Chapter 2

Background

In this chapter we give several basic results which serve as an introduction and help to understand the general framework of these notes. None of them is necessary to follow the analysis which is developed in the other chapters, so we do not include complete proofs which can be founded elsewhere. Our goal is to introduce the definition of hyperbolicity which is discussed at the end of the chapter.

2.1 Fundamental solutions

A linear differential operator is a finite sum

$$P = \sum p_\alpha \partial_x^\alpha \tag{2.1.1}$$

where the coefficients p_α are complex numbers. We will also consider systems, in which case the p_α matrices. We consider equations of the form

$$Pu = f. \tag{2.1.2}$$

It is convenient to use the language of distributions and we recall here several basic facts and notations. For any open subset Ω of $\mathbb{R}^n$, $C_0^\infty(\Omega)$ is the space of C^∞ functions φ on Ω with compact support in Ω, that is such that $\varphi(x) = 0$ when $x \notin K$, for some compact subset $K \subset \Omega$ which depends on φ. A distribution on Ω is a continuous linear form on $C_0^\infty(\Omega)$. The space of distributions on Ω is denoted by $\mathscr{D}'(\Omega)$ and the pairing $\mathscr{D}' \times C_0^\infty$ is denoted by

$$\langle u, \varphi \rangle. \tag{2.1.3}$$

If Ω' is an open subset of Ω, restricting φ to $C_0^\infty(\Omega')$ defines the restriction operator $u \mapsto u_{|\Omega'}$ from $\mathscr{D}'(\Omega)$ to $\mathscr{D}'(\Omega')$. The support of a distribution u is the smallest (relatively) closed set $F \subset \Omega$ (i.e. the intersection of all closed subsets F) such that the restriction of u to $\Omega \backslash F$ vanishes. The set of distributions on Ω with compact support in Ω is denoted by $\mathscr{E}'(\Omega)$. If $u \in \mathscr{E}'(\Omega)$ the pairing (2.1.3) can be extended to $C^\infty(\mathbb{R}^n)$ by the identity

$$\langle u, \varphi \rangle = \langle u, \chi\varphi \rangle \tag{2.1.4}$$

where $\chi \in C_0^\infty(\Omega)$ is equal to 1 on a relatively compact open set Ω' such that $\operatorname{supp} u \subset \Omega' \subset \overline{\Omega}' \subset \Omega$. The extension is independent of the choice of χ.

The convolution is defined for C^∞ functions φ and ψ on $\mathbb{R}^n$, one of them with compact support, by the formula

$$\varphi \star \psi(x) = \int \varphi(x-y)\psi(y)dy = \int \varphi(y)\psi(x-y)dy. \tag{2.1.5}$$

It extends to distributions and C^∞ functions, one of them with compact support, replacing the integral by the pairing:

$$u \star \psi(x) = \langle u, \psi(x - \cdot) \rangle \tag{2.1.6}$$

and $u \star \psi \in C^\infty(\mathbb{R}^n)$. It has compact support if both u and ψ have compact support. If $\varphi \in C_0^\infty(\mathbb{R}^n)$ then

$$\int (u \star \psi)(x)\varphi(x)dx = \langle u, \check{\psi} \star \varphi \rangle \tag{2.1.7}$$

where $\check{\psi}(x) = \psi(-x)$. This allows to define the convolution of two distributions u and v, one with compact support, by the formula

$$\langle u \star v, \varphi \rangle = \langle u, \check{v} \star \varphi \rangle \tag{2.1.8}$$

with the obvious definition of $\check{v}$:

$$\langle \check{v}, \psi \rangle = \langle v, \check{\psi} \rangle. \tag{2.1.9}$$

Lemma 2.1.1. *If u and v are distributions on $\mathbb{R}^n$, one at least with compact support, then*

$$\operatorname{supp}(u \star v) \subset \operatorname{supp} u + \operatorname{supp} v. \tag{2.1.10}$$

The derivatives of distributions are defined by

$$\langle \partial_{x_j} u, \varphi\rangle = -\langle u, \partial_{x_j}\varphi\rangle. \tag{2.1.11}$$

