DELIGNE RIEMANN ROCH AND CUBIC STRUCTURE

(Joint work with Ted Chinburg, George Pappas)

NOTATION: O **is the ring of integers of a number field** K, X **is a regular, projective, flat arithmetic surface over** $SpecO$ **with structure map** $f : X \to SpecO$. **As in Ted's talk we suppose that a finite group** G **acts tamely on** X **and that the quotient** $Y = X/G$ **is also regular In addition, here we shall assume that** G **is *abelian*.**

We have seen in previous lectures how to determine various Euler characteristics associated to the de Rham complex of X**. Our aim here is to explicitly determine**

$$Euler\,(O_X) \in Cl\,(\mathbf{Z}G)$$

without the ambiguities **of the previous Riemann Roch theorems (which are annihilated by** $|G|$**).**

KEY IDEAS

• (Class groups and algebra) use of the notion of cubic structure of $\mathbf{Z}G$**-modules. Here we shall restrict down to the third term in the** γ**-filtration of the character ring. This will have the effect of enabling us to describe Euler characteristics in terms of branch locus data. We have of course already seen such phenomena in Ted's talks.**

• use of the Deligne-Riemann-Roch theorem phrased in terms of the Deligne pairing of Y**-line bundles. A crucial feature here is that the Riemann-Roch isomorphism is functorial. This will enable us to carry out our calculations with 2 dimensional resolvents using local bases.**

• the theorem of the cube for the Jacobian of a curve translates into the important fact that the determinant of cohomology has cubic structure.

PHILOSOPHY

We have seen that the Euler characteristic of a De Rham complex is closely related to that of a 1-cycle to which we can apply Fröhlich methods.

Here we introduce a method which will enable us to determine the Euler characteristics of 2-cycles. To do this we develop a "calculus" for computing resolvents in 2-dimensional situations.

Our results may be thought of, in the first instance, as a 2 dimensional version of L. McCulloh's description of the Galois structure of the ring of integers of a tame abelian relative extension of number fields.

GAMMA FILTRATIONSLet R_G **be the ring of virtual complex characters of** G, **and put**

$$I_G = \ker\left(\deg : R_G \to \mathbf{Z}\right).$$

Recall that for degree 1 character χ_i **and** $n > 0$

$$\gamma^n\left(\chi_i - 1\right) = \begin{cases} \left(\chi_i - 1\right) \textbf{ if } n = 1 \\ 0 \textbf{ if } n > 1. \end{cases}$$

Since every character is a sum of degree 1 characters we see that F_G^n, **the** n**th term in the** γ**-filtration of** R_G, **is** I_G^n.

Write $G^{(3)} = G \times G \times G$. **For** $S \subset \{1,2,3\}$ **we define group the homomorphism** $\Delta_S : G \to G^{(3)}$ **by the rule**

$$\Delta_S\left(g\right)_k = \begin{cases} 1 \textbf{ if } k \notin S \\ g \textbf{ if } k \in S. \end{cases}$$

Dualising we get $\Delta_S^D : R_{G^{(3)}} \to R_G$ **where for** $g \in G$ **and abelian characters** χ_i **of** G

$$\Delta_S^D\left(\chi_1 \otimes \chi_2 \otimes \chi_3\right)\left(g\right) = \textstyle\prod_{i \in S} \chi_i\left(g\right).$$

We put these together to form

$$\nabla : R_{G^{(3)}} \to R_G \textbf{ with } \nabla = -\textstyle\sum_S \left(-1\right)^{|S|} \Delta_S^D$$

so that LEMMA 1

$$\nabla\left(\chi_1 \otimes \chi_2 \otimes \chi_3\right)\left(g\right) = \textstyle\prod_{i=1}^3 \left(\chi_i\left(g\right) - 1\right)$$

and therefore $\operatorname{Im} \nabla = F_G^3$.

