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School of Computing

Scientific Computation Group

Space-Time Residual Distribution Schemes on Moving Meshes

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Hyperbolic Conservation Laws

Consider the scalar conservation law

$$\partial_t u + \nabla \cdot \mathbf{f} \;=\; 0 \qquad \text{or} \qquad \partial_t u + \mathbf{a} \cdot \nabla u \;=\; 0$$

on a domain Ω.

- $\mathbf{a} = \frac{\partial \mathbf{f}}{\partial u}$ is the advection velocity for the flow.
- $u(\mathbf{x}, 0)$ is specified.
- $u(\mathbf{x}, t)$ is specified on inflow boundaries.

Residual Distribution

Integrating the conservation law gives

$$\partial_t u + \nabla \cdot \mathbf{f} \ = \ 0 \qquad \longrightarrow \qquad \int_\Omega \partial_t u + \nabla \cdot \mathbf{f} \ d\mathbf{x} \, dt \ = \ 0$$

- u can be continuous or discontinuous.
- Attempt to integrate the equations exactly.
- Distribute the integrals between the unknowns.
- For conservation, apply Gauss' divergence theorem.

This is related to the finite element approach.

Residual Distribution

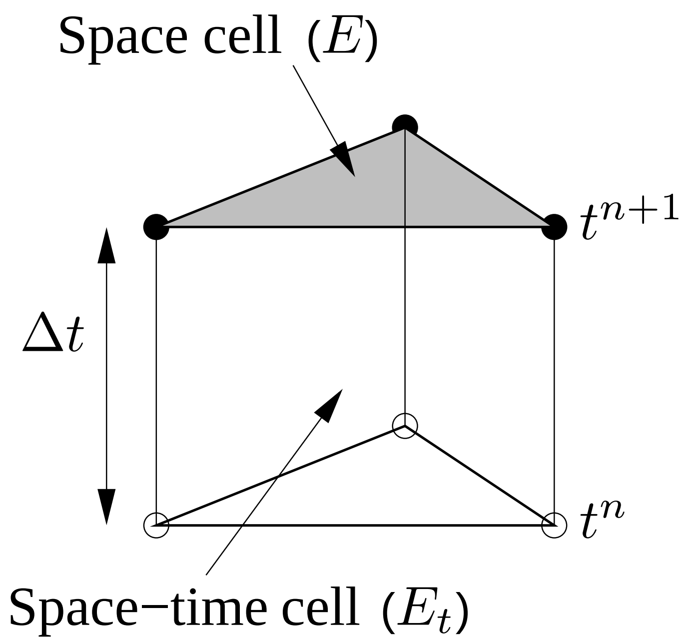
For a space-time mesh element (E_t), consider

$$\phi_{E_t} = \int_{E_t} \partial_t u + \nabla \cdot \mathbf{f} \; d\mathbf{x}\, dt$$

- For simplicity, assume that u is stored at mesh nodes and varies linearly in space and in time.
- In simple cases ϕ can be evaluated exactly using an appropriate conservative linearisation.
  - This leads to schemes with nice properties.
  - Conservation can be imposed in other cases.

Residual Distribution

- It is simplest to treat time and space slightly differently.
- Integrating over a space-time prism gives



$$\begin{aligned}\phi_{E_t} &= \int_E u^{n+1} - u^n \; d\mathbf{x} \; + \int_{t^n}^{t^{n+1}} \int_E \nabla \cdot \mathbf{f} \; d\mathbf{x}\, dt \\ &= \int_E u^{n+1} - u^n \; d\mathbf{x} \; - \int_{t^n}^{t^{n+1}} \oint_{\partial E} \mathbf{f} \cdot d\mathbf{n}\, dt\end{aligned}$$

Residual Distribution

The aim is to solve the equations given by

$$\partial_t u + \nabla \cdot \mathbf{f} \;=\; 0 \qquad \longrightarrow \qquad \sum_{E_t | i \in E_t} \beta_i^{E_t} \, \phi_{E_t} \;=\; 0 \qquad \forall \text{ nodes } i$$

This is done iteratively, at each time level, by:

- distributing each residual ϕ_{E_t} to adjacent nodes;
- carefully choosing the distribution coefficients $\beta_i^{E_t}$;
- applying a simple pseudo-time-stepping algorithm,

$$(u_i)^{(m+1)} \;=\; (u_i)^{(m)} \;-\; \frac{\Delta\tau}{|S_i|} \sum_{E_t | i \in E_t} \beta_i^{E_t} \, \phi_{E_t}$$

Scheme Design

Ideally, a residual distribution scheme would have the following properties.

- Conservative: for discontinuity capturing.
- Positive: to prohibit unphysical oscillations.
- Linearity Preserving: for accuracy.
- Continuous: for convergence of the iteration.
- Compact: for efficiency (and parallelism).
- Upwind: for physical realism.

Some Scalar Schemes

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For d-dimensional linear advection, assume that

- the spatial mesh is composed of simplices,
- the space-time mesh is prismatic,
- u varies linearly within each simplex and in time,

and write each element residual in the form

$$\phi_{E_t} = \sum_{i \in E_t} k_i u_i \quad \text{where} \quad k_i = \frac{\Delta t}{2d} \, \tilde{\mathbf{a}} \cdot \mathbf{n}_i \pm \frac{|E|}{d+1}$$

is an upwinding parameter.

