Stabilized Residual Distribution for Shallow Water Simulations

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Numerical approximations of
hyperbolic systems with source terms
and applications

Castro-Urdiales, September 2009

Acknowledgements

Rémi Abgrall (INRIA, U. Bordeaux I)

Residual Distribution schemes, initial work on SWE

Andreas Bollermann (RWTH Aachen)

Shallow water simulations with dry states

Survey

Residual Distribution

Date back to ideas of (P.L.Roe, *Num. Meth. Fluid Dyn. 1982*) : decompose local numerical error (fluctuation) in signals sent to nodes to evolve local value of the solution

- Multidimensional upwinding (80' and 90'). Roe (Michigan U. Ann Arbor), Deconinck (von Karman I.), Hubbard (Leeds U.), Napolitano (Politec. Bari) :
  1. Decomposition of Q-linear form in decoupled hyperbolic components
  2. Each scalar hyperbolic component discretized using MU technique

  Well adapted to steady supersonic, MU in sub-critical case with inexact decompositions (formal continuation), Roe linearization, no unsteady.
- Last 10 years. The above plus Abgrall (INRIA), Barth (NASA), Shu (Brown U.) :
  1. High order for time-dependent (consistent treatment of time derivative)
  2. Conservation without Roe linearization
  3. General construction of non-oscillatory schemes for steady/unsteady
  4. More than second order and dicontinuous approximation

  With generalization the idea of MU and the characteristic decompositions are playing a smaller role (matrix formulation)

Survey (cont'd)

Application to SWE (source terms)

Several publications on the subject :

- (P.Garcia-Navarro, M.E.Hubbard, A.Priestley *JCP 121,1995*), (H.Paillere, G.Degrez, H.Deconinck *I.J.N.M.F 26,1998*) : decompositions and MU for SWE. No accent source terms, non-conservative, 1^{st} order for unsteady.
- (M.E.Hubbard, M.J.Baines *JCP 138,1997*) : *ad-hoc* conservative correction, optimal decompositions for MU and hyperbolic elliptic splitting, source terms via both residual and pointwise approach. No unsteady, nothing on well-balancedness.
- (P.Brufau, P.Garcia-Navarro *JCP 186,2003*) : MU treatment of bed slope included in previously developed decompositions, C-property without/with dry states. Non-conservative, 1^{st} order for unsteady, no general framework for C-property.

Survey (cont'd)

Application to SWE (source terms)

- (M.Ricchiuto, R.Abgrall, H.Deconinck *JCP 222,2007*)
- (M.Ricchiuto, A.Bollermann HYP08, Maryland 2008), (M.Ricchiuto, A.Bollermann *JCP 228,2009*)

1. General conservative approach
2. No wave decompostition/MU : matrix approach, *positivity preserving* central scheme
3. More general analysis for C-property and accuracy in precence of sources
4. Time dependent

Of course benefitting of previous work on both RD and SWE subjects !

The talk summarizes the content of these papers, describing step by step the construction of the discretization.

Outline of the talk

Part 1. General framework, RD prototype

Part 2. RD related issues : construction of a high order scheme

Part 3. SWE related issues : C-property and depth positivity

MATHEMATICAL SETTING AND NOTATION

Mathematical setting and basic notation

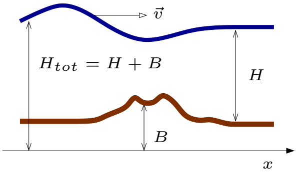
$$\partial_t u + \nabla \cdot \mathcal{F}(u) + \mathcal{S}(u, x, y) = 0$$

or

$$r(u, x, y) = 0$$

Example : SWE

$$\partial_t \overbrace{\begin{bmatrix} H \\ H\vec{v} \end{bmatrix}}^{u} + \nabla \cdot \underbrace{\begin{bmatrix} H\vec{v} \\ H\vec{v} \otimes \vec{v} + \frac{g}{2} H^2 \mathbf{I} \end{bmatrix}}_{\mathcal{F}(u)} + \overbrace{gH\nabla \begin{bmatrix} 0 \\ B \end{bmatrix}}^{\mathcal{S}(u,x,y)} = 0$$



Mathematical setting and basic notation

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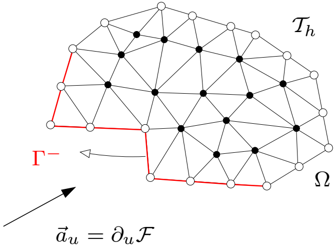

Some notation

- $\mathcal{T}_h = \bigcup T$ space triangulation
- Unknowns : $u_i^n \approx u(t^n, x_i, y_i)$ collocated solution values
- u_h continuous piecewise polynomial interpolation of a quantity u

$$u_h = \sum_{i \in \mathcal{T}_h} \psi_i \, u_i$$

- The ψ_is are continuous basis functions
- In the talk only linear interpolation (however most of the content generalizes)

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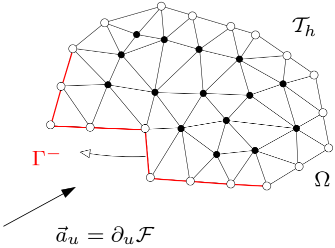

Some notation (cont'd)

$r_h(u_h, x, y)$ discrete approximation of the operator based on

- Polynomial approximation of the unknown u_h
- Discrete approximation of (nonlinear) flux $\mathcal{F}_h(u_h)$ and source $\mathcal{S}_h(u_h, x, y)$
- Disrete approximation of the time derivative

Example :

$$r_h(u_h, x, y) = \frac{u_h^{n+1} - u_h^n}{\Delta t} + \nabla \cdot \mathcal{F}_h^n + \mathcal{S}_h(u_h^n, x, y)$$

with $\mathcal{F}_h^n \sum_i \psi_i \mathcal{F}(u_i)$ and $\mathcal{S}_h(u_h^n, x, y) = \sum_i \psi_i \mathcal{S}(u_i^n, x_i, y_i)$.

$$\partial_t u + \nabla \cdot \mathcal{F}(u) + \mathcal{S}(u,x,y) = 0$$

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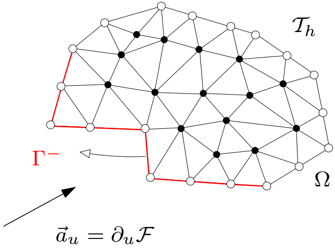


Some notation (cont'd)

- Cell residual :

$$\Phi^T(u_h) = \int_T r_h(u_h,x,y)\,dx$$

- Cell fluctuation :

$$\phi^T(u_h) = \int_T \left(\nabla \cdot \mathcal{F}_h(u_h) + \mathcal{S}_h(u_h,x,y)\right)\,dx$$

They coincide when considering the steady state.

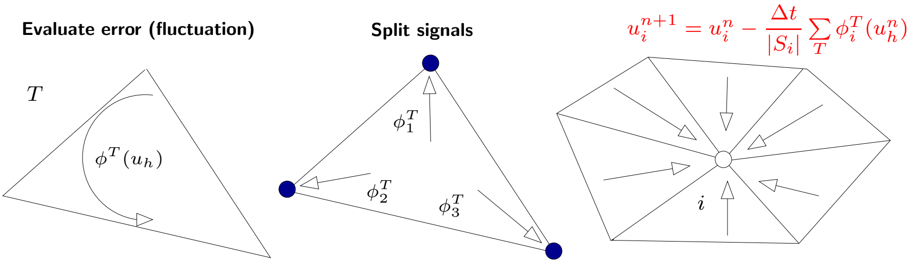
THE GENERAL PROTOTYPE

General prototype : steady case

Residual Distribution (P.L.Roe, *Num. Meth. Fluid Dyn. 1982*)

On every triangle T define split residuals $\phi_j^T(u_h)$ such that

$$\sum_{j\in T}\phi_j^T(u_h)=\phi^T(u_h)$$



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assemble and evolve :

$$\left.\begin{array}{l} u_i^{n+1}=u_i^n-\dfrac{\Delta t}{|S_i|}\displaystyle\sum_{T|i\in T}\phi_i^T(u_h^n) \\ \\ \text{repeat until steady state ....}\end{array}\right\}\equiv \sum_{T|i\in T}\phi_i^T(u_h)=0$$

Conservation, LW theorem for RD and $\nabla\cdot\mathcal{F}(u)=0$

Under some (standard) continuity assumptions on $\phi^T(u_h)$ and $\phi_i^T(u_h)$ the discrete solution u_h converges (if convergent !) to a weak solution u of the continuous problem, provided that (Abgrall, Roe *J.Sci.Comput.* 19, 2003) :

$$\phi^T(u_h)=\oint_{\partial T}\mathcal{F}_h(u_h)\cdot\hat{n}\,dl$$

for *some* continuous spatial approximation of the flux $\mathcal{F}$

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$$\phi^T(u_h)=\oint_{\partial T}\mathcal{F}_h(u_h)\cdot\hat{n}\,dl$$

for *some* continuous spatial approximation of the flux $\mathcal{F}$

At least $\mathcal{F}_h=\sum_{i\in\mathcal{T}_h}\mathcal{F}_i\,\psi_i$

Conservation

Definition (steady homogeneous case : $\nabla \cdot \mathcal{F}(u) = 0$)

Conservation is equivalent to the following condition :

$$\sum_{j \in T} \phi_j^T(u_h) = \oint_T \mathcal{F}_h(u_h) \cdot \hat{n}\, dl$$

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How to get it (1). "Traditional choice" : Roe linearization

Look for a parameter vector $\mathbf{z}$ such that if $\mathcal{F}_h(u_h) = \mathcal{F}(\mathbf{z}_h)$, with $\mathbf{z}_h$ linear, then

$$\oint_T \mathcal{F}_h(u_h) \cdot \hat{n}\, dl = \int_T \nabla \cdot \mathcal{F}(\mathbf{z}_h)\, dx = \int_T \partial_{\mathbf{z}} \mathcal{F}(\mathbf{z}_h) \cdot \nabla \mathbf{z}_h = |T| \partial_{\mathbf{z}} \mathcal{F}(\overline{\mathbf{z}}) \cdot \nabla \mathbf{z}_h|_T$$

with $\overline{\mathbf{z}}$ a simple (arithmetic) average of the nodal values in element T.

1. direct use of the quasi-linear form for : wave decompositions, multi-D upwinding
2. mainly Euler perfect gas (Deconinck, Roe, Struijs *Comp.&Fluids* 22, 1993)
3. the alternative is to evaluate exactly the mean value Jacobian

$$\overline{\partial_{\mathbf{z}} \mathcal{F}} = \frac{1}{|T|} \int_T \partial_{\mathbf{z}} \mathcal{F}(\mathbf{z}_h)\, dx$$

for a given set of variables $\mathbf{z}$.

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for some continuous discrete approximation of the flux.

How to get it (2) : "traditional choice", an approximation

Proposed in (Abgrall, Barth *SISC* 24, 2002) :

$$\overline{\partial_{\mathbf{z}} \mathcal{F}} = \sum_{q=1}^{\mathsf{G}_p} \omega_q \partial_{\mathbf{z}} \mathcal{F}(\mathbf{z}_h(x_q)) + R_q(\mathsf{G}_p, h, \mathbf{z})$$

for a given set of variables $\mathbf{z}$.

