Residual based schemes for shallow water flows: application to tsunami propagation & urban floods

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Melosh Medal Lecture

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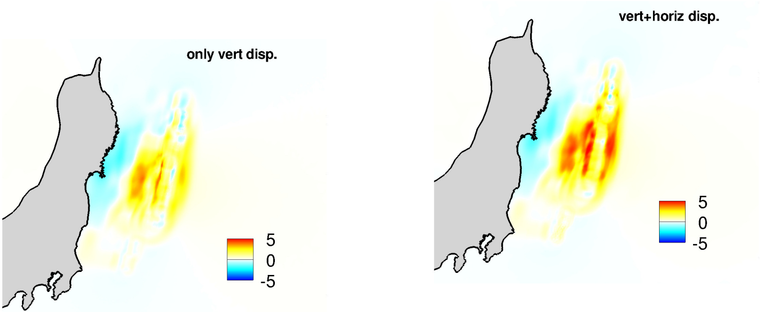
Duke University, Durham (NC)

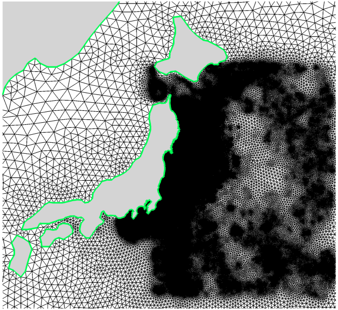
Setting: shallow water modelling for hazard prediction

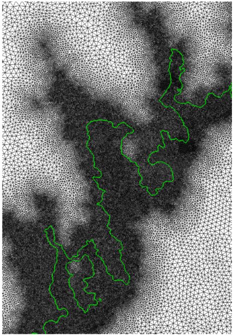
Initial water elevation due to seabed deformation

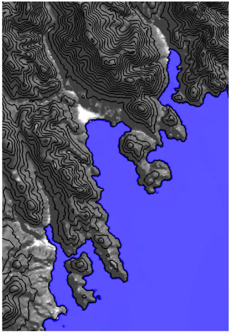


Model by [Satake et a., *Bull. Seismol. Soc. Am.* 2012], courtesy of BRGM Orleans.

Setting: shallow water modelling for hazard prediction

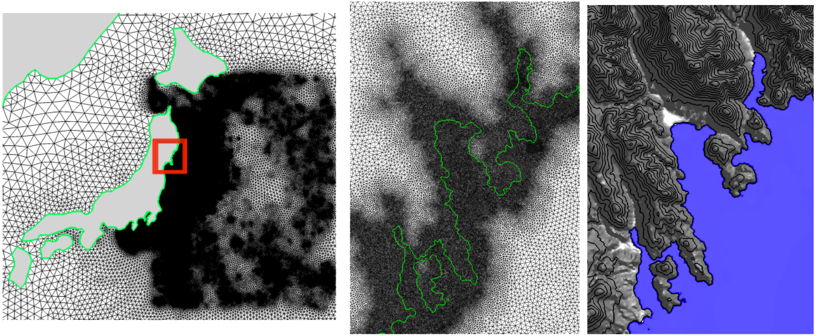






Grid, and close up view of the Iwate prefecture (initial state).

Setting: shallow water modelling for hazard prediction



Grid, and close up view of the Iwate prefecture (initial state).

Setting: shallow water modelling for hazard prediction

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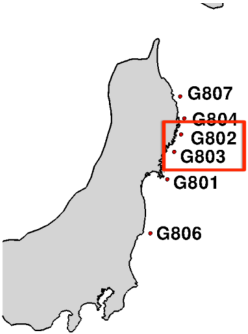
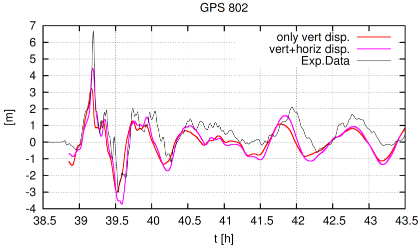
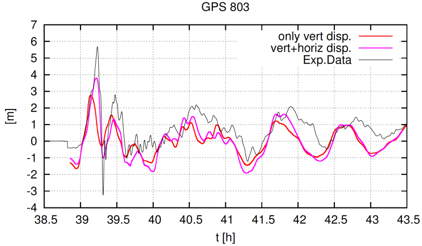
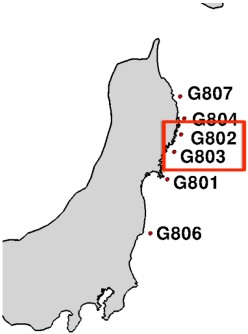
Signals in GPS buoys

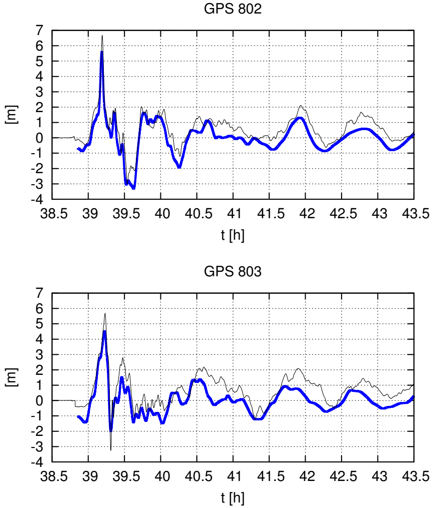






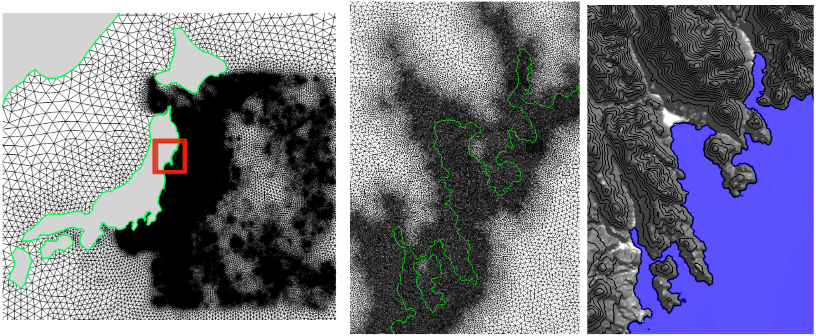
Signals in GPS buoys





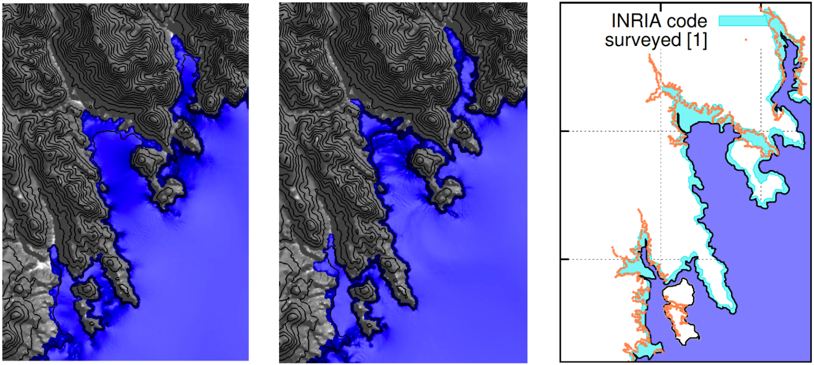
Finer mesh

Setting: shallow water modelling for hazard prediction



Grid, and close up view of the Iwate prefecture (initial state).

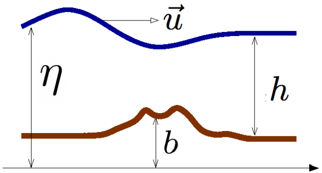
Setting: shallow water modelling for hazard prediction



Iwate prefecture: $t = 40$ min (left), $t = 50$ min (center), runup plot (right).

These problems can be studied using the shallow water model[1]

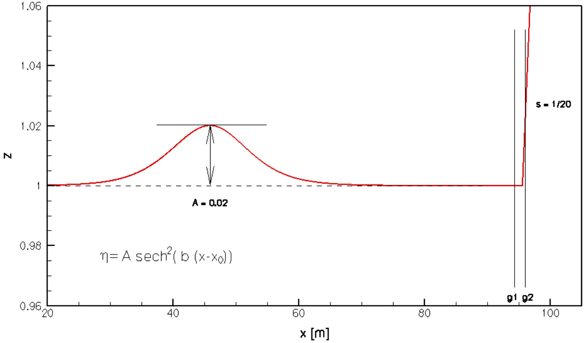
$$\vec{q} = h\vec{u}$$
$$p_h = g\frac{h^2}{2}$$
$$\vec{f} = \varphi_f^q \|\vec{q}\|\vec{q} = \frac{n^2\|\vec{q}\|}{h^{10/3}}\vec{q}$$
$$\eta = h + b$$

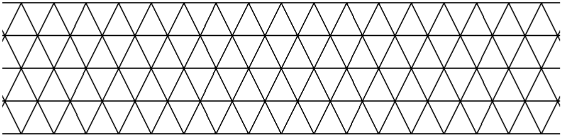


$$\partial_t h + \nabla \cdot \vec{q} = 0$$
$$\partial_t \vec{q} + \nabla \cdot (\vec{u} \otimes \vec{q}) + \nabla p_h + gh\nabla b + gh\vec{f} = 0$$

[1] possibly in curvilinear coord.s for large scale simulations as e.g. for the Tohoku case

Setting: shallow water modelling for hazard prediction





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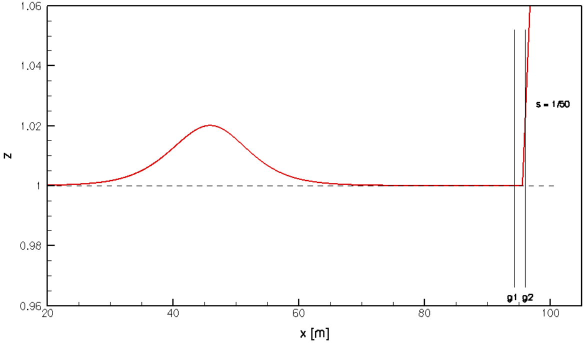