∇ **therefore induces a map** ∇'

$$\begin{array}{ccc} R_{G^{(3)}} & \overset{\nabla}{\to} & R_G \\ & \nabla' \searrow & \uparrow inc \\ & & F_G^3 \end{array}$$

Let J_f be finite ideles of large number field F containing K. Presently we shall see that on applying $Hom_{\Gamma_K}(-, J_f)$ we get a diagram of classgroups.

$$\begin{array}{ccc} Cl\left(OG^{(3)}\right) & \overset{\nabla^*}{\leftarrow} & Cl\left(OG\right) \\ & \nabla'^* \nwarrow & \downarrow \delta \\ & & Cl^{(3)}\left(OG\right) \end{array}$$

Thus at this stage the term $Cl^{(3)}(OG)$ has yet to be properly defined - but where we can see that it will be represented by idele valued character functions on F_G^3.

HYPOTHESES AND NOTATION

Suppose that irred. components of the fibres of Y are smooth and

that bad fibres are reduced with normal crossings. b =branch locus of X/Y (as previously). Now suppose that b is supported over a single maximal ideal $v \in Spec(O)$ and that $b = b_1 \cup b_2$ with b_i irred. divisors. Let B_i denote an irred. divisor of X above b_i ; then the cotangent space of the generic point ξ_i of B_i is a G-stable line; let ϕ_i denote the asso'd character of the inertia group I_i of ξ_i; then ϕ_i is a *faithful abelian v-adic* character of I_i. Moreover by standard theory the inertia group of any crossing point $b_1 \cap b_2$ is the direct sum

$$I = I_1 \oplus I_2.$$

We put $e_i = |I_i|$ and write $\widehat{I}$ for group of abelian v-adic characters of I. For any $\phi \in \widehat{I}$ we can write

$$\phi = \phi_1^{a_1}.\phi_2^{a_2} \text{ with } 0 \leq a_i < e_i.$$

STICKELBERGER MAPS

Define

$$r_i : \mathbf{Z} \bmod (e_i) \to [0, e_i) \cap \mathbf{Z}$$

to be the map characterised by $r_i(z) \equiv z \bmod (e_i)$. Then for $\phi = \phi_1^{a_1}.\phi_2^{a_2}$ we define

$$\theta_i : \widehat{I} \to \frac{1}{e_i}\mathbf{Z}$$

by the rule

$$\theta_i(\phi) = \frac{r_i(a_i)}{e_i}.$$

For $1 \leq i, j \leq 2$, we define

$$\theta_i.\theta_j(\phi) = \theta_i(\phi)\,\theta_j(\phi) = \frac{r_i(a_i)\,r_j(a_j)}{e_i e_j}$$

We then extend $\theta_i.\theta_j$ to R_I by linearity.

MAIN THEOREM

S'pose $(6, |G|) = 1$. The image of 2.Euler(O_X) in $Cl^{(3)}(\mathbf{Z}G)$ is represented by $\sigma : F_G^3 \to J_f$ where for $\psi \in F_G^3$ and for a finite place w of F

$$\sigma(\psi)_w = \begin{cases} 1 \text{ if } w \nmid p \\ Nv^{Q(\psi_w|_I)+L(\psi_w|_I)} \text{ if } w \mid p \end{cases}$$

where for $\phi \in \widehat{I}$

$$Q(\phi) = \textstyle\sum_{i,j=1}^{2} \theta_i.\theta_j(\phi)\,(b_i, b_j)_Y$$

$$L(\phi) = \textstyle\sum_{i=1}^{2} \theta_i(\phi)\left(b_i, \omega_{Y/\mathbf{Z}}\right)_Y$$

REMARK. Later on we shall see later that we lose absolutely nothing by restricting to $Cl^{(3)}(\mathbf{Z}G)$. CUBIC STRUCTURES. The

key-idea here is to look at OG classes whose Fröhlich representatives have special symmetries when restricted to the third term of the γ-filtration. DEFINITION For an abelian group A, a homo-

morphism $f : R_{G^{(3)}} \to A$ is called *cubic* if 1. (commutativity) for

$\sigma \in S_3$,

$$f(\chi_1 \otimes \chi_2 \otimes \chi_3) = f(\chi_{\sigma 1} \otimes \chi_{\sigma 2} \otimes \chi_{\sigma 3});$$