1. a LW theorem applies as soon as the quadrature error is below the truncation error
2. provides a quasi-linear form for upwinding and wave decomposition
3. expensive in presence of shocks

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How to get it (3) : more general approach

Proposed in (Csik, Ricchiuto, Deconinck *JCP* 179, 2002), (Ricchiuto, Csik, Deconinck *JCP* 209, 2005)

1. Decouple the evaluation of the cell residual from the distribution :

$$\phi^T(u_h) = \sum_{\text{edges} \in \partial T} \int_{\text{edge}} \mathcal{F}_h(x) \cdot \hat{n}\, dl = \sum_{\text{edges} \in \partial T} l_{\text{edge}} \sum_{q=1}^{\mathsf{G}_p} \omega_q \mathcal{F}_h(x_q) \cdot \hat{n}_{\text{edge}}$$

   for *any* continuous $\mathcal{F}_h$ of choice
2. arbitrary averages are used to evaluate the Jacobians needed for upwinding
3. results dependon the choice of $\mathcal{F}_h$ (*e.g.* $\mathcal{F}_h = \sum_i \psi_i \mathcal{F}_i$, or $\mathcal{F}_h = \mathcal{F}(\mathbf{z}_h)$ for some $\mathbf{z}$)
4. degree of freedom that can be exploited (*e.g.* exact preservation of steady contacts).

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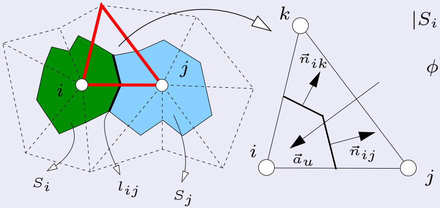
Exact boundary integration $\Rightarrow \phi^T = \oint_{\partial T} \mathcal{F}(\mathbf{z}_h)\hat{n}\, dl = \int_T \nabla \cdot \mathcal{F}(\mathbf{z}_h) dx = \int_T \partial_{\mathbf{z}} \mathcal{F} \cdot \nabla \mathbf{z}_h\, dx$

EXAMPLES OF RD SCHEMES

Examples of RD schemes

First order FV schemes in RD form



$$|S_i|\frac{u_i^{n+1}-u_i^n}{\Delta t} = \sum_j |l_{ij}|H(u_i,u_j,\vec{n}_{ij}) = -\sum_{T|i\in T}\phi_i^T$$

$$\phi_i^T = \frac{\mathcal{F}_i+\mathcal{F}_j}{2}\cdot\vec{n}_{ij} + \mathcal{D}_{\vec{n}_{ij}}(u_i,u_j) + \frac{\mathcal{F}_i+\mathcal{F}_k}{2}\cdot\vec{n}_{ij} + \mathcal{D}_{\vec{n}_{ik}}(u_i,u_k)$$

$$\phi_i^T+\phi_j^T+\phi_k^T = \sum_{\text{edges}}|l|_{\text{edge}}\frac{\mathcal{F}_1^{\text{edge}}+\mathcal{F}_2^{\text{edge}}}{2}\cdot\hat{n}_{\text{edge}}$$

SUPG scheme as RD scheme

SUPG scheme with mass lumping and explicit Euler time stepping (no B.C.s)

$$|S_i|\frac{u_i^{n+1}-u_i^n}{\Delta t} = -\sum_{T|i\in T}\left\{\int_T \psi_i\nabla\cdot\mathcal{F}_h(u_h)dx + \int_T \vec{a}_u\cdot\nabla\psi_i\,\tau\,\nabla\cdot\mathcal{F}_h(u_h)dx\right\}$$

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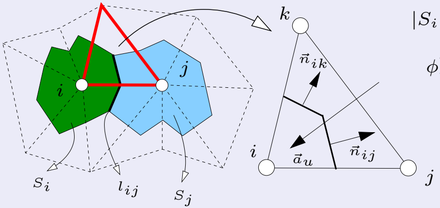

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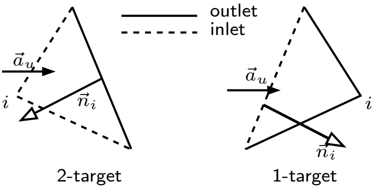
Examples : Multidimensional Upwind schemes



Inlet and outlet

In the scalar case, 2 configurations are possible :

1. 2 downstream nodes (2-target)
2. 1 downstream node (1-target)

$$\vec{a}_u \cdot \vec{n}_i < 0 \Longrightarrow \text{node } i \text{ is upstream}$$
$$\vec{a}_u \cdot \vec{n}_i > 0 \Longrightarrow \text{node } i \text{ is downstream}$$

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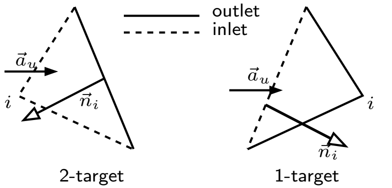

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MU schemes : definition and examples

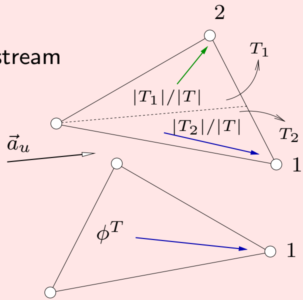
A scheme is MU if

1. in 2-target triangles $\phi_i^T = 0$, if i is upstream
2. in 1-target triangles $\phi_i^T = \phi^T$, $\phi_j^T = \phi_k^T = 0$ if i is downstream
3. Example : LDA scheme

$$\phi_i^{\text{LDA}}(u_h) = \beta_i^{\text{LDA}} \phi^T(u_h)$$

where

$$\beta_i^{\text{LDA}} = \frac{(\vec{a}_u \cdot \vec{n}_i)^+}{\sum\limits_{j \in T} (\vec{a}_u \cdot \vec{n}_j)^+}$$



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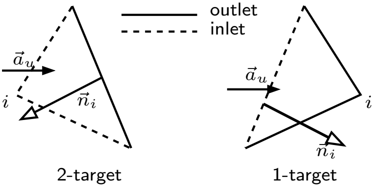

MU schemes : definition

A scheme is MU if

1. in 2-target triangles $\phi_i^T = 0$, ϕ_j^T and $\phi_k^T \neq 0$ if i is upstream
2. in 1-target triangles $\phi_i^T = \phi^T$, $\phi_j^T = \phi_k^T = 0$ if i is downstream

MU schemes : remarks

- they provide a better resolution of multi-D flows
- used in conjunction with wave decompositions can be applied to systems
- have good stability properties and fast convergence to steady state
- hard to generalize (need decompositions, rely on geometrical identities only valid for a linear approximation)
- hard to analyze (systems) and expensive

PART 2

RD SPECIFIC ISSUES : accuracy, positivity, convergence

CONSTRUCTION OF A HIGH ORDER RD SCHEME

The accuracy of RD schemes

- Error estimates built on variational formulation and stability analysis not available (lack coercivity proof)
- Accuracy is characterized by truncation error estimates (Abgrall, *JCP* 167, 2001), (Ricchiuto, Abgrall, Deconinck, *JCP* 222, 2007)

THE ACCURACY OF RD SCHEMES

- w smooth classical solution : $\nabla \cdot \mathcal{F}(w) + \mathcal{S}(w,x,y) = 0$
- w_h, $\mathcal{F}_h(w_h)$, $\mathcal{S}_h(w_h,x,y)$ the discrete linear approximation of w and of the corresponding exact flux and source
- φ a $C^1_0(\Omega)$ functions, and φ_h its discrete approximation

TRUNCATION ERROR (THE STEADY SCHEME IS $\sum_{T|\, i\in T} \phi_i^T(u_h) = 0$)

$$\mathcal{E}(w_h) := \sum_{i\in\mathcal{T}_h} \varphi_i \Big(\sum_{T|\, i\in T} \phi_i^T(w_h) \Big)$$

The accuracy of RD schemes

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Truncation error (the steady scheme is $\sum_{T|\, i\in T} \phi_i^T(u_h) = 0$)

$$\mathcal{E}(w_h) := \sum_{i\in\mathcal{T}_h} \varphi_i \Big(\sum_{T|\, i\in T} \phi_i^T(w_h) \Big)$$

Guiding principle

Under which condition the $\mathcal{RD}$ scheme equivalent to the Galerkin scheme plus terms introducing and error (formally) within the one of the Galerkin approx.

$$\mathcal{E}(w_h) = \overbrace{\int_\Omega \varphi_h \left(\nabla \cdot \mathcal{F}_h(w_h) + \mathcal{S}_h(w_h,x,y)\right) dx}^{I \,\equiv\, \mathcal{E}^{\text{Galerkin}}} + \overbrace{\frac{1}{3} \sum_{T\in\mathcal{T}_h} \sum_{i,j\in T} (\varphi_i - \varphi_j)(\phi_i^T - \phi_i^{\text{Gal}})}^{II}$$

with ϕ_i^{Gal} elemental contribution of the standard (continuous) Galerkin discretization

The accuracy of RD schemes

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- φ a $C_0^1(\Omega)$ functions, and φ_h its discrete approximation

Truncation error (the steady scheme is $\sum\limits_{T|\, i\in T} \phi_i^T(u_h) = 0$)

$$\mathcal{E}(w_h) := \sum_{i\in\mathcal{T}_h} \varphi_i \Big(\sum_{T|\, i\in T} \phi_i^T(w_h) \Big)$$

Final result

If the (continuous) spatial approximations are 2nd order accurate, then one has the global estimate

$$|\mathcal{E}(w_h)| \leq \big(C_0'(\mathcal{T}_h, w) \|\nabla\varphi\|_\infty + C_1'(\mathcal{T}_h, w)\|\varphi\|_\infty \big)\ h^2$$

provided that (in 2D) $\forall i \in T$ and $\forall T \in \mathcal{T}_h$

$$|\phi_i^T(w_h)| \leq C''(\mathcal{T}_h, w) h^3 = \mathcal{O}(h^3)$$

The accuracy of RD schemes

- w smooth classical solution : $\nabla \cdot \mathcal{F}(w) + \mathcal{S}(w,x,y) = 0$
- w_h, $\mathcal{F}_h(w_h)$, $\mathcal{S}_h(w_h,x,y)$ the discrete linear approximation of w and of the corresponding exact flux and source

A design criterion

The condition $\phi_i^T(w_h) = \mathcal{O}(h^3)$ gives a design criterion. In particular, since

$$\begin{aligned}
\phi^T(w_h) &= \int_T (\nabla \cdot \mathcal{F}_h(w_h) + \mathcal{S}_h(w_h,x,y))\, dx \\
&= \oint_{\partial T} (\mathcal{F}_h(w_h) - \mathcal{F}(w)) \cdot \hat{n}\, dl + \int_T (\mathcal{S}_h(w_h,x,y) - \mathcal{S}(w,x,y))\, dx \\
&= \mathcal{O}(\mathcal{F}_h(w_h) - \mathcal{F}(w)) \times \mathcal{O}(|\partial T|) + \mathcal{O}(\mathcal{S}_h(w_h,x,y) - \mathcal{S}(w,x,y)) \times \mathcal{O}(|T|) \\
&= \mathcal{O}(h^2) \times \mathcal{O}(h) + \mathcal{O}(h^2) \times \mathcal{O}(h^2) \\
&= \mathcal{O}(h^3)
\end{aligned}$$