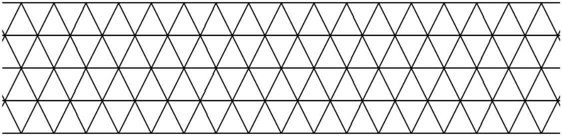

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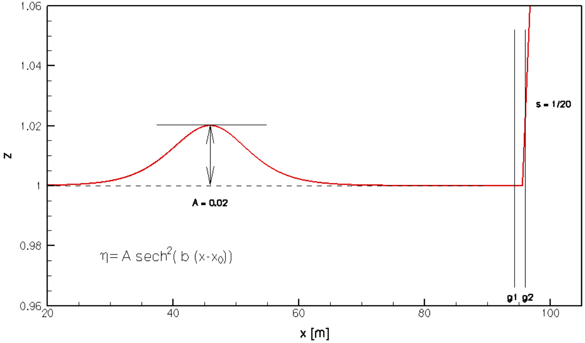

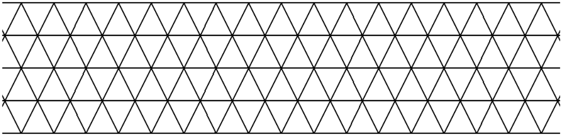

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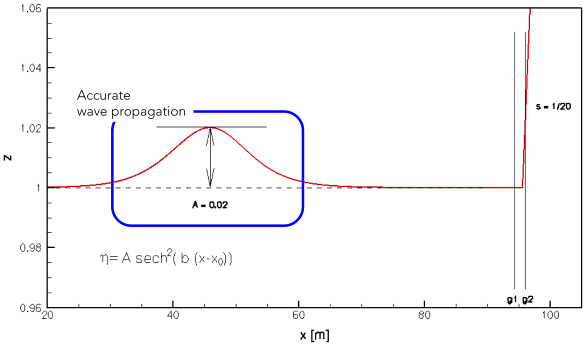

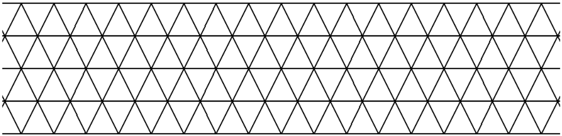

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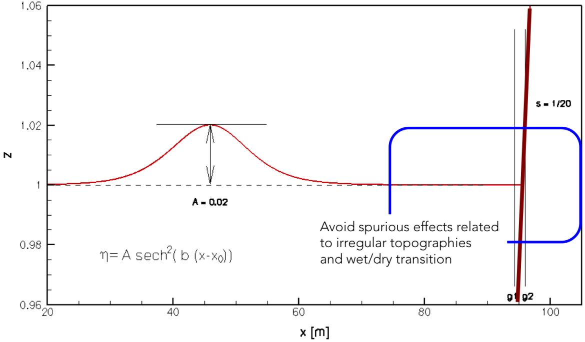

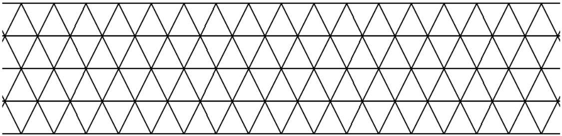

Joint effort over the last years with the members of my Inria team, and with many colleagues :

- R. Abgrall, U. Zurich (Switzerland)
- L. Arpaia, former PhD at Inria, now BRGM (France)
- A. Bollerman, former PhD at RWTH Aachen (Germany)
- H. Deconinck, von Karman Institute (Belgium)
- M. Hubbard, U. Nottingham (UK)
- M. Kazolea, Inria BSO (France)
- D. Sarmany, former post-doc at Leeds U. (UK)

Residual based shallow water modelling

Joint effort over the last years, acknowledging funding of

- PIA-ANR TANDEM (coordinated by the French CEA)
- EU-ERANET MIDWEST (coordinated by Inria)
- EDF (Electricité de France)
- Région Nouvelle Aquitaine
- Inria and Université de Bordeaux
- BGS IT&E GMBH (German SME)

Residual based shallow water modelling

Shallow water papers :

Application of conservative RD to the solution of the SWEs on unstructured meshes. *J.Comput.Phys.* 222, 2007
(with R. Abgrall and H. Deconinck)

Stabilized RD for SW simulations. *J.Comput.Phys.* 228, 2009
(with A. Bollermann)

On the C-property and generalized C-property of RD for the SWEs. *J.Sci.Comp.* 48, 2011

Unconditionally stable space-time discontinuous RD for SW flows. *J.Comput.Phys.* 253, 2014
(with D. Sarmany and M. Hubbard)

An explicit residual based approach for SW flows. *J.Comput.Phys*. 280, 2015

r-adaptation for SW: conservation, well balancedness, efficiency. *Computers & Fluids*, 160, 2018
(with L. Arpaia)

Residual distribution and finite volumes for SW in adaptive moving curvilinear coordinates, in preparation
(with L. Arpaia)

Residual distribution papers :

Explicit Runge-Kutta RD for time dependent problems: Second order case. *J.Comput.Phys.* 229, 2010
(with R. Abgrall)

Discontinuous upwind RD: A route to unconditional positivity and high order accuracy. *Computers & Fluids* 46, 2011
(with M. Hubbard)

An ALE formulation for explicit Runge-Kutta RD. *J.Sci.Comp.* 63, 2014
(with L. Arpaia and R. Abgrall)

Outline

- Shallow water equations and well balancing
- Finite volumes and fluctuation splitting
- MultiD: well balanced via residual distribution
- Including mesh movement and curvilinear coordinates
- Wetting-drying
- Efficient time stepping with mass matrices
- Applications and examples
- Conclusion and perspectives

Setting: shallow water modelling for hazard prediction

$$\vec{q} = h\vec{u}$$

$$p_h = g\frac{h^2}{2}$$

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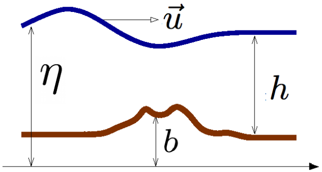


$$\partial_t h + \nabla \cdot \vec{q} = 0$$

$$\partial_t \vec{q} + \nabla \cdot (\vec{u} \otimes \vec{q}) + \nabla p_h + gh\nabla b + gh\vec{f} = 0$$

Simplified model

$$\partial_t u + \partial_x \mathcal{F}(u) + \gamma \partial_x b(x) = 0$$

- For this kind of problem, consistency w.r.t. $u = \text{const}$ has no meaning as this is a state that cannot occur
- So, what is a good notion of consistency here ?

Well balanced via fluctuation splitting

Simplified model

$$\partial_t u + \partial_x \mathcal{F}(u) + \gamma \partial_x b(x) = 0$$

Simple example: If γ is constant the state

$$\mathcal{F}(u(x)) + \gamma b(x) = \eta_0 = c^t$$

defines an exact steady equilibrium

> Consistency w.r.t. η seems like a better notion than consistency with respect to constant values of u

Well balanced via fluctuation splitting

Simplified model

$$\partial_t u + \partial_x \mathcal{F}(u) + \gamma \partial_x b(x) = 0$$

Simple example: If γ is constant the state

$$\mathcal{F}(u(x)) + \gamma b(x) = \eta_0 = c^t$$

defines an exact steady equilibrium

Well-balancedness or C-property is defined as the ability of preserving this equilibrium EXACTLY at the discrete level[2]

[2] Bermudez and Vazquez, *Computers & Fluids* 1994

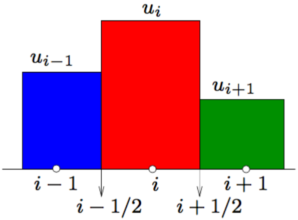
Well balanced via fluctuation splitting

Vast literature on designing well balanced numerical methods

- Bermudez and Vazquez, *Computers & Fluids* 1994
- **FV:** Hubbard and Garcia Navarro, *J.Comput.Phys.* 2000; Audusse et al, *SIAM SISC* 2004; M.J. Castro et al, *Math. & Computer Mod.* 2006, Noelle et al, *J.Comput.Phys.* 2007; Xing & Shu *Adv.Wat.Res.* 2011, etc.
- **DG:** Xing & Zhang *J.Sci.Comput.* 2013 ; Duran & Marche *Computers & Fluids* 2014; Xing *J.Comput.Phys.* 2014, etc.
- **cG + stabilization and RD:** Hauke *CMAME* 1998; Hubbard & Baines *J.Comput.Phys.* 1997; Brufau & Garcia Navarro *J.Comput.Phys.* 2003, Pasquetti et al *ICOSAHOM* 2014; Azerat et al, *SINUM* 2017; etc.
- etc.
- etc. etc.

Conservative approximation ($\widehat{\cdot}$ denotes numerical fluxes/source)

$$\Delta x_i \frac{du_i}{dt} + \widehat{\mathcal{F}}_{i+1/2} - \widehat{\mathcal{F}}_{i-1/2} + \gamma\widehat{\Delta b}_i = 0$$

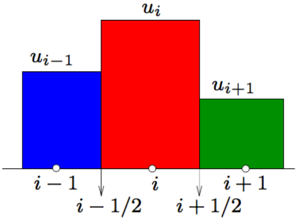


How to define $\widehat{\Delta b}_i$?

Well balanced via fluctuation splitting

Conservative approximation in fluctuation form

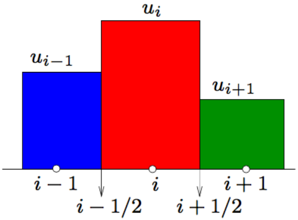
$$\Delta x_i \frac{du_i}{dt} + \widehat{\mathcal{F}}_{i+1/2} - \mathcal{F}(u_i) + \mathcal{F}(u_i) - \widehat{\mathcal{F}}_{i-1/2} + \gamma\widehat{\Delta b}_i = 0$$



Well balanced via fluctuation splitting

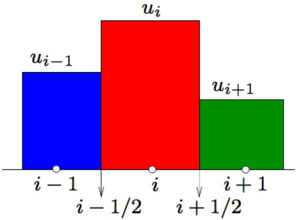
Conservative approximation in fluctuation form

$$\Delta x_i \frac{du_i}{dt} + \underbrace{\widehat{\mathcal{F}}_{i+1/2} - \mathcal{F}(u_i)}_{\psi_i^{i+3/2}} + \underbrace{\mathcal{F}(u_i) - \widehat{\mathcal{F}}_{i-1/2}}_{\psi_i^{i-1/2}} + \gamma\widehat{\Delta b}_i = 0$$



Conservative approximation in fluctuation form

$$\Delta x_i \frac{du_i}{dt} + \underbrace{\widehat{\mathcal{F}}_{i+1/2} - \mathcal{F}(u_i)}_{\psi_i^{i+1/2}} + \underbrace{\mathcal{F}(u_i) - \widehat{\mathcal{F}}_{i-1/2}}_{\psi_i^{i-1/2}} + \gamma\widehat{\Delta b}_i = 0$$



