5.72) \quad \mathcal{L}_0(a, d\varphi, \partial_\theta)U_{n+1} + \mathcal{L}_1(a, U_0, \partial_{t,x,\theta})U_n = F_n, \quad n \geq 0,$$

where

$$\begin{aligned} \mathcal{L}_0(a, d\varphi, \partial_\theta) &= \left( \sum_{j=0}^d \partial_j \varphi A_j(a, 0) \right) \partial_\theta + E(a) \\ \mathcal{L}_1(a, v, \partial_{t,x,\theta}) &= L_1(a, 0, \partial_{t,x}) + B(a, d\varphi, v) \partial_\theta, \end{aligned}$$

with

$$L_1(a, 0, \partial_{t,x}) = \sum_{j=0}^d A_j(a, 0) \partial_j, \quad B(a, d\varphi, v) = \sum_{j=0}^d \partial_j \varphi v \cdot \nabla_u A_j(a, 0).$$

Moreover,

$$\begin{aligned} F_0 &= \nabla^2 f(a, 0)(U_0, U_0) \\ F_n &= 2\nabla^2 f(a, 0)(U_0, U_n) - \sum_{j=0}^d \partial_j \varphi U_n \cdot \nabla_u A_j(a, 0) \partial_\theta U_0 + G_{n-1} \end{aligned}$$

where  $G_{n-1}$  depends only on  $(U_0, \dots, U_{n-1})$ .

The operator  $\mathcal{L}_0$  has constant coefficient in  $\theta$ . Using Fourier series,

$$(5.73) \quad U(t, x, \theta) = \sum_{\alpha \in \mathbb{Z}} \hat{U}_\alpha(t, x) e^{i\alpha\theta},$$

$$\mathcal{L}_0(a, d\varphi, \partial_\theta)U = \sum_{\alpha} i\mathcal{L}(a, \alpha d\varphi) \hat{U}_\alpha(t, x) e^{i\alpha\theta}$$

where  $\mathcal{L}(a, \tau, \xi)$  is the symbol (5.10) associated to the operator  $L(a, 0, \partial)$ . For  $\mathcal{L}_0$  to have a nontrivial kernel, at least one phase  $\alpha\varphi$  must be characteristic. Changing  $\varphi$  is necessary, we assume that this occurs for  $\alpha = 1$ . Next, the analysis is quite different depending on whether  $E = 0$  or not. In the former case, all the harmonics are characteristic; in the latter, the harmonics are not expected to be characteristic in general, except  $\alpha = -1$  if one considers real valued solutions. The following assumptions are satisfied in virtually all cases of application. They give a convenient framework for the construction of asymptotic solutions.

**Assumption 5.21.**  $\varphi \in C^\infty(\bar{\Omega})$  is a characteristic phase of constant multiplicity for  $L(a, 0, \partial)$  on a neighborhood  $\Omega$  of  $(0, \underline{x})$ .

If  $E \neq 0$ , we further assume that  $\det(\sum \partial_j \varphi A_j)$  does not vanish on  $\Omega$  and that for all  $\alpha$ , either  $\alpha\varphi$  is a characteristic phase of constant multiplicity for  $L(a, 0, \partial)$  or  $\det \mathcal{L}(a, 0, \alpha d\varphi)$  does not vanish on  $\bar{\Omega}$ .

**Example 5.22.** In the constant coefficient case,  $a(t, x) = \underline{a}$  constant, for planar phase  $\varphi = \tau t + \xi x$ , the conditions when  $E \neq 0$  read

$$(5.74) \quad \det(\tau A_0 + \sum \xi_j A_j - iE) = 0, \quad \det(\tau A_0 + \sum \xi_j A_j) \neq 0.$$

Denote by  $Z \subset \mathbb{Z}$  the set of indices  $\alpha$  such that  $\alpha\varphi$  is a characteristic phase. If  $E = 0$ , then  $Z = \mathbb{Z}$  and by homogeneity,  $\mathcal{L}(a, 0, \alpha d\varphi) = \alpha \mathcal{L}(a, 0, d\varphi)$ . On the other hand, if  $E \neq 0$ , by continuity, the assumption above implies that  $\det(\sum \partial_j \varphi A_j - i \frac{1}{\alpha} E) \neq 0$  on  $\bar{\Omega}$  if  $|\alpha|$  is large. Therefore the set  $Z \subset \mathbb{Z}$  of indices  $\alpha$  such that  $\alpha\varphi$  is a characteristic phase is finite. In both cases, there are projectors  $P_\alpha$ ,  $Q_\alpha$  and partial inverse  $R_\alpha$  such that for all  $\alpha \in \mathbb{Z}$ .

$$(I - Q_\alpha) \mathcal{L}(a, 0, \alpha d\varphi) = 0, \quad \mathcal{L}(a, 0, \alpha d\varphi) P_\alpha = 0$$

$$R_\alpha \mathcal{L}(a, 0, \alpha d\varphi) = I - P_\alpha, \quad P_\alpha R_\alpha = 0, \quad R_\alpha (I - Q_\alpha) = 0.$$

In particular, if  $\alpha \notin Z$ ,  $P_\alpha = 0$ ,  $Q_\alpha = \text{Id}$  and  $R_\alpha = (\mathcal{L}(a, 0, \alpha d\varphi))^{-1}$ .

**Remark 5.23.** When  $E = 0$ , then for  $\alpha = 0$ , one has  $P_0 = \text{Id}$ ,  $Q_0 = 0$  and  $R_0 = 0$ . When  $\alpha \neq 0$ , by homogeneity one can choose  $P_\alpha = P_1$ ,  $Q_\alpha = Q_1$  and  $R_\alpha = \alpha^{-1}R_1$ . This is systematically assumed in the exposition below.

In both cases ( $Z$  finite or  $E = 0$  with the particular choice above), Assumption 5.21 implies that the matrices  $P_\alpha$ ,  $Q_\alpha$  and  $R_\alpha$  are uniformly bounded for  $(t, x) \in \Omega$  and  $\alpha \in \mathbb{Z}$ . Thus one can define projectors  $\mathcal{P}$  and  $\mathcal{Q}$  on the kernel and on the image of  $\mathcal{L}_0(a, \partial_\theta)$  respectively, and a partial inverse  $\mathcal{R}$ . In the Fourier expansion (5.73),  $\mathcal{P}$  is defined by the relation

$$(5.75) \quad \mathcal{P}U = \sum_{\alpha} P_{\alpha} \hat{U}_{\alpha}(t, x) e^{i\alpha\theta}$$

with similar definitions for  $\mathcal{Q}$  and  $\mathcal{R}$ .

With these notations, the cascade (5.71) (5.72) is analyzed as in the linear case. It is equivalent to

$$(5.76) \quad \mathcal{P}U_0 = U_0, \quad (I - \mathcal{Q})\mathcal{L}_1(a, U_0, \partial)\mathcal{P}U_0 = (I - \mathcal{Q})F_0,$$

and for  $n \geq 1$ :

$$(5.77) \quad \begin{aligned} (I - \mathcal{Q})\mathcal{L}_1(a, U_0, \partial)\mathcal{P}U_n \\ = (I - \mathcal{Q})(F_n - \mathcal{L}_1(a, U_0, D)(I - \mathcal{P})U_n), \end{aligned}$$

$$(5.78) \quad (I - \mathcal{P})U_n = \mathcal{R}(F_{n-1} - \mathcal{L}_1(a, \partial)U_{n-1}).$$

**Remark 5.24.** As in Remark 5.6, the choice of projectors  $P_\alpha$ ,  $Q_\alpha$  is free. The natural choice is to take spectral projectors, that is  $P_\alpha(t, x)$  orthogonal with respect to sht scalar product defined by  $A_0(a(t, x))$  and  $Q_\alpha = A_0(\text{Id} - P_\alpha)A_0^{-1} = \text{Id} - P_\alpha^*$ .

### 5.2.2 The structure of the profile equations I: the dispersive case

We consider here the case where  $E \neq 0$ . In this case the set  $Z$  of characteristic harmonics is finite, and the polarization condition in (5.76) reads

$$(5.79) \quad U_0(t, x, \theta) = \sum_{\alpha \in Z} \hat{U}_{0,\alpha}(t, x) e^{i\alpha\theta}, \quad P_\alpha \hat{U}_{0,\alpha} = \hat{U}_{0,\alpha}$$

and the equation is a coupled system for the  $\{\hat{U}_{0,\alpha}\}_{\alpha \in Z}$ :

$$(5.80) \quad (I - Q_\alpha)L_1(a, 0, \partial)P_\alpha \hat{U}_{0,\alpha} = (I - Q_\alpha) \sum_{\beta+\gamma=\alpha} \hat{\Gamma}_{\alpha,\beta,\gamma}(\hat{U}_{0,\beta}, \hat{U}_{0,\gamma})$$

with

$$(5.81) \quad \hat{\Gamma}_{\alpha,\beta,\gamma}(U, V) = -i\gamma B(a, d\varphi, U)V + \nabla^2 f(a)(U, V).$$

The linear analysis applies to the left hand side in (5.80). It is hyperbolic and is a transport equation if  $\alpha\varphi$  is a regular phase, as explained in Lemmas 5.7 and 5.9. Choosing bases in the ranges of the  $P_\alpha$ , it can be made explicit as in (5.34). In general, the quadratic term in the right hand side of (5.80) does not vanish, so that the system for  $U_0$  appears as a *semilinear* hyperbolic system.

**Example 5.25 (Maxwell - anharmonic Lorentz equations).** Consider the system

$$(5.82) \quad \partial_t B + \text{curl} E = 0, \quad \partial_t E - \text{curl} B = -\partial_t P, \quad \varepsilon^2 \partial_t^2 P + P + V(P) = \gamma E$$

where  $V$  is at least quadratic. We apply the general computations above, that is to asymptotic expansions of order  $O(\varepsilon)$ . The dispersion relation is the same as for Maxwell-Lorentz. Consider an optical planar phase  $\varphi = \omega t + kx$  which satisfies the dispersion relation

$$(5.83) \quad |k|^2 = \omega^2 \left(1 + \frac{\gamma}{1 - \omega^2}\right) := \mu^2(\omega).$$

The set of the harmonics which satisfies the dispersion relations is  $Z = \{-1, 0, +1\}$ . The polarization conditions are for harmonic  $+1$  and  $-1$  are

$$(5.84) \quad \hat{E}_{\pm 1} \in k^\perp, \quad \hat{B}_{\pm 1} = -\frac{1}{\omega} k \times \hat{E}_{\pm 1}, \quad \hat{P}_{\pm 1} = \frac{\gamma \hat{E}_{\pm 1}}{1 - \omega^2}.$$

Physical solutions correspond to real valued fields, that is satisfy the conditions  $\hat{E}_{-1} = \overline{\hat{E}_1}$ . The polarization conditions for the harmonic 0 (the mean field) reduce to

$$(5.85) \quad \hat{P}_0 = \gamma \hat{E}_0.$$

The profile equations equations read

$$(5.86) \quad \begin{cases} \mu'(\omega) \partial_t \hat{E}_1 - \frac{k}{|k|} \cdot \partial_x \hat{E}_1 = i \frac{\gamma \omega^2}{2|k|(1 - \omega^2)} \left( V_2(\hat{E}_0, \hat{E}_1) \right)_\perp \\ \partial_t \hat{B}_0 + \text{curl} \hat{E}_0 = 0, \quad (1 + \gamma) \partial_t \hat{E}_0 - \text{curl} \hat{B}_0 = 0 \end{cases}$$

where  $V_2$  is the quadratic part of  $V$  at the origin and  $A_\perp$  is the projection of  $A$  on  $k^\perp$ .

**Remarks 5.26. 1) (Rectification).** When only the harmonics  $-1, 0, +1$  are present, one expects from equation (5.80) a quadratic coupling of  $\hat{U}_1$  and  $\hat{U}_{-1}$  as a source term for the propagation of  $\hat{U}_0$ , implying that a mean field  $\hat{U}_0$  can be created by oscillatory waves. This phenomenon is called *optical rectification*. It is not present in the case of (5.86) where in addition the propagation equation for  $\hat{U}_0$  is linear. Rectification occurs in optics, but for different equations.

**2) (Generation of harmonics).** For nonisotropic crystals (see (2.18), (2.19), the dispersion relation is non isotropic in  $k$ , and for special values of  $k$  the harmonic  $(2\omega, 2k)$  can be characteristic, see [Don]. In this case, the analogue of (5.86) couples  $\hat{E}_1$  and  $\hat{E}_2$ . This *nonlinear* phenomenon is used in physical devices for *doubling frequencies*.

**Example 5.27. (Generic equations for cubic nonlinearities).** When  $V$  is cubic, the system (5.86) is linear. In this case, the regime of weakly nonlinear optics is not reached for amplitudes  $O(\varepsilon)$ , but for amplitudes  $O(\varepsilon^{\frac{1}{2}})$ . All the computations can be carried out with the prefactor  $\sqrt{\varepsilon}$  in front of the sum in (5.69) in place of  $\varepsilon$  (see [Don, DoRa1]). For instance, consider the cubic model (2.21) with no damping and oscillations with amplitudes of order  $\varepsilon^{\frac{1}{2}}$ . With mean field  $\hat{E}_0 = 0$ , the transport equation for  $\hat{E}_1$  are of the form :

$$(5.87) \quad \partial_t \hat{E}_1 + v_g \partial_x \hat{E}_1 = i\delta \left( |\hat{E}_1|^2 \hat{E}_1 + \frac{1}{2} (\hat{E}_1 \cdot \hat{E}_1) \overline{\hat{E}_1} \right).$$

The solutions of this system can be computed explicitly using the conservations

$$(\partial_t + v_g \partial_x) |\hat{E}_1|^2 = 0, \quad (\partial_t + v_g \partial_x) (\hat{E}_1 \times \overline{\hat{E}_1}) = 0.$$

The solutions are determined from their initial data  $\hat{E}_1|_{t=0} = \hat{A} \in k^\perp$  :

$$\hat{E}_1(t, x) = e^{itI(x-v_g t)} R(t\phi(x-v_g t)) \hat{A}(x-v_g t)$$

where  $R(s)$  is the rotation of angle  $s$  in the plane  $k^\perp$ , oriented by the direction of  $k$  and

$$I = \frac{3\delta}{2} |\hat{A}|^2, \quad \phi = \frac{i\delta}{2|k|} k \cdot (\hat{A} \times \overline{\hat{A}}).$$

Two physical *nonlinear* phenomena are described by this formula:

- the *polarization of the electric field rotates* at the speed  $\phi$  in the plane  $k^\perp$ ;
- when incorporated to the phase  $(\omega t + kx)/\varepsilon$ , the term  $tI$  corresponds to a *self modulation of the phase, depending on the intensity* of the field.

### 5.2.3 The structure of the profile equation II : the nondispersive case; the generic Burger's equation

We now assume that  $E = 0$ . In this case the set of characteristic frequencies is  $Z = \mathbb{Z}$ . By homogeneity, there are only two different possibilities:

$$\begin{cases} \alpha = 0 : & P_0 = \text{Id}, \quad Q_0 = 0, \quad R_0 = 0, \\ \alpha \neq 0 : & P_\alpha = P_1, \quad Q_\alpha = Q_1, \quad R_\alpha = R_1. \end{cases}$$

It is convenient to split a periodic function  $U(\theta)$  into its mean value  $\underline{U} = \mathbb{E}U = \hat{U}_0$  and its oscillating part  $U^* = \mathbb{O}_s U = U - \underline{U}$ . Dropping several subscripts 1 in  $P_1$  etc, the equation (5.76) reads

$$\begin{cases} PU_0^* = U_0^* \\ (I - Q)L_1(a, 0, \partial)PU_0^* = (I - Q)\mathbb{O}_s(\Gamma(U_0, U_0)) \\ L_1(a, 0, \partial)\underline{U}_0 = \mathbb{M}(\Gamma(U_0, U_0)) \end{cases}$$

with

$$\Gamma(U, V) = \nabla^2 f(a)(U_0, U_0) - B(a, d\varphi, U)\partial_\theta V.$$

**Remarks 5.28. 1) The case of conservation laws.** For balance laws (3.2), the matrices  $A_j(u)$  are the Jacobian matrices  $\nabla f_j(u)$  of the fluxes  $f_j$ . Thus

$$(5.88) \quad \begin{aligned} B(d\varphi, U)\partial_\theta U &= \frac{1}{2}\partial_\theta \left( \sum \partial_j \varphi \nabla_u^2 f_j(0)(U, U) \right) \\ &:= \frac{1}{2}\partial_\theta (b(d\varphi)(U, U)) \end{aligned}$$

implying that

$$\mathbb{M}(B(d\varphi, U)\partial_\theta U) = 0.$$

For conservation laws, the source term  $f$  vanishes and the equation for  $\underline{U}_0$  decouples and reduces to the linearized equation  $L_1(0, \partial_{t,x})\underline{U}_0 = 0$ . In particular,  $\underline{U}_0 = 0$  if its initial value vanishes and the equation for  $U_0^*$  reduces to

$$(5.89) \quad \begin{cases} PU_0^* = U_0^* \\ (I - Q)L_1(a, 0, \partial)PU_0^* + \frac{1}{2}(I - Q)\partial_\theta (b(d\varphi)(U_0^*, U_0^*)) = 0. \end{cases}$$

**Proposition 5.29. (Self interaction coefficient for simple modes)**  
*Suppose that*

a)  $\lambda(a, u, \xi)$  is a simple eigenvalue of  $A_0^{-1}(a, u) \sum \xi_j A_j(a, u)$  for  $(a, u, \xi)$  close to  $(\underline{a}, 0, \underline{\xi})$ , with eigenvector  $R(a, u, \xi)$  normalized by the condition  ${}^t R A_0 R = 1$  (which means that  $R$  is unitary for the scalar product defined by  $A_0$ );

b) the phase  $\varphi$  is a solution of the eikonal equation

$$(5.90) \quad \partial_t \varphi + \lambda(a(t, x), 0, \partial_x \varphi) = 0$$

for  $(t, x)$  close to  $(0, \underline{x})$ , with  $\partial_x \varphi(0, \underline{x}) = \underline{\xi}$ .

Then,

i) the polarization condition  $PU_0^* = U_0^*$  reads

$$(5.91) \quad U_0^*(t, x, \theta) = \sigma(t, x, \theta)r(t, x), \quad r(t, x) = R(a(t, x), 0, \partial_x \varphi(t, x)).$$

ii) the equation for  $\sigma$  deduced from the equation for  $U_0^*$  reads

$$(5.92) \quad \partial_t \sigma + \mathbf{v}_g \cdot \partial_x \sigma + b_0(\underline{U}_0) \partial_\theta \sigma + \gamma \partial_\theta \sigma^2 = \mathbb{O}_s F(\underline{U}_0, \sigma)$$

where  $\mathbf{v}_g = \partial_\xi \lambda(a, 0, \partial_x \varphi)$  is the group velocity as in (5.19), the self interaction coefficient  $\gamma$  is

$$(5.93) \quad \gamma = \frac{1}{2} r \cdot \nabla_u \lambda(a, 0, \partial_x \varphi).$$

The term  $b_0$  is linear in  $\underline{U}_0$  and  $F$  is at most quadratic.

In particular, for systems of balance laws,  $b_0 = 0$ . For systems of conservation laws and  $a = \underline{a}$  constant, with a planar phase  $\varphi = \omega t + \xi \cdot x$ , the equation reduces to

$$(5.94) \quad \partial_t \sigma + \mathbf{v}_g \cdot \partial_x \sigma + \gamma \partial_\theta \sigma^2 = 0.$$

The coefficient  $\gamma$  vanishes exactly when the eigenvalue  $\lambda(\underline{a}, \cdot, \xi)$  is linearly degenerate at  $u = 0$ .

*Proof.* We proceed as in Remark ??, noticing that by symmetry, the right kernel of  $-\lambda A_0 + \sum \xi_j A_j$  is generated by  ${}^t R$ . The equation is therefore

$${}^t r L_1(a, 0, \partial_{t,x})(\sigma r) = \mathbb{O}_s {}^t r \Gamma(\underline{U}_0 + \sigma r, \underline{U}_0 + \sigma r)$$

The left hand side is  $(\partial_t + \mathbf{v}_g \cdot \partial_x) \sigma + c \sigma$  as explained in Lemma 5.9. Similarly, differentiating in  $u$  the identity  $(-\lambda A_0 + \sum \xi_j A_j) r = 0$ , and multiplying on the left by  ${}^t r$  implies that

$${}^t r B(a, d\varphi, v) = v \cdot \nabla_u \lambda(a, 0, \partial_x \varphi) {}^t r A_0.$$

This implies the proposition.  $\square$

**Remark 5.30 (Generic equations).** Under the assumptions of Proposition 5.29, the equations for  $U_0 = \underline{U}_0 + \sigma r$  consist of an hyperbolic system for  $\underline{U}_0$  coupled with a transport equation for  $\sigma$ . In the genuinely nonlinear case, the transport equation is a *Burger's equation* which therefore appears as the *generic model for the propagation of the profile of nonlinear oscillations*.

#### 5.2.4 Approximate and exact solutions

**Proposition 5.31.** *The Cauchy problem for (5.76) is locally well posed.*

*Proof.* The symmetry of the original problem reflects into a symmetry of the profile equation

$$(5.95) \quad (I - Q)\mathcal{L}_1(a, V, \partial_{t,x,\theta})PU = (I - Q)F$$

This has been seen in Lemma 5.7 for the  $\partial_{t,x}$  part. The proof is similar for  $(I - Q)BP\partial_\theta$ . Thus all the machinery of symmetric hyperbolic systems can be used, implying that the Cauchy problem is locally well posed.  $\square$

In the cascade of the equations for  $U_n$ , only the the first one for  $U_0$  is nonlinear. This implies that de domain of existence of solutions for the subsequent equations can be chosen independent of  $n$ . Therefore:

**Theorem 5.32 (WKB solutions).** *Suppose that  $a \in C^\infty$  and that  $\varphi$  is a  $C^\infty$  characteristic phase of constant multiplicity near  $(0, \underline{x})$  of  $L(a, 0, \partial)$ . Let  $\mathcal{P}$  be the projector (5.75). Given a neighborhood  $\omega$  of  $\underline{x}$  and a sequence of  $C^\infty$  functions  $H_n$  on a  $(\omega \times \mathbb{T})$  such that  $\mathcal{P}|_{t=0}H_n = H_n$ , there is a neighborhood  $\Omega$  of  $(0, \underline{x})$  and a unique sequence  $U_n$  of  $C^\infty$  solutions of (5.76) (5.77) (5.78) on  $\Omega \times \mathbb{T}$  such that*

$$(5.96) \quad \mathcal{P}U_n|_{t=0} = H_n.$$

We suppose below that such solutions of the profile equations are given. One can construct approximate solutions

$$(5.97) \quad u_{app,n}^\varepsilon(t, x) = \varepsilon \sum_{k=0}^n \varepsilon^k U_k(t, x, \varphi(t, x)/\varepsilon)$$

which satisfy the equation (5.68) with an error term of order  $O(\varepsilon^{n+1})$ . Using Borel's summation, one can also construct approximate solutions at order  $O(\varepsilon^\infty)$ . More precisely,  $err_n^\varepsilon = L(a, u_{app,n}^\varepsilon, \partial)u_{app,n}^\varepsilon - f(a, u_{app,n}^\varepsilon)$  satisfies

$$(5.98) \quad \varepsilon^{|\alpha|} \|\partial_{t,x}^\alpha err_n^\varepsilon\|_{L^\infty(\Omega)} \leq \varepsilon^{n+1} C_\alpha,$$



We look for exact solutions

$$(5.99) \quad u^\varepsilon = u_{app,n}^\varepsilon + v^\varepsilon.$$

The equation for  $v^\varepsilon$  has the form (4.49):

$$(5.100) \quad \tilde{A}_0(b, v) \partial_t v + \sum_{j=1}^d \tilde{A}_j(b, v) \partial_{x_j} v + \frac{1}{\varepsilon} E(a) v = \tilde{F}(b, v)$$

where  $b = (a, u_{app,n}^\varepsilon)$  is uniformly bounded and satisfies estimates (4.50) for  $\alpha > 0$ :

$$(5.101) \quad \varepsilon^{|\alpha|-1} \|\partial_{t,x}^\alpha b\|_{L^\infty(\Omega)} \leq C_\alpha.$$

The derivatives of  $F(b, 0) = \varepsilon^{-1} err_n^\varepsilon$  are controlled by (5.98). Moreover,  $A_j(b, v) = A_j(a, u_{app,n}^\varepsilon + v)$ . Therefore, we are in position to apply Theorem 4.21, localized on a suitable domain of determinacy, shrinking  $\Omega$  if necessary. Consider initial data  $h^\varepsilon$  which satisfy for  $|\alpha| \leq s$

$$(5.102) \quad \varepsilon^{|\alpha|} \|\partial_x^\alpha h\|_{L^2(\omega)} \leq \varepsilon^n C_\alpha.$$

**Theorem 5.33.** *If  $n > 1 + d/2$ , there are  $\varepsilon_1 > 0$  and a neighborhood  $\Omega$  of  $(0, \underline{x})$  such that for  $\varepsilon \in ]0, \varepsilon_1]$ , the Cauchy problem for (5.68) with initial data  $u_{app,n}^\varepsilon|_{t=0} + h^\varepsilon$ , has a unique solution  $u^\varepsilon \in C^0 H^s(\Omega)$  for all  $s$  and which satisfies*

$$(5.103) \quad \varepsilon^{|\alpha|} \|\partial_x^\alpha (u^\varepsilon(t) - u_{app,n}^\varepsilon(t))\|_{L^2} \leq \varepsilon^n C_\alpha.$$

In particular, this implies that

$$(5.104) \quad \|\partial_x^\alpha (u^\varepsilon(t) - u_{app,n}^\varepsilon(t))\|_{L^\infty} \leq \varepsilon^{n-\frac{d}{2}-|\alpha|} C_\alpha.$$

**Remark 5.34** (About domains of determinacy). All the analysis above is local and we have not been very careful about large domains  $\Omega$  where they would apply. This is a technical and difficult question, we just give some hints.

- The final comparison between  $u^\varepsilon$  and  $u_{app,n}^\varepsilon$  must be performed on a domain of determinacy  $\Omega$  of  $\omega$  for the system (5.68). For quasilinear equations, the domains of determinacy depend on the solution as explained in Section 3.6. However, for conical domains of the form (3.38), it is sufficient that the slope  $\lambda_*$  be strictly larger than  $\lambda_*(M)$  where  $M$  only involve a  $L^\infty$

bound of  $u_{app}^\varepsilon$ , since for  $\varepsilon$  small, the  $L^\infty$  norm of  $u^\varepsilon$  will remain smaller than  $M'$  with  $\lambda(M') \leq \lambda_*$ .

- If  $\Omega$  is contained in the domain of determinacy of  $\omega$  for the full system, then it is also contained in the domain of determinacy of the propagator  $L_\varphi$  if  $\varphi$  is smooth on  $\Omega$ . For domains of the form (3.38), it easily follows from the explicit formula (3.39). Thus, if  $\varphi$  is defined on  $\Omega$ , one can solve the profile equations as well as the equation for the residual on  $\Omega$  and Theorem 5.33 extends to such domains.

- There are special case where one can improve the general result above. Consider for instance a semi-linear wave equation in  $\mathbb{R}^3$  and the phase  $\varphi = t + |x|$ . Consider  $\Omega = \{(t, x) : 0 \leq t \leq \min\{T, R - |x|\}\}$ .  $\Omega$  is contained in the domain of determinacy of the ball  $\omega = \{x : |x| \leq R\}$  but  $\varphi$  is *not* smooth on  $\Omega$ . The transport operator associated to  $\varphi$  is  $\partial_t - \partial_r - 2/r$  with  $r = |x|$ . If the initial oscillatory data are supported away from 0, in the annulus  $\{R_1 \leq |x| \leq R\}$ , then the profiles can be constructed for  $t \leq R_1$ , since the rays launched from the support of the initial profiles do not reach the singular set  $\{x = 0\}$  before this time. Therefore, if  $T \leq R_1$ , and decreasing it necessary because of the nonlinearity of the first profile equation, one can construct profiles and approximate solutions on  $\Omega$ , whose support do not meet the set  $\{x = 0\}$  where the phase is singular. Next one can compare approximate and exact solutions on  $\Omega$ .

### 5.3 Strongly nonlinear expansions

In the previous section we discussed the standard regime of weakly nonlinear geometric optics. No assumption on the structure of the nonlinear terms were made. There are cases where these general theorems do not provide satisfactory results. Typically, this happens when interaction coefficients vanish because of the special structure of the equations. This implies that the transport equations are linear instead of being nonlinear. This phenomenon is called *transparency* in [Do]. This happens for instance in the following two cases:

- for systems of conservation laws and oscillations polarized on a linearly degenerate mode, since then the self interaction coefficient  $\gamma$  in (5.94) vanishes.
- for semilinear dispersive system with quadratic nonlinearity  $f(u, u)$  when the polarization projectors  $P_\alpha$  and  $Q_\alpha$  associated to the characteristic harmonic phases  $\alpha\varphi$ ,  $\alpha \in Z$ , satisfy

$$(5.105) \quad (I - Q_\alpha)f(P_\beta, P_\gamma) = 0, \quad (\alpha, \beta, \gamma) \in Z^3, \quad \alpha = \beta + \gamma.$$

The transparency condition is closely related to the *null condition* for quadratic interaction introduced to analyze global existence of smooth small solutions, as it means that some interactions of oscillations are not present. It happens to be satisfied in many examples from Physics. Note also, that the transparency condition is completely different from the degeneracy condition evoked in Remark 5.26 3) for cubic nonlinearities: there the nonlinear terms were absent of the main profile equation for *all* polarizations, because the nonlinearity was too weak. In the present case, the quadratic term is not identically zero; only certain quadratic interactions are killed.

To reach nonlinear regimes, one idea is to consider waves of larger amplitude or, equivalently, of higher energy. This program turns out to be very delicate. We do not give here a complete report on the available results but just give several hints and references. Two questions can be raised:

- What are the conditions for the construction of BKW solutions?
- When they exist, what are the conditions for their stability, i.e. when are they close to exact solutions?

It turns out that the answer to the second question is absolutely not automatic: there are (many) cases where one can construct BKW solutions, which define approximate solutions of the equation at any order  $\varepsilon^n$ , but which are strongly unstable, due to a *supercritical* nonlinearity. This emphasizes the interest of the stability results obtained in the weakly nonlinear regimes.

### 5.3.1 An example: two levels Maxwell-Bloch equations

We first give an example showing that the particular form of the equations play a fundamental role in the setting of the problem. Consider the system

$$(5.106) \quad \begin{cases} \partial_t B + \operatorname{curl} E = 0, & \partial_t E - \operatorname{curl} B = -\partial_t P, \\ \varepsilon^2 \partial_t P + P = \gamma_1 (N_0 + N) E, & \partial_t N = -\gamma_2 \partial_t P \cdot E. \end{cases}$$

Introducing  $Q = \varepsilon \partial_t P$  and  $u = (B, E, P, Q, N)$ , it falls into the general framework of semilinear dispersive equations (5.68), with quadratic nonlinearity:

$$L(\varepsilon \partial) u = \varepsilon \partial_t u + \sum_{j=1}^d \varepsilon A_j u + E u = q(u, u)$$

Consider BKW solutions (5.69) associated to a planar phase  $\varphi$ . The general theory for quadratic interaction concerns solutions of amplitude  $O(\varepsilon)$  ( $p = 1$  in (5.69)). Recall that the wave number  $\beta = d\varphi$  satisfies the eikonal equation  $\det L(i\beta) = 0$  and the Fourier coefficients of the principal term

$U_0 = \sum \hat{U}_{0,\nu} e^{i\nu\theta}$  satisfy the polarization condition  $\hat{U}_{0,\nu} = P(\nu\beta)\hat{U}_{0,\nu}$  where  $P(\xi)$  is the orthogonal projector on  $\ker L(i\xi)$ . The propagation equations for  $\hat{U}_{0,\nu}$  are a coupled system of hyperbolic equations :

$$(5.107) \quad \hat{L}_\nu \hat{U}_{0,\nu} = P(\nu\beta) \sum_{\nu_1+\nu_2=\nu} q(\hat{U}_{0,\nu_1}, \hat{U}_{0,\nu_2}),$$

When the characteristic phases  $\nu\varphi$  are regular with group velocity  $\mathbf{v}_\nu$ ,  $\hat{L}_\nu = \partial_t + \mathbf{v}_\nu \cdot \partial_x$ .

For the Maxwell-Bloch equations (5.106), this analysis is unsatisfactory for two reasons. First, for physically relevant choices of  $U_0$ , the interaction terms vanish. Thus the transport equations are linear and the nonlinear regime is not reached. Second, Maxwell-Bloch equations are supposed to be a refinement of cubic models in nonlinear optics, such as the anharmonic oscillator model which is discussed in Remarks 5.26 3). Both facts suggest that solutions of amplitude  $O(\sqrt{\varepsilon})$  could exist. Actually, BKW solutions of the equation with  $p = 1/2$ , are constructed in [Do]. Indeed, for equation (5.106), there is an easy trick, *a change of unknowns, which reduces the quadratic system to a cubic one*. Introduce the following inhomogeneous scaling of the amplitudes :

$$(B, E, P, Q) = \sqrt{\varepsilon}(\tilde{B}, \tilde{E}, \tilde{P}, \tilde{Q}), \quad N = \varepsilon\tilde{N}.$$

Then, the Maxwell-Bloch equations read

$$\begin{cases} \varepsilon \partial_t \tilde{B} + \varepsilon \operatorname{curl} \tilde{E} = 0, & \varepsilon \partial_t \tilde{E} - \varepsilon \operatorname{curl} \tilde{B} = -\tilde{Q}, \\ \varepsilon \partial_t \tilde{P} - \tilde{Q} = 0, & \varepsilon \partial_t \tilde{Q} + \Omega^2 P = \gamma_1 N_0 \tilde{E} + \varepsilon \gamma_1 \tilde{N} \tilde{E}, \\ \varepsilon \partial_t \tilde{N} = -\gamma_2 \tilde{Q} \cdot \tilde{E}. \end{cases}$$

This scaling agrees with the BKW solutions of [Do]. The question is to construct oscillatory solutions  $\tilde{u} = (\tilde{B}, \tilde{E}, \tilde{P}, \tilde{Q}, \tilde{N})$  of of amplitude  $O(1)$ . The difficulty is that the source term in the last equation is nonlinear and of amplitude  $O(1)$ . A tricky argument gives the answer. Consider the change of unknowns

$$n = \tilde{N} + \frac{\gamma_2}{\gamma_1 N_0} (\tilde{Q}^2 + \Omega^2 \tilde{P}^2).$$

Then the last equation is transformed into

$$\varepsilon \partial_t n = \varepsilon \frac{\gamma_2}{N_0} \tilde{N} \tilde{Q} \cdot \tilde{E} = \varepsilon (c_1 n - c_2 (\tilde{Q}^2 + \Omega^2 \tilde{P}^2)) \tilde{Q} \cdot \tilde{E}$$

and the source term is now  $O(\varepsilon)$ . Introducing  $u^\sharp := (\tilde{B}, \tilde{E}, \tilde{P}, \tilde{Q}, n)$ , the system is transformed to a system

$$(1.9) \quad L^\sharp(\varepsilon \partial_x) u^\sharp = \varepsilon f(u^\sharp)$$

where the key point is that the right hand side is  $O(\varepsilon)$ . For this equation, the *standard* regime of nonlinear geometric optics concerns  $O(1)$  solutions and thus one can construct BKW solutions and prove their stability.

This algebraic manipulation yields a result with physical relevance. The transport equations for the amplitude obtained in this are cubic, in accordance with the qualitative properties given by other isotropic models of nonlinear optics such as the anharmonic model or the Kerr model (see [Don]).