2. (associativity) for all $\chi, \phi, \psi, \xi \in \widehat{G}$

$$\begin{aligned} & f(\chi\phi \otimes \psi \otimes \xi) f(\chi \otimes \phi \otimes \xi) \\ & = f(\phi \otimes \psi \otimes \xi) f(\chi \otimes \phi\psi \otimes \xi). \end{aligned}$$

LEMMA 2. If $h : R_G \to A$, **then** $\nabla'^{*}(h)$ **is cubic. Proof: An uninspiring check of the identities. E.g. for (1) we use Lemma 1**

$$\nabla'^{*}(h)(\chi_1 \otimes \chi_2 \otimes \chi_3) = h\left(\prod_{i=1}^{3} (\chi_i - 1)\right).$$

DEFINITION For $w \in Spec(O)$ **and** $z \in O_w G^{(3)\times}$ **we call** $Det(z)$ ***cubic*** **if the map** $Det(z) : R_{G^{(3)}} \to O_{F_w}^{\times}$ **is cubic (in the above sense). We then let** T_w **denote the group of cubic elements in** $Det\left(O_w G^{(3)\times}\right)$.

Using Lemma 2 we can see that ∇ **induces a map** $Cl(\nabla)$

$$Cl(OG) \to \frac{Hom_{\Gamma_K}(R_{G^{(3)}}, J_f)}{T_0 \prod_{v<\infty} T_v} \overset{defn}{=} Cub(OG).$$

This then provides some motivation for DEFINITION The class $[L] \in Cl(OG)$ **of an** OG**-line bundle** L **is** ***cubic*** **iff**

$$Cl(\nabla)([L]) = 0.$$

THEOREM (Breen) Let G be the constant group scheme of G **over** O **and let** $\boldsymbol{G}^D$ **be the Cartier dual of G. Then there is a map**

$$\{cubic\ OG - bdles\} \to \mathbf{Biext}^1\left(\boldsymbol{G}^D, \boldsymbol{G}^D; G_m\right)$$

whose kernel is a homomorphic image of $H^1(O, \boldsymbol{G})$. **DEFINITION For brevity we now put** $D = T_0 \prod_{v<\infty} T_v$ **and consider the group** $\nabla'^{-1}(D) \subset Hom_{\Gamma_K}(F_G^3, J_f)$. **So the preceding diagram now gives following diagram (which defines** $Cl^{(3)}(OG)$**)**

$$\begin{array}{ccccc} Cl(OG) & \overset{\delta}{\to} & Cl^{(3)}(OG) & = & \frac{Hom_{\Gamma_K}\left(F_G^3, J_f\right)}{\nabla'^{-1}(D)} \\ & cl(\nabla) \searrow & \downarrow & & \downarrow \nabla' \\ & & Cub(OG) & = & \frac{Hom_{\Gamma_K}\left(R_{G^{(3)}}, J_f\right)}{D} \end{array}$$

In the special case when $O = \mathbf{Z}$ there are some very special features. Firstly THEOREM (Pappas)

When $O = \mathbf{Z}$, Biext$^1\left(\boldsymbol{G}^D, \boldsymbol{G}^D; G_m\right) = \{1\}$.