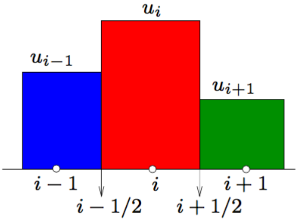
Conservation

$$\Delta x_i \frac{du_{i+1}}{dt} + \widehat{\mathcal{F}}_{i+3/2} - \widehat{\mathcal{F}}_{i+1/2} + \gamma\widehat{\Delta b}_{i+1} = 0$$

Well balanced via fluctuation splitting

Conservative approximation in fluctuation form

$$\Delta x_i \frac{du_i}{dt} + \underbrace{\widehat{\mathcal{F}}_{i+1/2} - \mathcal{F}(u_i)}_{\psi_i^{i+1/2}} + \underbrace{\mathcal{F}(u_i) - \widehat{\mathcal{F}}_{i-1/2}}_{\psi_i^{i-1/2}} + \gamma\widehat{\Delta b}_i = 0$$

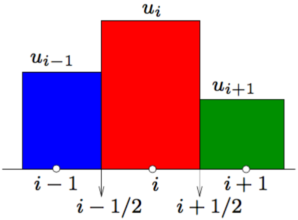


Conservation

$$\Delta x_i \frac{du_{i+1}}{dt} + \widehat{\mathcal{F}}_{i+3/2} - \mathcal{F}(u_{i+1}) + \underbrace{\mathcal{F}(u_{i+1}) - \widehat{\mathcal{F}}_{i+1/2}}_{\psi_{i+1}^{i+1/2}} + \gamma\widehat{\Delta b}_{i+1} = 0$$

Conservative approximation in fluctuation form

$$\Delta x_i \frac{du_i}{dt} + \underbrace{\widehat{\mathcal{F}}_{i+1/2} - \mathcal{F}(u_i)}_{\psi_i^{i+1/2}} + \underbrace{\mathcal{F}(u_i) - \widehat{\mathcal{F}}_{i-1/2}}_{\psi_i^{i-1/2}} + \gamma \widehat{\Delta b}_i = 0$$



Conservation

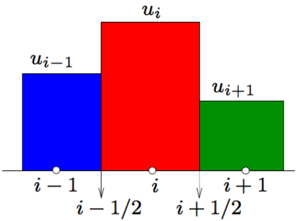
$$\psi_i^{i+1/2} + \psi_{i+1}^{i+1/2} = \mathcal{F}(u_{i+1}) - \mathcal{F}(u_i)$$

$$\psi_i^{i-1/2} + \psi_{i-1}^{i-1/2} = \mathcal{F}(u_i) - \mathcal{F}(u_{i-1})$$

Well balanced via fluctuation splitting

Conservative approximation in fluctuation form

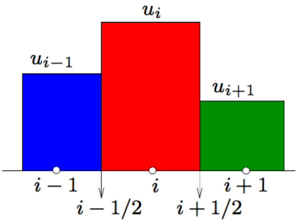
$$\Delta x_i \frac{du_i}{dt} + \psi_i^{i+1/2} + \psi_i^{i-1/2} + \gamma\widehat{\Delta b}_i = 0$$



Well balanced

Conservative approximation in fluctuation form

$$\Delta x_i \frac{du_i}{dt} + \psi_i^{i+1/2} + \psi_i^{i-1/2} + \gamma\widehat{\Delta b}_i = 0$$



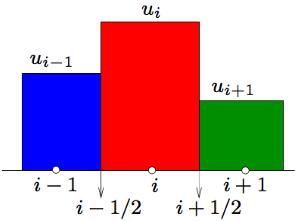
Well balanced

$$\gamma\widehat{\Delta b}_i = \gamma\widehat{\Delta b}_i^{\,i+1/2} + \gamma\widehat{\Delta b}_i^{\,i-1/2}$$

Well balanced via fluctuation splitting

Conservative approximation in fluctuation form

$$\Delta x_i \frac{du_i}{dt} + \psi_i^{i+1/2} + \psi_i^{i-1/2} + \gamma\widehat{\Delta b}_i = 0$$



Well balanced

$$\gamma\widehat{\Delta b}_i = \gamma\widehat{\Delta b}_i^{i+1/2} + \gamma\widehat{\Delta b}_i^{i-1/2}$$

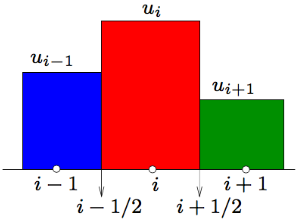
$$\gamma\widehat{\Delta b}_i^{i+1/2} + \gamma\widehat{\Delta b}_{i+1}^{i+1/2} = \gamma(b(x_{i+1}) - b(x_i))$$

$$\psi_i^{i+1/2} + \psi_{i+1}^{i+1/2} = \mathcal{F}(u_{i+1}) - \mathcal{F}(u_i) \quad \text{(reminder)}$$

Well balanced via fluctuation splitting

Conservative approximation in fluctuation form

$$\Delta x_i \frac{du_i}{dt} + \underbrace{\psi_i^{i+1/2} + \gamma\widehat{\Delta b}_i^{i+1/2}}_{\phi_i^{i+1/2}} + \underbrace{\psi_i^{i-1/2} + \gamma\widehat{\Delta b}_i^{i-/2}}_{\phi_i^{i-1/2}} = 0$$

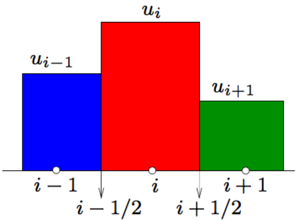


Well balanced

Well balanced via fluctuation splitting

Conservative approximation in fluctuation form

$$\Delta x_i \frac{du_i}{dt} + \phi_i^{i+1/2} + \phi_i^{i-1/2} = 0$$



Well balanced

Design $\gamma\widehat{\Delta b}_i^{i\pm1/2}$ so that

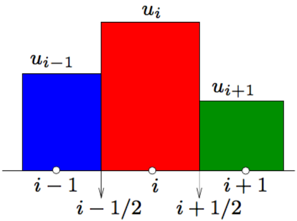
$$\phi_i^{\pm1/2} = \psi_i^{i\pm1/2} + \gamma\widehat{\Delta b}_i^{i\pm1/2} = 0$$

If $\eta_i := \mathcal{F}(u_i) + \gamma b(x_i) = \eta_0 = c^t \ ,\forall\, i$

Well balanced via fluctuation splitting

Conservative approximation in fluctuation form

$$\Delta x_i \frac{du_i}{dt} + \phi_i^{i+1/2} + \phi_i^{i-1/2} = 0$$



Well balanced

If we set

$$\phi^{i+1/2} := \int_{x_i}^{x_{i+1}} (\partial_x \mathcal{F} + \gamma \partial_x b)$$
$$= \mathcal{F}(u_{i+1}) - \mathcal{F}(u_i) + \gamma(b(x_{i+1}) - b(x_i))$$
$$= \eta_{i+1} - \eta_i$$

All schemes of the form $\phi_i^{i+1/2} := \beta_i^{i+1/2}\phi^{i+1/2}$ are well balanced !

Well balanced via fluctuation splitting

Conservative approximation in fluctuation form

$$\Delta x_i \frac{du_i}{dt} + \phi_i^{i+1/2} + \phi_i^{i-1/2} = 0$$

Example:

$$\widehat{\mathcal{F}}_{i+1/2} = \frac{\mathcal{F}(u_i) + \mathcal{F}(u_{i+1})}{2} \Rightarrow \psi_i^{i+1/2} = \frac{\mathcal{F}(u_{i+1}) - \mathcal{F}(u_i)}{2}$$

$$\gamma \widehat{\Delta b}_i^{i+1/2} = \frac{\gamma(b(x_{i+1}) - b(x_i))}{2}$$

$$\phi_i^{i+1/2} = \frac{1}{2}\phi^{i+1/2}$$

Conservative approximation in fluctuation form

$$\Delta x_i \frac{du_i}{dt} + \phi_i^{i-1/2} + \phi_i^{i-1/2} = 0$$

Example:

$$\widehat{\mathcal{F}}_{i+1/2} = \frac{\mathcal{F}(u_i) + \mathcal{F}(u_{i+1})}{2} - \frac{|\partial_u \mathcal{F}|^{\mathsf{Roe}}_{i+1/2}}{2}(u_{i+1} - u_i)$$

$$\gamma\widehat{\Delta b}_i^{i+1/2} = \frac{\gamma(b(x_{i+1}) - b(x_i))}{2} - \frac{\mathsf{sign}(\partial_u \mathcal{F}^{\mathsf{Roe}}_{i+1/2})}{2}\gamma(b(x_{i+1}) - b(x_i))$$

$$\phi_i^{i+1/2} = \frac{1 - \mathsf{sign}(\partial_u \mathcal{F}^{\mathsf{Roe}}_{i+1/2})}{2}\phi^{i+1/2}$$

Well balanced via residual distribution in 2D

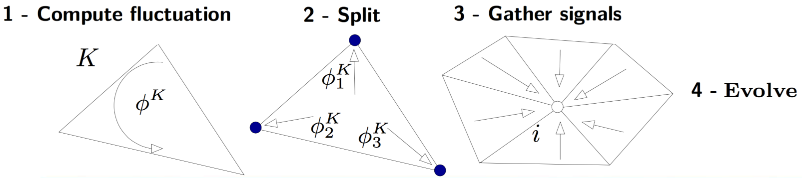
Residual distribution

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) + S = 0$$

Multidimensional fluctuation splitting

$$|C_i| \frac{du_i}{dt} + \sum_{K \ni} \phi_i^K = 0$$

$$\sum_{j \in K} \phi_j^K = \phi^K := \int_K (\nabla \cdot \boldsymbol{\mathcal{F}}(u) + S)$$



Residual distribution

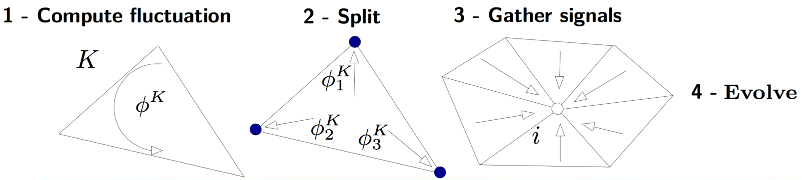
$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) + S = 0$$

Some remarks:

- The approximation is continuous in space. For this talk P^1 finite elements:

$$u = \sum_K \sum_{j \in K} \varphi_j u_j(t) \quad \varphi_j \text{ piecewise linear continuous basis fcns}$$