In particular, differential operators (2.1.1) act on distributions

$$\langle Pu, \varphi\rangle = \langle u, P^t\varphi\rangle \tag{2.1.12}$$

where, in the scalar case

$$P^t = \sum p_\alpha(-\partial_x)^\alpha \tag{2.1.13}$$

An important remark is the link between convolution and differentiation :

Lemma 2.1.2. *If u and v are distributions on $\mathbb{R}^n$, one at least with compact support, then*

$$\partial_{x_j}(u \star v) = (\partial_{x_j} u) \star v = u \star (\partial_{x_j} v). \tag{2.1.14}$$

In particular,

$$P(u \star v) = (Pu) \star v = u \star (Pv). \tag{2.1.15}$$

All this extends to functions and distributions with values in finite dimensional vector spaces and to systems. Given a finite dimensional vector space $\mathbb{E}$, the space $\mathscr{D}'(\Omega; \mathbb{E})$ of distributions on Ω with values in $\mathbb{E}$ is the dual space of $C_0^\infty(\Omega; \mathbb{E}')$, where the duality (2.1.3) extends for smooth u the pairing

$$\int \langle u(x), \varphi(x)\rangle_{\mathbb{E}\times\mathbb{E}'} dx. \tag{2.1.16}$$

If the coefficients p_α of P are linear mappings from $\mathbb{E}$ to another space $\mathbb{F}$, then P acts from $\mathscr{D}'(\Omega; \mathbb{E})$ to $\mathscr{D}'(\Omega; \mathbb{F})$ according to the identity (2.1.12) with now

$$P^t = \sum p^t_\alpha(-\partial_x)^\alpha \tag{2.1.17}$$

as $p^t_\alpha \in \mathcal{L}(\mathbb{F}' : \mathbb{E}')$ and P^t acts from $C_0^\infty(\Omega; \mathbb{F}')$ into $C_0^\infty(\Omega; \mathbb{E}')$.

The convolution can also be extended to vector valued distributions, provided that the product is properly defined. For example, if $u \in \mathscr{D}'(\mathbb{R}^n; (\mathbb{E}; \mathbb{F})$ and $v \in \mathscr{D}'(\mathbb{R}^n; \mathbb{E})$ one of them with compact support, the convolution $u \star v$ is well defined as a distribution in $\mathscr{D}'(\mathbb{R}^n; \mathbb{F})$. This will be clear in the applications below.

The following general result is due to Ehrenpreis [Ehr] and Malgrange [Mal] (see Theorem 7.3.10 in [Hör]). It is out of the scope of these notes but we mention it, without proof, because of its importance.

Theorem 2.1.3. *For any polynomial* (2.1.1) $P \neq 0$*, there is a distribution* $E \in \mathscr{D}'(\mathbb{R}^n)$ *such that* $P(\partial)E = \delta$*.*

If P *is a system with coefficients in* $\mathcal{L}(\mathbb{E})$ *and such that* $\det P \neq 0$*, then there is a distribution* $E \in \mathscr{D}'(\mathbb{R}^n; \mathcal{L}(\mathbb{E}))$ *such that* $P(\partial)E = \delta \mathrm{Id}_{\mathbb{E}}$*.*

As a consequence, under the assumptions of the theorem, when $f \in \mathscr{E}'(\mathbb{R}^n)$, $u = E \star f$ is a solution of the equation $P(\partial)u = f$.

