Chapter 1

Examples

1.1 Transport equations

$$(\partial_t + v \cdot \partial_x)u = f, \qquad u_{|t=0} = h. \tag{1.1.1}$$

$$u(t,x) = h(x - tv) + \int_0^t f(s, x - (t-s)v)ds. \tag{1.1.2}$$

$$u(t,x) = h(x - tv) + \int_0^t f(t-s, x - sv)ds. \tag{1.1.3}$$

1.2 1-D wave equation

$$(\partial_t^2 - \partial_x^2)u = f, \qquad u_{|t=0} = h_0, \quad \partial_t u_{|t=0} = h_1. \tag{1.2.1}$$

Let $v = (\partial_t - \partial_x)u$. Then $(\partial_t + \partial_x)v = 0$ and $v_{|t=0} = h_1 - \partial_x h_0$ so that

$$v(t,x) = (h_1 - \partial_x h_0)(x - t) + \int_0^t f(t-s, x-s)ds.$$

Thus

$$u(t,x) = h_0(x+t) + \int_0^t v(t-s, x+s)ds.$$

The contribution of h_1 in last integral is

$$\int_0^t h_1(x + 2s - t)dt = \frac{1}{2}\int_{-t}^t h_1(x-y)dy.$$

The contribution of h_0 is

$$h_0(x+t) - \frac{1}{2}\int_{-t}^{t} \partial_x h_0(x-y)dy = \frac{1}{2}\big(h_0(x+t) - h_0(x-t)\big).$$

The contribution of f is

$$\int_{\{0\leq s'\leq t\}} f(t-s-s', x-s+s')ds'ds = \frac{1}{2}\int_{\{|y|\leq s\leq t\}} f(t-s, x-y)dsdy.$$

$$\begin{aligned} u(t,x) = &\frac{1}{2}\big(h_0(x+t) - h_0(x-t)\big) + \frac{1}{2}\int_{-t}^{t} h_1(x-y)dy \\ &+ \frac{1}{2}\int_{\{|y|\leq s\leq t\}} f(t-s, x-y)dsdy. \end{aligned} \tag{1.2.2}$$

1.3 The multi-D wave equation

$$(\partial_t^2 - \Delta_x)u = f, \qquad u_{|t=0} = h_0, \quad \partial_t u_{|t=0} = h_1. \tag{1.3.1}$$

Fourier in x

$$(\partial_t^2 + |\xi|^2)\hat{u} = \hat{f}, \qquad \hat{u}_{|t=0} = \hat{h}_0, \quad \partial_t \hat{u}_{|t=0} = \hat{h}_1. \tag{1.3.2}$$

$$\begin{aligned} \hat{u}(t,\xi) = &\cos(t|\xi|)\hat{h}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{h}_1(\xi) \\ &+ \int_0^t \frac{\sin((t-s)|\xi|)}{|\xi|}\hat{f}(t-s,\xi)ds. \end{aligned} \tag{1.3.3}$$

Compute the inverse Fourier transform of $\cos(t|\xi|)$ and $\frac{\sin(t|\xi|)}{|\xi|}$. Let

$$E(t,x) = \frac{1}{(2\pi)^d}\int e^{ix\cdot\xi}\frac{\sin(t|\xi|)}{|\xi|}d\xi. \tag{1.3.4}$$

Then (formally)

$$\begin{aligned} u(t,x) = &\int \partial_t E(t,y)h_0(x-y)dy + \int E(t,y)h_1(x-y)dy \\ &+ \int_{\{0\leq s\leq t\}} E(s,y)f(t-s,x-y)dsdy. \end{aligned} \tag{1.3.5}$$

Example 1. In 1-D

$$E(t,x) = \frac{1}{2}1_{[-t,t]}(x).$$

Proof : compute the Fourier transform of $1_{[-t,t]}$. We recover the formula above.