- There is no additional reconstruction involved
- There is no Riemann problem involved (cf. examples that follow)

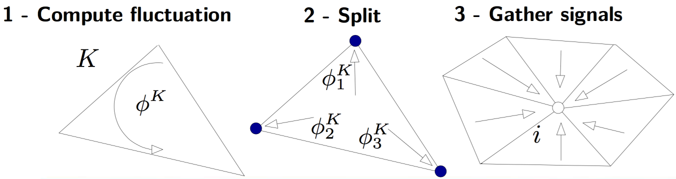


Residual distribution

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) + S = 0$$

Some examples: central splitting

$$\phi_i^K = \frac{1}{3}\phi^K$$



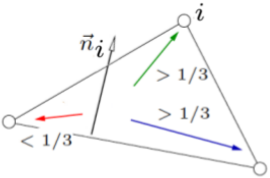
4 - Evolve

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) + S = 0$$

Some examples: central plus streamline dissipation

Streamline dissipation stabiilzation[3]

$$\phi_i^K = \frac{1}{3}\phi^K + \int_K \partial_u \boldsymbol{\mathcal{F}} \cdot \nabla \varphi_i \, \boldsymbol{\tau}_{\mathsf{stab}} \left(\nabla \cdot \boldsymbol{\mathcal{F}}(u) + S\right)$$
$$= \Big(\frac{1}{3} + \frac{1}{|K|}(\partial_u \boldsymbol{\mathcal{F}})_K \cdot \vec{n}_i \, \boldsymbol{\tau}_{\mathsf{stab}}\Big)\phi^K$$

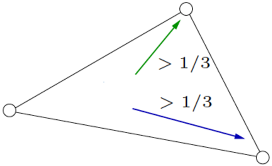


[3]Brooks and Hughes, *CMAME* 1982

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) + S = 0$$

SOME EXAMPLES: MULTIDIMENSIONAL UPWINIDING

Multidimensional upwind fluctuation splitting[4]



$$\phi_i^K = \beta_i^{\mathsf{LDA}} \phi^K\,, \quad \beta_i^{LDA} = (\partial_u \boldsymbol{\mathcal{F}} \cdot \vec{n}_i)^+ \boldsymbol{\tau}_{\mathsf{LDA}}$$

[4] Roe 1986, Deconinck et al., 1993

Residual distribution

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) + S = 0$$

A framework for well balanced in multi dimensions

$$|C_i| \frac{du_i}{dt} + \sum_{K \ni} \phi_i^K = 0$$

$$\sum_{j \in K} \phi_j^K = \phi^K := \int_K \left(\nabla \cdot \boldsymbol{\mathcal{F}}(u) + S \right)$$

Well balanced (MR, JCP 2015)

We consider a steady equilibrium associated to a *set of invariants* v (in space). Any *scheme of the form* $\phi_i^K = \beta_i^K \phi^K$ *with bounded coefficient (matrix)* β_i^K will be exactly well balanced provided that the evaluation of ϕ^K is done

1. by approximating $v = \sum_j \varphi_j v_j$
2. with exact quadrature

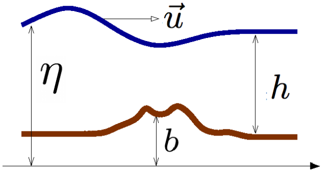
Shallow water examples

$$\vec{q} = h\vec{u}$$

$$p_h = g\frac{h^2}{2}$$

$$\vec{f} = \varphi_f^q \|\vec{q}\|\vec{q} = \frac{n^2\|\vec{q}\|}{h^{10/3}}\vec{q}$$

$$\eta = h + b$$



$$\partial_t h + \nabla \cdot \vec{q} = 0$$

$$\partial_t \vec{q} + \nabla \cdot (\vec{u} \otimes \vec{q}) + \nabla p_h + gh\nabla b + gh\vec{f} = 0$$

Lake at rest

- Steady equilibrium defined by the invariant $v = [\eta, \vec{q}]^t$.
- In particular we have $v(x,y) = v_0 = [\eta_0, 0]^t$ (no flow, but multiD)

How to handle this

1. Approximation: quite natural

$$h = \sum_i \varphi_i h_i \quad \text{and} \quad b = \sum_i \varphi_i b_i \Longrightarrow \eta = \sum_i \varphi_i \eta_i = \eta_0$$

2. Quadrature : easy

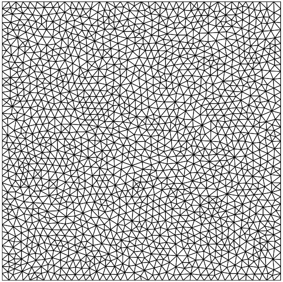
$$\phi^K = \underbrace{\oint_{\partial K} \begin{bmatrix} 0 \\ p_h \end{bmatrix} \vec{n}}_{\text{Integrate } h^2 \text{ polynomial exactly}} + \int_K gh \begin{bmatrix} 0 \\ \nabla b \end{bmatrix} = \int_K \begin{bmatrix} 0 \\ \nabla p_h \end{bmatrix} + \int_K gh \begin{bmatrix} 0 \\ \nabla b \end{bmatrix}$$

$$\underbrace{=}_{\nabla p_h = gh\nabla h} \int_K gh \begin{bmatrix} 0 \\ \nabla \eta \end{bmatrix} = 0$$

Shallow water examples

Lake at rest

- Steady equilibrium defined by the invariant $v = [\eta,\, \vec{q}]^t$.
- In particular we have $v(x,y) = v_0 = [\eta_0,\, 0]^t$ (no flow, but multiD)



Lake at rest

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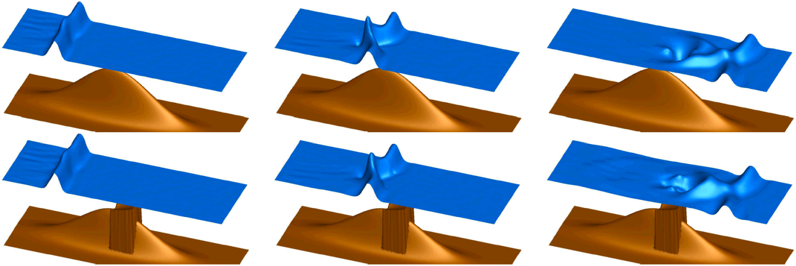


Constant energy

- Steady equilibrium defined by the invariant $v = [\mathcal{E} = \eta + \|\vec{u}\|^2/2g,\, \vec{q}]^t$.
- In particular we have $v(x,y) = v_0 = [\mathcal{E}_0,\, \vec{q}_0]^t$ (moving, but 1D + no $\vec{f}$)
- Assume that it makes sens to look at this in 2D ....
- Let $\vec{q}$ and $\vec{q}^{\perp}$ the flux components parallel and orthogonal to $\vec{q}_0$:

$$\partial_t h + \nabla\cdot(\vec{q}+\vec{q}^{\perp}) = 0$$

$$\partial_t(\vec{q}+\vec{q}^{\perp}) + \cancel{(\vec{u}\cdot\nabla)\vec{q}} - \cancel{(\vec{u}^{\perp}\cdot\nabla)\vec{q}^{\perp}} + \frac{gh}{gh-\|\vec{u}\|^2}\cancel{(gh\nabla\mathcal{E} - \vec{u}\vec{u}\cdot\nabla\mathcal{E})}$$
$$+ \frac{gh}{gh-\|\vec{u}\|^2}\cancel{\left(\frac{\vec{u}}{gh}\vec{u}\cdot(\nabla\vec{q}\cdot\vec{u}) - \frac{\|\vec{u}\|^2}{gh}(\nabla\vec{q})^t\cdot\vec{u}\right)} = \underline{gh\frac{\vec{u}^{\perp}\cdot\nabla b}{gh-\|\vec{u}\|^2}\vec{u}^{\perp}}$$

When setting $v = v_0$ on an unstructured mesh
The bathymetry cannot have cross-flow variations !!

Shallow water examples

Constant energy

- Steady equilibrium defined by the invariant $v = [\mathcal{E} = \eta + \|\vec{u}\|^2/2g,\ \vec{q}]^t$.
- In particular we have $v(x,y) = v_0 = [\mathcal{E}_0,\ \vec{q}_0]^t$ (moving, but 1D + no $\vec{f}$)

How to handle this

1. Approximation: less trivial

$$\mathcal{E} = \sum_i \varphi_i \mathcal{E}_i = \mathcal{E}_0\,, \quad \vec{q} = \sum_i \varphi_i \vec{q}_i = \vec{q}_0\,, \qquad \text{and} \qquad b = ?$$

   - Passing from $\mathcal{E}$ to physical var.s : solution of non-linear algebraic eq.[1]
   - Simply expanding b on the same basis will not work as $\nabla b \cdot \hat{q}^{\perp} \neq 0$
   - We assume some analytical approximation of b is available

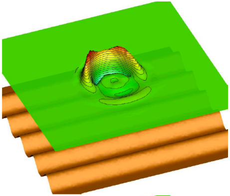
2. Quadrature: complex but (MR, *JCP* 2015)

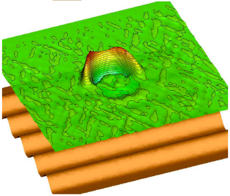
   Proposition
   Schemes with $\phi_i^K = \beta_i^K \phi^K$ with bounded coefficients β_i^K preserve exactly the initial steady equilibrium for exact quadrature.
   For approximate integration, the truncation error is $\epsilon \leq C\Delta x^{1+\min(p_f, p_v)}$ where p_f and p_v are the line and surface quadrature orders.

