A 1D Stabilized Finite Element Model for Non-hydrostatic Wave Breaking and Run-up

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Abstract We present a stabilized finite element model for wave propagation, breaking and run-up. Propagation is modelled by a form of the enhanced Boussinesq equations, while energy transformation in breaking regions is captured by reverting to the shallow water equations and allowing waves to locally converge into discontinuities. To discretize the system we propose a non-linear variant of the stabilized finite element method of (Ricchiuto and Filippini, *J.Comput.Phys.* 2014). To guarantee monotone shock capturing, a non-linear mass-lumping procedure is proposed which locally reverts the third order finite element scheme to the first order upwind scheme. We present different definitions of the breaking criterion, including a local implementation of the convective criterion of (Bjørkavåg and Kalisch, *Phys.Letters A* 2011), and discuss in some detail the implementation of the shock capturing technique. The robustness of the scheme and the behaviour of different breaking criteria is investigated on several cases with available experimental data.

1 Modelling Approach and Main Objectives

When arriving in the near shore region, waves are relatively long, with a ratio water-depth over wavelength $\sigma^2 \ll 1$. When approaching the shoreline the wave steepens and non-linear effects start to become dominating up to the moment in which the wave breaks ($\varepsilon = A/d \sim 1$), with important production of vorticity, and with potential

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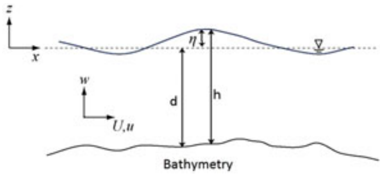

Fig. 1 Depth averaged equations: notation

energy transformation and dissipation. The first phase of the process can be modelled by the a properly chosen set of non-hydrostatic equations, such as for example, the Boussinesq equations, or other type of dispersive models [2]. The treatment of wave breaking is more delicate. Several approaches exist, see [10] for a recent review. The extensive study of [4], indicate that the energy transformation in the breaking region can be modelled by the dissipation across nonlinear discontinuities of hyperbolic models such as the shallow water equations. This is confirmed by the numerical results of [5, 10, 13, 18, 19]. For this reason, we use a hybrid model reverting from the enhanced Boussinesq to the shallow water equations in properly defined breaking regions.

To discretize the equations, we start from the stabilized finite element approach of [16], which has a very interesting potential in terms of providing low dispersion errors and very high accuracy on unstructured adaptive meshes. Here, we propose a new nonlinear variant of the method. In our approach, the third order finite element scheme is reverted to the first upwind scheme across discontinuities via nonlinear mass-lumping procedure. The objective of this paper is to present the hybrid modelling approach, and in particular the definition of the breaking detection algorithm, and the discussion of the discontinuity capturing methodology, and in particular of the choice of the mass-lumping limiter. Concerning the first aspect, we consider the hybrid criteria of [19], and [10], and a novel local implementation of the convective criterion of [3]. The mass-lumping limiter is instead chosen based on the requirement that smooth extrema should be preserved, and is based on a smoothness sensor. The model obtained is extensively tested. The behaviour of different breaking models is studied on several cases allowing comparisons with experimental data.

2 Hybrid Equations for Wave Breaking Treatment

To simulate wave propagation, we start from the following system, based on the enhanced Boussinesq equations in the form proposed in [14] (cf. Fig. 1):

$$\begin{cases} \partial_t \eta + \partial_x q = 0 & \text{(1a)} \\ \partial_t q + \partial_x (uq) + gh\partial_x \eta + ghC_f u = f_{\text{break}}(x,t)\mathscr{D}(\eta, q) & \text{(1b)} \\ \mathscr{D}(\eta, q) = Bd^2 \partial_{xxt} q + \beta g d^3 \partial_{xxx}\eta + \dfrac{1}{3} d\partial_x d \partial_{xt} q + 2\beta g d^2 \partial_x d \partial_{xx}\eta & \text{(1c)} \end{cases}$$

with $\eta = \eta(x, t)$ the wave height, $q = hu$ the discharge, $h = \eta + d$ the local height of the water column, $u = u(x, t)$ the depth-averaged velocity, and with $d = d(x)$ the depth w.r.t. an average still water level. The term $\mathscr{D}(\eta, q)$ represents the dispersive effects, with B and β obtained by optimizing the linear dispersion relation. The flag f_{break} assumes the value 1 in the Boussinesq regions, and 0 in breaking fronts, and allows to revert to the hyperbolic shallow water equations. We consider here three breaking criteria.