Some Scalar Schemes

Since the k_i sum to zero in an element

$$\phi_{E_t} = \sum_{i \in E_t} k_i (u_i - u_{in}) \,, \qquad u_{in} = (\sum_{i \in E_t} {k_i}^+)^{-1} (\sum_{i \in E_t} {k_i}^+ u_i - \phi_{E_t})$$

- The N scheme (linear, positive)

$$\phi_i^{E_t} = \beta_i^{E_t} \phi_{E_t} = {k_i}^+ (u_i - u_{in})$$

- The LDA scheme (linear, linearity preserving)

$$\phi_i^{E_t} = \beta_i^{E_t} \phi_{E_t} = (\sum_{i \in E_t} {k_i}^+)^{-1} {k_i}^+ \phi_{E_t}$$

Some Scalar Schemes

To achieve positivity *and* linearity preservation:

- The PSI scheme limits the distribution coefficients of the N scheme:

$$\beta_i^{E_t} \quad \longleftarrow \quad \frac{(\beta_i^{E_t})^+}{\sum_{k\in E_t}(\beta_k^{E_t})^+} \qquad \Rightarrow \qquad \beta_i^{E_t} \in [0,1]$$

- Blended schemes use weighted averages:

$$\phi^{\mathrm{Blend}} \;=\; \theta\,\phi^{\mathrm{N}} + (1-\theta)\,\phi^{\mathrm{LDA}} \qquad\qquad \theta \in [0,1]$$

  - This is more robust and flexible but may not be positive for the most common choices of θ.

Issues with Continuity

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Forcing continuity at element faces can be restrictive.

- It is difficult to change the representation locally, within mesh elements, since it has a knock-on effect on neighbours. This interferes with:
  - conservation, particularly at boundaries;
  - h- and p-adaptivity;
  - the limiting of high order schemes for positivity;
  - the stability of time-dependent schemes.
- Discontinuous flow cannot be represented exactly.

Discontinuous Representation

If u is allowed to be discontinuous then

$$\int_{\Omega} \partial_t u + \nabla \cdot \mathbf{f} \; d\mathbf{x}\, dt$$

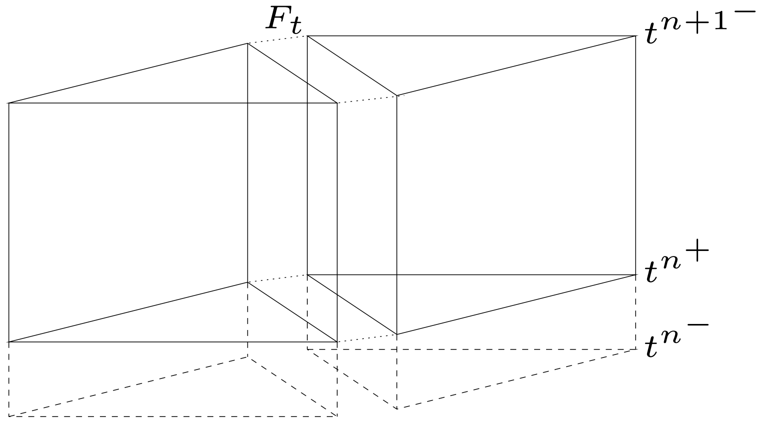
$$= \sum_{E_t} \int_{E} u^{n+1^-} - u^{n^+} \; d\mathbf{x} \; - \sum_{E_t} \int_{t^n}^{t^{n+1}} \oint_{\partial E} \mathbf{f} \cdot d\mathbf{n}\, dt$$

$$+ \sum_{F_t} \int_{t^n}^{t^{n+1}} \int_{F_t} [\mathbf{f}] \cdot d\mathbf{n}\, dt \; + \sum_{E} \int_{E} u^{n^+} - u^{n^-} \; d\mathbf{x}$$

It is possible to consider the face integrals separately.

$$\int_\Omega \partial_t u + \nabla \cdot \mathbf{f} \; d\mathbf{x}\, dt = \sum_{E_t} \phi_{E_t} + \sum_{F_t} \psi_{F_t} + \sum_{E} \psi_{E}$$

- The ψ are simply integrals over an interface of the flux difference across it.
- The ψ_{F_t} will be ignored here.



Discontinuities in Time

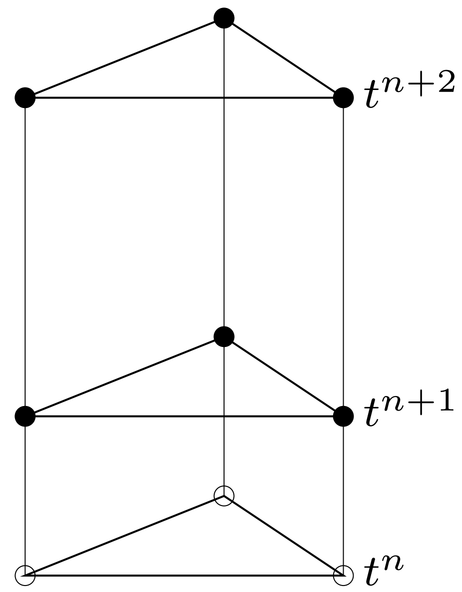
Integrating across temporal discontinuities gives

$$\psi_E = \int_E [u] \, d\mathbf{x}$$

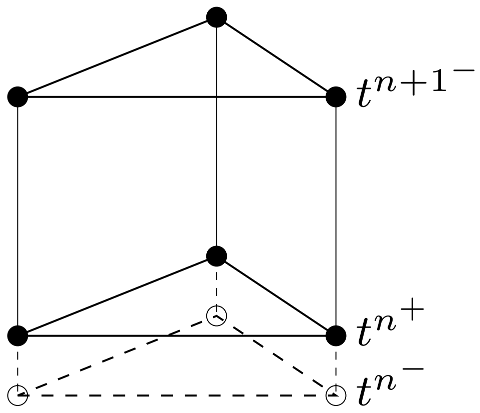
- Upwinding always distributes forward in time.
- This removes the necessity for the past shield condition on the space-time distribution.
- Schemes can now be positive for *any* time-step.