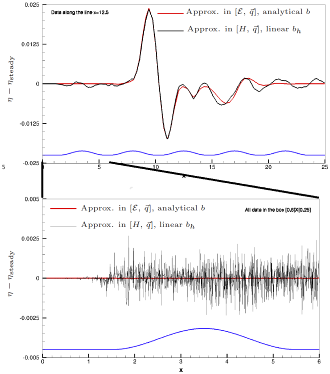
[1] Noelle, Xing, Shu, *JCP* 2007

Shallow water examples

Constant energy

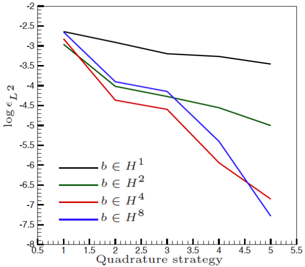






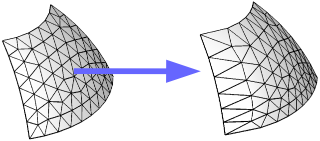
Shallow water examples

Constant energy



	$Q_3 - u(v_h, b)$	$Q_4 - u(v_h, b)$	$Q_3 - u_h$
25/50	1.452714e-07	3.698282e-10	3.35738e-04
25/100	9.508237e-09	4.450410e-12	8.85116e-05
rate	3.947	6.399	1.930
25/200	6.584230e-10	4.688134e-14	2.36592e-05
rate	3.865	6.591	1.913

Adaptive ALE in curvilinear coordinates



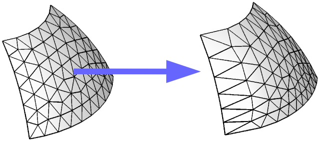
$\mathcal{A}_0 \subset \mathcal{S}^2$ $\qquad$ $\mathcal{A} \subset \mathcal{S}^2$

Extension to SWEs in **ALE** framework and curvilinear coords in (Arpaia and Ricchiuto, SIAM-GS 2017) and (Arpaia and Ricchiuto, in preparation).

$$\frac{\partial}{\partial t}\left(\sqrt{G} J_A \begin{bmatrix} h \\ hu^i \end{bmatrix}\right) + J_A \frac{\partial}{\partial x^j}\left(\mathsf{F}^j - \sqrt{G}\sigma^j \mathsf{u}\right) = \sqrt{G} J_A \mathsf{S}$$

$$\mathsf{F}^j = \sqrt{G}\begin{bmatrix} hu^j \\ T^{ij} \end{bmatrix}, \quad \mathsf{S} = -\begin{bmatrix} 0 \\ G^{ij} g h \dfrac{\partial b}{\partial x^j} \end{bmatrix} - \begin{bmatrix} 0 \\ \Gamma^i_{jk}\left(T^{jk} - hu^j \sigma^k\right) \end{bmatrix}$$

Adaptive ALE in curvilinear coordinates



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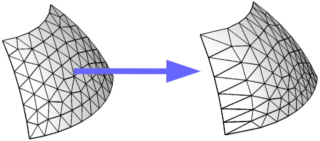
$$\frac{\partial}{\partial t}\left(\sqrt{G}J_A\begin{bmatrix} h \\ hu^i \end{bmatrix}\right) + J_A\frac{\partial}{\partial x^j}\left(\mathsf{F}^j - \sqrt{G}\sigma^j\mathsf{u}\right) = \sqrt{G}J_A\mathsf{S}$$

G	Jacobian of metric tensor (curv. coord.)
$J_A = \det(\partial x_i/\partial x_j^0)$	ALE coord. transformation Jacobian

Adaptive ALE in curvilinear coordinates

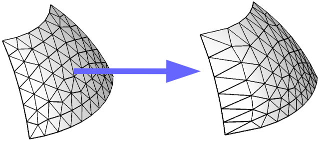


$\mathcal{A}_0 \subset \mathcal{S}^2$ $\qquad$ $\mathcal{A} \subset \mathcal{S}^2$

- (Discrete) geometric conservation in curvilinear coordinates:

Classical characterization by Thomas & Lombard (AIAA J., 1979) in Cartesian coordinates and in absence of sources:

$$\frac{\partial}{\partial t}(J_A u) + J_A \frac{\partial}{\partial x^j}(F_j(u) - \sigma^j u) = 0 \quad \underbrace{\Longrightarrow}_{u \text{ constant in space/time}} \quad \frac{\partial}{\partial t} J_A - J_A \frac{\partial}{\partial x^j} \sigma^j = 0$$



$\mathcal{A}_0 \subset \mathcal{S}^2$ $\qquad$ $\mathcal{A} \subset \mathcal{S}^2$

- (Discrete) geometric conservation in curvilinear coordinates:

In curvilinear coord.s and/or with sources, not all constant states are admissible. Schemes designed by combining geometric conservation and well balancing !

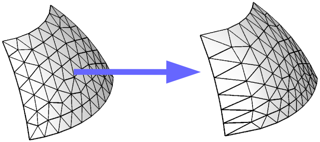
Example

$$\frac{\partial}{\partial t}(\sqrt{G}J_A h) = -\,J_A\frac{\partial}{\partial x^j}(\sqrt{G}hu^j - \sqrt{G}\sigma^j h)$$

$$\frac{\partial}{\partial t}(\sqrt{G}J_A hu^i) = -\,J_A\frac{\partial}{\partial x^j}(\sqrt{G}T^i j - \sqrt{G}\sigma^j hu^i) - \sqrt{G}J_A S$$

$$\frac{\partial}{\partial t}(\sqrt{G}J_A b) = -\,J_A\frac{\partial}{\partial x^j}(\sqrt{G}\sigma^j b) \quad \textbf{ALE remap}$$

Adaptive ALE in curvilinear coordinates



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- (Discrete) geometric conservation in curvilinear coordinates:

In curvilinear coord.s and/or with sources, not all constant states are admissible.
Schemes designed by combining geometric conservation and well balancing !

Example, setting $u^i = 0$ in the RHS

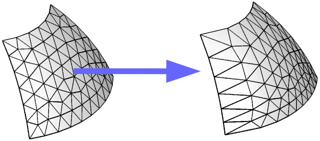
$$\frac{\partial}{\partial t}(\sqrt{G} J_A h) = J_A \frac{\partial}{\partial x^j}(\sqrt{G}\sigma^j h)$$

$$\frac{\partial}{\partial t}(\sqrt{G} J_A h u^i) = - J_A \frac{\partial}{\partial x^j}(\sqrt{G} G^{ij} g \frac{h^2}{2}) - \sqrt{G} J_A G^{ij} g h \frac{\partial b}{\partial x^j} - \Gamma^i_{jk} G^{jk} g \frac{h^2}{2}$$

$$\frac{\partial}{\partial t}(\sqrt{G} J_A b) = - J_A \frac{\partial}{\partial x^j}(\sqrt{G}\sigma^j b)$$

ALE remap

Adaptive ALE in curvilinear coordinates



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Schemes designed by combining geometric conservation and well balancing !

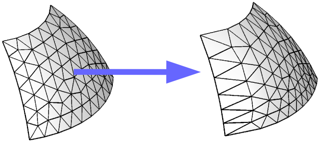
Example, summing the ALE remap for b with mass conservation

$$\frac{\partial}{\partial t}(\sqrt{G}J_A\eta) = J_A\frac{\partial}{\partial x^j}(\sqrt{G}\sigma^j\eta)$$

$$\frac{\partial}{\partial t}(\sqrt{G}J_A hu^i) = -J_A\frac{\partial}{\partial x^j}(\sqrt{G}G^{ij}g\frac{h^2}{2}) - \sqrt{G}J_A G^{ij}gh\frac{\partial b}{\partial x^j} - \Gamma^i_{jk}G^{jk}g\frac{h^2}{2}$$

$$\frac{\partial}{\partial t}(\sqrt{G}J_A b) = -J_A\frac{\partial}{\partial x^j}(\sqrt{G}\sigma^j b)$$ **ALE remap**

Adaptive ALE in curvilinear coordinates



$\mathcal{A}_0 \subset \mathcal{S}^2$ $\qquad$ $\mathcal{A} \subset \mathcal{S}^2$

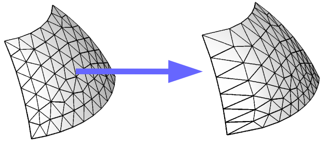
- (Discrete) geometric conservation in curvilinear coordinates:

In curvilinear coord.s and/or with sources, not all constant states are admissible.
Schemes designed by combining geometric conservation and well balancing !

Example, if $\eta = \eta^0 =$ const in the RHS

$$
\begin{aligned}
\frac{\partial}{\partial t}(\sqrt{G} J_A \eta) &= 0 \\
\frac{\partial}{\partial t}(\sqrt{G} J_A h u^i) &= -\sqrt{G} J_A G^{ij} \Big(\frac{\partial}{\partial x^j}\big(g \frac{h^2}{2}\big) + gh \frac{\partial b}{\partial x^j} \Big) \\
&\quad - J_A \Big(\frac{\partial}{\partial x^j}(\sqrt{G} G^{ij}) + \Gamma^i_{jk} G^{jk} \Big) g \frac{h^2}{2}
\end{aligned}
$$

Adaptive ALE in curvilinear coordinates



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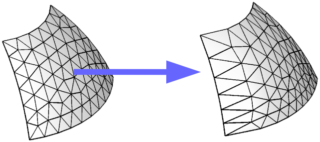
Example, if $\eta = \eta^0 =$ const in the RHS

$$\frac{\partial}{\partial t}(\sqrt{G}J_A\eta) = 0$$

$$\frac{\partial}{\partial t}(\sqrt{G}J_A hu^i) = -\sqrt{G}J_A G^{ij}\overbrace{\left(\frac{\partial}{\partial x^j}(g\frac{h^2}{2}) + gh\frac{\partial b}{\partial x^j}\right)}^{=0 \Rightarrow \text{well balanced condition}}$$

$$-J_A \underbrace{\left(\frac{\partial}{\partial x^j}(\sqrt{G}G^{ij}) + \Gamma^i_{jk}G^{jk}\right)}_{=0 \text{ Ricci's Lemma, and metric properties}} g\frac{h^2}{2}$$

Adaptive ALE in curvilinear coordinates



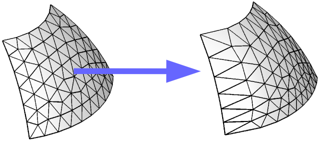
- (Discrete) geometric conservation in curvilinear coordinates.

  Clever combination of
  - (Discrete) geometric conservation
  - well balanced
  - ALE remap
  - Metric properties of the sphere

Constraints to be embedded in the discrete evaluation of

$$\phi^K = \int_K \left\{ J_A \frac{\partial}{\partial x^j} \left(\mathsf{F}^j - \sqrt{G}\sigma^j \mathsf{u} \right) - \sqrt{G} J_A \mathsf{S} \right\}$$

Adaptive ALE in curvilinear coordinates



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Extension to SWEs in **ALE** framework and curvilinear coords in (Arpaia and Ricchiuto, SIAM-GS 2017) and (Arpaia and Ricchiuto, in preparation).