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$$\phi_i^T = \beta_i^T \phi^T$$

with β_i^T uniformly bounded distribution coeff.s, have a $\mathcal{O}(h^2)$ truncation error

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Linearity preserving RD discretization

Our prototype has become :

$$\sum_{T|i\in T} \beta_i^T \phi^T(u_h) = 0 \, , \quad \phi^T(u_h) = \oint_{\partial T} \mathcal{F}_h(u_h) \cdot \hat{n} \, dl + \int_T \mathcal{S}_h(u_h, x, y) \, dx$$

1. How do we define β_i^T ?
2. How do we define $\mathcal{F}_h$ and $\mathcal{S}_h$?

POSITIVE HIGH ORDER NONLINEAR RD

Positive coefficient schemes (Spekreijse, *Math.Comp.* 49, 1987)

Scalar homogeneous case: if we can recast our prototype iterative model as

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{|S_i|} \sum_{T|\, i\in T} \overbrace{\sum_{j\in T} c_{ij}(u_i^n - u_j^n)}^{\phi_i^T(u_h)}$$

then we can easily prove that $\min_j u_j^n \leq u_i^{n+1} \leq \max_j u_j^n$ provided that

$$c_{ij} \geq 0 \quad \text{and} \quad \Delta t \leq \Delta t_{\text{lim}}$$

Positivity preserving discretizations

Positive coefficient schemes (Spekreijse, *Math.Comp.* 49, 1987)

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{|S_i|} \sum_{T|\, i\in T} \overbrace{\sum_{j\in T} c_{ij}(u_i^n - u_j^n)}^{\phi_i^T(u_h)}$$

$\min_j u_j^n \leq u_i^{n+1} \leq \max_j u_j^n$ provided that $c_{ij} \geq 0$ and $\Delta t \leq \Delta t_{\text{lim}}$

Construction

For $\nabla \cdot \mathcal{F}(u) = 0$ consider the *Lax-Friederich's* splitting

$$\phi_i^{\text{LF}}(u_h) = \frac{1}{3}\phi^T(u_h) + \alpha_{\text{LF}} \sum_{j\in T}(u_i - u_j)$$

positive coefficient scheme for α_{LF} large enough. Very simple, however

$$\beta_i^{\text{LF}} = \frac{1}{3} + \frac{\alpha_{\text{LF}} \sum\limits_{j\in T}(u_i - u_j)}{\phi^T(u_h)}$$

is in general unbounded !!

Positive coefficient schemes (Spekreijse, *Math.Comp.* 49, 1987)

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{|S_i|} \sum_{T|\, i\in T} \overbrace{\sum_{j\in T} c_{ij}(u_i^n - u_j^n)}^{\phi_i^T(u_h)}$$

$\min_j u_j^n \leq u_i^{n+1} \leq \max_j u_j^n$ provided that $c_{ij} \geq 0$ and $\Delta t \leq \Delta t_{\mathsf{lim}}$

Construction

$$\phi_i^{\mathsf{LF}}(u_h) = \frac{1}{3}\phi^T(u_h) + \alpha_{\mathsf{LF}} \sum_{j\in T}(u_i - u_j)\,, \quad \beta_i^{\mathsf{LF}} = \frac{1}{3} + \frac{\alpha_{\mathsf{LF}} \sum\limits_{j\in T}(u_i - u_j)}{\phi^T(u_h)}$$

Idea : apply a limiter to β_i^{LF}. So we define the Limited Lax-Friedrich's distribution by

$$\beta_i^{\mathsf{LLF}} = \frac{\psi(\beta_i^{\mathsf{LF}})}{\sum\limits_{j\in T}\psi(\beta_j^{\mathsf{LF}})}\,, \quad \psi(r) \text{ limiter function}$$

The scaling on the denominator guarantees that $\sum_j \beta_j^{\mathsf{LLF}} = 1$.

Positivity preserving discretizations

Positive coefficient schemes (Spekreijse, *Math.Comp.* 49, 1987)

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{|S_i|} \sum_{T|\, i\in T} \overbrace{\sum_{j\in T} c_{ij}(u_i^n - u_j^n)}^{\phi_i^T(u_h)}$$

$\min_j u_j^n \leq u_i^{n+1} \leq \max_j u_j^n$ provided that $c_{ij} \geq 0$ and $\Delta t \leq \Delta t_{\text{lim}}$

Construction

$$\phi_i^{\text{LLF}}(u_h) = \beta_i^{\text{LLF}} \phi^T(u_h)\,, \quad \beta_i^{\text{LLF}} = \frac{\psi(\beta_i^{\text{LF}})}{\sum\limits_{j\in T} \psi(\beta_j^{\text{LF}})}$$

Main properties :

1. β_i^{LLF} is uniformly bounded : the scheme has a $\mathcal{O}(h^2)$ truncation error
2. Provided that $\psi(r) \geq 0$ and $\dfrac{\psi(r)}{r} \geq 0$ then $\dfrac{\beta_i^{\text{LLF}}}{\beta_i^{\text{LF}}} \geq 0$, hence

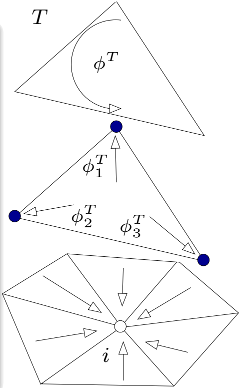
$$\phi_i^{\text{LLF}} = \beta_i^{\text{LLF}} \phi^T = \overbrace{\frac{\beta_i^{\text{LLF}}}{\beta_i^{\text{LF}}}}^{\gamma_i \geq 0} \overbrace{\beta_i^{\text{LF}} \phi^T}^{\phi_i^{\text{LF}}} = \sum_{j\in T} \overbrace{\gamma_i c_{ij}^{\text{LF}}}^{c_{ij}^{\text{LLF}}}(u_i - u_j)\,, \qquad c_{ij}^{\text{LLF}} = \gamma_i c_{ij}^{\text{LF}} \geq 0!!!!$$

Positivity preserving discretizations

Summary

1. Evaluate $\phi^T(u_h) = \oint_{\partial T} \mathcal{F}_h(u_h) \cdot \hat{n}\, dl + \int_T \mathcal{S}_h(u_h, x, y) dx$
2. Evaluate $\phi_i^{\mathsf{LF}}(u_h) = \frac{1}{3}\phi^T(u_h) + \alpha_{\mathsf{LF}} \sum_{j \in T} (u_i - u_j)$
3. Apply limiter : $\beta_i^{\mathsf{LLF}} = \dfrac{\max(0, \beta_i^{\mathsf{LF}})}{\sum_{j \in T} \max(0, \beta_j^{\mathsf{LF}})}$
4. Redistribute : $\phi_i^{\mathsf{LLF}}(u_h) = \beta_i^{\mathsf{LLF}} \phi^T(u_h)$
5. Evolve : $u_i^{n+1} = u_i^n - \dfrac{\Delta t}{|S_i|} \sum_{T | i \in T} \phi_i^{\mathsf{LLF}}(u_h^n)$

Repeat until steady state

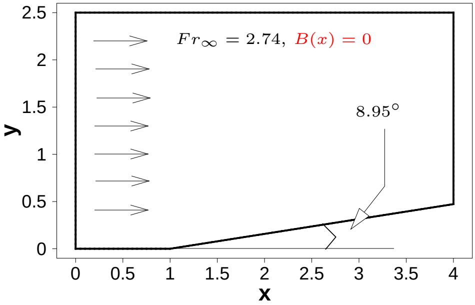


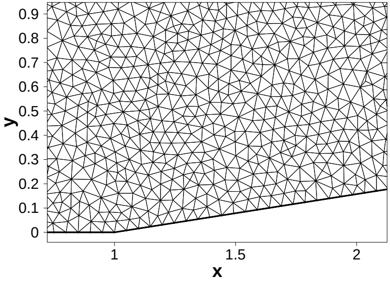
Remark : step 3 (limiting)

Either eq-by-eq or projection on eigenvectors of $\partial_u \mathcal{F} \cdot \vec{v}$ ($\vec{v}$ flow speed, arbitrary average).
Behavior : as in FV limiting (eq-by-eq more robust, projection sligthly better accuracy).

Example 1 : hydraulic jump over a restriction

Supercritical Flow : $Fr_\infty = 2.74,\ Fr_{\text{out}} > 1$





Solution : oblique hydraulic jump (shock)

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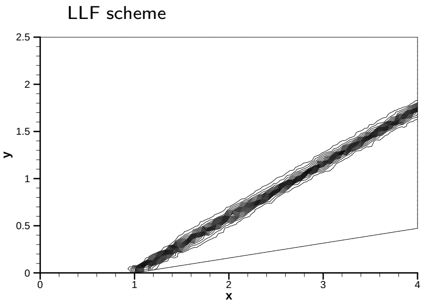

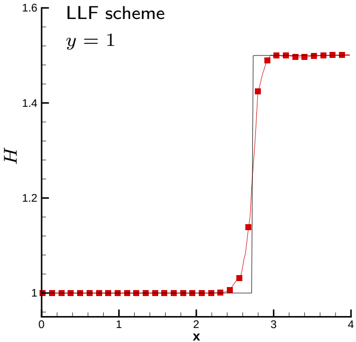

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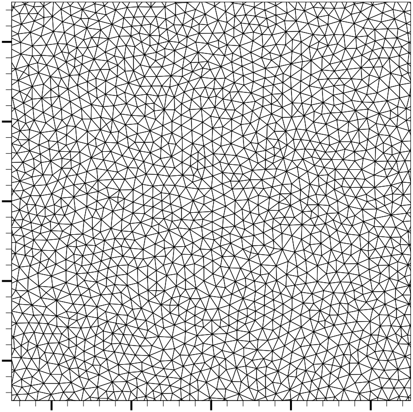
Example 2 : potential solution (Ricchiuto, Abgrall, Deconinck, *JCP* 222, 2007)

Consider a potential φ such that $\Delta\varphi = 0$, and take

$$H = \varphi + C_1 \quad \text{and} \quad \vec{v} = (\frac{\partial \varphi}{\partial y}, -\frac{\partial \varphi}{\partial x}) \quad \text{and} \quad B = \frac{1}{g}(C_2 - \frac{\|\nabla\varphi\|^2}{2}) - \varphi - C_1$$



Experiment : $\varphi = xy$

Exact solution : $H_{tot} = H + B$

$$H_{tot} = \frac{1}{g}(C_2 - \frac{x^2 + y^2}{2})$$

Plot : mesh (zoom)

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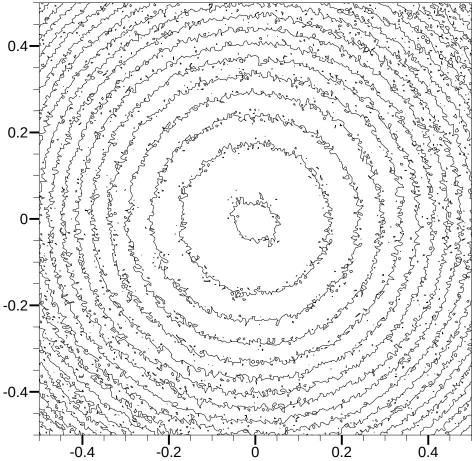

Experiment : $\varphi = xy$

Result of LLF scheme

Plot : $H_{tot} = H + B$

$$H_{tot} = \frac{1}{g}(C_2 - \frac{x^2 + y^2}{2})$$

What is the problem ?