2.2 Fourier-Laplace analysis

The Fourier transform is very useful to deal with constant coefficients equation. We denote it by $\mathscr{F}$ and often use the notation $\hat{u}$ for $\mathscr{F}u$. When $u \in L^1(\mathbb{R}^n)$ it is defined by

$$\hat{u}(\xi) = \mathscr{F}u(\xi) = \int e^{-i\xi\cdot x} u(x) dx. \tag{2.2.1}$$

We denote by $\mathscr{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing C^∞ function, by $\mathscr{S}'(\mathbb{R}^n) \subset \mathscr{D}'(\mathbb{R}^n)$ its dual space, the space of temperate distributions. The Fourier transform $\mathscr{F}$ acts from $\mathscr{S}$ to $\mathscr{S}$ and extends to $\mathscr{S}'$. It is an isomorphism from $\mathscr{S}$ to $\mathscr{S}$ and from $\mathscr{S}'$ to $\mathscr{S}'$. The inverse transform is defined by

$$u(x) = \frac{1}{(2\pi)^n} \int e^{i\xi\cdot x} \hat{u}(\xi) d\xi \tag{2.2.2}$$

when $\hat{u} \in L^1(\mathbb{R}^n)$. Also recall Plancherel's theorem which asserts that $\mathscr{F}$ is an isomorphism from L^2 to itself and that

$$\big\|u\big\|^2_{L^2} = \frac{1}{(2\pi)^n} \, \big\|\hat{u}\big\|^2_{L^2}. \tag{2.2.3}$$

The importance of the Fourier transform is reflected in the identities

$$\mathscr{F}T_h u = e^{-i\xi\cdot h}\mathscr{F}u, \quad \mathscr{F}(\partial_{x_j} u) = i\xi_j \mathscr{F}u, \quad \mathscr{F}(x_j u) = i\partial_{\xi_j}\mathscr{F}u \tag{2.2.4}$$

where T_h is the translation $T_h u(x) = u(x-h)$. In particular, if $P(\partial)$ is the differential operator (2.1.1)

$$u \in \mathscr{S}' \qquad \Rightarrow \qquad \mathscr{F}(Pu) = P(i\xi)\mathscr{F}u. \tag{2.2.5}$$

The polynomial $P(i\xi) = \sum p_\alpha (i\xi)^\alpha$ is called the *symbol* of $P(\partial)$. Clearly, the resolution of $P(\partial)u = f$ is linked to the inversion $1/P(i\xi)$.

When $u \in \mathscr{E}'(\mathbb{R}^n)$ has compact support, the definition (2.2.1) extends as

$$\hat{u}(\xi) = \langle u, e^{-i\xi\cdot} \rangle \tag{2.2.6}$$

(see (2.1.4)) and this formula shows that $\hat{u}$ extends to complex values $\xi \in \mathbb{C}^n$ and is holomorphic in ξ. This extension is called the *Fourier-Laplace transform* of u. Passing to complex frequencies ξ is a key point in these lectures. For instance, one would like to use Cauchy's theorem to move the integration path in (2.2.2) to a new path which avoids the zeros of $P(i\xi)$.

For ξ and η in $\mathbb{R}^n$, (2.2.6) can be read as

$$\hat{u}(\xi + i\eta) = \mathscr{F}(u_\eta)(\xi), \qquad u_\eta(x) = e^{\eta\cdot x} u(x). \tag{2.2.7}$$

This formula makes sense as soon as $u_\eta \in \mathscr{S}'(\mathbb{R}^n)$ and can be used to define the Fourier-Laplace transform of u for complex frequencies in $\mathbb{R} + i\Gamma$, where Γ is the set of $\eta \in \mathbb{R}^n$ such that $u_\eta \in \mathscr{S}'(\mathbb{R}^n)$ and one can characterize the corresponding set of Fourier Laplace transforms (see chapter 7. 4 in [Hör]). We give here an example of such a result which will be useful later on.

Theorem 2.2.1. *Suppose that Γ is an open convex cone in $\mathbb{R}^n \backslash \{0\}$ and that $\hat{u}$ is an holomorphic function on $\mathbb{R}^n - i\Gamma$ such that there are C and N such that*

$$\forall \zeta \in \mathbb{R}^n - i\Gamma, \qquad |\hat{u}(\zeta)| \le C(1 + |\zeta|)^N. \tag{2.2.8}$$

Then, $\hat{u}$ is the Fourier Laplace transform of a distribution $u \in \mathscr{S}'(\mathbb{R}^n)$ which is supported in the polar cone of Γ:

$$\Gamma^\circ = \{ x \in \mathbb{R}^n \ , \ \forall \eta \in \Gamma \ : \ \eta \cdot x \ge 0 \ \}. \tag{2.2.9}$$