### 5.3.2 A class of semilinear systems.

The analysis above has been extended to more general Maxwell-Bloch equations. In [JMR1], one considers the more general framework:

$$(5.108) \quad \begin{cases} L(\varepsilon \partial) u + \varepsilon f(u, v) = 0, \\ M(\varepsilon \partial) v + q(u, u) + \varepsilon g(u, v) = 0, \end{cases}$$

where  $f$  and  $g$  are smooth polynomial functions of their arguments and vanish at the origin,  $q$  is bilinear and

$$\begin{aligned} L(\varepsilon \partial) &:= \varepsilon \partial_t + \sum \varepsilon A_j \partial_{x_j} + L_0 := \varepsilon L_1(\partial) + L_0 \\ M(\varepsilon \partial) &:= \varepsilon \partial_t + \sum \varepsilon B_j \partial_{x_j} + M_0 := \varepsilon M_1(\partial) + M_0 \end{aligned}$$

are symmetric hyperbolic, meaning that the  $A_j$  and  $B_j$  are hermitian symmetric while  $L_0$  and  $M_0$  are skew adjoint. The main feature of this system is that the principal nonlinearity  $q(u, u)$  appears only on the second equation and depends only on the first set of unknowns  $u$ . The goal is to construct solutions satisfying

$$(5.109) \quad u^\varepsilon(t, x) \sim \sum_{n \geq 0} \varepsilon^n U_n(t, x, \varphi/\varepsilon), \quad v^\varepsilon(x) \sim \sum_{n \geq 0} \varepsilon^n V_n(t, x, \varphi/\varepsilon).$$

where  $\varphi = \omega t + kx$  is a linear phase. The profiles  $U_n(t, x, \theta)$  and  $V_n(t, x, \theta)$  are periodic in  $\theta$ . The difficulty comes from the  $O(1)$  interaction term in the second equation. We do not write the precise results of [JMR1] but we point out the different levels of compatibility that are needed to carry out the analysis:

*Level 1 : The transparency condition.* The eikonal equation is that  $\det L(i d\varphi) = 0$ . Denote by  $P(\beta)$  the orthogonal projector on  $\ker L(i\beta)$ . Similarly introduce  $Q(\beta)$  the orthogonal projector on  $\ker M(i\beta)$ . Assume

$$\det L(i\nu d\varphi) \neq 0 \quad \text{and} \quad \det M(i\nu d\varphi) \neq 0 \quad \text{for } \nu \text{ large.}$$

The transparency condition states that polarized oscillations  $U_0$  in  $\ker L(d\varphi\partial_\theta)$  produce oscillations  $q(U_0, U_0)$  in the range of  $M(d\varphi\partial_\theta)$ : For all integers  $\nu_1$  and  $\nu_2$  in  $\mathbb{Z}$  and for all vectors  $u$  and  $v$ ,

$$(5.110) \quad Q((\nu_1 + \nu_2)d\varphi) q\left(P(\nu_1 d\varphi)u, P(\nu_2 d\varphi)v\right) = 0.$$

When this condition is satisfied, one can write a triangular cascade of equations for the  $(U_n, V_n)$ . Using notations  $\mathcal{P}$  and  $\mathcal{Q}$  similar to (5.75) for the projectors on  $\ker L(d\varphi\partial_\theta)$  and  $\ker M(d\varphi\partial_\theta)$  respectively, and  $\mathcal{M}^{-1}$  for a partial inverse of  $M(d\varphi\partial_\theta)$ , the equations for  $U_0$  and  $\tilde{V}_0 = V_0 + \mathcal{M}^{-1}q(U_0, U_0)$  are

$$(5.111) \quad \begin{aligned} \mathcal{P}L_1(\partial)\mathcal{P}U_0 &= \mathcal{P}f(U_0, V_0), & \mathcal{P}U_0 &= U_0, \\ \mathcal{Q}M_1(\partial)\mathcal{Q}\tilde{V}_0 + \mathcal{D}(U_0, \partial_x)U_0 &= \mathcal{Q}G(U_0, V_0), & \mathcal{Q}\tilde{V}_0 &= \tilde{V}_0, \end{aligned}$$

where  $G$  is a nonlinear functional which involves the projectors  $\mathcal{P}$ ,  $\mathcal{Q}$  and the partial inverse  $\mathcal{M}^{-1}$ . There are similar equations for the terms  $(U_n, V_n)$ .

*Level 2 : Hyperbolicity of the profile equations.* This assumption can be made explicit (see again [JMR1]). When it is satisfied, one can construct BKW solutions and approximate solutions at any order.

*Level 3 : Stability of BKW solutions.* The linear and nonlinear, local in time, stability of the approximate solutions is proved under the following assumption:

For all  $\nu \in \mathbb{Z}$ , there is a constant  $C$  such that for all wave numbers  $\beta, \beta'$  and all vectors  $u$  and  $u'$

$$\left| Q(\beta')q(P(\nu d\varphi)u, P(\beta)u') \right| \leq C |\beta'_0 - \beta_0 - \nu\partial_t\varphi| |u| |u'|.$$

where  $\beta_0$  denotes the first component of  $\beta$ . It is shown that this condition implies both the conditions of levels 1 (easy) and 2 (more intricate).

*Level 4 : Normal form of the equation.* The condition above can be strengthened:

There is a constant  $C$  such that for all wave numbers  $\beta, \beta', \beta''$  and all vectors  $u$  and  $u'$

$$\left| Q(\beta'')q(P(\beta)u, P(\beta')u') \right| \leq C |\beta''_0 - \beta_0 - \beta'_0| |u| |u'|.$$

When this condition is satisfied, the system (5.108) is conjugate via a nonlinear pseudo-differential change of unknowns to a similar system with  $q = 0$ . Exceptionally, the change of variables may be local and not involve pseudo-differential operators, this is the case of Maxwell-Bloch equations.

### 5.3.3 Field equations and relativity

The construction of waves using asymptotic expansions is used in many areas of physics. In particular, in general relativity it is linked to the construction of nonlinear gravity waves (see [ChBr, CBGr, AlHu]). This also concerns field equations. An important new difficulty is that these equations are *not* hyperbolic, due to their gauge invariance. We mention the very important paper [Jea], where this difficulty is examined in detail, providing BKW expansions *and* rigorous justification of the asymptotic expansion for exact solutions, with applications to Yang-Mills equations and relativistic Maxwell's equations. For these equations, the transparency condition is satisfied so that the relevant nonlinear analysis concerns oscillations of large amplitude. An important feature of the paper [Jea] is that the compatibility condition is used both for the construction of asymptotic and exact solutions. The rigorous justification of asymptotic expansion given by Y.Choquet-Bruhat for Einstein's equations, seems to remain an open problem.

### 5.3.4 Linearly degenerate oscillations

For systems of conservation laws, the interaction coefficient  $\gamma$  in the Burger's transport equation (5.94) vanishes when the mode is linearly degenerate. We briefly discuss in this paragraph the construction and the stability of oscillatory waves

$$(5.112) \quad u^\varepsilon(t, x) \sim \underline{u}_0(t, x) + \sum_{n \geq p} (\sqrt{\varepsilon})^n U_n\left(t, x, \frac{\varphi_\varepsilon(t, x)}{\varepsilon}\right)$$

for symmetric hyperbolic systems of conservation laws

$$(5.113) \quad \partial_t f_0(u) + \sum_{j=1}^d \partial_{x_j} f_j(u) = 0.$$

The unperturbed state  $\underline{u}_0$  is a solution of (5.113) and the principal term of the phase  $\varphi_\varepsilon = \varphi_0 + \sqrt{\varepsilon}\varphi_1$  satisfies the eikonal equation

$$(5.114) \quad \partial_t \varphi_0 + \lambda(\underline{u}_0, \partial_x \varphi_0) = 0$$

where  $\lambda(u, \xi)$  is a *linearly degenerate* eigenvalue of constant multiplicity of the system.

The standard regime of weakly non linear geometric optics corresponds to  $p = 2$ . Because of the linear degeneracy, we consider now the case of larger amplitudes corresponding to the cases  $p = 1$  and  $p = 0$ .

- In space dimension  $d = 1$  and for  $p = 0$ , a complete analysis is given in [E] for gas dynamics and in [Hei, Sev, CoGu] in a general framework providing exact solutions which admit asymptotic expansions of the form.
- In any dimension, associated to linearly degenerate modes there always exist simple waves

$$u(t, x) = v(h(\xi \cdot x - \sigma t)),$$

that are exact solutions of (5.113), with  $v(s)$  a well-chosen curve defined for  $s \in I \subset \mathbb{R}$ ,  $\xi$  and  $h$  an arbitrary  $C^1(\mathbb{R}; I)$  function (cf [Maj]). Choosing  $h$  periodic with period  $\varepsilon$ , yields exact solutions satisfying (5.112) with  $p = 0$  and associated with the phase  $\varphi = \xi \cdot x - \sigma t$ . The question is to study the stability of such solutions: if one perturbs the initial data, do the solutions resemble to the unperturbed solutions? We give below partial answers, mainly taken from [CGM1, CGM2], but we also refer to [Che3] for extensions.

1. *Existence of BKW solutions for  $p = 1$ .* It is proved in [CGM1] that in general, one can construct asymptotic solutions (5.112) with  $p = 1$ , the main oscillation  $U_1^*$  being polarized along the eigenspace associated to  $\lambda$ . The average  $\underline{u}_1$  of  $U_1$  is an arbitrary solution of the linearized equation from (5.113) at  $\underline{u}_0$ . When  $\underline{u}_1 \neq 0$ , a correction  $\sqrt{\varepsilon}\varphi_1$  must be added to the phase  $\varphi_0$ , satisfying

$$(5.115) \quad \partial_t \varphi_1 + \partial_x \varphi_1 \cdot \nabla_\xi \lambda(\underline{u}_0, \partial_x \varphi_0) + \underline{u}_1 \cdot \nabla_u \lambda(\underline{u}_0, \partial_x \varphi_0) = 0$$

so that  $\varphi_\varepsilon = \varphi_0 + \sqrt{\varepsilon}\varphi_1$  satisfies the eikonal equation at the order  $O(\varepsilon)$  i.e.:

$$(5.116) \quad \partial_t \varphi_\varepsilon + \lambda(\underline{u}_0 + \sqrt{\varepsilon}\underline{u}_1, \partial_x \varphi_\varepsilon) = O(\varepsilon)$$

Moreover, the evolution of the main evolution  $U_1^*$  is nonlinear (in general) and coupled to the evolution of the average of  $U_2$ . The nonlinear regime is reached.

2. *Stability / Instability of BKW solutions.* In space dimension  $d = 1$ , the linear and nonlinear stability is proved using the notion of *good symmetrizer* cf [CoGu, Hei, Ser1]. When  $d > 1$ , the example of gas dynamics shows that the existence of a good symmetrizer does not suffice to control oscillations which are transversal to the phase and which can provoke *strong instabilities of Rayleigh type* cf [FSV, Gre].

As before, the question is to know whether the approximate solutions constructed by the BKW method are close to exact solutions, on some time



interval independent of  $\varepsilon$ . The main difficulty can be seen from two different angles:

- with  $u_{app}^\varepsilon = \underline{u}_0 + \sqrt{\varepsilon}U_1(t, x, \varphi_\varepsilon/\varepsilon) + \dots$ , the coefficients  $f'_j(u_{app}^\varepsilon)$  are not uniformly Lipschitz continuous, so that the usual energy method does not provide solutions on a uniform domain,

- the linearized equations involve a singular term in  $\varepsilon^{-\frac{1}{2}}D$ , which has no reason to be skew symmetric.

The first difficulty is resolved if one introduces the fast variable and look for exact solutions of the form

$$(5.117) \quad u^\varepsilon(t, x) = U^\varepsilon(t, x, \varphi_\varepsilon/\varepsilon).$$

The equation for  $U^\varepsilon$  reads

$$(5.118) \quad \sum_{j=0}^d A_j(U^\varepsilon) \partial_j U^\varepsilon + \frac{1}{\varepsilon} \sum_{j=0}^d \partial_j \varphi_\varepsilon A_j(U^\varepsilon) \partial_\theta U^\varepsilon = 0.$$

The construction of BKW solutions yields approximate solutions  $U_{app}^\varepsilon$ . The linearized equation is of the form

$$(5.119) \quad \mathcal{L}_c^\varepsilon + \frac{1}{\varepsilon} G_0 \partial_\theta + \frac{1}{\sqrt{\varepsilon}} (G_1 \partial_\theta + H_1) + C^\varepsilon,$$

where  $\mathcal{L}^\varepsilon$  is *symmetric hyperbolic, with smooth coefficients in the variables  $(t, x, \theta)$* . Moreover,  $G_0$  is symmetric and independent of  $\theta$ . On the other hand,  $G_1$  is symmetric but *does depend* on  $\theta$ , as it depends on  $U_1$ . Therefore the energy method after integration by parts reveals the singular term

$$(5.120) \quad \frac{1}{\sqrt{\varepsilon}} (D\dot{U}, \dot{U}) \quad \text{avec} \quad D := \frac{1}{2} (-\partial_\theta G_1 + H_1 + H_1^*).$$

This matrix can be computed explicitly: denoting by  $S(u)$  a symmetrizer of the system (5.113) and by  $\Sigma(u, \xi) := \sum \xi_j S(u) f'_j(u)$ ,

$$(5.121) \quad (D\dot{U}, \dot{U}) = D'(\partial_\theta U_1, \dot{U}, \dot{U}) + D''(\partial_\theta U_1, \dot{U}, \dot{U}).$$

with

$$\begin{aligned} 2D'(u, v, w) &= -((u \cdot \nabla_u \Sigma)v, w) + ((v \cdot \nabla_u \Sigma)u, w) + ((w \cdot \nabla_u \Sigma)u, v). \\ 2D''(u, v, w) &= ((v \cdot \nabla_u \lambda)Su, w) + ((w \cdot \nabla_u \lambda)Su_1, v). \end{aligned}$$

where  $\nabla_u \Sigma$ ,  $\nabla_u \lambda$  and  $S$  are taken at  $\underline{u}_0$  and  $\xi = \partial_x \varphi_0$ . Note that  $D(t, x, \theta)$  is a symmetric matrix. Since its average in  $\theta$  vanishes, it must vanish if it is nonnegative. Therefore:

the energy method provides uniform  $L^2$  estimates for the linearized equations if and only if  $D(t, x, \theta) \equiv 0$ .

Conversely, this condition is sufficient for the existence of Sobolev estimates and for the nonlinear stability (see [CGM1]):

**Theorem 5.35.** *If  $U_{app}^\varepsilon$  is a BKW approximate solution at order  $\varepsilon^n$  with  $n$  sufficiently high order and if the matrix  $D$  vanishes, there exist exact solutions  $U^\varepsilon$  of (5.118) such that  $U^\varepsilon - U_{app}^\varepsilon = O(\varepsilon^n)$ .*

The condition  $D = 0$  is very strong and unrealistic in general, except if  $U_1^* = 0$ , in which case we recover the standard scaling studied before. In gas dynamics, it is satisfied when the oscillations of  $U_1$  concern only the entropy and neither the velocity nor the density: this is what we call entropy waves, but in this case, one can go further and construct oscillations of amplitude  $O(1)$  as explained below. In the opposite direction, the Rayleigh instabilities studied in [FSV] for gas dynamics seem to be the general behavior that one should expect when  $D \neq 0$ .

3. *Entropy waves* In contrast with the one dimensional case, D. Serre [Ser1] has shown that the construction of  $O(1)$  solutions ( $p = 0$ ) for isentropic gas dynamics yields ill-posed equations for the cascade of profiles. The case of full gas dynamics is quite different: one can construct  $O(1)$  oscillations provided that the main term concerns only the entropy and not the velocity. Consider the complete system of gas dynamics expressed in the variables  $(p, v, s)$ , pressure, velocity and entropy:

$$(5.122) \quad \begin{cases} \rho (\partial_t v + (v \cdot \nabla_x) v) + \nabla_x p = 0 \\ \alpha (\partial_t p + (v \cdot \nabla_x) p) + \operatorname{div}_x v = 0 \\ \partial_t s + (v \cdot \nabla_x) s = 0 \end{cases}$$

with  $\rho(p, s) > 0$  and  $\alpha(p, s) > 0$ . In [CGM2], we prove the existence and the stability of non trivial solutions  $u^\varepsilon = (v^\varepsilon, p^\varepsilon, s^\varepsilon)$  of the form

$$(5.123) \quad \begin{aligned} v^\varepsilon(t, x) &= v_0(t, x) + \varepsilon V^\varepsilon(t, x, \varphi(t, x)/\varepsilon) \\ p^\varepsilon(t, x) &= p_0 + \varepsilon P^\varepsilon(t, x, \varphi(t, x)/\varepsilon) \\ s^\varepsilon(t, x) &= S^\varepsilon(t, x, \varphi(t, x)/\varepsilon) \end{aligned}$$

with  $V, P$  and  $S$  functions of  $(t, x, \theta)$ , periodic in  $\theta$  and admitting asymptotic expansions  $\sum \varepsilon^n V_n$  etc. The unperturbed velocity  $v_0$  satisfies the overdetermined system

$$\partial_t v_0 + (v_0 \cdot \nabla_x) v_0 = 0, \quad \operatorname{div}_x v_0 = 0,$$

for instance  $v_0$  can be a constant. Moreover,  $p_0$  is a constant and the phase  $\varphi$  is a smooth real valued function satisfying the eikonal equation  $\partial_t \varphi + (v_0 \cdot \nabla_x) \varphi = 0$ . The evolution of the principal term is governed by polarization conditions and a coupled system of propagation equations.

For instance consider the important example where  $v_0$  is constant (and we can take  $v_0 = 0$  by Galilean transformation). We choose a linear phase function solution of the eikonal equation  $\partial_t \varphi = 0$ , and rotating the axis we have  $\varphi(t, x) \equiv x_1$ . The components of the velocity are accordingly split into  $v = (v_1, w)$ . For the principal term, the polarization condition requires that the oscillation of the first component of  $V_0$  and  $P_0$ , denoted by  $V_{0,1}^*$  and  $P_0^*$ , vanish. Therefore

$$\begin{cases} v_1^\varepsilon(t, x) = \varepsilon \underline{V}_{0,1}(t, x) + O(\varepsilon^2), \\ w^\varepsilon(t, x) = \varepsilon W_0(t, x, x_1/\varepsilon) + O(\varepsilon^2), \\ p^\varepsilon(t, x) = p_0 + \varepsilon \underline{P}_0(t, x) + O(\varepsilon^2), \\ s^\varepsilon(t, x) = S(t, x, x_1/\varepsilon) + O(\varepsilon). \end{cases}$$

The profiles  $V_1(t, x)$ ,  $V_2(t, x, \theta)$ ,  $P(t, x)$  and  $S(t, x, \theta)$  satisfy

$$\left\{ \begin{array}{ll} \langle \rho(p_0, S) \rangle \partial_t \underline{V}_{0,1} + \partial_1 \underline{P} = 0, & V_{0,1}|_{t=0} = a_1(x), \\ \rho(p_0, S) (\partial_t W_0 + \underline{V}_{0,1} \partial_\theta W_0) + \nabla' \underline{P} = 0, & W_0|_{t=0} = a_2(x, \theta), \\ \langle \alpha(p_0, S) \rangle \partial_t \underline{P} + \partial_1 \underline{V}_{0,1} + \langle \text{div}' W \rangle = 0, & \underline{P}|_{t=0} = a_3(x), \\ \partial_t S + \underline{V}_{0,1} \partial_\theta S = 0, & S|_{t=0} = a_4(x, \theta). \end{array} \right.$$

where  $\langle U \rangle$  denotes the average in  $\theta$  of the periodic function  $U(\theta)$ , and  $\nabla'$  and  $\text{div}'$  denote the gradient and the divergence in the variables  $(x_2, \dots, x_d)$  respectively.

### 5.3.5 Fully nonlinear geometric optics

In this paragraph, we push the analysis one step further and consider an example where the amplitude of the oscillation is so large that the eikonal equation for the phase and the propagation equation for the profile are coupled. This example is taken from [Leb] (see also [Whi]) and concerns the Klein-Gordon equation

$$(5.124) \quad \varepsilon^2 \partial_t^2 u - \varepsilon^2 \partial_x^2 u + f(u) = 0$$

with  $f(u) = u^p$ ,  $p \in \mathbb{N}$ ,  $p > 1$  and odd. As in [Leb] one can consider much more general  $f$ , which may also depend on  $(t, x)$ , but for simplicity we restrict the exposition to this case. For such equations, the weakly-nonlinear regime concerns solutions of amplitude  $O(\varepsilon^{\frac{2}{p-1}})$ . Here we look for solutions of amplitude  $O(1)$ :

$$(5.125) \quad u^\varepsilon(t, x) = U^\varepsilon\left(t, x, \frac{\varphi(t, x)}{\varepsilon}\right), \quad U^\varepsilon(t, x, \theta) \sim \sum_{k \geq 0} \varepsilon^k U_k(t, x, \theta)$$

with  $U_k(t, x, \theta)$  periodic in  $\theta$ . The *compatibility or transparency condition* which is necessary for the construction is stated in Proposition 5.37 below.

Plugging (5.125) into (5.124) yields the following equations

$$\begin{aligned} \sigma^2 \partial_\theta^2 U_0 + f(U_0) &= 0, \\ \sigma^2 \partial_\theta^2 U_1 + \partial_u f(U_0) U_1 + T \partial_\theta U_0 &= 0, \\ \sigma^2 \partial_\theta^2 U_k + \partial_u f(U_0) U_k + T \partial_\theta U_{k-1} + \square U_{k-2} + R_k(x, U_0, \dots, U_{k-1}) &= 0. \end{aligned}$$

with  $\square = \partial_t^2 - \partial_x^2$ ,

$$(5.126) \quad \sigma^2 = (\partial_t \varphi)^2 - (\partial_x \varphi)^2, \quad T := 2\varphi'_t \partial_t - 2\varphi'_x \partial_x + \square \varphi.$$

Moreover, the  $R_k$  denote smooth functions of their arguments.

**Theorem 5.36.** *The profile equations together with initial conditions admit solutions  $\{U_k\}$ .*

We sketch the first part of the proof taken from [Leb].

1) Since  $\sigma$  is independent of  $\theta$ , the first equation is an o.d.e. in  $\theta$  depending on the parameter  $\sigma$ . For  $\sigma$  fixed, the solutions depend on two parameters. They are periodic in  $\theta$ , the period depending on the energy. Imposing the period equal to  $2\pi$  determines one of the parameter. Summing up, we see that

*the  $2\pi$  periodic in  $\theta$  solutions of the first equations are*

$$(5.127) \quad U_0(t, x, \theta) = V_0(t, x, \theta + \Theta(t, x)) = K(\sigma(t, x), \theta + \Theta(t, x))$$

with  $\Theta$  an arbitrary phase shift and  $K(\sigma, \theta) = \sigma^{\frac{2}{p-1}} G(\theta)$  where  $G(\theta)$  is the unique  $2\pi$ -periodic solution of  $\partial_\theta^2 G + G^p = 0$  satisfying  $G'(0) = 0$ ,  $G(0) > 0$ .

2) The second equation is a linear o.d.e.

$$(5.128) \quad \mathcal{L}U_1 = -T \partial_\theta U_0$$

where  $\mathcal{L}$  is the linearized operator at  $U_0$  from the first equation. The translation invariance implies that the kernel of  $\mathcal{L}$  is not trivial. Indeed, for fixed  $(t, x)$ ,  $\mathcal{L}$  is self adjoint and  $\ker \mathcal{L}$  is a one dimensional space generated by  $\partial_\theta V_0(t, x, \cdot + \Theta)$ . Therefore, the o.d.e. in  $U_1$  has a periodic solution if and only if the right hand side is orthogonal to the kernel. There holds

$$-T\partial_\theta U_0(t, x, \theta) = -\{T\partial_\theta V_0 + X(\Theta)\partial_\theta^2 V_0\}(t, x, \theta + \Theta)$$

where  $X = 2\varphi'_t \partial_t - 2\varphi'_x \partial_x$ . Therefore the integrability condition reads

$$(5.129) \quad \int_0^{2\pi} ((T\partial_\theta V_0)\partial_\theta V_0 + Z(\Theta)\partial_\theta^2 V_0\partial_\theta V_0) d\theta = \frac{1}{2}T \int_0^{2\pi} (\partial_\theta V_0)^2 d\theta = 0,$$

that is

$$(5.130) \quad \partial_t \varphi \partial_t J - \partial_x \varphi \partial_x J + \square \varphi J = 0,$$

where

$$J(t, x) := \int_0^{2\pi} (\partial_\theta K)^2 d\theta = \sigma^{\frac{4}{p-1}} \int_0^{2\pi} (\partial_\theta G(\theta))^2 d\theta = c((\partial_t \varphi)^2 - (\partial_x \varphi)^2)^{\frac{4}{p-1}}.$$

Therefore, (5.130) is a second order nonlinear equation in  $\varphi$ :

$$(5.131) \quad \alpha \partial_t^2 \varphi - 2\beta \partial_t \partial_x \varphi - \gamma \partial_x^2 \varphi = 0$$

with

$$\alpha = \frac{p-1}{2}\sigma^2 + (\partial_t \varphi)^2, \quad \beta = \partial_t \varphi \partial_x \varphi, \quad \gamma = \frac{p-1}{2}\sigma^2 - (\partial_x \varphi)^2.$$

Note that this equation is strictly hyperbolic for  $\sigma > 0$ . It is a substitute for the eikonal equation, but its nature is completely different :

*$\varphi$  is determined by the nonlinear wave equation (5.131)*

The compatibility condition (5.129) being satisfied, the solutions of (5.128) are

$$(5.132) \quad U_1(t, x, \theta) = \lambda_1 \partial_\theta U_0 - \mathcal{L}^{-1}T(\partial_\theta U_0) = \lambda_1 \partial_\theta U_0 + W_1.$$

where  $\lambda_1$  is an arbitrary function of  $(t, x)$  and  $\mathcal{L}^{-1}$  denotes the partial inverse onto  $\ker \mathcal{L}^\perp$  of the operator  $\mathcal{L} = \sigma^2 \partial_\theta^2 + f'(U_0)$ . Note that  $U_1(t, x, \theta)$  is also of the form  $V_1(t, x, \theta + \Theta(t, x))$ .

3) The third equation is of the form

$$(5.133) \quad \mathcal{L}U_2 = F_2 = T\partial_\theta U_1 + \frac{1}{2}f''(U_0)U_1^2 + \square U_0.$$

For this o.d.e in  $U_2$ , the integrability condition reads

$$(5.134) \quad \int_0^{2\pi} F_2(t, x, \theta) \partial_\theta U_0(t, x, \theta) d\theta = 0$$

Using (5.132), it is an equation for  $\lambda_1$  and  $\Theta$ .

**Proposition 5.37.** *[[Leb]] If the phase equation (5.130) is satisfied, then the condition (5.134) is independent of  $\lambda_1$  and reduces to a second order hyperbolic equation for  $\Theta$ .*

*Proof.* The term in  $\lambda_1^2$  is  $\frac{1}{2} \int_0^{2\pi} f''(U_0)(\partial_\theta U_0)^3 d\theta$ . The integral vanishes, since the translation invariance implies that

$$(5.135) \quad (\sigma^2 \partial_\theta^2 + f'(U_0)) \partial_\theta^2 U_0 = -f''(U_0)(\partial_\theta U_0)^2$$

implying the the right hand side is orthogonal to  $\partial_\theta U_0$ .

The linear term in  $\lambda_1$  is

$$\int_0^{2\pi} T(\lambda_1 \partial_\theta^2 U_0) \partial_\theta U_0 d\theta + \int_0^{2\pi} \lambda_1 f''(U_0)(\partial_\theta U_0)^2 W_1 d\theta$$

Using (5.135), the second integral is

$$- \int_0^{2\pi} \mathcal{L}(\partial_\theta^2 U_0) W_1 d\theta = \int_0^{2\pi} \partial_\theta^2 U_0 (T \partial_\theta U_0) d\theta$$

so that the linear term is

$$T \left( \int_0^{2\pi} \lambda_1 \partial_\theta^2 U_0 \partial_\theta U_0 d\theta \right) = T(0) = 0.$$

Therefore the condition (5.134) reduces to

$$(5.136) \quad \int_0^{2\pi} (T(\partial_\theta W_1) + \frac{1}{2}f''(U)(W_1)^2 + \square U_0) \partial_\theta U_0 d\theta = 0.$$

There holds

$$\begin{aligned} W_1(t, x, \theta) &= -\{\mathcal{L}_0^{-1}(T\partial_\theta V_0) + X(\Theta)\mathcal{L}_0^{-1}(\partial_\theta^2 V_0)\}(t, x, \theta + \Theta), \\ \square U_0(t, x, \theta) &= \{\square\Theta \partial_\theta V_0 + \partial_t \Theta A + \partial_x \Theta B + \square V_0\}(t, x, \theta + \Theta) \end{aligned}$$

where  $\mathcal{L}_0^{-1}$  denotes the partial inverse of  $\mathcal{L}_0 = \sigma^2 \partial_\theta^2 + f'(V_0)$ , and  $A$  and  $B$  depend only on  $V_0$ . Therefore, (5.136) is a second order nonlinear equation in  $\Theta$ , whose principal part is

$$\square \Theta \int_0^{2\pi} (\partial_\theta V_0)^2 d\theta + X^2 \Theta \int_0^{2\pi} \mathcal{L}_0^{-1}(\partial_\theta^2 V_0) \partial_\theta^2 V_0 d\theta.$$

□

4) Construction of BKW solutions. The phase  $\varphi$  is constructed from (5.131) and suitable initial conditions. The phase shift  $\Theta$  is given by the nonlinear hyperbolic equation and initial conditions. This completely determines the principal term  $U_0$ ,  $U_1$  up to the choice of  $\lambda_1$  and  $U_2$  is also determined from (5.133) up to a function  $\lambda_2 \partial_\theta U_0$  in the kernel of  $\mathcal{L}$ . The fourth equation in the cascade is  $\mathcal{L}U_3 = F_3$  and the compatibility condition is  $\int_0^{2\pi} F_3 \partial_\theta U_0 d\theta = 0$ . It is independent of  $\lambda_2$  and gives an equation for  $\lambda_1$ . The procedure goes on to any order and provides asymptotic solutions (5.125).

The bad point is that these asymptotic solutions are *unstable*. This has been used by G.Lebeau to prove that the Cauchy problem  $\square u + u^p$  is ill posed in  $H^s(\mathbb{R}^d)$  in the supercritical case, that is when  $1 < s < \frac{d}{2} - \frac{d}{p+1}$ .

**Theorem 5.38** ([Leb]). *The BKW solutions constructed in Theorem 5.36 are unstable*

We just give below an idea of the mechanism of instability. The linearized equation are

$$(5.137) \quad \varepsilon^2 (\partial_t^2 - \partial_x^2) \dot{u} - f'(u_{app}^\varepsilon) \dot{u} = \dot{f}.$$

The potential  $f'(u_{app}^\varepsilon)$  is a perturbation of  $\sigma^2 f'(G(\varphi/\varepsilon + \Theta))$ . Taking  $\varphi$  as a new time variable or restricting the evolution to times  $t \ll \sqrt{\varepsilon}$ , one can convince oneself that the relevant model is the equation:

$$(5.138) \quad \varepsilon^2 (\partial_t^2 - \partial_x^2) \dot{u} + f'(G(t/\varepsilon)) \dot{u} = \dot{f}.$$

Setting  $s = t/\varepsilon$  and performing a Fourier transform in  $x$ , reduces to the analysis of the Hill operator

$$(5.139) \quad \mathcal{M}_\lambda \dot{u} := (\partial_s^2 + f'(G(s)) + \lambda) \dot{u}, \quad \lambda = \varepsilon^2 \xi^2.$$

Let  $E_\lambda(s)$  denote the  $2 \times 2$  matrix which describes the evolution of  $(\dot{u}, \dot{u}')$  for the solutions of  $\mathcal{M}_\lambda \dot{u} = 0$ . Since the potential  $f'(G)$  is  $2\pi$ -periodic, the evolution for times  $s > 2\pi$  is given by

$$(5.140) \quad E_\lambda(s) = E_\lambda(s')(E_\lambda(2\pi))^k, \quad s = s' + 2k\pi.$$

Therefore, the behavior as  $s \rightarrow \infty$  is given by the iterates  $M_\lambda^k$ . and therefore depends only on the spectrum of  $\mathbb{M}_\lambda$ .

**Proposition 5.39** ([Leb]). *There are  $\mu_0 > 0$  and  $\lambda_0 > 0$  such that  $e^{\mu_0} > 1$  is an eigenvalue of  $\mathbb{M}_{\lambda_0}$  and for all  $\lambda \geq 0$  the real part of the eigenvalues of  $\mathbb{M}_\lambda$  are less than or equal to  $e^{\mu_0}$ .*

This implies that there are initial data such that the homogeneous equation (5.138) with  $\dot{f} = 0$  has solutions which grow like  $e^{\mu_0 t/\varepsilon}$ , thus which are larger than any given power of  $\varepsilon$  in times  $t = O(\varepsilon |\ln \varepsilon|)$ . This implies that expansion in power of  $\varepsilon$  is not robust under perturbations and eventually this implies the nonlinear instability of the approximate solutions. For the details, see [Leb].

## 5.4 Caustics

A major phenomenon in multidimensional propagation is *focusing*: this occurs when the rays of geometric optics accumulate and form an envelop. The phases satisfy eikonal equations,

$$(5.141) \quad \partial_t \varphi + \lambda(t, x, \partial_x \varphi) = 0.$$

and are given by Hamilton-Jacobi theory. Generically, that is for nonplanar planar phases, the rays are not parallel (and curved) and thus have an envelop which is called *the caustic set*.

When rays focus, amplitudes grow and, even in the linear case, one must change the asymptotic description. In the nonlinear case, the large amplitudes can be amplified by nonlinearities and therefore strongly nonlinear phenomena can occur.

### 5.4.1 Example : spherical waves

Consider for instance the following wave equation in space dimension  $d$

$$(5.142) \quad \square u^\varepsilon + F(\nabla_{t,x} u^\varepsilon) = 0$$



and weakly nonlinear oscillating solutions  $u^\varepsilon \sim \varepsilon U(t, x, \varphi/\varepsilon)$  associated to one of the two phases  $\varphi_\pm = t \pm |x|$ . The propagation of the oscillation of the principal amplitude  $U_0$ , is given by the following equation for  $V = \partial_\theta U_0$ :

$$(5.143) \quad 2\left(\partial_t \mp \frac{1}{|x|}x \cdot \partial_x \mp \frac{d-1}{|x|}\right)V + F(V\nabla\varphi_\pm) = 0.$$

- The rays of geometric optics are the integral curves of  $\partial_t \mp \partial_r$ , that is the lines  $x = y \mp t \frac{y}{|y|}$ . For  $\varphi_+$ , all the rays issued from the circle  $|y| = a$  cross at time  $t = a$  at  $x = 0$ . *This is the focusing case* (for positive times). The caustic set is  $\mathcal{C} = \{x = 0\}$ . For  $\varphi_-$ , the rays spread and do not intersect, *this is the defocusing case* (for positive times).

- For linear equations ( $F = 0$ ), *the local density of energy is preserved along the rays by the linear propagation*. This is a general phenomenon when the linear equations have a conserved energy. This means that if  $F = 0$ ,  $e(t, x, \theta) := |x|^{d-1} |V(t, x, \theta)|^2$  satisfies

$$(5.144) \quad (\partial_t \mp \partial_r)e = 0.$$

In the focusing case,  $|x| \rightarrow 0$  when one approaches the caustic set along a ray, and therefore the conservation of  $e$  implies that the intensity  $|V| \rightarrow \infty$ .

- For nonlinear equations,  $\tilde{V} = r^{\frac{d-1}{2}}V$  satisfies

$$(5.145) \quad (\partial_t \mp \partial_r)\tilde{V} + r^{\frac{d-1}{2}}F(r^{-\frac{d-1}{2}}V\nabla\varphi_\pm) = 0.$$

From here, it is clear that the nonlinearity can amplify or decrease the growth of the amplitude as  $r \rightarrow 0$  along the rays. We give several examples.

*Example 1 : Blow up mechanisms.*

Consider in space dimension  $d = 3$ , the equation (5.142) with  $F(\nabla u) = -|\partial_t u|^2 \partial_t u$ . For the solution of the focussing equation (5.143) the local energy  $e(t, x, \theta) := |x|^2 |V(t, x, \theta)|^2$  satisfies on the ray  $x = y - t \frac{y}{|y|}$ :

$$(5.146) \quad e(t, x, \theta) = \frac{e(0, y, \theta)}{1 - \frac{te(0, y, \theta)}{|x||y|}}, \quad y := x + t \frac{x}{|x|}.$$

Therefore, if  $V(0, y, \theta) \neq 0$ , even if it is very small,  $e$  and  $V$  blow up on the ray starting at  $y$ , *before reaching the caustic set*  $x = 0$ .

Two mechanisms are conjugated in this example. The first, is due to the nonlinearity  $-|V|^2V$ , as in the ordinary differential equation  $V' - |V|^2V = 0$ . But for this equation, the blow up time depends on the size of the data. It is very large when the data is small. The second mechanism is the amplification

caused by focusing, as in the linear transport equation  $\partial_t V - \partial_{|x|} V - \frac{1}{|x|} V = 0$ . Even if the data is small, it forces  $V$  to be very large before one reaches the caustic. Then the first mechanism is launched and the solution blows up very quickly.

Remark that the blow up occurs not only in  $L^\infty$  but also in  $L^1$ , *before the first time of focusing*. It proves that for solutions  $u^\varepsilon$  of (5.142), the principal term in the geometric optics expansion of  $\nabla u^\varepsilon$  blows up. This does not prove that  $\nabla u^\varepsilon$  itself becomes infinite for a fixed  $\varepsilon$ , but at least that it becomes arbitrarily large. This shows that focusing is an essential part of the mechanism which produces large solutions in a finite time bounded independently of the smallness of the data.

One can modify the example to provide an example where *the exact solutions is not extendable, even in the weak sense, after the first time of focusing*. Consider in dimension  $d = 5$ , the equation (5.142) with  $F(\nabla u) = (\partial_t u^\varepsilon)^2 - |\nabla_x u^\varepsilon|^2$  and initial data supported in the annulus  $\{1 < |x| < 2\}$  :

$$u^\varepsilon(0, x) = 0 \quad , \quad \partial_t u^\varepsilon(0, x) = U_1(x, |x|/\varepsilon) ,$$

Nirenberg's linearization,  $v^\varepsilon := 1 - \exp(-u^\varepsilon)$ , transform the equation into the linear initial value problem

$$\square v^\varepsilon = 0 \quad , \quad v^\varepsilon(0, x) = 0 \quad , \quad \partial_t v^\varepsilon(0, x) = U_1(x, |x|/\varepsilon) .$$

The solution  $v^\varepsilon$  is defined for all time and vanishes in the cone  $\mathcal{C} := \{|x| \leq 1 - t\}$ , but the domain of definition of  $u^\varepsilon$  is  $\Omega_\varepsilon$  defined as the connected open subset of  $\{v^\varepsilon < 1\}$ , which contains  $\mathcal{C}$ . Focusing make  $v^\varepsilon$  arbitrarily large on the focus line  $\{x = 0\}$ , or times  $t$  arbitrarily close to 1 as  $\varepsilon \rightarrow 0$ . One can choose data such that for all  $\delta > 0$ , if  $\varepsilon$  is small enough, then the ball  $B_\delta$  of radius  $\delta$  and centered at  $(t = 1, x = 0)$  (the first focusing point) is not contained in  $\Omega_\varepsilon$  for  $\varepsilon < \varepsilon_0$  and  $(\partial_t u^\varepsilon)^2 - |\nabla_x u^\varepsilon|^2$  is not integrable on  $\Omega_\varepsilon \cap B_\delta$  (up to the boundary  $\partial\Omega_\varepsilon$ ). In particular,  $u^\varepsilon$  *cannot* be extended to  $B_\delta$ , even as a *weak* solution. Thus the first focusing time is the largest common lifetime of the (weak) solutions  $u^\varepsilon$ .