CONVERGENCE AND UPWINDING

Smooth solutions and spurious modes

Scheme *formally* second order, and monotonicity preserving (L^∞-stable), however...

Behavior observed

- No code blow-up : $L^\infty-$stable process (global max/min bounded!)
- Poor iterative and grid convergence, spurious modes in smooth regions

Analysis (Abgrall, *JCP* 214, 2006) : smooth areas where $\phi^T = \mathcal{O}(h^3) \ll 1$

Linearize the nonlinear stady state system $\sum_{T|i\in T} \phi_i^T = 0$: $\mathsf{M}_h^* \mathbf{u} = B_h^*$

M_h^* hasn't full range : infinite solutions, hence spurious modes

Another way to see it (*Out the door, back through the window...*)

- The construction is based on the constraint $\phi_j^{\mathsf{LF}} \times \beta_j^{\mathsf{LLF}} \phi^T \geq 0$
- Upwinding not included in the process
- Locally can have "down-winding" or zero entries in equation (as central scheme and advection)

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Stabilization via streamline dissipation

A solution (Abgrall, *JCP* 214, 2006), (M.Ricchiuto, A.Bollermann *JCP 228,2009*)

In **smooth regions** (that is when $\phi^T \ll 1$) add streamline dissipation

$$\phi_i^{\mathrm{sd}} = \overbrace{\Theta(\phi^T)}^{\text{Smoothness sensor}} \underbrace{\int_T \Big(\partial_u \mathcal{F} \cdot \nabla \psi_i\Big) \, \tau \, \Big(\nabla \cdot \mathcal{F}_h + \mathcal{S}_h\Big)}_{\text{Streamline Dissipation}}$$

In practice... (Abgrall, Larat, Ricchiuto, Tavé, Comp.&Fluids 38, 2009)

- **1 point quadrature** guarantees dissipative character of ϕ_i^{sd}
- Simple diagonal expression for matrix τ
- Sensor : $\Theta(\phi^T) \approx 1$ if $\phi^T \ll 1$, otherwise $\Theta(\phi^T) = C\,h$. For the SWE :

$$\Theta(\phi^T) = \min(1,\, h^2 \frac{\|H\vec{v}\|_{L^\infty(T)}}{|\phi^H|}), \quad \phi^H = \oint_{\partial T} (H\vec{v})_h \cdot \hat{n} \, dl$$

$$\phi_i^{\mathrm{sd}} = \beta_i^{\mathrm{sd}} \phi^T$$

β_i^{sd} simple analytical expression and bounded ($\beta_i^{\mathrm{sd}} = \beta_i^{\mathrm{sd}}(u_h, \partial_u \mathcal{F}, \nabla \psi_i)$)

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SUMMARY

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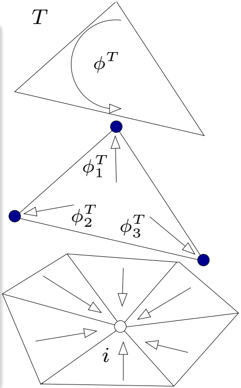

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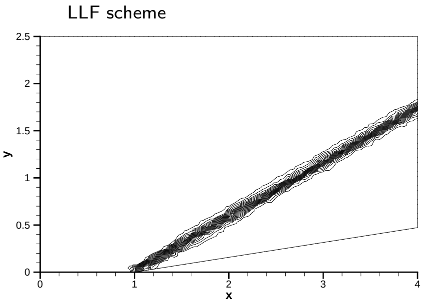

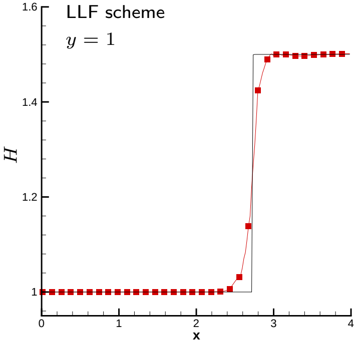

Solution : oblique hydraulic jump (shock)

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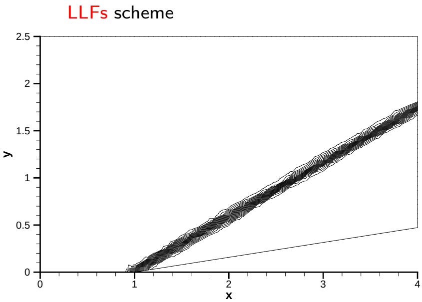

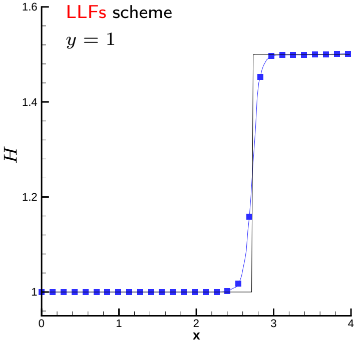

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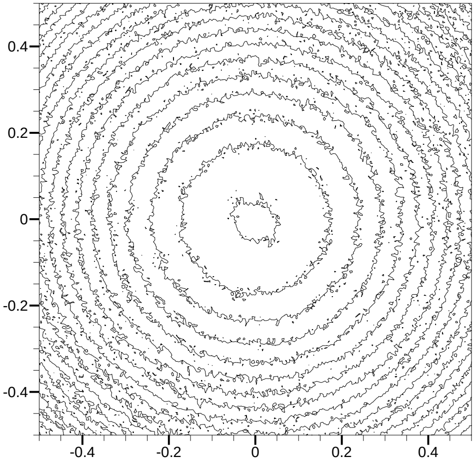


Experiment : $\varphi = xy$

Result of LLF scheme

Plot : $H_{tot} = H + B$

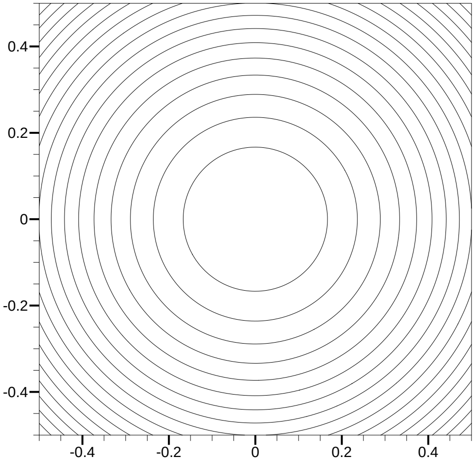
$$H_{tot} = \frac{1}{g}(C_2 - \frac{x^2 + y^2}{2})$$

Spurious modes ...

EXAMPLE 2 : POTENTIAL SOLUTION (RICCHIUTO, ABGRALL, DECONINCK, *JCP* 222, 2007)

Consider a potential φ such that $\Delta\varphi = 0$, and take

$$H = \varphi + C_1 \quad \text{and} \quad \vec{v} = \left(\frac{\partial\varphi}{\partial y}, -\frac{\partial\varphi}{\partial x}\right) \quad \text{and} \quad B = \frac{1}{g}\left(C_2 - \frac{\|\nabla\varphi\|^2}{2}\right) - \varphi - C_1$$



Experiment : $\varphi = xy$

Result of LLFs scheme

Plot : $H_{tot} = H + B$

$$H_{tot} = \frac{1}{g}\left(C_2 - \frac{x^2 + y^2}{2}\right)$$

No more spurious modes !!

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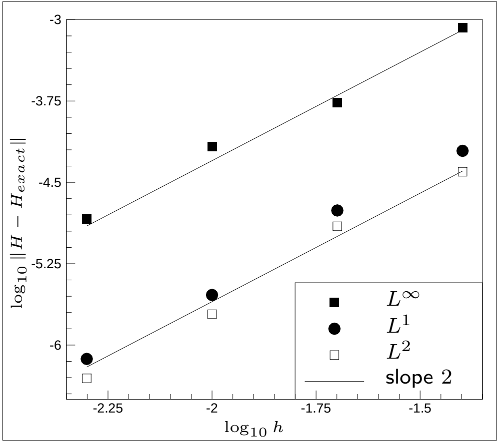

Experiment : $\varphi = xy$

Result of LLFs scheme :

Grid convergence plot

Get the proper convergence!!

HIGH ORDER RD FOR TIME DEPENDENT PROBLEMS

The time dependent case

$$\partial_t u + \nabla \cdot \mathcal{F}(u) + \mathcal{S}(u, x, y) = 0 \quad \text{or} \quad r(u) = 0$$

Consistent formulation

- The discrete model

$$|S_i| \frac{u_i^{n+1} - u_i^n}{\Delta t} + \sum_{T|i\in T} \phi_i^T(u_h^n) = 0$$

  is inconsistent in space (replacing explicit Euler by RK-k does not help).

- Set (high order ODE integrator)

$$r_h = \sum_{j=0}^{p} \alpha_j \frac{u_h^{n+1-j} - u_h^{n-j}}{\Delta t} + \sum_{j=0}^{q} \gamma_j \left(\nabla \cdot \mathcal{F}_h(u_h^{n+1-j}) + \mathcal{S}_h(u_h^{n+1-j}, x, y) \right)$$

  consistent schemes are in general defined by : given u_h^0

$$\sum_{T|i\in T} \Phi_i^T(r_h) = 0 \quad \text{with} \quad \sum_{j\in T} \Phi_j^T(r_h) = \Phi^T = \int_T r_h \, dx$$

- Truncation error analysis (Ricchiuto, Abgrall, Deconinck, *JCP* 222, 2007) : if

$$\Phi_i^T(r_h) = \beta_i^T \Phi^T \qquad \beta_i^T \text{ uniformly bounded}$$

  then TE=$\mathcal{O}(h^2)$ (at least second order ODE integrator).

The time dependent case

$$\partial_t u + \nabla \cdot \mathcal{F}(u) + \mathcal{S}(u,x,y) = 0 \quad \text{or} \quad r(u) = 0$$

Consistent formulation

- The discrete model

$$|S_i| \frac{u_i^{n+1} - u_i^n}{\Delta t} + \sum_{T|i\in T} \phi_i^T(u_h^n) = 0$$

  is inconsistent in space (replacing explicit Euler by RK-k does not help).

- Set (high order ODE integrator)

$$r_h = \sum_{j=0}^{p} \alpha_j \frac{u_h^{n+1-j} - u_h^{n-j}}{\Delta t} + \sum_{j=0}^{q} \gamma_j \left(\nabla \cdot \mathcal{F}_h(u_h^{n+1-j}) + \mathcal{S}_h(u_h^{n+1-j}, x, y) \right)$$

  consistent schemes are in general defined by : given u_h^0

$$\sum_{T|i\in T} \Phi_i^T(r_h) = 0 \quad \text{with} \quad \sum_{j\in T} \Phi_j^T(r_h) = \Phi^T = \int_T r_h \, dx$$

- Truncation error analysis (Ricchiuto, Abgrall, Deconinck, *JCP* 222, 2007) : if

$$\Phi_i^T(r_h) = \beta_i^T \Phi^T \qquad \beta_i^T \text{ uniformly bounded}$$

  then TE=$\mathcal{O}(h^2)$ (at least second order ODE integrator).