Example 2. In 3-D. E is the distribution

$$\langle E_t, \varphi \rangle = t \int_{S^2} \varphi(t\omega) d\omega. \tag{1.3.6}$$

Proof : compute the Fourier transform of this distribution

$$\hat{E}_t(\xi) = \langle E_t, e^{-ix\cdot\xi} \rangle = t \int_{S^2} e^{-it\omega\cdot\xi} d\omega.$$

This integral is invariant by rotation in ξ and we can assume that $\xi = (0,0,r)$, $r = |\xi|$. Write $\omega = (\cos\phi\,\sigma, \sin\phi)$ with $\phi \in]-\pi/2, \pi/2[$ and $\sigma \in S^1$. Then $d\omega = \frac{1}{2}(\cos\phi) d\sigma d\phi$. Hence

$$\hat{E}_t(\xi) = \frac{t}{2} \int e^{-itr\sin\phi} \cos\phi \; d\phi = \frac{\sin(tr)}{r}.$$

Example 3. In 5-D. E is the distribution

$$\langle E_t, \varphi \rangle = \frac{2t}{3} \int_{S^4} X\varphi(t\omega) d\omega, \qquad X = -x \cdot \partial_x + 3. \tag{1.3.7}$$

Proof : compute the Fourier transform, using that

$$X e^{-ix\cdot\xi} = (3 + ix\cdot\xi) e^{-ix\cdot\xi}$$

$$\hat{E}_t(\xi) = \langle E_t, e^{-ix\cdot\xi} \rangle = \frac{2t}{3} \int_{S^4} (3 + ix\cdot\xi) e^{-it\omega\cdot\xi} d\omega.$$

This integral is invariant by rotation in ξ and we can assume that $\xi = (0,0,r)$, $r = |\xi|$. Write $\omega = (\cos\phi\,\sigma, \sin\phi)$ with $\phi \in]-\pi/2, \pi/2[$ and $\sigma \in S^3$. Then $d\omega = \frac{3}{4}(\cos\phi)^3 d\sigma d\phi$. Hence

$$\begin{aligned} \hat{E}_t(\xi) &= \frac{t}{2} \int_{-\pi/2}^{\pi/2} e^{-itr\sin\phi} (3 + itr\sin\phi) \cos^3\phi \; d\phi \\ &= \frac{t}{2} \int_{-1}^{1} e^{-itrs} (3 + itrs)(1 - s^2)\phi \; ds. \end{aligned}$$

Integrating by parts we get

$$\begin{aligned} \hat{E}_t(\xi) &= \frac{t}{2} \int_{-1}^{1} e^{-itrs} (3 + \partial_s s)(1 - s^2) ds \\ &= \frac{t}{2} \int_{-1}^{1} e^{-itrs} ds = \frac{\sin(tr)}{r}. \end{aligned}$$

Example 3. In 2-D. We apply the formula in $\mathbb{R}^3$ for functions independent of x_3. Using the parametrization $(\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$ of S^2 we get

$$\begin{aligned}
\langle E_t, \varphi\rangle &= \frac{t}{4\pi}\int \varphi(t\cos\phi\cos\theta, t\cos\phi\sin\theta)\cos\phi d\theta d\phi \\
&= \frac{t}{2\pi}\int \varphi(tr\cos\theta, tr\sin\theta)\frac{r}{\sqrt{1-r^2}}drd\theta \\
&= \frac{t}{2\pi}\int_{\{|y|\leq 1\}} \varphi(ty)\frac{dy}{\sqrt{1-|y|^2}} = \frac{t}{2\pi}\int_{\{|y|\leq t\}} \varphi(y)\frac{dy}{\sqrt{t^2-|y|^2}}.
\end{aligned}$$

Thus the distribution E is the function

$$E(t,x) = \frac{t}{2\pi\sqrt{t^2-|y|^2}} \tag{1.3.8}$$

1.4 Gas dynamics

1.4.1 General Euler's equations

The equations of gas dynamics link the density ρ, the pressure p, the velocity $v = (v_1, v_2, v_3)$ and the total energy per unit of volume and unit of mass E through the equations:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho v) = 0 \\ \partial_t(\rho v_j) + \operatorname{div}(\rho v v_j) + \partial_j p = 0 \quad 1 \leq j \leq 3 \\ \partial_t E + \operatorname{div}(\rho E v + pv) = 0 \end{cases} \tag{1.4.1}$$