These examples taken from [JMR4, JMR7] illustrate that focusing and blow up can be created in the principal oscillations themselves. This is called *direct focusing* in [JMR4]. But nonlinear interactions make the problem much harder. Focusing and blow up can be created by phases not present in the principal term of the expansion, but which are generated after several interactions. This phenomenon is explored in detail in [JMR4] where it is called *hidden focusing*.

*Example 2 : Absorption of oscillations*

When combined with strongly dissipative mechanisms, focusing can lead to a complete absorption of oscillations, in finite time. The oscillations disappear when they reach the caustic set. The following example of such a behaviour is taken from [JMR6]. Consider in dimension  $d = 3$ , the dissipative wave equation

$$\square u + |\partial_t u|^2 \partial_t u = 0,$$

with oscillating initial data

$$u^\varepsilon(0, x) = \varepsilon U_0(x, |x|/\varepsilon) \quad , \quad \partial_t u^\varepsilon(0, x) = U_1(x, |x|/\varepsilon),$$

There are global weak solutions  $u^\varepsilon \in C^0([0, \infty[; H^1)$  with  $\partial_t u^\varepsilon \in C^0([0, \infty[; L^2) \cap L^4([0, \infty[ \times \mathbb{R}^4)$  (see [LiSt]). When  $U_0$  is even in  $\theta$  and  $U_1 = \partial_\theta U_0$ , the asymptotic expansion of  $u^\varepsilon$  is given by

$$u^\varepsilon(t, x) = \varepsilon U_0(t, x, (|x| + t)/\varepsilon) + \varepsilon^2 \dots$$

and  $V := \partial_\theta U_0$  satisfies the transport equation (5.145), which implies that the density of energy  $e(t, x, \theta) := |x|^2 |V(t, x, \theta)|^2$  satisfies (compare with (5.146)):

$$e(t, x, \theta) = \frac{e(0, y, \theta)}{1 + \frac{t e(0, y, \theta)}{|x| |y|}}, \quad y := x + t \frac{x}{|x|}.$$

If  $V(0, y, \theta) \neq 0$ , then along the ray  $x = y - t \frac{y}{|y|}$ ,

$$|V| \rightarrow +\infty, \quad \text{but} \quad e \rightarrow +0 \quad \text{as} \quad |x| \rightarrow 0.$$

This means that the amplitude  $V$  is infinite at the caustic, but the density of energy transported by the oscillations tends to zero, that is, is entirely dissipated. This suggests that the oscillations are absorbed when they reach the caustic set. This is rigorously proved in [JMR6].

### 5.4.2 Focusing before caustics

Example 1 of the preceding section illustrates a general phenomenon. Consider a semilinear equation and a phase  $\varphi$  solution of the eikonal equation (5.141) associated to simple eigenvalue  $\lambda(t, x, \xi)$  with initial value  $\varphi_0$ . The solution  $\varphi$  is constructed by the Hamilton Jacobi method : the graph of  $\partial_x \varphi$  at time  $t$  is the flow out  $\Lambda$  of  $\partial_x \varphi_0$  by  $H_\lambda$ , the hamiltonian field of  $\lambda$ . The phase  $\varphi$  is defined and smooth as long as the projection  $(x, \xi) \mapsto x$  from

$\Lambda \subset \mathbb{R}^d \times \mathbb{R}^d$  to  $\mathbb{R}^d$  is a diffeomorphism. This is equivalent to requiring that the flow of the  $X_\varphi$ , the projection of  $H_{\lambda|\Lambda}$ , is a diffeomorphism. This ceases to be true at the envelop of the integral curves of  $X_\varphi$ , which is called the caustic set  $\mathcal{C}$ . The transport equation for amplitudes given by geometric optics is of the form

$$(5.147) \quad \partial_t U_0 + \sum \partial_{\xi_j}(t, x, \partial_x \varphi) \partial_{x_j} U_0 + c(t, x) U_0 = F(U_0).$$

The coefficient  $c$  is precisely singular on the caustic set  $\mathcal{C}$  and becomes singular along the rays when one approaches  $\mathcal{C}$ . Therefore the discussion presented on the example (5.143) can be repeated in this more general context.

### 5.4.3 Oscillations past caustics

*The linear case*

In the linear theory, the exact solutions are smooth and bounded. This means that for high frequency waves, focusing creates large but not infinite intensities. This is a classical device to prove that mutli-dimensional hyperbolic equation do not preserve  $L^p$  norms except for  $p = 2$ . One idea to study the behaviour of oscillations near caustics, is to replace phase-amplitude expansions like (5.9) by *Lagrangian integrals* (see [Dui, Hö 1, Hö2]). For constant coefficient equations, they are of the form

$$(5.148) \quad (2\pi\varepsilon)^{-d} \iint e^{i\Phi(t,x,y,\xi)/\varepsilon} a(t, y) dy d\xi,$$

where the phase

$$\Phi(t, x, y, \xi) := -t\lambda(\xi) + (x - y) \cdot \xi + \psi(y)$$

is associated to a smooth piece of the characteristic variety given by the equation  $\tau + \lambda(\xi) = 0$  (see the representation (5.2)). These integrals are associated to Lagrangian manifolds,  $\Lambda \subset T^*\mathbb{R}^{1+d}$ , which are the union of the bicharacteristics

$$(5.149) \quad x = y + t\nabla_\xi \lambda(\xi), \quad \xi = d\psi(y), \quad \tau = -\lambda(\xi).$$

The projection in  $(t, x)$  of these lines are the rays of optics. For small times and as long as it makes sense, Hamilton Jacobi theory tells us that  $\Lambda$  is the graph of  $d\varphi$  where  $\varphi$  is the solution of the eikonal equation

$$\partial_t \varphi + \lambda(\partial_x \varphi) = 0, \quad \varphi|_{t=0} = \psi.$$

The projection from  $\Lambda$  to the base is nonsingular at points where the Hessian  $\partial_{y,\xi}^2 \Phi$  of  $\Phi$  with respect to the variables  $(y, \xi)$  is nonsingular. Therefore the caustic set  $\mathcal{C}$  associated to  $\Lambda$  is

$$(5.150) \quad \mathcal{C} := \{ (t, x) : \exists y : x = y + tg(y) \text{ and } \Delta(t, y) = 0 \},$$

where  $\Delta(t, y) = |\det \partial_{y,\xi}^2 \Phi|^{1/2}$  and  $g(y) = \nabla_{\xi} \lambda(d\psi(y))$ .

The local behaviour of integrals (5.148) is given by stationary phase expansions. Outside the caustic set, one recovers geometric optics expansions (5.9). The principal term is

$$(5.151) \quad \sum_{\{y | y+tg(y)=x\}} \frac{1}{\Delta(t, y)} i^{m(t,y)} a(t, y) e^{i\psi(y)/\varepsilon},$$

where  $2m(t, y)$  is the signature of  $\partial_{y,\xi}^2 \Phi$ . For small times, for each  $(t, x)$  there is one critical point  $y$ ,  $\psi(y) = \varphi(t, x)$  and  $m = 0$ . When one approaches the caustic set along the ray, the amplitude tends to infinity, as expected, since  $\Delta \rightarrow 0$ , but *after the caustic along that ray, one recovers a similar expansion, where the phase has experienced a shift equal to  $m\frac{\pi}{2}$* .

This explains how the geometric optics expansions (5.9) can be recovered from the integral representation (5.148) and how this representation resolves the singularity present in the geometric optics description. It also explains how geometric optics expansions are recovered *after* the caustic.

The detailed behavior of oscillatory integrals near the caustic is a complicated and delicate problem (see [Dui]). In particular, one issue is to get sharp  $L^\infty$  bounds, that is the maximum value of intensities, depending on  $\varepsilon$ . In the simplest case when the singularity of the projection of the Lagrangian  $\Lambda$  is a fold, there is good nonsingular asymptotic description using *Airy functions* (see e.g. [Lud1, Hö2]):

$$(5.152) \quad u^\varepsilon = \left\{ \varepsilon^{-\frac{1}{6}} a_\varepsilon(t, x) \text{Ai}(\sigma/\varepsilon^{\frac{2}{3}}) + \varepsilon^{\frac{1}{6}} b_\varepsilon(t, x) \text{Ai}'(\sigma/\varepsilon^{\frac{2}{3}}) \right\} e^{i\rho/\varepsilon}$$

where  $\sigma$  and  $\rho$  are two phase functions and the amplitudes  $a_\varepsilon$  and  $b_\varepsilon$  have asymptotic expansions  $\sum \varepsilon^n a_n$  and  $\sum \varepsilon^n b_n$  respectively. Moreover, the caustic set is  $\{\sigma = 0\}$ , and using the asymptotic expansions of the Airy function  $\text{Ai}$  and its derivative  $\text{Ai}'$ , one finds that  $u^\varepsilon = O(\varepsilon^\infty)$  in any compact subset of the shadow region  $\{\sigma > 0\}$  and that  $u^\varepsilon$  is the superposition of two wave trains (5.9) in any compact subset of the illuminated region  $\{\sigma < 0\}$  associated to the phases  $\rho \pm \frac{2}{3}(-\sigma)^{\frac{3}{2}}$ .

*The nonlinear case*

- The extension of the previous results to nonlinear equations is not easy and most questions remain open. An important difficulty is that products of oscillatory integral like (5.148) involve integrals with more and more variables, hinting that a similar representation of solutions of nonlinear equations would require an infinite number of variables. Thus this path seems difficult to follow, except if one has some argument to restrict the analysis to the principal term of the integrals.

This has been done for semi linear equations with constant coefficients in two cases: for *dissipative* nonlinearities or when the nonlinear source term is *globally Lipschitzian*. In these two cases, the structure of the equation implies that one can analyze the exact solution in low regularity spaces such as  $L^2$  or  $L^p$  and provide asymptotic expansions using oscillatory integrals with small errors *in these spaces only* (see [JMR7, JMR8]). The natural extension of (5.148) is

$$I^\varepsilon(A) = \frac{1}{(2\pi\varepsilon)^d} \iint e^{-i(t\lambda(\xi)+(x-y)\xi)} A(t, y, \psi(y)/\varepsilon) dy d\xi,$$

associated to profiles

$$A(t, y, \theta) = \sum_{n \neq 0} a_n(t, y) e^{in\theta}.$$

There are *transport equations for the profile A*, which are *o.d.e along the lifted rays  $y + tg(y)$  in the Lagrangian  $\Lambda$* . The phenomenon of absorption is interpreted as the property that  $A = 0$  past the caustic. It occurs if the dissipation is strong enough. For globally Lipschitzian nonlinearities or weak dissipation, the profile  $A$  is continued past the caustic, providing geometric optics approximations at leading order. Note that in the analogue of (5.151), the multiplication by  $i^m$  of the amplitude  $a$  has to be replaced by  $\mathcal{H}^m \mathcal{A}$ , where  $\mathcal{H}$  is the Hilbert transform on Fourier series

$$\mathcal{H}\left(\sum_{n \neq 0} a_n e^{i n \theta}\right) := \sum_{n \neq 0} i^{\text{sign } n} a_n e^{i n \theta}.$$

The very weak point of this analysis is that it leaves completely out the case of *conservative* nondissipative equations.

- In the same spirit, the use of representations such as (5.152) using Airy functions is not obvious, since a product of Airy function is not an Airy function. Such an approach is suggested by J.Hunter and J.Keller in [HuKe2]

as a technical device to match the geometric optics expansions before and after the caustics. Another output of their analysis is a formal classification of the qualitative properties of weakly nonlinear geometric optics, separating linear and nonlinear propagation, and linear and nonlinear effects of the caustic. R.Carles has rigorously explained this classification, mainly for spherical waves, for the wave equations

$$(5.153) \quad (\partial_t^2 - \Delta_x)u + a|\partial_t u|^{p-1}\partial_t u = 0, \quad p > 1; a \in \mathbb{C},$$

( $a > 0, a < 0, a \in i\mathbb{R}$  corresponding to the dissipative, accretive and conservative case, respectively) or the semiclassical nonlinear Schrödinger equation (NLS),

$$(5.154) \quad i\varepsilon\partial_t u + \frac{1}{2}\varepsilon^2\Delta_x u = \varepsilon^\alpha|u|^{2\sigma}u, \quad u|_{t=0} = f(x)e^{-|x|^2/2\varepsilon}.$$

with  $\alpha \geq 1, \sigma > 0$ , .

In [Car1], for  $1 < p < 2$ , the reader can find an  $L^\infty$  description near  $x = 0$  of radial waves in  $\mathbb{R}^3$  for (5.153). As expected the profiles and the solutions are unbounded (uniformly in  $\varepsilon$ ) and *new amplitudes* (of size  $\varepsilon^{1-p}$ ) must be added to the one predicted in [HuKe2] as correctors near the caustic in order to have a uniform approximation in  $L^\infty$ . In particular, this gives the correct evaluation of the exact solutions at the caustic set  $\{x = 0\}$ .

Concerning NLS, R.Carles has investigated all the behaviors with different powers  $\alpha \geq 1$  and  $\sigma > 0$ .

- When  $\alpha > d\sigma$  and  $\alpha > 1$ , the propagation of the main term ignores the nonlinearity, outside the caustic and at the caustic.

- When  $\alpha = 1 > d\sigma$ , the propagation before and after the caustic follows the rule of weakly nonlinear optics, and the matching at the focal point is like in the linear case.

- When  $\alpha = d\sigma > 1$ , the nonlinear effects take place only near the focal point. The remarkable new point is that *the transition between the amplitudes before and after the focal point is given a scattering operator*.

We refer the reader to [Car2, Car3, Car4] for details.

## 6 Diffractive optics

The regime of geometric optics is not adapted to the propagation of laser beams on distances that are much larger than the width of the beam. The standard descriptions that can be found in physics text book involve Schrödinger equations, used to model the diffraction of light in the direction transversal to beam along long distances. The goal of this section is to clarify the nature and range of validity of the approximations leading to Schrödinger like equations, using simple models and examples.

### 6.1 The origin of Schrödinger equation

#### 6.1.1 A linear example

Consider again a symmetric hyperbolic linear system (5.1) with constant coefficients and eigenvalues of constant multiplicities. Consider oscillatory initial data (5.3). The solution is computed by Fourier synthesis as explained in (5.4), revealing integrals

$$(6.1) \quad \begin{aligned} u_p^\varepsilon(t, x) &= \frac{1}{(2\pi)^{d/2}} \int e^{i\{y(k+\varepsilon\xi) - t\lambda_p(k+\varepsilon\xi)\}/\varepsilon} \Pi_p(k + \varepsilon\xi) \hat{h}(\xi) d\xi \\ &= e^{i(kx - t\omega_p)/\varepsilon} a_p^\varepsilon(t, x) \end{aligned}$$

The error estimates (5.6) show that the approximation of geometric optics is valid as long as  $t = o(1/\varepsilon)$ . To study times and distances of order  $1/\varepsilon$ , insert the second order Taylor expansion

$$\lambda_p(k + \varepsilon\xi) = \omega_p + \varepsilon\xi \mathbf{v}_p + \varepsilon^2 S_p(\xi) + O(\varepsilon^3 |\xi|^3),$$

with

$$(6.2) \quad S_p(\xi) := \frac{1}{2} \sum_{j,k} \partial_{\xi_j, \xi_k}^2 \lambda_p(k) \xi_j \xi_k,$$

into the integral (6.1) to find

$$a_p^\varepsilon(t, x) = \frac{1}{(2\pi)^{d/2}} \int e^{i(x\xi - t\xi \mathbf{v}_p)} e^{-i\varepsilon t S_p(\xi)} \Pi_p(k) \hat{h}(\xi) d\xi + O(\varepsilon^2 t) + O(\varepsilon)$$

Introduce the slow variable  $T := \varepsilon t$  and

$$(6.3) \quad a_{p,0}(T, t, y) := \frac{1}{(2\pi)^{d/2}} \int e^{i(x\xi - t\xi \mathbf{v}_p)} e^{-iT S_p(\xi)} \Pi_p(k) \hat{h}(\xi) d\xi$$



Then

$$(6.4) \quad \left\| a_p^\varepsilon(t, \cdot) - a_{p,0}(\varepsilon t, t, \cdot) \right\|_{H^s(\mathbb{R}^d)} \leq C(\varepsilon^2 t + \varepsilon) \|h\|_{H^{s+3}(\mathbb{R}^d)}.$$

The profile  $a_{p,0}$  satisfies the polarization conditions  $a_{p,0} = \Pi_p(k)a_{p,0}$  and a pair of partial differential equations:

$$(6.5) \quad (\partial_t + \mathbf{v}_p \cdot \nabla_x) a_{p,0} = 0, \quad \text{and} \quad (i\partial_T + S_p(\partial_x)) a_{p,0} = 0.$$

The first is the transport equation of geometric optics and the second is the Schrödinger equation, with *evolution governed by the slow time  $T$* , which we were looking for. These together with the initial condition

$$a_{p,0}(0, 0, y) = f(y)$$

suffice to uniquely determine  $a_{p,0}$ .

Note that for  $t = o(1/\varepsilon)$  one has  $T \rightarrow 0$ , and setting  $T = 0$  in (6.3) one recovers the approximation of geometric optics. A typical solution of the Schrödinger equation has spatial width which grows linearly in  $T$  (think of Gaussian solutions). Thus the typical width of our solution  $u^\varepsilon$  grows linearly in  $\varepsilon t$  which is consistent with the geometric observation that the wave vectors comprising  $u|_{t=0} = h e^{ikx/\varepsilon}$  make an angle  $O(\varepsilon)$  with  $k$ . The approximation (6.4) clearly presents three scales; the wavelength  $\varepsilon$ , the lengths of order 1 on which  $f$  varies, and, the lengths of order  $1/\varepsilon$  traveled by the wave on the time scales of the variations of  $B$  with respect to the slow time  $T$ .

In contrast to the case of the geometric optics expansion, the analysis above does not extend to nonplanar phases  $\varphi(t, x)/\varepsilon$ . Note that the rays associated to nonlinear phases focus or spread in times  $O(1)$ . The case of focusing of rays and its consequences for nonlinear waves was briefly discussed in Section 5. On the other hand, with suitable convexity hypotheses on the wave fronts, nonlinear transport equations along spreading rays yield nonlinear geometric optics approximations valid globally in time (see [God]). In the same way, one does not find such Schrödinger approximations for linear phases when the geometric approximations are not governed by transport equations. The classic example is conical refraction.

However, the analysis does extend to phases  $(kx - \omega t + \psi(\varepsilon t, \varepsilon x))/\varepsilon$  with *slowly varying* differential, observing that for times  $t \lesssim \varepsilon^{-1}$  the rays remain almost parallel. We refer to [Hu1, Dum2] for these extensions.

**Conclusion 6.1.** *The Schrödinger approximations provide diffractive corrections for times  $t \sim 1/\varepsilon$  to solutions of wavelength  $\varepsilon$  which are adequately described by geometric optics with parallel rays for times  $t \sim 1$ .*

*These key features, parallel rays and three scales are commonly satisfied by laser beams. The beam is comprised of virtually parallel rays. A typical example with three scales would have wavelength  $\sim 10^{-6}m$ , the width of the beam  $\sim 10^{-3}m$ , and the propagation distance  $\sim 1m$ .*

### 6.1.2 Formulating the ansatz.

For simplicity, we study solutions of semilinear symmetric hyperbolic systems with constant coefficients and nonlinearity which is of order  $J$  near  $u = 0$ . The quasilinear case can be treated similarly. Consider

$$(6.6) \quad L(\varepsilon\partial) = \varepsilon A_0 \partial_t + \sum_{j=1}^d \varepsilon A_j \partial_{x_j} + E$$

with self-adjoint matrices  $A_j$ ,  $A_0$  being definite positive, and with  $E$  skew-adjoint. The nonlinear differential equation to solve is

$$(6.7) \quad L^\varepsilon(\partial_x) u^\varepsilon + F(u^\varepsilon) = 0,$$

where  $u^\varepsilon$  is a family of  $\mathbb{C}^N$  valued functions. The nonlinear interaction term  $F$  is assumed to be a smooth nonlinear function of  $u$  and of order  $J \geq 2$  at the origin in the sense that the Taylor expansion at the origin satisfies

$$(6.8) \quad F(u) = \Phi(u) + O(|u|^{J+1}),$$

where  $\Phi$  is a homogeneous polynomial of degree  $J$  in  $u, \bar{u}$ .

*Time of nonlinear interaction and amplitude of the solution.*

The amplitude of nonlinear waves is crucially important. Our solutions have amplitude  $u^\varepsilon = O(\varepsilon^p)$  where the exponent  $p > 0$  is chosen that the nonlinear term  $F(u) = O(\varepsilon^{pJ})$  affects the principal term in the asymptotic expansion for times of order  $1/\varepsilon$ . The time of nonlinear interaction is comparable to the times for the onset of diffractive effects.

The time of nonlinear interaction is estimated as follows. Denote by  $S(t)$  the propagator for the linear operator  $L^\varepsilon$ . Then in  $L^2(\mathbb{R}^d)$ ,  $\|S(t)\| \leq C$ . The Duhamel representation

$$u(t) = S(t) u(0) - \int_0^t S(t - \sigma) F(u(\sigma)) d\sigma$$

suggests that the contribution of the nonlinear term at time  $t$  is of order  $t\varepsilon^{pJ}$ . For the onset of diffraction,  $t \sim 1/\varepsilon$  so the accumulated effect is expected to

be  $O(\varepsilon^{pJ-2})$  (note that the coefficient of  $\partial_t$  is  $\varepsilon$ ). For this to be comparable to the size of the solution we chose  $p$  so that  $pJ - 2 = p$ :

$$(6.9) \quad p = \frac{2}{J-1}.$$

*The basic ansatz.*

It has three scales

$$(6.10) \quad u^\varepsilon(x) = \varepsilon^p a(\varepsilon, \varepsilon t, \varepsilon x, t, x, \varphi(t, x)/\varepsilon),$$

where  $\varphi(t, x) = kx - t\omega$  and  $a(\varepsilon, T, X, t, x, \theta)$  is periodic in  $\theta$  and has an expansion:

$$(6.11) \quad a = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \dots$$

Since this expansion is used for times  $t \sim 1/\varepsilon$ , in order for the correctors  $\varepsilon a_1$  and  $\varepsilon^2 a_2$  to be smaller than the principal term for such times, one must control the growth of the profiles in  $t$ . The most favorable case would be that they are uniformly bounded for  $\varepsilon t \leq T_0$ , but this is not always the case as it will appear in the analysis below. Therefore we *impose the weaker condition of sublinear growth*:

$$(6.12) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \sup_{T, X, y, \theta} |a_1(T, X, t, x, \theta)| = 0.$$

This condition plays a central role in the analysis to follow.

### 6.1.3 First equations for the profiles.

Plugging (6.10) into the equation, ordering in powers of  $\varepsilon$  yields the equations

$$(6.13) \quad \mathcal{L}(\beta \partial_\theta) a_0 = 0,$$

$$(6.14) \quad \mathcal{L}(\beta \partial_\theta) a_1 + L_1(\partial_{t,x}) a_0 = 0,$$

$$(6.15) \quad \mathcal{L}(\beta \partial_\theta) a_2 + L_1(\partial_{t,x}) a_1 + L_1(\partial_{T,X}) a_0 + \Phi(a_0) = 0.$$

with  $\beta = d\varphi = (-\omega, k)$  and  $\mathcal{L}(\beta \partial_\theta) = L_1(\beta) \partial_\theta + E$ .

*Analysis of equations (6.13) and (6.14)*

These first two equations are those met in the geometric optics regime. They are analyzed using Fourier series expansions in  $\theta$ . We make the following assumptions, satisfied in many applications:

**Assumption 6.2.** *i)  $\beta$  satisfies the dispersion relation*

$$(6.16) \quad \det(L_1(\beta) - iE) = 0.$$

*Denote by  $Z$  the set of harmonics  $n \in \mathbb{Z}$  such that  $\det(L_1(n\beta) - iE) = 0$ .*

*ii) If  $E \neq 0$ ,  $\det(L_1(\beta)) \neq 0$ , so that  $Z$  is finite.*

*iii) For  $n \in Z$ ,  $n \neq 0$ , the point  $n\beta$  is a regular point in the characteristic in the sense of Definition 5.3.*

As in Section 5 we denote by  $\mathcal{P}$  and  $\mathcal{Q}$  projectors on the kernel and the image of  $\mathcal{L}(\beta\partial_\theta)$  respectively and by  $\mathcal{R}$  the partial inverse such that

$$(6.17) \quad \mathcal{R}\mathcal{L}(\beta\partial_\theta) = \text{Id} - \mathcal{P}, \quad \mathcal{P}\mathcal{R} = 0, \quad \mathcal{R}(\text{Id} - \mathcal{Q}) = 0.$$

They are defined on Fourier series  $\sum \hat{a}_n e^{in\theta}$  by projectors  $P_n$  and  $Q_n$  and partial inverses  $R_n$  acting on the Fourier coefficients.

The first equation asserts that the principal profile is polarized along the kernel of  $\mathcal{L}(\beta\partial_\theta)$ :

$$(6.18) \quad a_0 = \mathcal{P}a_0 \quad \Leftrightarrow \quad \forall n, \hat{a}_{0,n} = P_n \hat{a}_{0,n}.$$

The second equation is equivalent to

$$(6.19) \quad (\text{Id} - \mathcal{Q})L_1(\partial_{t,x})\mathcal{P}a_0 = 0$$

$$(6.20) \quad (\text{Id} - \mathcal{P})a_1 = -\mathcal{R}L_1(\partial_{t,x})\mathcal{P}a_0.$$

*Analysis of equation (6.15)*

This is the new part of the analysis. Applying the projectors  $\mathcal{Q}$  and  $\text{Id} - \mathcal{Q}$  and using (6.20) the equation is equivalent to

$$(6.21) \quad -\mathcal{T}(\partial_{t,x})a_1 = \mathcal{S}(\partial_{T,X}, \partial_{t,x})a_0 + (\text{Id} - \mathcal{Q})\Phi(a_0)$$

$$(6.22) \quad (\text{Id} - \mathcal{P})a_2 = -\mathcal{R}(L_1(\partial_{t,x})a_1 + L_1(\partial_{T,X})a_0 + \Phi(a_0)).$$

with

$$(6.23) \quad \mathcal{T}(\partial_{t,x}) = (\text{Id} - \mathcal{Q})L_1(\partial_{t,x})\mathcal{P}$$

$$(6.24) \quad \mathcal{S}(\partial_{T,X}, \partial_{t,x}) = (\text{Id} - \mathcal{Q})L_1(\partial_{T,X})\mathcal{P} - (\text{Id} - \mathcal{Q})L_1(\partial_{t,x})\mathcal{R}L_1(\partial_{t,x})\mathcal{P}.$$

**Conclusion 6.3.** *One has to determine the range of  $\mathcal{T}(\partial_{t,x})$ , acting in a space of functions satisfying the sublinear growth condition (6.12), so that  $a_0$  will be determined by the polarization condition (6.19), the propagation equation (6.21) in the intermediate variables, and the additional equation:*

$$(6.25) \quad \mathcal{S}(\partial_{T,X}, \partial_{t,x})a_0 - (\text{Id} - \mathcal{Q})\Phi(a_0) \in \text{Range}\mathcal{T}(\partial_{t,x}).$$

In Section 6.2 we investigate this question in different situation with increasing order of complexity. Before that, we make several general remarks about the operator  $\mathcal{S}$ .

### 6.1.4 The operator $\mathcal{S}$ as a Shödinger operator

The linear operators  $\mathcal{T}(\partial_{t,x})$  and  $\mathcal{S}(\partial_{T,X}, \partial_{t,x})$  act term by term on Fourier series in  $\theta$ . On the  $n$ -th coefficient, we obtain the operators

$$(6.26) \quad \widehat{\mathcal{T}}_n(\partial_{t,x}) = (\text{Id} - Q_n)L_1(\partial_{t,x})P_n,$$

$$(6.27) \quad \widehat{\mathcal{S}}_n(\partial_{T,X}, \partial_{t,x}) = (\text{Id} - Q_n)L_1(\partial_{T,X})P_n \\ - (\text{Id} - Q_n)L_1(\partial_{t,x})R_nL_1(\partial_{t,x})P_n.$$

Following Assumption 6.2, for  $n \neq 0$ ,  $n\beta$  is a regular point in the characteristic variety  $\mathcal{C}$ , meaning that near  $n\beta$ ,  $\mathcal{C}$  is given by the equation

$$(6.28) \quad \tau + \lambda_n(\xi) = 0$$

where  $\lambda_n$  is an eigenvalue of constant multiplicity. In particular  $-n\omega + \lambda_n(nk) = 0$ .

**Proposition 6.4.** *i) Let  $X_n(\partial_{t,x}) = \partial_t + \mathbf{v}_n \cdot \nabla_x$  with  $\mathbf{v}_n = \nabla_\xi \lambda_n(nk)$ . Then*

$$(6.29) \quad \widehat{\mathcal{T}}_n(\partial_{t,x}) = (\partial_t + \mathbf{v}_n \cdot \nabla_x)J_n, \quad J_n := (\text{Id} - Q_n)A_0P_n.$$

*ii) Let*

$$(6.30) \quad S_n(\partial_x) := \frac{1}{2} \sum_{j,k} \partial_{\xi_j \xi_k}^2 \lambda_n(nk) \partial_{x_j} \partial_{x_k}.$$

*Then*

$$(6.31) \quad \widehat{\mathcal{S}}_n(\partial_{T,X}, \partial_{t,x}) = \left( \partial_T + \mathbf{v}_n \cdot \nabla_X - iS_n(\partial_x) \right) J_n + \widehat{\mathcal{R}}_n(\partial_{t,x})X_n(\partial_{t,x})$$

*where  $\widehat{\mathcal{R}}_n$  is a first order operator which is specified below.*

*Proof.* Consider  $n = 1$  and drop the indices  $n$ . The first part is proved in Lemma 5.9. Recall that differentiating once the identity

$$\left( -i\lambda(\xi)A_0 + \sum_j i\xi_j A_j + E \right) P(\xi) = 0$$

yields

$$\left( -i\lambda(\xi)A_0 + \sum_j i\xi_j A_j + E \right) \partial_{\xi_p} P(\xi) = -i(-\partial_{\xi_p} \lambda(\xi)A_0 + A_p)P(\xi).$$

Multiplying on the left by  $\text{Id} - Q(k)$  implies

$$(6.32) \quad (\text{Id} - Q(k))(-\partial_{\xi_p} \lambda(k)A_0 + A_p)P(k) = 0,$$

$$(6.33) \quad (\text{Id} - P(k))\partial_{\xi_p} P(k) = -iR(k)(-\partial_{\xi_p} \lambda(k)A_0 + A_p)P(k).$$

The first identity implies (6.29).

Differentiating another time the identity and multiplying on the left by  $\text{Id} - Q(k)$  implies that

$$\begin{aligned} (\text{Id} - Q)(-\partial_{\xi_p} \lambda A_0 + A_p)\partial_{\xi_q} P + (\text{Id} - Q)(-\partial_{\xi_q} \lambda A_0 + A_q)\partial_{\xi_p} P \\ = \partial_{\xi_p, \xi_q}^2 \lambda (\text{Id} - Q)A_0 P \end{aligned}$$

where the functions are evaluated at  $\xi = k$ . Using (6.32), one can replace  $\partial_{\xi_p} P$  by  $(\text{Id} - P)\partial_{\xi_p} p$  in the identity above. With (6.33), this implies that

$$\begin{aligned} (\text{Id} - Q)(-\partial_{\xi_p} \lambda A_0 + A_p)R(-\partial_{\xi_q} \lambda A_0 + A_q)P \\ + (\text{Id} - Q)(-\partial_{\xi_q} \lambda A_0 + A_q)R(-\partial_{\xi_p} \lambda A_0 + A_p)P \\ = i\partial_{\xi_p, \xi_q}^2 \lambda (\text{Id} - Q)A_0 P. \end{aligned}$$

Together with the identity

$$L_1(\partial_t, \partial_x) = A_0(\partial_t + \mathbf{v} \cdot \nabla_x) + \sum_{p=1}^d (A_p - \partial_{\xi_p} \lambda A_0)\partial_{x_p} = A_0 X(\partial_{t,x}) + L_1'(\partial_x)$$

this implies that

$$\begin{aligned} (\text{Id} - Q)L_1(\partial_{t,x})RL_1(\partial_{t,x})P &= (\text{Id} - Q)X(\partial_{t,x})RL_1(\partial_{t,x})P \\ &+ (\text{Id} - Q)L_1'(\partial_x)RX(\partial_{t,x})P \\ &+ \frac{i}{2}\lambda (\text{Id} - Q)A_0 P \sum \partial_{\xi_p, \xi_q}^2 \partial_{x_p} \partial_{x_q} \end{aligned}$$

implying (6.31).  $\square$

**Corollary 6.5.** *If  $a(T, X, t, x)$  satisfies the polarization condition  $a = P_n a$  and the transport equation  $X(\partial_{t,x} a) = 0$ , then*

$$\widehat{\mathcal{S}}_n(\partial_{T,X}, \partial_{t,x})a = J_n(\partial_T + \mathbf{v} \cdot \nabla_X + iS_n(\partial_x))a.$$

The conclusion is that for all frequency  $n$ , the operator  $\widehat{\mathcal{S}}_n$  acting on the polarized vectors  $a$  is the Shrödinger operator  $\partial_T + \mathbf{v} \cdot \nabla_X + iS_n(\partial_x)$ , that is, for functions independent of  $X$ , the operator (6.5).

## 6.2 Construction of solutions

We consider first the simplest case:

### 6.2.1 The nondispersive case with odd nonlinearities and odd profiles.

We assume here that in equation (6.7),  $E = 0$  and  $F(u)$  is an odd function of  $u$  with nontrivial cubic term, that is with  $J = 3$  so that we choose  $p = 1$  according to the rule (6.9). We look for solutions

$$(6.34) \quad u^\varepsilon(t, x) \sim \varepsilon \sum_{n \geq 0} \varepsilon^n a_n(\varepsilon t, t, x, \varphi(t, x)/\varepsilon),$$

where  $\varphi(t, x) = kx - t\omega$  and the profiles  $a_n(T, t, x, \theta)$  are *periodic and odd in  $\theta$* .

We assume that  $\varphi$  is a characteristic phase and more precisely we assume that  $d\varphi$  is a *regular point* of the characteristic variety  $\det L(\tau, \xi) = 0$ . We introduce projectors  $P$  and  $Q$  on the kernel and on the image of  $L(d\varphi)$  respectively and we denote by  $R$  the partial inverse of  $L(d\varphi)$  with properties as above. We denote by  $X(\partial_{t,x}) = \partial_t + \mathbf{v} \cdot \nabla_x$  the transport operator and similarly by  $S(\partial_{t,x})$  the second order operator (6.2) or (6.30) associated to  $d\varphi$ . By homogeneity, all the harmonics  $n\varphi$ ,  $n \neq 0$  are characteristic and one can choose the projectors  $P_n$  to be equal to  $P$ ,  $Q_n = Q$ . Moreover, all the transport fields have the same speed and  $X_n(\partial_{t,x}) = X(\partial_{t,x})$ . Similarly,  $S_n(\partial_x) = S(\partial_x)$ .

Because we *postulate* that  $a_0$  is odd in  $\theta$ , its zero-th harmonic (its mean value) vanishes. Therefore, the polarization condition (6.20) reduces to

$$(6.35) \quad a_0(T, t, x, \theta) = Pa_0(T, t, x, \theta),$$

and the first condition equation (6.19) reads

$$(6.36) \quad (\partial_t + \mathbf{v}\nabla_x)a_0(T, t, x, \theta) = 0.$$

Because we also postulate that  $a_1$  is odd, (6.20) is equivalent to

$$(6.37) \quad (\text{Id} - P)a_1(T, t, x, \theta) = -\partial_\theta^{-1}RL_1(\partial_{t,x})Pa_0$$

where  $\partial_\theta^{-1}$  is the inverse of  $\partial_\theta$  acting on functions with vanishing mean value. Next, the equation (6.21) is

$$(6.38) \quad \begin{aligned} -(\partial_t + \mathbf{v}\nabla_x)(\text{Id} - Q)A_0Pa_1 &= (\text{Id} - Q)\Phi(a_0) + \partial_T(\text{Id} - Q)A_0Pa_0 \\ &\quad -(\text{Id} - Q)L_1(\partial_t, x)\partial_\theta^{-1}RL_1(\partial_{t,x})P, \end{aligned}$$

where we have used the following remark :

$$(6.39) \quad \text{if } \Phi \text{ is odd in } u \text{ and if } a_0 \text{ is odd in } \theta, \text{ then } \Phi(a_0) \text{ is odd in } \theta,$$

implying that the mean value of  $\Phi(a_0)$  vanishes and that  $\mathcal{Q}\Phi(a_0) = Q\Phi(a_0)$ .