Proof. For $\eta \in -\Gamma$, $\hat{u}_\eta(\xi) = \hat{u}(\xi + i\eta)$ is a temperate distribution and one can define $u_\eta = \mathscr{F}^{-1}\hat{u}_\eta$. Moreover the mapping $\eta \mapsto u_\eta$ is smooth and $\partial_{\eta_j}\hat{u}_\eta = i\partial_{\xi_j}\hat{u}_\eta$. Therefore, $\partial_{\eta_j} u_\eta = x_j u_\eta$ and thus $\partial_{\eta_j}(e^{-\eta\cdot x} u_\eta) = 0$. This shows that there is a distribution $u \in \mathscr{D}'$ such that

$$\forall \eta \in -\Gamma, \quad u_\eta = e^{\eta\cdot x} u \in \mathscr{S}'. \tag{2.2.10}$$

By construction (2.2.6) is satisfied and $\hat{u}$ is the Fourier-Laplace transform of u on $\mathbb{R}^n - i\Gamma$.

The bound (2.2.8) implies that the family u_η is bounded in $\mathscr{S}'$. Since $u_\eta \to u$ in $\mathscr{D}'$ when $\eta \to 0$, this implies that $u \in \mathscr{S}'$.

Moreover, if $x_0 \in \mathbb{R}^n \backslash \Gamma^\circ$ there is $\eta_0 \in -\Gamma$ such that $\eta_0 \cdot x_0 > 0$. Hence, for $\varphi \in C_0^\infty$ supported in a small enough neighborhood of x_0, $e^{-t\eta_0 \cdot x}\varphi \to 0$ in $\mathscr{S}$ when $t \to +\infty$. Hence

$$\big\langle u, \varphi \big\rangle = \big\langle u_{t\eta_0}, e^{-t\eta_0 \cdot x}\varphi \big\rangle \to 0$$

implying that $\langle u, \varphi \rangle = 0$ and thus that u vanishes near x_0. The proof of the theorem is complete. □

2.3 A necessary condition for the well posedness of the Cauchy problem

Consider a system of order m

(2.3.1)
$$P(\partial) = \sum_{|\alpha| \leq m} p_\alpha \partial_\alpha$$

with coefficients $p_\alpha \in \mathcal{L}(\mathbb{E})$. The symbol is $P(i\xi)$ and the *principal symbol* is

(2.3.2)
$$P_m(\xi) = \sum_{|\alpha| = m} \xi^\alpha p_\alpha$$

The *noncharacteristic Cauchy Problem* is associated to a *noncharacteristic hyperplane* $H = \{x : \nu \cdot x = 0\}$ with $\det P_m(\nu) \neq 0$. The usual problem can be stated as

(2.3.3)
$$Pu = f, \qquad X^k u_{|H} = g_k \ for \ 0 \leq k < m$$

and X is a vector field transversal to H[1]

In this formulation, it is implicitely assumed that the solution is smooth enough in the direction X so that the traces of $X^k u$ are well defined, possibly in a space of distributions. When the Cauchy data g_k vanish, extending u by 0 one one side of H,

(2.3.4)
$$Pu = f, \qquad \mathrm{supp} f \subset H_+, \quad \mathrm{supp} u \subset H_+$$

where H_+ denotes the half space $\{\nu \cdot x > 0\}$. A minimal form for the well posed-ness of the Cauchy problem is the condition that

(WP)
$$\begin{cases} \textit{for all } f \in C_0^\infty(H_+), \textit{ the equation } Pu = f \textit{ has a unique} \\ \textit{solution } u \in \mathscr{D}'(\mathbb{R}^n) \textit{ with support contained in } H_+. \end{cases}$$

Using functional analysis, this condition requires a-priori estimates:

[1] In coordinates where $\nu \cdot x = x_1$, one usually takes $X = \partial_{x_1}$.