This is a degenerate two-layer scheme.

- The distribution is much simpler in the discontinuity.



Continuous Double Layer



Discontinuous in time

A positive, linearity preserving, distribution is

$$\psi^E_{i,n^-} \;=\; 0 \qquad\qquad \psi^E_{i,n^+} \;=\; \beta^E_{i,n^+}\,\psi_E \;=\; \frac{|E|}{d+1}\left(u_i^{n^+} - u_i^{n^-}\right)$$

The Distribution

The aim is to solve the equations given by

$$\partial_t u + \nabla \cdot \mathbf{f} \;=\; 0 \qquad \longrightarrow \qquad \sum_{E_t | i \in E_t} \beta_i^{E_t}\, \phi_{E_t} \;+\; \sum_{E | i \in E} \beta_{i,n^+}^{E}\, \psi_E \;=\; 0$$

This is done iteratively, at each time level, by:

- distributing the ϕ_{E_t} and ψ_E to adjacent vertices;
- choosing the distribution coefficients, $\beta_i^{E_t}$ and β_{i,n^+}^{E};
- applying a simple pseudo-time-stepping algorithm,

$$(u_i)^{(m+1)} \;=\; (u_i)^{(m)} \;+\; \frac{\Delta\tau}{|S_i|}\Big(\sum_{E_t | i \in E_t} \beta_i^{E_t}\, \phi_{E_t} + \sum_{E | i \in E} \beta_{i,n^+}^{E}\, \psi_E \Big)$$

Results: Linear Advection (2D)

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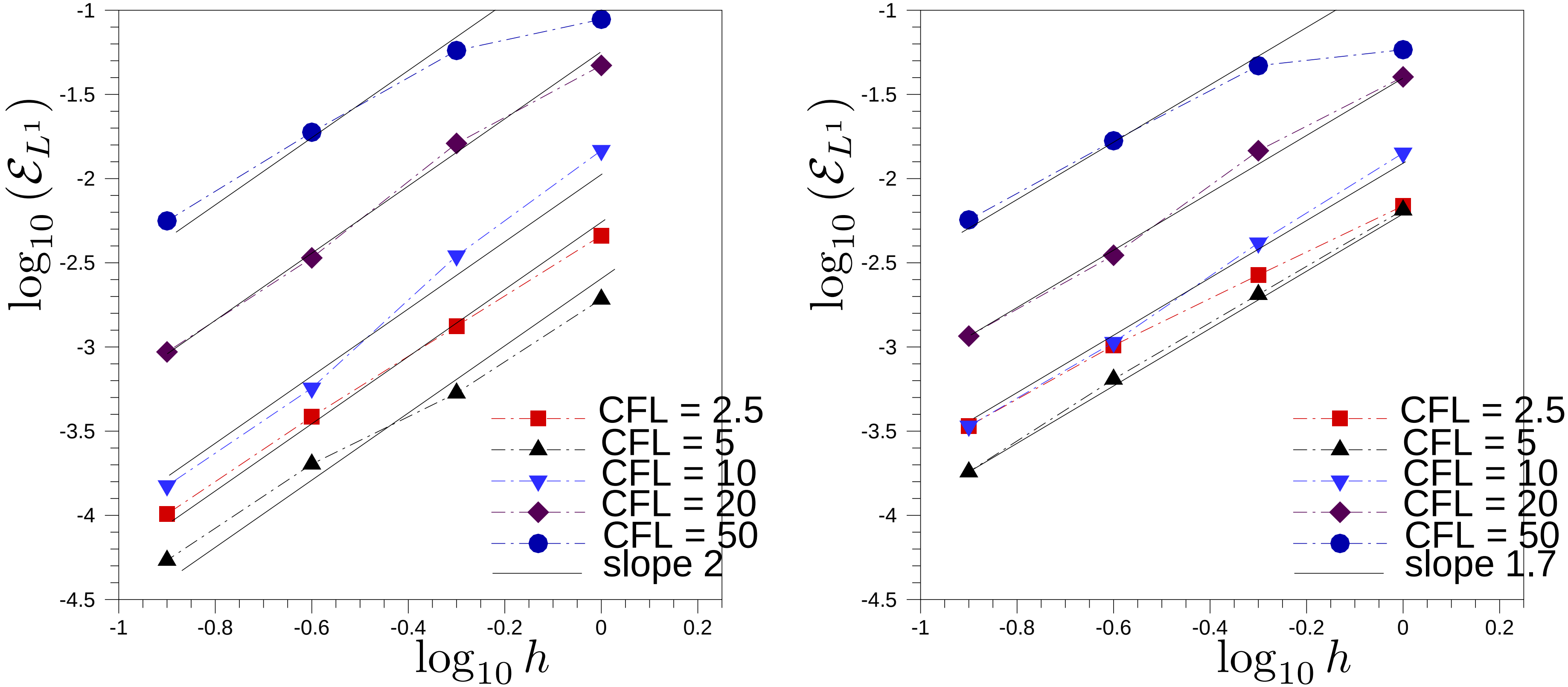

Mesh convergence for constant advection of a smooth profile: L^1 error for the ST LDA (left) and ST LDA-N (right) schemes.