Next we note that the *scalar* operator  $X(\partial_{t,x})$  commutes with all the constant coefficient operators in the right hand side of (6.38) so that (6.36) implies that the right hand side  $f_0$  of (6.38) satisfies

$$(6.40) \quad X(\partial_{t,x})f_0 = 0,$$

and the equation for  $\tilde{a}_1 = (\text{Id} - Q)A_0Pa_1$  reduces to

$$(6.41) \quad X(\partial_{t,x})\tilde{a}_1 = f_0.$$

By (6.40),  $f_0$  is constant along the rays of  $X$ , and thus  $\tilde{a}_1$  is affine on these rays. Therefore,

**Lemma 6.6.** *If  $f_0$  satisfies (6.40), the equation (6.41) has a solution with sublinear growth in  $t$ , if and only if  $f_0 = 0$ . In this case, the solutions  $\tilde{a}_1$  are bounded in time.*

Therefore, we see that the equation (6.38) has a solution  $a_1$  with sublinear growth in  $t$ , if and only if the following *two* equations are satisfied:

$$(6.42) \quad \begin{aligned} &(\text{Id} - Q)\Phi(a_0) + \partial_T(\text{Id} - Q)A_0Pa_0 \\ &- (\text{Id} - Q)L_1(\partial_t, x)\partial_\theta^{-1}RL_1(\partial_{t,x})P = 0, \end{aligned}$$

$$(6.43) \quad (\partial_t + \mathbf{v}\nabla_x)Pa_1 = 0.$$

Moreover, using Proposition 6.4, we see that for  $a_0$  satisfying (6.36), the first equation is equivalent to

$$(6.44) \quad \partial_Ta_0 + \partial_\theta^{-1}S(\partial_x)a_0 = R_0\Phi(a_0)$$

where  $R_0$  is the inverse of  $(\text{Id} - Q)A_0P$  from the image of  $(\text{Id} - Q)$  to the image of  $P$ . On the harmonic 1,  $\partial_\theta^{-1} = -i$ , and we recover a Schrödinger equation as in (6.5), with now a source term.

**Proposition 6.7 (Existence of the principal profile).** *Given a Cauchy data  $h_0(x, \theta)$ , in  $H^s(\mathbb{R}^d \times \mathbb{T})$ , with  $s > \frac{d+1}{2}$ , odd in  $\theta$  and satisfying  $h_0 = Ph_0$ , there is  $T_* > 0$  such that the two times problem (6.36), (6.44) has a unique solution  $a_0 \in C^0([0, T_*] \times \mathbb{R}; H^s(\mathbb{R}^d \times \mathbb{T}))$ , odd in  $\theta$  and satisfying  $a_0 = Pa_0$  and*

$$(6.45) \quad a_0(0, 0, x, \theta) = h_0(x, \theta).$$



*Proof.* By the first equation we look for  $a_0$  as a function of  $x - t\mathbf{v}$

$$(6.46) \quad a_0(T, t, x, \theta) = a_0(T, x - t\mathbf{v}, \theta).$$

Then the equation for  $a_0(T, y, \theta)$  is :

$$(6.47) \quad \partial_T a_0 + \partial_\theta^{-1} S(\partial_y) a_0 + R_0 \Phi(a_0) = 0, \quad a_0(0, y, \theta) = h(y, \theta).$$

This equation is quite similar to classical nonlinear Schrödinger equations,  $\partial_\theta^{-1}$  being bounded and skew adjoint. Thus it has a local solution in  $H^s$ , if  $s$  is large enough. The only point is to check that all the terms in the equations preserve oddness.  $\square$

Next we remark that (6.37), (6.46) and (6.43) imply that  $a_1$  is of the form

$$(6.48) \quad a_1(T, t, x, \theta) = a_1(T, x - t\mathbf{v}, \theta)$$

with  $(I - P)a_1$  determined by  $a_0$  and thus odd in  $\theta$ .

In the expansion in powers of  $\varepsilon$  of the equation, the term coming after (6.15) is

$$\mathcal{L}(\beta \partial_\theta) a_3 + L_1(\partial_{t,x}) a_2 + L_1(D_{T,X}) a_1 + \Phi'(a_0) a_1 + \Psi(a_0) = 0,$$

where  $\Psi$  is the quintic term in the Taylor expansion of  $F$  (recall that  $F$  is odd). Projecting with  $(\text{Id} - Q)$  and  $Q$ , using that  $(\text{Id} - P)a_1$  is known, that  $Pa_1$  satisfies (6.43) and that  $(\text{Id} - P)a_2$  is given by (6.22), we see that the equation is equivalent to

$$\begin{aligned} -(\partial_t + \mathbf{v}\nabla_x)(\text{Id} - Q)A_0Pa_2 &= (\partial_T + \partial_\theta^{-1}S(\partial_x))(\text{Id} - Q)A_0Pa_1 \\ &\quad + (\text{Id} - Q)\Phi'(a_0)Pa_1 + (\text{Id} - Q)b_0 \end{aligned}$$

where  $b_0$  is determined by  $a_0$ , together with an explicit determination of  $(\text{Id} - P)a_3$  in terms of  $a_0, a_1, a_2$ . In particular,  $b_0(T, t, x, \theta) = b_0(T, x - t\mathbf{v}, \theta)$  with  $b_0$  odd in  $\theta$ . The right hand side  $f_1$  satisfies  $(\partial_t + \mathbf{v}\nabla_x)f_1 = 0$ . Therefore, using Lemma 6.6, we see that  $a_2$  has a sublinear growth in  $t$  if and only if

$$(6.49) \quad (\partial_t + \mathbf{v}\nabla_x)Pa_2 = 0,$$

and  $Pa_1(T, t, x, \theta) = Pa_1(T, x - t\mathbf{v}, \theta)$  is therefore determined by the equation

$$(6.50) \quad \partial_T a_1 + \partial_\theta^{-1} S(\partial_y) a_1 + R_0 \Phi'(a_0) a_1 + R_0 b_0 = 0.$$

The equations for the higher profiles are derived in similar fashion. We see that the profiles  $a_n$  in (6.34) have the form  $a_n(T, t, x, \theta) = a_n(T, x - t\mathbf{v}, \theta)$ , so that

$$(6.51) \quad u^\varepsilon(t, x) \sim \varepsilon \sum_{n \geq 0} \varepsilon^n a_n(\varepsilon t, x - t\mathbf{v}, \varphi(t, x)/\varepsilon),$$

The  $(I - P)a_a$  are determined explicitly in terms of  $(a_0, \dots, a_{n-1})$  and the slow dynamics for  $Pa_n$  is determined by an equation of the form

$$(6.52) \quad \partial_T a_n + \partial_\theta^{-1} S(\partial_y) a_n + R_0 \Phi'(a_0) a_n + R_0 b_{n-1} = 0$$

with  $b_{n-1}$  explicitly given in terms of  $(a_0, \dots, a_{n-1})$ .

**Theorem 6.8 (Existence of profiles at all order).** *Suppose that for all  $n \in \mathbb{N}$ , initial data  $h_n \in \mathcal{S}(\mathbb{R}^d \times \mathbb{T})$  are given, odd in  $\theta$  and satisfying  $Ph_n = h_n$ . Then there is  $T_* \in ]0, \infty]$  and a unique sequence  $\{a_n\}_{n \geq 0}$  of functions in  $\mathcal{S}([0, T_*] \times \mathbb{R}^d \times \mathbb{T})$ , odd in  $\theta$  and satisfying the profile equations and the initial conditions  $Pa_n(0, y, \theta) = Ph_n(y, \theta)$*

Given such a sequence of profiles, Borel's theorem constructs

$$a^\varepsilon(T, y, \theta) \sim \sum_{n \geq 0} \varepsilon^n a_n(T, y, \theta)$$

in the sense that for all  $s$  and  $m \in \mathbb{N}$ , there holds for all  $\varepsilon \in ]0, 1]$  :

$$\left\| a^\varepsilon - \sum_{j \leq m} \varepsilon^j a_j \right\|_{H^s([0, T_*] \times \mathbb{R}^d \times \mathbb{T})} \leq C_{s, m} \varepsilon^{m+1}.$$

Then

$$u_{app}^\varepsilon(t, x) := \varepsilon a^\varepsilon(\varepsilon t, x - t\mathbf{v}, \varphi(t, x)/\varepsilon)$$

are approximate solutions of (6.7) in the sense that

$$L(\varepsilon \partial_{t, x}) u_{app}^\varepsilon + F(u_{app}^\varepsilon) = r^\varepsilon(\varepsilon t, x - t\mathbf{v}, \varphi(t, x)/\varepsilon)$$

with  $r^\varepsilon \sim 0$  in the sense above.

Using Theorem 4.21 ([Gu2], see also [Don, DJMR]), one can produce exact solutions which have the asymptotic expansion (6.51). Consider initial data

$$h^\varepsilon(x) \sim u_{app}^\varepsilon(0, x) \sim \varepsilon \sum_{n \geq 0} \varepsilon^n a_n(0, x, \varphi(0, x)/\varepsilon).$$

**Theorem 6.9 (Exact solutions).** *If  $\varepsilon$  is small enough, the Cauchy problem for (6.7) with initial data  $h^\varepsilon$  has a unique solution  $u^\varepsilon \in \mathcal{S}([0, T_*]/\varepsilon] \times \mathbb{R}^d)$  which satisfies for all  $s, m$  and  $\varepsilon$ :*

$$\|u^\varepsilon - u_{app}^\varepsilon\|_{H^s} \leq C_{s, m} \varepsilon^m.$$

### 6.2.2 Dispersive equations; the Schrödinger equation as a generic model for diffractive optics

Consider now the case of a dispersive equation (6.7) with  $E \neq 0$  in (6.6). Assume that the matrices  $A_j$  are real symmetric and that  $E$  is real skew symmetric, as usual in physical applications. Therefore the characteristic polynomial  $p(\tau, \xi) = \det(\tau A_0 + \sum \xi_j A_j - iE)$  is real, and thus equal to  $\det(\tau A_0 + \sum \xi_j A_j + iE)$ . This implies that  $p(-\tau, -\xi) = (-1)^N p(\tau, \xi)$  so that the characteristic variety  $\mathcal{C} = \{p = 0\}$  is symmetric i.e.  $-\mathcal{C} = \mathcal{C}$ .

We fix a planar phase  $\varphi$  with  $\beta = d\varphi = (-\omega, k)$  satisfying Assumption 6.2. We assume that only the harmonics  $n\beta$  with  $n \in \{-1, 0, 1\}$  are characteristic. We further assume that the nonlinearity  $F$  is odd, with non-trivial cubic term  $\Phi$ . Again, we look for solutions  $u^\varepsilon$  of the form (6.34), with profiles  $a_n(T, t, x, \theta)$  which are odd in  $\theta$ , so that the characteristic phase 0 never shows up in the computations. All these assumptions are realistic in applications.

Note that the symmetry assumption of the characteristic variety implies that the eigenvalues  $\lambda_\pm$  which describe the characteristic variety  $\mathcal{C}$  near  $\pm\beta$  (see (6.28)) satisfy the property that

$$(6.53) \quad \lambda_-(\xi) = -\lambda_+(-\xi).$$

In particular, the group velocities  $\mathbf{v}_\pm = \nabla \lambda_\pm(\pm k)$  are equal. We denote their common value by  $\mathbf{v}$ .

The projector  $\mathcal{P}$  on  $\ker \mathcal{L}(\beta \partial_\theta)$  acting on odd functions reduces to

$$\mathcal{P} \left( \sum_{k \in 1+2\mathbb{Z}} \hat{a}_k e^{ik\theta} \right) = P_- \hat{a}_{-1} e^{-i\theta} + P_+ \hat{a}_1 e^{i\theta}$$

where  $P_\pm$  are projectors on  $\ker \mathcal{L}(\pm i\beta)$ . Since  $\mathcal{L}(-i\beta) = \overline{\mathcal{L}(i\beta)}$ , one can choose  $P_- = \overline{P_+}$ , and choose similarly projectors  $Q_- = \overline{Q_+}$  on the images of  $\mathcal{L}(\pm i\beta)$  and partial inverses  $R_- = \overline{R_+}$ .

All what have been done in the nondispersive case can be repeated in this framework: one can construct asymptotic solutions

$$(6.54) \quad u^\varepsilon(t, x) \sim \varepsilon \sum_{n \geq 0} \varepsilon^n \mathbf{a}_n(\varepsilon t, x - t\mathbf{v}, \varphi(t, x)/\varepsilon),$$

with profiles  $\mathbf{a}_n$  which are odd in  $\theta$ . They are determined inductively. The Fourier coefficients  $\widehat{\mathbf{a}}_{n,k}$  for  $k \neq \pm 1$  and  $(\text{Id} - P_\pm) \widehat{\mathbf{a}}_{n,\pm 1}$  are given by explicit formulas and  $P_\pm \widehat{\mathbf{a}}_{n,\pm 1}$  are given by solving a Schrödinger equation.

In particular, the main term contains only the harmonics  $\pm 1$

$$(6.55) \quad \mathbf{a}_0(T, t, x) = \widehat{\mathbf{a}}_{0,-1}(T, t, x)e^{-i\theta} + \widehat{\mathbf{a}}_{0,+1}(T, t, x)e^{i\theta}$$

satisfying the polarization conditions:

$$(6.56) \quad \widehat{\mathbf{a}}_{0,\pm 1}(T, t, x) = P_{\pm} \widehat{\mathbf{a}}_{0,\pm 1}(T, t, x)$$

and the propagation equations

$$(6.57) \quad \partial_T \widehat{\mathbf{a}}_{0,\pm 1} \mp iS(\partial_y) \widehat{\mathbf{a}}_{0,\pm 1} + B_{\pm} \widehat{\Phi(\mathbf{a}_0)}_{\pm 1} = 0$$

where  $B_{\pm}$  denotes the partial inverse of  $(\text{Id}Q_{\pm})A_0P_{\pm}$  on the image of  $P_{\pm}$ . Denoting by  $\widetilde{\Phi}(u, v, w)$  the symmetric trilinear mapping such that  $\Phi(u) = \widetilde{\Phi}(u, u, u)$ , the harmonic one of  $\Phi(\mathbf{a}_0)$  is

$$(6.58) \quad \widehat{\Phi(\mathbf{a}_0)}_1 = \widetilde{\Phi}(\widehat{\mathbf{a}}_{0,-1}, \widehat{\mathbf{a}}_{0,1}, \widehat{\mathbf{a}}_{0,1}),$$

and there is a similar formula for  $\widehat{\Phi(\mathbf{a}_0)}_{-1}$ .

*This is the usual cubic Shrödinger equation for nonlinear optics that can be found in physics books (see e.g. [Blo, BoWo, Boy, NeMo, PaPu]).*

The BKW solutions provide approximate solutions of the equations, which can be converted into exact solutions, exactly as in Theorem 6.9. This mathematical analysis was first made in [Don]. We refer to this paper for explicit example of coming from various models in nonlinear optics, as discussed in Section 2.

### 6.2.3 Rectification

In the two analyses above, two facts were essential :

- Only regular phases were present. In applications, this excludes the harmonic 0. This was made possible by assuming that both the nonlinearity  $F$  and the profiles  $a$  were odd. When  $F$  is not odd, for instance when  $F$  is quadratic, the harmonic 0 is expected to be present. In general, and certainly for nondispersive equations, 0 is not a regular point of the characteristic variety, implying that there will be no simple analogue of Lemma 6.6. The creation of a nontrivial mean value field by nonlinear interaction of waves is called *rectification*.

- All the characteristic harmonics  $k\beta$  are regular have the same group velocities  $\mathbf{v}_k$ . If not, the Fourier coefficients  $\widehat{a}_{n,k}$  are expected to travel at the different speeds  $\mathbf{v}_k$ , so that the right hand side in equations like (6.21) or (6.37) have no definite propagation speed. In this case too, Lemma 6.6 must be revisited.

We now give the general principles which yield to the construction of the principal profile  $a_0$ , refereeing to [JMR9, Lan1] for precise statements and proofs.

For a general equation (6.7) with  $F$  satisfying (6.10), we look for solutions of the form

$$(6.59) \quad u^\varepsilon(t, x) = \varepsilon^p a^\varepsilon(\varepsilon t, t, x, \varphi/\varepsilon)$$

with  $a^\varepsilon = a_0 + \varepsilon a_1 + \varepsilon^2 a_2$  and  $\varphi$  a planar phase with  $\beta = d\varphi$  satisfying Assumption 6.2. The polarization condition (6.18) and the first equation (6.19) for  $a_0$  is that its Fourier coefficients  $\widehat{a}_{0,k}$  satisfy  $P_k \widehat{a}_{0,k} = \widehat{a}_{0,k}$  and

$$(6.60) \quad H_0(\partial_{t,x}) \widehat{a}_{0,0} = 0,$$

$$(6.61) \quad (\partial_t + \mathbf{v}_k \cdot \nabla_x) \widehat{a}_{0,k} = 0, \quad k \neq 0,$$

where  $H_0 = (\text{Id} - Q_0)L_1(\partial_{t,x})P_0$ . Moreover, the components  $(\text{Id} - P_k)\widehat{a}_{1,k}$  of the Fourier coefficients of  $a_1$  are given by (6.20):

$$(6.62) \quad \widehat{a}_{1,k} = -R_k L_1(\partial_{t,x})P_k \widehat{a}_{0,k}.$$

Next, the equations for  $P_k \widehat{a}_{1,k}$  deduced from (6.21) read

$$(6.63) \quad H_0(\partial_{t,x})P_0 \widehat{a}_{1,0} + \widehat{f}_{0,0} = 0$$

$$(6.64) \quad (\partial_t + \mathbf{v}_k \cdot \nabla_x)P_k \widehat{a}_{1,k} + \widehat{f}_{0,k} = 0, \quad k \neq 0,$$

where

$$\widehat{f}_{0,k} = (\text{Id} - Q_k)(A_0 \partial_T - L_1(\partial_{t,x})R_k L_1(\partial_{t,x}))P_k \widehat{a}_{0,k} + (\text{Id} - Q_k)\widehat{\Phi}(a_0)_k.$$

We recall that the main problem is *to find conditions on the  $\widehat{f}_{0,k}$  which imply that the equations for  $P_k \widehat{a}_{1,k}$  have solutions with sublinear growth in time.* This is what we briefly discuss below.

1. *Large time asymptotics for homogeneous hyperbolic equations.*

The operator  $H_0$  is symmetric hyperbolic (on the image of  $P_0$ ) with constant coefficients. For instance, in the nondispersive case,  $E = 0$ ,  $P = \text{Id}$  and  $H_0 = L_1$ . This leads one to consider the large time asymptotics of solutions of

$$(6.65) \quad H(\partial_{t,x})a = f, \quad a|_{t=0} = h$$

when  $H$  is a symmetric hyperbolic constant coefficient first order system. The characteristic variety  $\mathcal{C}_H$  is a homogeneous algebraic manifold. *It may*

contain hyperplanes, say  $\mathcal{C}_\alpha = \{(\tau, \xi), \tau + \mathbf{c}_\alpha \cdot \xi = 0\}$ . This *does* happen for instance for the Euler equation. The *flat* part  $\mathcal{C}_{flat}$  of  $\mathcal{C}_H$  is the union of the  $\mathcal{C}_\alpha$ . We denote by  $\pi_\alpha(\xi)$  the spectral projector on  $\ker H(-\mathbf{c}_\alpha \cdot \xi, \xi)$  and by  $\pi_\alpha(D_x)$  the corresponding Fourier multiplier. The following lemma is easily obtained by using the Fourier transform of the equation.

**Lemma 6.10.** *Suppose that  $h \in \mathcal{S}(\mathbb{R}^d)$  and  $a$  is the solution of the initial value problem  $H(\partial_{t,x})a = 0$ ,  $a|_{t=0} = h$ . Then*

$$\lim_{t \rightarrow \infty} \left\| a(t, \cdot) - \sum (\pi_\alpha(D_x)h)(\cdot - t\mathbf{c}_\alpha) \right\|_{L^\infty(\mathbb{R}^d)} = 0.$$

2. *Large time asymptotics for  $\Phi(a_0)$ .*

For harmonics  $k \neq 1$ , the transport equation (6.61) implies that  $\widehat{a}_{0,k}$  is a function of  $x - t\mathbf{v}_k$ . In the dispersive case, there is a finite number of frequencies  $k$  in play; in the nondispersive case, all the speeds  $\mathbf{v}_k$  are equal. Thus, in all cases, a finite number of speeds  $\mathbf{v}_k$  occur, and the Fourier coefficients of  $a_0$  either are the sum of rigid waves moving at the speed  $\mathbf{c}_\alpha$  or  $\mathbf{v}_k$  and of a spreading wave which tends to 0 at infinity. The Fourier component  $\widehat{\Phi(a_0)}_{,k}$  appears as a sum of products of such waves. Two principles can be observed:

- the product of rigid waves moving at different speeds tends to 0 at infinity;
- the product of a spreading wave and anything bounded, also tends to 0 at infinity.

This implies that for all  $T$ , the coefficients  $\widehat{f}_{0,k}$  have the form

$$(6.66) \quad g(t, x) = \sum g_j(x - t\mathbf{w}_j) + g_0(t, x)$$

where the set of speeds is  $\{\mathbf{w}_j\} = \{\mathbf{v}_k\} \cup \{\mathbf{c}_\alpha\}$ , and  $g_0(t, \cdot)$  tends to 0 in  $H^s$  as  $t \rightarrow \infty$ , uniformly in  $T$ .

3. *Large time asymptotics for inhomogeneous hyperbolic equations.*

Using the equation on the Fourier side, one proves the following estimate:

**Lemma 6.11.** *Consider  $g$  as in (6.66). The solution of  $H(\partial_{t,x})a = g$ ,  $a|_{t=0} = 0$  satisfies*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \left\| a(t, \cdot) - t \sum_{\alpha} \sum_{\{j: \mathbf{w}_j = \mathbf{c}_\alpha\}} \pi_\alpha(D_x)g_j(\cdot - t\mathbf{c}_\alpha) \right\|_{L^\infty(\mathbb{R}^d)} = 0.$$

In particular, there is a solution with sublinear growth if and only if

$$(6.67) \quad \pi_\alpha(\partial_x)g_j = 0 \quad \text{when } \mathbf{w}_j = \mathbf{c}_\alpha.$$

Applied to  $H = \partial_t + \mathbf{v}_k \cdot \nabla_x$ , this lemma implies that the equation  $(\partial_t + \mathbf{v}_k \cdot \nabla_x)a = g$  has a solution with sublinear growth if and only if

$$(6.68) \quad g_j = 0 \quad \text{when } \mathbf{w}_j = \mathbf{v}_k.$$

These principles can be applied to the equations (6.63) (6.64) to derive (necessary and) sufficient conditions for the existence of solutions with sublinear growth. These conditions give the desired equations for  $a_0$ . Their form depend on the relations between the speeds  $\mathbf{v}_k$  and  $\mathbf{c}_\alpha$ . To give an example, we consider a particular, simple case and refer to [JMR9, Lan1] for more general statements.

Consider a nondispersive system. Denote by  $\mathbf{v}$  the common value of the propagation speeds  $\mathbf{v}_k$  for  $k \neq 0$ . Assume that there is one hyperplane in the characteristic variety of  $L_1$ , with speed  $\mathbf{c}$  and projector  $\pi$ . The principal profile  $a_0$  is split into its mean value  $\langle a_0 \rangle = \underline{a}_0$  and its oscillation  $a_0^*$ . The oscillation satisfies the polarization and propagation condition

$$(6.69) \quad a_0^* = Pa_0^* \quad a_0^*(T, t, x, \theta) = a_0^*(T, x - t\mathbf{v}, \theta).$$

Moreover, the mean value satisfies

$$(6.70) \quad L_1(\partial_{t,x})\underline{a}_0 = 0$$

and

$$(6.71) \quad \lim_{t \rightarrow \infty} \sup_{T \in [0, T_*]} \|\underline{a}_0(T, t, \cdot) - \pi(D_x)\underline{a}_0(T, t, \cdot)\|_{L^\infty} = 0$$

$$(6.72) \quad \pi(D_x)\underline{a}_0(T, t, x) = \pi(D_x)\underline{a}_0(T, x - t\mathbf{c}).$$

The propagation equations link  $\pi\underline{a}_0$  and  $a_0^*$  as follows:

$$(6.73) \quad A_0 \partial_T \underline{a}_0 + \pi(\partial_y) \langle \Phi(\underline{a}_0 + \delta a_0^*) \rangle = 0,$$

$$(6.74) \quad (\partial_T + \partial_\theta^{-1} S(\partial_y) a_0^* + R_0(\Phi(\delta \underline{a}_0 + a_0^*)))^* = 0,$$

with  $\delta = 1$  when  $\mathbf{c} = \mathbf{v}$  and  $\delta = 0$  otherwise. There is no ‘‘Schrödinger’’ term in the equation for  $\underline{a}$  since the eigenvalue  $\xi \cdot \mathbf{c}$  is flat.

The property (6.71) implies that  $\underline{a}_0 \sim \pi(D_x)\underline{a}_0$  for large times  $t = T/\varepsilon$ .

This brief discussion explains how one can find the first profile  $a_0$  and correctors  $a_1$  and  $a_2$  in order to cancel out the three first terms (6.13), (6.14), (6.15). We refer to [JMR9, Lan1] for more precise statements and proofs. We note that the sublinear growth of  $a_1$  does not allow one to continue the expansion and get approximations at all order. In [JMR9] examples are given showing that this is sharp.

### 6.3 Other diffractive regimes

We briefly mention several other problems, among many, which have been studied in the diffractive regime.

The case of *slowly varying phases* is studied in [Dum1, Dum2], after the formal analysis given in [Hu1]. This concerns expansions of the form

$$u^\varepsilon(t, x) \sim \varepsilon^p a(t, x, \psi(t, x)/\varepsilon, \varphi(t, x)/\varepsilon^2)$$

or equivalent formulations that can be deduced by rescaling.

In the next section, we study *multi-phase* expansions in the regime of geometric optics. This can also be done in the diffractive regime.

The case of *transparent* nonlinearities can be also studied, in the spirit of what is done in the regime of geometric optics. See [JMR1, CoLa1] for Maxwell-Bloch equations.

The Schrödinger equation is “generic” in diffractive optics. However, in many applications, there are different relevant multi-scale expansions which yield different dispersive equations. Among them KdV in dimension  $d = 1$ , Davey-Stewartson equations (see e.g. [Col1, CoLa2]), K-P equations, (see e.g. [Lan2]), Zakharov system (see e.g. [CoCo]).



## 7 Wave interaction and multi-phase expansions

The aim of this section is to discuss a framework which describes wave interaction. We restrict ourselves here to the regime of geometric optics. The starting point is to consider multi-phases asymptotic expansion

$$(7.1) \quad u^\varepsilon(t, x) \sim \varepsilon^p \mathbf{u}^\varepsilon(t, x, \Phi(t, x)/\varepsilon)$$

where  $\Phi = (\varphi_1, \dots, \varphi_m)$  and possibly  $\mathbf{u}^\varepsilon \sim \sum \varepsilon^n \mathbf{u}_n$ . Points to discuss are : what are the periodicity conditions in  $\theta$  for the profiles  $\mathbf{u}$ ; in which function spaces can we look for the profiles; in which framework can we prove convergence? We start with examples and remarks showing that the phase generation by nonlinear interaction can yield very complicated situations, and indicating that restrictive assumptions on the set of phases are necessary for the persistence and stability of representations like (7.1). After a short digression concerning the general question of representation of oscillations, the main outcome of this section are:

- under an assumption of weak coherence on the phases, there are natural equations for the main profile which can be solved, yielding approximate solutions of the equation, that is modulo errors which tends to zero as  $\varepsilon$  tends to 0 without any rate of convergence.
- the stability of such approximate solution requires stronger assumptions (in general) called strong coherence.
- the strong coherence framework applies to physical situations and thus provides a rigorous justification of several asymptotic representation of solutions.

The wave interaction problem is also of crucial importance in the regime of diffractive optics. Another difficulty appears here : because the intensities are weaker, to be visible the interaction requires longer intervals of time. We will not tackle this problem in these notes and refer for instance to [Dum1, Dum2, JMR9, Lan1, BCL, CoLa1, CoLa2, Les2] among many other.

### 7.1 Examples of phase generation

#### 7.1.1 Generation of harmonics

The first occurrence of nonlinear interaction of oscillations is the creation of harmonics. Consider the elementary example

$$\partial_t u^\varepsilon + c \partial_x u^\varepsilon = (u^\varepsilon)^2 \quad u^\varepsilon|_{t=0} = a(x) \cos(x/\varepsilon).$$

For  $0 \leq t < \|a\|_{L^\infty}$ , the solution is  $u^\varepsilon(t, y) = \mathbf{u}(t, x, (x - ct)/\varepsilon)$  with

$$\mathbf{u}(t, x, \theta) = \frac{a(x) \cos \theta}{1 - ta(x) \cos \theta} = \sum_{n=1}^{\infty} t^{n-1} a(x)^n (\cos(\theta))^n$$

Thus the solution is a periodic function of the phase  $\theta = (x - ct)/\varepsilon$ , where all the harmonics  $n\theta$  are present, while the initial data has only the harmonics 1 and  $-1$ .

### 7.1.2 Resonance and phase matching ; weak vs strong coherence

Given wave packets with phases  $\varphi_1$  and  $\varphi_2$ , nonlinearities introduce terms with phases  $\nu_1\varphi_1 + \nu_2\varphi_2$ . These oscillations can be propagated : this is *resonance* or *phase matching* or not. To illustrate these phenomenon, consider the system

$$(7.2) \quad \begin{cases} \partial_t u_1 + c_1 \partial_x u_1 = 0, \\ \partial_t u_2 + c_2 \partial_x u_2 = 0, \\ \partial_t u_3 = u_1 u_2, \end{cases}$$

with initial data

$$u_j(0, x) = a_j(x) e^{i\psi_j/\varepsilon}, \quad j \in \{1, 2, 3\}.$$

We assume that  $0 \neq c_1 \neq c_2 \neq 0$ . Then for  $j \in \{1, 2\}$ ,

$$u_j = a_j(x - c_j t) e^{i\varphi_j/\varepsilon}, \quad \varphi_j(t, x) = \psi_j(x - c_j t)$$

and

$$(7.3) \quad \begin{aligned} u_3(t, x) &= a_3(x) e^{i\psi_3(x)} \\ &+ \int_0^t a_1(x - c_1 s) a_2(x - c_2 s) e^{i(\varphi_1(s, x) + \varphi_2(s, x))/\varepsilon} ds. \end{aligned}$$

The behavior of  $u_3^\varepsilon$  depends on the existence of critical points for the phase  $\varphi_1 + \varphi_2$ .

- *Resonance or phase matching* occur when  $\varphi_1 + \varphi_2$  is characteristic for the third field  $\partial_t$ , that is when

$$(7.4) \quad \partial_t(\varphi_1 + \varphi_2) = 0.$$

In this case,

$$u_3(t, x) = a_3(x) e^{i\psi_3(x)} + e^{i(\psi_1(x) + \psi_2(x))/\varepsilon} \int_0^t a_1(x - c_1 s) a_2(x - c_2 s) ds.$$

showing that the oscillations of  $u_1$  and  $u_2$  interact constructively and bring a contribution of order 1 to  $u_3$ .

- *Weak coherence.* If

$$(7.5) \quad \partial_t(\varphi_1 + \varphi_2) \neq 0 \quad \text{almost everywhere.}$$

then the integral in (7.3) tends to 0 as  $\varepsilon$  tends to 0. It is a *corrector* with respect to the principal term  $a_3 e^{i\psi_3/\varepsilon}$ . The oscillations of  $u_1$  and  $u_2$  do not interact constructively to modify the principal term of  $u_3$ . Note that there is no general estimate of the size of the corrector: it is  $O(\sqrt{\varepsilon})$  if the phase has only nondegenerate critical points, it is  $O(\varepsilon^{1/p})$  when there are degenerate critical points of finite order.

- *Strong coherence.* Suppose that

$$(7.6) \quad \partial_t(\varphi_1 + \varphi_2) \neq 0 \quad \text{everywhere}$$

In this case, and in this case only, one can perform direct integrations by part in the integral and the corrector is  $O(\varepsilon)$ . Indeed, if the coefficients  $a$  are smooth, repeated integrations by part yield a complete asymptotic expansion in powers of  $\varepsilon$ .

**Conclusion 7.1.** *Complete asymptotic expansion in powers of  $\varepsilon$  can exist only if all the phases in play are either resonant or strongly coherent. Otherwise, the convergence to zero of correctors can be arbitrarily slow.*

**Remark 7.2.** Resonance is a rare phenomenon In the example, the phases  $\varphi_1$  and  $\varphi_2$  satisfy

$$(\partial_t + c_1 \partial_x) \varphi_1 = 0, \quad (\partial_t + c_2 \partial_x) \varphi_2 = 0, \quad \partial_t(\varphi_1 + \varphi_2) = 0$$

Thus

$$0 = (\partial_t + c_2 \partial_x) \partial_t(\varphi_1 + \varphi_2) = \partial_t(\partial_t + c_2 \partial_x) \varphi_1 = c_1(c_1 - c_2) \psi_1''(y - c_1 t),$$

implying that  $\psi_1'' = 0$ . Thus  $\psi_1$  must be a plane phase and  $\varphi_1 = \alpha_1(x - c_1 t) + \beta_1$  and similarly  $\varphi_2 = \alpha_2(x - c_2 t) + \beta_2$ . The resonance occurs if and only if  $\alpha_1 c_1 + \alpha_2 c_2 = 0$ , showing that not only the phases must be planar but also the coefficients must be suitably chosen.

### 7.1.3 Strong coherence and small divisors

Consider initial data for (7.2) which are periodic in  $y/\varepsilon$ . Thus, for  $j = 1, 2$ :

$$u_j(0, x) = \sum_n a_{j,n}(x) e^{in x/\varepsilon}, \quad u_j(t, x) = \sum_n a_{j,n}(x - c_j t) e^{in \varphi_j/\varepsilon}$$

with  $\varphi_j = x - c_j t$ . Suppose that  $u_3(0, x) = 0$ . Then  $u_3$  is the sum of integrals

$$I_{m,n} = \left( \int_0^t a_{1,m}(x - c_1 s) a_{2,n}(x - c_2 s) e^{-i(mc_1 + nc_2)s/\varepsilon} ds \right) e^{i(m+n)x/\varepsilon}.$$

Resonances occur when  $(m, n) \in \mathbb{Z}^2$  are such that

$$(7.7) \quad mc_1 + nc_2 = 0.$$

This shows that the presence of resonances also depend on *arithmetical conditions*. For instance, if  $c_1/c_2 \notin \mathbb{Q}$ , there will be no resonant interaction.

Since  $\partial_t(m\varphi_1 + n\varphi_2)$  is a constant, the phases  $m\varphi_1 + n\varphi_2$  are either resonant or strongly coherent, yielding integrals which are either  $O(1)$  or  $O(\varepsilon)$ . More precisely, in the nonresonant case

$$I_{m,n} = \varepsilon \frac{O(1)}{mc_1 + nc_2}.$$

However, the convergence of the series  $\sum I_{m,n}$  depends on the smallness of the denominators  $mc_1 + nc_2$ .

**Conclusion 7.3.** *Interaction phenomena also depend on arithmetic properties of the spectrum of the oscillations. Moreover, the behavior of correctors may also involve small divisors problems.*

#### 7.1.4 Periodic initial data may lead to almost periodic solutions

Consider the following dispersive system

$$(7.8) \quad \begin{cases} \partial_t u_1 + \partial_x u_1 + \frac{1}{\varepsilon} u_2 = 0, \\ \partial_t u_2 - \partial_x u_2 - \frac{1}{\varepsilon} u_1 = 0. \end{cases}$$

The phenomenon which we describe below also occurs for nondispersive system, but in dimension  $d \geq 2$ . Consider an initial data with planar phase

$$(7.9) \quad u_j(0, x) = \sum a_{j,n} e^{inx/\varepsilon}.$$

The solution reads

$$(7.10) \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \sum_n \alpha_n \begin{pmatrix} i \\ n + \omega_n \end{pmatrix} e^{i(ny + \omega_n t)/\varepsilon} + \beta_n \begin{pmatrix} n + \omega_n \\ i \end{pmatrix} e^{i(ny - \omega_n t)/\varepsilon},$$

with  $\omega_n = \sqrt{n^2 + 1}$ . This solution is periodic in  $y/\varepsilon$ . However, the set  $\{\omega_n\}_{n \in \mathbb{Z}}$  of time frequencies is not contained in a finitely generated  $\mathbb{Z}$ -module. The solution is not periodic in time, it is only almost periodic.

**Conclusion 7.4.** *Even for periodic initial data, the solution cannot always be written in the form (7.1)  $\mathbf{u}(x, \Phi(x)/\varepsilon)$  with a finite number of phases  $\Phi_1, \dots, \Phi_m$  and periodic profiles  $\mathbf{u}(x, \theta)$ . The introduction of almost periodic functions can be compulsory.*

### 7.1.5 A single initial phase can generate a space of phases of infinite dimension over $\mathbb{R}$

Consider (7.8) with initial data

$$(7.11) \quad u_j(0, x) = \sum a_{j,n} e^{inx^2/\varepsilon}.$$

We have simply replaced the planar phases  $nx$  by the curved phases  $\psi_n(x) = nx^2/2$ . The outgoing phases are the solutions of the eikonal equation

$$\partial_t \varphi_n^\pm = \pm \sqrt{1 + (\partial_y \varphi_n^\pm)^2}, \quad \varphi_n^\pm(0, y) = \psi_n(y).$$

All the phases  $\varphi_n^\pm$  are defined and smooth on  $\Omega = \{(t, x); x > |t|, |t| < 1\}$ . One can check that the phases  $\varphi_n$  are independent not only over  $\mathbb{Q}$ , but also over  $\mathbb{R}$ .

**Conclusion 7.5.** *A one dimensional vector space of initial phases can generate a space of phases of infinite dimension over  $\mathbb{R}$ . In this case, any description (7.1) must include an infinite number of phases, whatever the properties imposed on the spectrum of the profiles.*

### 7.1.6 Multidimensional effects

We already have discussed the effects of focusing in Section 5.4: the linear amplification can be augmented by nonlinear term and produce blow up. We briefly discuss below two other examples taken from [JMR4].

