Non-oscillatory high-order Residual Distribution Schemes for the Euler equations

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Survey

Residual Distribution

Date back to ideas of (P.L.Roe, *Num. Meth. Fluid Dyn. 1982*) : decompose local numerical error (fluctuation) in signals sent to nodes to evolve local value of the solution

- Multidimensional upwinding (80' and 90'). **Roe** **(Michigan U. Ann Arbor)**, **Deconinck** **(von Karman I.)**, **Hubbard** **(Leeds U.)**, **Napolitano** **(Politec. Bari)** :
  1. Decomposition of Q-linear form in decoupled hyperbolic components
  2. Each scalar hyperbolic component discretized using MU technique

  Well adapted to steady supersonic, MU in sub-critical case with inexact decompositions (formal continuation), Roe linearization, no unsteady.
- Last 10 years. ... plus **Abgrall** **(INRIA)**, **Barth** **(NASA)**, **Shu** **(Brown U.)** :
  1. High order for time-dependent (consistent treatment of time derivative)
  2. Conservation without Roe linearization
  3. General construction of non-oscillatory schemes for steady/unsteady
  4. More than second order and dicontinuous approximation

  With generalization the idea of MU and the characteristic decompositions are playing a smaller role (matrix formulation)

Outline

Overview

Model problem and objective

Steady conservation laws

$$\nabla \cdot \boldsymbol{\mathcal{F}}(\mathbf{u}) = 0 \quad \text{on} \quad \Omega \subset \mathbb{R}^d \tag{1}$$

Generalities :

- Usual assumptions on the problem (hyperbolicity, etc.)
- Given BCs on inflow boundaries (Dirichlet)
- talk for $d = 2$ but everything goes similarly for $d = 3$.

High order issue

How do we get the higher order ?

1. Need higher degree polynomial representation : how is it built ?
2. What are the degrees of freedom (unknowns) used to achieve that ?

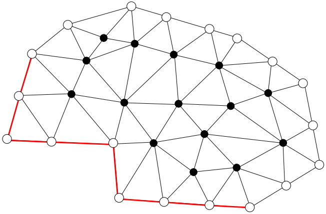
Cell centered discontinuous. Unknowns : averages and/or derivatives/moments.

- high order finite difference (swedish school, H.C. Yee)
- ENO/WENO Finite Volume
- Spectral Volume/Difference (Z.J.Wang)
- DG

Node centered continuous. Unknowns : pointwise values.

- Galerkin Least Squares and/or SUPG : Galerkin + dissipation + shock capturing (T.J.Hughes ++)
- RDS : non linear version of SUPG

Efficiency : number of dofs



- v : vertices (nodes)
- T : triangles (tetrahedra in 3d)
- e : edges
- f : faces (3d)

Euler relation for simplicia gives :

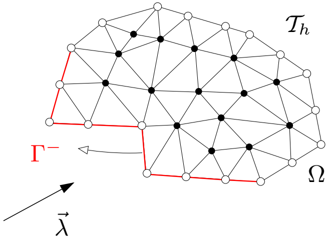
$$\text{in 2d} : \begin{cases} n_T \approx 2\, n_v \\ n_e \approx 3\, n_v \end{cases} \qquad \text{in 3d} : \begin{cases} n_T \approx 6\, n_v \\ n_f \approx 10\, n_v \\ n_e \approx 7\, n_v \end{cases}$$

We can estimate the number of dofs needed.

	2D		3D	
Order	Discontinuous	Continuous	Discontinuous	Continuous
2	$6n_v$	n_v	$24n_v$	n_v
3	$12n_v$	$4n_v$	$40n_v$	$8n_v$
4	$20n_v$	$9n_v$	$80n_v$	$27n_v$

Framework for scalar $\mathcal{CL}$s

$$\begin{aligned} \nabla \cdot \boldsymbol{\mathcal{F}}(u) &= 0 \quad \text{in} \quad \Omega \\ u &= g \quad \text{on} \quad \Gamma^- \\ \vec{\lambda}(u) &= \partial_u \boldsymbol{\mathcal{F}}(u) \end{aligned} \tag{2}$$



Some notations...

- Consider $\mathcal{T}_h$ triangulation of Ω (can do with quads...)
- Unknowns (Degrees of Freedom, DoF) : $u_i \approx u(M_i)$
- $M_i \in \mathcal{T}_h$ a given set of nodes (vertices +other dofs)
- u_h : **continuous** polynomial interpolation $u_h = \sum_i \psi_i \, u_i$

We consider here P^k Lagrange triangles

Overview

Residual Distribution ($\mathcal{RD}$)

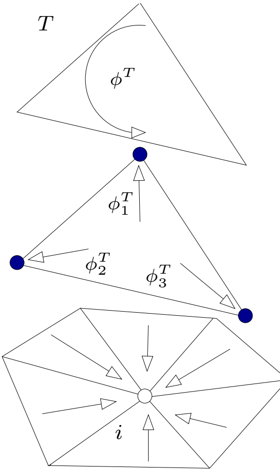
1. $\forall T \in \mathcal{T}_h$ compute : $\phi^T = \int_T \nabla \cdot \mathcal{F}_h(u_h)$

2. Distribution : $\phi^T = \sum_{i \in T} \phi_i^T$

   Distribution coeff.s : $\phi_i^T = \beta_i^T \phi^T$

3. Compute nodal values : solve algebraic system

$$\sum_{T|i \in T} \phi_i^T = 0, \quad \forall\, i \in \mathcal{T}_h \tag{3}$$



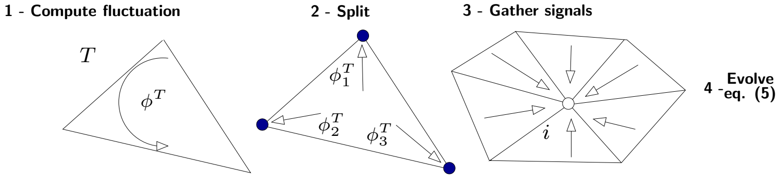
Simplified variant

Seek the limit $n \to \infty$ of

$$u_i^{n+1} = u_i^n - \omega_i \sum_{T|\, i\in T} \phi_i^T \quad \xrightarrow{n\to\infty} \quad \sum_{T|\, i\in T} \phi_i^T = 0 \tag{4}$$

The idea of Residual Distribution or Fluctuation Splitting

- Fluctuations & Signals (Roe, *Num.Meth.Fluid Dyn.*, 1982)
- Given an initial guess, nodal values evolve according to signals "proportional" to cell residuals (Roe's fluctuation)



Structural conditions

Conservation or LW theorem : convergence (if ....) to weak solution ?

Accuracy : characterization of the error, choice of ϕ_i^T

Oscillations : high order monotonicity preserving schemes

Conservation

Conservation, LW theorem for RD and $\nabla \cdot \boldsymbol{\mathcal{F}}(u) = 0$

Under some (standard) continuity assumptions on ϕ^T and ϕ_i^T the discrete solution u_h converges (if ..... !) to a weak solution of the continuous problem, provided that (Abgrall, Roe *J.Sci.Comput.* 19, 2003) :

$$\phi^T(u_h) = \oint_{\partial T} \boldsymbol{\mathcal{F}}_h(u_h) \cdot \hat{n}\, dl$$

for *some* continuous approximation of the flux $\boldsymbol{\mathcal{F}}_h$. That is

at least $\boldsymbol{\mathcal{F}}_h = \sum_{i \in \mathcal{T}_h} \boldsymbol{\mathcal{F}}_i \psi_i$

Conservation

Definition (steady homogeneous case : $\nabla \cdot \boldsymbol{\mathcal{F}}(u) = 0$)

Conservation is equivalent to the following condition :

$$\sum_{j \in T} \phi_j^T(u_h) = \oint_T \boldsymbol{\mathcal{F}}_h(u_h) \cdot \hat{n}\, dl$$

for some continuous discrete approximation of the flux.

Conservation

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How to get it (1). "Traditional choice" : Roe linearization

Look for a parameter vector $\mathbf{z}$ such that if $\boldsymbol{\mathcal{F}}_h(u_h)=\boldsymbol{\mathcal{F}}(\mathbf{z}_h)$, with $\mathbf{z}_h$ linear

$$\oint_T \boldsymbol{\mathcal{F}}_h(u_h)\cdot\hat{n}\,dl=\int_T \nabla\cdot\boldsymbol{\mathcal{F}}(\mathbf{z}_h)\,dx=\int_T \partial_{\mathbf{z}}\boldsymbol{\mathcal{F}}(\mathbf{z}_h)\cdot\nabla\mathbf{z}_h=|T|\partial_{\mathbf{z}}\boldsymbol{\mathcal{F}}(\overline{\mathbf{z}})\cdot\nabla\mathbf{z}_h|_T$$

with $\overline{\mathbf{z}}$ a simple (arithmetic) average of the nodal values in element T.

1. direct use of quasi-linear form : wave decompositions, multi-D upwinding
2. mainly Euler perfect gas (Deconinck, Roe, Struijs *Comp.&Fluids* 22, 1993)
3. the alternative is to evaluate **exactly** the mean value Jacobian

$$\overline{\partial_{\mathbf{z}}\boldsymbol{\mathcal{F}}}=\frac{1}{|T|}\int_T \partial_{\mathbf{z}}\boldsymbol{\mathcal{F}}(\mathbf{z}_h)\,dx$$

for a given set of variables $\mathbf{z}$.

Conservation

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Conservation is equivalent to the following condition :

$$\sum_{j \in T} \phi_j^T(u_h) = \oint_T \boldsymbol{\mathcal{F}}_h(u_h) \cdot \hat{n} \, dl$$

for some continuous discrete approximation of the flux.

How to get it (2) : "traditional choice", an approximation

Proposed in (Abgrall, Barth *SISC* 24, 2002) :

$$\overline{\partial_{\mathbf{z}} \boldsymbol{\mathcal{F}}} = \sum_{q=1}^{\mathsf{G}_p} \omega_q \partial_{\mathbf{z}} \boldsymbol{\mathcal{F}}(\mathbf{z}_h(x_q)) + R_q(\mathsf{G}_p, h, \mathbf{z})$$

for a given set of variables $\mathbf{z}$.

1. a LW theorem applies when quadrature error is below the truncation error
2. provides a quasi-linear form for upwinding and wave decomposition
3. **expensive** in presence of shocks

Conservation

Definition (steady homogeneous case : $\nabla \cdot \boldsymbol{\mathcal{F}}(u) = 0$)

Conservation is equivalent to the following condition :

$$\sum_{j \in T} \phi_j^T(u_h) = \oint_T \boldsymbol{\mathcal{F}}_h(u_h) \cdot \hat{n}\, dl$$

for some continuous discrete approximation of the flux.

How to get it (3) : more general approach

Proposed in (Csik, Ricchiuto, Deconinck *JCP* 179, 2002), (Ricchiuto, Csik, Deconinck *JCP* 209, 2005)

1. Decouple evaluation of the cell residual from distribution :

$$\phi^T(u_h) = \sum_{\text{edges} \in \partial T} \int_{\text{edge}} \boldsymbol{\mathcal{F}}_h(x) \cdot \hat{n}\, dl = \sum_{\text{edges} \in \partial T} l_{\text{edge}} \sum_{q=1}^{\mathsf{G}_p} \omega_q \boldsymbol{\mathcal{F}}_h(x_q) \cdot \hat{n}_{\text{edge}}$$

   for *any* continuous $\boldsymbol{\mathcal{F}}_h$ of choice
2. arbitrary averages to evaluate the Jacobians needed for upwinding
3. results depend on the choice of $\boldsymbol{\mathcal{F}}_h$ (*e.g.* $\boldsymbol{\mathcal{F}}_h = \sum_i \psi_i \boldsymbol{\mathcal{F}}_i$, or $\boldsymbol{\mathcal{F}}_h = \boldsymbol{\mathcal{F}}(\mathbf{z}_h)$ for some $\mathbf{z}$)
4. degree of freedom that can be exploited (*e.g.* exact preservation of steady contacts, or see SW talk tomorrow).

Conservation

How to understand the conservation constraint ?