Lemma 2.3.1. *If the condition WP is satisfied, then for all $f \in C^\infty$ with support in H the equation $Pu = f$ has a unique solution $u \in C^\infty$ with support in H_+. Moreover, if $\underline{x}$ is a point in H_+, there are constants C, R and s such that for all $u \in C^\infty$ with support in H_+:*

$$|u(\underline{x})| \le C \sup_{|\alpha|\le s,\ |x|\le R} |\partial^\alpha Pu(x)|. \tag{2.3.5}$$

Proof. See Lemma 12.3.2 in Hörmander [Hör] and the estimate (12.3.3) which follows. □

Theorem 2.3.2. *Suppose that the estimate* (2.3.5) *is satisfied. Then, there is a real γ_0 such that*

$$\det P(i(\xi + \tau\nu)) \neq 0 \quad if \quad (\tau, \xi) \in \mathbb{C} \times \mathbb{R}^n,\ \operatorname{Im}\tau < -\gamma_0. \tag{2.3.6}$$

Proof. Choose a function $\chi \in C^\infty(\mathbb{R})$ supported in $t > 0$ and such that $\chi = 1$ for $t > \frac{1}{2}\nu \cdot \underline{x} := \underline{t}$. Consider

$$u(x) = \chi(\nu \cdot x) e^{i(\xi+\tau\nu)\cdot x} r$$

with $\xi \in \mathbb{R}^n$, $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau < 0$, $\det P(i(\xi+\tau\nu)) = 0$ and $r \neq 0$ satisfying $P(i(\xi + \tau\nu))r = 0$. In particular, $Pu = 0$ when $t > \underline{t}$ and the right hand side of (2.3.5) is

$$O\big(e^{-\underline{t}\operatorname{Im}\tau}(1 + |\tau|^2 + |\xi|^2)^{s/2}\big)|r|$$

On the other hand $u(\underline{x}) = |r|e^{-2\underline{t}\operatorname{Im}\tau}|r|$ and (2.3.5) implies that, with a new constant C,

$$e^{-\underline{t}\operatorname{Im}\tau} \le C(1 + |\tau|^2 + |\xi|^2)^{s/2}. \tag{2.3.7}$$

Then the theorems follows immediately from the next lemma. □

Lemma 2.3.3. *Let $p(\tau, \xi)$be a polynomial in (τ, ξ). Suppose that there are C and M such that for all $\xi \in \mathbb{R}^n$ the roots of $p(\tau, \xi)$ in $\{im\tau < 0\}$ satisfy*

$$e^{-\operatorname{Im}\tau} \le C(1 + |\tau|^2 + |\xi|^2)^M.$$

Then, there is γ such that for all $\xi \in \mathbb{R}^n$, the roots of $p(\tau, \xi)$ are located in $\{\operatorname{Im}\tau \ge -\gamma\}$.

Proof. Let $\mu(\sigma) = \sup(-\operatorname{Im}\tau)$ where the supremum is taken for the $(\tau, \xi) \in \mathbb{C} \times \mathbb{R}^d$ such that $p(\tau, \xi) = 0$, $\operatorname{Im}\tau < 0$ and $|(\tau, \xi)| = \sigma$. The assumption implies that $\mu(\sigma)$ grows at most logarithmically as $\sigma \to \infty$. By a lemma on sub-algebraic functions (see e.g. Corollary A.2.6 in [Hör]), this implies that μ is bounded and the lemma follows. □

2.4 Hyperbolic polynomials

The following definition is motivated by Theorem 2.3.2.

Definition 2.4.1. *A polynomial* $p(\xi) = \sum_{|\alpha|\le m} p_\alpha \xi^\alpha$ *with principal part* $p_m = \sum_{|\alpha|=m} p_\alpha \xi^\alpha$ *is said to be hyperbolic in the direction* ν *if* $p_m(\nu) \neq 0$ *and there is* γ_0 *such that* $p(i\tau\nu + \tilde{\xi}) \neq 0$ *for all* $\xi \in \mathbb{R}^{1+d}$ *and all real* $\tau < -\gamma_0$.