Consider the system of conservation laws

$$\partial_t U + \nabla \cdot \mathbf{F} \;=\; 0 \qquad \text{or} \qquad \partial_t U + \mathbf{A} \cdot \nabla U \;=\; 0$$

on a domain Ω, where $\mathbf{A}$ gives the flux Jacobians.

$$\Phi_{E_t} \;=\; \int_E U^{n+1^-} - U^{n^+} \, d\mathbf{x} \;-\; \int_{t^n}^{t^{n+1}} \oint_{\partial E} \mathbf{F} \cdot d\mathbf{n} \, dt$$

$$\Psi_{F_t} \;=\; \int_{t^n}^{t^{n+1}} \int_{F_t} [\mathbf{F}] \cdot d\mathbf{n} \, dt \qquad \Psi_E \;=\; \int_E U^{n^+} - U^{n^-} \, d\mathbf{x}$$

can all (with care) be evaluated exactly, decomposed and distributed to the element vertices.

Nonlinear Systems

The element-based residuals take the form

$$\Phi_{E_t} = \sum_{i \in E_t} \mathbf{K}_i \, U_i \quad \text{where} \quad \mathbf{K}_i = \frac{\Delta t}{2d} \, \widetilde{\mathbf{A}} \cdot \mathbf{n}_i \pm \frac{|E|}{d+1} \mathbf{I}$$

The face-based residuals can be written

$$\Psi_{F_t} = \sum_{i \in F_t} \mathbf{K}_i \left({U_i}^R - {U_i}^L \right) \quad \text{where} \quad \mathbf{K}_i = \frac{\Delta t}{2d} \, \widehat{\mathbf{A}}_i \cdot \mathbf{n}$$

$$\Psi_E = \frac{|E|}{d+1} \sum_{i \in E} \left(U_i^{n^+} - U_i^{n^-} \right)$$

The $\mathbf{K}_i$ can be diagonalised to get the ${\mathbf{K}_i}^+$.

Source Terms

Include the source term in the element residual:

$$\Phi_{E_t} \;=\; \int_{E_t} \partial_t U + \nabla \cdot \mathbf{F} - S \;\, d\mathbf{x}\, dt$$

- With shallow water flows, care is needed to ensure that, when $b+d$ is constant,

$$\int_E \nabla \left(\frac{gd^2}{2} \right) \, d\mathbf{x} \;=\; -\int_E gd\,\nabla b \; d\mathbf{x}$$

- The conservative schemes apply as before.
- Discontinuities in space are more challenging.

Results: Euler Equations

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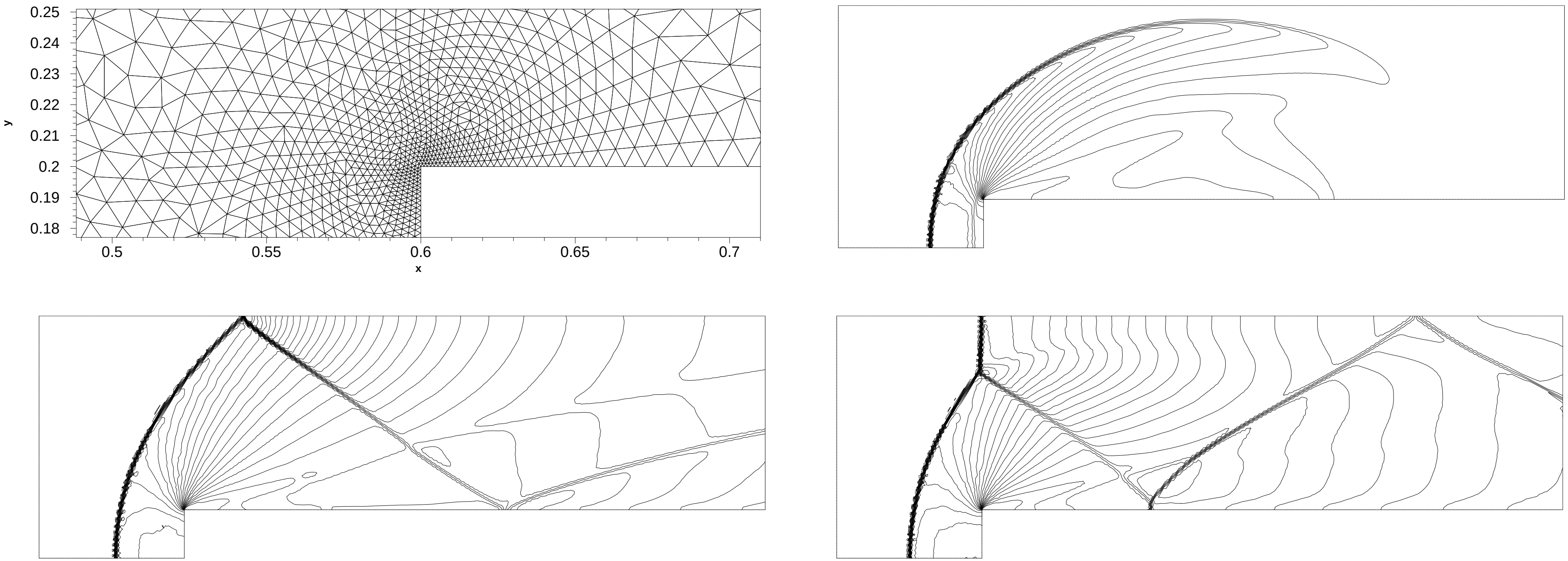

Supercritical backward facing step: ST LDA-N scheme; density contours with $M_\infty = 3.0$ and $\mathrm{CFL}_{\max} = 12.5$.