Moreover, $E = e + |v|^2/2$ where e is the specific internal energy. The variables ρ, p and e are linked by a state law. For instance, e can be seen as a function of ρ and p and one can take $u = (\rho, v, p) \in \mathbb{R}^5$ as unknowns. The second law of thermodynamics introduces two other dependent variables, the entropy S and the temperature T so that one can express p, e and T as functions $\mathcal{P}$, $\mathcal{E}$ and $\mathcal{T}$ of the variables (ρ, S), linked by the relation

$$d\mathcal{E} = \mathcal{T}dS + \frac{\mathcal{P}}{\rho^2}d\rho\,. \tag{1.4.2}$$

One can choose $u = (\rho, v, S)$ or $\widetilde{u} = (p, v, S)$ as unknowns. The equations read (for smooth solutions):

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho v) = 0 \\ \rho(\partial_t v_j + v\cdot\nabla v_j) + \partial_j p = 0 \quad 1 \leq j \leq 3 \\ \partial_t S + v\cdot\nabla S = 0 \end{cases} \tag{1.4.3}$$

with p given a a given function $\mathcal{P}$ of (ρ, S) or ρ function of (p, S).

Perfect gases. They satisfy the condition

$$\frac{p}{\rho} = RT, \tag{1.4.4}$$

where R is a constant. The second law of thermodynamics (1.4.2) implies that

$$d\mathcal{E} = \frac{\mathcal{P}}{R\rho} dS + \frac{\mathcal{P}}{\rho^2} d\rho$$

thus

$$\frac{\partial \mathcal{E}}{\partial S} = \frac{\mathcal{P}}{R\rho}, \quad \frac{\partial \mathcal{E}}{\partial \rho} = \frac{\mathcal{P}}{\rho^2} \qquad \text{and} \qquad \rho \frac{\partial \mathcal{E}}{\partial \rho} - R \frac{\partial \mathcal{E}}{\partial S} = 0$$

Therefore, the relation between e, ρ and S has the form

$$e = \mathcal{E}(\rho, S) = F\big(\rho\, e^{S/R}\big). \tag{1.4.5}$$

Thus the temperature $T = \mathcal{T}(\rho, S) = \frac{\partial}{\partial S}\mathcal{E} = G\big(\rho\, e^{S/R}\big)$ with $G(s) = \frac{s}{R} F'(s)$. This implies that T is a function of e:

$$T = \Psi(e) = \frac{1}{R} G\big(F^{-1}(e)\big). \tag{1.4.6}$$

A particular case of this relation is when Ψ is linear, meaning that e is proportional to T:

$$e = CT, \tag{1.4.7}$$

with C constant. In this case

$$\frac{1}{R} s F'(s) = C F(s), \qquad \text{thus} \qquad F(s) = \lambda s^{RC}.$$

This implies that $e$and p are linked to ρ and S by

$$e = \rho^{\gamma - 1} e^{C(S - S_0)}, \qquad p = (\gamma - 1)\rho^{\gamma} e^{C(S - S_0)} = (\gamma - 1)\rho e, \tag{1.4.8}$$

with $\gamma = 1 + RC$.

1.4.2 The isentropic system

When S is constant the system (1.4.3) reduces to

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0 \\ \rho(\partial_t v_j + v \cdot \nabla v_j) + \partial_j p = 0 \quad 1 \le j \le 3 \end{cases} \tag{1.4.9}$$

with ρ and p linked by a state law, $p = \mathcal{P}(\rho)$. For instance, $p = c\rho^{\gamma}$ for perfect gases satisfying (1.4.8).

1.4.3 Acoustics

By linearization of (1.4.9) around a constant state $(\underline{\rho}, \underline{v})$, one obtains the equations

$$\begin{cases} (\partial_t + \underline{v} \cdot \nabla)\rho + \underline{\rho}\,\mathrm{div} v = f \\ \underline{\rho}(\partial_t + \underline{v} \cdot \nabla) v_j + \underline{c}^2 \partial_j p = g_j \quad 1 \le j \le 3 \end{cases} \tag{1.4.10}$$

where $\underline{c}^2 := \frac{d\mathcal{P}}{d\rho}(\underline{\rho})$. Changing variables x to $x - t\underline{v}$, reduces to

$$\begin{cases} \partial_t \rho + \underline{\rho}\,\mathrm{div} v = f \\ \underline{\rho}\partial_t v + \underline{c}^2 \nabla p = g. \end{cases} \tag{1.4.11}$$