COROLLARY. Since $H^1\left(Spec\left(\mathbf{Z}\right), \boldsymbol{G}\right) = \{1\}$, by Breen there are no non trivial cubic classes. Since $\ker Cl\left(\nabla\right)$ consists of cubic classes, we see that $Cl\left(\nabla\right)$, and hence δ, is injective. $\square$

Secondly, again with $O = \mathbf{Z}$, we have a much sharper description of $\nabla'^{-1}\left(D\right)$. To this end for $w \in Spec\left(O\right)$ we define

$$H_w = \nabla'^{-1}\left(T_w\right)$$

so that by definition

$$\nabla'^{-1}\left(D\right) = \nabla'^{-1}\left(T_0 \textstyle\prod_{v<\infty} T_v\right) \supseteq H_0 \textstyle\prod_{v<\infty} H_v.$$

THEOREM If $O = \mathbf{Z}$ and if $(6, |G|) = 1$, then

$$H_0 = Hom_{\Gamma_{\mathbf{Q}}}\left(F_G^3, \mathbf{Q}^{c\times}\right)$$

$$\nabla'^{-1}\left(D\right) = H_0 \textstyle\prod_{p<\infty} H_p.$$

REMARK From the explicit trivialisation of a bundle on $OG^{(3)}$, which

comes with a cubic structure, one could construct a further invariant in a relative K-group. However, the essential point here is to know whether particular classes (arising from the cohomology of X) have cubic structure or not. We now wish to apply this to the geometric situation we considered at the start of the lecture. Recall: we have a G-cover $\pi : X \to Y$ with branch locus $b = b_1 \cup b_2$ supported over v. We let $\widetilde{Y} = Y \otimes \mathbf{Z}G$, so that on an open U of Y, $O_{\widetilde{Y}}\left(U\right) = O_Y\left(U\right)G$. DELIGNE PAIRING. Let L, M be $\widetilde{Y}$-line

bundles. Then the Deligne symbol $\langle L, M\rangle$ is the OG-line bundle generated locally (in the etale topology) by symbols $\langle l, m\rangle$ for rational sections l resp. m of L resp. M with *disjoint divisors* and with relations

$$\langle l, fm\rangle = f\left(\mathbf{div}\left(l\right)\right)\langle l, m\rangle$$

for a rational function f on $\widetilde{Y}$ with disjoint support from l. We then have canonical isomorphisms

$$\langle L, M_1 \otimes M_2 \rangle \cong \langle L, M_1 \rangle \otimes \langle L, M_2 \rangle$$

$$\langle L, M \rangle \cong \langle M, L \rangle$$

which we henceforth regard as identifications. EXAMPLE. With these identifications for any line bundle L, $\left\langle L, O_{\widetilde{Y}} \right\rangle = OG$; and for rational sections l resp. y of L,resp. $O_{\widetilde{Y}}$, $\langle l, y \rangle \in OG^{\times}$. THEOREM. (DRR) Write $\widetilde{f} : \widetilde{Y} \to Spec(OG)$ for the structure map, and let $\omega_{\widetilde{Y}/OG}$ be the asso'd canonical divisor. Then for a $\widetilde{Y}$-line bundle L there is a ***functorial*** isomorphism of OG-bundles

$$\det\left(R\widetilde{f}_*L\right)^2 \det\left(R\widetilde{f}_*\widetilde{Y}\right)^{-2} \cong \langle L, L \rangle \left\langle L, \omega_{\widetilde{Y}/OG} \right\rangle^{-1}$$

APPLICATION: • Put $L = \pi_* O_X$ which is locally free over $O_Y G$ by tameness;

• ignore the term $\det\left(R\widetilde{f}_*\widetilde{Y}\right)$ which will have the form $M \otimes_O OG$ and in particular will have trivial class when $O = \mathbf{Z}$;

• work out the two Deligne symbols $\langle O_X, O_X \rangle, \left\langle O_X, \omega_{\widetilde{Y}/OG} \right\rangle$.

In the sequel we shall in fact ignore the latter symbol, which gives the linear term in the Main Theorem, and instead concentrate on the term $\langle O_X, O_X \rangle$ which yields the quadratic term.

STRATEGY.