Results: Shallow Water Flow

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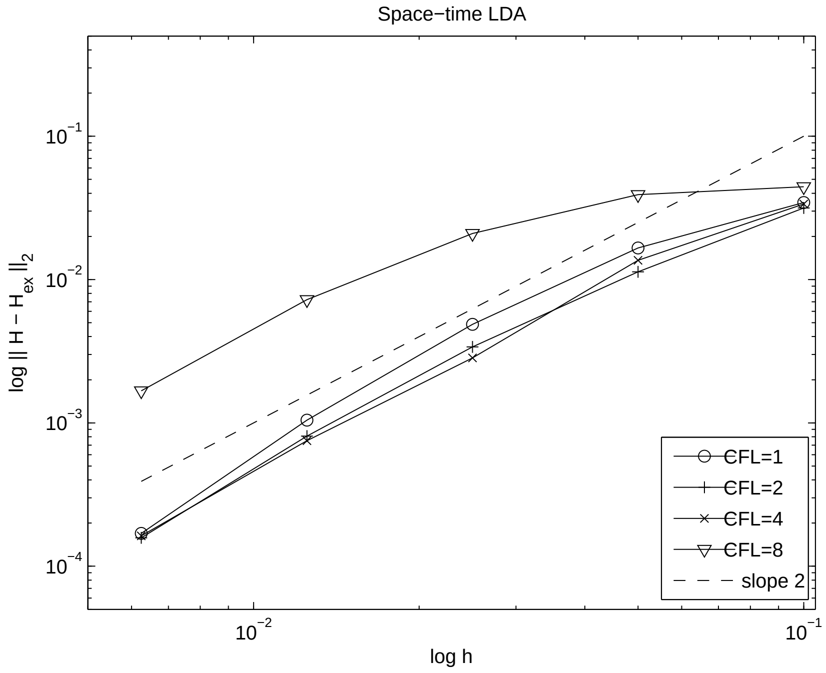

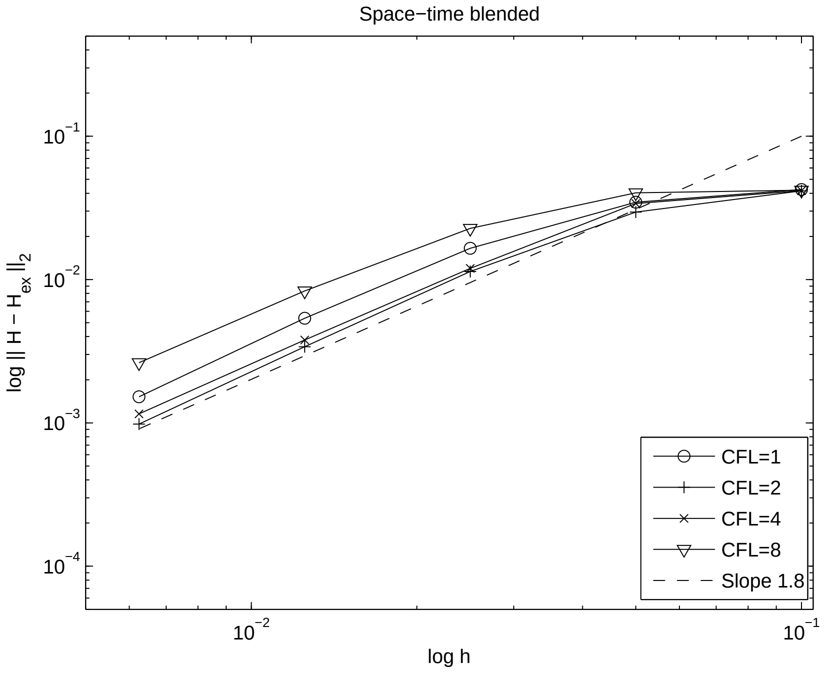

Travelling vortex (exact solution), mesh convergence:
ST LDA (left) and ST LDA-N (right) schemes.