Compute the fundamental solution in $D = 3$. Assume for simplicity that $\underline{\rho} = 1$, $\underline{c} = 1$, the general case following easily. In the Fourier side we have to compute $e^{itA(\xi)}$ where

$$A = \begin{pmatrix} 0 & {}^t\xi \\ \xi & 0 \end{pmatrix}.$$

Note that

$$A^2 = \begin{pmatrix} |\xi|^2 & 0 \\ 0 & \xi^t\xi \end{pmatrix}, \qquad A^3 = |\xi|^2 A,$$

so that

$$e^{itA} = I + \sum_{p \ge 0} \frac{(it)^{2p+2}}{(2p+2)!} |\xi|^{2p} A^2 + \sum_{p \ge 0} \frac{(it)^{2p+1}}{(2p+1)!} |\xi|^{2p} A$$

$$e^{itA} = \mathrm{Id} + \frac{\cos(t|\xi|) - 1}{|\xi|^2} A^2 + \frac{\sin(t|\xi|)}{|\xi|} iA$$

The inverse Fourier transform of $\frac{\sin(t|\xi|)}{|\xi|}$ is E_t given at (1.3.6). The inverse Fourier transform of $\frac{1-\cos(t|\xi|)}{|\xi|^2}$, F_t, satisfies $\partial_t F_t = E_t$,

$$\langle F_t, \varphi \rangle = \int_{0 \le s \le t} s\varphi(x + s\omega) ds d\omega = \int_{\{|y| \le t\}} \varphi(x+y) \frac{dy}{|y|}. \tag{1.4.12}$$

1.5 Maxwell's equations

1.5.1 General equations

The general Maxwell's equations read:

$$\begin{cases} \partial_t D - c\,\mathrm{curl} H = -j, \\ \partial_t B + c\,\mathrm{curl} E = 0, \\ \mathrm{div} B = 0, \\ \mathrm{div} D = q \end{cases} \tag{1.5.1}$$

where D is the electric displacement, E the electric field vector, H the magnetic field vector, B the magnetic induction, j the current density and q is the charge density; c is the velocity of light. They also imply the charge conservation law:

$$\partial_t q + \mathrm{div} j = 0. \tag{1.5.2}$$

To close the system, one needs *constitutive equations* which link E, D, H, B and j.

Equations in vacuum

Consider here the case $j = 0$ and $q = 0$ (no current and no charge) and

$$D = \varepsilon E, \qquad B = \mu H, \tag{1.5.3}$$

where ε is the dielectric tensor and μ the tensor of magnetic permeability.

In vacuum, ε and μ are scalar and constant. After some normalization the equation reduces to

$$\begin{cases} \partial_t E - \mathrm{curl} B = 0, \\ \partial_t B + \mathrm{curl} E = 0, \\ \mathrm{div} B = 0, \\ \mathrm{div} E = 0. \end{cases} \tag{1.5.4}$$

The first two equations imply that $\partial_t \mathrm{div} E = \partial_t \mathrm{div} B = 0$, therefore the constraints $\mathrm{div} E = \mathrm{div} B = 0$ are satisfied at all time if they are satisfied at time $t = 0$. This is why one can "forget" the divergence equation and focus on the evolution equations

$$\begin{cases} \partial_t E - \mathrm{curl} B = 0, \\ \partial_t B + \mathrm{curl} E = 0, \end{cases} \tag{1.5.5}$$

Moreover, using that curl curl $= -\Delta \mathrm{Id} + \mathrm{grad}\,\mathrm{div}$, for divergence free fields the system is equivalent to the wave equation :

$$\partial_t^2 E - \Delta E = 0. \tag{1.5.6}$$

1.5.2 Crystal optics

With $j = 0$ and $q = 0$, we assume in (1.5.3) that μ is scalar but that ε is a positive definite symmetric matrix. In this case the system reads:

$$\begin{cases} \partial_t(\varepsilon E) - \mathrm{curl} B = 0, \\ \partial_t B + \mathrm{curl} E = 0, \end{cases} \tag{1.5.7}$$

plus the constraint equations $\mathrm{div}(\varepsilon E) = \mathrm{div} B = 0$ which are again propagated from the initial conditions. One can choose coordinate axes so that ε is diagonal:

$$\varepsilon^{-1} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \tag{1.5.8}$$

with $\alpha_1 > \alpha_2 > \alpha_3$.

