From Roe's scheme to Fluctuation splitting, residual distribution and stabilized finite elements

Hyperbolic problems and a peek at dispersive wave propagation.

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Inria Bordeaux - Sud-Ouest

Winter school Nonlinear dispersive waves:
theory, numerics and applications
February 16-21, 2014

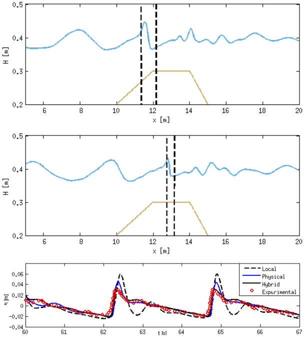
Context 1/2

Near shore hydrodynamics

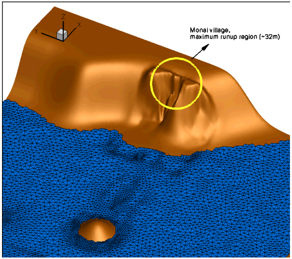
Bonneton,Chazel,Lannes,Marche,Tissier - Delis,Kazolea,Synolakis - Kirby,Grilli,et al (FUNWAVE-TVD) - Smit,Zijlema et al (SWASH) - Ricchiuto et al - etc.

(Ribbed channel clip)

Propagation :
enhanced Boussinesq
(or other dispersive model)



Breaking :
coupling with SW
(or similar hydrostatic)



Runup/flooding :
full SW
(or similar hydrostatic)

Context 2/2

Numerics: keywords

1. time dependent
2. smooth waves : high accuracy
3. wave grouping : low dispersion error
4. steep fronts (bores, h-jumps) : non-oscillatory
5. wet/dry : positivity preservation
6. geometrical flexibility : mesh adaptation (w.r.t. bathymetry and solution)
7. unstructured meshes

Need a high accuracy (low dispersion) discontinuity capturing method

A bit of first hand HISTORICAL PERSPECTIVE ...

High order schemes for fluid dynamics ..

Bram van Leer

the Arthur B. Modine Professor of aerospace engineering at the University of Michigan, in Ann Arbor.

He specialises in Computational fluid dynamics (CFD), fluid dynamics, and numerical analysis

where he has made substantial contributions. (wikipedia)

1. - Upwind high resolution methods for compressible flow: from donor cell to residual distribution, *Commun.Comput.Phys.* 1(2), 2006
2. - History of CFD - PART II, *AIAA Fluid Dynamics award*, 2010

Historical perspective : schemes for CFD 1/4

Godunov's theorem (S.K. Godunov, *Math.Sb* 47, 1959)

If an advection scheme preserves the monotonicity of the solution, it is at most first order accurate

The 70s' run for high order shock capturing schemes gives its first ripe fruits by the end of the decade

1. V.P. Kolgan's reports with limited linear recontrution : 1972
2. Boris, Book and Zalesak's work on FCT is out : 1973-79
3. van Leer's Toward the Ultimate Conservative Difference Scheme I-V papers appeared : 1979

The way out of Godunov's theorem is found : nonlinear schemes

Historical perspective : schemes for CFD 2/4

The challenge of multidimensional nonlinear limiters

1. A. Harten, *J.Comput.Phys* 49, 1983 and *SINUM* 21, 1984 : TVD conditions
2. Goodman and LeVeque, *Math.Comp.* 45, 1985 : TVD in 2D = first order

From mid-80s to end-90s

Non-oscillatory FV approaches

ENO/WENO schemes (Harten, Osher, Engquist and Chakravarthy),
TVB schemes (Shu *Math.Comp.* 1987),
positive coefficient schemes (Spekreijse, *Math.Comp* 1987, Barth *VKI LS* 1994)

Multidimensional discretization frameworks

Central/LW/SUPG approaches (Jameson, Morton, Ni, Lerat, Hughes),
Rotated and transverse Riemann solvers (Davis *JCP* 1984, LeVeque *JCP* 1988),
Roe's Fluctuation Splitting 1987 (in 2D)
Cockburn and Shu's papers on Discontinuous Galerkin (starting 1988)

Historical perspective : schemes for CFD 3/4

Discontinuous Galerkin

Smart and elegant combination of existing tools (local approximation, Galerkin projection, Riemann solvers, limiters) to automatically generate arbitrary order schemes for conservation laws.

Instant hit : many followers in appl.math. (hyperbolic guys) and engnrg. communities

Multidimensional upwind differencing

A **more fundamental and robust** approach [...] due to Roe (1986), is that of the "genuinely multidimensional" upwind schemes. These may be regarded as **the true multi-D generalization** of 1-D fluctuation splitting [...] These methods are best formulated on simplex-type (finite-element) grids and include **newly developed, compact limiters** for avoiding oscillations

Excerpt from Upwind high resolution methods for compressible flow: from donor cell to residual distribution, *Commun.Comput.Phys.* 1(2), 2006

Roe's fluctuation splitting in the scientific community

An **entirely new approach** is proposed, with its own set of "physically relevant" discretization rules and numerical constraints, with a **completely new meaning and use of nonlinear limiters**.

Despite the large interest[1], approach that never really conquered the CFD community :

1. new vocabulary and formalism take time to root
2. good results for interesting problems with this approach have taken time to surface

It is however considered today a possible alternative to higher order WENO Finite Volumes and DG...

[1] The list is quite long : Univrsity of Michigan (Roe-van Leer), VKI (Deconinck), Universiy of Reading (Baines-Hubbard), Politecnico di Bari (Napolitano and co.), ICASE (van Leer-Roe-Sidilkover), NASA (Barth-Wood-Kleb), Brown University (Shu), UTIAS Toronto (Groth), University of Lisbon (Gato), INRIA-Université Bordeaux (Abgrall) and several others ...

The talk 1/2

What have I brought for you ..

- high order schemes : some general principle
- unstructured grids : some general principles that are more general
- dispersive equations : my perspective

The talk 2/2

Part I

- Fluctuation form of FV schemes
- Design properties for fluctuation splitting/residual distribution : steady case
  1. Conservation
  2. Stability and upwinding
  3. Consistency (accuracy) conditions
  4. Discontinuity capturing

Part II

- Residual based schemes and time dependent problems : accuracy issues
- High order accurate schemes via consistent mass matrices
- Shallow Water issues : steady states, wetting/drying
- Dispersive equations and residual based
- Discrete dispersion analysis

BCs are neglected throughout the talk. A zero on the RHS most often means $=b.c.$ *terms*

Finite Volume schemes and Fluctuations in 1D

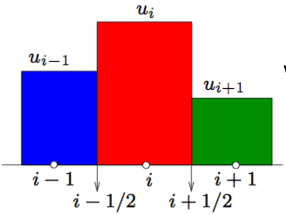
Finite Volume schemes and Fluctuations in 1D

Starting point : conservation law

$$\partial_t u + \partial_x \mathcal{F}(u) = 0$$

Conservative FV :

$$\Delta x_i \frac{\Delta u_i}{\Delta t} + \widehat{\mathcal{F}}(u^L_{i+1/2}, u^R_{i+1/2}) - \widehat{\mathcal{F}}(u^L_{i-1/2}, u^R_{i-1/2}) = 0$$



With

$$\frac{\Delta u_i}{\Delta t} = \frac{u_i^{n+1} - u_i^n}{t^{n+1} t^n}$$

Finite Volume schemes and Fluctuations in 1D

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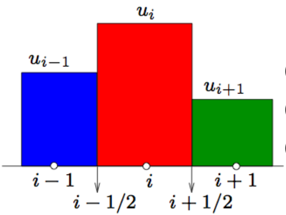

(consistent) $\widehat{\mathcal{F}}(u,u) = \mathcal{F}(u)$

(continuous) $\|\widehat{\mathcal{F}}(v,w) - \mathcal{F}(u)\| \leq L_{\mathcal{F}} \min(\|v-u\|, \|z-u\|)$

(E-stable. Monotone, etc.)

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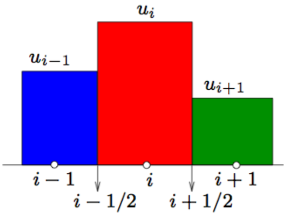

$$\Delta x_{i+1} \frac{\Delta u_{i+1}}{\Delta t} + \widehat{\mathcal{F}}_{i+3/2} - \widehat{\mathcal{F}}_{i+1/2} = 0$$

$$\Delta x_{i-1} \frac{\Delta u_{i-1}}{\Delta t} + \widehat{\mathcal{F}}_{i-1/2} - \widehat{\mathcal{F}}_{i-3/2} = 0$$

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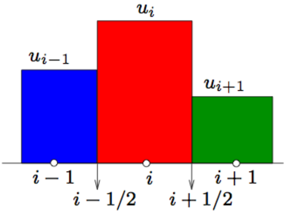


$$\Delta x_{i+1} \frac{\Delta u_{i+1}}{\Delta t} + \widehat{\mathcal{F}}_{i+3/2} - \widehat{\mathcal{F}}_{i+1/2} = 0$$

$$\Delta x_{i-1} \frac{\Delta u_{i-1}}{\Delta t} + \widehat{\mathcal{F}}_{i-1/2} - \widehat{\mathcal{F}}_{i-3/2} = 0$$

Discrete conservation : flux cancellation at interfaces

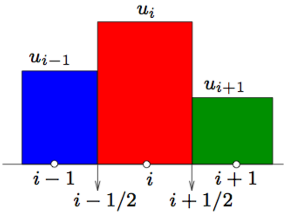
Finite Volume schemes and Fluctuations in 1D

Starting point : conservation law

$$\partial_t u + \partial_x \mathcal{F}(u) = 0$$

Conservative FV :

$$\Delta x_i \frac{\Delta u_i}{\Delta t} + (\widehat{\mathcal{F}}_{i+1/2} - \mathcal{F}_i) + (\mathcal{F}_i - \widehat{\mathcal{F}}_{i-1/2}) = 0$$



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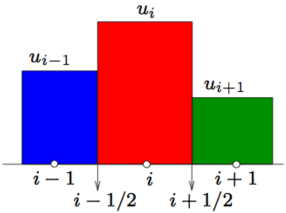
Conservative FV :

$$\Delta x_i \frac{\Delta u_i}{\Delta t} + \underbrace{(\widehat{\mathcal{F}}_{i+1/2} - \mathcal{F}_i)}_{\phi_i^{i+1/2}} + \underbrace{(\mathcal{F}_i - \widehat{\mathcal{F}}_{i-1/2})}_{\phi_i^{i-1/2}} = 0$$



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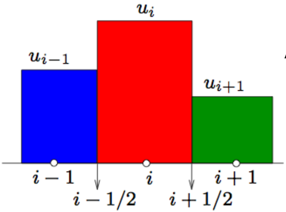

At $i + 1/2$ conservation is

$$\phi_i^{i+1/2} + \phi_{i+1}^{i+1/2} = (\widehat{\mathcal{F}}_{i+1/2} - \mathcal{F}_i) + (\mathcal{F}_{i+1} - \widehat{\mathcal{F}}_{i+1/2})$$

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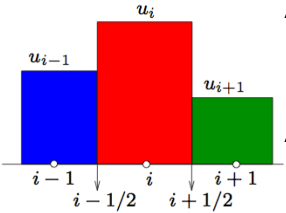

At $i+1/2$ conservation is

$$\phi_i^{i+1/2} + \phi_{i+1}^{i+1/2} = \mathcal{F}_{i+1} - \mathcal{F}_i := \phi^{i+1/2}$$

At $i-1/2$ conservation is

$$\phi_i^{i-1/2} + \phi_{i-1}^{i-1/2} = \mathcal{F}_i - \mathcal{F}_{i-1} := \phi^{i-1/2}$$

Finite Volume schemes and Fluctuations in 1D

Starting point : conservation law

$$\partial_t u + \partial_x \mathcal{F}(u) = 0$$

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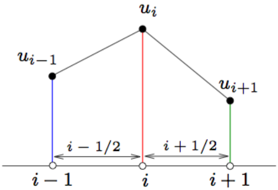

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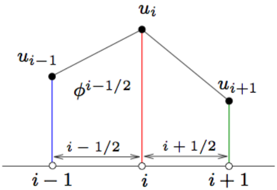

$$\phi^{i-1/2} := \int_{i-1}^{i} \partial_x \mathcal{F} \qquad \text{(fluctuation)}$$

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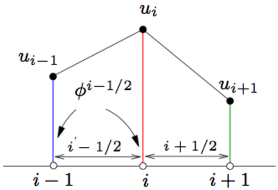

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$$\phi_i^{i-1/2} = \mathcal{F}_i - \widehat{\mathcal{F}}_{i-1/2} \qquad \text{(splitting)}$$

$$\phi_{i-1}^{i-1/2} = \widehat{\mathcal{F}}_{i-1/2} - \mathcal{F}_{i-1} \quad \text{(splitting)}$$

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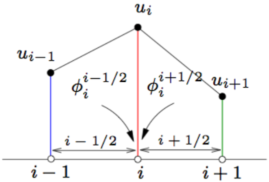

$$\phi^{i-1/2} := \int_{i-1}^{i} \partial_x \mathcal{F}, \qquad \phi^{i+1/2} := \int_{i}^{i+1} \partial_x \mathcal{F}$$

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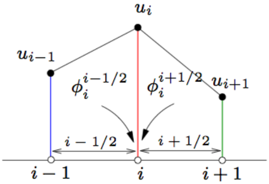

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Example 1 : Roe (upwind) scheme for advection $(\mathcal{F}(u) = au)$

Finite Volume schemes and Fluctuations in 1D

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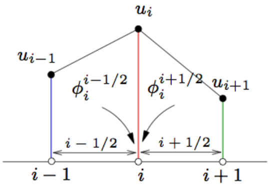

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Example 1 : Roe (upwind) scheme for advection ($\mathcal{F}(u) = au$)

$$\widehat{\mathcal{F}}_{i+1/2} = \frac{\mathcal{F}_i + \mathcal{F}_{i+1}}{2} - \frac{|a|}{2}(u_{i+1} - u_i)\ , \quad \widehat{\mathcal{F}}_{i-1/2} = \frac{\mathcal{F}_i + \mathcal{F}_{i-1}}{2} - \frac{|a|}{2}(u_i - u_{i-1})$$

$$\phi_i^{i+1/2} = \frac{1 - \text{sign}(a)}{2}(\mathcal{F}_{i+1} - \mathcal{F}_i) = \frac{1 - \text{sign}(a)}{2}\phi^{i+1/2}$$

$$\phi_i^{i-1/2} = \frac{1 + \text{sign}(a)}{2}(\mathcal{F}_i - \mathcal{F}_{i-1}) = \frac{1 + \text{sign}(a)}{2}\phi^{i-1/2}$$

Finite Volume schemes and Fluctuations in 1D

Conservative FV :

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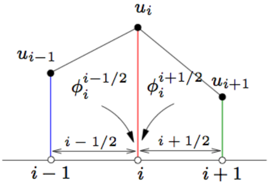

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Example 2 : Lax-Wendroff scheme

$$\widehat{\mathcal{F}}_{i+1/2} = \frac{\mathcal{F}_i + \mathcal{F}_{i+1}}{2} - \frac{a\Delta t}{2\Delta x}(au_{i+1} - au_i) \,, \quad \widehat{\mathcal{F}}_{i-1/2} = \frac{\mathcal{F}_i + \mathcal{F}_{i-1}}{2} - \frac{a\Delta t}{2\Delta x}(au_i - au_{i-1})$$

$$\phi_i^{i+1/2} = \frac{1 - \mathsf{CFL}}{2}(\mathcal{F}_{i+1} - \mathcal{F}_i) = \frac{1 - \mathsf{CFL}}{2}\phi^{i+1/2}$$

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Finite Volume schemes and Fluctuations in 1D

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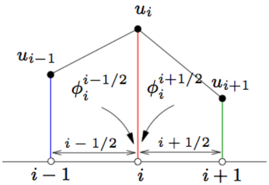

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Example 3 : Lax-Friedrich's scheme

$$\widehat{\mathcal{F}}_{i+1/2} = \frac{\mathcal{F}_i + \mathcal{F}_{i+1}}{2} - \alpha_{\mathsf{LF}}(u_{i+1} - u_i)\ , \quad \widehat{\mathcal{F}}_{i-1/2} = \frac{\mathcal{F}_i + \mathcal{F}_{i-1}}{2} - \alpha_{\mathsf{LF}}(u_i - u_{i-1})$$

$$\phi_i^{i+1/2} = \frac{1}{2}(\mathcal{F}_{i+1} - \mathcal{F}_i) + \alpha_{\mathsf{LF}}(u_i - u_{i+1}) = \frac{1}{2}\phi^{i+1/2} + \alpha_{\mathsf{LF}}(u_i - u_{i+1})$$

$$\phi_i^{i-1/2} = \frac{1}{2}(\mathcal{F}_i - \mathcal{F}_{i-1}) + \alpha_{\mathsf{LF}}(u_i - u_{i-1}) = \frac{1}{2}\phi^{i-1/2} + \alpha_{\mathsf{LF}}(u_i - u_{i-1})$$

Finite Volume schemes and Fluctuations in 1D

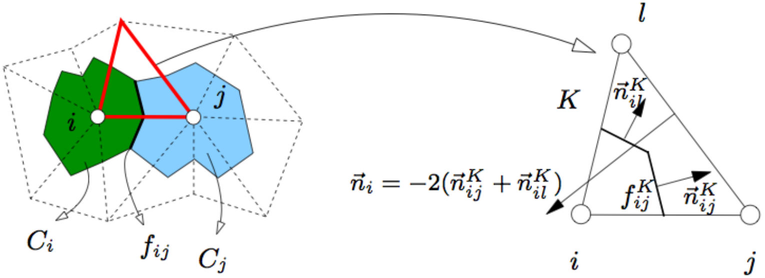
Nothing new so far !!!