Results: Shallow Water Flow

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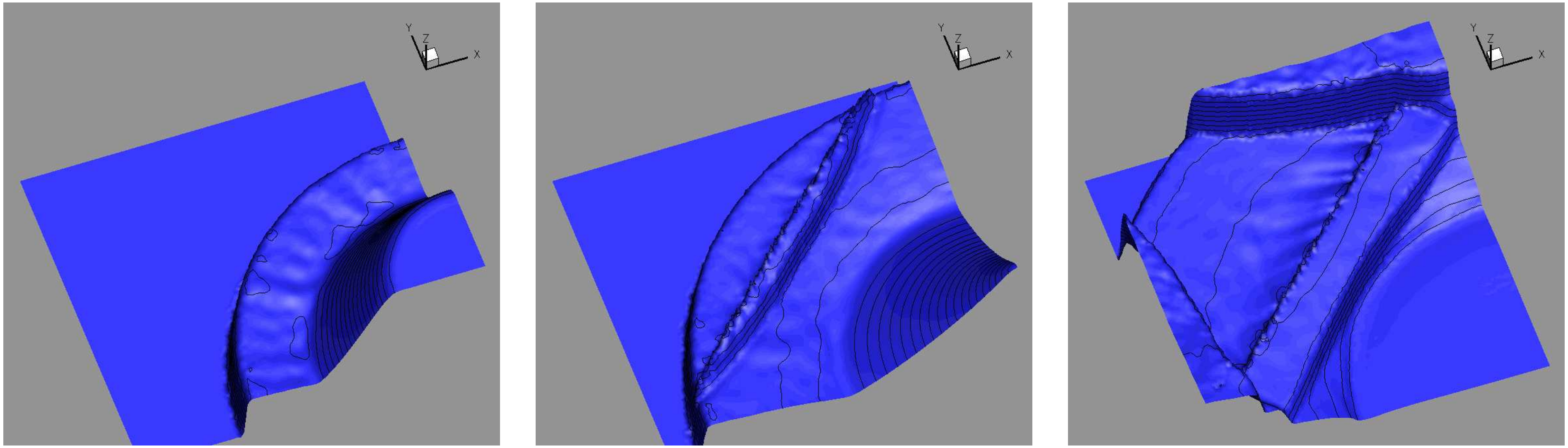

Circular dam break, discontinuous bed, unstructured mesh, free surface: stabilised LLF scheme.

Results: Shallow Water Flow

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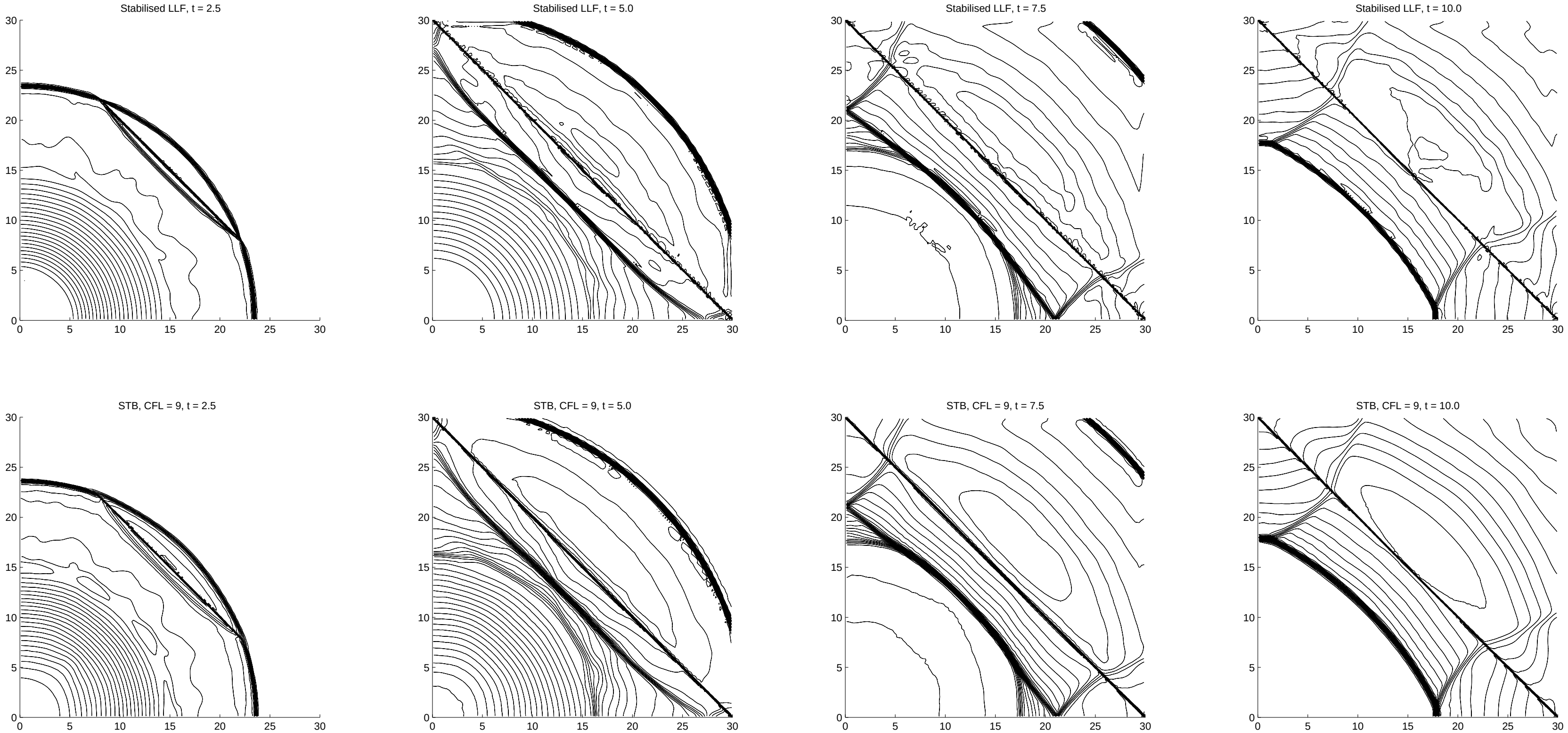
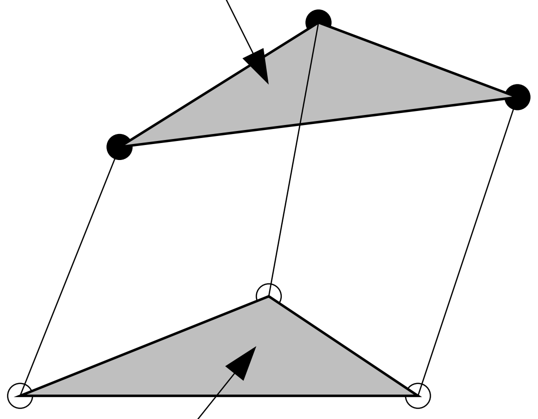


Circular dam break, discontinuous bed, locally refined mesh, free surface: stabilised LLF scheme, $\mathrm{CFL}_{\max} = 0.8$ (top); ST LDA-N scheme, $\mathrm{CFL}_{\max} = 9.0$ (bottom).