Consider $w \in Spec\,(O)$ and a given horizontal divisor D of Y_w. We call a rational function f on X_w ***a generic normal integral basis of*** O_{X_w} ***w.r.t.*** D, if the divisor of f, as a section of the $\widetilde{Y}_w$-bundle O_X, has image on Y_w which is horizontal and disjoint from D. THEOREM. For any given such D, we can find a generic normal integral basis of O_{X_w} w.r.t. D. KEY-POINT. Suppose now that for each $w \in Spec\,(O)$, we select 2 generic normal integral bases x_w, y_w with disjoint divisors, then

$$\langle O_{X_w}, O_{X_w} \rangle = O_w G \langle x_w, y_w \rangle .$$

Comparing bases for $w = (0)$ with bases at a maximal ideal v, we write

$$\langle x_0, y_0 \rangle = \lambda_v \langle x_v, y_v \rangle$$

for some $\lambda_v \in K_v G^{\times}$. And so the class of $\langle O_X, O_X \rangle$ in $Cl^{(3)}(OG)$ is represented by the character function

$$\theta \mapsto \prod_v Det(\lambda_v)(\theta) \text{ for } \theta \in F_G^3.$$

So now we must find a way to determine the λ_v. For this we need to use some resolvents - in the form of various eigen sections for the action of G.

First we need some notation. For simplicity in this section we shall assume K contains sufficient roots of unity. Let $\psi \in \widehat{G}$; then via the homomorphism $\psi : O_Y G \to O_Y$, a $\widehat{Y}$-vector bundle L pulls back to a Y bundle $(L \mid \psi)$. By the functorailty of the Deligne symbol we find

$$(\langle x, y \rangle \mid \psi) = \langle (x \mid \psi), (y \mid \psi) \rangle$$

We then use linearity to define

$$(\langle x, y \rangle \mid \theta) \text{ for all } \theta \in R_G.$$

THEOREM. For any 2 disjoint rational sections x, y of O_{X_w} and for any $\theta \in F_G^3$

$$(\langle x, y \rangle \mid \theta) \in K_w^{\times}$$

Furthermore, if X_w/Y_w is etale, then we have $(\langle x, y \rangle \mid -) \in H_w$.

Recall: $h \in H_w$ means $\nabla' h \in Det\left(O_w G^{(3)\times}\right)$ and has cubic symmetries.

In fact building on the proof of the latter part of the theorem we are able to reduce to the "totally ramified" case where $G = I$. Henceforth we now suppose this to be the case and, by what

has gone before, we know that the class of the Deligne symbol $\langle O_X, O_X \rangle$ is represented by the character function

$$\theta \mapsto (\langle x_0, y_0 \rangle \mid \theta) \prod_{w<\infty} (\langle x_w, y_w \rangle \mid \theta)^{-1}$$

and so by the theorem it is also represented by

$$\theta \mapsto \begin{cases} 1 \text{ away from } v \\ (\langle x_v, y_v \rangle \mid \theta)^{-1} \end{cases}$$

for $\theta \in F_G^3$.

SPECIAL GENERATORS.

Recall that we can write each $\phi \in \widehat{I}$ **as** $\phi = \phi_1^{a_1} \phi_2^{a_2}$. **We now set**

$$x_1 = (x_v \mid \phi_1 \otimes 1), \quad x_2 = (x_v \mid 1 \otimes \phi_2)$$

$$y_1 = (y_v \mid \phi_1 \otimes 1), \quad y_2 = (y_v \mid 1 \otimes \phi_2)$$

Again one finds that each pair x_i, y_i **has disjoint** $Y-$**divisors as sections of the line bundle** $(O_X \mid \phi_1 \otimes 1)$ **etc. Write each such divisor as div**$_Y(x_i)$. **Note that each of** x_i, y_i **is also a rational function over** X **and so as such has a divisor which we denote div**$_X(x_i)$. **The two divisors are related as follows: let** A_i **denote the union of the distinct divisors on** X **over** b_i, **then LEMMA 3. (a) div**$_X(x_i) =$

$A_i + H_i$, **div**$_X(y_i) = A_i + J_i$ **with** H_i, J_i **disjoint horizontal divisors.**

(b) div$_Y(x_i) = H_i$, **div**$_Y(y_i) = J_i$. **WARNING: From here on we**

shall identify a G**-stable horizontal divisor on** X **with its image on** Y.