The time dependent case

$$\partial_t u + \nabla \cdot \mathcal{F}(u) + \mathcal{S}(u,x,y) = 0 \quad \text{or} \quad r(u) = 0$$

Consistent formulation

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The time dependent case

$$\partial_t u + \nabla \cdot \mathcal{F}(u) + \mathcal{S}(u, x, y) = 0 \quad \text{or} \quad r(u) = 0$$

Consistent constructions in (Caraeni, Fuchs, *Th.Comp.Fluid Dyn.* 15, 2002), (Abgrall, Mezine, *JCP* 188, 2003), (Ricchiuto, Csik, Deconinck, *JCP* 209, 2005)

LLFs for unsteady problems (M.Ricchiuto, A.Bollermann *JCP 228,2009*)

The time dependent case

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The time dependent case

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4. Evaluate limited distribution coefficients : $\beta_i^{\mathsf{LLF}} = \dfrac{\max(0, \beta_i^{\mathsf{LF}})}{\sum\limits_{j \in T} \max(0, \beta_j^{\mathsf{LF}})}, \quad \beta_j^{\mathsf{LF}} = \Phi_i^{\mathsf{LF}} / \Phi^T$

The time dependent case

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5. Add upwind bias : $\Phi_i^{\mathsf{sd}} = \Theta(\Phi^T)\int\limits_T(\partial_u\mathcal{F}\cdot\nabla\psi_i\,\tau)\mathbf{r_h}\,dx \approx \beta_i^{\mathsf{sd}}\Phi^T$

The time dependent case

$$\partial_t u + \nabla \cdot \mathcal{F}(u) + \mathcal{S}(u,x,y) = 0 \quad \text{or} \quad r(u) = 0$$

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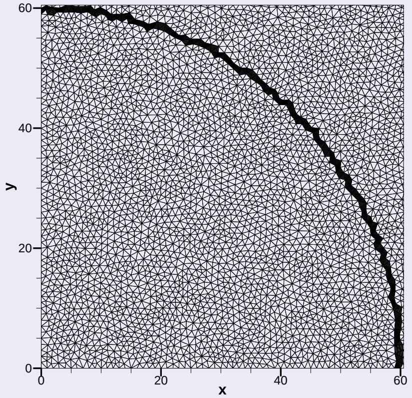
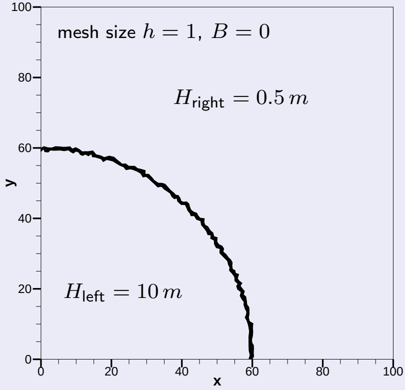
6. Solve for u_h^{n+1} the nonlinear system $\sum\limits_{T | i \in T} (\beta_i^{\mathsf{LLF}} + \beta_i^{\mathsf{sd}}) \Phi^T = 0\,, \ \forall\, i$

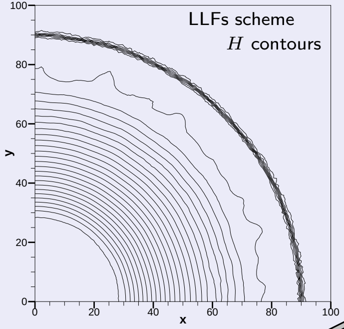
Example 1. Circular dam break : mesh and initial solution

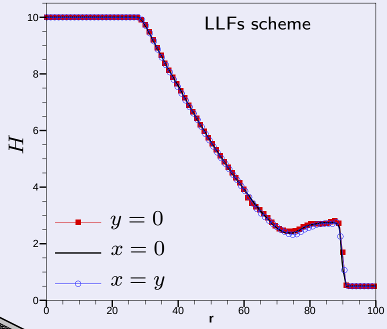


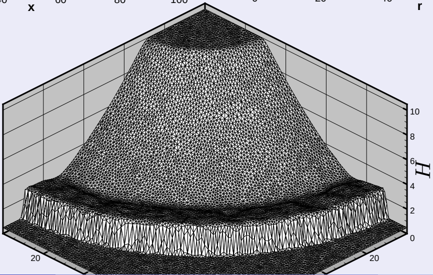


The time dependent case

Example 1. Circular dam break : LLFs result







The time dependent case

Example 2. Vortex transport (M.Ricchiuto, A.Bollermann *JCP 228,2009*)

If $\mathbf{p} = [H, v_x, v_y]^t$, $\mathbf{p}_0 = [H_0, \vec{v}_0]^t$, and $\vec{v}_\infty = (v_\infty, 0)$ exact solution of the form

$$\mathbf{p} = \mathbf{p}_0(x - \vec{v}_\infty t)$$

with

$$H_0(r_c) = H_\infty + \begin{cases} \dfrac{1}{g}\left(\dfrac{\Gamma}{\omega}\right)^2 (h(\omega\, r_c) - h(\pi)) & \text{if} \quad \omega\, r_c \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

and

$$\vec{v}_0 = \vec{v}_\infty + \begin{cases} \Gamma(1 + \cos(\omega\, r_c))\,(y_c - y, x - x_c) & \text{if} \quad \omega\, r_c \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

with $h(\cdot)$ analytically known.

EXAMPLE 2. VORTEX TRANSPORT (M.RICCHIUTO, A.BOLLERMANN *JCP 228,2009*)

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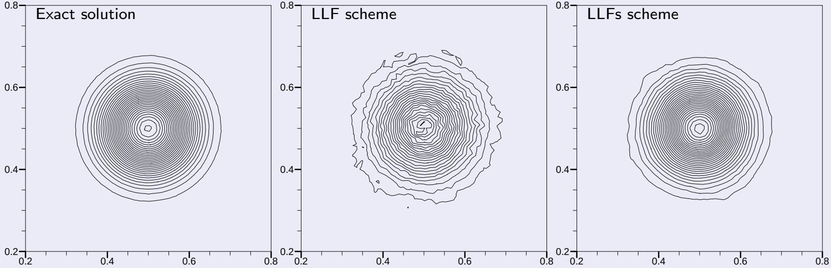
H at time $t = 1$ (meshsize $h = 1/80$, 40 pts. through core)



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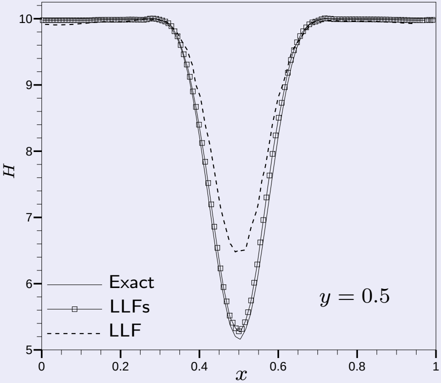
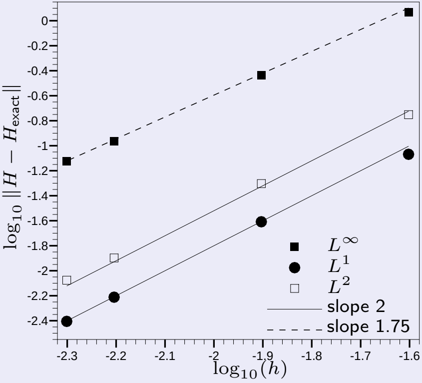
H at time $t = 1$, cut through the vortex and grid convergence (LLFs scheme)





PART 3
SHALLOW WATER EQUATIONS
Well-balancedness, water-height positivity

Well-balancedness : on the choice of $\mathcal{F}_h$ and $\mathcal{S}_h$

Objectives

- Define discrete operators based on (at least) second order approximation
- Preserve EXACTLY some steady state invariants ... case by case analysis
- Let $\mathbf{v}$ be a set of invariants

Proposition (Ricchiuto, Abgrall, Deconinck, *JCP* 222, 2007)

High order RD schemes preserve EXACTLY the steady state $\mathbf{v} = \mathbf{v}_0$ provided that the continuous discrete flux and source $\mathcal{F}_h$ and $\mathcal{S}_h$ are such that

$$\phi^T(\mathbf{v}_0) = \int_T \left(\nabla \cdot \mathcal{F}_h(u_h(\mathbf{v}_0)) + \mathcal{S}_h(u_h(\mathbf{v}_0), x, y)\right) dx = 0$$

Remarks

- Ok, but we still don't know who are $\mathcal{F}_h$ and $\mathcal{S}_h$ ...
- Similar residual based approach in FV in (Noelle, Xing, Shu, *JCP* 226, 2007)

Well-balancedness : on the choice of $\mathcal{F}_h$ and $\mathcal{S}_h$

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Trivially in this case high order schemes reduce to :

$$\mathcal{M}\left(\mathbf{u}^{n+1} - \mathbf{u}^n\right) = 0$$

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Invariants

The steady SWE can be recast as (set $\vec{v}^\perp = (-v_y, v_x)$)

$$\partial_x(H\,v_x) + \partial_y(H\,v_y) = 0$$

$$\vec{v}\,(\partial_x(H\,v_x) + \partial_y(H\,v_y)) + gH\nabla\left(\frac{\vec{v}\cdot\vec{v}}{2g} + H + B\right) + \overbrace{H\vec{v}^\perp(\partial_y v_x - \partial_x v_y)}^{\text{curl term}} = 0$$

Focus on steady irrotational flows with

$$\begin{aligned} q_x &= H\,v_x = \text{const} = q_x^0 \\ q_y &= H\,v_y = \text{const} = q_y^0 \\ \mathcal{I} &= \frac{\vec{v}\cdot\vec{v}}{2g} + H + B = \text{const} = \mathcal{I}^0 \end{aligned}$$

Proposition (Ricchiuto, Abgrall, Deconinck, *JCP* 222, 2007)

Well-balancedness : on the choice of $\mathcal{F}_h$ and $\mathcal{S}_h$

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Focus on steady irrotational flows with

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\end{aligned}
$$

Proposition (Ricchiuto, Abgrall, Deconinck, *JCP* 222, 2007)

For EXACT QUADRATURE, on structured grids high order RD preserve EXACTLY grid aligned quasi-1D solutions

$$\mathbf{v} = [q_x,\, q_y,\, \mathcal{I}]^t = [q_0,\, 0,\, \mathcal{I}_0]^t = \mathbf{v}_0$$

provided that $\mathcal{F}_h = \mathcal{F}(\mathbf{v}_h)$, $\mathcal{S}_h = \mathcal{S}(\mathbf{v}_h, x, y)$

Well-balancedness : on the choice of $\mathcal{F}_h$ and $\mathcal{S}_h$

Invariants

Focus on steady irrotational flows with

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$$q_y = H\, v_y = \text{const} = q_y^0$$
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Proposition (Ricchiuto, Abgrall, Deconinck, *JCP* 222, 2007)

Proof (sketch) : for EXACT QUADRATURE

$$\phi^T = \oint_{\partial T} \mathcal{F}(\mathbf{v}_h)\cdot\hat{n}\, dl + \int_T \mathcal{S}(\mathbf{v}_h, x, y) dx = \int_T \Big(A_x \partial_x \mathbf{v}_h + A_y \partial_y \mathbf{v}_h + \widetilde{\mathcal{S}}(\mathbf{v}_h, x, y)\Big)\, dx$$

$$= |T|(\overline{A}_x, \overline{A}_y)\nabla \mathbf{v}_h|_T + |T|\Big(\overset{0}{\frac{g\overline{H}}{g\overline{H} - 2\overline{v}\cdot\overline{v}}\overline{v}^{\perp}\overline{v}^{\perp}} \cdot \nabla B_h|_T \Big)$$

where A_x and A_y are not the components of $\partial_{\mathbf{v}}\mathcal{F}$. If $\mathbf{v}_i = \mathbf{v}_0\ \forall i \Rightarrow \nabla \mathbf{v}_h = 0$ (exact interpolation of constant). The red term is the remainder of the curl terms and simply vanishes on structured grids for grid-aligned quasi-1d solutions.