- Discrete geometric conservation in curvilinear coordinates
- Mass conservation vs ALE remap of the bathymetric, cf (Arpaia and Ricchiuto, Computers & Fluids, 2018)
- Adaptive mesh movement

Wetting drying

Wetting drying

Main issues

- Well balancedness and ∇b in partially wet cells
- Non-negativity of h
- Velocity approximation and singularity for $h \ll 1$

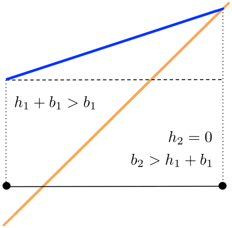
MODIFIED APPROXIMATION OF THE BATHYMETRY

Generalzation of the so-called modified hydrostatic reconstruction, see (Chen-Noelle, *SINUM* 2017) for a review.

Contribution of the bathymetry term in the residual:

$$\int_K gh\nabla b = |K|\, gh_K \sum_{j\in K} b_j \nabla\varphi_j$$

$$b_j = b(x_j)$$



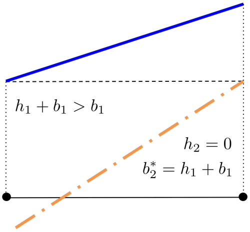
Modified approximation of the bathymetry

Generalzation of the so-called modified hydrostatic reconstruction, see (Chen-Noelle, *SINUM* 2017) for a review.

Contribution of the bathymetry term in the residual:

$$\int_K gh\nabla b = |K|\, gh_K \sum_{j\in K} b_j^*\nabla\varphi_j$$

$$b_j^* = \min\left(b(x_j), \max_{l\in K|h_l>0}(h_l+b_l)\right)$$



See (Brufau & Garcia-Navarro, *JCP* 2003) and (Ricchiuto & Bollermann, *JCP* 2009), (Ricchiuto, *JCP* 2015).

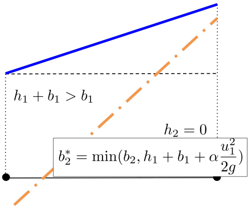
Modified approximation of the bathymetry

Generalzation of the so-called modified hydrostatic reconstruction, see (Chen-Noelle, *SINUM* 2017) for a review.

Contribution of the bathymetry term in the residual:

$$\int_K gh\nabla b = |K|\, gh_K \sum_{j\in K} b_j^* \nabla\varphi_j$$

$$b_j^* = \min\left(b(x_j), \max_{l\in K|h_l>0}(h_l + b_l + \alpha\|\vec{u}_l\|/2g)\right)$$



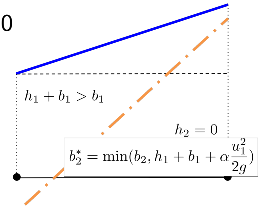
Generalizing the modified H-reconstruction of (Gallardo et al, *JCP* 2007).

Wetting drying

Modified approximation of the bathymetry

Generalzation of the so-called modified hydrostatic reconstruction, see (Chen-Noelle, *SINUM* 2017) for a review.

In all cases for η constant in the wet region, and $\vec{u}$=0 we have



$$\int_K gh\nabla b + \int_K gh\nabla h = 0$$

holds exactly. So dry areas do not perturb the initial equilibrium !

Positivity preserving distribution

Relies on principles dating back to

- A. Harten, *JCP* 1983 (LED and TVD conditions)
- S.P. Spekreijse, *Math. Comp.* 1987 (positive coeff. schemes)
- Roe, *ICASE rep.* 1990, Deconinck et al. *CAF* 1993 (limited distribution)

Positivity preserving distribution

LED scheme: Local Extremum Diminishing.

$$|C_i|\frac{du_i}{dt} + \sum_{K\ni i} \phi_i^K = 0$$

$$\phi_i^K = \sum_{j\in K} c_{ij}(u_i - u_j)\ , \quad c_{ij} \geq 0$$

- Local maxima are non increasing $(du_i/dt \leq 0)$
- Local maxima are non decreasing $(du_i/dt \geq 0)$

Positivity preserving distribution

Two step construction given ϕ^K

1. Define a positive coefficient first order distribution. Example:

$$(\phi_i^K)^{O1} = \frac{\phi^K}{3} + \alpha_K \sum_{j \in K} (u_i - u_j)$$

2.

Positivity preserving distribution

Two step construction given ϕ^K

1. Define a positive coefficient first order distribution. Example:

$$(\phi_i^K)^{O1} = \frac{\phi^K}{3} + \alpha_K \sum_{j \in K} (u_i - u_j) \Longleftarrow \text{first order and LED not well balanced}$$

2.

Positivity preserving distribution

Two step construction given ϕ^K

1. Define a positive coefficient first order distribution. Example:

$$(\phi_i^K)^{O1} = \frac{\phi^K}{3} + \alpha_K \sum_{j\in K}(u_i - u_j)$$

2. apply a (multiple entries) bounded positive limiter:

$$\phi_i^K = \frac{\theta}{\sum\limits_{j\in K}\theta}\phi^K\,, \quad \theta_i = \max\left(0, (\phi_i^K)^{O1}\phi^K\right)$$

Step 2. can be performed eq. by eq. or projecting on a relevant space (characteristic var.s, primitive var.s, etc). On the scalar level one can show that

$$\phi_i^K = \gamma_i(\phi_i^K)^{O1}\,, \;\; \gamma_i \in [0,1]$$

POSITIVITY PRESERVING DISTRIBUTION

When applied eq. by eq., we can show that

$$|C_i|\frac{dh_i}{dt} = -\sum_{K\ni i}\sum_{j\in K} c_{ij}^K(\vec{u})h_j$$

with $c_{ii}^K = \gamma_i c_{ii}^{O1} \geq 0$ and $c_{ij}^K = \gamma_i c_{ij}^{O1} \leq 0$. Integrated with explicit Euler this leads to a classical positivity preservation result (under a Δt constraint):

$$h_i^{n+1} = (1 - \frac{\Delta t \sum\limits_{K\ni i} c_{ii}^K}{|C_i|})h_i^n + \sum_{K\ni i}\sum_{j\in K} |c_{ij}^K| h_j^n$$

Bounded computation of the velocity

A key issue. Unbounded velocities in wet/dry cells can occur due to division by a small depth:

$$(\vec{q}_i^{\,n+1})_{\text{update}} \to \vec{u}_i^{\,n+1} = \begin{cases} \dfrac{(\vec{q}_i^{\,n+1})_{\text{update}}}{h_i^{n+1}} & \text{if } h_i^{n+1} > c_h \Delta x^2 \\ \alpha_i \dfrac{(\vec{q}_i^{\,n+1})_{\text{update}}}{h_i^{n+1}} & \text{otherwise} \end{cases} \to \vec{q}_i^{\,n+1} = h_i^{n+1} \vec{u}_i^{\,n+1}$$

with $\alpha_i = \min(1, h_i^{n+1} U_i / \|\vec{q}_i^{\,n+1}\|)$, where U_i is a local estimate of an upper bound for the velocity norm, e.g.

$$U_i = \max_{K \ni i} \max \left(\max_{\substack{j \in K \\ j \text{ is wet}}} \|u_j^n\|, \max_{j \in K} \sqrt{g h_j^n} \right)$$

(Implicit-explicit) time stepping for RD

Upwind FEM and mass matrix

Let's look at

$$\partial_t u + a\partial_x u + \gamma\partial_x b = 0$$

Consider the streamline upwind finite element method (no or periodic BCs)

$$\int_\Omega \varphi_i \partial_t u + \int_\Omega \varphi_i (a\partial_x u + \gamma\partial_x b) + \sum_K \int_K a\partial_x\varphi_i\, \tau\, u_t + \sum_K \int_K a\partial_x\varphi_i\, \tau\, (a\partial_x u + \gamma\partial_x b) = 0$$

I promise (no cheating) this is almost exactly the same as the first order upwind splitting, if the stabilization parameter is taken as $\tau = \Delta x_K/2|a|$.

Upwind FEM and mass matrix

Let's look at

$$\partial_t u + a\partial_x u + \gamma \partial_x b = 0$$

take $a > 0$ and consider the stabilized finite element method ($\tau = \Delta x_K / 2|a|$)

$$\int_\Omega (\varphi_i + \frac{\Delta x}{2}\partial_x \varphi_i)\partial_t u + a(u_i - u_{i-1}) + \gamma(b_i - b_{i-1}) = 0$$

this is almost the same as the first order upwind splitting. What we miss to get high order is a mass matrix...

The RD case is very similar.

Fully explicit residual based schemes

To develop the main idea we consider stabilized finite elements writing[5]

$$\int_\Omega \varphi_i(\partial_t u + \nabla\cdot\mathcal{F} + S) + \sum_{K\ni i}\int_K \gamma_i(\partial_t u + \nabla\cdot\mathcal{F} + S) = 0$$

with the consistency condition $\sum_{j\in K}\gamma_j = 0$

The analogy with RD implies that

$$\int_K (\varphi_i + \gamma_i)\partial_t u = \sum_{j\in K} m_{ij}^{\mathsf{RD}} \frac{du_j}{dt}$$

$$\int_K (\varphi_i + \gamma_i)(\nabla\cdot\mathcal{F} + S) = \beta_i^K \phi^K$$

and allows to construct explicit forms of mass matrices.

[5] Ricchiuto and Abgrall, *JCP* 2010

How to do fully explicit

Step 1 : Consider a semi-discrete explicit time approximation

$$r^{n+1} = \sum_{l\geq 0} \alpha_l \frac{\Delta^{n+1-l}u}{\Delta t} + \sum_{l\geq 1} \theta_l (\nabla \cdot \mathcal{F} + S)^{n+1-l}$$

Step 2 : write the unstabilized formulation (m^{G} the Galerkin mass matrix):

$$\sum_{K\ni i}\sum_{j\in K} m^{\mathsf{G}}_{ij} \sum_{l\geq 0} \alpha_l \frac{\Delta^{n+1-l}u_j}{\Delta t} + \sum_{K\ni i} \int_K \varphi_i \sum_{l\geq 1} \theta_l (\nabla \cdot \mathcal{F} + S)^{n+1-l} = 0$$

Step 3 : stabilize with a modified residual in which u^{n+1} is replaced by some explicit predictor:

$$\sum_{K\ni i}\sum_{j\in K} m^{\mathsf{G}}_{ij} \sum_{l\geq 0} \alpha_l \frac{\Delta^{n+1-l}u_j}{\Delta t} + \sum_{K\ni i} \int_K \varphi_i \sum_{l\geq 1} \theta_l (\nabla \cdot \mathcal{F} + S)^{n+1-l} +$$

$$+ \sum_{K\ni i} \int_K \gamma_i \sum_{l\geq 1} \hat{\alpha}_l \frac{\Delta^{\widehat{n+1-l}}u}{\Delta t} + \sum_{K\ni i} \int_K \gamma_i \sum_{l\geq 1} \theta_l (\nabla \cdot \mathcal{F} + S)^{n+1-l} = 0$$

(Implicit-explicit) time stepping for RD

How to do fully explicit

Lumping the Galerkin mass matrix and recasting as an error correction we get

$$|C_i|\left\{\sum_{l\geq 0}\alpha_l\frac{\Delta^{n+1-l}u_i}{\Delta t}-\sum_{l\geq 1}\hat{\alpha}_l\frac{\Delta^{\widehat{n+1-l}}u_i}{\Delta t}\right\}=$$
$$-\sum_{K\ni i}\left\{\sum_{j\in K}m_{ij}^{\mathsf{RD}}\sum_{l\geq 1}\hat{\alpha}_l\frac{\Delta^{\widehat{n+1-l}}u_j}{\Delta t}+\beta_i^K\sum_{l\geq 0}\theta_l\phi^K(u^{n+1-l})\right\}$$

How to choose $\Delta^{\widehat{n+1-l}}u_j$????