Finite Volume schemes and Fluctuations in 2D

The multi-D case. Starting point : conservation law

$$\partial_t u + \nabla \cdot \mathcal{F}(u) = 0$$

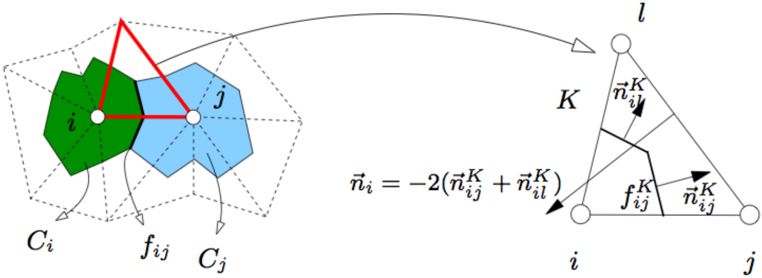
Finite Volume schemes and Fluctuations in 2D



The FV scheme reads

$$|C_i| \frac{du_i}{dt} + \sum_j \int_{f_{ij}} \widehat{\mathcal{F}} \cdot \hat{n}\, dl = 0$$

Finite Volume schemes and Fluctuations in 2D

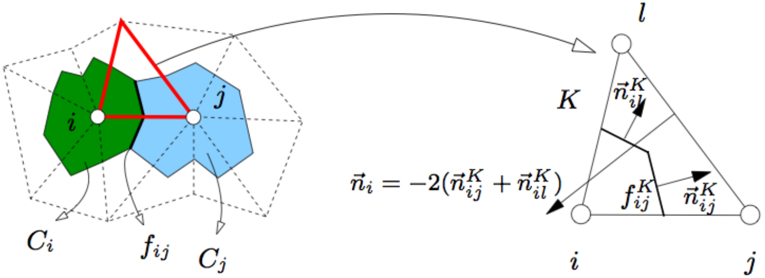


The FV scheme reads

$$|C_i|\frac{du_i}{dt} + \sum_{K\,|\,i\in K}\sum_{j\in K}\widehat{\mathcal{F}}_{ij}\cdot\vec{n}_{ij}^K = 0$$

Discrete conservation

$$\widehat{\mathcal{F}}_{ij}\cdot\vec{n}_{ij}^K + \widehat{\mathcal{F}}_{ji}\cdot\vec{n}_{ji}^K = 0$$

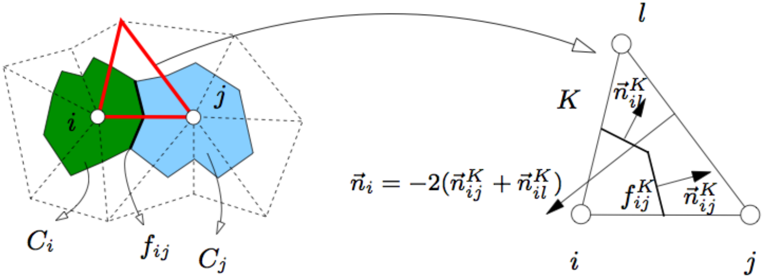


Using the identity $\sum_K \sum_j \vec{n}_{ij}^K = 0$

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \underbrace{\sum_{j\in K} (\widehat{\mathcal{F}}_{ij} - \mathcal{F}_i)\cdot \vec{n}_{ij}^K}_{\phi_i^K} = 0$$

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \phi_i^K = 0$$

Finite Volume schemes and Fluctuations



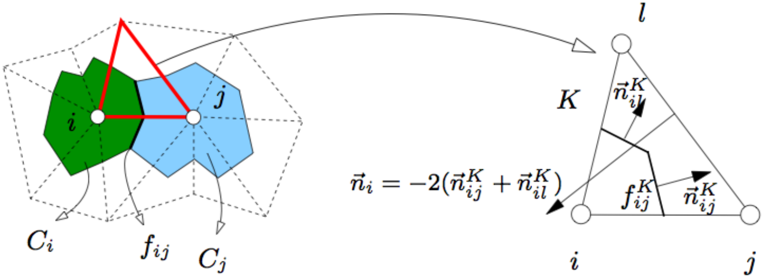
The FV scheme reads

$$|C_i|\frac{du_i}{dt} + \sum_{K\,|\,i\in K} \phi_i^K = 0\,, \qquad \phi_i^K = \sum_{j\in K} (\widehat{\mathcal{F}}_{ij} - \mathcal{F}_i)\cdot \vec{n}_{ij}^K$$

Discrete conservation

$$\widehat{\mathcal{F}}_{ij}\cdot\vec{n}_{ij}^K + \widehat{\mathcal{F}}_{ji}\cdot\vec{n}_{ji}^K = 0 \Longrightarrow \sum_{j\in K}\phi_j^K = \frac{1}{2}\sum_{j\in K}\mathcal{F}_i\cdot\vec{n}_j := \phi^K$$

Finite Volume schemes and Fluctuations



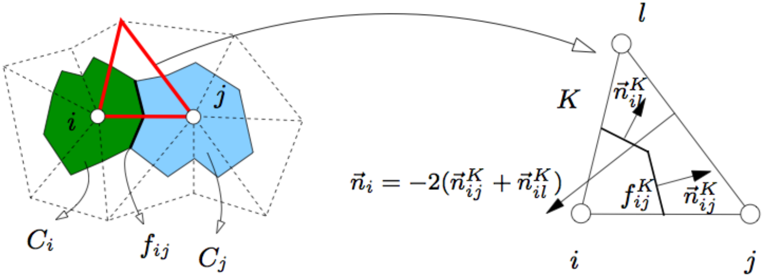
The FV scheme reads

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \phi_i^K = 0\ , \quad \phi_i^K = \sum_{j\in K} (\widehat{\mathcal{F}}_{ij} - \mathcal{F}_i) \cdot \vec{n}_{ij}^K$$

Discrete conservation ($\mathcal{F}_h$ continuous P^1 finite element approx.)

$$\sum_{j\in K} \phi_j^K = \phi^K = \int_K \nabla \cdot \mathcal{F}_h$$

Finite Volume schemes and Fluctuations



The FV scheme reads ($\mathcal{F}_h$ continuous P^1 finite element approx.)

$$\phi^K = \int_K \nabla \cdot \mathcal{F}_h \, , \qquad \overbrace{\sum_{j \in K} \phi_i^K = \phi^K}^{\text{Discrete conservation}}$$

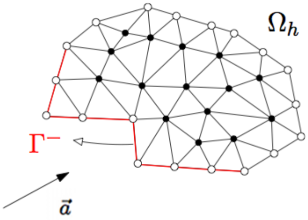
$$|C_i| \frac{du_i}{dt} + \sum_{K|i \in K} \phi_i^K = 0 \, , \qquad \phi_i^K = \sum_{j \in K} (\widehat{\mathcal{F}}_{ij} - \mathcal{F}_i) \cdot \vec{n}_{ij}^K$$

... but it's still the same guy .. !!

Residual Distribution framework (steady)

Residual Distribution framework (steady)

$$\begin{aligned} \nabla \cdot \mathcal{F}(u) &= 0 \quad \text{in} \quad \Omega \\ u &= g \quad \text{on} \quad \Gamma^- \qquad (1) \\ \vec{a}(u) &= \partial_u \mathcal{F}(u) \end{aligned}$$



Some notations...

- Consider Ω_h tesselation of Ω
- Unknowns (Degrees of Freedom, DoF) : $u_i \approx u(M_i)$
- $M_i \in \Omega_h$ a given set of nodes (vertices +other dofs)
- u_h : **continuous** polynomial interpolation (FE) $u_h = \sum_i \psi_i \, u_i$

Residual Distribution framework (steady)

For $n \geq 0$, until steady state do :

For all $K \in$ mesh do

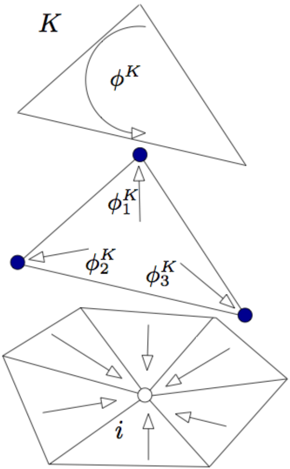
1. compute cell residual $\phi^K = \oint_{\partial K} \boldsymbol{\mathcal{F}}_h(u_h^n) \cdot \hat{n}\, dl$

2. distribute cell residual $\phi^K = \sum_{i \in K} \phi_i^K$

For all $i \in$ mesh do

3. evolve $|C_i| \dfrac{u_i^{n+1} - u_i^n}{\Delta t} = - \sum_{K | i \in K} \phi_i^K(u_h^n)$

$$\Longrightarrow \sum_{K | i \in K} \phi_i^K(u_h^n) = 0$$



Structural conditions

Stability. which form of stability (energy/entropy, equivalent algebraic condition, convergence ?), choice of ϕ_i^K

Accuracy. characterization of the error, choice of ϕ_i^K

Oscillations. monotonicity preserving schemes, choice of ϕ_i^K

What can we say about the stability of this method ?

First : what is stability ?

What can we say about the stability of this method ?

First : what is stability ?

Recall that we are solving steady state equations with by means of iterations

$$u_i^{n+1} = u_i^n - \omega_i \sum_{K|i\in K} \phi_i^K(u_h^n), \quad \omega_i = \frac{\Delta t}{|C_i|}$$

For h fixed (mesh), what can be said about the convergence to the steady solution we seek ?

What can we say about the stability of this method ?

First : what is stability ?

Abstractly, for h fixed we look at the convergence of (with ω a scalar,*e.g.* $\min_i \omega_i$)

$$\mathsf{u}^{n+1} = \mathsf{u}^n - \omega(A_h\, \mathsf{u}^n - f)$$

What can we say about the stability of this method ?

First : what is stability ?

Abstractly, for h fixed we look at the convergence of (with ω a scalar,*e.g.* $\min_i \omega_i$)

$$\mathsf{u}^{n+1} = \mathsf{u}^n - \omega(A_h\,\mathsf{u}^n - f)$$

A condition for convergence with $n \to \infty$ but h fixed

$$\|(\mathsf{I} - \omega A_h)\mathsf{u}\|^2 \leq r\|\mathsf{u}\|^2\,, \quad \forall \mathsf{u} \text{ and with } r < 1$$

which is equivalent to

$$\mathsf{u}^t A_h \mathsf{u} \geq \frac{1-r}{2\omega}\|\mathsf{u}\|^2 + \frac{\omega}{2}\|A_h\,\mathsf{u}\|^2 \geq C_h\|\mathsf{u}\|^2 \geq 0 \quad \forall\,\mathsf{u}$$

Coercivity ...

Weaker stability...?

Stability and energy

Consider the steady limit of

$$\partial_t u + \vec{a} \cdot \nabla u = 0$$

Semi-discrete counterpart

$$|C_i| \frac{du_i}{dt} + \sum_{K \,|\, i \in K} \phi_i^K = 0$$

Energy budget

The equivalent of the quantity $\mathsf{u}^t A_h \mathsf{u}$ seen in the previous slides is

$$\begin{aligned} \mathsf{u}^t A_h \mathsf{u} &\equiv \sum_{i \in \Omega_h} u_i \sum_{K \,|\, i \in K} \phi_i^K \\ &= \sum_{K \in \Omega_h} \sum_{i \in K} u_i \phi_i^K = \sum_{K \in \Omega_h} \phi_K^{\mathcal{E}} \end{aligned}$$

Stability and energy

Starting from

$$|C_i| \frac{du_i}{dt} + \sum_{K \,|\, i \in K} \phi_i^K = 0$$

Energy budget

$$\sum_{i \in \Omega_h} |C_i| u_i \frac{du_i}{dt} + \sum_{K \in \Omega_h} \phi_K^{\mathcal{E}} = 0$$

Stability and energy

Starting from

$$|C_i|\frac{du_i}{dt} + \sum_{K\,|\,i\in K} \phi_i^K = 0$$

Energy budget

$$\int_{\Omega_h} \frac{d\mathcal{E}_h}{dt} + \sum_{K\in\Omega_h} \phi_K^{\mathcal{E}} = 0$$

with the *energy density*

$$\mathcal{E} = \frac{u^2}{2}$$

and with $\mathcal{E}_h = \sum_{i\in\Omega_h} \mathcal{E}_i\psi_i$ (piecewise linear)

Stability and energy

Saying that

$$0 < \mathsf{u}^t A_h \mathsf{u} \equiv \sum_{K \in \Omega_h} \phi_K^{\mathcal{E}}$$

is equivalent to

Energy stability

$$\int_{\Omega_h} \frac{d\mathcal{E}_h}{dt} = - \sum_{K \in \Omega_h} \phi_K^{\mathcal{E}} \leq 0$$

with the *energy density*

$$\mathcal{E} = \frac{u^2}{2}$$

and with $\mathcal{E}_h = \sum_{i \in \Omega_h} \mathcal{E}_i \psi_i$ (piecewise linear)

Stability and energy

Saying that

$$0 < \mathsf{u}^t A_h \mathsf{u} \equiv \sum_{K \in \Omega_h} \phi_K^{\mathcal{E}}$$

is equivalent to

Energy stability (modulo boundary conditions)

$$\int_{\Omega_h} \frac{d\mathcal{E}_h}{dt} = -\int_{\partial\Omega_h} \mathcal{E}_h \, \vec{a} \cdot \hat{n} \, dl \; - \; \delta^{\mathcal{E}} \, , \quad \delta^{\mathcal{E}} \geq 0$$

what one would like is to find that

$$\phi_K^{\mathcal{E}} = \int_{\partial K} \mathcal{E}_h \, \vec{a} \cdot \hat{n} \, dl + \delta_K^{\mathcal{E}} \, , \quad \delta_K^{\mathcal{E}} \geq 0$$

Stability and upwinding

Consider again the steady limit of

$$\partial_t u + \vec{a} \cdot \nabla u = 0$$

A geometrical view of advection...

Stability and upwinding

Consider again the steady limit of

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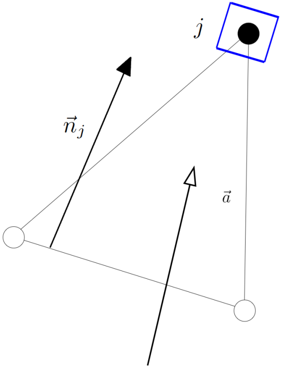
A geometrical view of advection...



1-target triangle
The inlet region is an edge
1 node downstream : 1 target

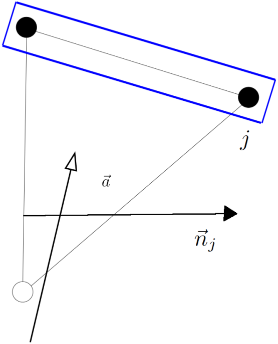
$$k_j = \frac{\vec{a} \cdot \vec{n}_j}{2} > 0$$

Stability and upwinding

Consider again the steady limit of

$$\partial_t u + \vec{a} \cdot \nabla u = 0$$

A geometrical view of advection...



2-target triangle
The outlet region is an edge
2 nodes downstream : 2 targets

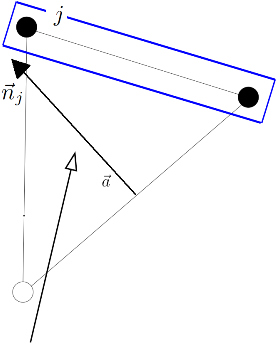
$$k_j = \frac{\vec{a} \cdot \vec{n}_j}{2} > 0$$

Stability and upwinding

Consider again the steady limit of

$$\partial_t u + \vec{a} \cdot \nabla u = 0$$

A geometrical view of advection...



2-target triangle
The outlet region is an edge
2 nodes downstream : 2 targets

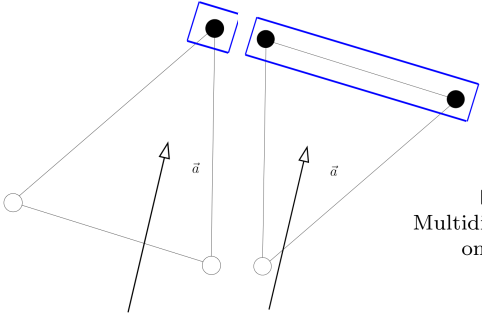
$$k_j = \frac{\vec{a} \cdot \vec{n}_j}{2} > 0$$

Stability and upwinding

Consider now the semi-discrete RD advection equation :

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \phi_i^K = 0$$

A geometrical view of advection...



Multidimensional Upwinding (MU)
Multidimensional Upwind (MU) schemes only split ϕ^K to downstream nodes, *i.e.* those for which $k_j > 0$.

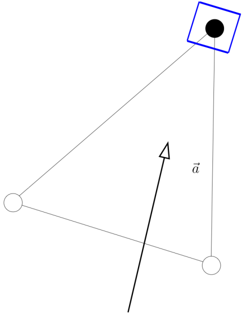
1. All MU schemes reduce to the same in the 1-target case
2. All MU scheme reduce to the upwind scheme in 1D

Stability and upwinding

Consider now the semi-discrete RD advection equation :

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \phi_i^K = 0$$

A geometrical view of advection...



Multidimensional Upwinding (MU)

Multidimensional Upwind (MU) schemes

In 1-target elements, (assume node 1 is downstream)

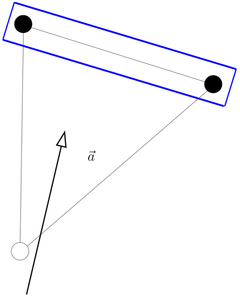
$$\phi_1^K = \phi^K$$
$$\phi_2^K = 0$$
$$\phi_3^K = 0$$

Stability and upwinding

Consider now the semi-discrete RD advection equation :

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \phi_i^K = 0$$

A geometrical view of advection...