Finite volume

- Numerical flux $f_{i+1/2}$
- $u_i^{n+1} = u_i^n - \dfrac{\Delta t}{\Delta x}\big(f_{i+1/2} - f_{i-1/2}\big)$
- Continuous $f_{i+1/2} = \hat{f}(u_i, u_{i+1})$
- Similarly in 2D/3D

RD schemes

- Split residual ϕ_i^T
- Conservation $\sum_{j\in T} \phi_j^T = \oint_{\partial T} \boldsymbol{\mathcal{F}}_h(u_h) \cdot \vec{n} dl$

Conservation

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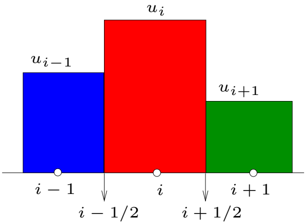
What is the link ?

Simple analogy

Finite volumes 1D

- $\Delta x \dfrac{u_i^{n+1} - u_i^n}{\Delta t} = -\big(f_{i+1/2} - f_{i-1/2}\big)$
- Conservation : consistent numerical flux $f_{i+1/2} = \hat{f}(u_i, u_{i+1})$



$$f^{\text{Roe}}_{i+1/2} = \frac{f(u_i) + f(u_{i+1})}{2} - \frac{|\lambda_{\text{Roe}}|}{2}(u_{i+1} - u_i)$$

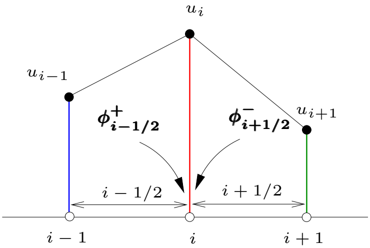
$$f^{\mathbf{z}}_{i+1/2} = \frac{f(u_i) + f(u_{i+1})}{2} - \frac{|\lambda_{\mathbf{z}}|}{2}(\mathbf{z}_{i+1} - \mathbf{z}_i)$$

$$f_{i+1/2} = \frac{1 + \text{sgn}(\lambda_{\text{Av}})}{2} f(u_i) + \frac{1 - \text{sgn}(\lambda_{\text{Av}})}{2} f(u_{i+1})$$

Simple analogy

Finite volumes 1D *vs* RD in 1D

- $\Delta x \dfrac{u_i^{n+1} - u_i^n}{\Delta t} = -\big(f_{i+1/2} - f_{i-1/2}\big) = -\big(\phi^-_{i+1/2} + \phi^+_{i-1/2}\big)$



$$f^{\text{Roe}}_{i+1/2} = \frac{f(u_i) + f(u_{i+1})}{2} - \frac{|\lambda_{\text{Roe}}|}{2}(u_{i+1} - u_i)$$

$$\phi^-_{i+1/2} = f_{i+1/2} - f(u_i) = \lambda^-_{\text{Roe}}(u_{i+1} - u_i)$$

$$\phi^+_{i-1/2} = f(u_i) - f_{i-1/2} = \lambda^+_{\text{Roe}}(u_i - u_{i-1})$$

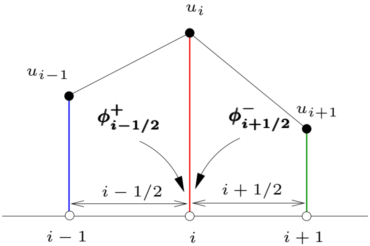
Splitting based on Roe linearization

$$\phi^+_{i-1/2} = \frac{f(u_i) - f(u_{i-1})}{2} - \frac{|\lambda_{\text{Roe}}|}{2}(u_{i-1} - u_i)$$
$$= \frac{1}{2}(\lambda_{\text{Roe}} + |\lambda_{\text{Roe}}|)(u_i - u_{i-1}) = \lambda^+_{\text{Roe}}(u_i - u_{i-1})$$

Simple analogy

Finite volumes 1D *vs* RD in 1D

- $\Delta x \dfrac{u_i^{n+1} - u_i^n}{\Delta t} = -\left(f_{i+1/2} - f_{i-1/2}\right) = -\left(\phi^-_{i+1/2} + \phi^+_{i-1/2}\right)$



$$f^{\mathbf{z}}_{i+1/2} = \frac{f(u_i) + f(u_{i+1})}{2} - \frac{|\lambda_{\mathbf{z}}|}{2}(\mathbf{z}_{i+1} - \mathbf{z}_i)$$

$$\phi^-_{i+1/2} = f_{i+1/2} - f(u_i) \approx \lambda^-_{\mathbf{z}}(\mathbf{z}_{i+1} - \mathbf{z}_i)$$

$$\phi^+_{i-1/2} = f(u_i) - f_{i-1/2} \approx \lambda^+_{\mathbf{z}}(\mathbf{z}_i - \mathbf{z}_{i-1})$$

Splitting based on approx. mean value Jacobian - $\lambda_{\mathbf{z}} = \sum_{q=1}^{\mathsf{G}_p} \omega_q \partial_{\mathbf{z}} f(\mathbf{z}(x_q))$

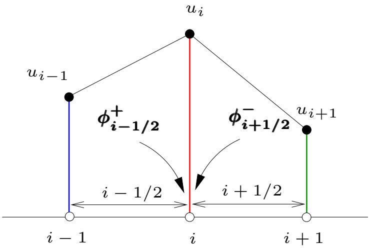
$$\phi^+_{i-1/2} = \frac{f(u_i) - f(u_{i-1})}{2} - \frac{|\lambda_{\mathbf{z}}|}{2}(\mathbf{z}_{i-1} - \mathbf{z}_i)$$

$$= \frac{1}{2}(\lambda_{\mathbf{z}} + |\lambda_{\mathbf{z}}| + R_q)(\mathbf{z}_i - \mathbf{z}_{i-1}) \overbrace{\approx}^{\mathsf{G}_p \gg 1} \lambda^+_{\mathbf{z}}(\mathbf{z}_i - \mathbf{z}_{i-1})$$

Simple analogy

Finite volumes 1D *vs* RD in 1D

- $$\Delta x \frac{u_i^{n+1}-u_i^n}{\Delta t} = -\big(f_{i+1/2}-f_{i-1/2}\big) = -\big(\phi^-_{i+1/2}+\phi^+_{i-1/2}\big)$$



$$f_{i+1/2} = \frac{1+\text{sgn}(\lambda_{\text{Av}})}{2} f(u_i) + \frac{1-\text{sgn}(\lambda_{\text{Av}})}{2} f(u_{i+1})$$

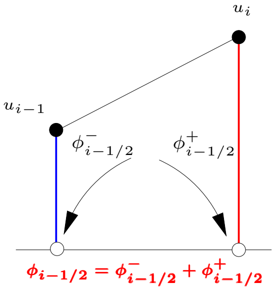
$$\phi^-_{i+1/2} = f_{i+1/2} - f(u_i)$$

$$\phi^+_{i-1/2} = f(u_i) - f_{i-1/2}$$

Flux vector splitting

$$\begin{aligned}\phi^+_{i-1/2} &= \frac{f(u_i)-f(u_{i-1})}{2} - \frac{\text{sgn}(\lambda_{\text{Av}})}{2}\big(f(u_{i-1})-f(u_i)\big)\\ &= \beta^+_{i-1/2}\big(f(u_{i-1})-f(u_i)\big) \ \text{ with } \ \beta^+_{i-1/2} = \frac{1}{2}\big(1+\text{sgn}(\lambda_{\text{Av}})\big)\end{aligned}$$

Simple analogy



Finite volumes 1D

- $u_i^{n+1} = u_i^n - \dfrac{\Delta t}{\Delta x}\left(f_{i+1/2} - f_{i-1/2}\right)$

RDS : 1d flux difference splitter

- $u_i^{n+1} = u_i^n - \dfrac{\Delta t}{\Delta x}\left(\phi^-_{i+1/2} + \phi^+_{i-1/2}\right)$

Roe linearization

$$\phi_{i-1/2} = \lambda_{\mathsf{Roe}}(u_i - u_{i-1}) = f(u_i) - f(u_{i-1}) = \int_{x_{i-1}}^{x_i} \partial_x f(u)dx$$

Approx. mean value linearization

$$\phi_{i-1/2} = \lambda_{\mathbf{z}}(\mathbf{z}_i - \mathbf{z}_{i-1}) = \int_{x_{i-1}}^{x_i} \partial_x f(z)dx + \Delta x\, R_q$$

Flux vector splitting

$$\phi_{i-1/2} = \underbrace{(\beta^+_{i-1/2} + \beta^-_{i-1/2})}_{=1}(f(u_i) - f(u_{i-1})) = \int_{x_{i-1}}^{x_i} \partial_x f dx$$

Consistency with respect to the original PDE

- Studied by Taylor expansion : OK if 1D or if the mesh has geometrical symetries, not the general case . . .
  Note that Taylor expansion says that all (most) FV scheme are not even consistent !
- More mathematical arguments : weak form

Consistency with respect to the original PDE

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  Note that Taylor expansion says that all (most) FV scheme are not even consistent !
- More mathematical arguments : weak form

Weak form

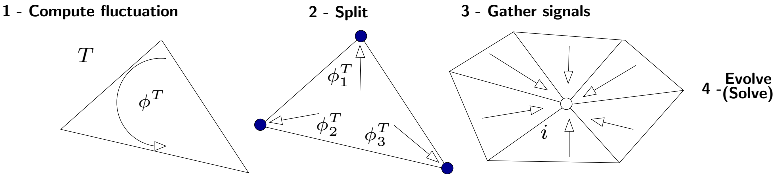
$$\text{div } \mathcal{F}(u) = 0 + \text{ BCs} \longleftrightarrow \int_\Omega \nabla\varphi \cdot \mathcal{F}(u) dx + \text{boundary terms} = 0.$$

Allows to consider easily discontinuities, more geometrical flexibility.

Conservation

Scheme :

- "Numerical" (approximation of) flux $\mathcal{F}_h(u_h)$ *continuous across edges*
- $\phi^T := \oint_{\partial T} \mathcal{F}_h(u_h) \cdot \vec{n} dl$
- $\sum_{T, M_i \in T} \phi_i^T = 0$



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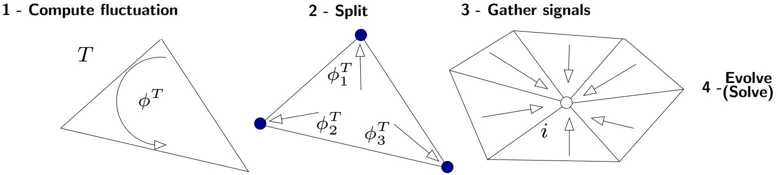
- $\sum_{vertices} \varphi(M_i) \times \left(\sum_{T, M_i \in T} \phi_i^T \right) = 0$ with $\varphi \in C_0^1(\Omega)$



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$$0 = \sum_i \varphi_i \left(\sum_{T \ni i} \phi_i^T \right) + \text{B.C.s} = \sum_T \sum_{j \in T} \varphi_j \phi_j^T + \text{B.C.s}$$

$$= \int_\Omega \varphi_h \nabla \cdot \mathcal{F}_h(u_h) + \sum_T \sum_{i,j \in T} (\varphi_i - \varphi_j)(\phi_j^T - \phi_j^{\text{Galerkin}}) + \text{B.C.s}$$

continuity of $\mathcal{F}_h$

$$= -\int_\Omega \nabla \varphi_h \cdot \mathcal{F}_h(u_h) + \sum_T \sum_{i,j \in T} (\varphi_i - \varphi_j)(\phi_j^T - \phi_j^{\text{Galerkin}}) + \text{B.C.s}$$

$$\longrightarrow -\int \nabla \varphi \cdot \mathcal{F}(u) + \text{boundary terms}$$

Conclusion

- The "conservation" conditions are the correct ones
- Details : (Abgrall & Roe, *J.Sci.Comp.* 19, 2003) ,(Abgrall & Barth, *SISC* 24, 2002)
- Continuity of the interpolant
- Residuals : net flux balance

  1D:

$$\phi = f(u_{i+1}) - f(u_i)$$

  2D:

$$\phi^T = \oint_{\partial T} \boldsymbol{\mathcal{F}}_h(u_h) \cdot \vec{n}\, dl$$

- **Remark :** the choice of $\mathcal{F}_h$ is one of the degrees of freedom :
  1. $\mathcal{F}_h = \mathcal{F}(u_h)$
  2. $\mathcal{F}_h = \mathcal{F}(p_h)$ for a given unknown (vector) p
  3. $\mathcal{F}_h = \sum_i \psi_i \mathcal{F}(u_i)$
  4. etc. etc.