(Implicit-explicit) time stepping for RD

Result (Ricchiuto and Abgrall, *JCP* 2010)

Proposition Given a time semi-discretization with a truncation error estimate of the type $|r^{n+1}| \leq c_t \Delta t^{k_t+1}$, for a P^k finite element approximation, the scheme verifies a consistency estimate of the type $\epsilon < C\Delta x^{k+1}$ provided that $k_t \geq k$, that β_i^K and γ_i are uniformly bounded, and that the modified residual verifies the lower order consistency estimate $|\widehat{r}^{n+1}| \leq c_t \Delta t_t^k$.

(Implicit-explicit) time stepping for RD

Example 1: RK2-RD scheme

$$r^{n+1} = \frac{u^{n+1} - u^n}{\Delta t} + \frac{1}{2}(\nabla \cdot \mathcal{F} + S)^n + \frac{1}{2}(\nabla \cdot \mathcal{F} + S)^*$$
$$\widehat{r}^{n+1} = \frac{u^{*} - u^n}{\Delta t} + \frac{1}{2}(\nabla \cdot \mathcal{F} + S)^n + \frac{1}{2}(\nabla \cdot \mathcal{F} + S)^*$$

The predicted u^* value can be obtained by a first explicit step without mass matrix:

$$|C_i| \frac{u_i^* - u_i^n}{\Delta t} = -\sum_{K \ni i} \beta_i^K \phi^K(u^n)$$

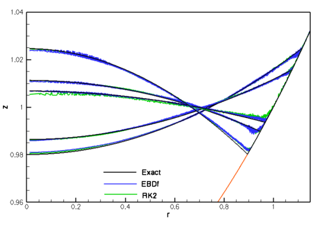
Example 2: EBDf-RD scheme

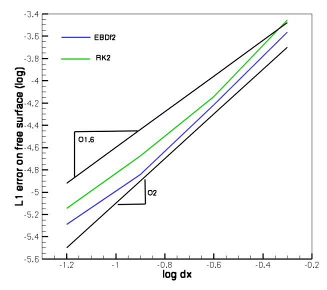
$$
\begin{aligned}
r^{n+1} &= \frac{3}{2}\frac{u^{n+1}-u^{n}}{\Delta t} - \frac{1}{2}\frac{u^{n}-u^{n-1}}{\Delta t} + (\nabla\cdot\mathcal{F}+S)^{*} \\
\widehat{r}^{\,n+1} &= \frac{u^{n}-u^{n-1}}{\Delta t} + (\nabla\cdot\mathcal{F}+S)^{*}
\end{aligned}
$$

The * value is now he time extrapolated one $x^* = 2x^n - x^{n-1}$.

Explicit RD: Thacker oscillations

Thacker oscillations





(Implicit-explicit) time stepping for RD

Implicit-explicit EBDf-RD

For stiff problems, an IMEX version of EBDf can be constructed :

$$\sum_{K\ni i}\sum_{j\in K} m_{ij}^{\mathsf{G}} \frac{3}{2}\frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\Delta t} =$$

$$-\sum_{K\ni i}\left\{\sum_{j\in K} m_{ij}^{\mathsf{RD}}\frac{u_j^n-u_j^{n-1}}{\Delta t}+\beta_i^K\left(\nabla\cdot\mathcal{F}+S\right)^*+\sum_{j\in K} m_{ij}^{\mathsf{RD}} f_j^*\right\}$$

- $m_{ij}^{\mathsf{RD}} f_j^*$ is an approximation of $m_{ij}^{\mathsf{RD}} f_j^{n+1} = m_{ij}^{\mathsf{G}} f_j^{n+1} + m_{ij}^{\mathsf{Stab}} f_j^{n+1}$
- We will keep the full A-stable implicit unstabilized form :$m_{ij}^{\mathsf{G}} f_j^{n+1}$
- We will still use the modified residual for the stabilization: $m_{ij}^{\mathsf{Stab}} f_j^*$

Implicit-explicit EBDF-RD

Putting back together all the terms we end with the implicit update:

$$\sum_{K\ni i}\sum_{j\in K} m_{ij}^{\mathsf{G}}\left\{\frac{3}{2}\;\frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\Delta t}+f^{n+1}-f^*\right\}=$$

$$-\sum_{K\ni i}\left\{\sum_{j\in K} m_{ij}^{\mathsf{RD}}\frac{u_j^n-u_j^{n-1}}{\Delta t}+\beta_i^K\left(\nabla\cdot\mathcal{F}+S\right)^*+\sum_{j\in K} m_{ij}^{\mathsf{RD}} f_j^*\right\}$$

Implicit phase

The implicit equation

$$|C_i|\left\{\frac{3}{2}\frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\Delta t}+f^{n+1}-f^*\right\}=-R_i$$

is **solved analytically**

(Implicit-explicit) time stepping for RD

Implicit-explicit EBDf-RD

Putting back together all the terms we end with the implicit update:

$$\sum_{K\ni i}\sum_{j\in K} m_{ij}^{\mathsf{G}}\left\{\frac{3}{2}\;\frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\Delta t}+f^{n+1}-f^*\right\}=$$

$$-\sum_{K\ni i}\left\{\sum_{j\in K} m_{ij}^{\mathsf{RD}}\frac{u_j^n-u_j^{n-1}}{\Delta t}+\beta_i^K\left(\nabla\cdot\mathcal{F}+S\right)^*+\sum_{j\in K} m_{ij}^{\mathsf{RD}} f_j^*\right\}$$

(Linear) Stability

By Kreiss (1962) and Strang (1964) (cf. book by Richtmyer & Morton (1967)):

If the discrete system

$$u^{n+1}=C(\Delta t)u^n$$

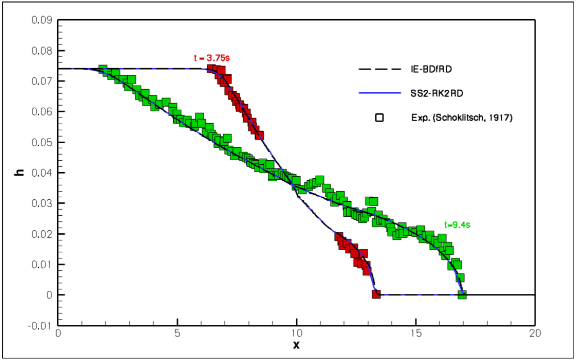
is stable, that given a bounded operator $Q(\Delta t)$*, the discrete perturbed system*

$$u^{n+1}=C(\Delta t)u^n+\Delta t Q(\Delta t)u^n$$

is also stable

Some applications

Dam break experiment by Schoklitsch (1917)

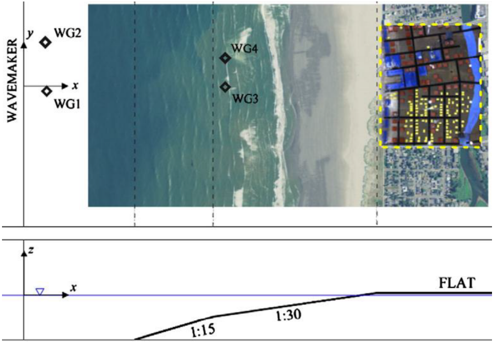


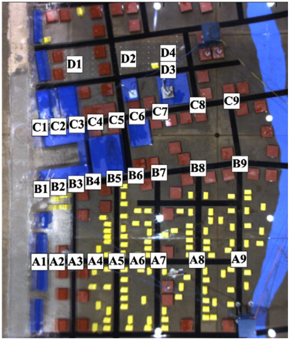
- Friction dominated
- Friction coefficient "blow up" at the wet/dry point

Seaside experiment

Wave tank reproduction of Seaside (Oregon)

Full details in (Park et al, *Coast.Eng.* 2013)





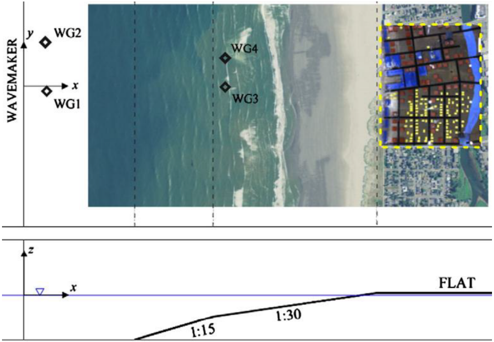
Wave tank reproduction of Seaside (Oregon)

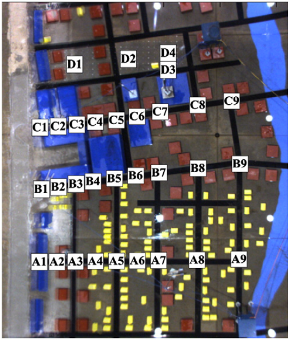
Seaside back

Seaside top

Wave tank reproduction of Seaside (Oregon)

Full details in (Park et al, *Coast.Eng.* 2013)