Multidimensional Upwinding (MU)

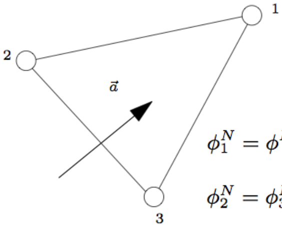
Multidimensional Upwind (MU) schemes

In 2-targets elements (assume node 1 is upstream)

$$\phi_1^K = 0$$

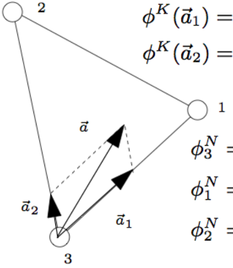
$$\phi_2^K + \phi_3^K = \phi^K$$

Example 1 : Roe's optimal N scheme[1]



$$\phi_1^N = \phi^K$$

$$\phi_2^N = \phi_3^N = 0$$



$$\phi^K(\vec{a}_1) = \int_K \vec{a}_1 \cdot \nabla u_h$$

$$\phi^K(\vec{a}_2) = \int_K \vec{a}_2 \cdot \nabla u_h$$

$$\phi_3^N = 0$$

$$\phi_1^N = \phi^K(\vec{a}_1)$$

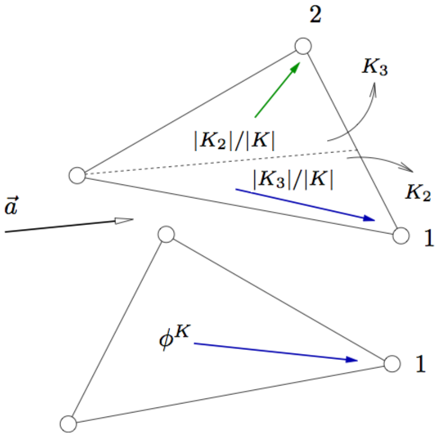
$$\phi_2^N = \phi^K(\vec{a}_2)$$

[1](Roe Cranfield U.Tech.Rep., 1987 ; Roe, Sidilkover *SINUM*, 1992)

Example 2 : the LDA scheme

$$\phi_i^{\mathsf{LDA}}(u_h) = \beta_i^{\mathsf{LDA}} \phi^K(u_h)$$

$$\beta_i^{\mathsf{LDA}} = k_i^+ \Big(\sum_{j \in K} k_j^+ \Big)^{-1}$$



Stability and MU

The following properties can be easily shown :

1. MU schemes, 1-target (Deconinck, Ricchiuto *Enc.Comput.Mech.*, 2007)

$$\phi_K^{\mathcal{E}} = \int_{\partial K} \mathcal{E}_h \, \vec{a} \cdot \hat{n} \, dl + \delta_K^{\mathcal{E}} \, , \qquad \delta_K^{\mathcal{E}} \geq 0$$

2. N scheme energy stable (Barth, NASA 1996 ; Abgrall, Barth *SISC*, 2002)
3. LDA scheme, 2-targets (Deconinck, Ricchiuto *Enc.Comput.Mech.*, 2007)

$$\phi_{\mathsf{LDA}}^{\mathcal{E}} = \underbrace{\big(\sum_{j \in K} k_j^+\big) \left(\frac{u_{out}^2}{2} - \frac{u_{in}^2}{2} \right)}_{\substack{\text{NRG balance} \\ \text{along streamline}}} + \delta_{\mathsf{LDA}}^{\mathcal{E}} \, , \qquad \delta_{\mathsf{LDA}}^{\mathcal{E}} \geq 0$$

Multidimensional upwinding does the job ...

Stability, upwinding, and dissipation

1. FV scheme (1st order upwind) NRG stable (Barth, NASA 1996 ; Abgrall, Barth *SISC*, 2002), also E-flux schemes by (Osher *SINUM*, 1984)

2. Streamline upwind finite element scheme SUPG (Hughes, Brooks *CMAME*, 1982) :

$$\int_{\Omega_h} \psi_i \nabla \cdot \boldsymbol{\mathcal{F}}_h(u_h) + \sum_{K\in\Omega_h} \int_K \vec{a}(u_h)\cdot\nabla\psi_i\,\tau\,\vec{a}(u_h)\cdot\nabla u_h = 0$$

can be written as the RD scheme with[2]

$$\phi_i^K = \beta_i^{\mathsf{SUPG}}\phi^K \quad \text{with} \quad \beta_i^{\mathsf{SUPG}} = \frac{1}{3} + \frac{k_i}{|K|}\tau$$

[2]simplest case of P^1 approx

Stability, upwinding, and dissipation

1. FV scheme (1st order upwind) NRG stable (Barth, NASA 1996 ; Abgrall, Barth *SISC*, 2002), also E-flux schemes by (Osher *SINUM*, 1984)

2. Streamline upwind finite element scheme SUPG (Hughes, Brooks *CMAME*, 1982) :

$$\phi^{\mathcal{E}}_{\mathsf{SUPG}} = \oint_{\partial K} \frac{u_h^2}{2}\,\vec{a}\cdot\hat{n}\,dl + \underbrace{\int_K \vec{a}\cdot\nabla u_h\,\tau\,\vec{a}\cdot\nabla u_h}_{\text{Streamline dissipation } \geq 0}$$

Stability, upwinding, and dissipation

1. FV scheme (1st order upwind) NRG stable (Barth, NASA 1996 ; Abgrall, Barth *SISC*, 2002), also E-flux schemes by (Osher *SINUM*, 1984)

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3. Lax-Friedrich's/Rusanov scheme

$$\phi_i^{\mathsf{LF}} = \int_K \psi_i \vec{a}\cdot\nabla u_h + \alpha_{\mathsf{LF}} \sum_{j\in K} (u_i - u_j)$$

Stability, upwinding, and dissipation

1. FV scheme (1st order upwind) NRG stable (Barth, NASA 1996 ; Abgrall, Barth *SISC*, 2002), also E-flux schemes by (Osher *SINUM*, 1984)

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3. Lax-Friedrich's/Rusanov scheme (P^1 case)

$$\phi^{\mathcal{E}}_{\mathsf{LF}} = \oint_{\partial K} \frac{u_h^2}{2}\,\vec{a}\cdot\hat{n}\,dl + \frac{\alpha_{\mathsf{LF}}}{3}\sum_{i,j\in K}(u_i - u_j)^2$$

Stability, upwinding, and dissipation

Upwinding has beneficial effect in terms of energy stability

Design criteria : accuracy, what is the truncation error ?

- By Taylor expansion : only on structured meshes
- Error analysis based on variational form :
  1. variational form unclear for RD (only for FV and stabilized FEM)
  2. NRG stability not enough (too weak, for FV and stabilized FEM stronger stability can be shown)

Design criteria : what is the truncation error ?

Idea : use the same principles of the TE analysis in a 'weak' formalism ...

$$\int_\Omega \nabla\varphi \cdot \boldsymbol{\mathcal{F}}(u)dx + \text{BCs} = 0 \longleftrightarrow \int_\Omega \nabla\varphi \cdot \boldsymbol{\mathcal{F}}_h(u_h)dx + \text{BCs} = \varepsilon_h$$

with u a smooth exact (classical) solution

This gives a consistency estimate..

What is ε_h ?

What do we have ... ?

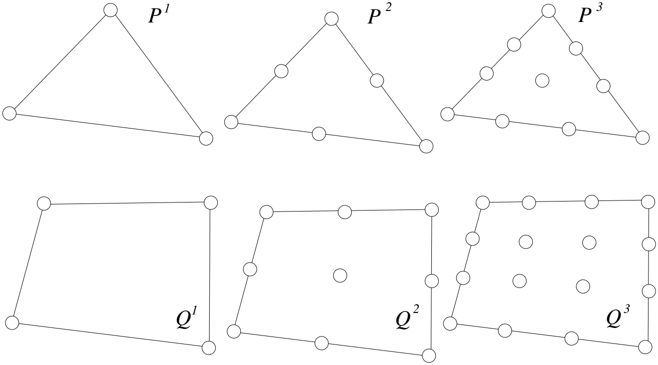
Consider

1. $w \in H^{k+1}$ smooth solution : $\nabla \cdot \boldsymbol{\mathcal{F}}(w) = \partial_u \boldsymbol{\mathcal{F}}(w) \cdot \nabla w = 0$
2. $w - w_h = O(h^{k+1})$, $\boldsymbol{\mathcal{F}}(w) - \boldsymbol{\mathcal{F}}_h(w_h) = O(h^{k+1})$ in L^2 from approximation theory, see *e.g.* (Ern, Guermond Springer, 2004)
3. $\nabla(w - w_h) = O(h^k)$, $\nabla \cdot (\boldsymbol{\mathcal{F}}(w) - \boldsymbol{\mathcal{F}}_h(w_h)) = O(h^k)$ in L^2 from approximation theory, see *e.g.* (Ern, Guermond Springer, 2004)

with w_h a continuous polynomial approximation of degree k (*e.g* standard Lagrange elements)

Design criteria, consistency analysis

Continuous Lagrange elements



What do we do ... ?

Consider

1. $w \in H^{k+1}$ smooth solution : $\nabla \cdot \boldsymbol{\mathcal{F}}(w) = \partial_u \boldsymbol{\mathcal{F}}(w) \cdot \nabla w = 0$
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3. $\nabla(w - w_h) = O(h^k)$, $\nabla \cdot (\boldsymbol{\mathcal{F}}(w) - \boldsymbol{\mathcal{F}}_h(w_h)) = O(h^k)$ in L^2 from approximation theory, see *e.g.* (Ern, Guermond Springer, 2004)

Take the steady RD scheme

$$\sum_{K|i \in K} \phi_i^K(u_h) = 0$$

approximating $\nabla \cdot \boldsymbol{\mathcal{F}}$ in node i

WHAT DO WE DO ... ?

Consider

1. $w \in H^{k+1}$ smooth solution : $\nabla \cdot \boldsymbol{\mathcal{F}}(w) = \partial_u \boldsymbol{\mathcal{F}}(w) \cdot \nabla w = 0$
2. $w - w_h = O(h^{k+1})$, $\boldsymbol{\mathcal{F}}(w) - \boldsymbol{\mathcal{F}}_h(w_h) = O(h^{k+1})$ in L^2 from approximation theory, see *e.g.* (Ern, Guermond Springer, 2004)
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Formally replace the nodal values of u_h, computed by the scheme, with those of the exact solution w,
exaclty as done in finite difference TE analysis

What do we do ... ?

Consider

1. $w \in H^{k+1}$ smooth solution : $\nabla \cdot \boldsymbol{\mathcal{F}}(w) = \partial_u \boldsymbol{\mathcal{F}}(w) \cdot \nabla w = 0$
2. $w - w_h = O(h^{k+1})$, $\boldsymbol{\mathcal{F}}(w) - \boldsymbol{\mathcal{F}}_h(w_h) = O(h^{k+1})$ in L^2 from approximation theory, see *e.g.* (Ern, Guermond Springer, 2004)
3. $\nabla(w - w_h) = O(h^k)$, $\nabla \cdot (\boldsymbol{\mathcal{F}}(w) - \boldsymbol{\mathcal{F}}_h(w_h)) = O(h^k)$ in L^2 from approximation theory, see *e.g.* (Ern, Guermond Springer, 2004)

We obtain

$$\sum_{K|i \in K} \phi_i^K(w_h) \neq 0$$

since of course the nodal values of the exact solution w do not verify the discrete equations

What do we do ... ?

Consider

1. $w \in H^{k+1}$ smooth solution : $\nabla \cdot \boldsymbol{\mathcal{F}}(w) = \partial_u \boldsymbol{\mathcal{F}}(w) \cdot \nabla w = 0$
2. $w - w_h = O(h^{k+1})$, $\boldsymbol{\mathcal{F}}(w) - \boldsymbol{\mathcal{F}}_h(w_h) = O(h^{k+1})$ in L^2 from approximation theory, see *e.g.* (Ern, Guermond Springer, 2004)
3. $\nabla(w - w_h) = O(h^k)$, $\nabla \cdot (\boldsymbol{\mathcal{F}}(w) - \boldsymbol{\mathcal{F}}_h(w_h)) = O(h^k)$ in L^2 from approximation theory, see *e.g.* (Ern, Guermond Springer, 2004)

Given φ a $C_0^r(\Omega)$ class function, r large enough, define

$$\epsilon_h := \sum_{i \in \Omega_h} \varphi_i \sum_{K | i \in K} \phi_i^K(w_h)$$

A global measure of how much the discrete equations differ from the continuous one

What do we do … ?

Estimate ϵ_h (Abgrall, Roe *J.Sci.Comp.*, 2003 ; Ricchiuto, Abgrall, Deconinck *J.Comput.Phys*, 2007)

$$\epsilon_h = \sum_{K\in\Omega_h} \sum_{i\in K} \varphi_i \phi_i^K(w_h) = \epsilon_a + \epsilon_d$$

$$\epsilon_a = \underbrace{-\int_{\Omega_h} \nabla\varphi_h \cdot (\boldsymbol{\mathcal{F}}_h(w_h) - \boldsymbol{\mathcal{F}}(w))}_{\text{approximation error}} \rightarrow \|\epsilon_a\| \leq C'_a\, h^{k+1}$$

$$\epsilon_d = \underbrace{\sum_{K\in\Omega_h} \sum_{i,j\in K} \frac{\varphi_i - \varphi_j}{n^K_{\mathsf{DoF}}} (\phi_i^K(w_h) - \phi_i^{\mathsf{G}}(w_h))}_{\text{distribution error}}$$

$$\rightarrow \|\epsilon_d\| \leq C'\, h^{-1} \mathsf{sup}_K \sup_{i\in K} \|\phi_i^K(w_h)\| + C'''_a\, h^{k+1}$$

What do we do ... ?

Estimate ϵ_h (Abgrall, Roe *J.Sci.Comp.*, 2003 ; Ricchiuto, Abgrall, Deconinck *J.Comput.Phys*, 2007)

$$\|\epsilon_h\| \leq C_a\, h^{k+1} + C'\, h^{-1} \mathsf{sup}_K \sup_{i \in K} \|\phi_i^K(w_h)\|$$

For a polynomial approximation of degree k,
a sufficient condition to have a $\|\epsilon_h\| \leq C\, h^{k+1}$ is (in 2d)

$$\phi_i^K(w_h) = \mathcal{O}(h^{k+2})\,, \quad \forall\, K \in \Omega_h\,,\, \forall\, i \in K$$

A local TE condition ..

Design criteria, high order schemes

For a polynomial approximation of degree k,
a sufficient condition to have a $\|\epsilon_h\| \leq C\, h^{k+1}$ is (in 2d)

$$\phi_i^K(w_h) = \mathcal{O}(h^{k+2})\,, \quad \forall\, K \in \Omega_h\,,\, \forall\, i \in K$$

High order prototype 1 (FEM-like)

$$\phi_i^K(u_h) = \int_K \omega_i^K \nabla \cdot \boldsymbol{\mathcal{F}}_h(u_h)\,, \quad \|\omega_i^K\| < C < \infty$$

High order prototype 2 (std RD)

$$\phi_i^K(u_h) = \beta_i^K \oint_{\partial K} \boldsymbol{\mathcal{F}}_h(u_h) \cdot \hat{n}\, dl = \beta_i^K \phi^K(u_h)\,, \quad \|\beta_i^K\| < C < \infty$$

LDA scheme (P^1) elements

Distribution coeff. :

$$\beta_i^{\mathsf{LDA}} = k_i^+ \Big(\sum_{j \in K} k_j^+ \Big)^{-1}$$

SUPG

Test fcn :

$$\omega_i^K = \psi_i + \vec{a}(u_h) \cdot \nabla \psi_i \, \tau \,, \;\; \vec{a}(u_h) = \partial_u \boldsymbol{\mathcal{F}}(u_h)$$

Nonlinear high order schemes

So far we have

1. A "stability" criterion requiring an upwind bias (other stabilization strategies mentioned later if time ..)
2. An accuracy (consistency) criterion requiring bounded weights in the residual splitting

In other words ..

In other words

The scheme should read (let's stick to RD-like schemes, or P^1 approximation)

$$|C_i| \frac{u_i^{n+1} - u_i^n}{\Delta t} + \sum_{K \,|\, i \in K} \beta_i^K \phi^K = 0$$

1. with β_i^K larger for downstream nodes (upwinding, stability) ;
2. with uniformly bounded β_i^K (consistency).

How about discontinuity capturing ?

Discontinuity capturing : positivity

$$|C_i|\frac{du_i}{dt} = -\sum_{K\,|\,i\in K}\phi_i^K$$

Positive coefficient scheme (Spekreijse, *Math.Comp.* 49, 1987)

A scheme for which

$$\phi_i^K = \sum_{\substack{j\in K\\ j\neq i}} c_{ij}^K(u_i - u_j) \quad \text{with} \quad c_{ik}^K \geq 0$$

s said to be LED (Local Extremum Diminishing[1])

[1]... look at the sign of the time derivative !!!!!

Discontinuity capturing : positivity

$$|C_i|\frac{u_i^{n+1}-u_i^n}{\Delta t} = -\sum_{K\,|\,i\in K}\sum_{\substack{j\in K\\ j\neq i}} c_{ij}^K(u_i^n - u_j^n)$$

Positive coefficient scheme (Spekreijse, *Math.Comp.* 49, 1987)

When combined with Explicit Euler time integration[1] the LED property leads to

$$u_i^{n+1} = \sum_j \overline{c}_{ij}u_j^n$$

where

$$\sum_j \overline{c}_{ij} = 1 \quad \text{and} \quad \overline{c}_{ij} \geq 0 \;\; \Big(\text{provided} \;\; \frac{\Delta t}{|C_i|}\sum_j c_{ij}^K \leq 1\Big)$$

In this case the scheme is said (by abuse of language) to be positive

A positive scheme verifies the discrete max principle

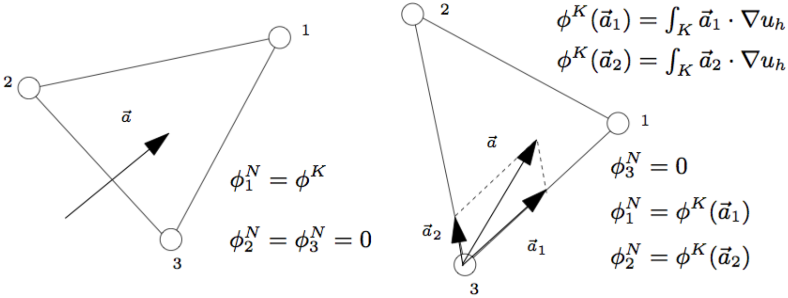
$$\min_j u_j^n \leq u_i^{n+1} \leq \max_j u_j^n$$

Generalization of TVD and monotonicity analysis of Harten (A. Harten *J.Cumput.Phys* 1983)

[1] and in general with a boundedness preserving time integration scheme, see (Gottlieb,Shu,Tadmor *SIAM Review* 2001 - Hundsdorfer,Ruuth *Math.Comp.* 2005)

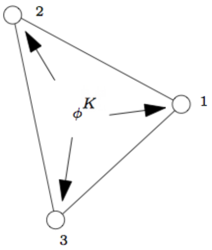
Positive schemes : examples

Example 1 : Roe's optimal N scheme[1]



[1] (Roe Cranfield U.Tech.Rep., 1987 ; Roe, Sidilkover *SINUM*, 1992)

Example 2 : Lax-Friedrich's distribution



$$\phi_i^{\mathsf{LF}} = \int_K \psi_i \nabla \cdot \boldsymbol{\mathcal{F}}_h + \alpha_{\mathsf{LF}} \sum_{j \in K} (u_i - u_j)$$

for positivity (scalar case)

$$\alpha_{\mathsf{LF}} \geq h_K \sup_{x \in K} \|\partial_u \boldsymbol{\mathcal{F}}(u_h(x))\|$$

Bad news ... (Godunov)

All linear positive (LED) schemes are first order accurate ...