Well-balancedness : on the choice of $\mathcal{F}_h$ and $\mathcal{S}_h$

Invariants

Focus on steady irrotational flows with

$$q_x = H\, v_x = \text{const} = q_x^0$$
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Proposition (Ricchiuto, Abgrall, Deconinck, *JCP* 222, 2007)

Catch.

- The set $\mathbf{v}_h$ actually depends on the bed height B.
- However, if $B_h = \sum_{i\in\mathcal{T}_h} B_i\psi_i$ then interpolating $\mathbf{v}_h$ to evaluate the flux is equivalent to the interpolation of

$$\widetilde{\mathbf{v}} = \begin{bmatrix} \dfrac{\vec{v}\cdot\vec{v}}{2g} + H \\ q_x \\ q_y \end{bmatrix}$$

Well-balancedness : on the choice of $\mathcal{F}_h$ and $\mathcal{S}_h$

Approximate quadrature

In 2d exact quadrature impractical for this representation. Errors very small, however well above machine zero. Behavior : $\mathcal{E} = C\,h^p$ with p depending on the quadrature formulas

$$\oint_{\partial T} \mathcal{F}_h(\mathbf{v}_h)\cdot\hat{n}\,dl \approx \sum_{\text{edges}} |l_{\text{edge}}| \sum_{j=1}^{p_f} \omega_j \mathcal{F}(\mathbf{v}_h(x_j))\cdot\hat{n}$$

$$\int_T H(\mathbf{v}_h)\nabla B_h\,dx \approx |T| \sum_{j=1}^{p_v} \omega_j H(\mathbf{v}_h(x_j))\;\nabla B_h|_T$$

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$$\int_T H(\mathbf{v}_h)\nabla B_h\,dx \approx |T| \sum_{j=1}^{p_v} \omega_j H(\mathbf{v}_h(x_j))\ \nabla B_h|_T$$

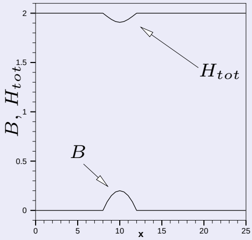
Example

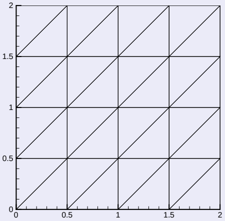
$q_x = 4.42$

$q_y = 0$

$\mathcal{I} = 22.06605$

$$B = \begin{cases} 0.2 - \frac{(x-10)^2}{20} & \text{if } x \in [8,12] \\ 0 \text{ otherwise} \end{cases}$$





Well-balancedness : on the choice of $\mathcal{F}_h$ and $\mathcal{S}_h$

Approximate quadrature

In 2d exact quadrature impractical for this representation. Errors very small, however well above machine zero. Behavior : $\mathcal{E} = C\, h^p$ with p depending on the quadrature formulas

$$\oint_{\partial T} \mathcal{F}_h(\mathbf{v}_h) \cdot \hat{n}\, dl \approx \sum_{\text{edges}} |l_{\text{edge}}| \sum_{j=1}^{p_f} \omega_j \mathcal{F}(\mathbf{v}_h(x_j)) \cdot \hat{n}$$

$$\int_T H(\mathbf{v}_h) \nabla B_h\, dx \approx |T| \sum_{j=1}^{p_v} \omega_j H(\mathbf{v}_h(x_j))\; \nabla B_h|_T$$

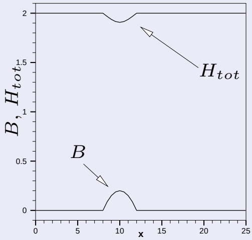
Example

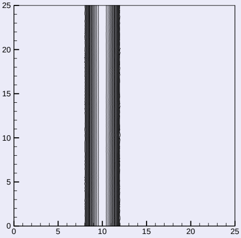
$q_x = 4.42$

$q_y = 0$

$\mathcal{I} = 22.06605$

$$B = \begin{cases} 0.2 - \frac{(x-10)^2}{20} & \text{if } x \in [8, 12] \\ 0 \text{ otherwise} \end{cases}$$





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	$p_f = 2,\ p_v = 4$	$p_f = 3,\ p_v = 4$
25/50	5.422332e-09	2.032884e-09
25/100	3.545746e-10	5.934772e-11
25/200	1.513017e-11	1.855520e-12
rate	4.25	5.05

Table: L^2 error on H at time $t = 0.5$, LLFs scheme

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$$\int_T H(\mathbf{v}_h)\nabla B_h\,dx \approx |T| \sum_{j=1}^{p_v} \omega_j H(\mathbf{v}_h(x_j))\;\nabla B_h|_T$$

Example

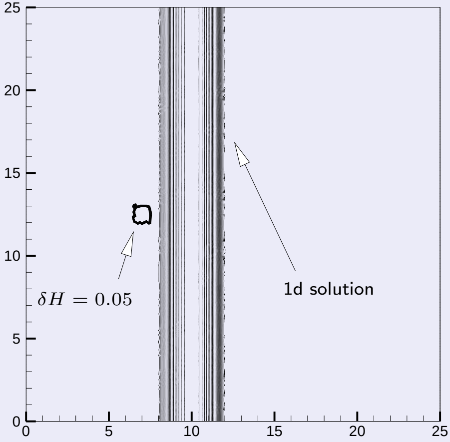
Same problem, but with

$$\mathcal{F}_h = \mathcal{F}(\mathbf{p}_h)$$
$$\mathcal{S}_h = \mathcal{S}(\mathbf{p}_h, x, y)$$
$$\mathbf{p}_h = [H_h, \vec{v}_h]^t$$

	$p_f = 2,\ p_v = 1$	$p_f = 3,\ p_v = 1$
25/50	1.019986e-04	1.019898e-04
25/100	2.730489e-05	2.730430e-05
25/200	6.713026e-06	6.712989e-06
rate	1.95	1.95

Table: L^2 error on H at time $t = 0.5$, LLFs scheme. Exact quadratue already for $p_f = 2,\ p_v = 1$

PERTURBATION (NOELLE, XING, SHU, *JCP* 226, 2007)



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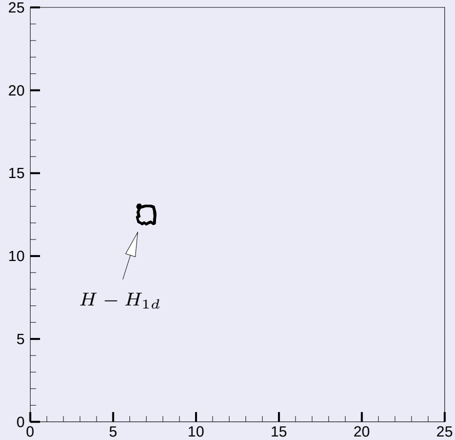

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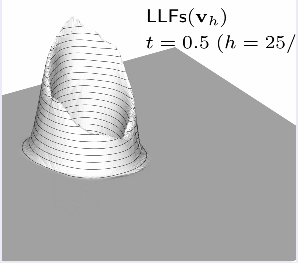

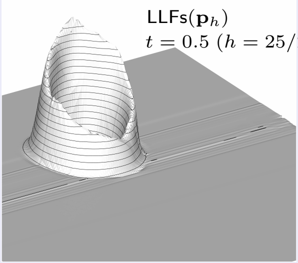

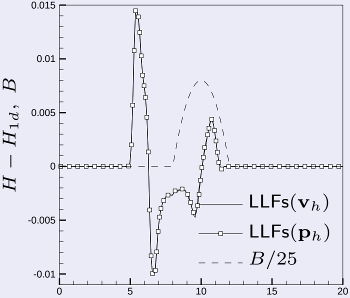

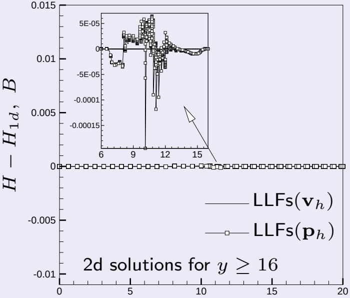

Well-balancedness : on the choice of $\mathcal{F}_h$ and $\mathcal{S}_h$

Unstructured meshes

$$\partial_x(H\,v_x) + \partial_y(H\,v_y) = 0$$

$$\vec{v}\,(\partial_x(H\,v_x) + \partial_y(H\,v_y)) + gH\nabla\left(\frac{\vec{v}\cdot\vec{v}}{2g} + H + B\right) + H\vec{v}^{\perp}(\partial_y v_x - \partial_x v_y) = 0$$

Even with exact quadrature, on unstructured grids the curl term in the second equation pops up giving nonzero residuals.

Perturbation

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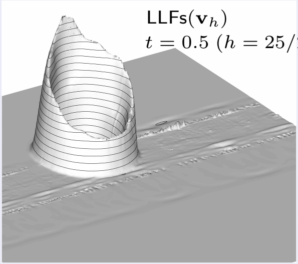
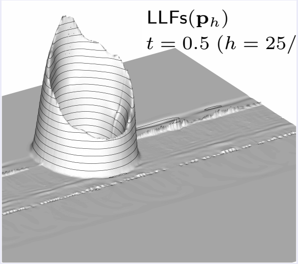


LLFs($\mathbf{v}_h$)

$t = 0.5$ ($h = 25/200$)



LLFs($\mathbf{p}_h$)

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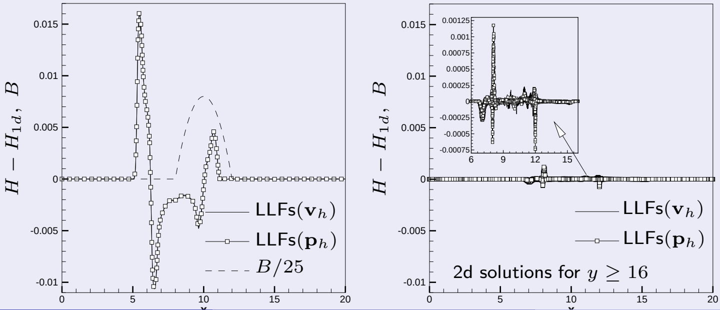

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Remarks

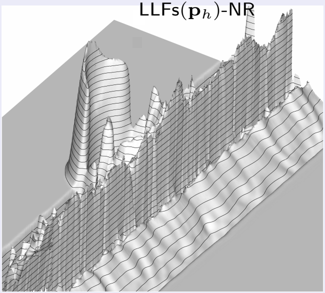
- numerical errors in unperturbed region measure error on $H\vec{v}^{\perp}(\partial_y v_x - \partial_x v_y)$
- magnitude of error very small, however only within truncation

EXAMPLE OF NON-RESIDUAL SCHEME

If we take the scheme

$$\sum_{T|i\in T}\left((\beta_i^{\mathsf{LLF}}+\beta_i^{\mathsf{sd}})\int_T\left(\frac{u_h^{n+1}-u_h^n}{\Delta t}+\nabla\cdot\mathcal{F}_h(\mathbf{p}_h^{n+1/2})\right)dx+\frac{1}{3}\int_T\mathcal{S}_h(\mathbf{p}_h^{n+1/2},x,y)dx\right)=0$$

LLFs($\mathbf{p}_h$)-NR



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Unstructured meshes : lake at rest

Particular case of previous equilibrium obtained for $q_x = q_y = 0$:

$$v_x = 0$$
$$v_y = 0$$
$$\mathcal{I} = H + B = \text{const}$$

Proposition (Ricchiuto, Abgrall, Deconinck, *JCP* 222, 2007)

High order RD preserve EXACTLY steady state lake at rest solutions provided that the quadrature is exact with respect to $\mathcal{F}_h = \mathcal{F}(\mathbf{q}_h)$, $\mathcal{S}_h = \mathcal{S}_h(\mathbf{q}_h, x, y)$, with $\mathbf{q}_h$ any state vector containing $H + B$ as a variable.