1.5.3 Laser - matter interaction

Still with $j = 0$ and $q = 0$ and B proportional to H, say $B = H$, the interaction light-matter is described through the relation

$$D = E + P \tag{1.5.9}$$

where P is the polarization field. P can be given explicitly in terms of E, for instance in the *Kerr nonlinearity* model:

$$P = |E|^2 E. \tag{1.5.10}$$

In other models P is given by an evolution equation:

$$\frac{1}{\omega^2}\partial_t^2 P + P - \alpha |P|^2 P = \gamma E \tag{1.5.11}$$

harmonic oscillators when $\alpha = 0$ or *anharmonic oscillators* when $\alpha \neq 0$.

In other models, P is given by *Bloch's equation* which come from a more precise description of the physical interaction of the light and the electrons at the quantum mechanics level.

With $Q = \partial_t P$, the equations (1.5.1) (1.5.11) can be written as a first order 12×12 system:

$$\begin{cases} \partial_t E - \mathrm{curl} B + Q = 0, \\ \partial_t B + \mathrm{curl} E = 0, \\ \partial_t P - Q = 0, \\ \partial_t Q + \omega^2 P - \omega^2 \gamma E - \omega^2 \alpha |P|^2 P = 0. \end{cases} \tag{1.5.12}$$

1.6 Magneto-hydrodynamics

1.6.1 A model

The equations of isentropic magnetohydrodynamics (MHD) appear in basic form as

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 \\ \partial_t (\rho u) + \mathrm{div}(\rho u^t u) + \nabla p + H \times \mathrm{curl} H = 0 \\ \partial_t H + \mathrm{curl}(H \times u) = 0 \end{cases} \tag{1.6.1}$$

$$\mathrm{div} H = 0, \tag{1.6.2}$$

where $\rho \in \mathbb{R}$ represents density, $u \in \mathbb{R}^3$ fluid velocity, $p = p(\rho) \in \mathbb{R}$ pressure, and $H \in \mathbb{R}^3$ magnetic field. With $H \equiv 0$, (1.6.1) reduces to the equations of isentropic fluid dynamics.

Equations (1.6.1) may be put in conservative form using identity

$$H \times \mathrm{curl} H = (1/2) \mathrm{div}(|H|^2 I - 2H^t H)^{\mathrm{tr}} + H \mathrm{div} H \tag{1.6.3}$$

together with constraint (1.6.2) to express the second equation as

$$\partial_t (\rho u) + \mathrm{div}(\rho u^t u) + \nabla p + (1/2) \mathrm{div}(|H|^2 I - 2H^t H)^{\mathrm{tr}} = 0. \tag{1.6.4}$$

They may be put in symmetrizable (but no longer conservative) form by a further change, using identity

$$\mathrm{curl}(H \times u) = (\mathrm{div} u) H + (u \cdot \nabla) H - (\mathrm{div} H) u - (H \cdot \nabla) u \tag{1.6.5}$$

together with constraint (1.6.2) to express the third equation as

$$\partial_t H + (\mathrm{div} u) H + (u \cdot \nabla) H - (H \cdot \nabla) u = 0. \tag{1.6.6}$$

1.7 Elasticity

The linear wave equation in an elastic homogeneous medium is a second order constant coefficients 3×3 system

$$\partial_t^2 v - \sum_{j,k=1}^{3} A_{j,k} \partial_{x_j} \partial_{x_k} v = f \tag{1.7.1}$$

where the $A_{j,k}$ are 3×3 real matrices. In anisotropic media, the form of the matrices $A_{j,k}$ is complicated (it may depend upon 21 parameters). The basic hyperbolicity condition is that

$$A(\xi) := \sum \xi_j \xi_k A_{j,k} \tag{1.7.2}$$

is symmetric and positive definite for $\xi \neq 0$.

In the isotropic case

$$\sum_{j,k=1}^{3} A_{j,k} \partial_{x_j} \partial_{x_k} v = 2\lambda \Delta_x v + \mu \nabla_x (\mathrm{div}_x v). \tag{1.7.3}$$

The hyperbolicity condition is that $\lambda > 0$ and $2\lambda + \mu > 0$.