A system $P(\partial_x)$ (2.1.1) *is said to be hyperbolic in the direction* $\nu \in \mathbb{R}^{1+d}$ *if the characteristic determinant is* $\det P(i\xi)$.

Theorem 2.4.2. *Suppose that* p *is hyperbolic in the direction* ν. *Then,*

i) p_m *is hyperbolic in the direction*ν, *which is equivalent to the conditions that for all* $\xi \in \mathbb{R}^n$, *the roots in* τ *of* $p_m(\tau\nu + \xi) = 0$ *are real;*

ii) p *is also hyperbolic in the direction* $-\nu$; *In particular, there is* γ_1 *such that the roots in* τ *of* $p(\xi + \tau\nu) = 0$ *are located in* $|\mathrm{Im}\,\tau| \le \gamma_1$;

iii) if Γ *denotes the component of* ν *in the open set* $\{p(\xi) \neq 0\}$, *then* Γ *is an open convex cone in* $\mathbb{R}^{1+d}$ *and* p *is hyperbolic in any direction* $\vartheta \in \Gamma$;

iv) There are constants $c > 0$ *and* γ *such that for all* $\zeta \in \mathbb{C}^n$

$$\mathrm{Im}\,\zeta \in -i\gamma\nu - i\Gamma \quad \Rightarrow \quad |p(\zeta)| \ge c. \tag{2.4.1}$$

Proof. (See Gårding [Går] or chapter 12 in Hörmander [Hör]).

a) Note that the assumption is equivalent to the condition

$$\xi \in \mathbb{R}^n,\ \tau \in \mathbb{C},\ \mathrm{Im}\,\tau < -\gamma_0 \quad \Rightarrow \quad p(\xi + \tau\nu) \neq 0. \tag{2.4.2}$$

Moreover,

$$p_m(\xi + \tau\nu) = \lim_{t\to\infty} p(t\xi + t\tau\nu).$$

The roots of $p(t\xi + t\tau\nu)$ are located in $\mathrm{Im}\,\tau \ge -\gamma_0/t$ and by continuity of the roots of polynomials, the roots of $p_m(\xi + \tau\nu)$ are located in $\mathrm{Im}\,\tau \ge 0$, as well as the roots of $p_m(\xi - \tau\nu) = (-1)^m p_m(-\xi + \tau\nu)$, thus these roots are real. This proves i).

b) Note that $p_m(-\nu) = (-1)^m p_m(\nu) \neq 0$. Write the polynomial in τ, $p(\xi + \tau\nu)$, as

$$p(\xi + \tau\nu) = p_m(\nu)\big(\tau^m - c(\xi)\tau^{m-1} + \dots\big)$$

where c is the sum of the roots τ_j. By assumption $\mathrm{Im}\,c \ge -m\gamma_0$. Moreover $c(\xi)$ is linear in ξ and therefore $\mathrm{Im}\,c$ is a constant. Hence

$$\mathrm{Im}\,\tau_j = \mathrm{Im}\,c - \sum_{k\neq j} \mathrm{Im}\,\tau_k \le \mathrm{Im}\,c + (m-1)\gamma_0. \tag{2.4.3}$$

This proves ii). Using i) we see that

$$\operatorname{Im} c = -\operatorname{Im} p_{m-1}(\nu)/p_m(\nu)$$

where $p_{m-1}(\xi) = \sum_{|\alpha|=m-1} p_\alpha \xi^\alpha$ is the homogeneous part of degree $m-1$ of m. Thus, the inequality above provides us with an explicit bound for γ_1.

c) For $\xi \in \mathbb{R}^n$, remember that all the roots in τ of $p_m(\xi + \tau\nu) = 0$ are real by i) and introduce

$$\Gamma_1 = \{\xi \in \mathbb{R}^n \ : \ p_m(\xi + \tau\nu) = 0 \ \Rightarrow \ \tau < 0\},$$

$$\Gamma_0 = \{\xi \in \mathbb{R}^n \ : \ p_m(\xi + \tau\nu) = 0 \ \Rightarrow \ \tau \neq 0\}.$$