Nonlinear high order schemes

Good news ... limiters

We know the answer to this limitation since more than 40 years now : we need nonlinear schemes.

Let us introduce a limiter.. somewhere

Where does the limiter $\ell(\cdot)$ come in

Recall that one prototype of a high order scheme is obtained as the steady state of

$$|C_i| \frac{u_i^{n+1} - u_i^n}{\Delta t} + \sum_{K | i \in K} \beta_i^K \phi^K(u_h), \quad \|\beta_i^K\| \leq C < \infty$$

Where does the limiter $\ell(\cdot)$ come in

Recall that one prototype of a high order scheme is obtained as the steady state of

$$|C_i|\frac{u_i^{n+1}-u_i^n}{\Delta t}+\sum_{K|i\in K}\beta_i^K\phi^K(u_h)\,,\ \ \|\beta_i^K\|\leq C<\infty$$

For linear positive coefficient schemes

$$\phi_i^{\mathsf{P}}(u_h)=\sum_{j\in K}c_{ij}^K(u_i-u_j)\,,\ \ c_{ij}^K\geq 0$$

Formally we have

$$\beta_i^{\mathsf{P}}(u_h)=\frac{\sum\limits_{j\in K}c_{ij}^K(u_i-u_j)}{\phi^K(u_h)}\quad \text{in general unbounded !}$$

Where does the limiter $\ell(\cdot)$ come in

Recall that one prototype of a high order scheme is obtained as the steady state of

$$|C_i|\frac{u_i^{n+1}-u_i^n}{\Delta t}+\sum_{K|i\in K}\beta_i^K\phi^K(u_h)\,,\ \ \|\beta_i^K\|\leq C<\infty$$

For linear positive coefficient schemes

$$\phi_i^{\mathsf{P}}(u_h)=\sum_{j\in K}c_{ij}^K(u_i-u_j)\,,\ \ c_{ij}^K\geq 0$$

Formally we have

$$\beta_i^{\mathsf{P}}(u_h)=\frac{\sum\limits_{j\in K}c_{ij}^K(u_i-u_j)}{\phi^K(u_h)}\quad \text{in general unbounded !}$$

Why not applying a limiter to get a bounded coefficient ? .. !!!

Where does the limiter $\ell(\cdot)$ come in

Linear positive coefficient schemes

$$\phi_i^{\mathsf{P}}(u_h) = \sum_{j \in K} c_{ij}^K (u_i - u_j) \, , c_{ij}^K \geq 0 \; ; \quad \beta_i^{\mathsf{P}}(u_h) = \frac{\phi_i^{\mathsf{P}}(u_h)}{\phi^K(u_h)} \text{ unbounded}$$

$$\beta_i^{\mathsf{LP}}(u_h) = \frac{\ell\big(\beta_i^{\mathsf{P}}(u_h)\big)}{\sum\limits_{j \in K} \ell\big(\beta_j^{\mathsf{P}}(u_h)\big)} \text{ limited distribution coefficient}$$

Nonlinear high order schemes

Where does the limiter come in

Linear positive coefficient schemes

$$\phi_i^{\mathsf{P}}(u_h) = \sum_{j\in K} c_{ij}^K(u_i - u_j)\,, c_{ij}^K \geq 0\,; \quad \beta_i^{\mathsf{P}}(u_h) = \frac{\phi_i^{\mathsf{P}}(u_h)}{\phi^K(u_h)} \text{ unbounded}$$

$$\beta_i^{\mathsf{LP}}(u_h) = \frac{\ell\big(\beta_i^{\mathsf{P}}(u_h)\big)}{\sum\limits_{j\in K}\ell\big(\beta_j^{\mathsf{P}}(u_h)\big)} \text{ limited distribution coefficient}$$

Provided $\ell(r) \geq 0$ and $\dfrac{\ell(r)}{r} \geq 0$ we have

$$\phi_i^{\mathsf{LP}}(u_h) = \beta_i^{\mathsf{LP}}\phi^K = \underbrace{\frac{\beta_i^{\mathsf{LP}}}{\beta_i^{\mathsf{P}}}}_{\gamma_i^{\mathsf{P}}\geq 0}\phi_i^{\mathsf{P}} = \sum_{j\in K} c_{ij}^{\mathsf{LP}}(u_i - u_j)\,, \quad c_{ij}^{\mathsf{LP}} = \gamma_i^{\mathsf{P}} c_{ij}^K \geq 0!$$

High order RD scheme

For $n \geq 0$, until steady state do :

For all $K \in$ mesh do

1. compute cell residual $\phi^K = \oint_{\partial K} \boldsymbol{\mathcal{F}}_h(u_h) \cdot \hat{n}\, dl$
2. compute linear positive distribution $\phi_i^{\mathsf{P}} = \sum_j c_{ij}^K (u_i - u_j)$
3. limit $\beta_i^{\mathsf{P}} = \phi_i^{\mathsf{P}} / \phi^K \;\; \rightarrow \;\; \beta_i^{\mathsf{LP}} = \ell(\beta_i^{\mathsf{P}}) / \left(\sum_j \ell(\beta_j^{\mathsf{P}}) \right)$
4. distribute cell residual $\phi_i^K = \beta_i^{\mathsf{LP}} \phi^K$

For all $i \in$ mesh do

5. evolve $|C_i| \dfrac{u_i^{n+1} - u_i^n}{\Delta t} = - \sum_{K \,|\, i \in K} \phi_i^K(u_h^n)$

Limiter

The simplest possible choice is

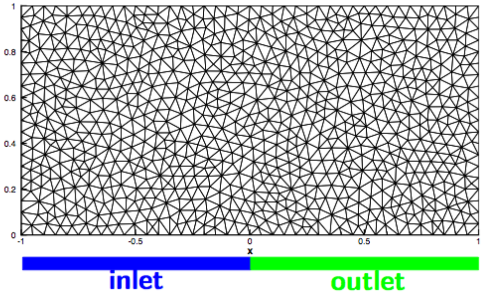
$$\ell(r) = \max(0, r)$$

Examples

Rotational advection

Scalar example : $\vec{a} \cdot \nabla u = 0$ with $\vec{a} = (y, 1 - x)$ and bcs

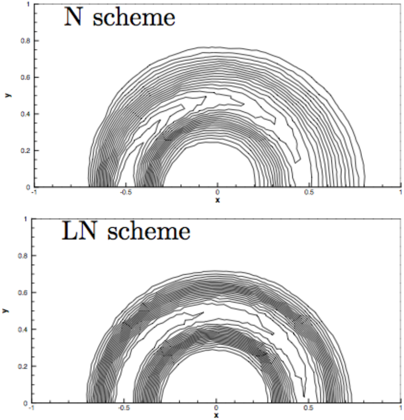
$$u_{\text{in}} = \begin{cases} \cos(2\pi(x+0.5))^2 & \text{if} \quad x \in [-0.75, -0.25] \\ 0 & \text{otherwise} \end{cases}$$

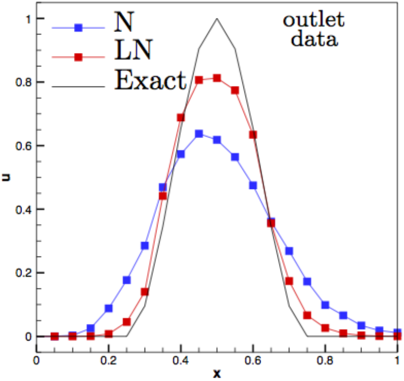


Examples (cont'd)

Rotational advection

N and Limited N (LN) schemes

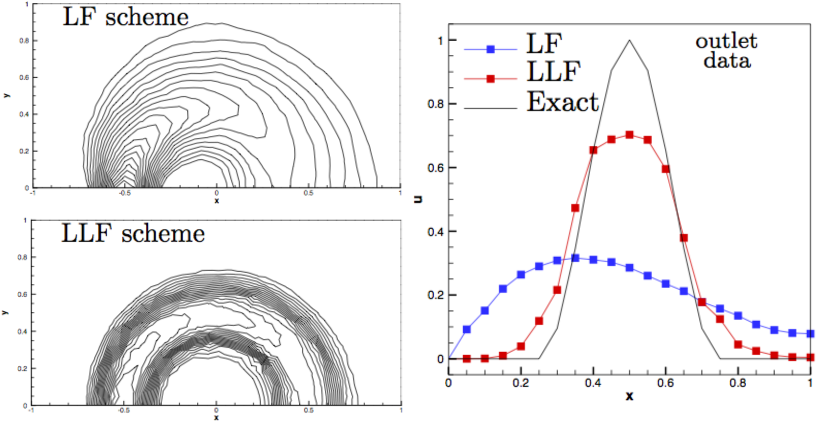




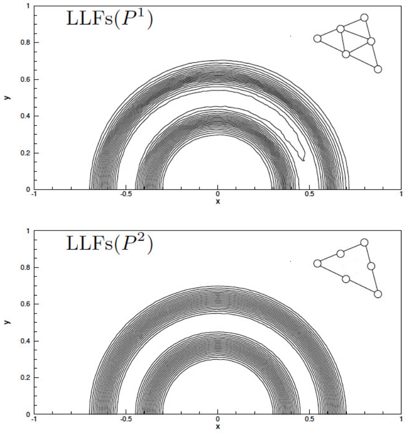
Examples (cont'd)

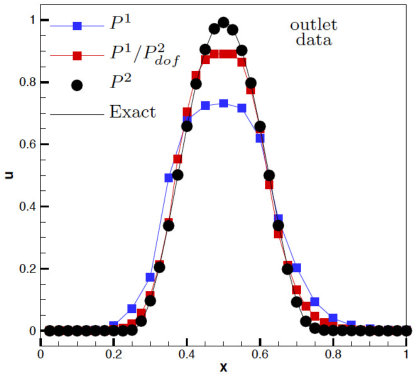
Rotational advection

LF and Limited LF (LLF) schemes

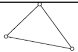
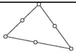
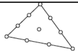


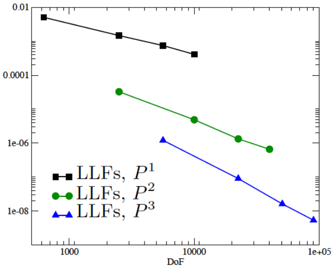
Higher order nonlinear Lax Friedrich's scheme





Higher order nonlinear Lax Friedrich's scheme

			
h	$\epsilon_{L^2}(P^1)$	$\epsilon_{L^2}(P^2)$	$\epsilon_{L^2}(P^3)$
$1/25$	0.50493E-02	0.32612E-04	0.12071E-05
$1/50$	0.14684E-02	0.48741E-05	0.90642E-07
$1/75$	0.74684E-03	0.13334E-05	0.16245E-07
$1/100$	0.41019E-03	0.66019E-06	0.53860E-08
	$\mathcal{O}^{\text{ls}}_{L^2}$ =**1.790**	$\mathcal{O}^{\text{ls}}_{L^2}$ =**2.848**	$\mathcal{O}^{\text{ls}}_{L^2}$ =**3.920**

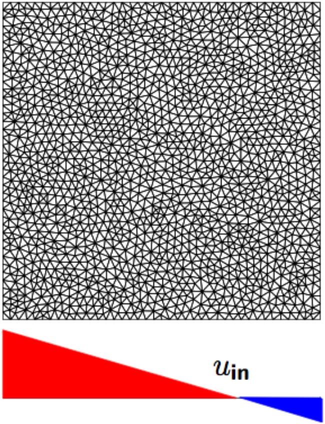


Examples (cont'd)

Burger's equation

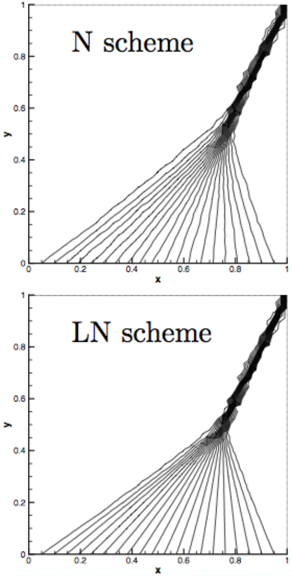
Scalar example : $\nabla \cdot \boldsymbol{\mathcal{F}}(u) = 0$ with $\boldsymbol{\mathcal{F}}(u) = (u,\ \frac{u^2}{2})$ and bcs

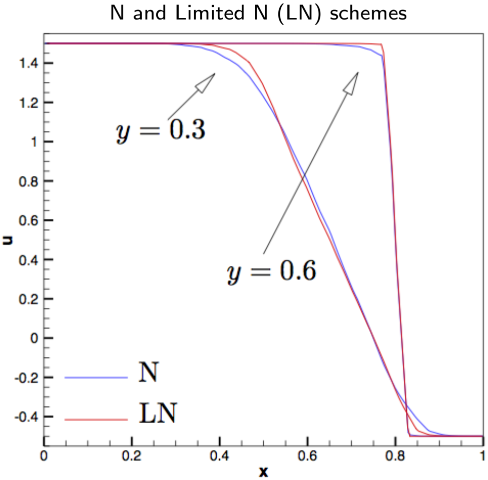
$$u(x, y = 0) = 1.5 - 2x$$



Examples (cont'd)

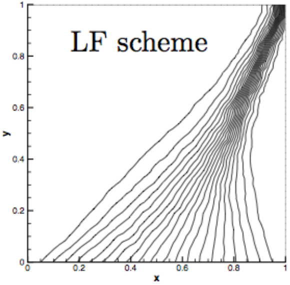
Burger's equation

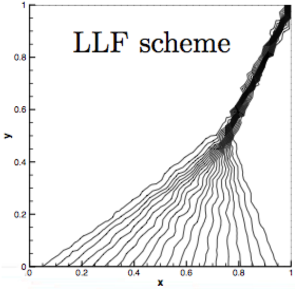


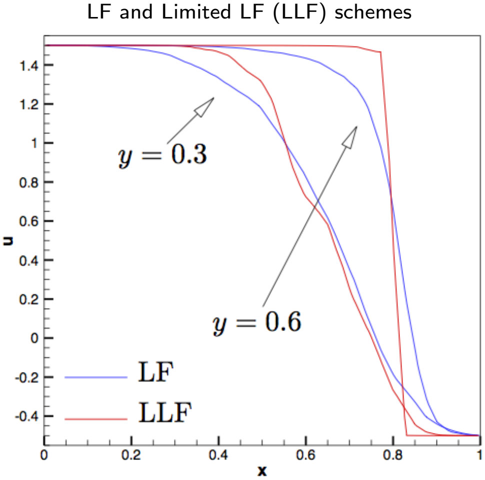


Examples (cont'd)

Burger's equation







Remarks on extension to systems

Historical perspective

Two approaches (Roe *J.Comput.Phys*, 1986 ; Nishikawa, Rad, Roe AIAA Conf. 2001) and (van der Weide, Deconinck *Comput.Fluid Dyn.*, Wiley 1996)

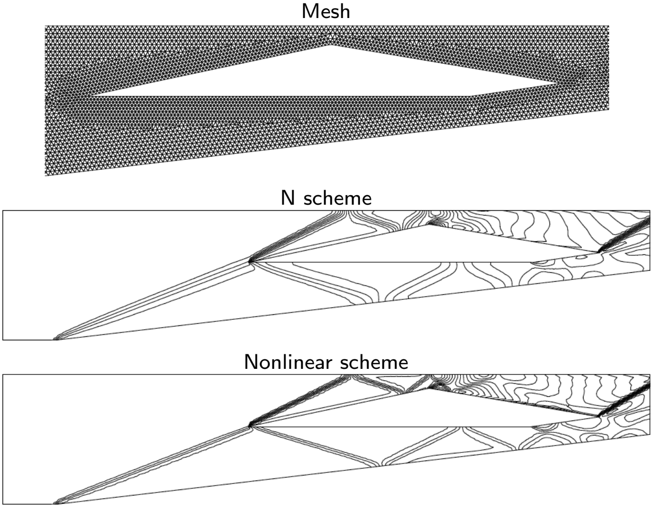
1. Local projection (wave decomposition) of the *continuous PDE* to obtain (possibly decoupled) scalar equations discretized independently
2. Formal matrix generalization in which the scalar flux vector is replaced by a tensor and the $k_i = \vec{a} \cdot \vec{n}_i/2$ coefficients become matrix flux Jacobians

Practical implementation

Hybrid of the two (Abgrall, Mezine *J.Comput.Phys*, 2004 ; Ricchiuto, Csik, Deconinck *J.Comput.Phys*, 2005) :

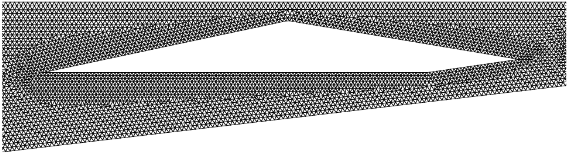
- Matrix formulation for linear first order schemes
- Projection onto characteristic directions to obtain scalar residuals to work with for the limiting procedure (similar to FV limiting on characteristic var.s)

Example 1 : Mach 3.6 scramjet inlet (Euler, perfect gas)

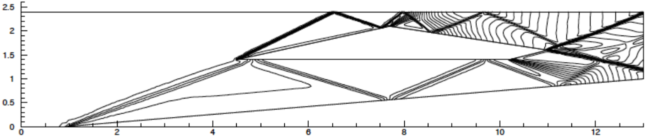


Example 1 : Mach 3.6 scramjet inlet (Euler, perfect gas)

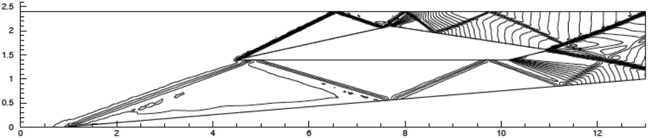
Mesh



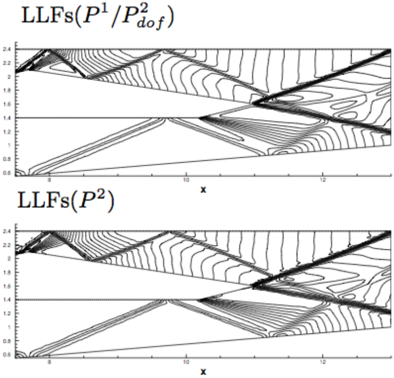
LLFs scheme : P^1 on conformally refined mesh

