Metrics without isometries are generic

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Abstract

We prove that for any compact manifold of dimension greater than 1, the set of pseudo-Riemannian metrics having a trivial isometry group contains an open and dense subset of the space of metrics.

Keywords: metrics without isometries; space of pseudo-Riemannian metrics
Mathematics Subject Classification (2010): 53C50

Let V be a compact manifold and $\mathcal{M}_{p,q}$ be the set of smooth pseudo-Riemannian metrics of signature (p,q) on V (we suppose that it is not empty). In [2] D'Ambra and Gromov wrote: "everybody knows that $\mathrm{Is}(V,g) = \mathrm{Id}$ for generic pseudo-Riemannian metrics g on V, for $\dim(V) \geq 2$." Nevertheless, as far as we know, no proof of this fact is available. The purpose of this short article is to give a proof of this result in the case where V is compact and to precise the meaning of the word generic. Let us recall that it is known since the work of Ebin [3] (see also [4]) that the set of Riemannian metrics without isometries on a compact manifold is open and dense. We prove:

Theorem 1 *If V is a compact manifold such that $\dim(V) \geq 2$ then the set $\mathcal{G} = \{g \in \mathcal{M}_{p,q} \mid \mathrm{Is}(g) = \mathrm{Id}\}$ contains a subset that is open and dense in $\mathcal{M}_{p,q}$ for the C^∞-topology.*

This result is optimal in the sense that $\mathcal{G}$ is not always open as we showed in [5]. The particularity of the Riemannian case lies in the fact that the natural action of the group of smooth diffeomorphisms of V, denoted by $\mathrm{Diff}(V)$, on $\mathcal{M}_{n,0}$ is proper. Furthermore, Theorem 4.2 of [5] says that when this action is proper then $\mathcal{G}$ is an open subset of $\mathcal{M}_{p,q}$. The idea of proof is therefore to find a big enough subset of $\mathcal{M}_{p,q}$ invariant by $\mathrm{Diff}(V)$ on which the action is proper. We have decided to be short rather than self-contained, in particular we are going to use several results from our former work [5].

In the following $\mathcal{M}_{p,q}$ will be endowed with the C^∞-topology without further mention of it and by a perturbation we will always mean an arbitrary small perturbation.

For any $g \in \mathcal{M}_{p,q}$ we denote by Scal_g its scalar curvature and by M_g the maximum of Scal_g on V. Let $\mathcal{F}_V$ be the set of pseudo-Riemannian metrics g such that $\mathrm{Scal}_g^{-1}(M_g)$ contains a (non trivial) geodesic. The big set we are looking for is actually the complement of $\mathcal{F}_V$.

Proposition 2 *The set $\mathcal{O}_V = \mathcal{M}_{p,q} \setminus \mathcal{F}_V$ is an open dense subset of $\mathcal{M}_{p,q}$ invariant by the action of $\mathrm{Diff}(V)$ and the restriction of the action of $\mathrm{Diff}(V)$ to $\mathcal{O}_V$ is proper.*

Proof. The set $\mathcal{O}_V$ is clearly invariant. Let $g \in \mathcal{M}_{p,q}$ and $x_0 \in V$ realizing the maximum of Scal_g. It is easy to find a perturbation with arbitrary small support increasing the value of $\mathrm{Scal}_g(x_0)$. Repeating these deformation on smaller and smaller neighborhood of x_0 we find a perturbation of g such that the maximum of the scalar curvature is realized by only one point (see [3] p. 35 for a similar construction). Hence $\mathcal{O}_V$ is dense in $\mathcal{M}_{p,q}$.

Let us see now that $\mathcal{F}_V$ is closed. Let g_n be a sequence of metrics of $\mathcal{F}_V$ converging to g_∞. For any $n \in \mathbb{N}$ there exists a g_n-geodesic γ_n such that Scal_{g_n} is constant and equal to $M_{g_n} = \max_{x\in V} \mathrm{Scal}_{g_n}(x)$ on it. As the sequence of exponential maps converges to the exponential map of g_∞ and as V is compact we see that (up to subsequences) the sequence of geodesics γ_n converges to a g_∞-geodesic γ_∞. As $\mathrm{Scal}_{g_n} \to \mathrm{Scal}_{g_\infty}$ we know that Scal_{g_∞} is constant along γ_∞ and its value is necessarily M_{g_∞}. Hence $g_\infty \in \mathcal{F}_V$ and $\mathcal{F}_V$ is closed.