Note that $\Gamma_1 \subset \Gamma_0 = \{p_m \neq 0\}$. By continuity of roots of polynomials, Γ_1 is open. Moreover, for $\xi \in \overline{\Gamma}_1$ the roots are in $\{\tau \leq 0\}$ and thus $\overline{\Gamma}_1 \cap \Gamma_0 = \Gamma_1$. This shows that Γ_1 is open and relatively closed in $\{p_m \neq 0\}$. Since $\nu \in \Gamma_1$ because for $\xi = \nu$ the zeros are $\tau = -1$. This implies that Γ_1 contains the component of ν in $\{p_m \neq 0\}$, that is Γ.

For $\xi \in \gamma_1$ and $t \in]0,1]$, let $\xi_t = t\xi + (1-t)\nu$. Then

$$p_m(\xi_t + \tau\nu) = t^m p_m(\xi + \sigma\nu), \quad \sigma = (1 - t + \tau)/t$$

and $\sigma < 0$ implies that $\tau < t - 1 \leq 0$. Hence the segment $[\nu, \xi]$ is contained in $\Gamma_1 \subset \Gamma_0$ implying that Γ_1 is contained in the component of ν in Γ_0. This shows that $\Gamma_1 = \Gamma$ is star shaped with respect to ν and that

$$\vartheta \in \Gamma, \ p_m(\vartheta + \tau\nu) = 0 \ \Rightarrow \ \tau < 0. \tag{2.4.4}$$

Equivalently, since $p_m(\vartheta)$ and $p_m(\nu)$ do not vanish,

$$\vartheta \in \Gamma, \ p_m(\sigma\vartheta + \nu) = 0 \ \Rightarrow \ \sigma < 0. \tag{2.4.5}$$

d) We prove that if $\vartheta \in \Gamma$ then

$$\xi \in \mathbb{R}^n, \ \operatorname{Im}\tau < -\gamma_0, \ \operatorname{Im}\sigma \leq 0 \quad \Rightarrow \quad p(\xi + \tau\nu + \sigma\vartheta) \neq 0. \tag{2.4.6}$$

This follows from (2.4.2) when σ is real. Therefore, for all $(\xi, \tau) \in \mathbb{R}^n \times \mathbb{C}$ with $\operatorname{Im}\tau < -\gamma_0$ the polynomial in σ, $p(\xi + \tau\nu + \sigma\theta)$ has no real real roots and hence the number m_- of roots in $\{\operatorname{Im}\sigma < 0\}$ is independent of (ξ, τ). We compute this number for $\xi = 0$ and $\tau = -i\lambda$ with λ real large. Rescaling $\sigma = i\lambda\tilde{\sigma}$, m_- is the number of roots in $\{\operatorname{Re}\tilde{\sigma} < 0\}$ of

$$\lambda^{-m} p(-i\lambda\nu + \lambda\tilde{\sigma}\vartheta).$$

When λ tends to $+\infty$, this polynomial converges to $(-i)^m p_m(\nu - \tilde{\sigma}\vartheta)$, which ha all its roots in $\operatorname{Re}\tilde{\sigma} > 0$ by (2.4.5). Hence $m_- = 0$ and (2.4.6) is proved.

e) Since Γ is open, for $\vartheta \in \Gamma$ there is $\varepsilon > 0$ such that $\vartheta' = \vartheta - \varepsilon\nu \in \Gamma$. Thus (2.4.6) applied to ϑ' implies that

$$p(\xi + (\tau - \varepsilon\sigma)\nu + \sigma\vartheta) \neq 0$$

when $\operatorname{Im}\tau < -\gamma_0$ and $\operatorname{Im}\sigma \leq 0$. If

$$\operatorname{Im}\sigma < -\gamma_0/\varepsilon \tag{2.4.7}$$

then $\tau = \varepsilon\sigma$ satisfies $\operatorname{Im}\tau < -\gamma_0$ and $p(\xi + \sigma\vartheta) \neq 0$. This shows that p is hyperbolic in the direction ϑ.