We can now exhibit our "special generators". For $i = 1, 2$ **we first put**

$$T_i = 1 + x_i + x_i^2 + ... + x^{e_i - 1}$$

$$S_i = 1 + y_i + y_i^2 + ... + y^{e_i - 1}$$

$$T = T_1 T_2, \quad S = S_1 S_2.$$

The strategic gain with such generators is: LEMMA 4.

$$(T \mid \phi_1^{a_1}\phi_2^{a_2}) = x_1^{r_1(a_1)} x_2^{r_2(a_2)}.$$

Then S, T are generic normal bases of O_{X_v} with disjoint divisors. So now it suffices to determine the Deligne symbols $(\langle T, S\rangle \mid \theta)$ for $\theta \in F_G^3$,i.e. we must determine $\left(\langle T, S\rangle \mid \nabla\phi^{\underline{a}} \otimes \phi^{\underline{b}} \otimes \phi^{\underline{c}}\right)$ for $\phi^{\underline{a}}, \phi^{\underline{b}}, \phi^{\underline{c}} \in \widehat{I}$.

In fact the following shows that we need only determine the valuations of these elements in $K_v^\times$: PROPOSITION 1

$$Hom\left(F_G^3, O_v^\times\right) \subset H_v.$$

As our next preparatory step we show PROPOSITION 2

For all $i, j \quad \langle x_i, y_j\rangle^{e_i} \in K_v^\times$ and

$$v\,\langle x_i, y_j\rangle^{e_i e_j} = -\left(b_i, b_j\right)_Y$$

Proof. For first part note that $\langle x_i, y_j\rangle^{e_i} = x_i^{e_i}\left(\mathbf{div}_Y\left(y_j\right)\right) \in K_v^\times$.

For the 2nd part suppose for simplicity that $i \neq j$. Then, as above,

$$v\left\langle x_i^{e_i}, y_j^{e_j}\right\rangle = v\left(x_i^{e_i}\left(\mathbf{div}_Y\left(y_j^{e_j}\right)\right)\right) \overset{L3}{=} v\left(x_i^{e_i}\left(e_j J_j\right)\right).$$

As $H_i \cap J_j = \emptyset$ this valuation is equal to the intersection number

$$\left(e_i A_i + e_i H_i, e_j J_j\right)_Y = \left(e_i A_i, e_j J_j\right)_Y = \left(b_i, e_j J_j\right)_Y$$

As $i \neq j$, we know that $y_j^{e_j}$ is a non-zero function on b_i. Since $\mathbf{div}_{Y_w}\left(y_j^{e_j}\right) = b_j + e_j J_j$ it follows that

$$\left(b_i, e_j J_j\right)_Y = -\left(b_i, b_j\right)_Y. \quad \square$$

PROOF OF MAIN THEOREM.

Recall that we are now required to show that

$$v\left(\langle T,S\rangle \mid \nabla\left(\phi^{\underline{a}}\otimes\phi^{\underline{b}}\otimes\phi^{\underline{c}}\right)\right)$$
$$=\sum_{i,j}\left(b_i,b_j\right)_Y \theta_i.\theta_j\left(\nabla\left(\phi^{\underline{a}}\otimes\phi^{\underline{b}}\otimes\phi^{\underline{c}}\right)\right)$$

Sketch proof of this equality.