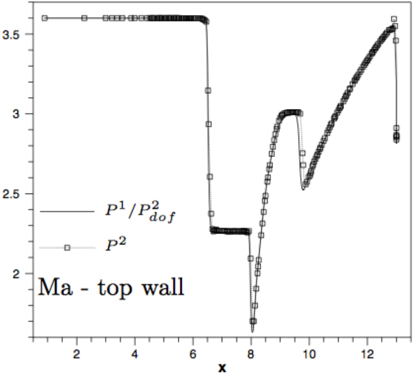


LLFs scheme : P^2

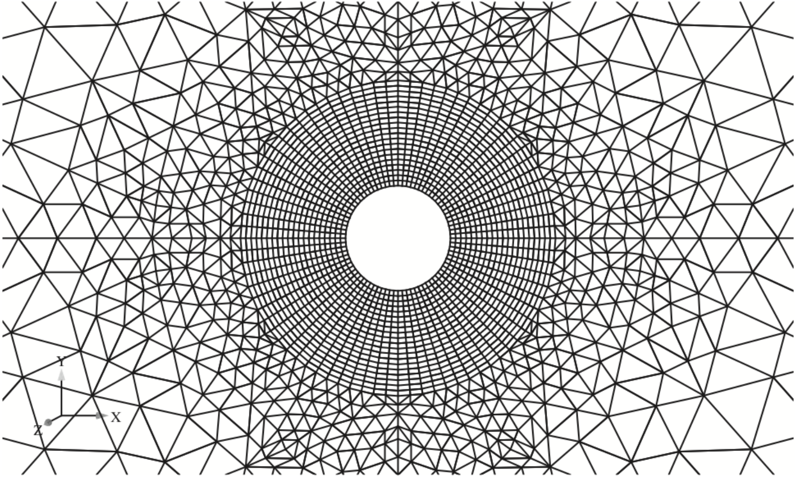


Scramjet inlet

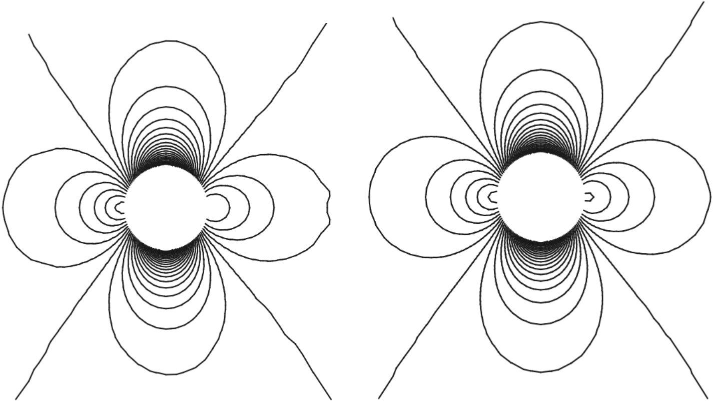




Euler equations : subsonic cylinder

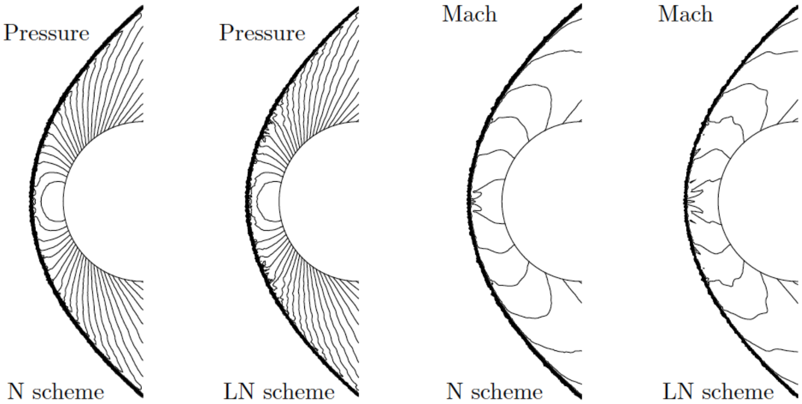


Euler equations : subsonic cylinder



Conformally refined $P^1 - Q^1$ (left) *vs* $P^2 - Q^2$ (right)

Example 2 : Mach 10 bow shock (Euler, perfect gas)



PART II

Time dependent problems and a peek at dispersive equations

Time dependent advection

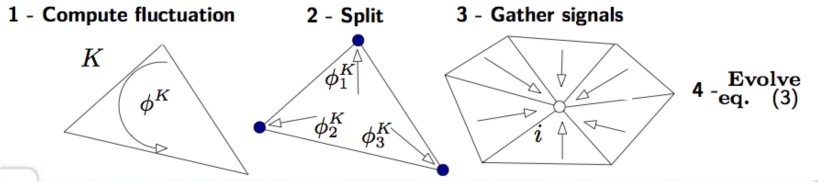
What is the problem with the time dependent case ..?

$$\partial_t u + \vec{a} \cdot \nabla u = 0 \qquad \text{on} \quad \Omega \times [0, T_f] \subset \mathbb{R}^2 \times \mathbb{R}^+$$

Time dependent advection

What is the problem with the time dependent case ..?

$$|C_i|\frac{du_i}{dt} + \sum_{K\,|\,i\in K} \beta_i^K \phi^K(u_h) = 0 \qquad \textbf{(3)}$$

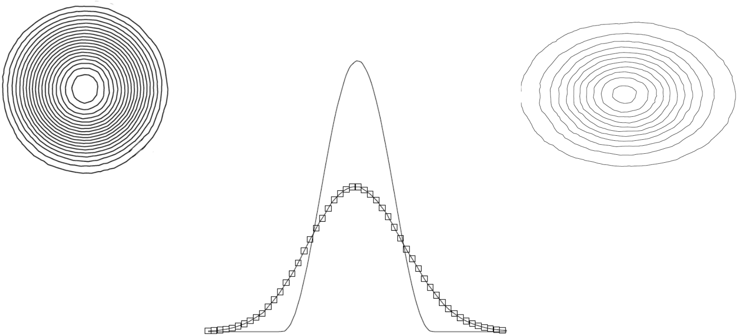


Time dependent advection

What is the problem with the time dependent case ..?

$$|C_i|\frac{du_i}{dt} + \sum_{K\,|\,i\in K} \beta_i^K \phi^K(u_h) = 0$$

EXAMPLE : transport of a smooth profile (LN scheme + RK2)



Time dependent advection

$$|C_i|\frac{du_i}{dt} + \sum_{K\,|\,i\in K} \beta_i^K \phi^K(u_h) = 0$$

- This guy is in general only first order accurate in space, whatever the finite element approximation (P^1, P^2, etc) ;
- Obviously, using higher order time integration does *not* help, since it is the spatial discretization that is wrong

Consistency analysis ...

$$|C_i|\frac{du_i}{dt} + \sum_{K\,|\,i\in K}\beta_i^K\phi^K = 0$$

Time continuous consistency analysis[3] (P^1 triangles to fix ideas)

[3] Deconinck-Ricchiuto *Enc.Comput.Mech.* 2007

Consistency analysis ...

$$|C_i| \frac{du_i}{dt} + \sum_{K \,|\, i \in K} \beta_i^K \phi^K = 0$$

Time continuous consistency analysis[3] (P^1 triangles to fix ideas)

(*i*) Let $w(t,x,y)$ be a regular exact solution : $\partial_t w + \vec{a} \cdot \nabla w = 0$, $w_i(t) = w(t, x_i, y_i)$

(*ii*) Let $\phi^K(w_h)$ the quantity obtained when formally replacing the nodal values of the numerical solution by the w_is

(*iii*) For $\varphi \in C_0^1$ be a compactly supported smooth function with $\varphi_i = \varphi(x_i, y_i)$

(*iv*) define the integral truncation error for a $\mathcal{LP}$ scheme

$$\epsilon := \Big| \sum_{i \in \Omega_h} \varphi_i |C_i| \frac{dw_i}{dt} + \sum_{K \,|\, i \in K} \varphi_i \beta_i^K \phi^K(w_h) \Big|$$

$$= \Big| \sum_{K \in \Omega_h} \sum_{j \in K} \varphi_j \Big(\frac{|K|}{3} \frac{dw_j}{dt} + \beta_j^K \phi^K(w_h) \Big) \Big|$$

[3] Deconinck-Ricchiuto *Enc.Comput.Mech.* 2007

Time dependent advection

Consistency analysis ...

$$\epsilon = \Big| \int_{\Omega} \left(\varphi_h \partial_t (w_h - w) - (w_h - w)\vec{a} \cdot \nabla \varphi_h\right)$$

$$+ \sum_{K \in \Omega_h} \sum_{i,j \in K} (\varphi_j - \varphi_i)\Big(|C_j| \frac{dw_j}{dt} + \beta_j^K \phi^K(w_h)\Big)$$

$$- \sum_{K \in \Omega_h} \sum_{i,j \in K} (\varphi_j - \varphi_i) \int_K \psi_j \left(\partial_t (w_h - w) + \vec{a} \cdot \nabla (w_h - w)\right)\Big|$$

$$\leq C_1 h^2 + C_2 h^{-1} \sup_{\substack{K \in \Omega_h \\ j \in K}} \Big| \frac{|K|}{3} \frac{dw_j}{dt} + \beta_j^K \phi^K(w_h)\Big|$$

To have $\epsilon < h^2$ we need the satisfaction of a local truncation error condition :

$$\sup_{\substack{K \in \Omega_h \\ j \in K}} \Big| \frac{|K|}{3} \frac{dw_j}{dt} + \beta_j^K \phi^K(w_h)\Big| \leq C h^3$$

Time dependent advection

Consistency analysis ...

Pushing it a bit more :

$$\Big|\frac{|K|}{3}\frac{dw_j}{dt}+\beta_j^K\phi^K(w_h)\Big| = \Big|\frac{|K|}{3}\frac{dw_j}{dt}+\beta_j^K\int_K \vec{a}\cdot\nabla w_h\Big|$$

$$= \Big|\frac{|K|}{3}\frac{dw_j}{dt}-\beta_j^K\int_K \partial_t w_h+\beta_j^K\int_K \big(\partial_t(w_h-w)+\vec{a}\cdot\nabla(w_h-w)\big)\Big|$$

$$\leq \Big|\frac{|K|}{3}\frac{dw_j}{dt}-\beta_j^K\int_K \partial_t w_h\Big| + Ch^3$$

The h^2 consistency condition is

$$\Big|\frac{|K|}{3}\frac{dw_j}{dt}-\beta_j^K\int_K \partial_t w_h\Big| \leq Ch^3$$

This is in general not true...

How do we get around this problem ?

High order schemes for time dependent problems

$$\Delta x \frac{du_i}{dt} + a(u_i - u_{i-1}) = 0$$

How do we make it second order in space ?

$$\Delta x \frac{du_i}{dt} + a(u_i - u_{i-1}) = 0$$

How do we make it second order in space ?

Finite Volume/Difference guy answers : enlarge the stencil

$$\Delta x \frac{du_i}{dt} + \frac{3}{2} a(u_i - u_{i-1}) - \frac{1}{2} a(u_{i-1} - u_{i-2}) = 0$$

High order schemes for time dependent problems

$$\Delta x \frac{du_i}{dt} + a(u_i - u_{i-1}) = 0$$

How do we make it second order in space ?

Finite Element guy answers : do not forget the mass matrix !

$$m_{ii-1} \frac{du_{i-1}}{dt} + m_{ii} \frac{du_i}{dt} + m_{ii+1} \frac{du_{i+1}}{dt} + a(u_i - u_{i-1}) = 0$$

Step 1

$$\partial_t u + a\partial_x u = 0$$

The Galerkin FEM discretization reads :

$$\int_{\Omega_h} \psi_i \partial_t u_h + \frac{a}{2}(u_{i+1} - u_{i-1}) = 0$$

or equivalently (set $\phi^{i+1/2} = a(u_{i+1} - u_i)$, $\phi^{i-1/2} = a(u_i - u_{i-1})$)

$$\int_{\Omega_h} \psi_i \partial_t u_h + \frac{1}{2}\phi^{i-1/2} + \frac{1}{2}\phi^{i+1/2} = 0$$

Step 1

The (P^1) Galerkin mass matrix is obtained as

$$\int_{\Omega_h} \psi_i \partial_t u_h = \frac{\Delta x}{6}\frac{du_{i-1}}{dt} + \frac{2\Delta x}{3}\frac{du_i}{dt} + \frac{\Delta x}{6}\frac{du_{i+1}}{dt}$$

As a result, we get the **fourth order scheme** (w.r.t. Δx)

$$\frac{\Delta x}{6}\frac{du_{i-1}}{dt} + \frac{2\Delta x}{3}\frac{du_i}{dt} + \frac{\Delta x}{6}\frac{du_{i+1}}{dt} + \frac{a}{2}(u_{i+1} - u_{i-1}) = 0$$

STEP 2

$$\partial_t u + a\partial_x u = 0$$

Can we find a Petrov-Galerkin test function ω_i which yields :

$$\int_{\Omega_h} \omega_i \partial_t u_h + a(u_i - u_{i-1}) = 0 \quad ???$$

The answer is yes, but the solution is not unique !

Step 3(a)

$$\partial_t u + a\partial_x u = 0$$

SUPG scheme of Hughes and co-workers :

$$\overbrace{\int_{\Omega_h} \psi_i(\partial_t u_h + a\partial_x u_h)}^{\text{Galerkin}} + \overbrace{\int_{\Omega_h} a\partial_x\psi_i\,\tau\,(\partial_t u_h + a\partial_x u_h)}^{\text{Streamline dissipation terms}} = 0$$

For $\tau = \Delta x/(2|a|)$ one easily shows that $(a > 0)$

$$\int_{\Omega_h} (\psi_i + a\partial_x\psi_i\,\tau\,)a\partial_x u_h = \int_{\Omega_h} (\psi_i + \Delta x\frac{\text{sign}(a)}{2}\partial_x\psi_i)a\partial_x u_h = a(u_i - u_{i-1})$$

Step 3(a)

$$\partial_t u + a\partial_x u = 0$$

SUPG scheme of Hughes and co-workers. For the choice of the test function

$$\omega_i = \varphi_i + \Delta x \frac{\mathsf{sign}(a)}{2} \partial_x \varphi_i$$

we obtain the **third order accurate scheme** (w.r.t. Δx)

$$\frac{5\Delta x}{12}\frac{du_{i-1}}{dt} + \frac{2\Delta x}{3}\frac{du_i}{dt} - \frac{\Delta x}{12}\frac{du_{i+1}}{dt} + a(u_i - u_{i-1}) = 0$$

Step 3(b)

$$\partial_t u + a\partial_x u = 0$$

Another example : pure residual based approach

$$\frac{\max(0,\ a)}{a}\int_{i-1}^{i}(\partial_t u_h + a\partial_x u_h) + \frac{\min(0,\ a)}{a}\int_{i}^{i+1}(\partial_t u_h + a\partial_x u_h) = 0$$

corresponding to the test fcn

$$\omega_i = \begin{cases} \dfrac{1+\mathsf{sign}(a)}{2} & \text{if } x \in (x_{i-1},\ x_i) \\ \dfrac{1-\mathsf{sign}(a)}{2} & \text{if } x \in (x_i,\ x_{i+1}) \\ 0 & \text{otherwise} \end{cases} \qquad \text{(piecewise constant)}$$

All calculations done, this leads to the second order scheme $(a > 0)$

$$\frac{\Delta x}{2}\frac{du_{i-1}}{dt} + \frac{\Delta x}{2}\frac{du_i}{dt} + a(u_i - u_{i-1}) = 0$$

Reverse engineering a scheme ...

For the steady limit of

$$\partial_t u + \vec{a} \cdot \nabla u = 0$$

high order fluctuation splitting/residual distribution give the steady state algebraic system

$$\sum_{K \,|\, i \in K} \beta_i^K \phi^K = 0$$

People started to look for test functions ω_i such that

$$\int_K \omega_i\big|_K \vec{a} \cdot \nabla u_h = \beta_i^K \phi^K = \beta_i^K \int_K \vec{a} \cdot \nabla u_h$$

Reverse engineering a scheme ...

Time dependent solutions of

$$\partial_t u + \vec{a} \cdot \nabla u = 0$$

would be now sought by integrating in time

$$\sum_{K\,|\,i\in K} \int_K \omega_i\big|_K \partial_t u_h + \sum_{K\,|\,i\in K} \beta_i^K \phi^K = 0$$

With a "*consistent mass matrix*" stemming from the first integral, consistency being intended as

$$\int_K \omega_i\big|_K \vec{a} \cdot \nabla u_h = \beta_i^K \phi^K = \beta_i^K \int_K \vec{a} \cdot \nabla u_h$$

Where the real trouble is ...

Reverse engineering a scheme ...

The reverse engineering papers : finite element analogies, space-time formulations, geometrical constructions, and some imagination ....

- Maerz & Degrez, VKI PR 9617, 1996
- Ferrante & Deconinck VKI PR 9708, 1997
- Hubbard & Roe *IJNMF* 33, 2000
- Caraeni & Fuchs *Computers&Fluids* 4-5, 2005 (from a PhD defended in 2000)
- Csik & Deconinck *IJNMF* 2002
- Abgrall & Mezine *J.Comput.Phys.* 188, 2003
- Ricchiuto & Csik & Deconinck *J.Comput.Phys.* 209, 2005
- De Palma *et al.* *J.Comput.Phys.* 208, 2005
- Ricchiuto & Bollermann *J.Comput.Phys.* 228, 2009
- Ricchiuto & Abgrall *J.Comput.Phys.* 16, 2010
- Hubbard and Ricchiuto *Computers&Fluids* 46, 2011
- Bonfiglioli and Paciorri *IJCFD* 27, 2013
- and more ...