Moving Meshes

- The mesh at the new time may differ from that at the old time.
- Integrating over a distorted space-time prism gives



$$\begin{aligned}\phi_{E_t} &= \int_{E^{n+1}} u^{n+1}\,d\mathbf{x} - \int_{E^n} u^n\,d\mathbf{x} - \int_{t^n}^{t^{n+1}} \oint_{\partial E} \mathbf{f}_t \cdot d\mathbf{n}_t\,dt \\ &= \int_{E^{n+1}} u^{n+1}\,d\mathbf{x} - \int_{E^n} u^n\,d\mathbf{x} - \int_{t^n}^{t^{n+1}} \oint_{\partial E} (\mathbf{f} - u\mathbf{v}) \cdot d\mathbf{n}\,dt\end{aligned}$$

Moving Meshes

For d-dimensional linear advection, the element residual can still be written in the form

$$\phi_{E_t} \approx \sum_{i \in E_t} k_i u_i \qquad \text{where} \qquad k_i^* = \frac{\Delta t}{2d} (\tilde{\mathbf{a}}^* - \mathbf{v}_i^*) \cdot \mathbf{n}_i^* \pm \frac{|E^*|}{d+1}$$

- The superscript $\cdot^*$ indicates the time level.
- The mesh velocity should not be averaged because it may not satisfy $\nabla \cdot \mathbf{v} = 0$.
- The distribution schemes can be applied as before.

Implementation

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- The mesh nodes are moved during pseudo-time-stepping, according to

$$\mathbf{x}_i^{(m+1)} = \mathbf{x}_i^{(m)} + \frac{\sum_{E|i\in E} M_E \, \mathbf{x}_E^{(m)}}{\sum_{E|i\in E} M_E}$$

- A surface area monitor, $M_E = |E|\,(1+\alpha|\nabla u|_E^2)^{\frac{1}{2}}$ is interleaved with a Laplacian smoother.

- The pseudo-time-stepping is continued after the movement is stopped, with mesh velocities