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  4. etc. etc.

Issue of accuracy

What is the truncation error, which condition on the residuals ?

- By Taylor expansion : leads to nowhere, need symetry in the mesh.
- Again 'weak' form.
  Basically define an *integral truncation error*

$$\int_\Omega \nabla\varphi \cdot \boldsymbol{\mathcal{F}}(u)dx + \text{BCs} = 0 \longleftrightarrow \int_\Omega \nabla\varphi \cdot \boldsymbol{\mathcal{F}}_h(u_h)dx + \text{BCs} = \varepsilon_h$$

What is ε_h ?

Evaluation of the truncation error.

- Let w be a smooth *classical* solution :
  1. $\nabla \cdot \boldsymbol{\mathcal{F}}(w) = \partial_u \boldsymbol{\mathcal{F}}(w) \cdot \nabla w = 0$ everywhere in Ω
  2. $w - w_h = O(h^2)$, $\boldsymbol{\mathcal{F}}(w) - \boldsymbol{\mathcal{F}}_h(w_h) = O(h^2)$ in suitable norms

Basic idea : equivalent equation (integral truncation error)
(details in (Abgrall, *JCP* 167, 2001), (Ricchiuto, Abgrall, Deconinck, *JCP* 222, 2007))

$$\epsilon_h := \sum_i \varphi_i \left(\sum_{T \ni i} \phi_i^T(w_h) \right) = \ ?$$

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Basic idea : equivalent equation (integral truncation error)
(details in (Abgrall, *JCP* 167, 2001), (Ricchiuto, Abgrall, Deconinck, *JCP* 222, 2007))

$$\epsilon_h = \sum_i \varphi_i \Big(\sum_{T \ni i} \phi_i^T(w_h) \Big) = \sum_T \sum_{j \in T} \varphi_j \phi_j^T(w_h)$$

$$= -\overbrace{\int_\Omega \nabla \varphi \cdot (\boldsymbol{\mathcal{F}}_h(w_h) - \boldsymbol{\mathcal{F}}(w))}^{\textit{approximation error}} + \overbrace{\sum_T \sum_{i,j \in T} (\varphi_i - \varphi_j)(\phi_j^T(w_h) - \phi_j^{\text{Galerkin}}(w_h))}^{\textit{distribution error}}$$

Evaluation of the truncation error.

- Let w be a smooth *classical* solution :
  1. $\nabla \cdot \boldsymbol{\mathcal{F}}(w) = \partial_u \boldsymbol{\mathcal{F}}(w) \cdot \nabla w = 0$ everywhere in Ω
  2. $w - w_h = O(h^2)$, $\boldsymbol{\mathcal{F}}(w) - \boldsymbol{\mathcal{F}}_h(w_h) = O(h^2)$ in suitable norms

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$$\begin{aligned}|\epsilon_h| &\leq O(h^2) + O\big(\#\text{elements}\big) \times O(h) \times O(\phi_j^T(w_h)) \\ &= O(h^2) + O(h^{-2}) \times O(h) \times O(\phi_j^T(w_h)) \\ &= O(h^2) + O(h^{-1}) \times O(\phi_j^T(w_h))\end{aligned}$$

Sufficient condition to have $O(h^2)$ global error ϵ_h

$$\phi_j^T(w_h) = O(h^3) \text{ (local error)}$$

Final remark, why steady problems

- Consider again w, a smooth *classical* solution :
  1. $\nabla \cdot \boldsymbol{\mathcal{F}}(w) = \partial_u \boldsymbol{\mathcal{F}}(w) \cdot \nabla w = 0$ everywhere in Ω
  2. $w - w_h = O(h^2)$, $\boldsymbol{\mathcal{F}}(w) - \boldsymbol{\mathcal{F}}_h(w_h) = O(h^2)$ in suitable norms

The element residual can be estimated :

$$\begin{aligned}
\phi^T(w_h) &= \oint_{\partial T} \boldsymbol{\mathcal{F}}_h(w_h) \cdot \vec{n}\, dl \\
&= \int_{\partial T} \big(\boldsymbol{\mathcal{F}}_h(w_h) - \boldsymbol{\mathcal{F}}(w)\big) \cdot \vec{n} dl \qquad (\,\nabla \cdot \boldsymbol{\mathcal{F}}(w) = 0\,) \\
&= O(h) \times O(h^2) \qquad (\,\boldsymbol{\mathcal{F}}(w) - \boldsymbol{\mathcal{F}}_h(w_h) = O(h^2)\,) \\
&= O(h^3)
\end{aligned}$$

The LP condition for second order of accuracy

$$\phi_i^T(w_h) = \beta_i^T \; \phi^T(w_h)$$

$$\beta_i^T := \beta_i^T(w_h) \text{ uniformly bounded} \Longrightarrow |\epsilon_h| = O(h^2).$$

Summary

So far ...

Conservation :

$$\phi^T = \oint_T \boldsymbol{\mathcal{F}}_h(u_h)\cdot\vec{n}\,dl\,, \quad \sum_{j\in T}\phi_j^T = \phi^T$$

2^{nd} order accuracy : second order approximation in space ($\boldsymbol{\mathcal{F}}_h(u_h)$)

$$\phi_i^T = \beta_i^T\,\phi^T, \quad \sum_{j\in T}\beta_j^T = 1$$

$\beta_i^T := \beta_i^T(u_h)$ uniformly bounded $\Longrightarrow$ $|\text{error}| = O(h^2)$.

Nodal equation :

$$u_i^{n+1} = u_i^n - \omega_i \sum_{T|\,i\in T}\phi_i^T \xrightarrow{n\to\infty} \sum_{T|\,i\in T}\beta_i^T \oint_T \boldsymbol{\mathcal{F}}_h(u_h)\cdot\vec{n}\,dl = 0$$

Second order positive RD schemes

Positive coefficient schemes (Spekreijse, *Math.Comp.* 49, 1987)

If we can recast our prototype iterative model as

$$u_i^{n+1} = u_i^n - \omega_i \sum_{T \,|\, i \in T} \overbrace{\sum_{j \in T} c_{ij}(u_i^n - u_j^n)}^{\phi_i^T}$$

then we can easily prove that $\min_j u_j^n \leq u_i^{n+1} \leq \max_j u_j^n$ provided that

$$c_{ij} \geq 0 \quad \text{and} \quad \Delta t \leq \Delta t_{\text{lim}}$$

Unfortunately such schemes are first order accurate .....
.... unless $c_{ij} = c_{ij}(u_h)$ (Godunov's theorem !!)

Positivity preserving discretizations

Positive coefficient schemes (Spekreijse, *Math.Comp.* 49, 1987)

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{|S_i|} \sum_{T|\, i \in T} \overbrace{\sum_{j \in T} c_{ij}(u_i^n - u_j^n)}^{\phi_i^T}$$

$\min_j u_j^n \leq u_i^{n+1} \leq \max_j u_j^n$ provided that $c_{ij} \geq 0$ and $\Delta t \leq \Delta t_{\text{lim}}$

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$\min_j u_j^n \leq u_i^{n+1} \leq \max_j u_j^n$ provided that $c_{ij} \geq 0$ and $\Delta t \leq \Delta t_{\text{lim}}$

High order schemes : example

For example consider the (simple) *Lax-Friederich's* splitting

$$\phi_i^{\text{LF}}(u_h) = \frac{1}{3}\phi^T(u_h) + \alpha_{\text{LF}} \sum_{j\in T}(u_i - u_j)$$

with $c_{ij}^{\text{LF}} \geq 0$ for α_{LF} large enough, and 1st order accurate. Indeed :

$$\beta_i^{\text{LF}} = \frac{1}{3} + \frac{\alpha_{\text{LF}} \sum_{j\in T}(u_i - u_j)}{\phi^T}$$

is in general unbounded !!

Positivity preserving discretizations

High order schemes : example

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$c_{ij}^{\mathsf{LF}} \geq 0$ but β_i^{LF} is in general unbounded !!

Positivity preserving discretizations

High order schemes : example

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$$\phi_i^{\mathsf{LF}}(u_h) = \frac{1}{3}\phi^T(u_h) + \alpha_{\mathsf{LF}} \sum_{j\in T}(u_i - u_j)$$

$$c_{ij}^{\mathsf{LF}} \geq 0 \quad \text{but} \quad \beta_i^{\mathsf{LF}} \quad \text{is in general unbounded !!}$$

Idea : apply a limiter to β_i^{LF}, define the Limited Lax-Friedrich's distribution

$$\beta_i^{\mathsf{LLF}} = \frac{\psi(\beta_i^{\mathsf{LF}})}{\sum\limits_{j\in T} \psi(\beta_j^{\mathsf{LF}})}\,, \qquad \psi(r) \text{ limiter function}$$

The scaling on the denominator guarantees that $\sum_j \beta_j^{\mathsf{LLF}} = 1$.

Positivity preserving discretizations

High order schemes : example

$$\phi_i^{\mathsf{LLF}}(u_h) = \beta_i^{\mathsf{LLF}}\phi^T(u_h)\,, \quad \beta_i^{\mathsf{LLF}} = \frac{\psi(\beta_i^{\mathsf{LF}})}{\sum\limits_{j\in T}\psi(\beta_j^{\mathsf{LF}})}$$

1. β_i^{LLF} is uniformly bounded : the scheme has a $\mathcal{O}(h^2)$ truncation error
2. Provided that $\psi(r)\geq 0$ and $\dfrac{\psi(r)}{r}\geq 0$ then $\dfrac{\beta_i^{\mathsf{LLF}}}{\beta_i^{\mathsf{LF}}}\geq 0$, hence

$$\phi_i^{\mathsf{LLF}} = \beta_i^{\mathsf{LLF}}\phi^T = \overbrace{\frac{\beta_i^{\mathsf{LLF}}}{\beta_i^{\mathsf{LF}}}}^{\gamma_i\geq 0}\overbrace{\beta_i^{\mathsf{LF}}\phi^T}^{\phi_i^{\mathsf{LF}}} = \sum_{j\in T}\overbrace{\gamma_i c_{ij}^{\mathsf{LF}}}^{c_{ij}^{\mathsf{LLF}}}(u_i-u_j)\,, \qquad c_{ij}^{\mathsf{LLF}} = \gamma_i c_{ij}^{\mathsf{LF}} \geq 0!!!!$$

Summary (2)

So far ...

Conservation : $\phi^T = \oint_T \boldsymbol{\mathcal{F}}_h(u_h)\cdot\vec{n}\,dl\,,\quad \sum_{j\in T}\phi_j^T = \phi^T$

2^{nd} order and positivity : second order approximation in space ($\boldsymbol{\mathcal{F}}_h(u_h)$)

$$\phi_i^T = \beta_i^T\,\phi^T \quad \text{with} \quad \beta_i^T = \frac{\psi(\beta_i^{\text{LO}})}{\sum_{j\in T}\psi(\beta_j^{\text{LO}})}$$

$\psi(r)$ positive sign preserving limiter, β_i^{LO} from a 1st order positive scheme.