A more dramatic multidimensional effect is *instantaneous blow up*, which may occur when the initial phase  $\varphi$  has a stationary point. This is illustrated in the following example. Consider in dimension  $d = 3$

$$(7.12) \quad \begin{cases} (\partial_t^2 - \Delta_x)u^\varepsilon = 0, & u^\varepsilon|_{t=0} = 0, & \partial_t u^\varepsilon|_{t=0} = h(|x|^2)e^{i|x|^2/\varepsilon}, \\ \partial_t v^\varepsilon = |\partial_t u^\varepsilon|^2 |v^\varepsilon|^2, & v^\varepsilon|_{t=0} = v_0(x). \end{cases}$$

Then

$$u^\varepsilon(t, 0) = th(t)e^{it^2/\varepsilon} \quad \partial_t u^\varepsilon = 2i\varepsilon^{-1}t^2 h(t)e^{it^2/\varepsilon} + O(1).$$

Therefore, if  $h(0) = 1$ ,  $|\partial_t u^\varepsilon| \geq \varepsilon^{-1} t^2$  on an interval  $[0, T_0]$  and if  $v_0(0) = 1$ , and

$$v(t, 0) \geq \frac{1}{1 - t^5/5\varepsilon^2}.$$

This proves that the maximal time of existence of bounded solutions is  $T(\varepsilon) \lesssim \varepsilon^{\frac{2}{5}}$ , and there is no uniform domain of existence of solutions.

When combined with phase interaction, the focusing effects can come from linear combinations of the phases, before the initial phase focus. More precisely, the initial data can launch regular phases  $\varphi_j$ ; nonlinear interaction creates new phases  $\psi = \sum n_j \varphi_j$ , and one among them may have a critical point at  $t = 0$ , producing instantaneous explosion. This phenomenon is called *hidden focusing* in [JMR4] and illustrated with the following example

$$\begin{cases} (\partial_t^2 - \Delta_x)u^\varepsilon = 0, & u^\varepsilon|_{t=0} = 0, & \partial_t u^\varepsilon|_{t=0} = e^{ix_1/\varepsilon}, \\ (\partial_t^2 - c\Delta_x)v^\varepsilon + a\partial_1 v^\varepsilon = 0, & v^\varepsilon|_{t=0} = \varepsilon e^{ix_1/\varepsilon}, & \partial_t v^\varepsilon|_{t=0} = b(0, x)e^{ix_1/\varepsilon}, \\ (\partial_t^2 - 3\Delta_x)w^\varepsilon = f^\varepsilon, & w^\varepsilon|_{t=0} = 0, & \partial_t w^\varepsilon|_{t=0} = e^{ix_1/\varepsilon}, \\ \partial_t z^\varepsilon = |\partial_{x_1} w^\varepsilon|^2 |z^\varepsilon|^2, & z^\varepsilon|_{t=0} = 0, & \end{cases}$$

with

$$f^\varepsilon = \frac{1}{(b(t, x))^2} (\partial_t \bar{u}^\varepsilon + \partial_{x_1} \bar{u}^\varepsilon)^3 (\partial_t u^\varepsilon - \partial_{x_1} u^\varepsilon) (\partial_t v^\varepsilon)^2.$$

The phases in play are  $\varphi_1 = x - t$ ,  $\varphi_2 = x + t$  and the solutions  $\varphi_3$  and  $\varphi_4$  of the eikonal equation for  $\partial_t^2 - c\Delta_x$  with initial value  $x_1$ . If  $c(0, 0) = 2$ ,  $\varphi_3 = x + 2t + 0(t^2 + |tx|)$  and  $\varphi_4 = x - 2t + 0(t^2 + |tx|)$ . In particular the phase  $\phi = -2\varphi_2 + \varphi_1 + 2\varphi_3 = O(t^2 + |tx|)$  which is revealed by the interaction  $f^\varepsilon$  has a critical point at the origin, producing blow up in the fourth equations as in the previous example:

**Proposition 7.6** ([JMR4]). *One can choose the functions  $a(t, x)$ ,  $b(t, x)$  and  $c(t, x)$  such that  $c(0, 0) = 2$  and the maximal time of existence of bounded solutions is  $T(\varepsilon) \lesssim \varepsilon^{\frac{1}{6}}$ .*

## 7.2 Description of oscillations

In this section we deviate slightly from the central objective of nonlinear optics, and evoke the more general question of how one can model oscillations.

### 7.2.1 Group of phases, group of frequencies

In the representation (7.1),  $\Phi$  is thought as a vector valued function of  $(t, x)$ , for instance  $\Phi = (\varphi_1, \dots, \varphi_m)$  and the phases are linear combinations

$\varphi = \sum \alpha_j \varphi_j = \alpha \cdot \Phi$ ; the important property of the set of  $\alpha$  is that it must be an additive group, to account for wave interaction. As explained in the examples above, this group is not necessarily finitely generated, for instance in the case of almost periodic oscillations. This leads to the following framework:

**Assumption 7.7.**  $\Lambda$  is an Abelian group (called the group of frequencies) and we have an injective homomorphism from  $\Lambda$  to  $C^\infty(\bar{\Omega}; \mathbb{R})$  where  $\Omega$  be an open set in  $\mathbb{R}^{1+d}$ . We denote it by  $\alpha \mapsto \alpha \cdot \Phi$ .

The set  $\mathcal{F} = \Lambda \cdot \Phi$  is an additive subgroup of  $C^\infty(\bar{\Omega}; \mathbb{R})$  (the group of phases) assumed to contain no nonzero constant function.

We also assume that for all  $\varphi \in \mathcal{F} \setminus \{0\}$ ,  $d\varphi \neq 0$  a.e. in  $\bar{\Omega}$ .

We denote by  $\alpha \cdot \Phi(t, x)$  the value at  $(t, x)$  of the phase  $\alpha \cdot \Phi$  and by  $\alpha \cdot d\Phi(t, x)$  its differential. The notation  $\alpha \mapsto \alpha \cdot \Phi$  for the homomorphism from  $\Lambda \mapsto C^\infty(\bar{\Omega}; \mathbb{R})$  is chosen to mimic the case where  $\Lambda \subset \mathbb{R}^m$  and  $\Phi \in C^\infty(\bar{\Omega}; \mathbb{R}^m)$ .

**Examples 7.8.** a) When  $\mathcal{F}$  is finitely generated, then one can choose a basis over  $\mathbb{Z}$  and we are reduced to the case where  $\Lambda = \mathbb{Z}^p$ .

b) Another interesting case occurs when the  $\mathbb{R}$ -vector space generated by  $\mathcal{F}$ , called  $\tilde{\mathcal{F}}$ , is of finite dimension  $m$  over  $\mathbb{R}$ . Then one can choose a finite basis  $(\psi^\ell)$  of  $\tilde{\mathcal{F}}$  and denote by  $\cdot \Psi$  the mapping  $\beta \cdot \Psi = \sum_{1 \leq \ell \leq m} \beta_\ell \psi^\ell$  from  $\mathbb{R}^m$  to  $\tilde{\mathcal{F}}$ . Then, we have a group homomorphism  $M : \Lambda \mapsto \mathbb{R}^m$  such that

$$(7.13) \quad \alpha \cdot \Phi = (M\alpha) \cdot \Psi.$$

### 7.2.2 The fast variables

The question is to know what is the correct space  $\Theta$  for the fast variables  $\theta$  in the profiles  $\mathbf{u}(t, x, \theta)$ . In the periodic case,  $\Theta$  is a torus  $\mathbb{T}^m$ , the set of frequencies is  $\mathbb{Z}^m$ , and the link is clear in Fourier expansions: by the substitution  $\theta = \Phi/\varepsilon$ ,

$$(7.14) \quad e^{i\alpha\theta} \quad \mapsto \quad e^{i\alpha \cdot \Phi/\varepsilon}$$

transform the exponential into the desired oscillation. This approach extends to general abelian group, using the corresponding Fourier analysis. We refer for instance to [Wei, HeRo] for a detailed presentation of Fourier analysis on groups.

The group  $\Lambda$  is equipped with the discrete topology. Its *characters* are the homomorphisms from  $\Lambda$  to the unit circle  $S^1 \subset \mathbb{C}$ , and the set of characters

is another abelian group, denoted by  $\Theta$ , which is compact when equipped with the pointwise convergence. The duality theorem of Pontryagin-Van Kampen asserts that the group of characters of  $\Theta$  is  $\Lambda$ . We denote this duality by

$$(7.15) \quad (\alpha, \theta) \in \Lambda \times \Theta \quad \mapsto \quad e_\alpha(\theta) \in S^1.$$

This extends to the general setting, the functions  $e^{i\alpha\theta}$  for in the duality  $\alpha \in \mathbb{Z}^m$ ,  $\theta \in \mathbb{T}^m$ .

One advantage of working on a compact group  $\Theta$  is that there is a nice  $L^2$  Fourier theory. Equipped with the Haar probability  $d\theta$ , the  $\{e_\alpha, \alpha \in \Lambda\}$  form an orthonormal basis of  $L^2(\Theta)$  and any  $\mathbf{u} \in L^2(\Theta)$  has the Fourier series decomposition

$$\mathbf{u}(\theta) = \sum_{\alpha} \hat{\mathbf{u}}(\alpha) e_\alpha(\theta), \quad \hat{\mathbf{u}}(\alpha) = \int_{\Theta} \mathbf{u}(\theta) \overline{e_\alpha(\theta)} d\theta.$$

The *spectrum* of  $\mathbf{u}$  is the support of  $\hat{\mathbf{u}}$  in  $\Lambda$ .

### 7.2.3 Profiles and oscillations

Profiles are functions  $\mathbf{u}(t, x, \theta)$  on  $\Omega \times \Theta$ . The simplest example are trigonometric polynomials, that is finite linear combinations of the characters  $e_\alpha(\theta)$  with coefficients  $\hat{\mathbf{u}}_\alpha \in C^\infty(\bar{\Omega})$ .

We now present the intrinsic definition of the substitution  $\Phi/\varepsilon$  in place of  $\theta$  in profiles. For all  $(t, x) \in \bar{\Omega}$  and  $\varepsilon > 0$  the mapping  $\alpha \mapsto e^{i\alpha \cdot \Phi(t, x)/\varepsilon}$  is a character on  $\Lambda$ , and therefore defines  $p^\varepsilon(t, x) \in \Theta$  such that

$$(7.16) \quad e^{i\alpha \cdot \Phi(t, x)/\varepsilon} = e_\alpha(p^\varepsilon(t, x)).$$

For  $\mathbf{u} \in C^0(\bar{\Omega} \times \Theta)$  one can define the family of oscillating functions:

$$(7.17) \quad \mathbf{p}^\varepsilon(\mathbf{u})(t, x) = \mathbf{u}(t, x, p^\varepsilon(t, x)).$$

This coincides with the expected definition for trigonometric polynomial  $\mathbf{u} = \sum \hat{\mathbf{u}}_\alpha e_\alpha$ , since then

$$\mathbf{p}^\varepsilon(\mathbf{u})(t, x) = \sum \hat{\mathbf{u}}_\alpha(t, x) e^{i\alpha \cdot \Phi(t, x)/\varepsilon}.$$



### 7.2.4 Profiles in $L^p$ and associated oscillating families

The definition of the substitution can be extended to profiles  $\mathbf{u} \in L^p$ ,  $p < +\infty$ , not for a fixed  $\varepsilon$ , but in an asymptotic way. The result is a class of bounded families in  $L^p$ , modulo families which converge strongly to 0. The key observation is the following:

**Proposition 7.9.** *For  $\mathbf{u} \in C_0^0(\overline{\Omega} \times \Theta)$  and  $1 \leq p < +\infty$ , there holds*

$$\|\mathbf{u}\|_{L^p(\Omega \times \Theta)} = \lim_{\varepsilon \rightarrow 0} \|\mathbf{p}^\varepsilon(\mathbf{u})\|_{L^p(\Omega)}.$$

*Proof.*  $\mathbf{v} = |\mathbf{u}|^p \in C_0^0(\overline{\Omega} \times \Theta)$  and it is sufficient to prove that

$$\int_{\Omega \times \Theta} \mathbf{v}(t, x, \theta) dt dx d\theta = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \mathbf{p}^\varepsilon(\mathbf{v})(t, x) dt dx.$$

By density one can assume that  $\mathbf{v}$  is a trigonometric polynomial. In this case

$$\begin{aligned} \int_{\Omega} \mathbf{p}^\varepsilon(\mathbf{u})(t, x) dt dx &= \int_{\Omega} \sum \hat{\mathbf{u}}(t, x, \alpha) e^{i\alpha \cdot \Phi(t, x)/\varepsilon} dt dx \\ &\xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} \hat{\mathbf{u}}(t, x, 0) dt dx = \int_{\Omega \times \Theta} \mathbf{u}(t, x, \theta) dt dx d\theta, \end{aligned}$$

where the convergence follows from Lebesgue's lemma and the assumption that  $d(\alpha \cdot \Phi) \neq 0$ , a.e. if  $\alpha \neq 0$ .  $\square$

**Definition 7.10.** *Let  $\mathcal{L}_\varepsilon^p(\Omega)$  denote the space of families  $(u^\varepsilon)$  which are bounded in  $L^p(\Omega)$ . Two families  $(u^\varepsilon)$  and  $(v^\varepsilon)$  are equivalent if  $(u^\varepsilon - v^\varepsilon)$  tends to 0 in  $L^p(\Omega)$ . This is denoted by  $(u^\varepsilon) \sim (v^\varepsilon)$ .*

Consider the following semi-norm on  $\mathcal{L}_\varepsilon^p(\Omega)$ :

$$(7.18) \quad \|(u^\varepsilon)\|_p = \limsup_{\varepsilon \rightarrow 0} \|u^\varepsilon\|_{L^p(\Omega)}.$$

The set  $\mathcal{N}_\varepsilon^p$  of families  $(u^\varepsilon)$  such that  $\|(u^\varepsilon)\|_p = 0$  is the subspace of families  $(u^\varepsilon) \sim 0$  which converge strongly to zero in  $L^p(\Omega)$ . The quotient space  $\mathcal{L}_\varepsilon^p/\mathcal{N}_\varepsilon^p$  is space of bounded families modulo families which converge strongly to 0. With the norm (7.18) it is a Banach space.

For  $\mathbf{u} \in C_0^0(\overline{\Omega} \times \Theta)$  the family  $(u^\varepsilon) = (\mathbf{p}^\varepsilon(\mathbf{u}))$  is bounded in  $L^p(\Omega)$ . Seen in the quotient space  $\mathcal{L}_\varepsilon^p$ , Proposition 7.9 implies that  $\mathbf{p}^\varepsilon$  defined on  $C_0^0(\overline{\Omega} \times \Theta)$  is an isometry for the norms of  $L^p(\Omega \times \Theta)$  and  $\mathcal{L}_\varepsilon^p(\Omega)$ . Thus, by density  $\mathbf{p}^\varepsilon$  extends as an isometry  $\tilde{\mathbf{p}}^\varepsilon$  from  $L^p(\Omega \times \Theta)$  into  $\mathcal{L}_\varepsilon^p$ .

**Definition 7.11** ( $\mathcal{F}$ -oscillating families in  $L^p$ ).  $L_{\varepsilon, \mathcal{F}}^p$  denotes the image of  $L^p(\Omega \times \Theta)$  by  $\tilde{\mathbf{p}}^\varepsilon$ . A family  $(u^\varepsilon)$  in  $\mathcal{L}_\varepsilon^p$  is called  $\mathcal{F}$ -oscillating, if its image in the quotient  $L_\varepsilon^p$  belongs to  $L_{\varepsilon, \mathcal{F}}^p$ . The space of such families is denoted by  $\mathcal{L}_{\varepsilon, \mathcal{F}}^p \subset \mathcal{L}_\varepsilon^p$ .

Because  $\tilde{\mathbf{p}}^\varepsilon$  is an isometry, for  $(u^\varepsilon) \in \mathcal{L}_{\varepsilon, \mathcal{F}}^p$ , there is a unique  $\mathbf{u} \in L^p(\Omega \times \Theta)$  such that  $(u^\varepsilon) \in \tilde{\mathbf{p}}^\varepsilon(\mathbf{u})$ . This  $\mathbf{u}$  is called *the profile of the oscillating family*  $(u^\varepsilon)$  and we can think of it as

$$u^\varepsilon(t, x) \sim \mathbf{u}(t, x, p^\varepsilon(t, x))$$

where the  $\sim$  is taken in the sense that the difference converges to 0 in  $L^p$ , the substitution is exact if  $\mathbf{u}$  is continuous, and taken in the extended sense if  $\mathbf{u} \in L^p$ .

### 7.2.5 Multiscale weak convergence, weak profiles

**Proposition 7.12.** *If  $\Lambda$  is countable, then for all  $(u^\varepsilon) \in \mathcal{L}_\varepsilon^p(\Omega)$ ,  $1 < p < +\infty$ , there is a sequence  $\varepsilon_n \rightarrow 0$  and there is  $\mathbf{u} \in L^p(\Omega \times \Theta)$  such that for all trigonometric polynomial  $\mathbf{a}$ :*

$$(7.19) \quad \int_{\Omega} u^{\varepsilon_n}(t, x) \mathbf{p}^{\varepsilon_n}(\mathbf{a})(t, x) dt dx \xrightarrow{\varepsilon_n \rightarrow 0} \int_{\Omega \times \Theta} \mathbf{u}(t, x, \theta) \mathbf{a}(t, x, \theta) dt dx d\theta.$$

This  $\mathbf{u}$  is called the *weak profile with respect to the group of phases  $\mathcal{F}$*  of the family  $(u^{\varepsilon_n})$ . In other contexts it is also called the *two scale weak limit*.

If  $(u^\varepsilon) \in \mathcal{L}_{\varepsilon, \mathcal{F}}^p$  is  $\mathcal{F}$  oscillating with profile  $\mathbf{u}$ , then  $\mathbf{u}$  is also the weak profile of the family and no extraction of subsequence is necessary.

We do not investigate these notions further. They have been used in the context of geometric optics in [JMR7, JMR8], see also Section 7.6.4 below. They are also useful in other situations such as homogenisation.

### 7.3 Formal multi-phases expansions

The goal of this section is to show that the formal analysis of Section 5.2 can be carried out in the more general setting of multi-phases expansions. We consider the weakly nonlinear framework, that is equations of the form

$$(7.20) \quad A_0(a, \varepsilon u) \partial_t u + \sum_{j=1}^d A_j(a, \varepsilon u) \partial_{x_j} u + \frac{1}{\varepsilon} E(a) u = F(u),$$

where the coefficient  $a = a(t, x)$  is smooth and given. The matrices  $A_j$  and  $E$  are assumed to be smooth functions of their argument. Moreover, the  $A_j$

are self adjoint with  $A_0$  definite positive and  $E$  is skew adjoint. The source term  $F$  is a smooth function of its arguments.

### 7.3.1 Phases and profiles. Weak coherence

Following the general principles, we consider a group of phases  $\mathcal{F}$  and also introduce the vector space  $\tilde{\mathcal{F}}$  they span. The characteristic phases play the prominent role. We use the notations

$$(7.21) \quad \begin{cases} A_j(a) = A_j(a, 0), \\ L(a, \partial) = A_0(a)\partial_t + \sum A_j(a)\partial_{x_j} + E(a) = L_1(a, \partial) + E(a). \end{cases}$$

**Assumption 7.13 (Weak coherence).** *We are given an additive subgroup  $\mathcal{F}$  of  $C^\infty(\bar{\Omega}; \mathbb{R})$  such that the vector space  $\tilde{\mathcal{F}}$  generated by  $\mathcal{F}$  is finite dimensional over  $\mathbb{R}$ . We assume that*

- i) for all  $\varphi \in \tilde{\mathcal{F}}$ , either  $\det L(a(t, x), id\varphi(t, x))$  vanishes everywhere on  $\bar{\Omega}$  and the dimension  $\ker L(a(t, x), id\varphi(t, x))$  is independent of  $(t, x)$ , or the determinant  $\det L(a(t, x), id\varphi(t, x))$  is different from zero almost everywhere.*
- ii) for all  $\varphi \in \mathcal{F} \setminus \{0\}$ ,  $d\varphi(t, x) \neq 0$  for almost all  $(t, x) \in \bar{\Omega}$ .*

This condition is called *weak coherence*. The condition *i)* means that either  $\varphi$  is a characteristic phase of constant multiplicity in the sense of Definition (5.3), or it is characteristic almost nowhere. For the model (7.2), and  $\mathcal{F}$  generated by the phases  $\varphi_1$  and  $\varphi_2$ , weak coherence occurs when  $\alpha\partial_t\varphi_1 + \alpha_2\partial_t\varphi_2 \neq 0$  almost everywhere when  $(\alpha_1, \alpha_2) \neq (0, 0)$ . We also refer to Section 7.6.1 for generic examples in dimension  $d = 1$ .

In dimension  $d > 1$ , an example (which is indeed strongly coherent as explained below) is the case where  $a = \underline{a}$  is constant and  $\mathcal{F}$  is a group of planar phases  $\{\beta_0 t + \sum \beta_j x_j, \beta \in \Lambda \subset \mathbb{R}^{1+d}\}$ .

Taking a basis  $(\psi_1, \dots, \psi_m)$  for  $\tilde{\mathcal{F}}$ , then any phase  $\varphi \in \tilde{\mathcal{F}}$  is represented as  $\varphi = \beta \cdot \Psi = \sum \beta_j \psi_j$  with  $\beta \in \mathbb{R}^m$ . The subgroup  $\mathcal{F}$  corresponds to frequencies  $\beta$  in a sub group  $\Lambda \subset \mathbb{R}^m$ . There are three possibilities:

- The group  $\mathcal{F}$  (or  $\Lambda$ ) is finitely generated of rank  $p = m$ . Then one can choose a basis of  $\mathcal{F}$  over  $\mathbb{Z}$  which is also a basis of  $\tilde{\mathcal{F}}$  over  $\mathbb{R}$ . Then the oscillations are naturally represented as

$$(7.22) \quad u^\varepsilon(t, x) = \mathbf{u}(t, x, \Psi(t, x)/\varepsilon),$$

with  $\mathbf{u}(t, x, \theta)$   $2\pi$  periodic in  $(\theta_1, \dots, \theta_m)$ . The space for the variables  $\theta$  is  $\mathbb{T}^m = (\mathbb{R}/2\pi\mathbb{Z})^m$ .

- The group  $\mathcal{F}$  is finitely generated of rank  $p > m$ . Then one can choose a basis  $(\varphi_1, \dots, \varphi_p)$  of  $\mathcal{F}$  such that  $\mathcal{F} = \{\alpha \cdot \Phi = \sum \alpha_j \varphi_j; \alpha \in \mathbb{Z}^p\}$ . Writing the  $\varphi_j$  in the base  $(\psi_1, \dots, \psi_m)$  yields a  $m \times p$  matrix  $M$  as in (7.13), such that

$$(7.23) \quad \Lambda = M\mathbb{Z}^p, \quad \Phi(t, x) = {}^t M \Psi(t, x).$$

The oscillations can be represented under two forms:

$$(7.24) \quad u^\varepsilon(t, x) = \mathbf{u}(t, x, \Psi(t, x)/\varepsilon),$$

$$(7.25) \quad u^\varepsilon(t, x) = \mathbf{v}(t, x, \Phi(t, x)/\varepsilon),$$

with  $\mathbf{u}(t, x, Y)$  *quasi-periodic* in  $Y$  with spectrum contained in  $\Lambda$ , or with  $\mathbf{v}(t, x, \theta)$  *periodic* in  $(\theta_1, \dots, \theta_p)$ . In accordance with (7.23), the link between  $\mathbf{u}$  and  $\mathbf{v}$  is that

$$(7.26) \quad \mathbf{u}(t, x, Y) = \mathbf{v}(t, x, {}^t M Y).$$

Identifying periodic functions with functions on the torus, the variables  $\theta$  can be seen as living in  $\Theta = \mathbb{T}^p$ .

- The group  $\mathcal{F}$  is not finitely generated. Then the representation (7.22) involve profiles  $\mathbf{u}(t, x, Y)$  which are *almost periodic* in  $Y \in \mathbb{R}^m$ . Recall that the space of almost periodic functions of  $Y$ , with values in a Banach space  $B$  is the closure in  $L^\infty(\mathbb{R}^m; B)$  of the set of finite sums  $\sum e^{i\beta Y} b_\beta$ , with  $\beta \in \mathbb{R}^m$  and  $b_\beta \in B$ , called *trigonometric polynomials*.

The analogue of (7.25) involves a *Bohr compactification* of  $\mathbb{R}^m$  (see e.g. [HeRo, Wei]). The dual group of the discrete group  $\Lambda$  is a compact group  $\Theta$  (see Section 7.2.2). There is a natural mapping  $\pi : \mathbb{R}^m \mapsto \Theta$  defined as follows. A point  $Y \in \mathbb{R}^m$  is mapped to the character  $\alpha \mapsto e^{i\alpha \cdot Y}$ . In the quasi-periodic case,  $\pi = \varpi \circ {}^t M$  where  $\varpi$  is the natural map from  $\mathbb{R}^p$  to  $\mathbb{T}^p$ . The link between almost periodic profiles with spectrum contained in  $\Lambda$  and functions on the group  $\Theta$  is that

$$(7.27) \quad \mathbf{u}(t, x, Y) = \mathbf{v}(t, x, \pi Y).$$

Similarly, the mapping  $p^\varepsilon$  in (7.16) is  $p^\varepsilon = \pi(\Psi/\varepsilon)$ . In this setting, the oscillations can be written

$$(7.28) \quad u^\varepsilon(t, x) = \mathbf{v}(t, x, \pi(\Phi(t, x)/\varepsilon)),$$

with  $\mathbf{v}(t, x, \theta)$  defined on  $\Omega \times \Theta$ .

### 7.3.2 Phases and profiles for the Cauchy problem

We make the elementary but important remark that *giving initial oscillating data for the solution  $u^\varepsilon$  is not the same as giving initial data for the profile  $\mathbf{u}$* , since taking  $t = 0$  in (7.22) yields:

$$(7.29) \quad u^\varepsilon(0, x) = \mathbf{u}(0, x, \Psi(0, x)/\varepsilon).$$

Given initial oscillations

$$(7.30) \quad u^\varepsilon(0, x) = h^\varepsilon(x) \sim \mathbf{h}(x, \Psi_0(x)/\varepsilon)$$

we have to make the link between  $\Psi_0$  and  $\Psi$  and between  $\mathbf{u}$  and  $\mathbf{h}$  which *do not depend on the same set of fast variables*. The situation is that we have a group  $\mathcal{F}$  for the phases in  $\Omega$  and a group  $\mathcal{F}_0$  of initial phases. Clearly, the first relation is that  $\mathcal{F}_0 = \{\varphi|_{t=0}; \varphi \in \mathcal{F}\}$ . But this is not sufficient : we also want that for all  $\varphi_0 \in \mathcal{F}_0$  the solutions of the eikonal equation with initial value  $\varphi_0$  belong to  $\mathcal{F}$ . This is a very strong assumption indeed, but which is satisfied for instance within the framework of planar phases, but also in other interesting circumstances. The conditions can be summarized in the following assumption:

**Assumption 7.14 (Weak coherence for the Cauchy problem).** *Let  $\mathcal{F} = \Lambda \cdot \Psi \subset \tilde{\mathcal{F}} \subset C^\infty(\bar{\Omega}; \mathbb{R})$  satisfy Assumption 7.13. We denote by  $\mathcal{C}_\Lambda$  the set of frequencies  $\alpha \in \Lambda$  such that  $\det L(a, i\alpha \cdot \Psi) = 0$ . When  $\alpha \notin \mathcal{C}_\Lambda$  let  $P_\alpha = 0$  and when  $\alpha \in \mathcal{C}_\Lambda$  let  $P_\alpha(t, x)$  denote the spectral projector of*

$$A_0^{-1}(a) \left( \sum_{j=1}^d i\partial_{x_j}(\alpha \cdot \Psi) A_j(a) + E(a) \right)$$

*associated to the eigenvalue  $-i\partial_t(\alpha \cdot \Psi)$ .*

*With  $\bar{\omega} = \bar{\Omega} \cap \{t = 0\}$  let  $\mathcal{F}_0 \subset \tilde{\mathcal{F}}_0 \subset C^\infty(\bar{\omega})$  be the group and vector space of the restrictions to  $t = 0$  of the phases  $\varphi$  in  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$  respectively. We assume that for all  $\varphi_0 \in \tilde{\mathcal{F}}_0$ ,  $d\varphi_0 \neq 0$  almost every where on  $\bar{\omega}$ .*

*$\tilde{\mathcal{F}}_0$  is finite dimensional, of dimension  $m_0 \leq m$ ; choosing a basis, defines  $\Psi_0 \in C^\infty(\bar{\omega}; \mathbb{R}^{m_0})$  and a group  $\Lambda_0 \subset \mathbb{R}^{m_0}$  such that  $\mathcal{F}_0 = \Lambda_0 \cdot \Psi_0$ .*

*The mapping  $\varphi \mapsto \varphi|_{t=0}$  induces a surjective group homomorphism from  $\Lambda$  to  $\Lambda_0$ , called  $\rho$ , such that*

$$(7.31) \quad \alpha \cdot \Psi|_{t=0} = (\rho(\alpha)) \cdot \Psi_0.$$

*We assume that*

$$(7.32) \quad \forall \alpha_0 \in \Lambda_0, \quad \forall x \in \bar{\omega} : \quad \sum_{\alpha \in \rho^{-1}(\alpha_0)} P_\alpha(0, x) = \text{Id}.$$

Note that the sum in (7.32) has at most  $N$  terms different from zero, since different  $\alpha$ 's in  $\rho^{-1}(\alpha_0)$  correspond to different eigenvalues of  $A_0^{-1}(\sum i\partial_{x_j}(\alpha_0 \cdot \Psi_0)A_j + E)$ . That the sum is equal to the identity means that all the eigenvalues are present in the sum and thus correspond to a characteristic phase  $\alpha \cdot \Psi \in \mathcal{F}$  with initial value  $\alpha_0 \cdot \Psi_0$ .

**Lemma 7.15.** *For a trigonometric polynomial  $\mathbf{h}(x, Y_0) = \sum \widehat{\mathbf{h}}_{\alpha_0}(x)e^{i\alpha_0 \cdot Y_0}$ , define*

$$(7.33) \quad \mathbf{u}(0, x, Y) = \sum \widehat{P}_\alpha(0, x)\widehat{\mathbf{h}}_{\rho(\alpha)}(x)e^{i\alpha \cdot Y}$$

Then

$$(7.34) \quad \mathbf{u}(0, x, \Psi(0, x)/\varepsilon) = \mathbf{h}(x, \Psi_0(x)/\varepsilon).$$

We note that (7.33) also defines a trigonometric polynomial, since there are finitely many  $\alpha$  such that  $P_\alpha \neq 0$  and  $\rho(\alpha)$  belongs to the spectrum of  $\mathbf{h}$ . The identity (7.34) is an immediate consequence of (7.31) and (7.32).

This lemma, when extended to more general classes of profiles, will provide the initial data for  $\mathbf{u}$ , knowing the profile  $\mathbf{h}$  of the initial data.

### 7.3.3 Formal BKW expansions

Given a basis  $(\psi_1, \dots, \psi_m)$  of  $\widetilde{\mathcal{F}}$ , we look for solutions

$$(7.35) \quad u^\varepsilon \sim \sum_{n \geq 0} \varepsilon^n \mathbf{u}_n(t, x, \Psi(t, x)/\varepsilon)$$

with profiles  $\mathbf{u}_n(t, x, Y)$  which are periodic / quasi-periodic / almost periodic in  $Y \in \mathbb{R}^m$ . They have Fourier expansions

$$\mathbf{u}_n(t, x, Y) = \sum_{\alpha \in \Lambda} \widehat{\mathbf{u}}_{n,\alpha}(t, x) e^{i\alpha Y}.$$

and for the moment we leave aside the question of convergence of the Fourier series.

As in Section 5.2, one obtains a sequence of equations

$$(7.36) \quad \mathcal{L}(a, \partial_Y) \mathbf{u}_0 = 0$$

and for  $n \geq 0$

$$(7.37) \quad \mathcal{L}(a, \partial_Y) \mathbf{u}_{n+1} + \mathcal{L}_1(a, \mathbf{u}_0, \partial_{t,x,Y}) \mathbf{u}_n = F_n(\mathbf{u}_0, \dots, \mathbf{u}_n) = \mathbf{F}_n$$

where

$$(7.38) \quad \mathcal{L}(a, \partial_Y) = \sum_{j=1}^m L_1(a, d\psi_j) \partial_{Y_j} + E(a)$$

$$(7.39) \quad \mathcal{L}_1(a, v, \partial_{t,x,Y}) = L_1(a, \partial_{t,x}) + \sum_{j=1}^m v \cdot \nabla_v \tilde{L}_1(a, 0, d\psi_j) \partial_{Y_j}.$$

Here  $\tilde{L}_1(a, v, \tau, \xi)$  denotes the complete symbol  $\tau A_0(a, v) + \sum \xi_j A_j(a, v)$ . Moreover,  $\mathbf{F}_0 = F(\mathbf{u}_0)$  and for  $n > 0$ ,  $\mathbf{F}_n = F'(\mathbf{u}_0)\mathbf{u}_n +$  terms which depend only on  $(\mathbf{u}_0, \dots, \mathbf{u}_{n-1})$ .

The equations are analyzed on (formal) Fourier series. In particular,

$$\mathcal{L}(a, \partial_Y) e^{i\alpha Y} = e^{i\alpha Y} L(a, id(\alpha \cdot \Psi)), \quad \alpha \in \Lambda.$$

By Assumption 7.26:

1) either the phase  $\alpha \cdot \Psi$  is characteristic of constant multiplicity. and we can introduce smooth projectors  $P_\alpha(t, x)$  and  $Q_\alpha(t, x)$  on the kernel and the image of  $L(a(t, x), id(\alpha \cdot \Psi)(t, x))$  respectively, and the partial inverse  $R_\alpha$  such that  $R_\alpha(\text{Id} - Q_\alpha) = 0$ ,  $P_\alpha R_\alpha = 0$ ,  $R_\alpha \mathcal{L}(a, d(\alpha \cdot \Psi)) = 0$ .

2) or the phase  $\alpha \cdot \Psi$  is almost nowhere characteristic, and we *define*

$$(7.40) \quad P_\alpha = 0, \quad Q_\alpha = \text{Id}, \quad R_\alpha = (L(a(t, x), id(\alpha \cdot \Psi)(t, x)))^{-1}.$$

They define operators  $\mathcal{P}$ ,  $\mathcal{Q}$  and  $\mathcal{R}$ , acting on Fourier series (at least on formal Fourier series). The equation (7.36) reads

$$(7.41) \quad \mathbf{u}_0 = \mathcal{P}\mathbf{u}_0 \quad \Leftrightarrow \quad \forall \alpha, \quad \hat{\mathbf{u}}_{0,\alpha} \in \ker L(a, id(\alpha \cdot \Psi)).$$

The equation (7.37) is projected by  $\mathcal{Q}$  and  $\text{Id} - \mathcal{Q}$ ; the first part gives

$$(7.42) \quad (\text{Id} - \mathcal{P})\mathbf{u}_{n+1} = \mathcal{R}(\mathbf{F}_n - \mathcal{L}_1(a, \partial_{t,x,Y})\mathbf{u}_n)$$

and the second the equation

$$(7.43) \quad (\text{Id} - \mathcal{Q})(\mathcal{L}_1(a, \mathbf{u}_0, \partial_{t,x,Y})\mathbf{u}_n - \mathbf{F}_n) = 0.$$

In particular, the equation for the principal profile is

$$(7.44) \quad (\text{Id} - \mathcal{Q})(\mathcal{L}_1(a, \mathbf{u}_0, \partial_{t,x,Y})\mathcal{P}\mathbf{u}_0) = (\text{Id} - \mathcal{Q})F(\mathbf{u}_0).$$

Knowing  $\mathbf{u}_0$ , (7.42) with  $n = 0$  determines  $(\text{Id} - \mathcal{P})\mathbf{u}_1$ ; injecting in (7.43) for  $n = 1$  gives an equation for  $\mathcal{P}\mathbf{u}_1$  and so on by induction.

This is the general scheme at the formal level. To make it rigorous, we have to define the operators  $\mathcal{P}$ ,  $\mathcal{Q}$  and  $\mathcal{R}$  on function spaces and to solve the profile equations.

### 7.3.4 Determination of the main profiles

It is important that equation (7.44) depends only on the definition of the projectors  $\mathcal{P}$  and  $\mathcal{Q}$ , and not on the partial inverse  $\mathcal{R}$ . To fix the ideas, we assume from now on that the projectors  $P_\alpha(t, x)$  are the projectors on  $\ker \mathcal{L}(a(t, x), i\alpha)$  which are *orthogonal for the scalar product induced by  $A_0(a(t, x))$* , and that  $Q_\alpha = A_0(\text{Id} - P)A_0^{-1} = (\text{Id} - P_\alpha^*)$ .