$$\mathbf{v}_i = \frac{\mathbf{x}_i^{n+1} - \mathbf{x}_i^n}{\Delta t}$$

Results

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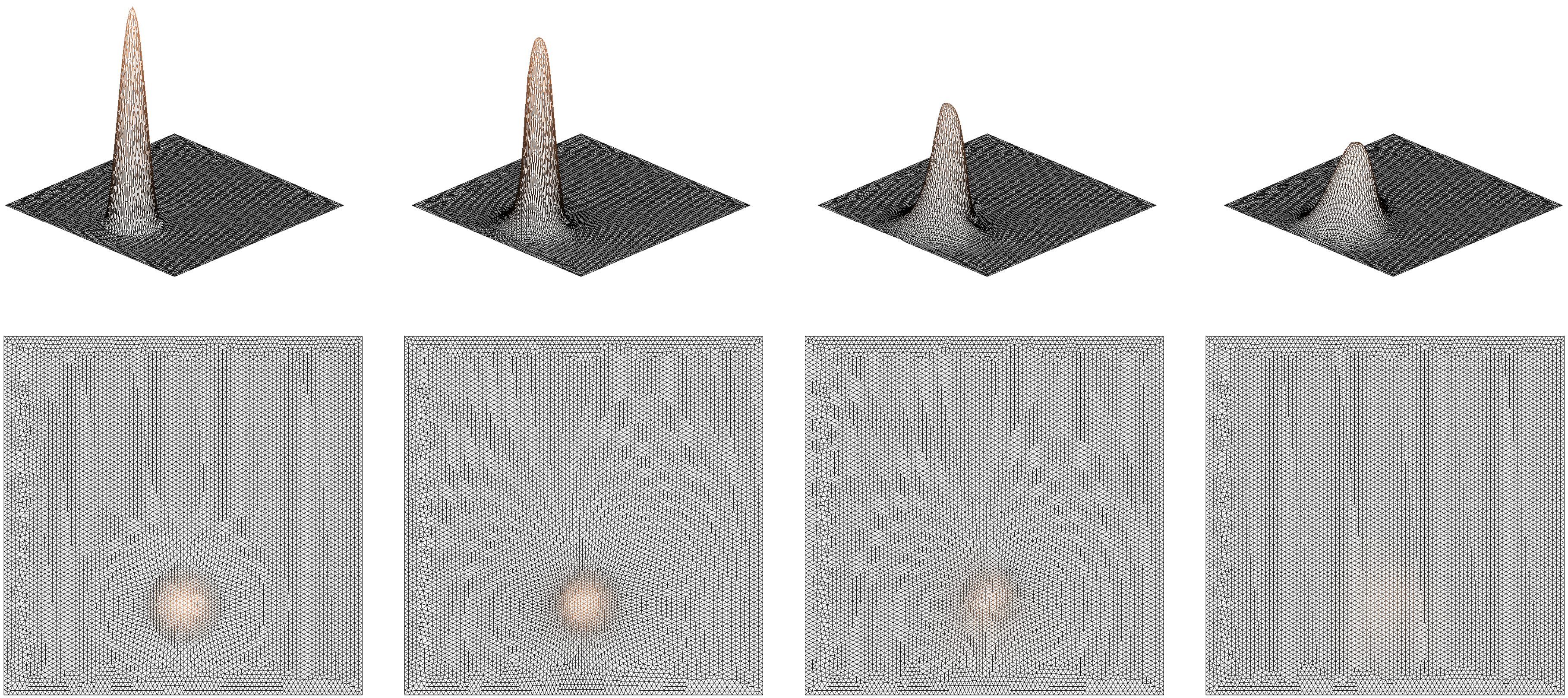
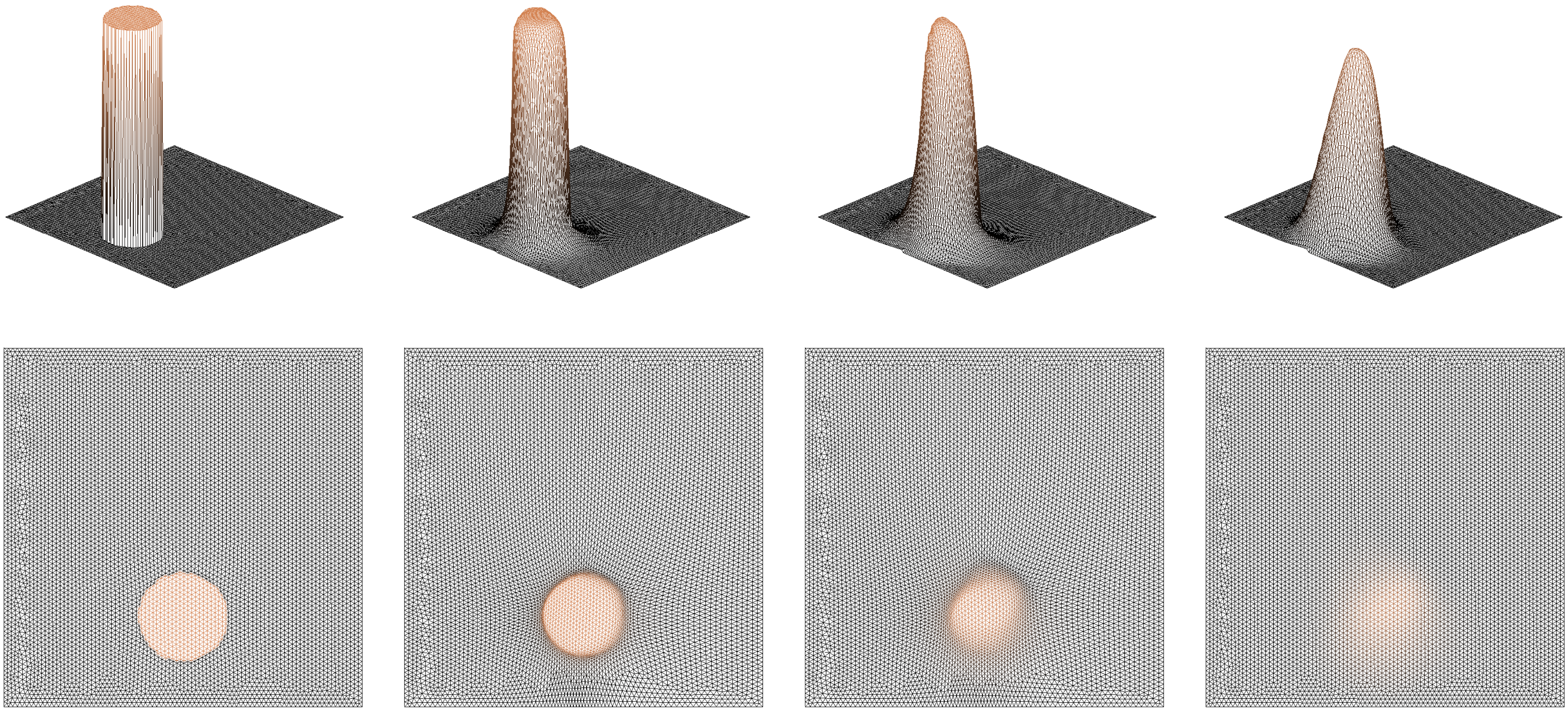


Rotating cosine-squared profile, space-time PSI scheme: moving mesh (0, 1, 5 revolutions), fixed mesh (5 revolutions).

Results



Rotating cylinder profile, space-time PSI scheme:
moving mesh (0, 1, 5 revolutions), fixed mesh (5 revolutions).

Current Issues

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For linear advection on fixed meshes the scheme is:

- positive for any time-step;
- conservative, linearity preserving, compact, upwind and continuous;
- second order accurate for smooth profiles.

It also gives good approximations to the Euler and shallow water equations, although it is not yet:

- easy to converge the inner iteration;
- as robust as the best flux-based schemes.

Current Issues

On moving meshes, the scheme is designed to retain all of the fixed mesh properties, but:

- these are only the first results;
- it's not yet clear whether imposing positivity constrains the time-step;
- second order accuracy relies on using appropriate quadrature to evaluate the residual.

At the moment, the moving mesh scheme for nonlinear systems only exists on paper.