Nodal equation : $u_i^{n+1} = u_i^n - \omega_i \sum_{T|\,i\in T}\phi_i^T \xrightarrow{n\to\infty} \sum_{T|\,i\in T}\beta_i^T \oint_T \boldsymbol{\mathcal{F}}_h(u_h)\cdot\vec{n}\,dl = 0$

Remark

The limiter is used to increase the accuracy : opposite of FV schemes

Summary (2)

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Conservation : $\phi^T = \oint_T \boldsymbol{\mathcal{F}}_h(u_h) \cdot \vec{n}\, dl\,, \quad \sum_{j \in T} \phi_j^T = \phi^T$

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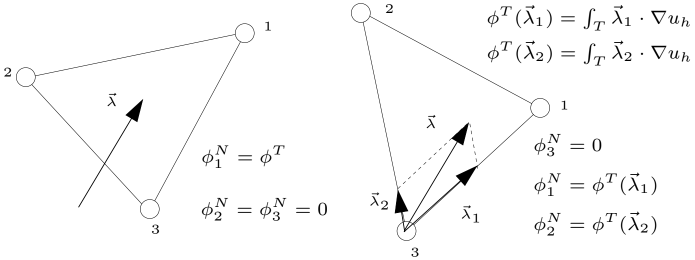
Nodal equation : $u_i^{n+1} = u_i^n - \omega_i \sum_{T|\, i \in T} \phi_i^T \overset{n \to \infty}{\longrightarrow} \sum_{T|\, i \in T} \beta_i^T \oint_T \boldsymbol{\mathcal{F}}_h(u_h) \cdot \vec{n}\, dl = 0$

Remark

The limiter is used to increase the accuracy : opposite of FV schemes

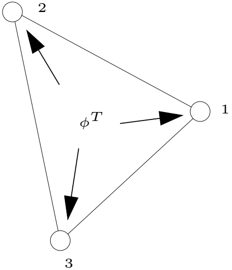
Examples



Positive schemes on P^1 meshes : the N scheme

1. Optimal Multi-D upwind scheme initially proposed by Roe
2. Positive and energy stable
3. Formal extensions on P^k meshes, $k \geq 2$, do exist but are much less effective (upwind character less pronounced, non-positive)

Examples (cont'd)



$$\phi_i^{\mathsf{LF}} = \frac{1}{3}\phi^T + \alpha_{\mathsf{LF}} \sum_{j\in T}(u_i - u_j)$$

for positivity (scalar case)

$$\alpha_{\mathsf{LF}} \geq \frac{1}{6}\, h \sup_{x\in T} \|\partial_u \boldsymbol{\mathcal{F}}(u_h(x))\|$$

Positive schemes on P^1 meshes : the LF scheme

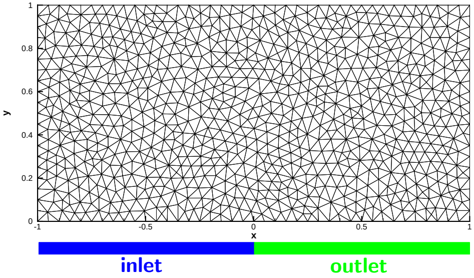
1. Central scheme : simple but extremely diffusive
2. Positive and energy stable
3. Extensions on P^k meshes trivial

Examples (cont'd)

Rotational advection

Scalar example : $\vec{\lambda}\cdot\nabla u = 0$ with $\vec{\lambda} = (y, 1-x)$ and boundary condition

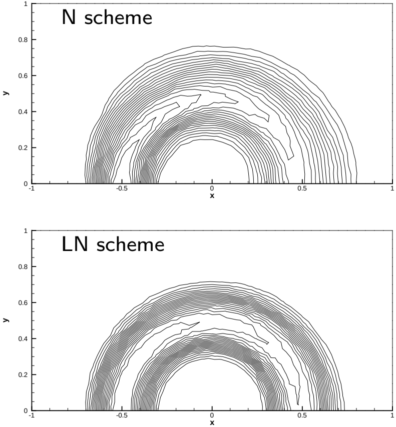
$$u_{\text{in}} = \begin{cases} \cos(2\pi(x+0.5))^2 & \text{if} \quad x \in [-0.75, -0.25] \\ 0 & \text{otherwise} \end{cases}$$

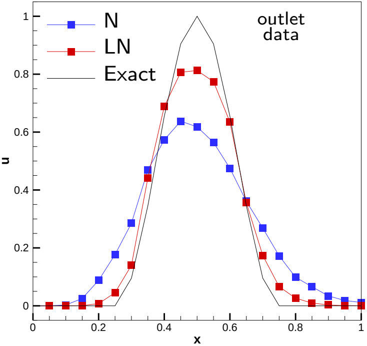


Examples (cont'd)

Rotational advection

N and Limited N (LN) schemes

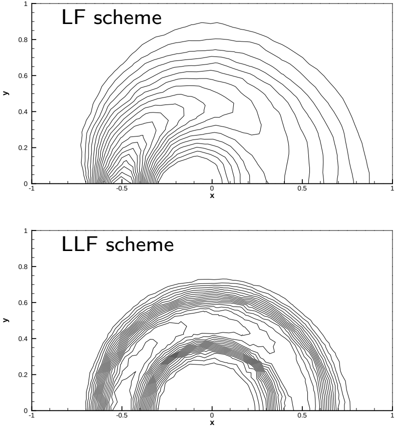


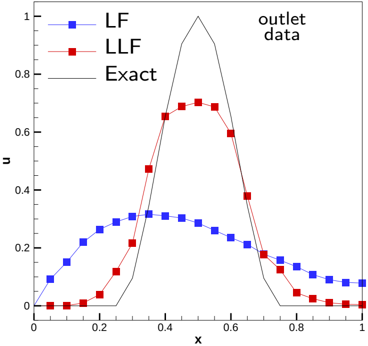


Examples (cont'd)

Rotational advection

LF and Limited LF (LLF) schemes



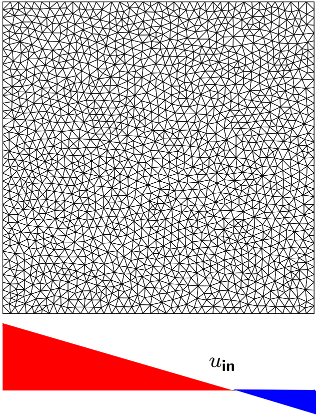


Examples (cont'd)

Burger's equation

Scalar example : $\nabla \cdot \mathcal{F}(u) = 0$ with $\mathcal{F}(u) = (u, \frac{u^2}{2})$ and boundary condition

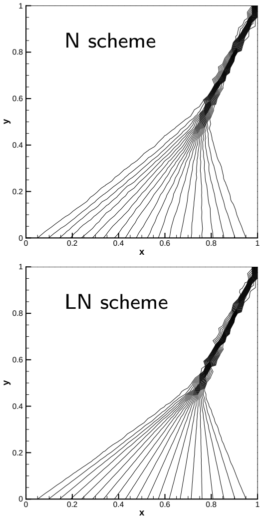
$$u_{\text{in}} = \frac{3}{2} - 2x$$

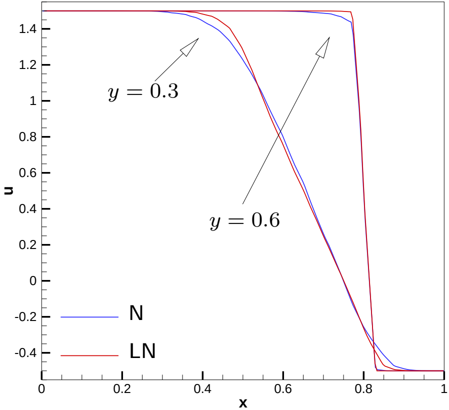


Examples (cont'd)

Burger's equation

N and Limited N (LN) schemes

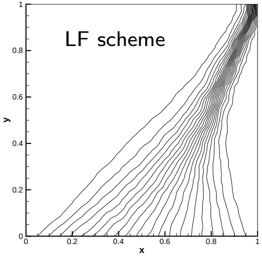


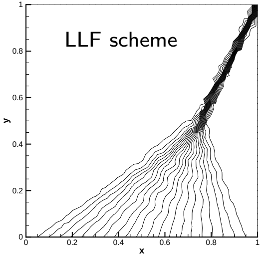


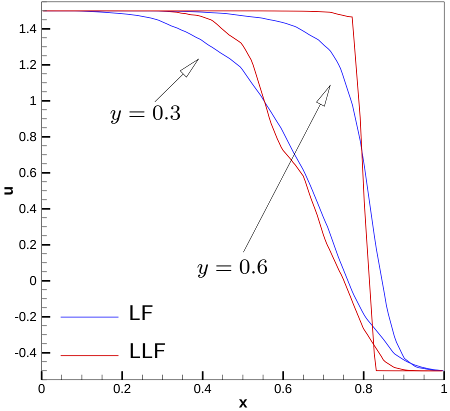
Examples (cont'd)

Burger's equation

LF and Limited LF (LLF) schemes







Summarizing

- Integral TE analysis leads to weighted residual prototype :

$$\sum_{T|i\in T} \beta_i^T \oint_{\partial T} \boldsymbol{\mathcal{F}}_h(u_h)\cdot\vec{n}\,dl = 0$$

- Use of positive coefficient schemes plus limiters to get bounded β_i^Ts
- Limiters used with opposite goal of FV : to *formally* increase accuracy

Too simple to be all of it ...

There is a catch ... which will be discussed for the higher order case

General framework and notation
The basic ingredients : second order case
General prototype
Conservation
Accuracy
Positivity
Examples
High order
Add ons
Prototype
Examples
Convergent schemes
Stabilization
Scalar results
Extension to systems
Examples
Conclusions

Overview

What is the added complexity ?

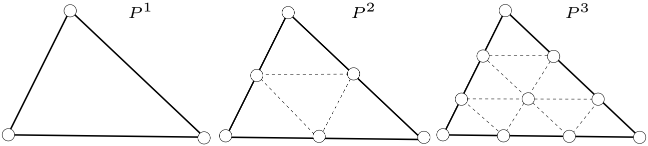
Questions to be answered

1. Polynomial approximation ?
2. Accuracy condition
3. Positive schemes

What is the added complexity ?

Questions to be answered

1. Approximation : standard P^k Lagrange finite elements



Other possibilities are being explored, *e.g.*

- Bezier polynomials (NURBS)
- std. Lagrange plus bubbles (for mass lumping + explicit time dependent)

What is the added complexity ?

Questions to be answered

1. Approximation : standard P^k Lagrange finite elements
2. Accuracy condition : generalization of integral TE analysis

What is the added complexity ?

Questions to be answered

1. Approximation : standard P^k Lagrange finite elements
2. Accuracy condition : generalization of integral TE analysis
3. Positive schemes : positive coefficient first order schemes on P^k meshes ?

Generalized prototype

1. $\forall T \in \mathcal{T}_h$ compute :

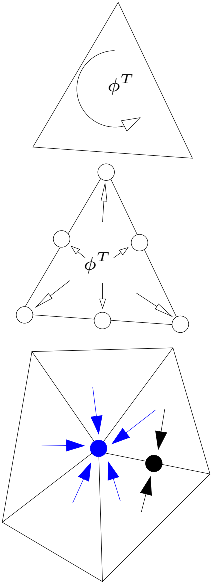
$$\phi^T = \int_T \nabla \cdot \boldsymbol{\mathcal{F}}_h(u_h)$$

2. Distribution : $\phi^T = \sum_{j \in T} \phi_j^T$

   Distribution coeff.s : $\phi_i^T = \beta_i^T \phi^T$

3. Compute nodal values : solve algebraic system

$$\sum_{T | i \in T} \phi_i^T = 0, \quad \forall i \in \mathcal{T}_h$$



Design properties

Structural conditions, basic properties

Under which conditions on the ϕ_i^Ts we get

- Correct weak solutions (if convergent with h)
- Formal k^{th} order of accuracy
- Monotonicity (discrete max priciple)
- Proper convergence ($k+1$ rates with h, and iterative convergence !)

Most (all) of the conditions seen in the P^1 case
generalize straightforwardly to P^k spatial approximation

Condition 1 : conservation

Lax-Wendroff theorem (Abgrall & Roe, *J.Sci.Comp.* 19, 2003)

(i)Technical assumptions, e.g. : continuity of ϕ_i^T, consistency of flux approximation ($\nabla \cdot \boldsymbol{\mathcal{F}}_h = 0$ and $\phi_i^T = 0$ if $u_h = c^t$).

(ii) If there is a $\boldsymbol{\mathcal{F}}_h$, continuous approximation of $\boldsymbol{\mathcal{F}}$ such that

$$\phi^T = \sum_{j\in T} \phi_j^T = \int_T \nabla \cdot \boldsymbol{\mathcal{F}}_h = \oint_{\partial T} \boldsymbol{\mathcal{F}}_h \cdot \hat{n} \qquad (5)$$

then

If a bounded sequence u_h, solution of scheme (3), converges (with h) to u $\Longrightarrow$ u is a weak solution of the problem.

Condition 1 : conservation

Remark. Conservation : 2 underlying conditions

1. Existence of continuous flux approximation $\boldsymbol{\mathcal{F}}_h$ such that

$$\phi^T = \int_T \nabla \cdot \boldsymbol{\mathcal{F}}_h = \oint_{\partial T} \boldsymbol{\mathcal{F}}_h \cdot \hat{n}$$

for example $\boldsymbol{\mathcal{F}}_h = \boldsymbol{\mathcal{F}}(u_h)$, but also $\boldsymbol{\mathcal{F}}_h = \sum_i \psi_i \boldsymbol{\mathcal{F}}_i$

2. "Consistency" relation

$$\sum_{j \in T} \phi_j^T = \phi^T$$

All as in the P^1 case

Condition 2 : accuracy

Truncation error analysis (Ricchiuto, Abgrall, Deconinck, *J.Comp.Phys* 222, 2007)

Error estimates built on variational formulation and stability analysis (coercivity) not available.