By the symmetry assumption for  $L$ , the projectors  $P_\alpha$  and  $Q_\alpha$  are uniformly bounded in  $\alpha$ . Moreover, smoothness in  $x$  for fixed  $\alpha$  follows from the constant multiplicity assumption. However, when studying the convergence properties of the Fourier series defining  $\mathcal{P}$  and  $\mathcal{Q}$ , we need a little bit more:

**Assumption 7.16 (Uniform coherence).** *The projectors  $P_\alpha$  and  $Q_\alpha$  are bounded as well as their derivatives, uniformly with respect to  $(t, x) \in \bar{\Omega}$  and  $\alpha \in \Lambda$ .*

**Example 7.17.** This condition is trivially satisfied if  $a(t, x) = \underline{a}$  is constant, since then the projectors are bounded and independent of  $(t, x)$ . As in Proposition 4.18, it also holds if the symbol  $\mathcal{L}(a(t, x), \eta)$  is symmetric hyperbolic in a direction  $\underline{\eta}$  with constant multiplicities in  $(t, x, \eta)$  for  $\eta \neq 0$ .

Several frameworks can be considered:

(H1) The group  $\mathcal{F}$  is finitely generated of rank  $p \geq m$ .

In this case, instead of the quasi-periodic representation (7.22) one uses the periodic one (7.25). With obvious modifications, the equation reads

$$(7.45) \quad (\text{Id} - \mathcal{Q})(\mathcal{L}_1(a, \mathbf{u}_0, \partial_{t,x,\theta})\mathcal{P}\mathbf{u}_0) = (\text{Id} - \mathcal{Q})F(\mathbf{u}_0).$$

The profiles are sought in Sobolev spaces  $H^s(\Omega \times \mathbb{T}^p)$ . One key argument for the convergence of Fourier series is Plancherel's theorem : for  $\mathbf{u}(x, \theta) = \sum \hat{\mathbf{u}}_\alpha(x)e^{i\alpha \cdot \theta}$  there holds

$$(7.46) \quad \|\mathbf{u}\|_{H^s(\omega \times \mathbb{T}^p)}^2 \approx \sum_\alpha \|\hat{\mathbf{u}}_\alpha\|_{H^s(\omega)}^2 + |\alpha|^{2s} \|\hat{\mathbf{u}}_\alpha\|_{L^2(\omega)}^2$$

The profile equation inherits symmetry from the original equation and can be solved by an iterative scheme, in the spirit of the general theory of symmetric hyperbolic systems (see Theorem 3.2).

(H2) The equation is semi-linear and the nonlinearity  $F$  is real analytic.

In this case, one can use Wiener's algebra of almost periodic functions

$$\mathbb{A}^s = \left\{ \mathbf{u}(x, Y) = \sum \hat{\mathbf{u}}_\alpha(x)e^{i\alpha Y}; \sum_\alpha \|\hat{\mathbf{u}}_\alpha\|_{H^s} < \infty \right\}.$$



The profile equation on Fourier series reads

$$(7.47) \quad (\text{Id} - Q_\alpha)(L_1(a, \partial_{t,x})P_\alpha \hat{\mathbf{u}}_{0,\alpha} = (\text{Id} - Q_\alpha)\hat{\mathbf{F}}_{0,\alpha}, \quad \alpha \in \Lambda.$$

For each  $\alpha$ ,  $(\text{Id} - Q_\alpha)(L_1(a, \partial_{t,x})P_\alpha$  is a hyperbolic system (Lemma 5.7). Assuming  $\mathbf{F}_0$  given, or given by an iterative process, the coefficients  $\hat{\mathbf{u}}_{0,\alpha}$  can be determined from their initial value. Because the projectors are uniformly bounded the equations (7.47) can be uniformly solved. Next, we note that because  $F$  is real analytic, it maps the space  $\mathbb{E}^s$  into itself if  $s > \frac{d}{2}$ .

To simplify the exposition we use the following notations and terminology:

**Notations 7.18.**  $\Omega$  is a truncated cone  $\{(t, x) : 0 \leq t \leq T, \lambda_* t + |x| \leq R\}$  with  $T \leq T_0$  and  $\lambda_*$  will be chosen large enough so that  $\Omega$  will be contained in the domain of determinacy of  $\omega = \{x : |x| \leq R\}$  whenever this is necessary (see the discussion in Section 3.6).

Recall also from this section  $u$  defined on  $\Omega$  is said to be continuous in time with values in  $L^2$  if its extension by 0 outside  $\Omega$  belongs to  $C^0([0, T_0]; L^2(\mathbb{R}^d))$ ; for  $s \in \mathbb{N}$ , we say that  $u$  is continuous with values in  $H^s$  if the derivatives  $\partial_x^\alpha u$  for  $|\alpha| \leq s$  are continuous in time with values in  $L^2$ . We denote these spaces by  $C^0 H^s(\Omega)$ .

There are analogous definitions and notations for functions defined on  $\Omega \times \mathbb{T}^p$  or  $\Omega \times \mathbb{R}^m$ .

**Theorem 7.19. (Existence of the principal profile under the uniform weak coherence assumption).** *Suppose that Assumptions 7.13, 7.16 and (H1) [resp. (H2)] are satisfied. For all initial data  $\mathcal{P}\mathbf{u}_0|_{t=0}$  given in  $H^s(\omega \times \mathbb{T}^p)$  with  $s > \frac{d+p}{2} + 1$  [resp.  $\mathbb{E}^s(\omega)$  with  $s > \frac{d}{2}$ ], there is  $T > 0$  such that the profile equations (7.41) (7.44) have a unique solution  $\mathbf{u}_0 \in C^0 H^s((\Omega \cap \{t \leq T\}) \times \mathbb{T}^p)$  [resp.  $C^0 \mathbb{A}^s(\Omega \cap \{t \leq T\})$ ]*

For details, we refer the reader to [JMR2, JMR3].

### 7.3.5 Main profiles for the Cauchy problem

The previous theorem solves the profile equations knowing the value of  $\mathbf{u}_0$  at time  $t = 0$ . When considering the Cauchy problem with oscillating data (7.30), one has to lift the initial profile  $\mathbf{h}$  to an initial condition for  $\mathbf{u}_0$ .

We consider the framework of Assumption 7.14 with group of phases  $\mathcal{F} = \Lambda \cdot \Psi$  and  $\mathcal{F}_0 = \Lambda_0 \cdot \Psi_0$ . We choose a left inverse  $\ell$  from  $\tilde{\mathcal{F}}_0$  to  $\tilde{\mathcal{F}}$  of the restriction map  $\rho : \varphi \mapsto \varphi|_{t=0}$ , so that

$$(7.48) \quad \tilde{\mathcal{F}} = \ell \tilde{\mathcal{F}}_0 \oplus \tilde{\mathcal{F}}_1, \quad \tilde{\mathcal{F}}_1 = \ker \rho.$$

We can choose accordingly a basis in  $\tilde{\mathcal{F}}_0$  and a basis in  $\tilde{\mathcal{F}}_1$ , defining accordingly functions  $\Psi_0 \in C^\infty(\omega; \mathbb{R}^{m_0})$  and  $\Psi_1 \in C^\infty(\Omega; \mathbb{R}^{m_1})$ . The former is lifted in  $\ell\Psi_0 \in C^\infty(\Omega; \mathbb{R}^{m_0})$ , so that the functions in  $\tilde{\mathcal{F}}$  can be represented as  $\alpha_0 \cdot \ell\Psi_0 + \alpha_1 \cdot \Psi_1$ , with  $\alpha_0 \in \mathbb{R}^{m_0}$  and  $\alpha_1 \in \mathbb{R}^{m_1}$ . The group  $\mathcal{F} \subset \tilde{\mathcal{F}}$  can be written in this decomposition showing that one can assume that

$$(7.49) \quad \Lambda \subset \Lambda_0 \times \Lambda_1, \quad \Lambda_1 \subset \mathbb{R}^{m_1}$$

and that for  $\alpha = (\alpha_0, \alpha_1) \in \Lambda$

$$(7.50) \quad \alpha \cdot \Psi = \alpha_0 \cdot \ell\Psi_0 + \alpha_1 \cdot \Psi_1$$

The Lemma 7.15 give the natural lifting  $\mathbf{h} \mapsto \mathbf{u}_{0|t=0}$  for trigonometric polynomials. To extend it to spaces of profiles, we need some assumptions to ensure the convergence of Fourier series.

**Assumption 7.20.** *With the notations of Assumption 7.14 we suppose that  $\Lambda_0$  is finitely generated.*

Choosing a  $\mathbb{Z}$ -basis in  $\mathcal{F}_0$ , we determines a matrix  $M_0$  such that  $\Lambda_0 = M_0\mathbb{Z}^{p_0}$ . Let  $\Phi_0 = {}^t M_0\Psi_0$ . We represent the initial oscillations with profiles  $\mathbf{h}(x, \theta_0)$  which are periodic in  $\theta_0$ :

$$(7.51) \quad \mathbf{h}(x, \theta_0) = \sum_{\alpha_0 \in \mathbb{Z}^{p_0}} \hat{\mathbf{h}}_{\alpha_0}(x) e^{i\alpha_0 \cdot \theta_0} \in H^s(\omega \times \mathbb{T}^{p_0}).$$

In accordance with (7.50) we define for  $\alpha_0 \in \mathbb{Z}^{p_0}$  and  $\alpha_1 \in \Lambda_1$

$$(7.52) \quad (\alpha_0, \alpha_1) \cdot \Phi = \alpha_0 \cdot \ell\Phi_0 + \alpha_1 \cdot \Psi_1, (M\alpha_0) \cdot \ell\Psi_0 + \alpha_1 \cdot \Psi_1.$$

and accordingly we look for profiles  $\mathbf{u}$  which are functions of  $(t, x) \in \Omega$ ,  $\theta_0 \in \mathbb{T}^{p_0}$  and  $Y_1 \in \mathbb{R}^{m_1}$ .

For fixed  $t$ , the profiles  $\mathbf{u}$  we will consider will live in spaces

$$(7.53) \quad \mathbb{E}^s := C_{pp}^0(\mathbb{R}^{m_1}; H^s(\omega \times \mathbb{T}^{p_0})).$$

of almost periodic functions of  $Y_1$  valued in  $H^s(\omega \times \mathbb{T}^{p_0})$ . This space is the closure in  $L^\infty(\mathbb{R}^{m_1}; H^s(\omega \times \mathbb{T}^{p_0}))$  of the space of trigonometric polynomials, that is finite sums

$$(7.54) \quad \mathbf{v}(x, \theta_0, Y_1) = \sum_{\alpha_1} \hat{\mathbf{v}}_{\alpha_1}(x, \theta_0) e^{i\alpha_1 \cdot Y_1}$$

with coefficients  $\hat{\mathbf{v}}_{\alpha_1} \in H^s(\omega \times \mathbb{T}^{p_0})$ . Each coefficient can be in its turn expanded in Fourier series in  $\theta_0$ .

Following the notations of Assumptions 7.14 and 7.20, we denote by  $\tilde{\mathcal{C}}_\Lambda$  the set of  $(\alpha_0, \alpha_1) \in \mathbb{Z}^{p_0} \times \Lambda_1$  such that  $(M_0\alpha_0, \alpha_1)$  belongs to  $\Lambda$  and the corresponding phase  $(\alpha_0, \alpha_1) \cdot \Phi$  is characteristic. With little risk of confusion, we denote by  $P_{(\alpha_0, \alpha_1)}$  and  $Q_{(\alpha_0, \alpha_1)}$  the associated projectors. For  $(\alpha_0, \alpha_1) \notin \tilde{\mathcal{C}}_\Lambda$ , we set  $P_{(\alpha_0, \alpha_1)} = 0$  and  $Q_{(\alpha_0, \alpha_1)} = \text{Id}$ .

The projectors  $\mathcal{P}$  and  $\mathcal{Q}$  are obviously defined on trigonometric polynomials,

$$\sum \hat{\mathbf{v}}_{\alpha_0, \alpha_1}(x) e^{i(\alpha_0 \cdot \theta_0 + \alpha_1 \cdot Y_1)}$$

**Lemma 7.21.** *The projector  $\mathcal{P}$  extends as a bounded projector in  $\mathbb{E}^0$ .*

*Proof.* For a trigonometric polynomial with coefficients  $\mathbf{v}_{\alpha_0, \alpha_1}$ , let  $\mathbf{w} = \mathcal{P}\mathbf{v}$  and define

$$\begin{aligned} V_{\alpha_0}(x, Y_1) &= \sum_{\alpha_1} \hat{\mathbf{v}}_{\alpha_0, \alpha_1}(x) e^{i\alpha_1 \cdot Y_1}, \\ W_{\alpha_0}(x, Y_1) &= \sum_{\alpha_1} \hat{\mathbf{w}}_{\alpha_0, \alpha_1}(x) e^{i\alpha_1 \cdot Y_1} = \sum_{\alpha_1} P_{(\alpha_0, \alpha_1)} \hat{\mathbf{v}}_{\alpha_0, \alpha_1}(x) e^{i\alpha_1 \cdot Y_1}. \end{aligned}$$

Because the number of  $\alpha_1$  such that  $(\alpha_0, \alpha_1) \in \tilde{\mathcal{C}}_\Lambda$  is at most  $N$  and the projectors are bounded, there holds

$$\begin{aligned} |W_{\alpha_0}(x, Y_1)|^2 &\leq NC \sum_{\alpha_1} |\hat{\mathbf{v}}_{\alpha_0, \alpha_1}(x)|^2 \\ &= NC \lim_{R \rightarrow \infty} \frac{1}{(2R)^{m_1}} \int_{[-R, R]^{m_1}} |V_{\alpha_0}(x, Y)|^2 dY. \end{aligned}$$

Thus, by Fatou's lemma,

$$\begin{aligned} \|\mathbf{w}(\cdot, \cdot, Y_1)\|_{L^2(\omega \times \mathbb{T}^{p_0})}^2 &\leq NC \lim_{R \rightarrow \infty} \frac{1}{(2R)^{m_1}} \int_{[-R, R]^{m_1}} \|\mathbf{w}(\cdot, \cdot, Y)\|_{L^2(\omega \times \mathbb{T}^{p_0})}^2 dY \\ &\leq NC \sup_{Y \in \mathbb{R}^{m_1}} \|\mathbf{w}(\cdot, \cdot, Y)\|_{L^2(\omega \times \mathbb{T}^{p_0})}^2. \end{aligned}$$

This proves that  $\mathcal{P}$  is bounded on trigonometric polynomials for the norm of  $\mathbb{E}^0$ . The lemma follows by density.  $\square$

Using Assumption 7.16, one can repeat the proof for derivatives and  $\mathcal{P}$  acts from  $\mathbb{E}^s$  to  $\mathbb{E}^s$ . There is a similar proof for the action of  $\text{Id} - \mathcal{Q}$ , thus for  $\mathcal{Q}$ . Finally, for  $\mathbf{h} \in H^s(\omega \times \mathbb{T}^{p_0})$ , one can construct

$$(7.55) \quad \boldsymbol{\ell}\mathbf{h}(x, \theta_0, Y_1) = \sum_{\alpha_0, \alpha_1} P_{(\alpha_0, \alpha_1)} \hat{\mathbf{h}}_{\alpha_0}(x) e^{i(\theta_0 \cdot \theta_0 + \alpha_1 \cdot Y_1)} \in \mathbb{E}^s$$

using again that for each  $\alpha_0$  there are at most  $N$  nonvanishing terms in the sum. Therefore:

**Proposition 7.22.** *Under Assumptions 7.14, 7.16 and 7.20, the projectors  $\mathcal{P}$  and  $\mathcal{Q}$  are well defined in the spaces  $\mathbb{E}^s$  for all  $s \geq 0$ . In addition, the lifting operator (7.55) is bounded from  $H^\infty(\omega \times \mathbb{T}^{p_0})$  into  $\mathbb{E}^s$  and  $(\text{Id} - \mathcal{P})\boldsymbol{\ell} = 0$ .*

Using the Notations 7.18, we can repeat

**Theorem 7.23. Existence of the principal profile for the Cauchy problem under the uniform weak coherence assumption).** *Suppose that Assumptions 7.14, 7.16 and 7.20 are satisfied. For all data  $\mathbf{h} \in H^s(\omega \times \mathbb{T}^{p_0})$  with  $s > \frac{d+p}{2} + 1$ , there is  $T > 0$  such that the profile equations (7.41) (7.44) have a unique solution  $\mathbf{u}_0 \in C^0\mathbb{E}^s((\Omega \cap \{t \leq T\}))$  satisfying*

$$(7.56) \quad \mathbf{u}_0(0, x, \theta_0, Y_1) = \boldsymbol{\ell}\mathbf{h}.$$

This is proved in [JMR4] when  $\Lambda_1 = \mathbb{R}$ , but the proof extends immediately to the general case considered here. It is based on the symmetry property inherited by the profile equation, and the fact that the projectors act in the proper spaces.

### 7.3.6 Approximate solutions at order $o(1)$ in $L^2$

When  $\mathbf{u}_0$  is determined, to solve the equation with an error  $O(\varepsilon)$ , it would be sufficient to determine  $\mathbf{u}_1$  such that

$$(7.57) \quad \mathcal{L}(a, \partial_Y)\mathbf{u}_1 = \mathcal{Q}(\mathbf{F}_0 - \mathcal{L}_1(a, \partial_{t,x,Y})\mathbf{u}_0).$$

that is, formally,

$$(7.58) \quad (\text{Id} - \mathcal{P})\mathbf{u}_1 = \mathcal{R}(\mathbf{F}_0 - \mathcal{L}_1(a, \partial_{t,x,Y})\mathbf{u}_0).$$

Obviously, the principal difficulty is to show that  $\mathcal{R}$  is a bounded operator on suitable spaces of profiles. The weak coherence Assumption 7.13 permits phases  $\alpha \cdot \Psi$  such that the determinant  $\det L(a, id\alpha \cdot \Psi)$  vanishes on a set of measure 0. In this case the inverse  $R_\alpha = (L(a, id\alpha \cdot \Psi))^{-1}$  is defined almost everywhere but is certainly not bounded. However, this is sufficient to obtain approximate solutions in spaces of low regularity. To give an idea of a possible result consider the Cauchy problem for (7.20) in the framework of Theorem 7.23.

**Theorem 7.24.** *Under the Assumptions of Theorem 7.23 suppose that  $\mathbf{h} \in H^s(\omega \times \mathbb{T}^{p_0})$  is given and  $\mathbf{u}_0 \in C^0\mathbb{E}^s(\Omega)$  is a solution of the main profile equation with initial data (7.56). Then, there is a family of functions  $u^\varepsilon \in C^0L^2(\Omega)$  such that*

$$\begin{aligned} u^\varepsilon(0, x) - \mathbf{h}(x, \Phi_0(x)/\varepsilon) &\rightarrow 0 && \text{in } L^2(\omega) \\ u^\varepsilon(t, x) - \mathbf{u}_0(t, x, \Phi(t, x)/\varepsilon) &\rightarrow 0 && \text{in } C^0L^2(\Omega) \end{aligned}$$

and

$$(7.59) \quad A_0(a, \varepsilon u^\varepsilon) \partial_t u^\varepsilon + \sum_{j=1}^d \varepsilon A_j(a, \varepsilon u^\varepsilon) \partial_{x_j} u^\varepsilon + \frac{1}{\varepsilon} E(a) u^\varepsilon = F(u^\varepsilon) + f^\varepsilon$$

with  $f^\varepsilon \rightarrow 0$  in  $L^2(\Omega)$ .

*Scheme of the proof. Step 1.* The profile  $\mathbf{f}_0 := \mathbf{F}_0 - \mathcal{L}_1(a, \partial_{t,x,Y}) \mathbf{u}_0$  is known and can be approximated by trigonometric polynomials: for all  $\delta > 0$ , there is a finite sum  $\mathbf{f}'_0 = \sum e^{i\alpha Y} \hat{\mathbf{f}}_\alpha$  such that  $\|\mathbf{f}_0 - \mathbf{f}'_0\| \leq \delta$ , in the space of profiles.

*Step 2.* One can modify the Fourier coefficient  $\hat{\mathbf{f}}_\alpha$ , and assume that they are smooth and *vanish* near the negligible set where  $\det L(a, d\alpha \cdot \Psi)$  vanishes. One can then ensure that

$$(7.60) \quad \|(\mathbf{f}_0 - \mathbf{f}'_0)(\cdot, \Phi(\cdot)/\varepsilon)\|_{L^2(\Omega)} \leq 2\delta$$

*Step 3.* The conditions on  $\hat{\mathbf{f}}_\alpha$  allow one to define  $R_\alpha \hat{\mathbf{f}}_\alpha$  and therefore an approximate first corrector  $\mathbf{u}'_1$  which is a smooth trigonometric polynomial.

Consider  $u^\varepsilon(t, x) = (\mathbf{u}_0 + \varepsilon \mathbf{u}'_1)(t, x, \Phi(t, x)/\varepsilon)$ . It satisfies

$$|u^\varepsilon(t, x) - \mathbf{u}_0(t, x, \Phi(t, x)/\varepsilon)| \leq \varepsilon K(\delta)$$

and satisfies the equation up to an error  $e^\varepsilon(t, x)$  which is  $(\mathbf{f}_0 - \mathbf{f}'_0)(t, x, \Psi(t, x)/\varepsilon)$  plus terms factored by  $\varepsilon$  and involving  $\mathbf{u}_1$  and  $\mathbf{u}_0$ . Thus

$$\|e^\varepsilon\|_{L^2} \leq 2\delta + \varepsilon K(\delta)$$

Choosing  $\delta = \delta(\varepsilon) \rightarrow 0$  sufficiently slowly so that  $\varepsilon K(\delta) \rightarrow 0$  proves the theorem.  $\square$

**Remark 7.25.** Because we truncate the Fourier coefficients in Step 2, estimates of the error  $f^\varepsilon$  in  $L^2$  (or in  $L^p$  but with  $p < \infty$ ) is the best that one can expect with this method.

### 7.3.7 Strong coherence

The difficulty caused by the phases such that  $\det L(a, id\varphi)$  vanishes on a small set, is not only a technical difficulty met in the definition of the corrector  $\mathbf{u}_1$ . The examples of Section 7.1.6 show that this problem can have severe consequences. To avoid it, the following condition is natural:

**Assumption 7.26 (Strong coherence).** *We are given an additive subgroup  $\mathcal{F}$  of  $C^\infty(\Omega; \mathbb{R})$  such that the vector space  $\tilde{\mathcal{F}}$  generated by  $\mathcal{F}$  is finite dimensional over  $\mathbb{R}$ . We assume that*

- i) for all  $\varphi \in \tilde{\mathcal{F}}$  the dimension of the kernel of  $\det L(a(t, x), id\varphi(t, x))$  is independent of  $(t, x) \in \bar{\Omega}$ ,*
- ii) for all  $\varphi \in \tilde{\mathcal{F}} \setminus \{0\}$ ,  $d\varphi(t, x) \neq 0$  for all  $(t, x) \in \bar{\Omega}$ .*

**Example 7.27.** An important example is the case where  $a = \underline{a}$  is constant and  $\mathcal{F}$  is a group of planar phases  $\{\beta_0 t + \sum \beta_j x_j, \beta \in \Lambda \subset \mathbb{R}^{1+d}\}$ . We refer to [JMR2, JMR4] for other examples and a detailed discussion of this assumption. We just mention here the case of phases generated by  $\{|x| \pm t, t\}$  in the case of wave equation, or Maxwell equations, or more generally in the case of spherically invariant systems.

In particular, i) implies that each phase  $\varphi \in \tilde{\mathcal{F}}$  is either characteristic everywhere or nowhere characteristic. This assumption implies that all the partial inverses  $R_\alpha$  are well defined everywhere on  $\bar{\Omega}$ . Therefore the Step 2 in the proof of Theorem 7.24 can be eliminated.

**Remark 7.28.** When the strong coherence assumption is satisfied, the approximation results in Theorem 7.24 can be improved, so that the errors tend to zero in  $L^\infty$ .

### 7.3.8 Higher order profiles, approximate solutions at all order

Under Assumption 7.26, for each  $\alpha \in \Lambda$  the partial inverse  $R_\alpha(t, x)$  is well defined everywhere on  $\bar{\Omega}$  and smooth. However, they are not necessarily uniformly bounded with respect to  $\alpha$  since the determinant  $\det L(a, id(\alpha \cdot \Psi))$  can be very small. The summation of the Fourier series for  $(Id - \mathcal{P})\mathbf{u}_1$  can be delicate and ultimately impossible in certain cases. This is a *small divisors problem* which seems difficult to solve in the general framework of almost periodic profiles.

**Assumption 7.29 (Small divisors conditions).** *Suppose that Assumption 7.26 is satisfied and that  $\mathcal{F}$  is finitely generated. Taking a basis  $(\varphi_1, \dots, \varphi_p)$ ,*

and representing  $\Lambda = M\mathbb{Z}^p$  as in (7.23), we assume that for all  $\beta \in \mathbb{N}^{1+d}$  there are constants  $c > 0$  and  $\nu$  such that for all  $\alpha \in \mathbb{Z}^d$  and for all  $(t, x) \in \bar{\Omega}$ :

$$(7.61) \quad |\partial_{t,x}^\beta R_{M\alpha}(t, x)| \leq C(1 + |\alpha|)^\nu.$$

**Example 7.30.** When  $a(t, x) = \underline{a}$  is constant and the phases are planar, the matrices  $R_{M\alpha}$  are independent of  $(t, x)$ . The condition (7.61) splits in two conditions: there are  $c$  and  $\nu$  such that, when  $M\alpha \cdot \Psi$  is not characteristic

$$(7.62) \quad |\det L(\underline{a}, id(M\alpha \cdot \Psi))| \geq c(1 + |\alpha|)^{-\nu},$$

and when  $M\alpha \cdot \Psi$  is characteristic, the nonvanishing eigenvalues  $\lambda(\alpha)$  of  $L(\underline{a}, id(M\alpha \cdot \Psi))$  satisfy

$$(7.63) \quad |\lambda(\alpha)| \geq c(1 + |\alpha|)^{-\nu}.$$

Under Assumption 7.29, for  $\mathbf{F} \in H^\infty(\Omega \times \mathbb{T}^p)$  the Fourier series defining  $\mathcal{R}\mathbf{F}$  converges and its sum belongs to  $\mathbf{F} \in H^\infty(\Omega \times \mathbb{T}^p)$ . This immediately implies the following:

**Proposition 7.31.** *Suppose that Assumption 7.29 is satisfied. If the  $\mathbf{u}_0 \in H^\infty(\Omega \times \mathbb{T}^p)$  is given, there is  $\mathbf{u}_1 \in H^\infty(\Omega \times \mathbb{T}^p)$  satisfying (7.57).*

All the ingredient being now prepared, one can construct BKW solutions with arbitrary small residual.

**Theorem 7.32.** [Complete asymptotic solution, in the uniformly strongly coherent case, with the small divisors condition]. *Suppose that  $\Omega \subset [0, T_0] \times \mathbb{R}^d$  is contained in the domain of determinacy of the initial domain  $\omega$ . Suppose that Assumptions 7.26, 7.16 and 7.29 are satisfied. Given initial data for  $\mathcal{P}\mathbf{u}_n|_{t=0}$  in  $H^\infty(\omega \times \mathbb{T}^p)$  there is  $T > 0$  and a sequence of profiles  $\mathbf{u}_n \in H^\infty((\Omega \cap \{t \leq T\}) \times \mathbb{T}^p)$  which satisfy (7.36) (7.37).*

We refer to [JMR2] for details. In this paper it is also proved that the small divisor condition is generically satisfied. For instance, for a constant coefficient system, the condition is satisfied for almost all choice of  $p$  characteristic planar phases.

Whence one has a BKW solution at all order, one can construct approximate solutions

$$(7.64) \quad u_{app,n}^\varepsilon(t, x) = \sum_{k=0}^n \varepsilon^k \mathbf{u}_k(t, x, \Phi(t, x)/\varepsilon).$$

which are solutions of the equation with error terms of order  $O(\varepsilon^n)$ . Using Borel's Theorem, one can construct approximate solutions at infinite order.

## 7.4 Exact oscillating solutions

In this section we briefly describe several methods which aim to construct exact solutions of (7.20) such that

$$(7.65) \quad u^\varepsilon(t, x) - \mathbf{u}_0(t, x, \Psi(t, x))/\varepsilon \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

where  $\mathbf{u}_0$  is a principal profile.

### 7.4.1 Oscillating solutions with complete expansions

The easiest case is when one has a BKW solution at all order, that is under Assumption 7.29. In this case, one can construct approximate solutions  $u_{app,n}^\varepsilon(t, x)$  as in (7.64). Then one can write the equation for  $u^\varepsilon - u_{app,n}^\varepsilon$ , and if  $n$  is large enough, one can use Theorem 4.21 and obtain a result analogous to Theorem 5.33. We refer to [JMR2] for details.

### 7.4.2 Prepared data, continuation of solutions

We consider oscillations belonging to a *finitely generated* group of phases  $\mathcal{F}$  which satisfies Assumption (7.26). We look for *exact* solutions of a nondispersive equation 7.20 (with  $E = 0$ ) of the form:

$$(7.66) \quad u^\varepsilon(t, x) = \mathbf{u}^\varepsilon(t, x, \Phi(t, x)/\varepsilon)$$

with  $\mathbf{u}^\varepsilon(t, x, \theta)$  periodic in  $\theta$ , and  $\Phi$  related to  $\Psi$  as in (7.23). For  $u^\varepsilon$  to solve (7.20) it is *sufficient* that  $\mathbf{u}^\varepsilon$  solve

$$(7.67) \quad L_1(a, \varepsilon \mathbf{u}^\varepsilon, \partial_{t,x}) \mathbf{u}^\varepsilon + \frac{1}{\varepsilon} \mathcal{L}(a, \varepsilon \mathbf{u}^\varepsilon, \partial_\theta) \mathbf{u}^\varepsilon = F(\mathbf{u}^\varepsilon)$$

with

$$(7.68) \quad L_1(a, v, \partial_{t,x}) = A_0(a, v) \partial_t + \sum_{j=1}^d A_j(a, v) \partial_{x_j}$$

$$(7.69) \quad \mathcal{L}(a, v, \partial_\theta) = \sum_{j=1}^p L_1(a, v, d\varphi_j) \partial_{\theta_j}.$$

This is precisely the type of equations (4.35) of (4.45) discussed in Section 4.1.



**Remark 7.33.** It is remarkable that the *condition* (4.39) for (7.69), which is that the rank of  $L(a(t, x), d\alpha \cdot \Phi)$  is constant, *is exactly condition i) of Assumption 7.26*. This indicates that the conditions of Section 4.1.3, which came from commutation requirements, are not only technical but deeply related to the focusing effects.

In order to apply Propositions 4.18 and 4.20, we are led to supplement Assumption 7.26 with the following

**Assumption 7.34.** *Assume that there is  $\underline{\psi} \in \tilde{\mathcal{F}}$ , such that the system  $L(a, \partial)$  is hyperbolic in the direction  $d\underline{\psi}$ .*

The next result follows directly from Proposition 4.20:

**Lemma 7.35.** *Under Assumptions 7.26, 7.16 and 7.34, the singular term  $\mathcal{L}(a(t, x), 0, \partial_\theta)$  satisfies the Assumption 4.12*

Therefore Theorem 4.14 applies to the Cauchy problem for (7.67) with initial data

$$(7.70) \quad \mathbf{u}^\varepsilon|_{t=0}(x, \theta) = \mathbf{g}^\varepsilon(x, \theta)$$

(see Remark 4.15). In this context, we have to assume that the *data are prepared* in the sense of Section 4.1.2, meaning that the times derivatives  $\partial_t^k \mathbf{u}^\varepsilon|_{t=0}$  which can be computed from the equation satisfy

$$(7.71) \quad \sup_{\varepsilon \in ]0, \varepsilon_0]} \left\| \partial_t^k \mathbf{u}^\varepsilon|_{t=0} \right\|_{H^{s-k}(\omega \times \mathbb{T}^p)} < \infty.$$

**Theorem 7.36.** *Suppose that  $\Omega \subset [0, T_0] \times \mathbb{R}^d$  is contained in the domain of determinacy of the initial domain  $\omega$ . Under Assumptions 7.26 and 7.34, for  $s > 1 + (d + p)/2$  and initial data satisfying (7.71), there is  $T > 0$  such that for  $\varepsilon \in ]0, \varepsilon_0]$  the Cauchy problem (7.67) (7.70) has a solution  $\mathbf{u}^\varepsilon$  in  $CH^s((\Omega \cap \{t \leq T\}) \times \mathbb{T}^p)$ , and the family  $\{\mathbf{u}^\varepsilon\}$  is bounded in this space.*

Moreover, by Theorem 7.19, the Cauchy problem for the principal profile is well posed.

**Theorem 7.37.** *Under the assumptions of Theorem 7.36 suppose that in addition  $\mathbf{g}^\varepsilon \rightarrow \mathbf{g}_0$  in  $H^s(\omega \times \mathbb{T}^p)$ . Then,  $\mathbf{u}^\varepsilon \rightarrow \mathbf{u}_0$  in  $CH^{s'}((\Omega \cap \{t \leq T\}) \times \mathbb{T}^p)$  for all  $s' < s$ , where  $\mathbf{u}^0$  satisfies the main profile equation with initial data  $\mathbf{g}_0$ .*

*Sketch of proof.* By compactness, one can extract a subsequence such that  $\mathbf{u}^\varepsilon \rightarrow \mathbf{u}_0$  in  $CH^{s'}((\Omega \cap \{t \leq T\}) \times \mathbb{T}^p)$  for all  $s' < s$ , implying that  $\mathbf{u}_0(0, x, \theta) = \mathbf{g}_0(x, \theta)$ . Passing to the limit in the equation (7.67) multiplied by  $\varepsilon$ , implies that  $\mathcal{L}(a, \partial_\theta)\mathbf{u}_0 = 0$ , thus that  $\mathbf{u}_0$  satisfies the polarization condition  $\mathcal{P}\mathbf{u}_0 = \mathbf{u}_0$ . Next, multiplying the equation by  $\mathcal{P}$  kills the singular term, and passing to the limit implies that  $\mathbf{u}_0$  satisfies the profile equation. By uniqueness of the limit, the full family converges.  $\square$

**Conclusion 7.38.** The first theorem means that we have solved the Cauchy problem for (7.20) with initial data

$$(7.72) \quad u^\varepsilon|_{t=0}(x) = \mathbf{g}^\varepsilon(x, \Phi(0, x)/\varepsilon)$$

and found the solution under the form (7.66). The second theorem implies the convergence in (7.65). To apply this method, the difficult part is to check the preparation conditions (7.71). This method is better adapted to *continuation problems* as in Theorem 4.5 when the oscillations are created not by the initial data but by source terms. We leave to the reader to duplicate Theorem 4.5 in the present context.

The assumptions of the theorem are satisfied when  $a$  is constant and the phases are planar; it also applies when there the space  $\tilde{\mathcal{F}}$  is strongly coherent and the system is hyperbolic with constant multiplicities in the direction  $d\psi$  of Assumption 7.34 (see Proposition 4.20).

### 7.4.3 Construction of solutions. General oscillating data.

Our goal here is to solve the Cauchy problem with more general oscillating data, in particular without assuming the preparation conditions (7.71). The idea is similar to the one developed in the previous section, but different. We consider initial data of the form

$$(7.73) \quad h^\varepsilon(x) = \mathbf{h}^\varepsilon(x, \Phi_0(x)/\varepsilon)$$

where  $\mathbf{h}^\varepsilon$  is a bounded family in  $H^s(\omega \times \mathbb{T}^{p_0})$ , and we look for solutions of the form

$$(7.74) \quad u^\varepsilon(t, x) = \mathbf{u}_b^\varepsilon(t, x, \Phi_b(t, x)/\varepsilon)$$

where the  $\Phi_b(0, x) = \Phi_0(x)$ . The  $\mathbf{u}_b^\varepsilon$  are sought in  $C^0H^s(\Omega \times \mathbb{T}^{p_0})$ .

In this section, we consider a nondispersive system (7.20) with  $E = 0$ . The strategy is to use Proposition 4.20 and Theorem 4.14, thanks to the following assumption:

**Assumption 7.39.** We are given a finitely generated subgroup  $\mathcal{F}_b \subset C^\infty(\bar{\Omega})$  generating a finite dimensional linear space  $\tilde{\mathcal{F}}_b$ , such that

- i) for all  $\psi \in \tilde{\mathcal{F}}_b \setminus \{0\}$  and all  $(t, x) \in \bar{\Omega}$ ,  $d_x \psi(t, x) \neq 0$ ,
- ii) for all  $\psi \in \tilde{\mathcal{F}}_b \setminus \{0\}$   $A_0^{-1}(a(t, x))L(a(t, x), d\psi)$  has eigenvalues independent of  $(t, x)$ ; moreover, the eigenprojectors belong to a bounded set in  $C^\infty(\bar{\Omega})$  independent of  $\psi \in \tilde{\mathcal{F}}$ .

Choose a  $\mathbb{R}$ -basis  $\{\psi_1, \dots, \psi_{m_b}\}$  of  $\tilde{\mathcal{F}}_b$  and a  $\mathbb{Z}$ -basis  $\{\varphi_1, \dots, \varphi_{p_b}\}$  of  $\mathcal{F}_b$ . Let  $M_b$  be the  $m_b \times p_b$  matrix such that  $\Phi_b = {}^t M \Psi_b$  where  $\Phi_b = {}^t(\varphi_1, \dots, \varphi_{p_b})$  and  $\Psi_b = {}^t(\psi_1, \dots, \psi_{m_b})$ . We look for solutions of (7.20) of the form (7.74). The equation for  $\mathbf{u}_b^\varepsilon$  reads

$$(7.75) \quad L_1(a, \varepsilon \mathbf{u}_b^\varepsilon, \partial_{t,x}) \mathbf{u}_b^\varepsilon + \frac{1}{\varepsilon} \mathcal{L}_b(a, \varepsilon \mathbf{u}_b^\varepsilon) \mathbf{u}_b^\varepsilon = F(\mathbf{u}_b^\varepsilon).$$

with

$$(7.76) \quad \mathcal{L}_b(a, v, \partial_\theta) = \sum_{j=1}^{p_b} L_1(a, v, d\varphi_j) \partial_{\theta_j}.$$

Under Assumption 7.39 and Proposition 4.20, we can apply Theorem 4.14 and Remark 4.15, which imply the following result:

**Theorem 7.40.** Suppose that  $\Omega \subset [0, T_0] \times \mathbb{R}^d$  is contained in the domain of determinacy of the initial domain  $\omega$ . Under Assumptions 7.39 for  $s > 1 + (d + p_b)/2$  initial data  $\mathbf{h}^\varepsilon$  bounded in  $H^s(\omega \times \mathbb{T}^{p_b})$ , there are  $T > 0$  and  $\varepsilon_0 > 0$  such that for  $\varepsilon \in ]0, \varepsilon_0]$  the Cauchy problem for (7.75) with initial data

$$(7.77) \quad \mathbf{u}_b^\varepsilon|_{t=0}(x, \theta_b) = \mathbf{h}^\varepsilon(x, \theta_b).$$

has a unique solution in  $C^0 H^s((\Omega \cap \{t \leq T\}) \times \mathbb{T}^{p_b})$ .