Remarks

- Independently on the time discretization almost all the techniques proposed involve a non diagonal mass matrix coupling all nodal values :

$$\sum_{K\in\Omega_h}\sum_{j\in K} m_{ij}^K \frac{du_j}{dt} + \sum_{K\,|\,i\in K} \beta_i^K \phi^K = 0$$

- As for stabilized FEM, after time discretization, independently on the explicit or implicit (or space-time) nature of the time integration chosen, one needs to solve a (non-symmetric and possibly nonlinear) algebraic system :

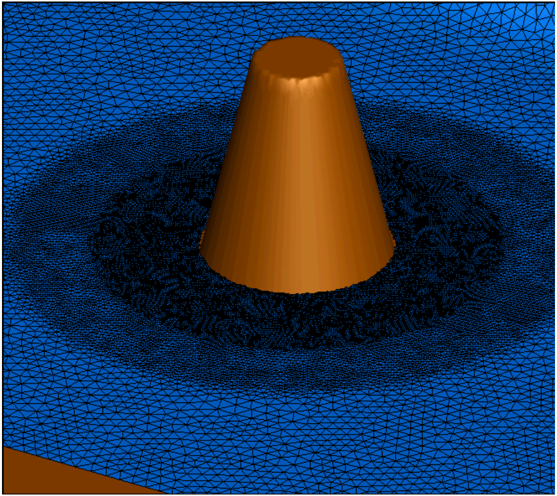
$$M(\mathsf{u}^{n+1})\mathsf{u}^{n+1} + \Delta t F(\mathsf{u}^{n+1}) = \Delta t G(\mathsf{u}^{n},\, \mathsf{u}^{n-1},\, \ldots)$$

- The construction seen for steady problems based on limiting of a monotonicity preserving scheme allows to ensure that $M(\mathsf{u}^{n+1})$ is an inverse monotone matrix[3], $(M(\mathsf{u}^{n+1}))^{-1}_{ij} \geq 0$, so that the properties of the spatial discretization (discrete maximum principle) are preserved (not shown) ;
- Almost all of these schemes do not allow simple explicit updates with the only one exception: the predictor-corrector approach proposed in (Ricchiuto-Abgrall, *J.Comput.Phys.* 2010)

[3] $m_{ii} > 0,\ m_{ij} \leq 0$ and irreducibly diagonally dominant

Briggs' experiment

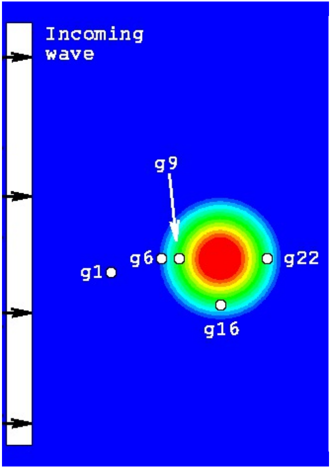
Reproducing one of the tests of[4]

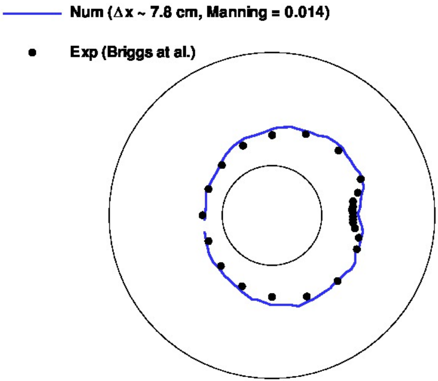


[4] Briggs et al Pure and Appl. Geophysics 1995

Briggs' experiment

Reproducing one of the tests of[5]

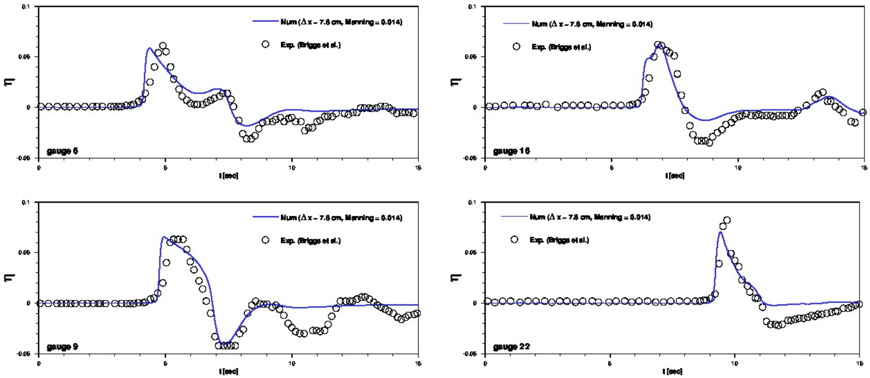




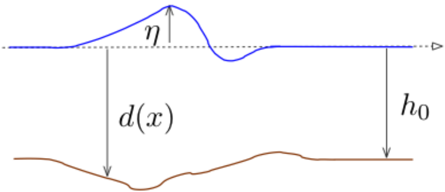
[5] Briggs et al Pure and Appl. Geophysics 1995

Time dependent problems : shallow water

Briggs' experiment

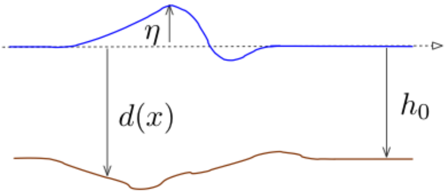


Specific issues related to shallow water



$$\partial_t \underbrace{\begin{bmatrix} h \\ q \end{bmatrix}}_{W} + \partial_x \underbrace{\begin{bmatrix} q \\ uq + \dfrac{gh^2}{2} \end{bmatrix}}_{F(W)} \underbrace{- gh\partial_x \begin{bmatrix} 0 \\ d \end{bmatrix}}_{S(W,x)} = 0\,, \qquad \left\{ \begin{array}{ll} h = & \eta + d \\ q = & hu \end{array} \right.$$

Specific issues related to shallow water



1. Nonlinear hyperbolic system of conservation laws : hydraulic jumps (steady shocks) and propagating bores (moving shocks), standard Riemann problems not involving dry states. Ok if discontinuity capturing scheme is used.

2. Dry states (flat bathymetry). Ok if positivity preserving scheme is used.

3. Variable bathymetry. Question : what do we do with the $\partial_x d$ term ?

Finite Volume schemes and Fluctuations in 1D

$$\partial_t u + a\partial_x u + g'(x) = 0$$

Conservative FV :

$$\Delta x_i \frac{\Delta u_i}{\Delta t} + \widehat{\mathcal{F}}_{i+1/2} - \widehat{\mathcal{F}}_{i-1/2} + S_i = 0$$

Roe scheme for advection with source

$$\widehat{\mathcal{F}}_{i+1/2} = \frac{\mathcal{F}_i + \mathcal{F}_{i+1}}{2} - \frac{|a|}{2}(u_{i+1} - u_i) \;, \quad S_i = ?????$$

Finite Volume schemes and Fluctuations in 1D

$$\partial_t u + a\partial_x u + g'(x) = 0$$

Conservative FV :

$$\Delta x_i \frac{\Delta u_i}{\Delta t} + \widehat{\mathcal{F}}_{i+1/2} - \widehat{\mathcal{F}}_{i-1/2} + S_i = 0$$

Roe scheme for advection with source

Brutal answer : if there is an invariant, discretize the equation for the invariant.

$$\partial_t \eta + a\partial_x \eta = 0\,, \quad \text{with} \quad \eta = u + \frac{g}{a}$$

$$\Delta x_i \frac{\Delta \eta_i}{\Delta t} + \widehat{\mathcal{F}}^{\eta}_{i+1/2} - \widehat{\mathcal{F}}^{\eta}_{i-1/2} = 0$$

$$\widehat{\mathcal{F}}^{\eta}_{i+1/2} = \frac{\mathcal{F}^{\eta}_i + \mathcal{F}^{\eta}_{i+1}}{2} - \frac{|a|}{2}(\eta_{i+1} - \eta_i)$$

WELL BALANCED of C-Property :
If $\eta_i(t=0) = \eta_0\ \forall i$, nothing happens, the invariant is preserved, and $u_i = \eta_0 - g_i/a$

Finite Volume schemes and Fluctuations in 1D

$$\partial_t u + a\partial_x u + g'(x) = 0$$

Conservative FV :

$$\Delta x_i \frac{\Delta u_i}{\Delta t} + \widehat{\mathcal{F}}_{i+1/2} - \widehat{\mathcal{F}}_{i-1/2} + \Delta x\, S_i = 0$$

Roe scheme for advection with source

How to use the Brutal answer :

$$\partial_t \eta + a\partial_x \eta = 0\,, \quad \text{with} \quad \eta = u + \frac{g}{a}$$

$$\Delta x_i \frac{\Delta \eta_i}{\Delta t} = \Delta x_i \frac{\Delta u_i}{\Delta t} \quad \text{and} \quad \widehat{\mathcal{F}}^{\eta}_{i+1/2} = \widehat{\mathcal{F}}_{i+1/2} + \frac{g_i + g_{i+1}}{2} - \frac{\text{sign}(a)}{2}(g_{i+1} - g_i)$$

$$S_i = \frac{g_{i+1} - g_{i-1}}{2\Delta x} + \frac{\text{sign}(a)}{2\Delta x}(g_i - g_{i-1}) - \frac{\text{sign}(a)}{2\Delta x}(g_{i+1} - g_i)$$

Finite Volume schemes and Fluctuations in 1D

$$\partial_t u + a\partial_x u + g'(x) = 0$$

Conservative FV/RD :

$$\Delta x_i \frac{\Delta u_i}{\Delta t} + \phi_i^{i-1/2} + \phi_i^{i+1/2} = 0$$

Roe scheme for advection with source

Residual based answer : use fluctuations. For Roe scheme we have seen that

$$\phi_i^{i\pm1/2} = \frac{1 \mp \mathsf{sign}(a)}{2}\phi^{i\pm1/2}$$

In the homogeneous case $\phi = \int \partial_x \mathcal{F} = -\int \partial_t u$. Similarly, we can take now[1]

$$\phi^{i+1/2} = -\int_i^{i+1} \partial_t u = \int_i^{i+1} (a\partial_x u + g'(x)) = a(u_{i+1} - u_i) + (g_{i+1} - g_i)$$

Along the discrete invariant state $u_i = \eta_0 - g_i/a$ we have $\phi^{i\pm1/2} = 0$ identically !!

[1] CAVEAT : we have assumed the same representation for u and g over the cell !!!!

Finite Volume schemes and Fluctuations in 1D

$$\partial_t u + a\partial_x u + g'(x) = 0$$

Conservative FV/RD :

$$\Delta x_i \frac{\Delta u_i}{\Delta t} + \phi_i^{i-1/2} + \phi_i^{i+1/2} = 0$$

Roe scheme for advection with source

Residual based answer : equivalent to

$$\Delta x_i \frac{\Delta u_i}{\Delta t} + \widehat{\mathcal{F}}_{i+1/2} - \widehat{\mathcal{F}}_{i-1/2} + \Delta x\, S_i = 0$$

$$S_i = \frac{g_{i+1} - g_{i-1}}{2\Delta x} + \frac{\mathsf{sign}(a)}{2\Delta x}(g_i - g_{i-1}) - \frac{\mathsf{sign}(a)}{2\Delta x}(g_{i+1} - g_i)$$

Same as before ..

First order Roe scheme for shallow water[1]

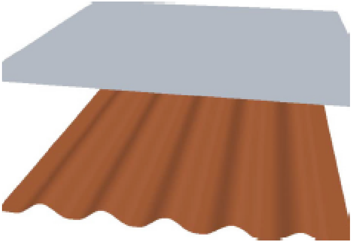
$$\partial_t W + \partial_x F(W) + S(W,x) = 0 \quad \Longrightarrow \quad \Delta x \frac{W_i^{n+1} - W_i^n}{\Delta t} + \hat{F}_{i+1/2} - \hat{F}_{i-1/2} + \Delta x S_i = 0$$

$$\hat{F}_{i+1/2} = \frac{F_{i+1} + F_i}{2} - \frac{|A_{i+1/2}|}{2}(W_{i+1} - W_i)$$

$$S_i = -g h_{i-1/2} \frac{\mathbb{I}_2 + \mathsf{sign}(A_{i-1/2})}{2} \begin{bmatrix} 0 \\ \dfrac{d_i - d_{i-1}}{\Delta x} \end{bmatrix} - g h_{i+1/2} \frac{\mathbb{I}_2 - \mathsf{sign}(A_{i+1/2})}{2} \begin{bmatrix} 0 \\ \dfrac{d_{i+1} - d_i}{\Delta x} \end{bmatrix}$$

Exact discrete invariant

The physical lake at rest state : $\eta_i = q_i = u_i = 0 \, \forall \, i$ and $h_i = d_i \, \forall \, i$.



[1] Bermudez-Vazquez *Computers&Fluids* 1994

First order Roe scheme for shallow water[1]

$$\partial_t W + \partial_x F(W) + S(W,x) = 0 \quad \Longrightarrow \quad \Delta x \frac{W_i^{n+1} - W_i^n}{\Delta t} + \hat{F}_{i+1/2} - \hat{F}_{i-1/2} + \Delta x S_i = 0$$

$$\hat{F}_{i+1/2} = \frac{F_{i+1} + F_i}{2} - \frac{|A_{i+1/2}|}{2}(W_{i+1} - W_i)$$

$$S_i = -g h_{i-1/2} \frac{\mathbb{I}_2 + \mathsf{sign}(A_{i-1/2})}{2} \left[\begin{array}{c} 0 \\ \frac{d_i - d_{i-1}}{\Delta x} \end{array} \right] - g h_{i+1/2} \frac{\mathbb{I}_2 - \mathsf{sign}(A_{i+1/2})}{2} \left[\begin{array}{c} 0 \\ \frac{d_{i+1} - d_i}{\Delta x} \end{array} \right]$$

Exact discrete invariant

The physical lake at rest state. To check this use the equivalence

$$\hat{F}_{i+1/2} - \hat{F}_{i-1/2} + \Delta x S_i = \frac{\mathbb{I}_2 + \mathsf{sign}(A_{i-1/2})}{2} \phi^{i-1/2} + \frac{\mathbb{I}_2 - \mathsf{sign}(A_{i+1/2})}{2} \phi^{i+1/2}$$

and note that along this state

$$\phi^{i+1/2} = \int_i^{i+1} (\partial_x F + S) = \left[\begin{array}{c} q_{i+1} - q_i \\ u_{i+1} q_{i+1} - u_i q_i + g \frac{h_{i+1}^2 - h_i^2}{2} \end{array} \right] - g h_{i+1/2} \left[\begin{array}{c} 0 \\ d_{i+1} - d_i \end{array} \right] = 0$$

[1] Bermudez-Vazquez *Computers&Fluids* 1994

Lake at rest and dry states[1]

Exact discrete invariant

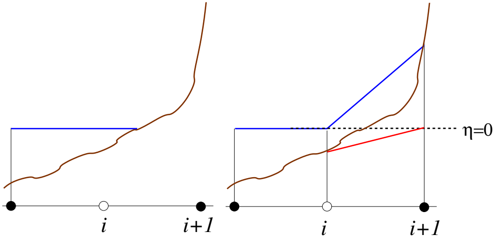
The physical lake at rest state. In presence of dry areas

$$\phi^{i+1/2} = \left[\begin{array}{c} 0 \\ -\frac{gh_i^2}{2} \end{array} \right] - \frac{gh_i}{2} \left[\begin{array}{c} 0 \\ d_{i+1} - d_i \end{array} \right] = -\frac{gh_i}{2} \left[\begin{array}{c} 0 \\ d_{i+1} \end{array} \right] \neq 0!$$

To cure the problem, set in these cells[2]

$$\phi^{i+1/2} = -\frac{gh_i}{2} \left[\begin{array}{c} 0 \\ h_i + \Delta_{i+1/2} \end{array} \right] \quad \text{with} \quad \Delta_{i+1/2} = \max(d_{i+1} - d_i,\, -h_i)$$

This allows to recover the $\phi = 0$ condition.



[1] Castro et al *Math. and Computer Modeling* 42 2005

[2] note that above the $\eta = 0$ line $d(x)$ becomes negative

Multidimensional case

$$\partial_t W + \nabla \cdot F(W) + S(W, x, y) = 0$$

Finite Volume

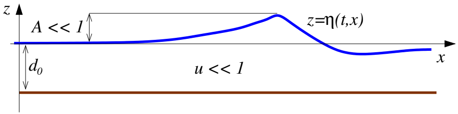
Need to add *artificial* contribution of the source term to the flux integrals on finite volume faces ...

RD/FEM

Source term naturally included in the residual, along invariants $\phi = 0$ works also in multi-D

Let's have a look at dispersive equations

Enhanced Boussinesq equations



Linearized MS equations[1]

$$\begin{cases} \partial_t \eta + d_0 \partial_x u = 0 \\ \partial_t u - (\beta + \frac{1}{3}) d_0^2 \partial_{xxt} u + g \partial_x \eta - \beta g d_0^2 \partial_{xxx} \eta = 0 \end{cases}$$

Dispersion relation

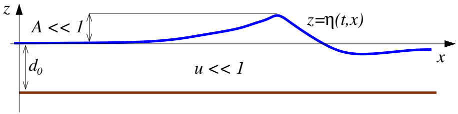
Let $C_0^2 = g d_0^2$

$$\omega^2 = \underbrace{(kC_0)^2}_{\text{Linearized Shallow Water}} \frac{1 + \beta (\mathrm{k} d_0)^2}{1 + B (\mathrm{k} d_0)^2}$$

Dispersion coeff. β chosen by minimizing error w.r.t. Airy theory.

[1] Madsen and Sørensen *Coastal Engineering* 1992, Schäffer and Madsen *Coastal Engineering* 1995

Enhanced Boussinesq equations



Linearized Nw equations[1]

$$\begin{cases} \partial_t \eta + d_0 \partial_x u + A_2 d_0^3 \partial_{xxx} u = 0 \\ \partial_t u + A_1 d_0^2 \partial_{xxt} u + g \partial_x \eta = 0 \end{cases}$$

with $A_1 = \alpha + \alpha^2/2,\ A_2 = A_1 + 1/3$

Dispersion relation

Let $C_0^2 = g d_0^2$

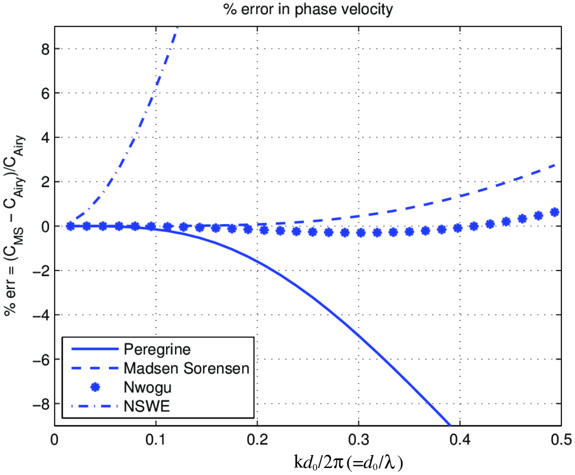
$$\omega^2 = \underbrace{(kC_0)^2}_{\text{Linearized Shallow Water}} \frac{1 - A_2 (\mathsf{k} d_0)^2}{1 - A_1 (\mathsf{k} d_0)^2}$$

Dispersion coeff. α chosen by minimizing error w.r.t. Airy theory.