1. w smooth pointwise solution : $\nabla \cdot \mathcal{F}(w) = \partial_u \mathcal{F}(w) \cdot \nabla w = 0$
2. $w - w_h = O(h^{k+1})$, $\mathcal{F}(w) - \mathcal{F}_h(w_h) = O(h^{k+1})$ in suitable norms
3. φ a $C_0^1(\Omega)$ class function

Truncation error

Condition 2 : accuracy

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Truncation error

$$\epsilon_h = \sum_i \varphi_i \Big(\sum_{T \ni i} \phi_i^T(w_h) \Big) = \sum_T \sum_{j \in T} \varphi_j \phi_j^T(w_h)$$

$$= -\overbrace{\int_\Omega \nabla\varphi \cdot (\boldsymbol{\mathcal{F}}_h(w_h) - \boldsymbol{\mathcal{F}}(w))}^{approximation\ \ error} + \overbrace{\sum_T \sum_{i,j\in T} (\varphi_i - \varphi_j)(\phi_j^T(w_h) - \phi_j^{\text{Galerkin}}(w_h))}^{distribution\ \ error}$$

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Truncation error

Final result : the global estimate

$$|\epsilon_h| \leq C'(\mathcal{T}_h, w)\|\nabla\varphi\|_\infty \, h^{k+1}$$

holds provided that (in 2D) the local error estimate is verified

$$|\phi_i^T(w_h)| \leq C''(\mathcal{T}_h, w)h^{k+2} = O(h^{k+2})$$

Condition 2 : accuracy

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3. φ a $C_0^1(\Omega)$ class function

Truncation error

As before we can estimate $\phi^T(w_h)$:

$$\phi^T(w_h) = \oint_{\partial T} \boldsymbol{\mathcal{F}}_h(w_h) \cdot \vec{n}\, dl \overbrace{=}^{\nabla\cdot\boldsymbol{\mathcal{F}}(w)=0} \oint_{\partial T} (\boldsymbol{\mathcal{F}}_h(w_h) - \boldsymbol{\mathcal{F}}(w)) \cdot \vec{n}\, dl$$

$$= O(\boldsymbol{\mathcal{F}}_h(w_h) - \boldsymbol{\mathcal{F}}(w)) \times O(|\partial T|) \overbrace{=}^{k+1^{\text{th}}\ \text{order approx.}} O(h^{k+1}) \times O(h) = O(h^{k+2})$$

Condition 2 : accuracy

Higher order accuracy

The condition $\epsilon_h = O(h^{k+1})$ is met if

- $\phi_i^T = \beta_i^T \, \phi^T$ with β_i^T uniformly bounded distribution coeff.s
- in general if $\phi_i^T(w_h) = O(h^{k+2})$

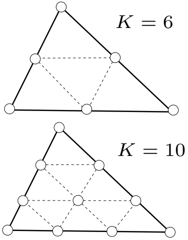
Condition 3 : monotonicity

Positive coefficient schemes (Spekreijse, *Math.Comp.* 49, 1987)

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{|S_i|} \sum_{T|\, i\in T} \overbrace{\sum_{j\in T} c_{ij}(u_i^n - u_j^n)}^{\phi_i^T}$$

$$\min_j u_j^n \leq u_i^{n+1} \leq \max_j u_j^n \quad \text{provided that } c_{ij} \geq 0 \quad \text{and} \quad \Delta t \leq \Delta t_{\text{lim}}$$

Extension of positive schemes on P^k meshes



- MultiD Upwind : either non positive, or formal and non-upwind and expensive
- We take the easiest possible choice (and least accurate)

$$\phi_i^{\text{LF}} = \frac{1}{K}\phi^T + \alpha_{\text{LF}} \sum_{j\in T}(u_i - u_j)$$

positive coefficient scheme for

$$\alpha_{\text{LF}} \geq \frac{1}{2K}\, h \sup_{x\in T} \|\partial_u \boldsymbol{\mathcal{F}}(u_h(x))\|$$

Nonlinear higher order schemes

We proceed as before

1. Evaluation of $\phi^T = \oint_{\partial T} \boldsymbol{\mathcal{F}}_h(u_h) \cdot \vec{n}\, dl$
2. Evaluation of $\phi_i^{\mathsf{LF}} = \frac{1}{K}\phi^T + \alpha_{\mathsf{LF}} \sum_{j \in T} (u_i - u_j)$
3. Limiting :

$$\beta_i^{\mathsf{LLF}} = \frac{\psi(\beta_i^{\mathsf{LF}})}{\sum_{j\in T} \psi(\beta_j^{\mathsf{LF}})} \overset{\text{in practice}}{\widehat{=}} \frac{\max(0, \beta_i^{\mathsf{LF}})}{\sum_{j\in T} \max(0, \beta_j^{\mathsf{LF}})} \quad \text{Generalized minmod}$$

4. Distribution : $\phi_i^{\mathsf{LLF}} = \beta_i^{\mathsf{LLF}} \phi^T$
5. Evolve : $u_i^{n+1} = u_i^n - \omega_i \sum_{T |\, i \in T} \phi_i^{\mathsf{LLF}} \xrightarrow{n\to\infty} \sum_{T |\, i \in T} \phi_i^{\mathsf{LLF}} = 0$

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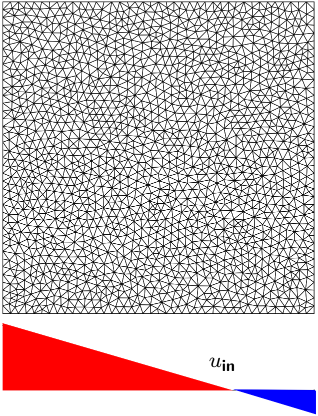
Examples (cont'd)

Burger's equation

Scalar example : $\nabla \cdot \boldsymbol{\mathcal{F}}(u) = 0$ with $\boldsymbol{\mathcal{F}}(u) = (u, \frac{u^2}{2})$ and boundary condition

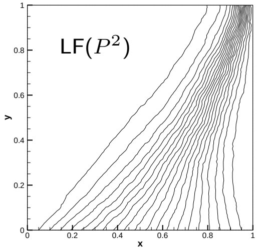
$$u_{\text{in}} = \frac{3}{2} - 2x$$

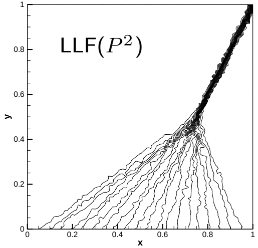


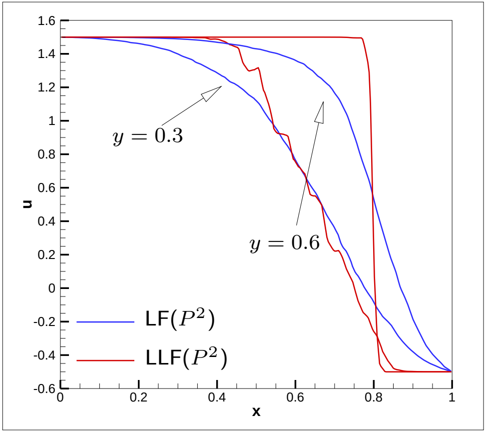
Examples (cont'd)

Burger's equation

LF and Limited LF (LLF) schemes on P^2 elements





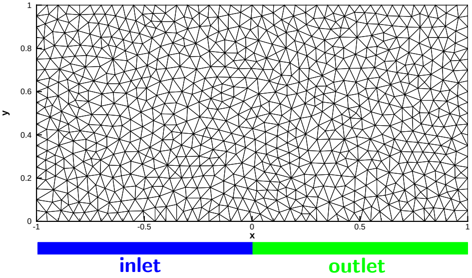


Examples (cont'd)

Rotational advection

Scalar example : $\vec{\lambda} \cdot \nabla u = 0$ with $\vec{\lambda} = (y, 1-x)$ and boundary condition

$$u_{\text{in}} = \begin{cases} \cos(2\pi(x+0.5))^2 & \text{if} \quad x \in [-0.75, -0.25] \\ 0 & \text{otherwise} \end{cases}$$



Examples (cont'd)

Rotational advection

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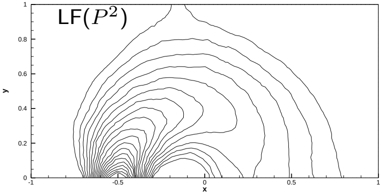
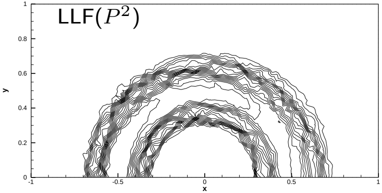
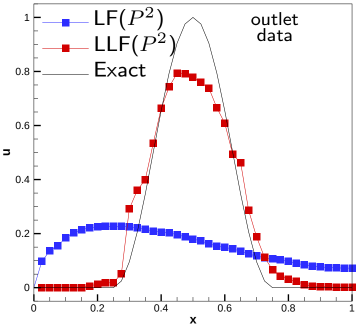


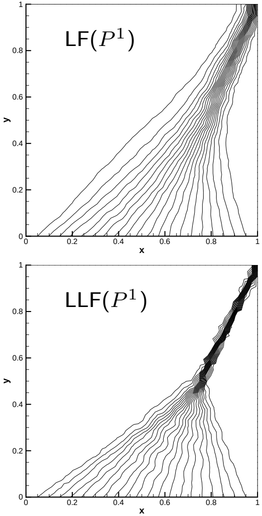


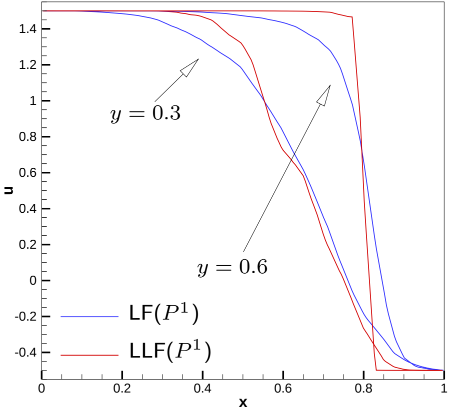


Examples (cont'd)

Burger's equation

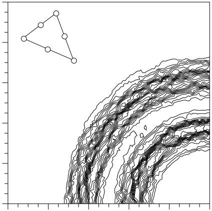
LF and Limited LF (LLF) schemes on P^1 elements





Overview

Smooth solutions and spurious modes



Symptoms

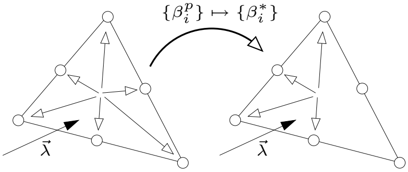
Shocks (nonlinear) : Monotone capturing. Kept in 1 or 2 cells, no staircases ;

Smooth sol.s Lack of smoothness, staircase structure ;

Contacts (linear) : Monotone capturing. Spread over several cells, and then same as smooth parts ;

Convergence Poor iterative convergence (smooth sol.s)
$\Rightarrow$ Poor grid convergence (1^{st} order at most)

Smooth solutions and spurious modes



Analysis (Abgrall, *JCP* 214, 2006) : smooth areas where $\phi^T = \mathcal{O}(h^{k+2}) \ll 1$

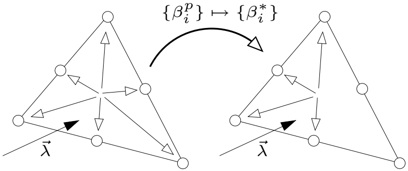
- Linearize the nonlinear stady state system $\sum_{T|i\in T} \phi_i^T = 0$: $\mathsf{M}_h^* \mathbf{u} = B_h^*$

M_h^* does not have full range : infinite solutions, hence spurious modes

Another way to see it (*Out the door, back through the window...*)

- The construction is based on the constraint $\phi_j^{\mathsf{LF}} \times \beta_j^{\mathsf{LLF}} \phi^T \geq 0$
- Upwinding not included in the process
- Locally can have "down-winding" or zero entries in equation (as central scheme and advection)

Smooth solutions and spurious modes



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"Stabilization" via streamline dissipation