**Remark 7.41.** It is important to note that the solutions  $u_b^\varepsilon$  have bounded derivatives in  $x$ , but not in  $t$ . In particular,  $u_b^\varepsilon$  has rapid oscillations in time.

This theorem allows one to solve the Cauchy problem for (7.20) with initial data (7.73) and initial phases  $\Phi_0(x) = \Phi_b(0, x)$ . To continue the analysis, we must make the connection with the framework elaborated in the previous sections for the Cauchy problem and the description of oscillations. The key is in the next result.

**Proposition 7.42.** *Assume that for  $\xi \neq 0$ ,  $A_0^{-1} \sum \xi_j A_j$  has at least two different eigenvalues. Suppose that  $\mathcal{F}_b \subset \tilde{\mathcal{F}}_b \subset C^\infty(\bar{\Omega})$  satisfy Assumption 7.39. Then, the space  $\tilde{\mathcal{F}}$  generated by  $\tilde{\mathcal{F}}_b$  and  $\psi_0 = t$  is strongly coherent that is satisfies Assumption 7.26.*

*Proof.* Let  $\psi = \tau t + \psi_b \in \tilde{\mathcal{F}}$ . Then the kernel of  $L(a(t, x), d\psi)$  is the kernel of  $\tau \text{Id} + A_0^{-1}(a)L(a, d\psi_b)$  and thus has constant dimension.

If  $d\psi$  vanishes at one point, then  $L(a(t, x), d\psi) = 0$  at this point, thus everywhere. Since  $A_0^{-1} \sum \xi_j A_j$  has at least two eigenvalues when  $\xi \neq 0$ , this implies that  $d_x \psi = d_x \psi_b$  vanishes everywhere, and hence  $\psi_b = 0$  and  $\psi = 0$ .  $\square$

Conversely, this clearly shows how we can use Theorem 7.40 in the context of the framework of Assumptions 7.14, 7.20 for the Cauchy problem. For the convenience of the reader we sum up the assumptions.

**Assumption 7.43.** *Suppose that:*

i)  $\mathcal{F}$  is a subgroup of a strongly coherent space  $\tilde{\mathcal{F}}$  satisfying Assumption 7.26.

ii) The mapping  $\rho : \varphi \mapsto \varphi|_{t=0}$  has a kernel in  $\tilde{\mathcal{F}}$  of dimension 1, generated by  $\underline{\psi}$ , such that  $\partial_t \underline{\psi}$  never vanishes on  $\bar{\Omega}$  and  $d\underline{\psi}$  is a direction of hyperbolicity of  $L(a, \partial_{t,x})$ . Moreover, the group  $\mathcal{F}_0 = \rho\mathcal{F}$  is finitely generated.

iii) The uniform coherence Assumption 7.16 is satisfied.

With these notations, let  $\tilde{\mathcal{F}}_b$  such that  $\tilde{\mathcal{F}} = \mathbb{R}\underline{\psi} \oplus \tilde{\mathcal{F}}_b$ . Let  $\ell$  a right inverse of  $\rho$  from  $\tilde{\mathcal{F}}_0 = \rho\tilde{\mathcal{F}}$  to  $\tilde{\mathcal{F}}_b$  and let  $\mathcal{F}_b = \ell\mathcal{F}_0$ .

**Proposition 7.44.** *If in addition, possibly after changing the time function,  $\underline{\psi}(t, x) = t$ , then  $\mathcal{F}_b \subset \tilde{\mathcal{F}}_b$  satisfy Assumption 7.39.*

*Proof.* If  $\underline{\psi} = t$ , saying that the dimension of the kernel of  $L(a(t, x), d(\tau t + \psi_b))$  is constant for all  $\tau$ , is equivalent to saying that the real eigenvalues of  $A_0^{-1}L(a(t, x), d\psi_b)$  have constant multiplicity. By hyperbolicity of  $d\underline{\psi}$  all the eigenvalues are real.  $\square$

**Remark 7.45.** In practice, when solving the Cauchy problem with oscillating initial data, one starts with a finitely generated group of phases  $\mathcal{F}_0 \subset C^\infty(\bar{\omega})$ . One constructs all the solutions of the eikonal equation with initial data in  $\mathcal{F}_0$ , and the group  $\mathcal{F}$  that they generate. The big constraint is to check that  $\mathcal{F}$  satisfies Assumption 7.26. The examples given in Section 7.1 show on one hand that this assumption is very strong, but on the

other hand that in the multi-dimensional case direct and hidden focusing effect require such strong conditions.

Under Assumption 7.43 one can solve both the Cauchy problem for data (7.73) by Theorem 7.40 and the principal profile equation by Theorem 7.23. Note that the principal profile is here a function  $\mathbf{u}_0(t, x, Y, \theta_b)$  with  $Y \in \mathbb{R}$ .  $Y$  is the placeholder for  $\psi/\varepsilon = t/\varepsilon$ .

**Theorem 7.46. (Rigorous justification of the geometric optics approximation).** *Suppose that  $\mathbf{h}^\varepsilon$  is a bounded family in  $H^s(\omega \times \mathbb{T}^{p_b})$  with  $s > 1 + (d + p_b)/2$  and that  $\mathbf{h}^\varepsilon \rightarrow \mathbf{h}^0$  in this space. Let  $\mathbf{u}_0$  be the solution of the main profile equation given by Theorem 7.23 corresponding to  $\mathbf{h}^0$ . Then the Cauchy problem for (7.20) with initial data (7.73) has a solution  $u^\varepsilon$  such that*

$$(7.78) \quad \limsup_{\varepsilon \rightarrow 0} \sup_{t, x} |u^\varepsilon(t, x) - \mathbf{u}_0(t, x, t/\varepsilon, \Phi_b(t, x)/\varepsilon)| = 0.$$

*Sketch of proof.* (See also [JMR4] and Section 7.5). As in Section 7.3.6, one can construct  $\mathbf{u}_{app}^\varepsilon(t, x, \theta_b)$  such that

$$\mathbf{u}_{app}^\varepsilon(t, x, \theta_b) - \mathbf{u}_0(t, x, t/\varepsilon, \theta_b) \rightarrow 0$$

in  $L^\infty$  and  $\mathbf{u}_{app}^\varepsilon(t, x, \theta_b)$  is an approximate solution of (7.75) in the sense that it satisfies the equation up to an error term which tends to 0 in  $C^0 H^{s-1}(\Omega \times \mathbb{T}^{p_b})$ . Then the stability property of the equation implies that  $\mathbf{u}_b^\varepsilon - \mathbf{u}_{app}^\varepsilon \rightarrow 0$  in  $C^0 H^{s-1}(\Omega \times \mathbb{T}^{p_b})$ .  $\square$

#### 7.4.4 The case of constant coefficients and planar phases

We illustrate the results of the previous sections in the important case where  $a$  is constant and the phases are planar. We consider here a quasilinear system

$$(7.79) \quad A_0(u) \partial_t u + \sum_{j=1}^d A_{x_j}(u) \partial_j u = F(u).$$

In the regime of weakly nonlinear optics, the solutions are  $u^\varepsilon = u_0 + O(\varepsilon)$  where  $u_0$  is a solution with no rapid oscillations. Here we choose  $u_0$  to be a constant, which we can take to be 0, assuming thus that  $F(0) = 0$ . We choose planar initial phases and quasi-periodic initial profiles. This yields to consider initial data of the form

$$(7.80) \quad u^\varepsilon|_{t=0}(x) = \varepsilon \mathbf{h}^\varepsilon(x, {}^t M_b x / \varepsilon)$$

where the  $\mathbf{h}^\varepsilon(x, \theta_b)$  are periodic in  $\theta_b$ , bounded in  $H^s(\mathbb{R}^d \times \mathbb{T}^{p_b})$  and  $M_b$  is a  $d \times p_b$  matrix. The initial phases are  $(M_b \alpha) \cdot x$  for  $\alpha \in \mathbb{Z}^{p_b}$ .

Looking for solutions of the form

$$(7.81) \quad u^\varepsilon(t, x) = \varepsilon \mathbf{u}_b^\varepsilon(t, x, {}^t M_b x / \varepsilon),$$

yields the equation

$$(7.82) \quad A_0(\varepsilon \mathbf{u}_b^\varepsilon) \partial_t \mathbf{u}_b^\varepsilon + \sum_{j=1}^d A_j(\varepsilon \mathbf{u}_b^\varepsilon) \partial_{x_j} \mathbf{u}_b^\varepsilon + \frac{1}{\varepsilon} \sum_{j=1}^{p_b} B_j(\varepsilon \mathbf{u}_b^\varepsilon) \partial_{\theta_j} \mathbf{u}_b^\varepsilon = G(\varepsilon \mathbf{u}_b^\varepsilon) \mathbf{u}_b^\varepsilon.$$

with symmetric matrices  $B_j$  which we do not compute explicitly here. We are in position to apply Theorem 4.2:

**Theorem 7.47.** *With assumptions as above, let  $\{\mathbf{h}^\varepsilon; \varepsilon \in ]0, \varepsilon_0]\}$  be a bounded family in  $H^s(\mathbb{R}^d \times \mathbb{T}^{p_b})$  with  $s > (d + p_b + 1)/2$ . There is  $T > 0$  such that for  $\varepsilon \in ]0, \varepsilon_0]$  the Cauchy problem for (7.82) with initial data  $\mathbf{h}_b^\varepsilon$  has a unique solution  $\mathbf{u}_b^\varepsilon \in C^0([0, T]; H^s(\mathbb{R}^d \times \mathbb{T}^{p_b}))$ ;*

The profile equation concerns profiles  $\mathbf{u}_0(t, x, Y, \theta_b)$  periodic in  $\theta_b$  and almost periodic in  $Y \in \mathbb{R}$ . The profiles satisfy the polarization condition (7.41):

$$(7.83) \quad \mathcal{P} \mathbf{u}_0 = \mathbf{u}_0$$

and the propagation equation (7.43):

$$(7.84) \quad (\text{Id} - \mathcal{Q})(L_1(\partial_t, \partial_x) \mathbf{u}_0 + \Gamma(\mathbf{u}_0, \partial_Y, \partial_{\theta_b}) \mathbf{u}_0 - G(0) \mathbf{u}_0) = 0$$

where

$$(7.85) \quad \Gamma(\mathbf{v}, \partial_Y, \partial_{\theta_b}) \mathbf{w} = \mathbf{v} \cdot \nabla_u A_0(0) \partial_Y \mathbf{w} + \sum_{j=1}^{p_b} \mathbf{u} \nabla_u B_j(0) \partial_{\theta_j} \mathbf{w}.$$

and the projectors  $\mathcal{P}$  and  $\mathcal{Q}$  are defined in Proposition 7.22. In particular,  $\mathcal{P}$  is the projector on the kernel of  $\mathcal{L}(\partial_Y, \partial_{\theta_b}) = A_0(0) \partial_Y + \sum B_j(0) \partial_{\theta_j}$ . The initial condition for  $\mathbf{u}_0$  is  $\ell \mathbf{h}$  as in (7.56).

**Theorem 7.48.** *With assumptions as in Theorem 7.47, suppose that  $\mathbf{h}^\varepsilon \rightarrow \mathbf{h}$  in  $H^s(\mathbb{R}^d \times \mathbb{T}^{p_b})$ . There is  $T' > 0$  such that*

*i) the profile equation with initial data  $\ell \mathbf{h}$  has a solution  $\mathbf{u}_0 \in C^0([0, T']; \mathbb{E}^s)$  with  $\mathbb{E}^s = C_{pp}^0(\mathbb{R}; H^s(\mathbb{R}^d \times \mathbb{T}^{p_b}))$ ;*

*ii) there holds*

$$(7.86) \quad \lim_{\varepsilon \rightarrow 0} \sup_{0 \leq t \leq \min\{T, T'\}} \|\mathbf{u}_b^\varepsilon(t, \cdot, \cdot) - \mathbf{u}_0(t, \cdot, t/\varepsilon, \cdot)\|_{H^{s-1}(\mathbb{R}^d \times \mathbb{T}^{p_b})} = 0.$$

## 7.5 Asymptotics of exact solutions

In this section we present the principles of different methods which can be used to prove that exact solutions have an asymptotic expansion.

### 7.5.1 The method of simultaneous approximations

*The principle of the method.* On one hand, there is an equation for exact solutions

$$(7.87) \quad \partial_t u^\varepsilon = M^\varepsilon(u^\varepsilon, \partial)u^\varepsilon$$

and on the other hand there is another equation for the profiles

$$(7.88) \quad \partial_t \mathbf{u} = \mathcal{M}(\mathbf{u}, \partial)\mathbf{u}.$$

The equations are supplemented with initial conditions. The profile  $\mathbf{u}$  may depend on more variables than the functions  $u^\varepsilon$ . Moreover there is a law  $\mathbf{u} \mapsto U^\varepsilon(\mathbf{u})$  which assigns a family of functions to a profile. Both equations are solved by iteration:

$$(7.89) \quad \partial_t u_{\nu+1}^\varepsilon = M^\varepsilon(u_\nu^\varepsilon, \partial)u_{\nu+1}^\varepsilon,$$

$$(7.90) \quad \partial_t \mathbf{u}_{\nu+1} = \mathcal{M}(\mathbf{u}_\nu, \partial)\mathbf{u}_{\nu+1}.$$

The principle of the method relies on the following elementary lemma:

**Lemma 7.49.** *Suppose that*

- 1) *the sequence  $u_\nu^\varepsilon$  converges to  $u^\varepsilon$ , uniformly in  $\varepsilon$  in a space  $X$ ;*
- 2) *the sequence  $\mathbf{u}_\nu$  converges to  $\mathbf{u}$  in a space  $\mathcal{X}$ ;*
- 3) *the mapping  $\mathbf{u} \mapsto \{U^\varepsilon(\mathbf{u}), \varepsilon \in ]0, \varepsilon_0]\}$  maps continuously  $\mathcal{X}$  to the space of bounded families in  $X$  equipped with the topology of uniform convergence;*
- 4) *for the initial term  $\nu = 0$  of the induction,  $u_0^\varepsilon - U^\varepsilon(\mathbf{u}_0) \rightarrow 0$  in  $X$ ,*
- 5) *for each  $\nu \geq 0$ , assuming that  $u_\nu^\varepsilon - U^\varepsilon(\mathbf{u}_\nu) \rightarrow 0$  in  $X$ , then  $u_{\nu+1}^\varepsilon - U^\varepsilon(\mathbf{u}_{\nu+1}) \rightarrow 0$  in  $X$ .*

*Then  $u^\varepsilon - U^\varepsilon(\mathbf{u}) \rightarrow 0$  in  $X$ .*

*Proof.* By 4) and 5), the property 6),  $u_\nu^\varepsilon - U^\varepsilon(\mathbf{u}_\nu) \rightarrow 0$  in  $X$ , is true for all  $\nu$ . By 1) for  $\delta > 0$ , there is  $\nu_0$  such that for all  $\nu \geq \nu_0$  and all  $\varepsilon \in ]0, \varepsilon_0]$ :

$$\|u^\varepsilon - u_\nu^\varepsilon\|_X \leq \delta.$$

Moreover, by 2) and 3), for  $\nu$  large enough, there holds

$$\|U^\varepsilon(\mathbf{u}) - U^\varepsilon(\mathbf{u}_\nu)\|_X \leq \delta.$$

Thus, choosing  $\nu$  such that both property are satisfied and using 6), we see that

$$\limsup_{\varepsilon \rightarrow 0} \|u^\varepsilon - U^\varepsilon(\mathbf{u})\|_X \leq 2\delta.$$

and the lemma follows.  $\square$

Of course, this lemma is just a general principle which can (and often must) be adapted to particular circumstances of cases under examinations.

**Remark 7.50.** In this strategy, the main step is the fifth, which amounts to proving that for a *linear equation* whose coefficients have given asymptotic expansions, the solution has an asymptotic expansion.

*Application to the oscillating Cauchy problem.*

This method has been applied in  $d = 1$  with spaces  $X$  of  $C^1$  functions and spaces of profiles  $\mathcal{X}$  of class  $C^1$  in all the variables and almost periodic in the fast variables (see [JMR10] and section 7.6.1 below).

It has been applied also in the multidimensional case  $d > 1$ , for coherent almost periodic oscillations in the Wiener Algebra and semi-linear equations (see Theorem 7.19 for the context and [JMR3] for precise results.)

We now briefly review how it applies to prove the second part of Theorem 7.46 as in [JMR4]. For simplicity, we consider the case of equation (7.79) and planar phases as in Theorem 7.48. The equations for exact solutions are given at (7.82) and for the profiles at (7.83) (7.84). The iterative scheme leads to linear problems

$$(7.91) \quad A_0(\varepsilon \mathbf{v}_b^\varepsilon) \partial_t \mathbf{u}_b^\varepsilon + \sum_{j=1}^d A_j(\varepsilon \mathbf{v}_b^\varepsilon) \partial_{x_j} \mathbf{u}_b^\varepsilon + \frac{1}{\varepsilon} \sum_{j=1}^{p_b} B_j(\varepsilon \mathbf{v}_b^\varepsilon) \partial_{\theta_j} \mathbf{u}_b^\varepsilon = G(\varepsilon \mathbf{v}_b^\varepsilon) \mathbf{v}_b^\varepsilon.$$

and

$$(7.92) \quad \begin{cases} \mathcal{P} \mathbf{u} = \mathbf{u} \\ (\text{Id} - \mathcal{Q})(L_1(\partial_t, \partial_x) \mathbf{u} + \Gamma(\mathbf{v}, \partial_Y, \partial_{\theta_b}) \mathbf{u} - G(0) \mathbf{v}) = 0. \end{cases}$$

The key point is to prove that if the  $\mathbf{v}^\varepsilon$  are bounded in  $C^0([0, T]; H^s(\mathbb{R}^d \times \mathbb{T}^{p_b}))$  if  $\mathbf{u} \in C^0([0, T], \mathbb{E}^s(\mathbb{R} \times \mathbb{R}^d \times \mathbb{T}^{p_b}))$  and if  $\mathbf{v}_b^\varepsilon(t, x, \theta_b) - \mathbf{v}(t, x, t/\varepsilon, \theta_b)$  tends to 0 in  $C^0([0, T]; H^{s-1}(\mathbb{R}^d \times \mathbb{T}^{p_b}))$ , then  $\mathbf{u}_b^\varepsilon(t, x, \theta_b) - \mathbf{u}(t, x, t/\varepsilon, \theta_b)$  tends to 0 in  $C^0([0, T]; H^{s-1}(\mathbb{R}^d \times \mathbb{T}^{p_b}))$ . This means that the problem of the asymptotic description of solutions is completely transferred to linear problems. Using approximation, one can restrict the analysis to the case where  $\mathbf{v}$  is a trigonometric polynomial, and in this case use BKW solutions.



### 7.5.2 The filtering method

This method has been introduced in [KM1, KM2], with motivations coming from fluid mechanics in particular with applications to the analysis of the low mach limit of Euler equations. It has been next extended to more general contexts such as (7.82) in [Sch1, Sch2, Sch3, Sch4] and then adapted to various other problems by many authors.

*The principle of the method.* The method applies to evolution equations where two time scales (at least) are present:

$$(7.93) \quad \partial_t u^\varepsilon + \frac{1}{\varepsilon} G u^\varepsilon + \Phi(u^\varepsilon) = 0, \quad u^\varepsilon|_{t=0} = h^\varepsilon,$$

where  $G$  generates a  $C^0$  group  $e^{-tG}$  which is uniformly bounded in a Hilbert space  $H_1$  and also uniformly bounded in  $H \subset H_1$ . Moreover,  $\Phi$  is a bounded family of continuous functionals from  $H$  to  $H_1$  which map bounded sets in  $H$  to bounded sets in  $H_1$ .

*It is assumed* that there is a preliminary construction which provides us with a family of solutions  $u^\varepsilon$  bounded in  $C^0([0, T]; H)$ . The question is to describe the asymptotic behavior of  $u^\varepsilon$ . Assume that the embedding  $H \subset H_1$  is compact and  $h^\varepsilon \rightarrow h$  in  $H$ . The question is to describe the fast oscillations in time of  $u^\varepsilon$ . The idea is very classical in the study of evolution systems, and can be related to themes like adiabatic limits or the construction of wave operators in scattering theory: set

$$(7.94) \quad v^\varepsilon(t) = e^{\frac{t}{\varepsilon}G} u^\varepsilon(t) \quad \Leftrightarrow \quad u^\varepsilon(t) = e^{-\frac{t}{\varepsilon}G} v^\varepsilon(t).$$

Thus  $v^\varepsilon$  is bounded in  $C^0([0, T]; H)$  and

$$(7.95) \quad \partial_t v^\varepsilon(t) = -e^{\frac{t}{\varepsilon}G} \Phi_\varepsilon(u^\varepsilon(t))$$

is bounded in  $C^0([0, T]; H_1)$ . Therefore,  $v^\varepsilon$  is compact in  $C^0([0, T]; H_1)$ , and up to the extraction of a subsequence

$$(7.96) \quad v^\varepsilon \rightarrow v \quad \text{in } C^0([0, T]; H_1), \quad v \in L^\infty([0, T]; H).$$

The asymptotic behavior of  $u^\varepsilon$  is given by

$$(7.97) \quad u^\varepsilon(t) - e^{-\frac{t}{\varepsilon}G} v(t) \rightarrow 0 \quad \text{in } C^0([0, T]; H_1).$$

Two scales of time are present in main term  $e^{-\frac{t}{\varepsilon}G} v(t)$ , yielding to introduce the profile

$$(7.98) \quad \mathbf{u}(t, Y) := e^{-YG} v(t)$$

so that

$$(7.99) \quad u^\varepsilon(t) = \mathbf{u}(t, t/\varepsilon).$$

By construction,  $\mathbf{u}$  satisfies the ‘‘polarization’’ condition

$$(7.100) \quad (\partial_Y + G)\mathbf{u}(t, Y) = 0.$$

Note that in (7.98),  $v$  appears as a parametrization of the profiles. The slow evolution of the profile is determined from the slow evolution of  $v$ , which is obtained by *passing to the weak limit* in the equation (7.95). Because we already know the approximation (7.97), the slow evolution reads

$$(7.101) \quad \partial_t v + \underline{\Phi}(v(t)) = 0$$

where  $\underline{\Phi}$  is the nonlinear operator defined by

$$(7.102) \quad \underline{\Phi}(v) = w - \lim_{\varepsilon \rightarrow 0} e^{\frac{t}{\varepsilon}G} \Phi(e^{-\frac{t}{\varepsilon}G} v)$$

*Application to the oscillatory Cauchy problem.*

We sketch the computation in the case of equation (7.82) and planar phases as in Theorem 7.48. The equation is of the form (7.93) with

$$G = \sum_{j=1}^{p_b} A_0^{-1} B_j(0) \partial_{\theta_j}$$

and a nonsingular term  $\Phi_\varepsilon$  given by:

$$\begin{aligned} A_0(0)\Phi_\varepsilon(\mathbf{u}) &= (A_0(\varepsilon\mathbf{u}) - A_0(0))\partial_t \mathbf{u} + \sum A_j(\varepsilon\mathbf{u})\partial_{x_j} \mathbf{u} \\ &\quad + \sum \frac{1}{\varepsilon} (B_j(\varepsilon\mathbf{u}) - B_j(0))\partial_{\theta_j} \mathbf{u} - G(\varepsilon\mathbf{u})\mathbf{u}. \end{aligned}$$

The fast evolution is made explicit in Fourier series in  $\theta_b$ :

$$(7.103) \quad e^{-YG} \left( \sum_{\alpha} \hat{v}_\alpha e^{i\alpha\theta_b} \right) = \sum_{\tau, \alpha} e^{i(\tau Y + \alpha\theta_b)} P_{\tau, \alpha} \hat{v}_\alpha$$

where the  $P_{\tau, \alpha}$  are the spectral projectors on  $\ker(\tau \text{Id} + \sum \alpha_j A_0^{-1} B_j)$ . In particular, Proposition 7.22 implies that  $e^{-tG}$  is bounded in the spaces  $H^s(\mathbb{R}^d \times \mathbb{T}^{p_b})$ .

In accordance with Theorem 7.47, consider a family of solutions  $\mathbf{u}_b^\varepsilon$  of (6.60) bounded in  $C^0([0, T]; H^s(\mathbb{R}^d \times \mathbb{T}^{p_b}))$ . Then  $\mathbf{v}^\varepsilon = e^{\frac{t}{\varepsilon}G} \mathbf{u}_b^\varepsilon(t)$  is bounded

in  $C^0([0, T]; H^s)$  and  $\partial_t \mathbf{v}^\varepsilon$  is bounded in  $C^0([0, T], H^{s-1})$ , so that extracting a subsequence,

$$\mathbf{v}^\varepsilon \rightarrow \mathbf{v} \quad \text{in } C^0([0, T], H_{loc}^{s'})$$

for all  $s' < s$ . This implies that

$$(7.104) \quad \left\| u_b^\varepsilon(t, \cdot) - \mathbf{u}_0(t, t/\varepsilon, \cdot) \right\|_{H_{loc}^{s'}} \rightarrow 0$$

where the profile  $\mathbf{u}_0$  is defined by

$$(7.105) \quad \mathbf{u}_0(t, Y, \cdot) = e^{-Y G} \mathbf{v}(t, \cdot).$$

The explicit form (7.103) of the evolution implies that  $\mathbf{u}_0$  belongs to a class  $C^0([0, T]; \mathbb{E}^{s'})$  (see Proposition 7.22) and satisfies the polarization condition  $\mathcal{P}\mathbf{u}_0 = \mathbf{u}_0$ .

The evolution of  $\mathbf{v}$  is obtained by passing to the limit in the equation (7.95) for  $\partial_t \mathbf{v}^\varepsilon$ . Using (7.104), the problem is reduced to the following computation:

$$(7.106) \quad \partial_t \mathbf{v}(t, \cdot) = -w - \lim_{\varepsilon \rightarrow 0} e^{\frac{t}{\varepsilon} G} \Phi_0(\mathbf{u}_0(t, t/\varepsilon, \cdot))$$

where

$$\Phi_0(\mathbf{u}) = \sum A_j(0) \partial_{x_j} \mathbf{u} + \Gamma(\mathbf{u}, \partial_{Y, \theta}) \mathbf{u} - G(0) \mathbf{u}.$$

and  $\Gamma$  is given by (7.85).

**Lemma 7.51.** *If  $\mathbf{f}$  is a profile in  $C^0([0, T]; \mathbb{E}^s)$ , then  $e^{\frac{t}{\varepsilon} G} \mathbf{f}(t, t/\varepsilon, \cdot)$  converges in the sense of distributions to  $\mathcal{P}\mathbf{f}(t, 0, \cdot)$ .*

*Proof.* It is sufficient to prove the convergence when  $\mathbf{f}$  is a trigonometric polynomial, that is a finite sum

$$\mathbf{f}(t, Y, x, \theta_b) = \sum_{\tau, \alpha} \hat{\mathbf{f}}_{\tau, \alpha}(t, x) e^{i\tau Y + \alpha \theta_b}.$$

In this case

$$e^{\frac{t}{\varepsilon} G} \mathbf{f}(t, t/\varepsilon, x, \theta_b) = \sum_{\lambda, \tau \alpha} e^{i(\tau - \lambda)t/\varepsilon} P_{\lambda, \alpha} \hat{\mathbf{f}}_{\tau, \alpha}(t, x) e^{i\alpha \theta_b}.$$

converges weakly to

$$\sum_{\tau \alpha} P_{\tau, \alpha} \hat{\mathbf{f}}_{\tau, \alpha}(t, x) e^{i\alpha \theta_b} = \mathcal{P}\mathbf{f}(t, 0, x, \theta_b).$$

□

**Corollary 7.52.** *The main profile satisfies the propagation equation*

$$\partial_t \mathbf{u}_0 + \mathcal{P}\Phi_0(\mathbf{u}_0) = 0,$$

that is the profile equation (7.84).

*Proof.* Let  $\mathbf{f} = \Phi_0(\mathbf{u}_0)$ . Then  $\partial_t \mathbf{v}(t, \cdot) = -\mathcal{P}\mathbf{f}(t, 0, \cdot)$  and

$$\partial_t \mathbf{u}_0(t, Y, \cdot) = e^{-YG} \partial_t \mathbf{v}(t, \cdot) = -e^{-YG} \mathcal{P}\mathbf{f}(t, 0, \cdot) = -\mathcal{P}\mathbf{f}(t, Y, \cdot)$$

where the last identity follows immediately from (7.103).  $\square$

**Remark 7.53.** By uniqueness of the profile equation, it follows that the limit is independent of the extracted subsequence so that the full sequence  $\mathbf{v}^\varepsilon$  converges and the approximation (7.104) holds for the complete family as  $\varepsilon \rightarrow 0$ .

## 7.6 Further examples

### 7.6.1 One dimensional resonant expansions.

In the one dimensional case  $d = 1$ , strong coherence assumptions such as Assumption 7.26 are not necessary. Weak coherence is sufficient for the construction of the main profiles (as in Theorem 7.23). The strong coherence conditions were used to construct solutions in Sobolev spaces. In 1-D, one can use different methods such as integration along characteristics and construct solutions in  $L^\infty$  (or  $W^{1,\infty}$  is the quasilinear case) avoiding the difficulty of commutations. Moreover, there is a special way to describe the interaction operators in 1-D which is interesting in itself.

For simplicity, we consider strictly hyperbolic nondispersive, semilinear systems, refereeing to [HuKe1, HMR, JMR10, Kal] for the quasilinear case. After a diagonalization and a change of dependent variables, the equations are

$$(7.107) \quad X_k u_k := \partial_t u_k + \lambda_k(t, x) \partial_x u_k = f_k(t, x, u_1, \dots, u_N),$$

for  $k \in \{1, \dots, N\}$ . The  $\lambda_k$  are real and  $\lambda_1 < \lambda_2 < \lambda_3$ . The  $f_k$  are smooth functions of the variables  $(t, x, u) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ .

The initial oscillations are associated to phases in a group  $\mathcal{F}_0 \subset C^\infty(\bar{\omega})$  or in the vector space  $\tilde{\mathcal{F}}_0$  they span, where  $\omega$  is a bounded interval in  $\mathbb{R}$ . We assume that  $\tilde{\mathcal{F}}_0$  is finite dimensional and that for all  $\varphi \in \tilde{\mathcal{F}}_0$ , the derivative

$\partial_x \varphi$  is different from zero almost everywhere. Thus the initial data will be or the form

$$(7.108) \quad u_k^\varepsilon(0, x) = h_k^\varepsilon(x) \sim \mathbf{h}_k(x, \Psi_0(x)/\varepsilon).$$

where we have chosen a basis  $\{\psi_{0,1}, \dots, \psi_{0,m}\}$  and  $\Psi_0 \in C^\infty(\bar{\omega}; \mathbb{R}^m)$  is the function with components  $\psi_{0,j}$ .

For all  $k$  we consider the solutions of the  $k$ -th eikonal equation

$$(7.109) \quad X_k \varphi = 0, \quad \varphi|_{t=0} \in \mathcal{F}_0.$$

It is one of the main feature of the 1D case, that the eikonal equation factors as a product of linear equations. In particular, the set of solutions of (7.109) is a group  $\mathcal{F}^k$ , isomorphic to  $\mathcal{F}_0$ , and all the phases are defined and smooth on  $\bar{\Omega}$  the domain of determinacy of  $\bar{\omega}$ . Similarly, the vector space  $\tilde{\mathcal{F}}^k$  spanned by  $\mathcal{F}^k$  is the set of solutions of  $X_k \varphi = 0$  with initial data in the vector space  $\tilde{\mathcal{F}}_0$ . They initial phase  $\psi_{0,j}$  are propagated by  $X_k$ , defining bases  $\{\psi_{k,1}, \dots, \psi_{k,m}\}$  of  $\tilde{\mathcal{F}}_k$  and functions  $\Psi_k \in C^\infty(\bar{\Omega}; \mathbb{R}^m)$  satisfying

$$X_k \Psi^k = 0, \quad \Psi^k|_{t=0} = \Psi_0.$$

The complete group of phases which are obtained by nonlinear interaction is  $\mathcal{F} = \mathcal{F}^1 + \dots + \mathcal{F}^N$ , generating the vector space  $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}^1 + \dots + \tilde{\mathcal{F}}^N$ . Note that, by construction,

$$(7.110) \quad \{\varphi \in \tilde{\mathcal{F}}; X_k \varphi = 0\} = \tilde{\mathcal{F}}^k.$$

We assume that  $\tilde{\mathcal{F}}$  satisfies the weak coherence assumption, which here means that

$$(7.111) \quad \forall \varphi \in \tilde{\mathcal{F}} \setminus \tilde{\mathcal{F}}^k : X_k \varphi \neq 0, \text{ a.e. on } \Omega.$$

To follow the general discussion in the previous section, we should introduce a basis of  $\tilde{\mathcal{F}}$ , thus a function  $\Psi$  and accordingly represent the solution as in (7.22). This can be done, but does not give the simplest and most natural description. Indeed, the diagonal form of the system and (7.110) imply that the projector  $\mathcal{P}$  has a simple form : the oscillation with phase  $\varphi \neq 0$  in  $\mathcal{F}$  is characteristic if it belongs to one of the  $\mathcal{F}^k$ , and then the projector  $P$  is simply the projector on the  $k$ -th axis (we admit here, or assume, that the index  $k$  is necessarily unique). Thus, the polarization condition (7.41) implies that the  $k$ -th component of the main term depends only on the phases in  $\tilde{\mathcal{F}}^k$  so that is main oscillation is of the form

$$(7.112) \quad u_k^\varepsilon(t, x) \sim \mathbf{u}_k(t, x, \Psi_k(t, x)/\varepsilon)$$

with  $\mathbf{u}_k(t, x, Y)$  periodic / quasi-periodic / almost-periodic in  $Y \in \mathbb{R}^m$ .

One should consider different copies  $Y^k \in \mathbb{R}^m$  of the variable  $Y$ , one for each component. However, *when replaced by  $\Psi^k/\varepsilon$ , the variables  $Y^k$  are not necessarily independent*. This is precisely the phenomenon of *resonance* or *phase matching*. As a result, the description of the projector  $\mathcal{P}$  is not immediate. This is the price of using the description (7.112).

**Definition 7.54.** *A resonance is an  $N$ -tuple  $(\varphi_1, \dots, \varphi_N)$  of functions such that  $X_k \varphi_k = 0$  for all  $k$  and  $\sum d\varphi_k = 0$ . The resonance is trivial when all the  $d\varphi_k$  are identically zero.*

Remark that the definition does not depend on the vector fields themselves but on the foliations they defined. The data of  $N$  foliations by curves in  $\mathbb{R}^2$  is called an  $N$ -web in differential geometry, and resonances correspond to abelian relations on webs (see [Poi, BlBo]). The existence of resonances is a rare phenomenon : *for an  $N$ -web, the dimension of the space of resonances (modulo constants functions) is equal at most  $(N - 1)(N - 2)/2$* . Moreover, for “generic” vector fields  $X_k$ , there are no nontrivial resonances. Therefore, in order to have resonances, first the vector fields  $X_k$  must be suitably chosen and second, the phases  $\varphi_k$  must also to be chosen carefully. The case  $N = 3$  is well illustrated by the example of Section 7.1.2. However, resonance is a very important phenomenon. Constant coefficient vector fields  $X_k = \partial_t - \lambda_k \partial_x$ , are resonant with maximal dimension. We refer to [JMR1] for a more general discussion of resonances and examples.

**Notations 7.55.**

1. We denote by  $\vec{\Psi}$  the function  $(\Psi^1, \dots, \Psi^N) \in C^\infty(\bar{\Omega}; (\mathbb{R}^m)^N)$ .
2. The resonances (within the set of phases under consideration) are described by the vector space  $R \subset (\mathbb{R}^m)^M$  of the  $\vec{\alpha} = (\alpha^1, \dots, \alpha^N)$  such that  $\vec{\alpha} \cdot \vec{\Psi} := \sum \alpha^k \cdot \Psi^k = 0$ .
3. The combinations of phases which are characteristic for  $X_k$  are described by the vector space  $R_k \subset (\mathbb{R}^m)^M$  of the  $\vec{\alpha}$  such that  $\vec{\alpha} \cdot \vec{\Psi} \in \tilde{\mathcal{F}}_k$ .  $R_k$  is the sum of  $R$  and  $A_k := \{\vec{\alpha} : \forall l \neq k, \alpha^l = 0\}$ . This sum is direct since  $\alpha^k \cdot \Psi^k = 0$  only if  $\alpha^k = 0$ . We denote by  $\rho_k$  the mapping

$$(7.113) \quad \rho_k : R_k \mapsto \mathbb{R}^m \quad \text{such that} \quad \vec{\alpha} \cdot \vec{\Psi} = \rho_k(\vec{\alpha}) \cdot \Psi^k.$$

4. We denote by  $\vec{Y} = (Y^1, \dots, Y^N)$  the variable in  $(\mathbb{R}^m)^N$ , dual to the  $\vec{\alpha}$ , and  $\vec{\alpha} \cdot \vec{Y} = \sum \alpha^k \cdot Y^k$ . The natural vector space for the placeholder of  $\vec{\Psi}$  in the profiles is  $V = R^\perp = \{\vec{Y} : \forall \vec{\alpha} \in R, \vec{\alpha} \cdot \vec{Y} = 0\}$ .
5. We denote by  $V_k \subset V$  the orthogonal of  $R_k$ .