We have shown that Γ is star-shaped with respect to ν and this is true for any direction of hyperbolicity, thus for all $\vartheta \in \Gamma$. Hence Γ is convex finishing the proof of *iii*).

f) According to (2.4.6), for all $\zeta \in \mathbb{R}^n - i\Gamma$ the roots τ_j of $p(\zeta + \tau\nu)$ are located in $\operatorname{Im}\tau \geq -\gamma_0$. Hence

$$|p(\zeta - i\gamma\nu| = |p_m(\nu)| \prod_j | - i\gamma - \tau_j| \geq |p_m(\nu)|(\gamma - \gamma_0)^m \tag{2.4.8}$$

if $\gamma > \gamma_0$. This implies *iv*) and the proof of the theorem is complete. □

2.5 Fundamental solutions of hyperbolic equations

Theorem 2.5.1. *Suppose that $P(\partial)$ is a differential system* (2.3.1) *on $\mathbb{E}$, which is hyperbolic in the direction ν. Γ. Then there is a fundamental solution E which is supported in the polar cone Γ° associated to the cone of hyperbolic directions $\Gamma(\nu)$. Moreover, there is a real γ such that $e^{-\gamma\nu\cdot x}E \in \mathscr{S}'(\mathbb{R}^n)$.*

Proof. We construct E as the inverse Fourier-Laplace transform of $P(i\xi)^{-1}$. We have

$$P(\partial)e^{\gamma\nu\cdot x}u = e^{\gamma\nu\cdot x}P(\partial + \gamma\nu)u \tag{2.5.1}$$

so that we can replace $P(\partial)$ by $P(D + \gamma\nu)$ and use (2.4.8) if γ is chosen large enough, which we assume from now on.

Because $P(i\xi)^{-1} = p^{-1}(\xi)Q(i\xi)$ where Q is the transposed of the matrix of cofactors of P, we see that $P(i\xi + \gamma\nu)$ is invertible when $\xi \in \mathbb{R}^n - i\Gamma$. Q is a polynomial in ξ of degree $m(N-1)$ where $N = \dim\mathbb{E}$ and that there is a constant C such that

$$\operatorname{Im}\xi \in -\Gamma \quad \Rightarrow \quad \left|P^{-1}(i\xi + \gamma\nu)\right| \leq C(|\gamma| + |\xi|)^{m(N-1)}. \tag{2.5.2}$$

Therefore, by Theorem 2.2.1, $P^{-1}(i\xi+\gamma\nu)$ is the Fourier-Laplace transform of a temperate distribution E_γ supported in Γ°. By construction, the Fourier transform of $P(D+\gamma\nu)E_\gamma$ is $P(i\xi+\gamma\nu)\hat{E}_\gamma(\xi) = \mathrm{Id}_{\mathbb{E}}$ so that $P(D+\gamma\nu)E_\gamma = \delta \mathrm{Id}_{\mathbb{E}}$, and $E = e^{\gamma\nu\cdot x}E_\gamma$ has the desired properties. □

Corollary 2.5.2. *Let P be hyperbolic in the direction ν. Then, for all $f \in \mathscr{D}'(\mathbb{R}^n)$ supported in $\{\nu \cdot x \geq 0\}$ the equation $P(D)u = f$ has a unique solution $u \in \mathscr{D}'(\mathbb{R}^n)$ supported in $\{\nu \cdot x \geq 0\}$. If f is C^∞, then the solution is C^∞. Moreover,*

$$\operatorname{supp} u \subset \operatorname{supp} f + \Gamma^\circ.$$

Proof. $u = E \star f$ has the desired property. That it is unique is a consequence of Holmgren uniqueness theorem. □

Remarks 2.5.3. 1. Since P is hyperbolic in the direction $-\nu$, there is another fundamental solution supported in $-\Gamma^\circ$.

2. By Holmgren uniqueness theorem, E is the unique fundamental solution supported in the half plane $\{\nu \cdot x \geq 0\}$.