A solution (Abgrall, *J.Comp.Phys.* 214, 2006)

Add upwind biasing/energy stabilizing Streamline Dissipation (SD) term

$$\phi_i^{\mathsf{LLFs}} = \beta_i^{\mathsf{LLF}}\phi^T + \theta(u_h)\int_T (\vec{\lambda}\cdot\nabla\psi_i)\,\tau_{\mathsf{SD}}\,(\vec{\lambda}\cdot\nabla u_h)$$

with ψ_i Lagrange basis fcn. of node i

Properties, implementation

1. How costly is it ?
2. What is $\theta(u_h)$?

"Stabilization" via streamline dissipation

Simplified integration

The role of the SD term is to add NRG dissipation, *i.e.* the bilinear form

$$b_{\mathsf{SD}}(\psi_i, u_h) = \theta(u_h) \int_T (\vec{\lambda} \cdot \nabla \psi_i)\, \tau_{\mathsf{SD}}\, (\vec{\lambda} \cdot \nabla u_h) \quad \text{is positive semidefinite}$$

In other words

$$b_{\mathsf{SD}}(u_h, u_h) = \theta(u_h) \int_T \tau_{\mathsf{SD}}\, (\vec{\lambda} \cdot \nabla u_h)^2 > 0 \;\; \text{whenever} \;\; \vec{\lambda} \cdot \nabla u_h = \nabla \cdot \boldsymbol{\mathcal{F}} \neq 0$$

“Stabilization” via streamline dissipation

Simplified integration

Idea (Abgrall, Larat, Ricchiuto *Comp.&Fluids* 38, 2009) : replace b_{SD} by

$$b^h_{\mathsf{SD}}(\psi_i, u_h) = \theta(u_h) \sum_{j=1}^{\mathsf{G}_p} \omega_j \, |T| (\vec{\lambda} \cdot \nabla \psi_i(x_j)) \tau_{\mathsf{SD}} (\vec{\lambda} \cdot \nabla u_h(x_j))$$

Choice of (x_j, ω_j) : constraints/properties

1. For a smooth exact solution w : $b^h_{\mathsf{SD}}(\psi_i, w_h) = O(h^{k+2})$ independently of (x_j, ω_j)!

Formal accuracy never spoiled independently of this choice

"Stabilization" via streamline dissipation

Simplified integration

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Choice of (x_j,ω_j) : constraints/properties

1. Formal accuracy never spoiled independently of this choice
2. Dissipation : since $\theta\geq 0$ and $\tau_{\mathsf{SD}}\geq 0$, the condition

$$b^h_{\mathsf{SD}}(u_h,u_h)\geq 0 \iff \vec{\lambda}\cdot\nabla u_h=0$$

is equivalent to

$$\vec{\lambda}\cdot\nabla u_h=0 \iff \vec{\lambda}\cdot\nabla u_h(x_j)=0\;\forall j$$

- We need a sufficient number of "non-colinear" points such that the hyper-plane $\vec{\lambda}\cdot\nabla u_h$ of dimension $k-1$ is uniquely defined.
- Only constraint on weights : $\omega_j\geq 0 \implies$ we take $\omega_j=1$!!

"Stabilization" via streamline dissipation

Simplified integration

Idea (Abgrall, Larat, Ricchiuto *Comp.&Fluids* 38, 2009) : replace b_{SD} by $(\omega_j = 1)$

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number of "non-colinear" points

2D			3D		
k	# simpl. points	exact	k	# simpl. points	exact
1	1	1	1	1	1
2	3	3	2	4	4
3	6	6	3	10	15

exact : exact integration formulas with non-negative weights from (Dunavant, *IJNME* 21, 1985), (Jinyun, *CMAME* 43, 1984), and (Keast, *CMAME* 55, 1986)

Not an incredible reduction in number of points..

"Stabilization" via streamline dissipation

Simplified integration

Idea (Abgrall, Larat, Ricchiuto *Comp.&Fluids* 38, 2009) : replace b_{SD} by ($\omega_j = 1$)

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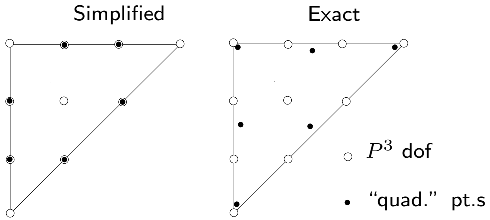
Real simplification : use available data (no recontruction). Example :



Exact : exact integration formula with non-negative weights from (Dunavant, *IJNME* 21, 1985)

"Stabilization" via streamline dissipation

Smoothness sensor θ

The sensor $\theta \geq 0$ has the role of switching off the SD term in shocks.
Definition used in all computations (Abgrall, *JCP* 214, 2006) :

$$\theta(u_h) = \min(1, \frac{\|\vec{\lambda}\|_T \|u_h\|_T h^2}{|\phi^T|})$$

Principle :

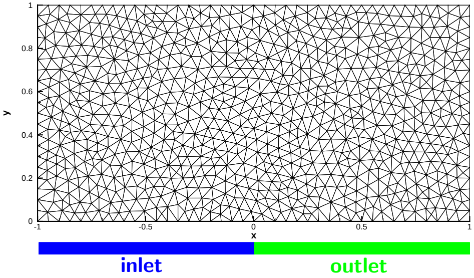
- Across shocks : $\phi^T = O(h) \Longrightarrow \theta = O(h)$
- Smooth solutions : $\phi^T = O(h^{k+2}) \Longrightarrow \theta = 1$

Examples (cont'd)

Rotational advection

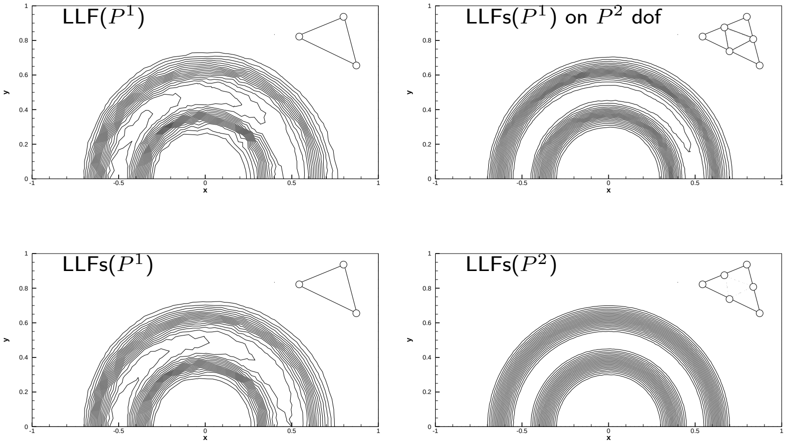
Scalar example : $\vec{\lambda} \cdot \nabla u = 0$ with $\vec{\lambda} = (y, 1-x)$ and boundary condition

$$u_{\text{in}} = \begin{cases} \cos(2\pi(x+0.5))^2 & \text{if } \quad x \in [-0.75, -0.25] \\ 0 & \text{otherwise} \end{cases}$$



Examples (cont'd)

Rotational advection : Limited LF (LLF) scheme



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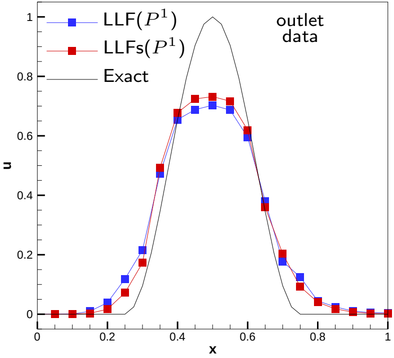
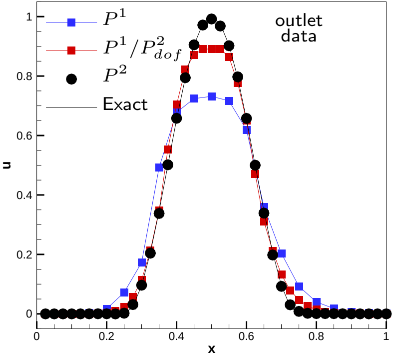
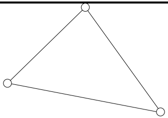
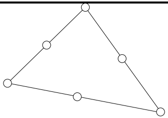
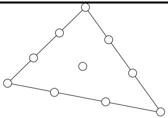
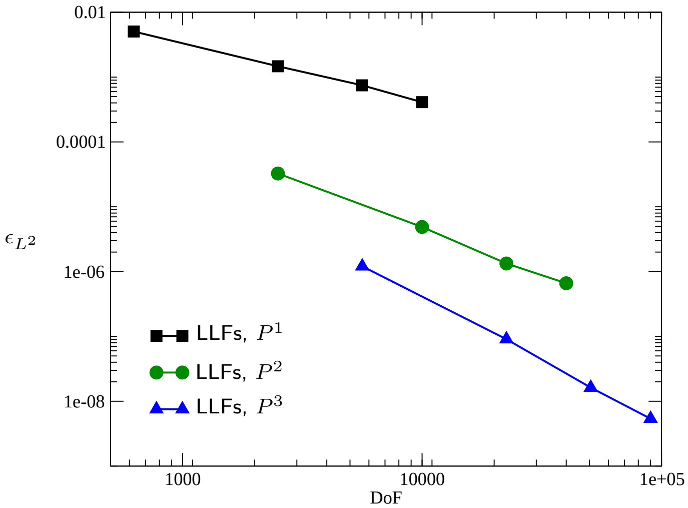




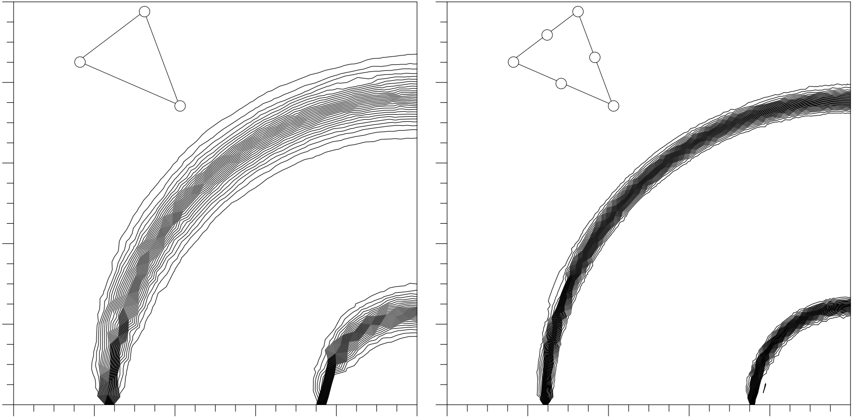
Grid convergence

			
h	$\epsilon_{L^2}(P^1)$	$\epsilon_{L^2}(P^2)$	$\epsilon_{L^2}(P^3)$
$1/25$	0.50493E-02	0.32612E-04	0.12071E-05
$1/50$	0.14684E-02	0.48741E-05	0.90642E-07
$1/75$	0.74684E-03	0.13334E-05	0.16245E-07
$1/100$	0.41019E-03	0.66019E-06	0.53860E-08
	$\mathcal{O}^{\text{ls}}_{L^2}=$ **1.790**	$\mathcal{O}^{\text{ls}}_{L^2}=$ **2.848**	$\mathcal{O}^{\text{ls}}_{L^2}=$ **3.920**

Grid convergence : error *vs* DoF



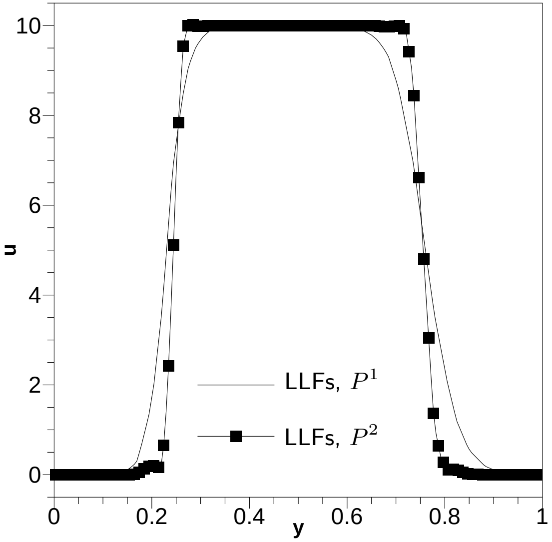
Rotation of a top hat



LLFs scheme, P^1 interpolation LLFs scheme, P^2 interpolation

Contact in spread on same numer of DoF (fewer cells in P^2 case)