By definition $v\left(\langle T,S\rangle \mid \nabla\left(\phi_1^{\underline{a}}\otimes\phi^{\underline{b}}\otimes\phi^{\underline{c}}\right)\right)$ **is equal to**

$$=v(\left(\langle T,S\rangle \mid \phi^{\underline{a}+\underline{b}+\underline{c}}-\phi^{\underline{a}+\underline{b}}..\right)$$

By the definition of T,S

$$=v(\left(\left\langle \prod_{i=1}^{2}T_i,\prod_{j=1}^{2}S_j\right\rangle \mid \phi^{\underline{a}+\underline{b}+\underline{c}}-\phi^{\underline{a}+\underline{b}}..\right)$$

$$=v\left(\left\langle\left(\prod_i T_i \mid \phi^{\underline{a}+\underline{b}+\underline{c}}\right),\left(\prod_j S_j \mid \phi^{\underline{a}+\underline{b}+\underline{c}}\right)\right\rangle..\right)$$

$$=v(\left\langle \prod_i x_i^{r_i(\underline{a}+\underline{b}+\underline{c})},\prod_j y_j^{r_j(\underline{a}+\underline{b}+\underline{c})}\right\rangle....)$$

$$=v\left(\prod_{i,j}\left\langle x_i,y_j\right\rangle^{r_i(\underline{a}+\underline{b}+\underline{c})r_j(\underline{a}+\underline{b}+\underline{c})}...\right)$$

$$=v\left(\prod_{i,j}\left\langle x_i,y_j\right\rangle^{r_i(\underline{a}+\underline{b}+\underline{c})r_j(\underline{a}+\underline{b}+\underline{c})}...\right)$$

$$=v\left(\prod_{i,j}\left\langle x_i,y_j\right\rangle^{e_ie_j\theta_i.\theta_j\left(\phi_1^{\underline{a}}\otimes\phi^{\underline{b}}\otimes\phi^{\underline{c}}\right)}...\right)$$

$$=v\left(\prod_{i,j}\left\langle x_i,y_j\right\rangle^{e_ie_j\theta_i.\theta_j\left(\nabla\phi_1^{\underline{a}}\otimes\phi^{\underline{b}}\otimes\phi^{\underline{c}}\right)}\right)$$

by Proposition 2

$$=\sum_{i,j}-\left(b_i,b_j\right)\theta_i.\theta_j\left(\nabla\left(\phi_1^{\underline{a}}\otimes\phi^{\underline{b}}\otimes\phi^{\underline{c}}\right)\right)\ .\quad\Box$$

For a character x **of** G **and for a locally free rank one** OG**-module** L, **let** $L_x=(L\mid x)$ **denote the induced locally free rank one** O**-module. For characters** x,y,z **of** G, **we have**

$$\nabla^*(L)_{x,y,z}=\frac{L_{xyz}L_xL_yL_z}{L_{xy}L_{yz}L_{xz}L_1}$$

[Thus I am now writing x, y, z in place of χ, ϕ, ψ!]

There is a canonical associativity isomorphism

$$\alpha : \nabla^* (L)_{xy,z,w} \nabla^* (L)_{x,y,w} \cong \nabla^* (L)_{y,z,w} \nabla^* (L)_{x,yz,w}$$

Proof. First note that

$$\nabla^* (L)_{xy,z,w} = \frac{L_{xyzw} L_{xy} L_z L_w}{L_{xyz} L_{zw} L_{wxy} L_1}$$

$$\nabla^* (L)_{x,y,w} = \frac{L_{xyw} L_x L_y L_w}{L_{xy} L_{yw} L_{wx} L_1}$$

$$\nabla^* (L)_{y,z,w} = \frac{L_{yzw} L_y L_z L_w}{L_{yz} L_{zw} L_{wy} L_1};$$

$$\nabla^* (L)_{x,yz,w} = \frac{L_{xyzw} L_x L_{yz} L_w}{L_{xyz} L_{yzw} L_{wx} L_1}$$

The result then follows by using the canonical isomorphisms, which we henceforth regard as identifications,

$$L_{xy} L_{xy}^{-1} = O \quad \text{and} \quad L_{wxy} L_{wxy}^{-1} = O$$

and by using

$$L_{yz} L_{yz}^{-1} = O \quad \text{and} \quad L_{yzw} L_{yzw}^{-1} = O.$$