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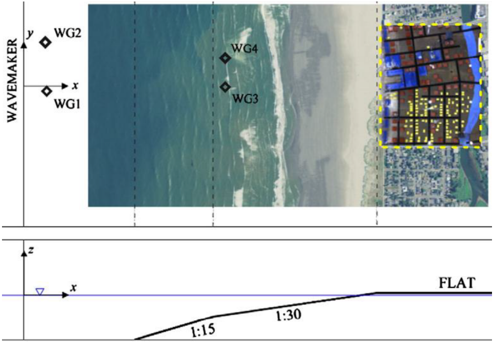
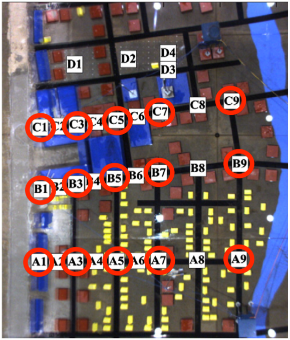
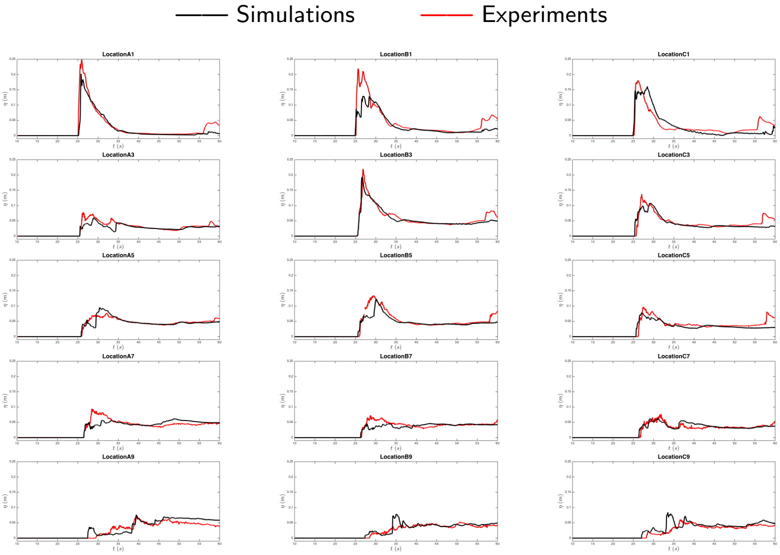
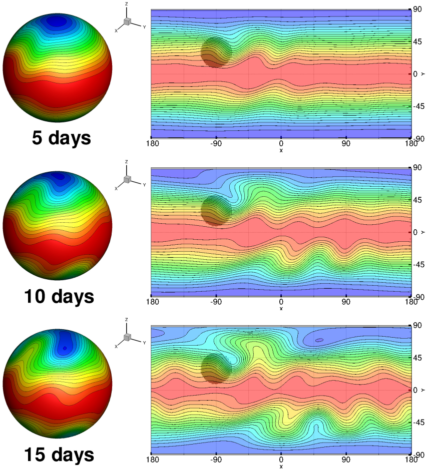


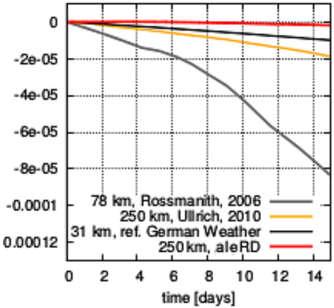


Seaside experiment



Case #5 from *[Williamson et al., 1992]* is a zonal flow perturbed by a mountain



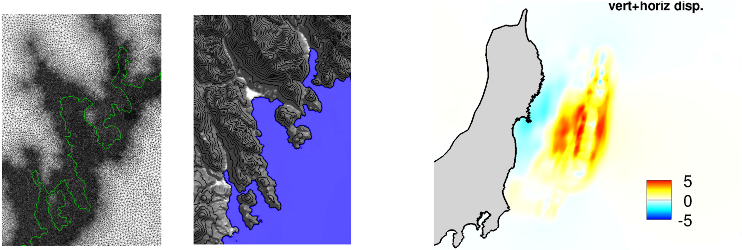


Relative energy error

Global zonal geostrophic flow

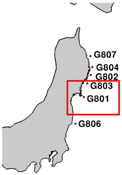
2011 Tohoku tsunami

- **Mesh 1**: $5\,km$ to $120\,m$
- **Mesh 2**: $15\,km$ to $360\,m$ (half the number of cells)
- Source: vertical + horizontal displacement (courtesy of BRGM)

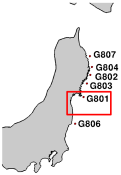


Left and center: Embedded mesh and initial state in the vicinity of Iwate prefecture.
Right: initial vertical displacement.

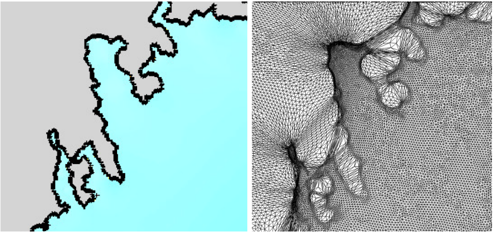
Adaptive ALE simulations



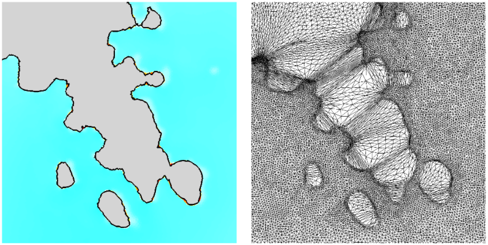
Adaptive ALE simulations



Shoreline adaptation



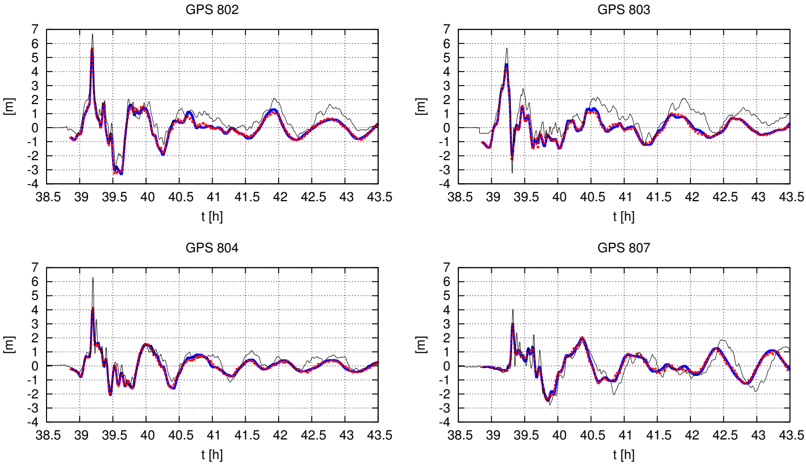
Initial shoreline (Iwate prefecture)



Initial shoreline (Miyagi prefecture)

ALE simulation against reference

— Mesh a (Reference) — Mesh b + ADAPT-ALE — Obs.Data



Flash floods in the city of Worms (Rhineland-Palatinate)

Collaboration with BGS IT&E (German SME):
A. Roland et al, EGU meeting 2018

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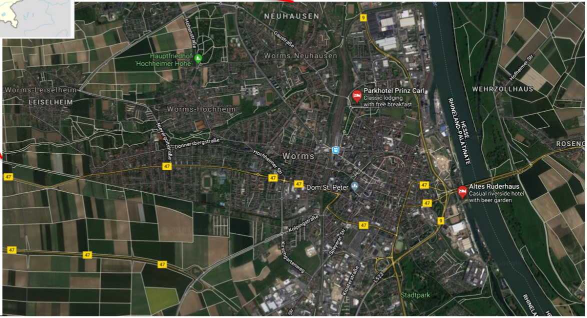
Worms, Germany



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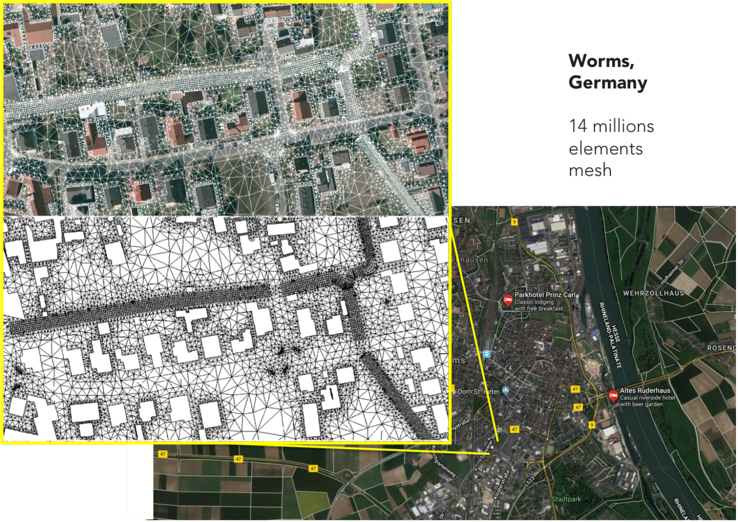

Worms, Germany

14 millions elements mesh

Flash floods in the city of Worms (Rhineland-Palatinate)

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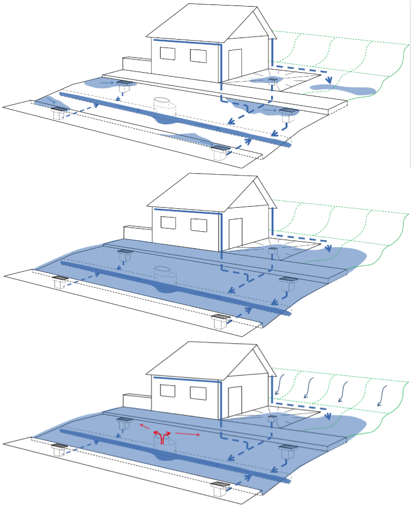

2-way coupling with sub-models for sewage system (Bernoulli-type models), and unresolved structures (pressure patches)

Flash floods in the city of Worms (Rhineland-Palatinate)

Collaboration with BGS IT&E (German SME):
A. Roland et al, EGU meeting 2018

Worms city movie

Flash floods in the city of Worms (Rhineland-Palatinate)

December 2nd, 2017:
results presented during a public audition in the city aiming at raising the population's awareness on the risks of floods, and on the importance of proper forecasting in the development of hazard reduction policies

Summary and outlook

- Residual distribution: genuinely multidimensional well balanced schemes
- general framework to solve balance laws on unstructured moving grids
- close relations to continuous stabilized finite elements (e.g. SUPG)
- mass matrix requires careful design of efficient time stepping (error correction approach)
- numerical results are extremely satisfactory/promising

ONGOING AND FUTURE WORK

- More on IMEX: stiffly stable RK, application to kinetic approximations
- Higher order (≥ 3) unsteady for complex/realistic applications ?
- Other constraints : e.g. energy/entropy conservation
- Immersed/embedded BCs: feasible for urban inundation?
- Dispersive waves
- etc. etc.

THX !!! ☺