Remarks

1. similar to hydrostatic reconstruction
2. it boils down to the criterion that over an element one has numerically

$$(\nabla H)_h = -(\nabla B)_h$$

on the lake at rest state. Same as in (P.Brufau, P.Garcia-Navarro *JCP 186,2003*)

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Well-balancedness : on the choice of $\mathcal{F}_h$ and $\mathcal{S}_h$

Well-balancedness : example

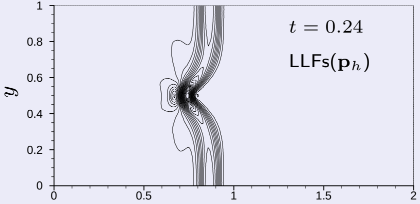
Take $B(x,y) = 0.8e^{-5(x-0.9)^2-50(y-0.5)^2}$ and the initial state $[H+B,\, v_x\, v_y] = [1,\, 0\,0]$, at time $t = 0.5$ the LLFs errors are

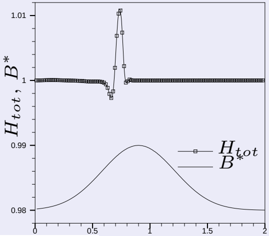
	L^∞	L^1	L^2
$e_{H_{tot}}$	8.955510e-17	2.605999e-17	3.183067e-17
e_u	1.567940e-18	2.485329e-19	3.201703e-19
e_v	1.432740e-18	1.789517e-19	2.327169e-19

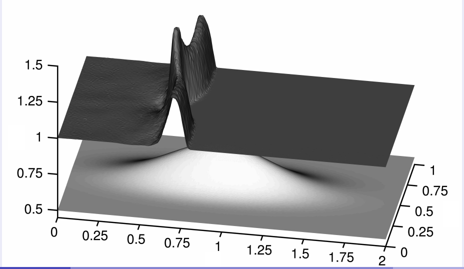
WELL-BALANCEDNESS : EXAMPLE

Perturbation of the initial solution (untructured mesh, $h = 1/100$) :

$$H + B = \begin{cases} 1.01 & \text{if} \quad 0.05 < x < 0.15 \\ 1 & \text{otherwise} \end{cases}$$







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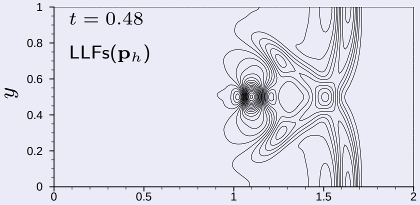

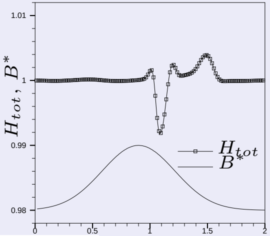

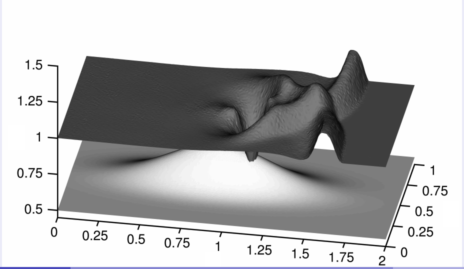

SHALLOW WATER EQUATIONS
Water-height positivity

Wetting and drying

Issues to be dealth with

1. Preserving the condition $H \geq 0$
2. Undefined flow speed
3. Definition of α_{LF}
4. C-property in front-cells

The first 3 points are tied one-another.

Wetting and drying

Proposition ($H \geq 0$ and LLF scheme) (M.Ricchiuto, A.Bollermann *JCP 228,2009*)

When limiting eq.-by-eq., the LLF scheme preserves the positivity of H, provided that

$$\alpha_{\mathsf{LF}} > h \sup_{x \in T} \|\vec{v}_h\| \quad \forall T$$

and under the (Crank-Nicholson) time-step limitation

$$\Delta t \leq 2\,\Delta t_{\text{lim}}\,, \quad \Delta t_{\text{lim}} = \min_{T \in \mathcal{T}_h} \frac{|T|}{3\alpha_{\mathsf{LF}}}$$

In practice (parameters used in all the computations of this talk)

Let L_{ref} be a reference length (*e.g.* $L_{\text{ref}} = \max_{i,j \in \mathcal{T}_h} \|x_i - x_j\|$), and $\overline{h} = h/L_{\text{ref}}$.

1. Switch to eq.-by-eq. limiting in front cells
2. $\forall j \in \mathcal{T}_h$ set : $\vec{v}_j = 0$ if $H_j \leq C_{\vec{v}} = \overline{h}^2$
3. Set (everywhere) : $\alpha_{\mathsf{LF}} = h \max_{j \in T}(\|\vec{v}_j\| + \sqrt{g\,H_j}) + \overline{h}^2$
4. In Φ_i^{sd} term set : $\Theta = \underbrace{\Theta(\Phi^T)}_{\text{std def}}\, e^{-\overline{h}\frac{H_\infty}{H^T_{\min}}}$ (with $H_\infty = \max_{i \in \mathcal{T}_h} H_0(x_i)$, $\quad H^T_{\min} = \min_{j \in T} H_j$)

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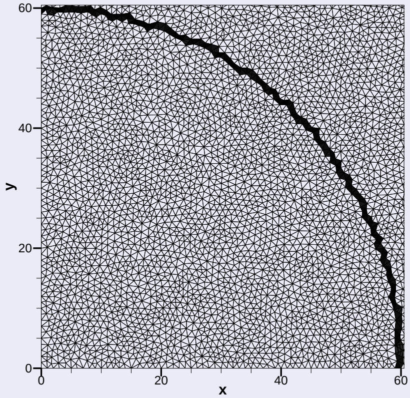
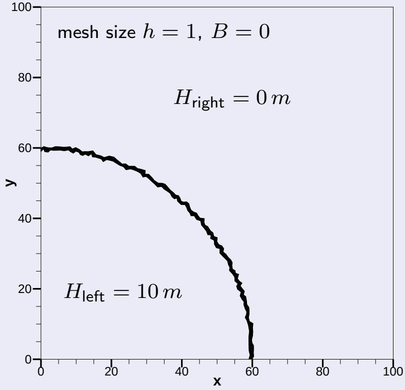
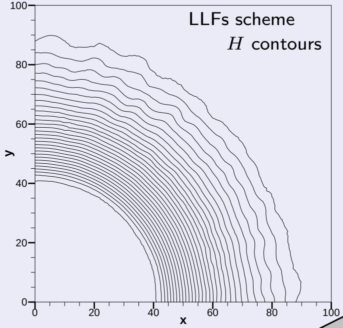
Nothing is done on H and $H\vec{v}$!

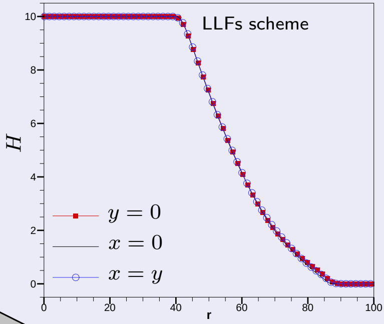
Example 1. Dam break on dry bed : mesh and initial solution

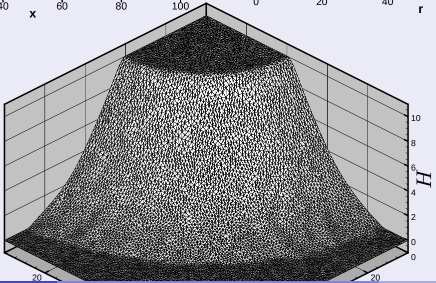




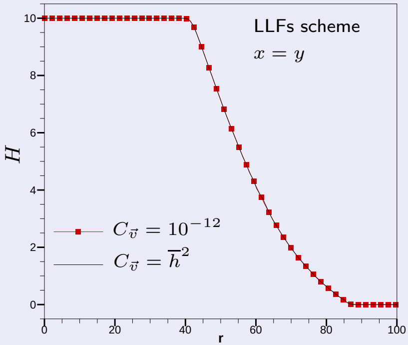
EXAMPLE 1. DAM BREAK ON DRY BED : LLFS RESULT

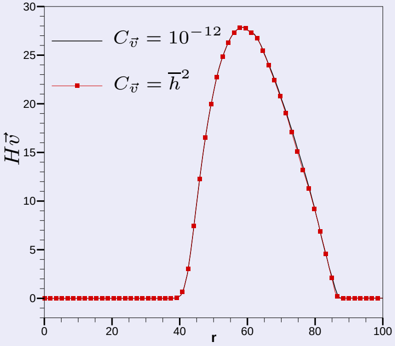






Example 1. Dam break on dry bed : LLFs result, influence of $C_{\vec{v}}$





C-property and front cells

We simply follow (P.Brufau, P.Garcia-Navarro *JCP 186,2003*).

C-property Preservation of $(H + B, \vec{v}) = (H_0, 0)$ state. Fix needed to get $\nabla(H + B)_h$ right in front cells with adverse slope.