[1] Nwogu *Coastal Engineering* 1994

Enhanced Boussinesq equations

Dispersion relations : models overview



Prescribed values of model coefficients

NEXT STEP : CONTINUOUS TO DISCRETE

- Influence of the scheme
- dissipation for given mesh size
- dispersion error for given mesh size
- Objective: do not pollute the dispersion of the model

Next step : continuous to discrete

Scalar model problem :

$$\partial_t u - \alpha \partial_{txx} u + a \partial_x u - \beta \partial_{xxx} u = 0$$

Dispersion relation :

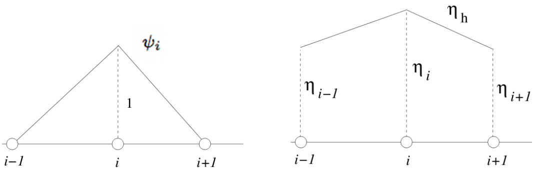
$$\omega = - \underbrace{\mathsf{k}a}_{\text{pure advection}} \frac{1 + \beta \mathsf{k}^2}{1 + \alpha \mathsf{k}^2} \qquad \text{dispersion}$$

P^1 FEM

- We consider a tasselation of the domain composed of non-overlapping elements;
- Unknowns at nodes: $\{\eta_i(t)\}_{i\geq 1}$ and $\{q_i(t)\}_{i\geq 1}$ ($\{u_i(t)\}_{i\geq 1}$ for model prob.);
- P^1 piecewise linear continuous approximation

$$\begin{aligned} \eta_h(t,x) &= \sum_{i\geq 1} \eta_i(t)\psi_i(x) = \sum_K \sum_{j\in K} \eta_j(t)\psi_j(x) \\ q_h(t,x) &= \sum_{i\geq 1} q_i(t)\psi_i(x) = \sum_K \sum_{j\in K} q_j(t)\psi_j(x) \\ u_h(t,x) &= \sum_{i\geq 1} u_i(t)\psi_i(x) = \sum_K \sum_{j\in K} u_j(t)\psi_j(x) \end{aligned} \tag{2}$$

- ψ_i are standard continuous piecewise linear finite element basis functions;



Continuous Galerkin (cG) for $\partial_t u - \alpha\partial_{txx} u + a\partial_x u - \beta\partial_{xxx} u = 0$

$$\int\limits_{\Omega_h} \psi_i \partial_t u_h + \int\limits_{\Omega_h} \alpha \partial_{xt} u_h \, \partial_x \psi_i - \int\limits_{\Omega_h} a u_h \, \partial_x \psi_i + \int\limits_{\Omega_h} \beta w_h^u \, \partial_x \psi_i = 0$$

$$\int\limits_{\Omega_h} \psi_i w_h^u + \int\limits_{\Omega_h} \partial_x u \partial_x \psi_i = 0$$

Continuous Galerkin (cG) for $\partial_t u - \alpha\partial_{txx} u + a\partial_x u - \beta\partial_{xxx} u = 0$

$$\frac{\Delta x}{6}\Big(\frac{du_{i-1}}{dt} + 4\frac{du_i}{dt} + \frac{du_{i-1}}{dt}\Big) - \frac{\alpha}{\Delta x}\Big(\frac{du_{i-1}}{dt} - 2\frac{du_i}{dt} + \frac{du_{i-1}}{dt}\Big) \\ + \frac{a}{2}(u_{i+1} - u_{i-1}) - \frac{\beta}{\Delta x^2}(u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}) = 0$$

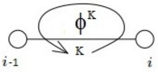
Continuous Galerkin (cG) for $\partial_t u - \alpha\partial_{txx}u + a\partial_x u - \beta\partial_{xxx}u = 0$

$$\frac{\Delta x}{6}\Big(\frac{du_{i-1}}{dt} + 4\frac{du_i}{dt} + \frac{du_{i-1}}{dt}\Big) - \frac{\alpha}{\Delta x}\Big(\frac{du_{i-1}}{dt} - 2\frac{du_i}{dt} + \frac{du_{i-1}}{dt}\Big)$$
$$+\frac{a}{2}(u_{i+1} - u_{i-1}) - \frac{\beta}{\Delta x^2}(u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}) = 0$$

FD2 scheme (same cost) :

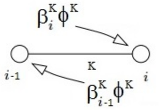
$$\Delta x\frac{du_i}{dt} - \frac{\alpha}{\Delta x}\Big(\frac{du_{i-1}}{dt} - 2\frac{du_i}{dt} + \frac{du_{i-1}}{dt}\Big)$$
$$+\frac{a}{2}(u_{i+1} - u_{i-1}) - \frac{\beta}{\Delta x^2}(u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}) = 0$$

Central Residual Distribution for $\partial_t u - \alpha\partial_{txx} u + a\partial_x u - \beta\partial_{xxx} u = 0$



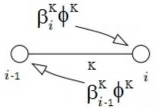
- Nodal equations :

$$\frac{1}{2}\Phi^K + \frac{1}{2}\Phi^{K+1} = 0$$



- With cell residuals

$$\Phi^K = \int_K \left(\partial_t u_{h|_K} - \alpha\partial_t w^u_{h|_K} + a\partial_x u_{h|_K} - \beta\partial_x w^u_{h|_K}\right) dx$$



- Same extra equations for $w^u \approx \partial_{xx} u$

Central Residual Distribution for $\partial_t u - \alpha \partial_{txx} u + a \partial_x u - \beta \partial_{xxx} u = 0$

$$\frac{\Delta x}{4}\left(\frac{du_{i-1}}{dt} + 2\frac{du_i}{dt} + \frac{du_{i-1}}{dt}\right) - \frac{\alpha}{4\Delta x}\left(\frac{du_{i-2}}{dt} - 2\frac{du_i}{dt} + \frac{du_{i-2}}{dt}\right)$$
$$+\frac{a}{2}(u_{i+1} - u_{i-1}) - \frac{\beta}{\Delta x^2}(u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}) = 0$$

FD2 scheme (cRD penta-diagonal system) :

$$\Delta x \frac{du_i}{dt} - \frac{\alpha}{\Delta x}\left(\frac{du_{i-1}}{dt} - 2\frac{du_i}{dt} + \frac{du_{i-1}}{dt}\right)$$
$$+\frac{a}{2}(u_{i+1} - u_{i-1}) - \frac{\beta}{\Delta x^2}(u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}) = 0$$

Stabilized Upwind Schemes (SUPG and uRD)

Schemes **cG** and **cRD** are centered approximations not well suited for the discretization of the hyperbolic (advection or shallow water) limit for which some form of upwinding is necessary to stabilize the system

We want to look at the properties of upwind stabilized variants of the centered schemes presented

Stabilized Upwind Schemes (SUPG and uRD)

Streamline Upwind Petrov-Galerkin stabilization[8] :

$$\mathcal{R}_i(u_h) + \sum_{\mathsf{K}\in\Omega_h} \int_K a\partial_x \psi_i^{\mathsf{K}} \tau_{\mathsf{K}} r^K = 0 \tag{3}$$

τ_{K} is the SUPG stabilization parameter:

$$\tau_{\mathsf{K}} = \frac{1}{\sum_{j\in\mathsf{K}} |a\partial_x \psi_j^K|}$$

and r^K the local residual vector

$$r^K = \partial_t u_{h|_K} - \alpha\partial_t w^u_{h|_K} + a\partial_x u_{h|_K} - \beta\partial_x w^u_{h|_K}$$

[8] T.J.R Hughes, G. Scovazzi and T. Tezduyar, *J.Sci.Comp.* 43 2010

Stabilized Upwind Schemes (SUPG and uRD)

The final form (P1 case with $\Phi^K = \int_K r^K$ for the above choice of τ_K):

$$\mathcal{R}_i(u_h) + \frac{\mathsf{sign}(a)}{2}\Phi^{\mathsf{K}_{i-1/2}} - \frac{\mathsf{sign}(a)}{2}\Phi^{\mathsf{K}_{i+1/2}} = 0 \qquad (4)$$

$\mathcal{R}_i$ is the centred part of the scheme: if $\mathcal{R}_i^{\mathsf{cG}} \longrightarrow$ **SUPG** scheme;

if $\mathcal{R}_i^{\mathsf{cRD}} \longrightarrow$ **uRD** scheme.

Upwinding on the advection direction (hyperbolic limit)

Stabilized Upwind Schemes (SUPG and uRD)

Difference w.r.t. std. Roe upwind scheme (in 1d ...)

- Same advection operator ($a > 0 \longrightarrow a(u_i - u_{i-1})$) !!
- Both are residual based generalizations of first order Roe scheme
- All terms upwinded at once (including high order differential terms)
- Non-diagonal mass matrices for the first order time derivative
- Linear algebraic system to invert, as for all other schemes (due to the presence of xt derivative)

Time continuous error analysis

The objective is to characterize

1. The differences in error (TE analysis)
2. The dispersion error (DE analysis)
3. Same for Boussinesq

Time continuous TE analysis

Brute force ...

Time continuous TE analysis

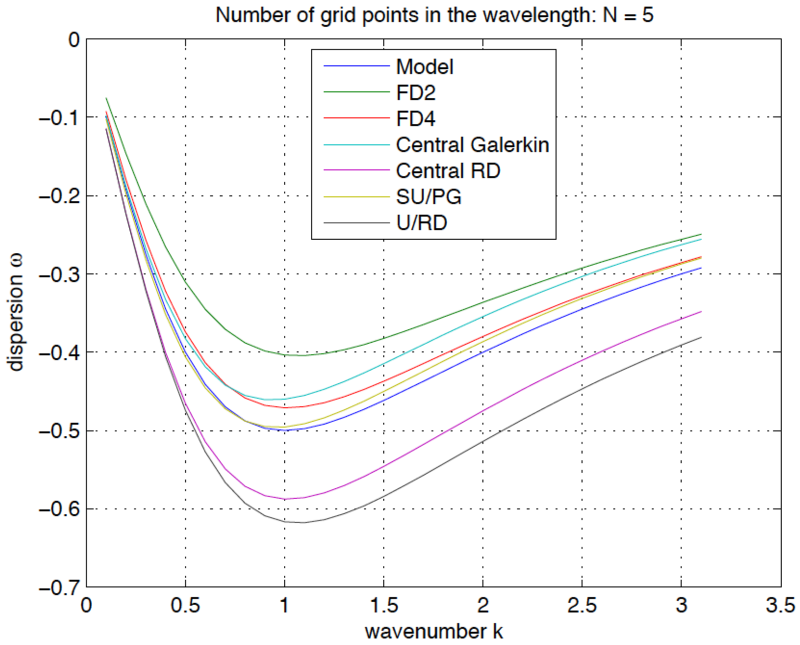
$$
\begin{aligned}
\mathsf{TE}_{\mathsf{cG}} &= \frac{\Delta x^2}{12} \partial_{xxxx} (\alpha \partial_t u + 2\beta \partial_x u) + \mathcal{O}(\Delta x^4) \\
\mathsf{TE}_{\mathsf{cRD}} &= \frac{\Delta x^2}{2} \partial_{xx} (\frac{1}{2} \partial_t u + \frac{1}{3} a \partial_t u - \frac{2}{3} \alpha \partial_{xxt} u - \frac{4}{3} \beta \partial_{xxt} u) + \mathcal{O}(\Delta x^3) \\
\mathsf{TE}_{\mathsf{SUPG}} &= \frac{\Delta x^2}{12} \partial_{xxxx} (\alpha \partial_t u + 2\beta \partial_x u) + \mathcal{O}(\Delta x^3) \\
\mathsf{TE}_{\mathsf{uRD}} &= \frac{\Delta x^2}{2} \partial_{xx} (\frac{1}{2} \partial_t u + \frac{1}{3} a \partial_t u - \frac{2}{3} \alpha \partial_{xxt} u - \frac{4}{3} \beta \partial_{xxt} u) + \mathcal{O}(\Delta x^3)
\end{aligned}
$$

Time continuous DE analysis

As in the continuous case :

1. Set $u_i(t) = u_0 e^{\nu t + \mathsf{j} \mathsf{k} x_i}$
2. Replace in the FD form of the scheme : $\dfrac{du_i(t)}{dt} = \nu\, u_i(t),\ u_{i\pm 1}(t) = e^{\pm \mathsf{j} \Delta x} u_i(t)$
3. Solve for $\nu = \xi + \mathsf{j}\ \omega$
4. $\omega = \omega(\mathsf{k}, \Delta x)$

Time continuous DE analysis



Lucky shot ?

Extension to Boussinesq models

Extension to Boussinesq models

Schemes easily extended, main differences are

- Auxiliary variables for MS : $\partial_{xx}\eta$ (and $\partial_x q$ for cRD and both upwind schemes)
- Auxiliary variables for Nw : $\partial_{xx}u$ (and $\partial_x u$ for cRD and both upwind schemes)
- Upwinding : scalar a replaced by A^K elemental average of shallow water flux Jacobian
- DE analysis : same main steps but leads to eigenvalue problem (ν from characteristic equation)
- $\omega = \omega(\mathsf{k}d_0, \mathsf{k}\Delta x)$

Error analysis : linearized MS model

$$\begin{cases} \partial_t \eta + d_0 \partial_x u = 0 \\ \partial_t u - (\beta + \frac{1}{3}) d_0^2 \partial_{xxt} u + g \partial_x \eta - \beta g d_0^2 \partial_{xxx} \eta = 0 \end{cases}$$

TE analysis : MS model - centered schemes

FD2 scheme.

$$\mathrm{TE}^{\eta}_{\mathsf{FD2}} = \frac{d_0 \Delta x^2}{6} \partial_{xxx} u_i + \mathcal{O}(\Delta x^4)$$

$$\mathrm{TE}^{u}_{\mathsf{FD2}} = \frac{\Delta x^2}{6} \partial_{xx} \left(-\frac{B d_0^2}{2} \partial_{x^2 t} u_i + g \partial_x \eta_i - \frac{3}{2} \beta g d_0^2 \partial_{xxx} \eta_i \right) + \mathcal{O}(\Delta x^4)$$

cG scheme.

$$\mathrm{TE}^{\eta}_{\mathsf{cG}} = \frac{\Delta x^4}{24} \partial_{xxxx} \left(\frac{1}{3} \partial_t \eta_i + \frac{d_0}{5} \partial_x u_i \right) + \mathcal{O}(\Delta x^6)$$

$$\mathrm{TE}^{u}_{\mathsf{cG}} = \frac{\Delta x^2}{12} \partial_{xxxx} \left(B d_0^2 \partial_t u_i - \beta g d_0^2 \partial_x \eta_i \right) + \mathcal{O}(\Delta x^4)$$

cRD scheme.

$$\mathrm{TE}^{\eta}_{\mathsf{cRD}} = \frac{\Delta x^2}{2} \partial_{xx} \left(\frac{1}{2} \partial_t \eta_i + \frac{d_0}{3} \partial_x u_i \right) + \mathcal{O}(\Delta x^4)$$

$$\mathrm{TE}^{u}_{\mathsf{cRD}} = \Delta x^2 \partial_{xx} \left(\frac{1}{4} \partial_t u_i - \frac{1}{3} B d_0^2 \partial_{xxt} u_i + \frac{1}{6} g \partial_x \eta_i - \frac{1}{4} \beta g d_0^2 \partial_{xxx} \eta_i \right) + \mathcal{O}(\Delta x^4)$$

TE analysis : MS model - centered schemes

FDWK scheme (FD2 on ∂_{xxx} and ∂_{xx} - FD4 on ∂_x)[1]

$$\begin{aligned}
\mathrm{TE}^{\eta}_{\mathsf{FDWK}} &= \frac{d_0 \Delta x^4}{30} \partial_{xxxxx} u_i + \mathcal{O}(\Delta x^6) \\
\mathrm{TE}^{u}_{\mathsf{FDWK}} &= \frac{\Delta x^2}{4} \partial_{xxxx} \left(\frac{1}{3} B d_0^2 \partial_t u_i + \beta g d_0^2 \partial_x \eta_i \right) + \mathcal{O}(\Delta x^4)
\end{aligned}$$

cG scheme.

$$\begin{aligned}
\mathrm{TE}^{\eta}_{\mathsf{cG}} &= \frac{\Delta x^4}{24} \partial_{xxxx} \left(\frac{1}{3} \partial_t \eta_i + \frac{d_0}{5} \partial_x u_i \right) + \mathcal{O}(\Delta x^6) \\
\mathrm{TE}^{u}_{\mathsf{cG}} &= \frac{\Delta x^2}{12} \partial_{xxxx} \left(B d_0^2 \partial_t u_i - \beta g d_0^2 \partial_x \eta_i \right) + \mathcal{O}(\Delta x^4)
\end{aligned}$$

cRD scheme.

$$\begin{aligned}
\mathrm{TE}^{\eta}_{\mathsf{cRD}} &= \frac{\Delta x^2}{2} \partial_{xx} \left(\frac{1}{2} \partial_t \eta_i + \frac{d_0}{3} \partial_x u_i \right) + \mathcal{O}(\Delta x^4) \\
\mathrm{TE}^{u}_{\mathsf{cRD}} &= \Delta x^2 \partial_{xx} \left(\frac{1}{4} \partial_t u_i - \frac{1}{3} B d_0^2 \partial_{xxt} u_i + \frac{1}{6} g \partial_x \eta_i - \frac{1}{4} \beta g d_0^2 \partial_{xxx} \eta_i \right) + \mathcal{O}(\Delta x^4)
\end{aligned}$$

[1](G. Wei, J.T. Kirby, *J. Waterway, Port, Coastal, and Ocean Engineering*, 1995)

TE analysis : MS model - upwind schemes

FDWK scheme (FD2 on ∂_{xxx} and ∂_{xx} - FD4 on ∂_x)[1]

$$\mathrm{TE}^{\eta}_{\mathsf{FDWK}} = \frac{d_0 \Delta x^4}{30} \partial_{xxxxx} u_i + \mathcal{O}(\Delta x^6)$$

$$\mathrm{TE}^{u}_{\mathsf{FDWK}} = \frac{\Delta x^2}{4} \partial_{xxxx} \left(\frac{1}{3} B d_0^2 \partial_t u_i + \beta g d_0^2 \partial_x \eta_i \right) + \mathcal{O}(\Delta x^4)$$

SUPG scheme.