Rotation of a top hat : outlet profile



Examples (cont'd)

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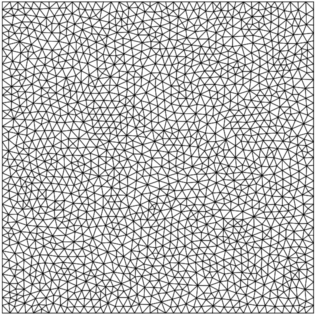
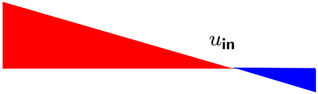
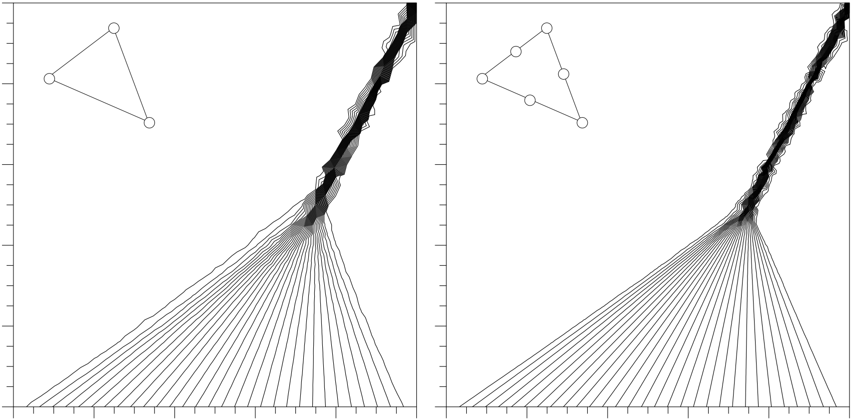




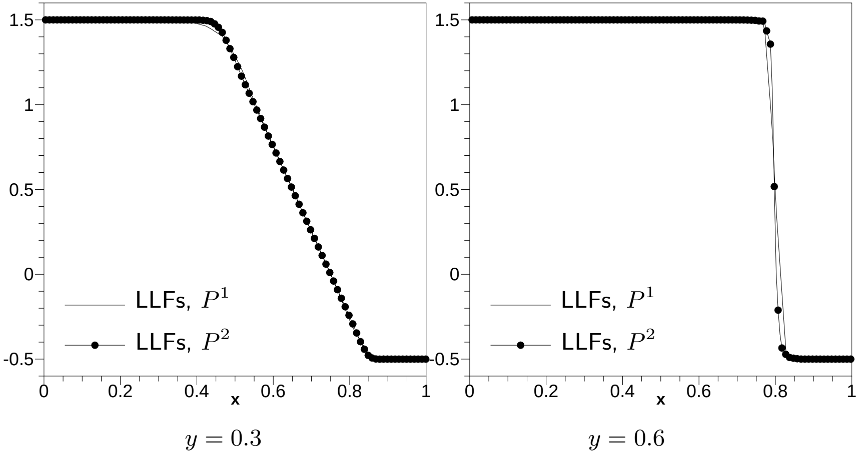
Numerical example : Burger's eq.n



LLFs scheme, P^1 interpolation   LLFs scheme, P^2 interpolation

Shock captured in 1 or 2 cells (more DoF in P^2 case)

Burger's eq.n : cuts at $y = 0.3$ and $y = 0.6$



Overview

Extension to systems

$$\nabla \cdot \overbrace{\begin{bmatrix} \rho\vec{u} \\ \rho\vec{u}\otimes\vec{u} + p\mathbf{I}_2 \\ \rho H\vec{u} \end{bmatrix}}^{\boldsymbol{\mathcal{F}}} = 0 \qquad + \qquad \text{BCs}$$

- Same framework : distribute residual **vector** $\phi^T = \oint_{\partial T} \boldsymbol{\mathcal{F}}_h \cdot \vec{n}\, dl$
- Conservation/LW theorem : satisfied as long as ϕ^T computed as above
- Accuracy : TE analysis formally identical, same accuracy conditions
- Streamline Dissipation term as in SUPG schemes (formal extension) plus simplified quadradure

A few remarks on

1. What is a positive coefficient for a system
2. How do we perform the limiting

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What is a positive coefficients scheme for a system ?

What is a positive coefficients scheme for a system ?

Wave decomposition (Abgrall, Mezine *JCP* 195, 2004)

Assume that the solution vector is a simple wave :

$$\mathbf{u} = u^k \mathbf{r}_k^\xi$$

with $\mathbf{r}_k^\xi$ and eigenvector of

$$K_\xi = \partial_u \boldsymbol{\mathcal{F}} \cdot \xi \quad \xi \in \mathbb{R}^2$$

For the LF scheme, and for a symmetric linear system, one can show that

$$\phi_i^{\text{LF}} = \left[\sum_{j \in T} c_{ij}^k (u_i^k - u_i^k) \right] \mathbf{r}_k^\xi \quad \text{with} \quad c_{ij}^k \geq 0$$

and that the wave strength u^k remains bounded (Abgrall, Mezine *JCP* 195, 2004).

What is a positive coefficients scheme for a system ?

Wave decomposition (Abgrall, Mezine *JCP* 195, 2004)

Framework justifies limiting via characteristic projection :

- let $\{\mathbf{r}_k^\xi\}_{k=1}^{\mathsf{n}_{\mathsf{eq.s}}}$ and $\{\mathbf{l}_k^\xi\}_{k=1}^{\mathsf{n}_{\mathsf{eq.s}}}$ be the right and left eigenvectors of K_ξ
- for $k=1, \mathsf{n}_{\mathsf{eq.s}}$ do
  1. define : $$\varphi^k = \mathbf{l}_k^\xi \cdot \phi^T \in \mathbb{R}$$
  2. for all dof $j \in T$ define : $$\varphi_j^{\mathsf{LF}} = \mathbf{l}_k^\xi \cdot \phi_j^{\mathsf{LF}} \in \mathbb{R}$$
  3. for all dof $j \in T$: $$\beta_j^{\mathsf{LLF}} = \frac{\psi(\beta_i^{\mathsf{LF}})}{\sum_{i\in T}\psi(\beta_i^{\mathsf{LLF}})} \quad \text{with } \beta_j^{\mathsf{LF}} = \varphi_j^{\mathsf{LF}}/\varphi^k$$
  4. for all dof $j \in T$: $$\varphi_j^{\mathsf{LLF}k} = \beta_j^{\mathsf{LLF}}\varphi^k$$
- for all dof $j \in T$ set :

$$\phi_j^{\mathsf{LLF}} = \sum_{k=1}^{\mathsf{n}_{\mathsf{eq.s}}} \varphi_j^{\mathsf{LLF}k}\mathbf{r}_k^\xi$$

What is a positive coefficients scheme for a system ?

Thermodynamics

Positivity of density and pressure

$$\rho \geq 0 \quad , \quad p \geq 0$$

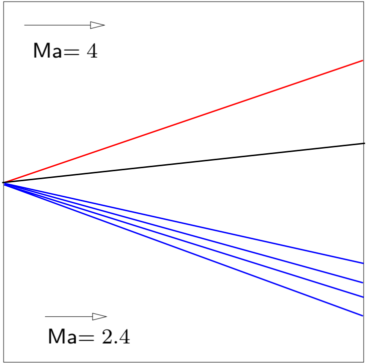
- satisfied by the LF scheme
- $\rho \geq 0$ satisfied by LLF is the limiting is performed eq.n by eq.n.

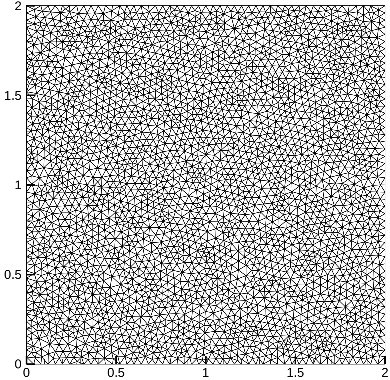
What is a positive coefficients scheme for a system ?

Projection or eq. by eq. ?

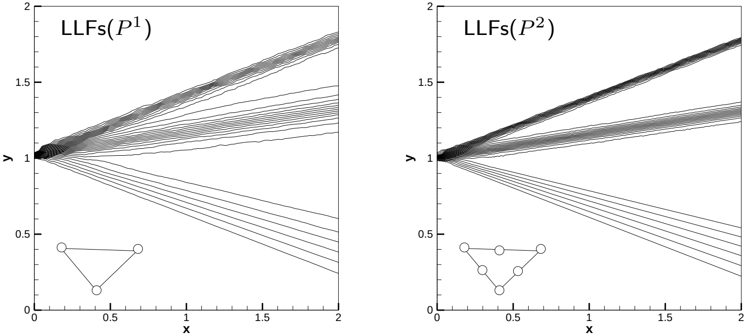
Behavior same of FV schemes plus slope limiters

Jet interaction



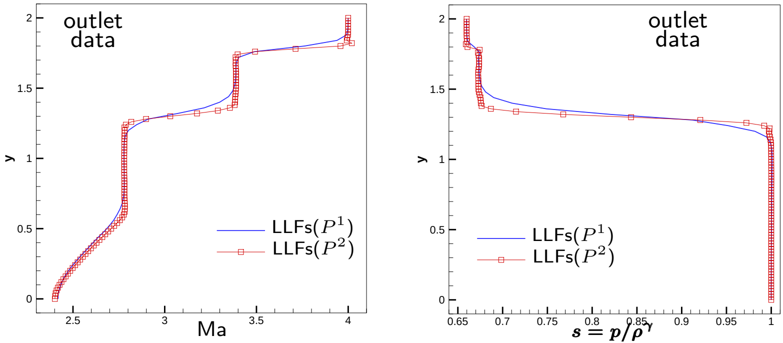


Jet interaction



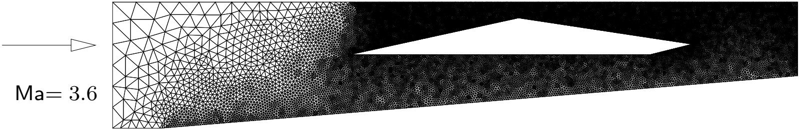
Mach number contour plot (30 levels)

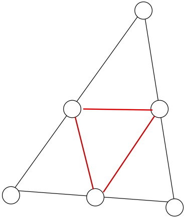
Jet interaction

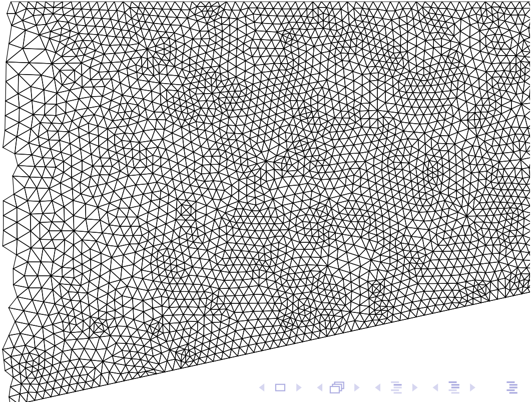


Mach and entropy at the outlet

Scramjet inlet

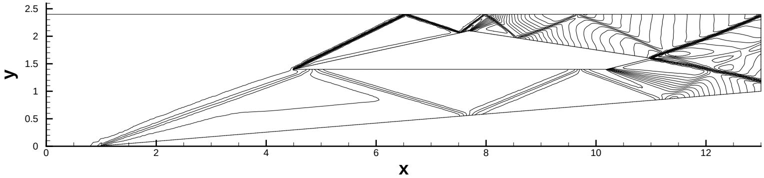




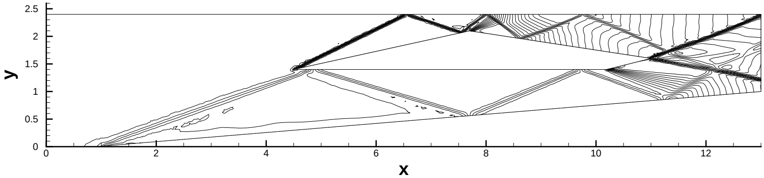


Scramjet inlet

LLFs(P^1/P^2_{dof})