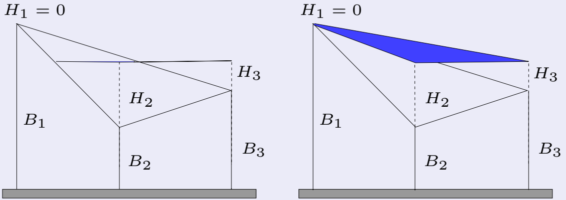


Figure: Front cell. Left: real situation. Right: numerical representation

Wetting and drying

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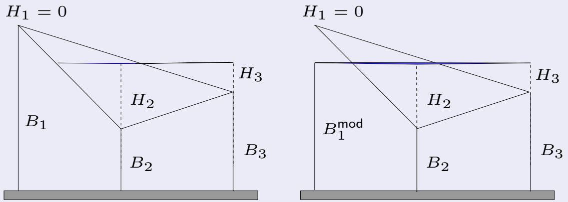
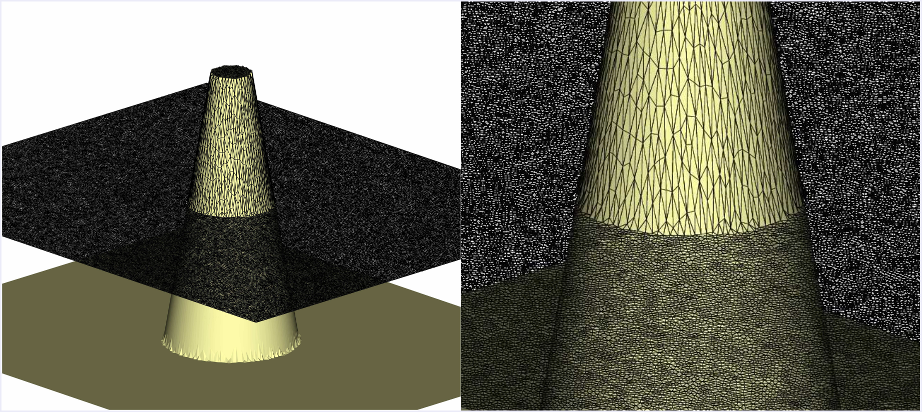


Figure: Front cell. Left: real situation. Right: numerical representation

1. Set $H_{\text{max}} = \max_{j|H_j>0}(H_j + B_j)$
2. For all dry nodes k : if $B_k > H_{\text{max}}$ then $B_k^{\text{mod}} := H_{\text{max}}$
3. Use ∇B_h^{mod} to compute $\phi^T(u_h)$

Example 2. Wave run up on a conical island (Hubbard, Dodd *Coast.Engrg.* 47, 2002)



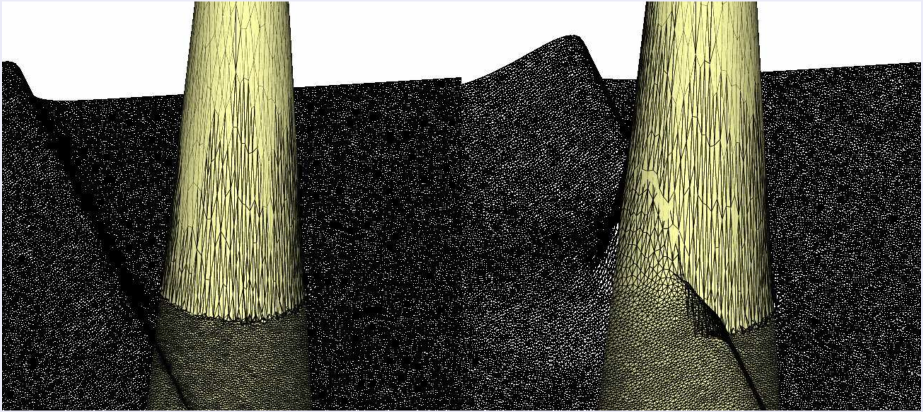
Example 2. Wave run up on a conical island (Hubbard, Dodd *Coast.Engrg.* 47, 2002)

	L^∞	L^1	L^2
$e_{H_{tot}}$	2.775558e-17	1.532978e-19	1.643908e-18
e_u	2.221603e-18	7.987680e-21	5.578502e-20
e_v	1.252903e-18	6.400735e-21	4.081257e-20

Table: Lake at rest solution : errors at time $t = 5$, LLFs scheme.

Example 2. Wave run up on a conical island (Hubbard, Dodd *Coast.Engrg.* 47, 2002)

Wave run-up simulation (Hubbard, Dodd *Coast.Engrg.* 47, 2002) with LLFs scheme.



Wetting and drying

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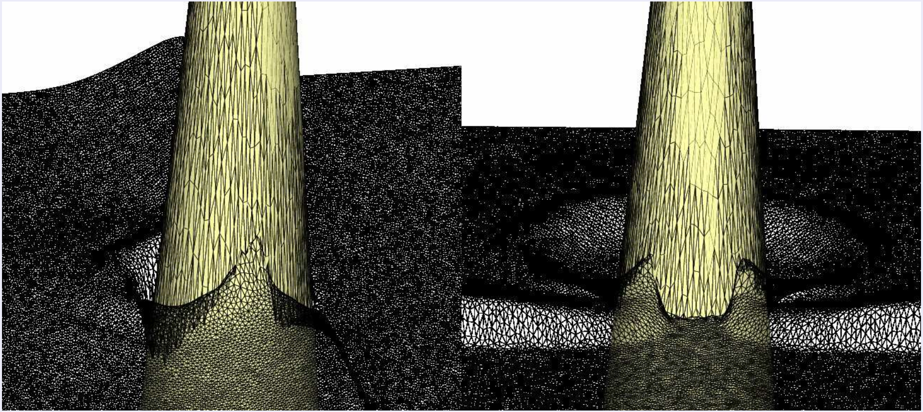

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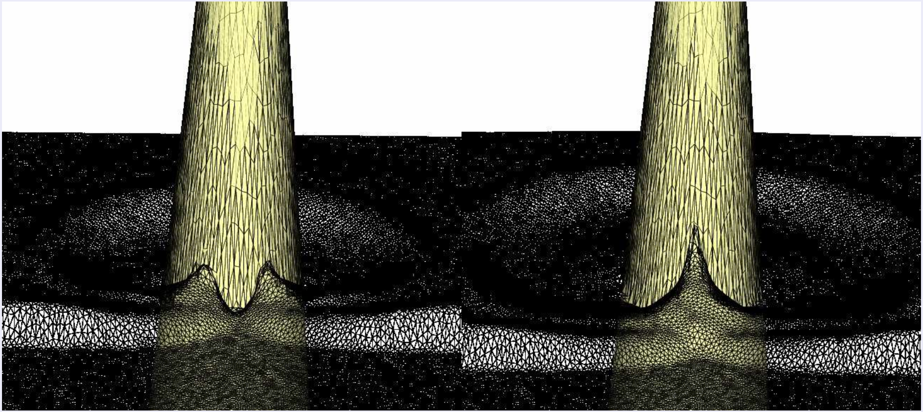


Example 3. Thacker's oscillations (Thacker *JFM* 107, 1981)

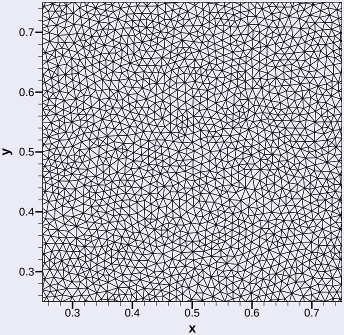
On $\Omega = [-2, 2]^2$ set :

$$B(x,y) = -H_0\left(1-(x^2+y^2)\right) = -H_0\left(1-r^2\right)$$

Two exact periodic solutions exist. A curved free-surface solution is obtained for

$$(H+B)(x,y,t=0) = H_0\left\{\Gamma - 1 - r^2\left(\Gamma^2-1\right)\right\}, \quad u=v=0$$

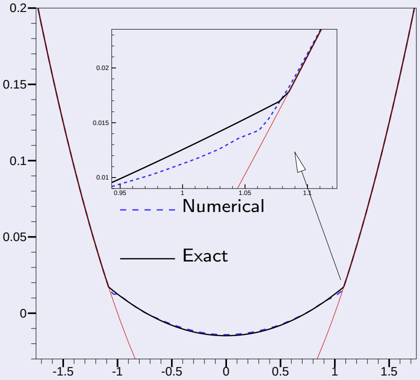
Γ a shape parameter. We show solutions obtained on an unstructured grid with $h \approx 4/100$



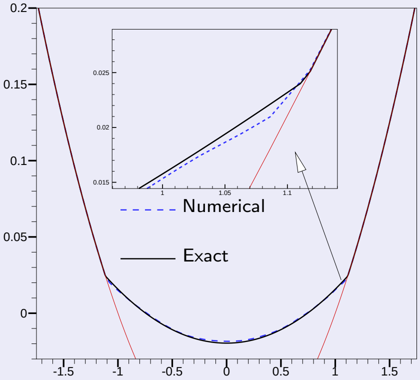
Wetting and drying

Example 3. Thacker's oscillations (Thacker *JFM* 107, 1981)

LLFs scheme, data along the line $y = 0$



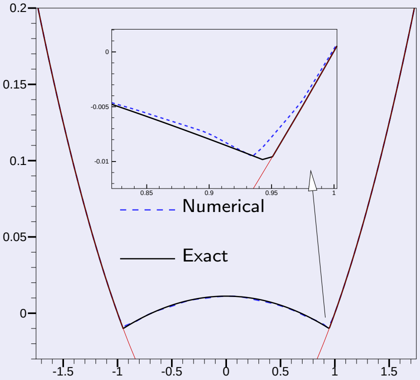
$t = 3T + T/3$



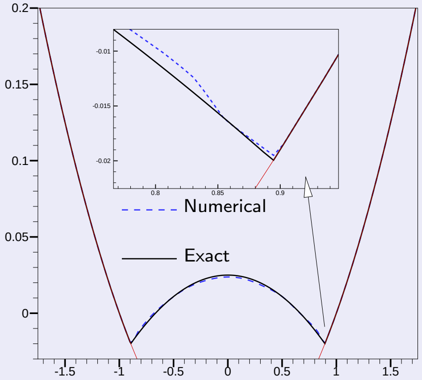
$t = 3T + T/2$

Example 3. Thacker's oscillations (Thacker *JFM* 107, 1981)

LLFs scheme, data along the line $y = 0$



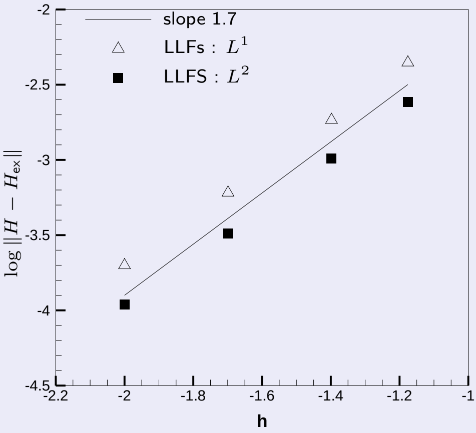
$t = 3T + T/2 + T/3$



$t = 4T$

Example 3. Thacker's oscillations (Thacker *JFM* 107, 1981)

Grid convergence : LLFs scheme, time $t = T$.



Conclusions

Summary : nonlinear limited LF scheme

- Second order for steady and time dependent problems
- C-property : easy for lake-at-rest, moving equilibria are tough on unstructured grids
- However, good balance between various terms in (perturbations stay small)
- Wetting-drying without any cut-off on conserved quantitites, second order on cases with dry areas (and continuous sol.s)

Main flaw

Implicit nonlinear scheme with $CFL = 2$ restriction ....

Improvements, ongoing work

1. Time dependent. Fully explicit RK-RD (with R.Abgrall), space-time with discontinuous time representation (with M.Hubbard)
2. Very high order (with R.Abgrall). Further reduction of TE (need exact C-property ?)
3. Stiff problems. Analogy with stabilized (continuous) Galerkin schemes : asymptotic analysis of scheme, projection consistent with asymptotic solutions

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