$$\mathrm{TE}^{\eta}_{\mathsf{SUPG}} = \frac{C_0 \Delta x^3}{2g} \partial_{xxx} \left(\frac{1}{3} \partial_t u_i - \frac{1}{2} B d_0^2 \partial_{xxt} u_i + \frac{1}{6} g \partial_x \eta_i - \frac{1}{3} \beta g d_0^2 \partial_{xxx} \eta_i \right) + \mathcal{O}(\Delta x^4)$$

$$\mathrm{TE}^{u}_{\mathsf{SUPG}} = \frac{\Delta x^2}{12} \partial_{xxxx} \left(B d_0^2 \partial_t u_i - \beta g d_0^2 \partial_x \eta_i \right) + \mathcal{O}(\Delta x^3)$$

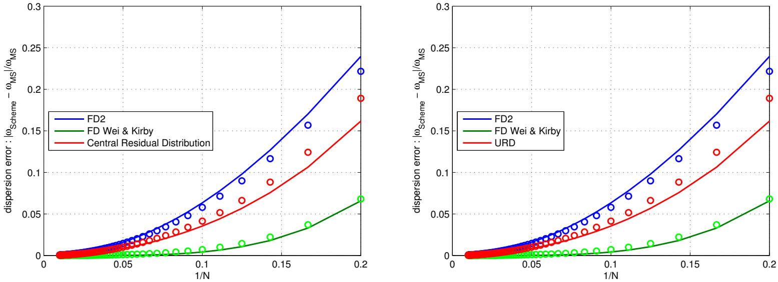
URD scheme.

$$\mathrm{TE}^{\eta}_{\mathsf{URD}} = \frac{\Delta x^2}{2} \partial_{x^2} \left(\frac{1}{2} \partial_t \eta_i + \frac{d_0}{3} \partial_x u_i \right) + \mathcal{O}(\Delta x^3)$$

$$\mathrm{TE}^{u}_{\mathsf{URD}} = \Delta x^2 \partial_{x^2} \left(\frac{1}{4} \partial_t u_i - \frac{1}{3} B d_0^2 \partial_{xxt} u_i + \frac{1}{6} g \partial_x \eta_i - \frac{1}{4} \beta g d_0^2 \partial_{xxx} \eta_i \right) + \mathcal{O}(\Delta x^3)$$

[1] (G. Wei, J.T. Kirby, *J. Waterway, Port, Coastal, and Ocean Engineering*, 1995)

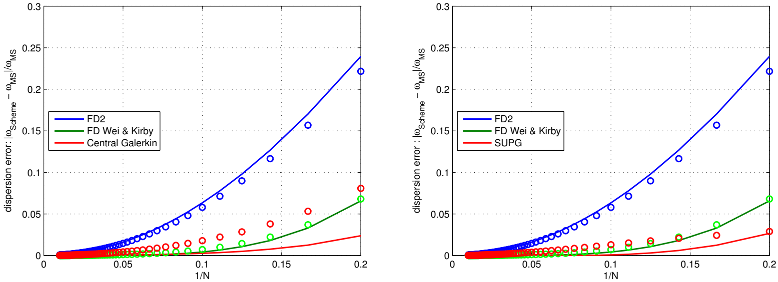
DE analysis : MS model - cRD and URD



Solid line : k$d_0 = 0.5$ - Circles k$d_0 = 2.6$

N : points per wavelength

DE analysis : MS model - cG and SUPG



Solid line : k$d_0 = 0.5$ - Circles k$d_0 = 2.6$

N : points per wavelength

Error analysis : linearized Nw model

$$\begin{cases} \partial_t \eta + d_0 \partial_x u + A_2 d_0^3 \partial_{xxx} u = 0 \\ \partial_t u + A_1 d_0^2 \partial_{xxt} u + g \partial_x \eta = 0 \end{cases}$$

TE analysis : Nw model - FDWK, cG, SUPG

FDWK scheme (FD2 on ∂_{xxx} and ∂_{xx} - FD4 on ∂_x)[1]

$$\begin{aligned}
\mathrm{TE}^{\eta}_{\mathsf{FDWK}} &= \frac{\Delta x^2}{4} A_2 d_0^3 \partial_{xxxxx} u_i + \mathcal{O}(\Delta x^4) \\
\mathrm{TE}^{u}_{\mathsf{FDWK}} &= \frac{\Delta x^2}{12} A_1 D_0^2 \partial_{xxxxt} u_i + \mathcal{O}(\Delta x^4)
\end{aligned}$$

cG scheme.

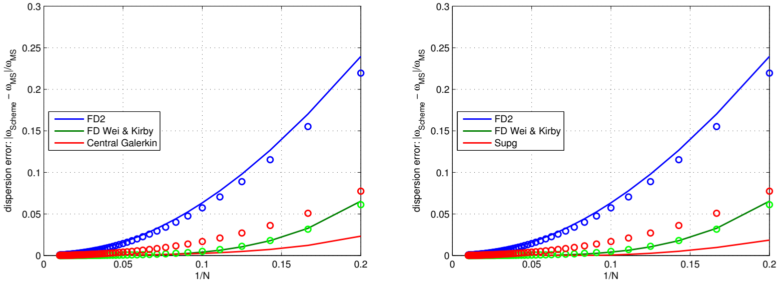
$$\begin{aligned}
\mathrm{TE}^{\eta}_{\mathsf{cG}} &= \frac{\Delta x^2}{12} A_2 d_0^3 \partial_{xxxxx} u_i + \mathcal{O}(\Delta x^4) \\
\mathrm{TE}^{u}_{\mathsf{cG}} &= \frac{\Delta x^2}{12} A_1 D_0^2 \partial_{xxxxt} u_i + \mathcal{O}(\Delta x^4)
\end{aligned}$$

SUPG scheme.

$$\begin{aligned}
\mathrm{TE}^{\eta}_{\mathsf{SUPG}} &= \frac{\Delta x^2}{12} A_2 d_0^3 \partial_{xxxxx} u_i + \mathcal{O}(\Delta x^3) \\
\mathrm{TE}^{u}_{\mathsf{SUPG}} &= \frac{\Delta x^2}{12} A_1 D_0^2 \partial_{xxxxt} u_i + \mathcal{O}(\Delta x^3)
\end{aligned}$$

[1](G. Wei, J.T. Kirby, *J. Waterway, Port, Coastal, and Ocean Engineering*, 1995)

DE analysis : Nw model



Solid line : k$d_0 = 0.5$ - Circles k$d_0 = 2.6$

N : points per wavelength

Time continuous error analysis

1. Stabilized FEM formulation best results in terms of accuracy
2. P^1 cG and SUPG similar (or better) accuracy than FDWK (FD4) in 1D
3. In 2D : excellent approximation of planar waves on regular grids
4. 2D unstructured : is P^1 enough ?

MS equations ...

$$\begin{cases} \partial_t \eta + \nabla \cdot \vec{q} = 0 \\ \partial_t q + \nabla \cdot (\vec{u} \otimes \vec{q}) + gH \nabla \eta + \vec{\psi} = 0 \end{cases}$$

where $\vec{\psi} \equiv (\psi_x, \psi_y)$ are the dispersive terms of the model which can be written as

$$\begin{cases} \psi_x = -Bh^2 \partial_{tx} \nabla \cdot \vec{q} - \frac{1}{6} h \partial_x h \partial_t \left(\nabla \cdot \vec{q} + \partial_x q_x \right) - \frac{1}{6} h \partial_y h \partial_{tx} q_y - \beta g h^2 \partial_x w^\eta \\ \psi_y = -Bh^2 \partial_{ty} \nabla \cdot \vec{q} - \frac{1}{6} h \partial_y h \partial_t \left(\nabla \cdot \vec{q} + \partial_y q_y \right) - \frac{1}{6} h \partial_x h \partial_{ty} q_x - \beta g h^2 \partial_y w^\eta \\ w^\eta = \nabla \cdot (h \nabla \eta) \end{cases}$$

Numerical examples : shelf

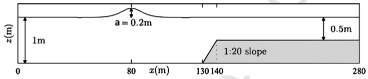
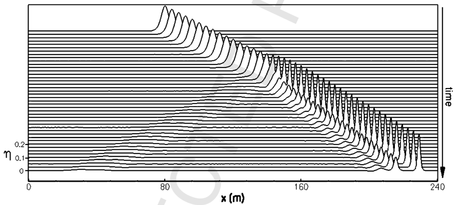
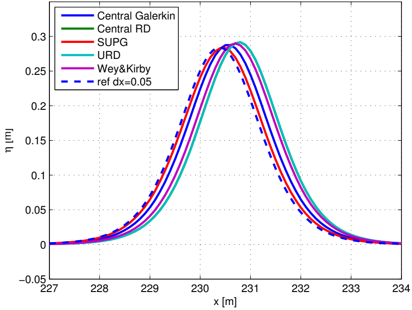


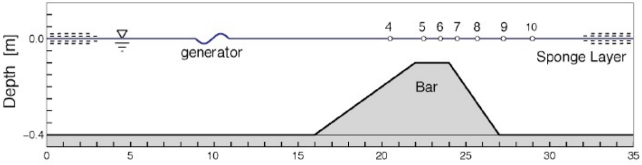
Fig. 12. Sketch of the submerged shelf test.

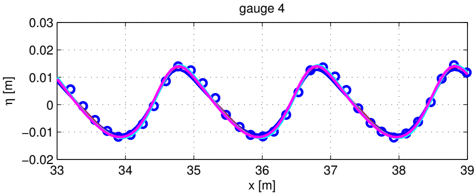




Tallest
wave
comparison

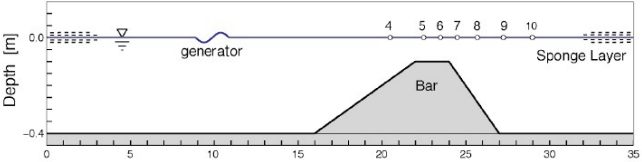
Numerical examples : submerged bar

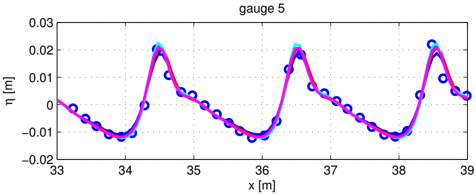




Experimental data (o), FDWK scheme (—), cG scheme (—), SUPG scheme (—), and URD scheme (—)

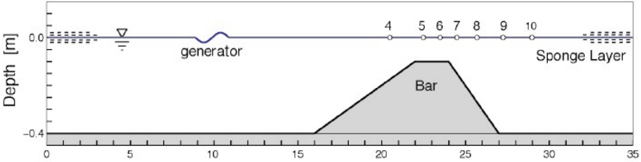
Numerical examples : submerged bar

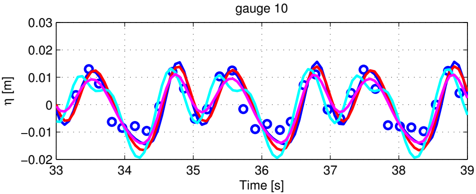




Experimental data (o), FDWK scheme (—), cG scheme (—), SUPG scheme (—), and URD scheme (—)

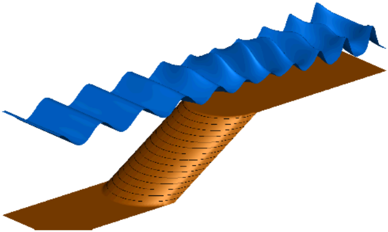
Numerical examples : submerged bar

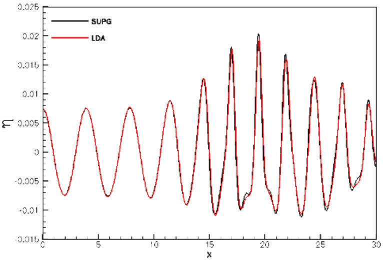


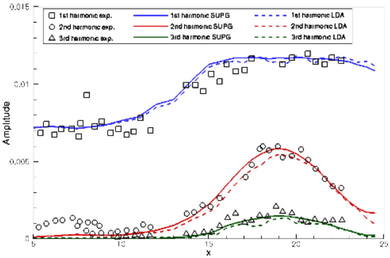


Experimental data (o), FDWK scheme (—), cG scheme (—), SUPG scheme (—), and URD scheme (—)

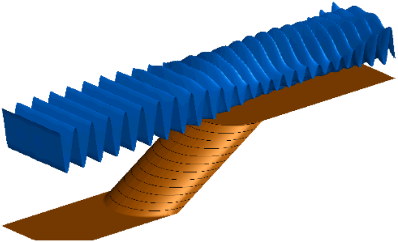
Numerical examples : circular shoal (hexagons) - $\Delta x = 0.1m$

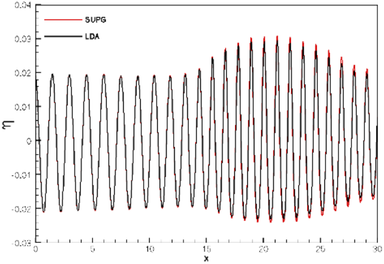


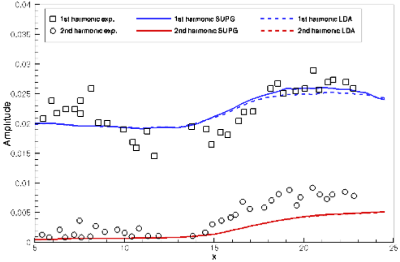




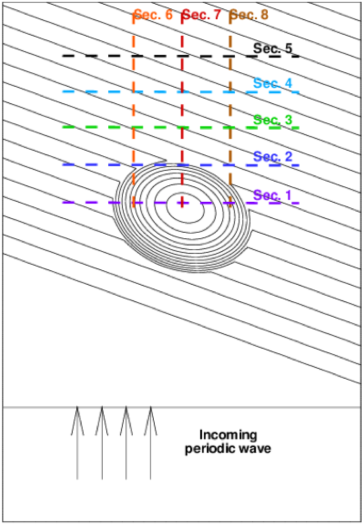
Numerical examples : circular shoal (hexagons) - $\Delta x = 0.1m$

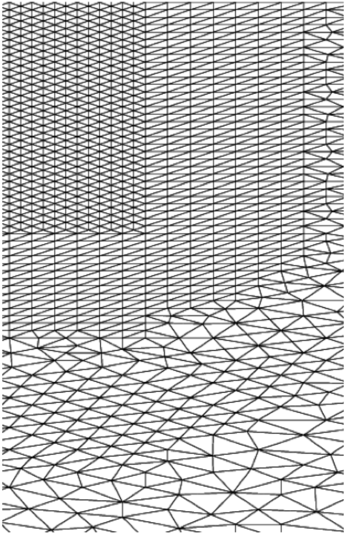




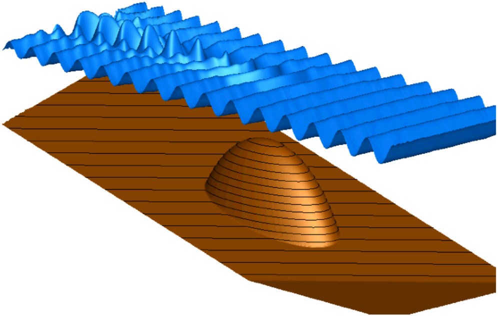


Numerical examples : elliptic shoal (unstructured)



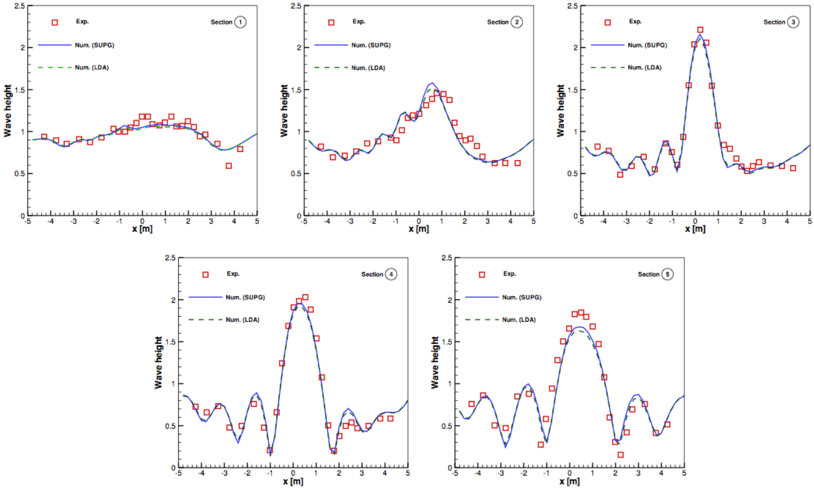


Numerical examples : elliptic shoal (unstructured)



(Ribbed channel clip)

Numerical examples : elliptic shoal (unstructured)



SUMMARY

Parti I

- Fluctuation form of FV schemes
- General design properties for fluctuation splitting/residual distribution :
  1. Conservation
  2. Stability and upwinding
  3. Consistency (accuracy) conditions
  4. Discontinuity capturing

Parti II

- Residual based schemes and time dependent problems : accuracy issues
- High order accurate schemes via consistent mass matrices
- Shallow Water issues : steady states, wetting/drying
- Dispersive equations and residual based
- Discrete dispersion analysis

PERSPECTIVES

What's the plan ...

- Analysis of continuous stabilized higher oder FEM approximation for dispersive eq.s ($k = 2, 3$, Lagrange and Bezier)
- Schemes : dispersion optimized schemes ? Structured grid residual based schemes, and seek generalization to unstructured
- Discontinuities. FEM-RD formalism (already done for Euler and SW) : use monotone spatial operators (RD) in variational context (mass matrices, high order derivatives etc)
- Grid adaptation : time dependent (moving fronts) based on ALE and ALE mapping if remeshing is necesary
- Time integration : Explicit (eBDF), Implicit (BDf) or space-time ?
- Green Naghdi equations on unstructured adaptive (moving) grids
- Uncertainty quantification + analysis of variance : assess models robustness
- etc. etc.

Thanks to ..

Collaborators

- A. Filippini (PhD, Inria), L. Arpaia (PhD, Inria) - stabilized FEM for Bouss/GN and SW (moving adaptive grids)
- S. Bellec (PhD, Inria) - exact solutions to Boussinesq eq.s, discrete asymptotics
- M. Colin (IPB and Inria) - depth averaged modeling in general
- R. Abgrall (Zurich University) - residual schemes
- P. Bonneton (EPOC Bordeaux) - Boussinesq++
- A.I. Delis (Tech. Univ. Crete) - Boussinesq,NLSW++
- P. Congedo (Inria) - UQ applied to depth average models
- A. Guardone (Politecnico di Milano) - ALE adaptation
- etc. ....