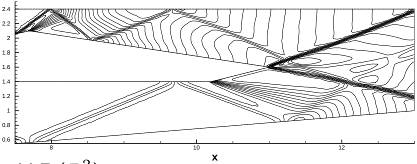


LLFs(P^2)

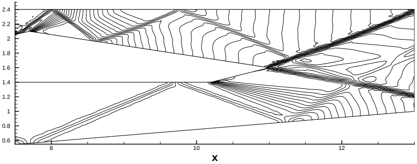


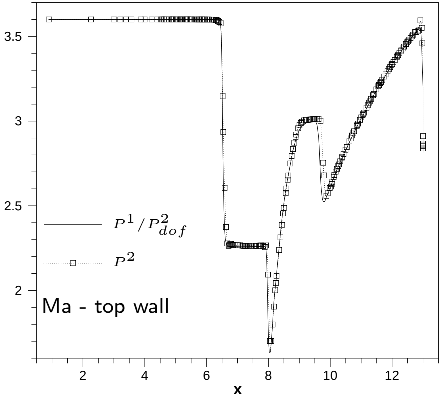
Scramjet inlet

LLFs(P^1/P^2_{dof})



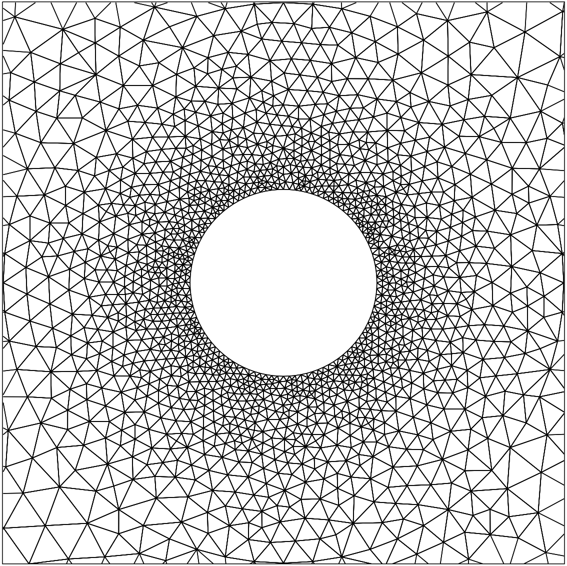
LLFs(P^2)





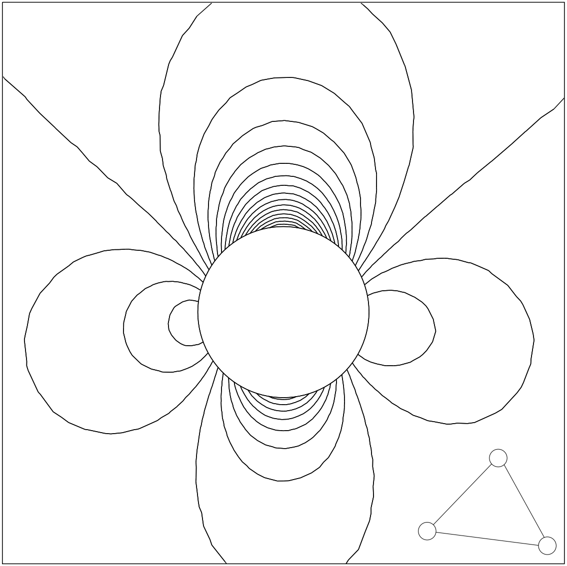
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
Mesh :
2719 nodes
5308 elements
100 nodes
on cylinder



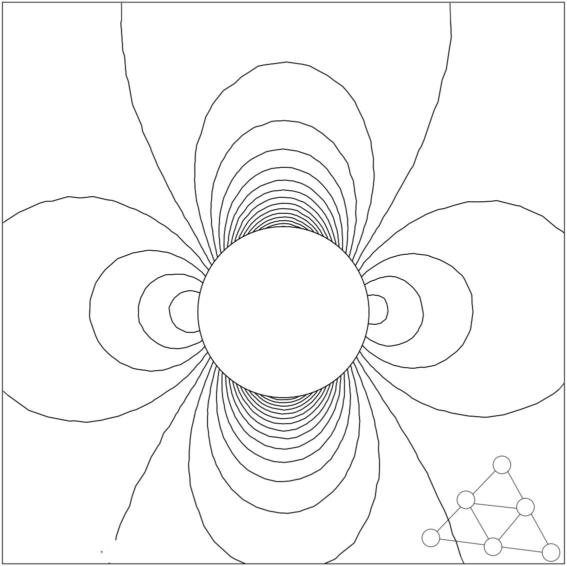
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
LLFs scheme
P^1 elements :
pressure



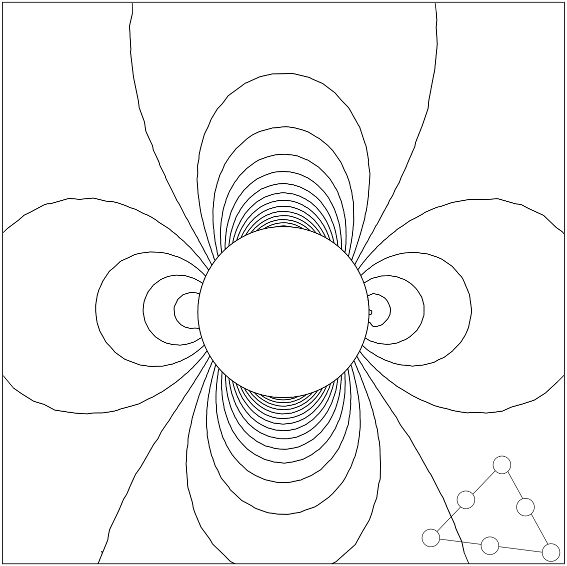
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
LLFs scheme
P^1 elements
P^2 conformal
sub-triangulation :
pressure



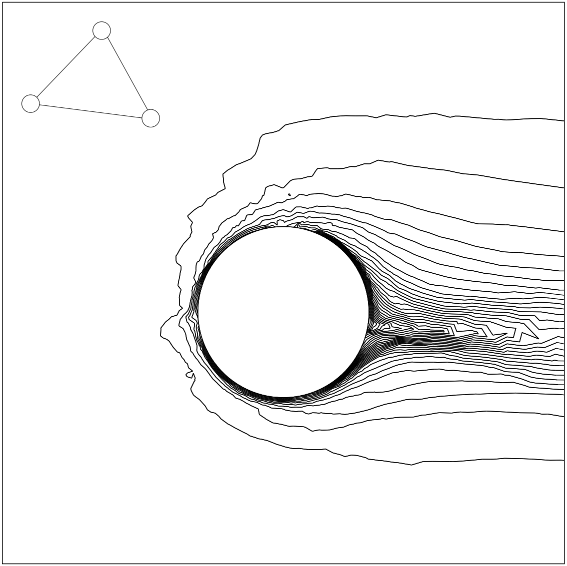
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
LLFs scheme
P^2 elements :
pressure
(linear
boundary
representation)



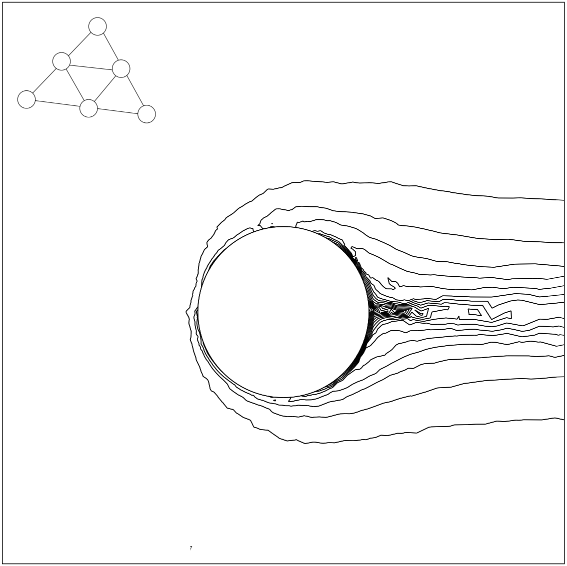
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
LLFs scheme
P^1 elements :
entropy



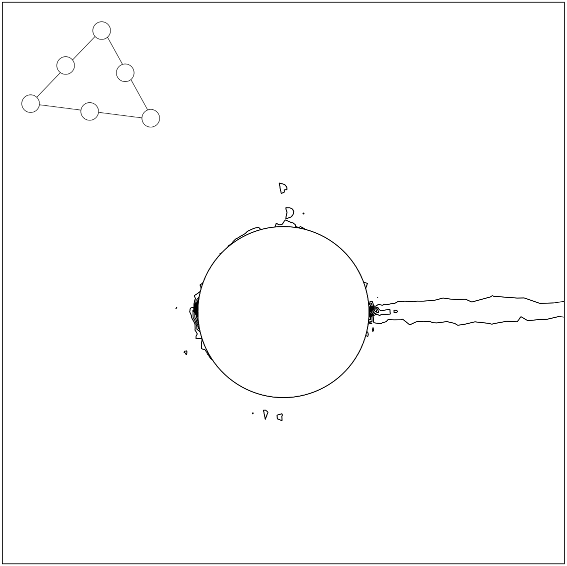
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
LLFs scheme
P^1 elements
P^2 conformal
sub-triangulation :
entropy



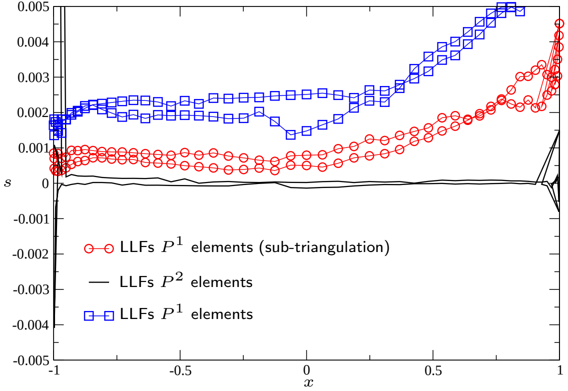
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
LLFs scheme
P^2 elements :
entropy
(linear
boundary
representation)

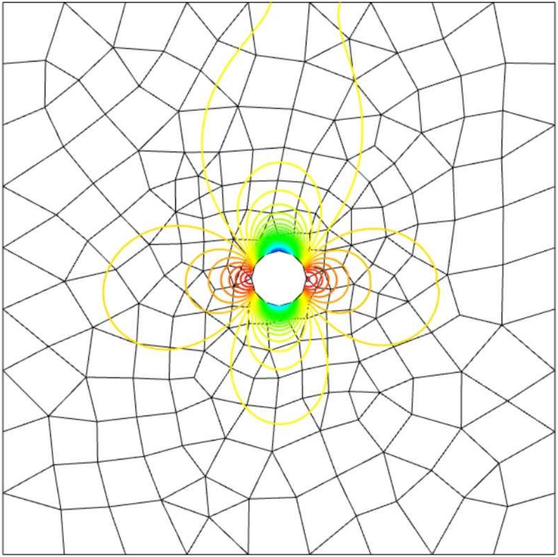


$Ma = 0.35$ cylinder flow : entropy distribution

$Ma = 0.35$
flow on cylinder
LLFs scheme :
entropy
on the cylinder



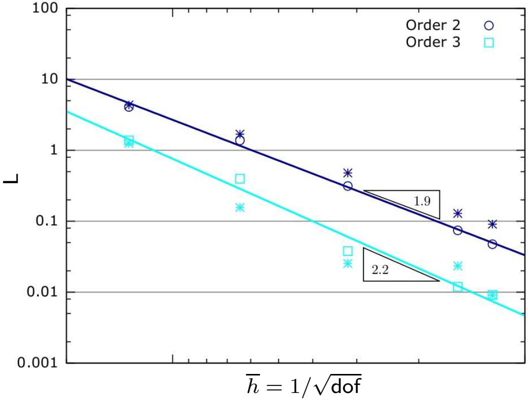
Example of hybrid gird



Grid convergence : Lift

Second order : ok
Third order :
Error much lower
Rate not too good

Linear representation of the boundary not enough.
Need isoparametric elements.



Overview

Conclusions

Topics not covered in the talk

- Time-dependent (some info in tomorrow's talk on Shallow Water)
- Source terms (see tomorrow's talk on Shallow Water)
- Viscous terms
- mesh/polynomial adaptation
- isoparametric and other polynomial choices
- application to other systems : NS, Shallow Water, MHD