Let us suppose that the action of $\mathrm{Diff}(V)$ on $\mathcal{M}_{p,q}$ is not proper (otherwise there is nothing to prove). It means (see [5]) that there exists a sequence of metrics $(g_n)_{n\in\mathbb{N}}$ converging to g_∞ and a non equicontinuous sequence of diffeomorphisms $(\phi_n)_{n\in\mathbb{N}}$ such that the sequence of metrics $(\phi_n^* g_n)$ converges to g'_∞. The proposition will follow from the fact that g_∞ or g'_∞ have to belong to $\mathcal{F}_V$.

We first remark that the sequence of linear maps $(D\phi_n(x_n))_{n\in\mathbb{N}}$ lies in $\mathrm{O}(p,q)$ up to conjugacy by a converging sequence. As the sequence $(\phi_n)_{n\in\mathbb{N}}$ is non equicontinuous, we know by [5, Proposition 2.3] that there exists a subsequence such that $\|D\phi_{n_k}(x_{n_k})\| \to \infty$ when $k \to \infty$. We deduce from the KAK decomposition of $\mathrm{O}(p,q)$ the existence of what are called in [6], see subsection 4.1 therein for details, strongly approximately stable vectors, more explicitly we have:

Fact 3 *For any sequence $(x_n)_{n\in\mathbb{N}}$ of points of V, there exist a sequence $(v_n)_{n\in\mathbb{N}}$ such that (up to subsequences):*

- $\forall n \in \mathbb{N},\ v_n \in T_{x_n}V$,
- $D\phi_n(x_n)v_n \to 0$,
- $v_n \to v_\infty \neq 0$

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points of V realizing the maximum of the function $\mathrm{Scal}_{\phi_n^* g_n}$. The manifold being compact, we can assume that this sequence is convergent to a point x_∞. We can also assume that the sequence $(\phi_n(x_n))_{n\in\mathbb{N}}$ converges to y_∞. Of course x_∞ (resp. y_∞) realizes the maximum of $\mathrm{Scal}_{g'_\infty}$ (resp. Scal_{g_∞}).

Let $(v_n)_{n\in\mathbb{N}}$ be a sequence given by Fact 3. Reproducing the computation p. 471 of [5], we see that the scalar curvature of g'_∞ is constant along the g'_∞-geodesic starting from x_∞ with speed v_∞ (by symmetry the scalar curvature of g_∞ is constant along a geodesic containing y_∞):

$$\begin{aligned}
\mathrm{Scal}_{g'_\infty}(\exp_{g'_\infty}(x_\infty, v_\infty)) - \mathrm{Scal}_{g'_\infty}(x_\infty) &= \lim_{n\to\infty} \mathrm{Scal}_{\phi_n^* g_n}(\exp_{\phi_n^* g_n}(x_n, v_n)) - \mathrm{Scal}_{\phi_n^* g_n}(x_n) \\
&= \lim_{n\to\infty} \mathrm{Scal}_{g_n}(\exp_{g_n}(D\phi_n(x_n, v_n)) - \mathrm{Scal}_{g_n}(\phi_n(x_n)) \\
&= \mathrm{Scal}_{g_\infty}(y_\infty) - \mathrm{Scal}_{g_\infty}(y_\infty) = 0.
\end{aligned}$$

Hence g_∞ and g'_∞ do not belong to $\mathcal{O}_V$.$\square$

It follows from Theorem 4.2 of [5] and Proposition 2 that $\mathcal{G} \cap \mathcal{O}_V$ is open. As $\mathcal{O}_V$ is dense we just have to show that $\mathcal{G}$ is dense in $\mathcal{O}_V$ in order to prove Theorem 1. Let g be a metric in $\mathcal{O}_V$, as we saw earlier we can perturb it in such a way that the maximum of Scal_g is realized by only one point p. This point is now fixed by isometry. We choose now U an open subset of V that do not contain p in its closure but whose closure is contained in some normal neighborhood O of p. According to [1, Theorem 3.1] by Beig et al., we can perturb again g in such a way that there are no local Killing fields on U. We choose the perturbation in order that $\mathrm{Scal}^{-1}(M_g) = \{p\}$. The new metric has now a finite isometry group (it is 0-dimensional and compact by Proposition 2 as the metric still lies in $\mathcal{O}_V$). Actually, the proof of Proposition 2 implies also that the set of germs of local isometries fixing p is itself compact. It means that any isometry of a perturbation of g with support not containing O can send a geodesic γ_1 starting from p only on a finite number of geodesics that do not depend on the perturbation. Therefore, it is easy to find a perturbation of the metric along γ_1 (with support away from O) in order to destroy these possibilities. Now, any isometry has to fix pointwise γ_1 (we chose a non symmetric perturbation). Repeating this operation for $n = \dim V$ geodesics $\gamma_1, \dots \gamma_n$ such that the vectors $\gamma_i'(0)$ span T_pV, we obtain a perturbation of g such that any of its isometries has to be the identity i.e. the perturbed metric is in $\mathcal{G}$.

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