6. The projection  $\pi_k : \vec{Y} \mapsto Y^k$  is surjective from  $V$  to  $\mathbb{R}^m$ , since if  $\alpha^k \cdot Y^k = 0$  for all  $\vec{Y} \in V$ , this implies that  $\alpha^k \cdot \Psi^k = 0$  and thus  $\alpha^k = 0$ . Its kernel  $\{\vec{Y} \in V : Y^k = 0\}$  is  $V_k$ . Therefore there is a natural isomorphism

$$(7.114) \quad V/V_k \leftrightarrow \mathbb{R}^m.$$

Equivalently, one can choose a lifting linear operator  $\pi_k^{(-1)}$  from  $\mathbb{R}^m$  to  $V$  such that  $\pi_k \pi_k^{(-1)} = \text{Id}$ . In particular, with (7.113), there holds

$$(7.115) \quad \forall \vec{\alpha} \in R_k, \forall Y^k \in \mathbb{R}^m, \quad \vec{\alpha} \cdot \pi_k^{(-1)} Y^k = \rho_k(\vec{\alpha}) \cdot Y^k.$$

With these notations, we can now describe the projectors. We consider profiles *which are almost periodic* : given a finite dimensional space  $E$ , denote by  $C_{pp}^0(\bar{\Omega} \times E)$  the space of functions  $\mathbf{u}(t, x, Y)$  which are almost periodic in  $Y \in E$ , that is the closure in  $L^\infty$  of finite sums  $\sum \hat{\mathbf{u}}_\alpha(t, x) e^{i\alpha \cdot Y}$ . Give such profiles  $\mathbf{u}_k(t, x, Y^k)$  which are almost periodic in  $Y^k$ , the nonlinear coupling  $f_k(t, x, \mathbf{u}_1, \dots, \mathbf{u}_N)$  appears as a function

$$(7.116) \quad \mathbf{f}(t, x, \vec{Y}), \quad (t, x) \in \Omega, \vec{Y} \in V,$$

which is almost periodic in  $\vec{Y}$ . Expanding  $\mathbf{f}_k$  in Fourier series, the projector  $\mathbf{E}_k$  must keep [resp. eliminate] the exponentials  $e^{i\vec{\alpha} \cdot \vec{Y}}$  when  $\vec{\alpha} \cdot \vec{Y} \in \tilde{\mathcal{F}}^k$ , [resp.  $\notin \tilde{\mathcal{F}}^k$ ]. Translated to the  $Y$  variables, this means that

$$(7.117) \quad \begin{cases} \mathbf{E}_k(e^{i\vec{\alpha} \cdot \vec{Y}}) = 0 & \text{when } \vec{\alpha} \notin R_k, \\ \mathbf{E}_k(e^{i\vec{\alpha} \cdot \vec{Y}}) = e^{i\rho_k(\vec{\alpha}) \cdot Y^k} & \text{when } \vec{\alpha} \in R_k, \end{cases}$$

This operator is linked to the *averaging operator with respect to  $V_k$*  acting on almost periodic functions :

$$(7.118) \quad \mathbf{M}_k \mathbf{f}(\vec{Y}) = \lim_{T \rightarrow \infty} \frac{1}{\text{vol}(TB_k)} \int_{TB_k} \mathbf{f}(\vec{Y} + \vec{Y}') d\vec{Y}'$$

where  $B_k$  is any open ball (or cube) in  $V_k$  and  $d\vec{Y}'$  a Lebesgue measure on  $V_k$ , which also measures the volume in the denominator. This average is invariant by translations in  $V_k$ , so that  $\mathbf{M}_k \mathbf{f}$  can be seen as a function on  $V/V_k$ . Using (7.114), it can be seen as a function of  $Y^k \in \mathbb{R}^m$ . Equivalently, using the lifting  $\pi_k^{(-1)}$ , we set

$$(7.119) \quad \mathbf{E}_k \mathbf{f}(Y^k) = \mathbf{M}_k \mathbf{f}(\pi_k^{(-1)} Y^k)$$

which is independent of the particular choice of  $\pi_k^{(-1)}$ .

**Proposition 7.56.** *Using the notations above, in the representation (7.112) of the oscillations, the profile equations read:*

$$(7.120) \quad \begin{aligned} (\partial_t + \lambda_k(t, x)\partial_x)\mathbf{u}_k(t, x, Y^k) \\ = \mathbf{E}_k f_k(t, x, \mathbf{u}_1(t, x, Y^1), \dots, \mathbf{u}_N(t, x, Y^N)) \end{aligned}$$

for  $k = 1, \dots, N$ , with initial data

$$(7.121) \quad \mathbf{u}_k(0, x, Y) = \mathbf{h}_k(x, Y).$$

**Example 7.57.** Consider the case  $N = 3$  with one initial phase  $\psi_0(x)$ . Let  $\psi_k$  be the solution of  $X_k\psi_k = 0$  with initial data equal to  $\psi_0$ . The resonant set is  $R = \{\alpha \in \mathbb{R}^3 : \sum_k \alpha_k \psi_k = 0\}$ . There are two cases:

- either  $R = \{0\}$ ; then averaging operators are defined by

$$(\mathbf{E}_1 F)(t, x, \theta_1) := \lim_{T \rightarrow +\infty} \frac{1}{T^2} \int_0^T \int_0^T F(\theta_1, \theta_2, \theta_3) d\theta_2 d\theta_3,$$

with similar definitions for  $\mathbf{E}_2$  and  $\mathbf{E}_3$ .

- or there is a nontrivial resonance  $\alpha \in R \setminus \{0\}$ , and

$$(\mathbf{E}_1 F)(t, x, \theta_1) := \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T F\left(\theta_1, -\frac{\alpha_1 \theta_1 + \alpha_3 \sigma}{\alpha_2}, \sigma\right) d\sigma,$$

with similar formulas for  $k = 2$  and  $k = 3$ . Since the vector fields are pairwise linearly independent, all three components  $\alpha_k$  of  $\alpha \in R \setminus \{0\}$  are not equal to 0. Thus the right hand side makes sense.

**Theorem 7.58** ([JMR10]). *Suppose that  $\Omega$  is contained in the domain of determinacy of the interval  $\omega \subset \mathbb{R}$ . Suppose that  $\mathbf{h}_k \in C_{pp}^0(\omega \times \mathbb{R}^m)$  and that  $\{u_{0,k}^\varepsilon\}_{\varepsilon \in ]0,1]}$  is a bounded family in  $L^\infty(\omega)$  such that*

$$(7.122) \quad u_{0,k}^\varepsilon(\cdot) - \mathbf{h}_k(\cdot, \Psi_0(\cdot)/\varepsilon) \rightarrow 0 \quad \text{in } L^1(\omega), \quad \text{as } \varepsilon \rightarrow 0.$$

Then there is  $T > 0$  such that :

i) for all  $\varepsilon \in ]0, 1]$ , the Cauchy problem for (7.107) with Cauchy data  $u_{0,k}^\varepsilon$  as a unique solution  $u^\varepsilon = (u_1^\varepsilon, \dots, u_N^\varepsilon)$  in  $C^0(\overline{\Omega_T})$ , where  $\Omega_T := \Omega \cap \{0 < t < T\}$ ,

ii) the profile equations (7.120) (7.121) have a unique solution  $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_N)$  with  $\mathbf{u}_k \in C_{pp}^0(\Omega_T \times \mathbb{R}^m)$ ,

iii)

$$(7.123) \quad u_k^\varepsilon(\cdot) - \mathbf{u}_k(\cdot, \Psi_k(\cdot)/\varepsilon) \rightarrow 0 \quad \text{in } L^1(\Omega_T), \quad \text{as } \varepsilon \rightarrow 0.$$



**Remarks 7.59.** In (7.122) (7.123) the  $L^1$  norm can be replaced by  $L^p$  norms for all finite  $p$ . There is a similar result in  $L^\infty$ , if the condition (7.111) is slightly reinforced, requiring that  $X_k\varphi \neq 0$  a.e. on all integral curves of  $X_k$  when  $\varphi \in \tilde{\mathcal{F}} \setminus \tilde{\mathcal{F}}_k$ .

2. A formal derivation of the profile equations is given in [HuKe1, MaRo, HMR]. Partial rigorous justifications were previously given by [Tar, Jol, Kal].

3. The results quoted above concern continuous solutions of semilinear first order systems, or  $C^1$  solutions in the quasilinear case. For weak solutions of quasilinear systems of conservation laws, which may present shocks, the validity of weakly nonlinear geometric optics is now proved in a general setting in [Che3] generalizing a result of [Sch4] (see also [DiPM]) The interaction of (strong) shock waves or contact discontinuities and small amplitude oscillations is described in [Cor1, Cor2, CoGu].

## 7.6.2 Generic phase interaction for dispersive equations

We give here a more explicit form of the profile equation (7.44) in cases that are important in applications to nonlinear optics.

We consider a constant coefficient dispersive system

$$(7.124) \quad L(\varepsilon\partial_{t,x})u = \varepsilon A_0\partial_t u + \sum_{j=1}^d \varepsilon A_j\partial_{x_j}u + Eu = F(u),$$

with  $F(u) = O(|u|^3)$ . We assume that the cubic part of  $F$  does not vanish and we denote it by  $F_3(u, u, u)$ . In this context, the weakly nonlinear regime concerns waves of amplitude  $O(\varepsilon^{\frac{1}{2}})$ .

We consider four characteristic planar phases  $\varphi_j$  which satisfy the resonance (or phase matching) condition:

$$(7.125) \quad \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 = 0.$$

We denote by  $\mathcal{F}$  the group that they generate. It has finite rank  $\leq 3$ . We make the following assumption (where the dispersive character is essential):

**Assumption 7.60.** *In  $\mathcal{F} \setminus \{0\}$ , the only characteristic phases are  $\pm\varphi_j$  for  $j \in \{1, 2, 3, 4\}$ . Moreover, we assume that the  $d\varphi_j$  are regular points of the characteristic variety.*

In our general framework, we would choose a basis of  $\mathcal{F}$ , for instance  $(\varphi_1, \varphi_2, \varphi_3)$ , and use profiles  $\mathbf{u}(t, x, \theta_1, \theta_2, \theta_3)$  periodic in  $\theta_1, \theta_2$  and  $\theta_3$ . However, the assumption above and the polarization condition (7.41) imply that

$\mathbf{u}_0$  has only 9 nonvanishing coefficients, and is better represented as

$$\begin{aligned} \mathbf{u}_0(t, x, \theta) = & \mathbf{u}_{0,0}(t, x) + \mathbf{u}_{0,1}(t, x, \theta_1) + \mathbf{u}_{0,2}(t, x, \theta_2) \\ & + \mathbf{u}_{0,3}(t, x, \theta_3) + \mathbf{u}_{0,4}(t, x, -\theta_1 - \theta_2 - \theta_3) \end{aligned}$$

with  $\mathbf{u}_{0,j}$  periodic of *one* variable with the only harmonics +1 and -1:

$$\mathbf{u}_{0,j}(t, x, \theta) = a_j(t, x)e^{i\theta} + \tilde{a}_j(t, x)e^{-\theta}.$$

We are interested in real solutions, meaning that  $\tilde{a}_j$  is the complex conjugate of  $a_j$ , with mean value equal to zero, meaning that  $\mathbf{u}_{0,0} = 0$ . This corresponds to waves of the form

$$(7.126) \quad u^\varepsilon(t, x) = \varepsilon^{\frac{1}{2}} \sum_{j=1}^4 a_j(t, x)e^{i\varphi_j/\varepsilon} + \varepsilon^{\frac{1}{2}} \sum_{j=1}^4 \bar{a}_j(t, x)e^{-i\varphi_j/\varepsilon}.$$

The  $a_j$  must satisfy the polarization condition

$$(7.127) \quad a_j \in \ker L(id\varphi_j).$$

Introducing the group velocity  $v_j$  associated to the regular point  $d\varphi_j$  of the characteristic variety, the dynamics of  $a_j$  is governed by the transport field  $\partial_t + v_j \cdot \nabla_x$ .

For a cubic interaction  $(\text{Id} - \mathcal{Q})F_3(\mathbf{u}_0, \mathbf{u}_0, \mathbf{u}_0)$ , the  $\alpha$ -th Fourier coefficient is

$$(7.128) \quad \sum_{\beta_1 + \beta_2 + \beta_3 = \alpha} (\text{Id} - Q_\alpha)F_3(\hat{\mathbf{u}}_{0,\beta_1}, \hat{\mathbf{u}}_{0,\beta_2}, \hat{\mathbf{u}}_{0,\beta_3})$$

Applied to the present case, this shows that the equations have the form

$$(7.129) \quad (\partial_t + v_1 \cdot \nabla_x)a_1 = r_1 F_3(\bar{a}_2, \bar{a}_3, \bar{a}_4),$$

$$(7.130) \quad (\partial_t + v_2 \cdot \nabla_x)a_2 = r_1 F_3(\bar{a}_3, \bar{a}_4, \bar{a}_1),$$

$$(7.131) \quad (\partial_t + v_3 \cdot \nabla_x)a_3 = r_1 F_3(\bar{a}_4, \bar{a}_1, \bar{a}_2),$$

$$(7.132) \quad (\partial_t + v_3 \cdot \nabla_x)a_4 = r_1 F_3(\bar{a}_1, \bar{a}_2, \bar{a}_3),$$

where the  $r_j$  are projectors on  $\ker L(id\varphi_j)$ .

Note that the condition  $\mathbf{u}_{0,0} = 0$  is consistent with the equation (7.44) for cubic nonlinearities.

This is the *generic form of four wave interaction*. There are analogues for three wave interaction. These settings cover fundamental phenomena such as *Raman scattering*, *Brillouin scattering* or *Rayleigh scattering*, see e.g. [Blo, Boy, NeMo, PaPu].

### 7.6.3 A model for Raman interaction

We consider here a simplified model, based on a one dimensional version of Maxwell-Bloch equation (see Section 2). The electric field  $E$  is assumed to have a constant direction, orthogonal to the direction of propagation.  $B$  is perpendicular both to  $E$  and the axis of propagation. The polarization  $P$  is parallel to  $E$ . This leads to the following set of equations

$$\begin{cases} \partial_t b + \partial_x e = 0, \\ \partial_t e + \partial_x b = -\partial_t \text{tr}(\Gamma \rho), \\ i\varepsilon \partial_t \rho = [\Omega, \rho] - e[\Gamma, \rho], \end{cases}$$

where  $e$  and  $b$  take their value in  $\mathbb{R}$  and  $\Gamma$  is a Hermitian symmetric matrix, with entries in  $\mathbb{C}$ . In this case, note that  $\text{tr}(\Gamma[\Gamma, \rho]) = 0$  and

$$\varepsilon \partial_t \text{tr}(\Gamma \rho) = \frac{1}{i} \text{tr}(\Gamma[\Omega, \rho]).$$

A classical model to describe Raman scattering in one space dimension (see e.g. [Boy, NeMo, PaPu]) uses a three level model for the electrons, with  $\Omega$  having three simple eigenvalues,  $\omega_1 < \omega_2 < \omega_3$ . Moreover, the states 1 and 2 have the same parity, while the state 3 has the opposite parity. This implies that the interaction coefficient  $\gamma_{1,2}$  vanishes. Finally, we assume that

$$\Omega = \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0 & 0 & \gamma_{1,3} \\ 0 & 0 & \gamma_{2,3} \\ \gamma_{3,1} & \gamma_{3,2} & 0 \end{pmatrix}.$$

As noticed in Section 5.3.2, the scaling of the waves requires some care for Maxwell-Bloch equations (see [JMR1]). The density matrix  $\rho$  is assumed to be a perturbation of the ground state

$$\underline{\rho} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The scaling for amplitudes is

$$(7.133) \quad \begin{aligned} e &= \varepsilon \tilde{e}, & b &= \varepsilon \tilde{b}, \\ \rho_{1,1} &= 1 + \varepsilon^2 \tilde{\rho}_{1,1}, & \rho_{1,k} &= \varepsilon \tilde{\rho}_{1,k} \quad \text{for } k = 2, 3, \\ \rho_{j,k} &= \varepsilon^2 \tilde{\rho}_{j,k} \quad \text{for } 2 \leq j, k \leq 3. \end{aligned}$$

The different scalings for the components of  $\rho$  is consistent with the fact that  $\rho$  must remain a projector, as implied by the Liouville equation for  $\rho$ .

Substituting in the equation, and neglecting  $O(\varepsilon^2)$  terms which do not affect the principal term of the expansions, yields the following model system for  $u = (\tilde{b}, \tilde{e}, \tilde{\rho}_{1,2}, \tilde{\rho}_{1,3}, \tilde{\rho}_{2,1}, \tilde{\rho}_{3,1})$

$$\begin{cases} \varepsilon \partial_t \tilde{b} + \varepsilon \partial_x \tilde{e} = 0, \\ \varepsilon \partial_t \tilde{e} + \varepsilon \partial_x \tilde{b} - i \omega_{3,1} (\gamma_{1,3} \tilde{\rho}_{3,1} - \gamma_{3,1} \tilde{\rho}_{1,3}) + i \varepsilon \omega_{3,2} (\gamma_{2,3} \tilde{\rho}_{3,1} \tilde{\rho}_{1,2} - \gamma_{3,2} \tilde{\rho}_{2,1} \tilde{\rho}_{1,3}) = 0, \\ \varepsilon \partial_t \tilde{\rho}_{1,3} - i \omega_{3,1} \tilde{\rho}_{1,3} + i \tilde{e} \gamma_{1,3} + i \varepsilon \tilde{e} \tilde{\rho}_{1,2} \gamma_{2,3} = 0, \\ \varepsilon \partial_t \tilde{\rho}_{3,1} + i \omega_{3,1} \tilde{\rho}_{3,1} - i \tilde{e} \gamma_{3,1} - i \varepsilon \tilde{e} \gamma_{3,2} \tilde{\rho}_{2,1} = 0, \\ \varepsilon \partial_t \tilde{\rho}_{1,2} - i \omega_{2,1} \tilde{\rho}_{1,2} + i \varepsilon \tilde{e} \tilde{\rho}_{1,3} \gamma_{3,2} = 0, \\ \varepsilon \partial_t \tilde{\rho}_{2,1} + i \omega_{2,1} \tilde{\rho}_{2,1} - i \varepsilon \tilde{e} \gamma_{2,3} \tilde{\rho}_{3,1} = 0. \end{cases}$$

This is of the form

$$(7.134) \quad L(\varepsilon \partial_{t,x}) u + \varepsilon f(u) = 0$$

where  $f$  is quadratic. As a consequence of the equality  $\gamma_{1,2} = 0$ , the linear operator  $L(\varepsilon \partial_{t,x})$  splits into two independent systems, where we dropped the tildes:

$$L_1(\varepsilon \partial_{t,x}) \begin{pmatrix} b \\ e \\ \rho_{1,3} \\ \rho_{3,1} \end{pmatrix} := \begin{cases} \varepsilon \partial_t b + \varepsilon \partial_x e, \\ \varepsilon \partial_t e + \varepsilon \partial_x b - i \omega_{3,1} (\gamma_{1,3} \tilde{\rho}_{3,1} - \gamma_{3,1} \tilde{\rho}_{1,3}), \\ \varepsilon \partial_t \rho_{1,3} - i \omega_{3,1} \rho_{1,3} + i e \gamma_{1,3}, \\ \varepsilon \partial_t \rho_{3,1} + i \omega_{3,1} \rho_{3,1} - i e \gamma_{3,1}, \end{cases}$$

and

$$L_2(\varepsilon \partial_{t,x}) \begin{pmatrix} \rho_{1,2} \\ \rho_{2,1} \end{pmatrix} := \begin{cases} \varepsilon \partial_t \rho_{1,2} - i \omega_{2,1} \rho_{1,2}, \\ \varepsilon \partial_t \rho_{2,1} + i \omega_{2,1} \rho_{2,1}. \end{cases}$$

The characteristic varieties of  $L_1$  and  $L_2$  are respectively

$$\begin{cases} \mathcal{C}_{L_1} = \{(\tau, \xi) \in \mathbb{R}^2; \xi^2 = \tau^2(1 + \chi(\tau))\}, & \chi(\tau) := \frac{2\omega_{3,1} |\gamma_{1,3}|^2}{(\omega_{3,1})^2 - \tau^2}, \\ \mathcal{C}_{L_2} = \{(\tau, \xi) \in \mathbb{R}^2; \tau = \pm \omega_{2,1}\}. \end{cases}$$

*Raman interaction occurs when a laser beam of wave number  $\beta_L = (\omega_L, \kappa_L) \in \mathcal{C}_{L_1}$  interacts with an electronic excitation  $\beta_E = (\omega_{2,1}, \kappa_E) \in \mathcal{C}_{L_2}$  to produce a scattered wave  $\beta_S = (\omega_S, \kappa_S) \in \mathcal{C}_{L_1}$  via the resonance relation*

$$\beta_L = \beta_E + \beta_S.$$

One further assumes that  $\beta_L \notin \mathcal{C}_{L_2}$ ,  $\beta_E \notin \mathcal{C}_{L_1}$  and  $\beta_S \notin \mathcal{C}_{L_2}$ .

We represent the oscillations using the two independent phases

$$\varphi_L(t, x) = \omega_L t + \kappa_L x, \quad \varphi_S(t, x) = \omega_S t + \kappa_S x.$$

The third phase  $\varphi_E$  is a linear combination of these two phases:

$$(7.135) \quad \varphi_E = \varphi_L - \varphi_S.$$

Accordingly, we use profiles  $\mathbf{u}$  depending on two fast variables  $\theta_L$  and  $\theta_S$ , so that

$$(7.136) \quad u^\varepsilon(t, x) \sim \mathbf{u}(t, y, \varphi_L/\varepsilon, \varphi_S/\varepsilon)$$

Because  $f$  is quadratic, the conditions on  $(\beta_L, \beta_S, \beta_E)$  show that the profile equations have solutions with spectrum satisfying

$$\text{spec}(\mathbf{u}) \subset \{\pm(1, 0), \pm(0, 1), \pm(1, -1)\}$$

We now restrict our attention to these solutions. It is convenient to call  $u_{\pm L}$ ,  $u_{\pm S}$  and  $u_{\pm E}$  the Fourier coefficients  $\hat{\mathbf{u}}_{\pm 1, 0}$ ,  $\hat{\mathbf{u}}_{0, \pm 1}$  and  $\hat{\mathbf{u}}_{\pm 1, \mp 1}$  respectively. Thus, for real solutions

$$\begin{aligned} u^\varepsilon(t, x) = & u_L(x) e^{i\varphi_L(t, x)/\varepsilon} + u_S(x) e^{i\varphi_S(t, x)/\varepsilon} + u_E(x) e^{i\varphi_E(t, x)/\varepsilon} \\ & u_{-L}(x) e^{-i\varphi_L(t, x)/\varepsilon} + u_{-S}(x) e^{-i\varphi_S(t, x)/\varepsilon} + u_{-E}(x) e^{-i\varphi_E(t, x)/\varepsilon} \end{aligned}$$

In addition, we use the notations  $b_L, e_L, \rho_{j, k, L}$  for the components of  $u_L$  etc.

**Lemma 7.61.** *The polarization conditions for the principal profile read*

$$(7.137) \quad \begin{aligned} b_{\pm E} = e_{\pm E} = \rho_{1, 3, \pm E} = \rho_{3, 1, \pm E} = 0, \\ \rho_{1, 2, \pm L} = \rho_{1, 2, \pm S} = \rho_{2, 1, \pm L} = \rho_{2, 1, \pm S} = 0, \\ \rho_{1, 2, -E} = \rho_{2, 1, E} = 0, \end{aligned}$$

and

$$(7.138) \quad b_L = -\frac{1}{\omega_L} \kappa_L e_L, \quad \rho_{1, 3, L} = \frac{\gamma_{1, 3}}{\omega_{3, 1} - \omega_L} e_L, \quad \rho_{3, 1, L} = \frac{\gamma_{3, 1}}{\omega_{3, 1} + \omega_L} e_L,$$

with similar formula when  $L$  is replaced by  $-L$  and  $\pm S$ . Furthermore, for real fields and hermitian density matrices, one has the relations

$$(7.139) \quad e_{-L} = \overline{e_L}, \quad e_{-S} = \overline{e_S}, \quad \rho_{2, 1, -E} = \overline{\rho_{1, 2, E}}.$$

Therefore, the profile equations for  $\mathbf{u}_0$  involve only  $(e_L, e_S)$  and  $\sigma_E := \rho_{1, 2, E}$ :

**Theorem 7.62.** *For real solutions, the evolution of the principal term is governed by the system*

$$(7.140) \quad \begin{cases} (\partial_t + v_L \partial_y) e_L + i c_1 e_S \sigma_E = 0, \\ (\partial_t + v_S \partial_y) e_S + i c_2 e_S \overline{\sigma_E} = 0, \\ \partial_t \sigma_E + i c_3 e_L \overline{e_S} = 0, \end{cases}$$

where  $v_L$  and  $v_S$  are the group velocities associated to the frequencies  $\beta_L$  and  $\beta_S$  respectively. The complete fields are recovered using the polarization conditions stated in the preceding lemma.

This is the familiar form of the equations relative to three waves mixing. We refer for example to [Boy, NeMo] for an explicit calculation of the constants  $c_k$ . The Raman instability (spontaneous or stimulated) is related to the amplification properties of this system.

#### 7.6.4 Maximal dissipative equations

For maximal dissipative systems, energy solutions exist and are stable. In this context the weakly nonlinear approximations can be justified in the energy norm. The new point is that profiles need not be continuous, they belong to  $L^p$  spaces, in which case the substitution  $\theta = \Psi/\varepsilon$  has to be taken in the sense of Definition 7.11.

In  $\mathbb{R}^{1+d}$ , consider the Cauchy problem:

$$(7.141) \quad \partial_t u + \sum_{j=1}^d A_j \partial_j u + f(u) = 0, \quad u|_{t=0} = h.$$

**Assumption 7.63.** *The matrices  $A_j$  are symmetric with constant coefficients and the distinct eigenvalues  $\lambda_k(\xi)$  of  $A(\xi) = \sum \xi_j A_j$  have constant multiplicity for  $\xi \neq 0$ . We denote by  $\Pi_k$  the orthogonal spectral projector on  $\ker A(\xi) - \lambda_k(\xi) \text{Id}$ .*

**Assumption 7.64** (Maximal monotonicity).  *$f$  is a  $C^1$  function from  $\mathbb{C}^N$  to  $\mathbb{C}^N$ , with  $f(0) = 0$  and there are  $p \in [1, +\infty[$  and constants  $0 < c < C < +\infty$  such that for all  $(t, x) \in \mathbb{R}^{1+d}$  and all  $u$  and  $v$  in  $\mathbb{C}^N$ ,*

$$|f(u)| \leq C |u|^p, \quad |\partial_{u, \bar{u}} f(u)| \leq C |u|^{p-1},$$

$$\text{Re} \left( (f(u) - f(v)) \cdot (\bar{u} - \bar{v}) \right) \geq c |u - v|^{p+1},$$

Solutions of the Cauchy problem (7.141) are constructed in [Str] (see also [LS] or [Ha]): we consider smooth domains  $\omega \subset \mathbb{R}^d$  and  $\Omega \subset [0 + \infty[ \times \mathbb{R}^d$  contained in the domain of determination of  $\omega$ . The key ingredient are the energy estimates which follow from the monotonicity assumptions:

$$(7.142) \quad \|u(t)\|_{L^2}^2 + c \int_0^t \|u(t')\|_{L^{p+1}}^{p+1} dt' \leq \|u(0)\|_{L^2}^2$$

and

$$(7.143) \quad \|u(t) - v(t)\|_{L^2}^2 + c \int_0^t \|u(t') - v(t')\|_{L^{p+1}}^{p+1} dt' \leq \|u(0) - v(0)\|_{L^2}^2.$$

**Proposition 7.65.** *For all  $h \in L^2(\omega)$ , there is a unique  $u \in C^0 L^2 \cap L^{p+1}(\Omega)$  solution of (7.141).*

We assume that the initial data have periodic oscillations

$$(7.144) \quad h^\varepsilon(x) \sim \mathbf{h}(x, \varphi_0(x)/\varepsilon)$$

with  $\mathbf{h}(x, \theta)$  periodic in  $\theta \in \mathbb{T}$  and  $\varphi_0 \in C^\infty(\bar{\omega})$  satisfying  $d\varphi_0(x) \neq 0$  for all  $x \in \bar{\omega}$ . For  $1 \leq k \leq K$ , we denote by  $\varphi_k$  the solution of the eikonal equation

$$\partial_t \varphi_k + \lambda_k(\partial_x \varphi_k) = 0, \quad \varphi_k|_{t=0} = \varphi_0.$$

We assume that the space generated by the phases is weakly coherent and that no resonance occur:

**Assumption 7.66.** *The  $\varphi_k$  are defined and smooth on  $\bar{\Omega}$  and  $d_x \varphi_k(x) \neq 0$  at every point. Moreover,*

*For all  $\alpha \in \mathbb{Z}^K$ ,  $\det L(d\varphi(t, x)) \neq 0$  a.e. on  $\Omega$ , except when  $\alpha$  belongs to one of the coordinate axis.*

Here  $L(\partial_t, \partial_x) = \partial_t + A(\partial_x)$ . In this setting, the profiles are functions  $\mathbf{u}(t, x, \theta)$  which are periodic in  $\theta \in \mathbb{R}^K$  and periodic in  $\theta$ . The projector  $\mathcal{P}$  is

$$\mathcal{P} \sum (\hat{\mathbf{u}}_\alpha(t, x) e^{i\alpha\theta}) = \sum P_\alpha (\hat{\mathbf{u}}_\alpha(t, x) e^{i\alpha\theta})$$

and  $P_\alpha$  is the orthogonal projector on  $\ker L(\alpha \cdot \Phi)$ . Introducing the averaging operator  $\underline{\mathbf{E}}$  with respect to all the variables  $\theta$  and  $\mathbf{E}_k$  with respect to the all the variables except  $\theta_k$ , there holds

$$(7.145) \quad \mathcal{P}\mathbf{u} = \underline{\mathbf{E}}\mathbf{u} + \sum_{k=1}^K \Pi_k(d\varphi_k) \mathbf{E}_k \mathbf{u}^*$$

with  $\mathbf{E}\mathbf{u} = \hat{\mathbf{u}}_0$  the average of  $\mathbf{u}$  and  $\mathbf{u}^* = \mathbf{u} - \hat{\mathbf{u}}_0$  its oscillation.

The polarization conditions  $\mathcal{P}\mathbf{u} = \mathbf{u}$  reads

$$(7.146) \quad \mathbf{u}(t, x, \theta) = \underline{\mathbf{u}}(t, x) + \sum_{k=1}^K \mathbf{u}_k^*(t, x, \theta_k), \quad \mathbf{u}_k^* = \tilde{\Pi}_k \mathbf{u}_k^*$$

where  $\tilde{\Pi}_k = \Pi_k(d\varphi_k)$  and  $\mathbf{u}_k^*$  is a periodic solution with vanishing mean value. The profile equations

$$\partial_t \mathbf{u} - \sum_{j=1}^d \mathcal{P}A_j \partial_j \mathbf{u} + \mathcal{P}f(\mathbf{u}) = 0,$$

decouple into a system

$$(7.147) \quad L(\partial_t, \partial_x) \underline{\mathbf{u}} + \mathbf{E}f(\mathbf{u}) = 0,$$

$$(7.148) \quad X_k \mathbf{u}_k^* + \tilde{\Pi}_k (\mathbf{E}_k f(\mathbf{u}))^* = 0$$

where  $X_k = \tilde{\Pi}_k L(\partial_t, \partial_x) \tilde{\Pi}_k$  is the propagator associated to the  $k$ -th eigenvalue. The initial conditions read

$$(7.149) \quad \underline{\mathbf{u}}|_{t=0}(x) = \underline{\mathbf{h}}(x),$$

$$(7.150) \quad \mathbf{u}_k^*|_{t=0}(x, \theta_k) = \tilde{\Pi}_k(0, x) \mathbf{h}^*(x, \theta_k) = 0$$

where  $\underline{\mathbf{h}}$  is its mean value and  $\mathbf{h}^* = \mathbf{h} - \underline{\mathbf{h}}$  its oscillation.

The equation (7.147) inherits the dissipative properties from the original equation:

**Lemma 7.67.** *If  $\mathbf{f}$  is a profile and  $\mathbf{v}$  satisfies the polarization condition (7.146), then*

$$\begin{aligned} \int_{\Omega} \underline{\mathbf{E}} \mathbf{f} \underline{\mathbf{v}} dt dx + \sum_{k=1}^K \int_{\Omega \times \mathbb{T}} \mathbf{E}_k \mathbf{f}^* \mathbf{v}_k^* dt dx d\theta_k \\ = \int_{\Omega \times \mathbb{T}^K} \mathbf{f} \mathbf{v} dt dx d\theta. \end{aligned}$$

*Proof.* Because  $\tilde{\Pi}_k$  is self adjoint and  $\tilde{\Pi}_k \mathbf{v}_k^* = \mathbf{v}_k^*$ ,

$$\int_{\Omega \times \mathbb{T}} \tilde{\Pi}_k \mathbf{E}_k \mathbf{f}^* \mathbf{v}_k^* dt dx d\theta_k = \int_{\Omega \times \mathbb{T}^K} \mathbf{f}(t, x, \theta) \mathbf{v}_k^*(t, x, \theta_k) dt dx d\theta.$$

Adding up implies the lemma.  $\square$



Similarly,

$$\begin{aligned} \int L(\partial_t, \partial_x) \underline{\mathbf{u}} \underline{\mathbf{u}} dx + \sum_{k=1}^K \int X_k \mathbf{u}_k^* \mathbf{u}_k^* dt dx d\theta_k \\ = \int L(\partial_t, \partial_x) \mathbf{u} \mathbf{u} dt dx d\theta. \end{aligned}$$

and the energy method implies the following

**Proposition 7.68.** *For all  $\mathbf{h} \in L^2(\omega \times \mathbb{T})$ , there is a unique  $\mathbf{u} \in C^0L^2 \cap L^{p+1}(\Omega \times \mathbb{T}^K)$  of the form (7.146) and satisfying the profile equations (7.147) (7.148). with initial data (7.149) (7.150).*

Knowing  $\mathbf{u}$ , one can construct

$$(7.151) \quad v^\varepsilon(t, x) \sim \mathbf{u}(t, x, \Phi(t, x)/\varepsilon)$$

not performing directly the substitution  $\theta = \Phi/\varepsilon$  in  $\mathbf{u}(t, x, \theta)$  since this function is not necessarily continuous in  $\theta$ , but in the sense of Definition 7.11: the family  $(v^\varepsilon)$  is  $\mathcal{F}$  oscillating with profile  $\mathbf{u}$ , where  $\mathcal{F}$  is the group generated by the  $\varphi_k$ . The asymptotic in (7.151) is taken in  $L^{p+1}(\Omega)$  and in  $C^0L^2(\Omega)$ .

A key observation is that since  $\mathbf{u}$  solves the profile equation, one can select  $v^\varepsilon$  satisfying approximately the original equation (in the spirit of Section 7.3.6:

**Proposition 7.69.** *There is a bounded family  $v^\varepsilon$  in  $C^0L^2(\Omega) \cap L^{p+1}(\Omega)$ ,  $\mathcal{F}$  oscillating with profile  $\mathbf{u}$  and such that*

$$\|L(\partial_t, \partial_x)v^\varepsilon + f(v^\varepsilon)\|_{L^{1+1/p}(\Omega)} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

Using the monotonicity of the equation, one can compare  $v^\varepsilon$  with the exact solution  $u^\varepsilon$  given by Proposition 7.65:

$$\begin{aligned} \|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^2}^2 + c \int_0^t \|u^\varepsilon(t') - v^\varepsilon(t')\|_{L^{p+1}}^{p+1} dt' \\ \leq \|u^\varepsilon(0) - v^\varepsilon(0)\|_{L^2}^2 + c \int_0^t \|Lv^\varepsilon + (v^\varepsilon)(t')\|_{L^{1+1/p}}^{1+1/p} dt' \end{aligned}$$

This implies the following result.

**Theorem 7.70.** *Suppose that  $\mathbf{h} \in L^2(\omega)$  and that the family  $h^\varepsilon$  satisfies (7.144) in  $L^2$ . Let  $\mathbf{u}$  denote the solution of the profile equation and  $(u^\varepsilon)$  the family of solutions of (7.141) with initial data  $(h^\varepsilon)$ . Then  $u^\varepsilon$  satisfies (7.151) in  $L^{p+1}(\Omega) \cap C^0L^2(\Omega)$ .*

This method is extended in [JMR7, JMR8] to analyze oscillations in the presence of caustics and in [Che2] to study oscillation near a diffractive boundary point. For nonlinear problem, no  $L^\infty$  description is known near caustics or diffractive boundary points. The method sketched above applies to energy solutions but provides only asymptotics in  $L^p$  spaces with  $p < \infty$ .

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