The simplest, due to Tonelli and Petti [19], is based on a local measure of non-linearity. Breaking regions are denoted as those for which $\varepsilon = |\eta|/|d| \geq \varepsilon_{\text{cr}}$, with $\varepsilon_{\text{cr}} \approx 0.8$. Once a breaking front has been detected, its end (*de-breaking*) is located as the point in the flow direction where ε is below ≈ 0.35 (see [10, 19] for more).

The second criterion, proposed in [10], uses a hybrid condition involving vertical velocity and slope. A point is flagged as breaking if *either* $|\partial_t \eta| > \gamma\sqrt{gh}$ *or* $|\partial_x \eta| > \tan\phi_{\text{cr}}$. The values γ and ϕ_{cr} may depend on the case simulated (see [10] for more).

Lastly, we consider a local implementation of the *convective* criterion of [3]. The idea is that breaking occurs when the free surface velocity is larger than the wave celerity. In [3] only simple cases have been considered for which at least the celerity is known a-priori. Here we proceed as follows:

1. Pre-flagging using the criterion of [10] with smaller γ and ϕ_{cr};
2. For every front (set of neighbouring pre-flagged nodes) locate crest and trough;
3. For every front evaluate celerity C_b and crest velocity u_S;
4. Final flagging: if $u_S \geq C_b$ set $f_{\text{break}} = 0$ for $x \in [x_{\min}, x_{\max}]$

Combining the relations $\partial_t \eta \approx -C_b \partial_x \eta$ and $\partial_t \eta = -\partial_x q$, we obtain $C_b \approx \partial_x q / \partial_x \eta$ which is implemented as $C_b = (q_{\text{crest}} - q_{\text{trough}})/(\eta_{\text{crest}} - \eta_{\text{trough}})$. To obtain u_S, vertical asymptotic expansions can be used to show that (see e.g. [3, 7]) $u_S = u - \alpha h^2 \partial_{xx} u$, with $\alpha = 1/3$ the analytical value. Here this constant is kept free, to account for the different wave shoaling provided by Boussinesq models, and to be able to correct wave under-shoaling [7]. The results reported are obtained with $\alpha = 2/3$. A parametric study is under way to understand the influence of this parameter for different Boussinesq equations. The definition of $[x_{\min}, x_{\max}]$, giving local position and width of the breaking region is the same used in [10, 18].

3 Discretization and Discontinuity Capturing

The numerical discretization follows the initial developments made in [16] where upwind stabilized residual based and finite element discretizations of the Boussinesq equations of [14] have been analyzed and tested on a large number of one and two-dimensional wave propagation problems. Already for P^1 interpolation, the results of [16] show a high potential of the approach in terms of providing low dispersion error and high accuracy with the flexibility of a natural unstructured mesh formulation.

Here we propose a discontinuity capturing method based on a nonlinear lumping of the mass matrix allowing to locally recover first order upwind flux differencing.

Set $\mathbf{W} = [\eta \quad q]^T$, $F(\mathbf{W}) = [q \quad (uq + g\frac{h^2}{2})]^T$, $S = [0 \ - gh\partial_x d]^T$, $D = [0 \ \mathscr{D}]^T$, $F_f = [0 \quad - ghC_f u]^T$, and $A = \frac{\partial F(\mathbf{W})}{\partial \mathbf{W}}$ the shallow water flux Jacobian. Let also $d\mathbf{W}_i/dt$ be the (continuous) time derivative of the value of $\mathbf{W}$ at node i, Δx the 1D mesh spatial spacing, I_2 the 2×2 identity matrix, and denote with superscripts $i \pm 1/2$ arithmetic cell-average values. The spatial discretization we propose reads:

$$\begin{aligned} \Delta x \frac{d\mathbf{W}_i}{dt} &+ \delta^{i-1/2}\{\frac{\Delta x}{6}[\frac{d\mathbf{W}_{i-1}}{dt} - \frac{d\mathbf{W}_i}{dt}] + \frac{\Delta x}{2}\mathrm{sign}(A^{i-\frac{1}{2}})\frac{d\mathbf{W}^{i-\frac{1}{2}}}{dt}\} \\ &+ \delta^{i+1/2}\{\frac{\Delta x}{6}[\frac{d\mathbf{W}_{i+1}}{dt} - \frac{d\mathbf{W}_i}{dt}] - \frac{\Delta x}{2}\mathrm{sign}(A^{i+\frac{1}{2}})\frac{d\mathbf{W}^{i+\frac{1}{2}}}{dt}\} \\ &+ \frac{\mathrm{I}_2 + \mathrm{sign}(A^{i-\frac{1}{2}})}{2}(F_i - F_{i-1} + \Delta x\, S^{i-\frac{1}{2}}) \\ &+ \frac{\mathrm{I}_2 - \mathrm{sign}(A^{i+\frac{1}{2}})}{2}(F_{i+1} - F_i + \Delta x\, S^{i+\frac{1}{2}}) = f_{\mathrm{break}_i} D_i + \mathscr{F}_{f\,i} \end{aligned} \tag{2}$$

One can distinguish the terms associated to the Galerkin approximation, and the stabilization terms, multiplied by the sign of the Jacobian A. These terms have been simplified using the properties of the P^1 finite element approximation, as detailed in [16]. The right hand side contains the contributions of friction and dispersive terms, also involving centred and upwind biasing contributions, and requiring the evaluation of auxiliary variables necessary for the high order derivatives. These terms are quite complex and we refer to [16] for details. Note that if the right hand side is zero, for $\delta^{i\pm 1/2} = 0$ we obtain the standard first order upwind flux difference scheme. Our implementation in this limit actually follows the well-balanced, positivity preserving upwind approach of e.g. [6], and it includes an entropy fix [9] to avoid problems in strongly accelerating regions with small water heights (cf. [1] for more). So, if $\delta_i = 0$ and $f_{\mathrm{break}_i} = 0$ the scheme is locally first-order, it preserves the positivity of the depth, and it is well-balanced. Whenever $f_{\mathrm{break}_i} = 1$, we automatically set $\delta_i = 1$. In this case, the resulting scheme is third-order accurate in space, as amply demonstrated in [16]. The main ingredient is the choice of the limiter $\delta(\mathbf{W})$. An extensive study and comparison of different limiters available in literature is provided in [1]. Many of these result in an over-dissipative method. An effective definition is based on the smoothness sensor

$$\sigma_i = \min(1, r_i)\,, \quad r_i = \frac{\dfrac{|\eta_i - \eta_{i-1}|}{\Delta x} + \dfrac{|\eta_i - \eta_{i+1}|}{\Delta x}}{\dfrac{|\eta_{i+2} - 4\eta_{i+1} + 6\eta_i - 4\eta_{i-1} + \eta_{i-2}|}{12\Delta x^2}}$$

with r_i the ratio between the magnitude of the first order derivative and the difference between a fourth and second order approximation of the second order derivative. In smooth regions, the denominator of r_i is of $\mathscr{O}(\Delta x^2)$ while the numerator is bounded, resulting in $\sigma = 1$. On a discontinuity, while the numerator is of an order $\mathscr{O}(1/\Delta x)$, the denominator is of an order $\mathscr{O}(1/\Delta x^2)$, giving $\sigma = \mathscr{O}(\Delta x)$. Finally, we have set

$\delta_i = \sigma_i$ if $\sigma_i \le 1/2$, and $\delta_i = 1$ otherwise. The typical result obtained for a standard Riemann problem is reported on Fig. 2 where the sensor proposed is compared to the Superbee and to the Monotonized Central limiter [12]. In the tests that follow, as in [15] we pre-multiply δ by an exponential function smoothly switching off high order terms in vicinity of dry fronts. For all the tests considered, time integration has been performed with the non-dissipative second-order Crank-Nicholson scheme.

4 Numerical Validation

4.1 Periodic Wave over a Submerged Bar

We consider the experiment of plunging breaking periodic waves over a submerged bar of Beji and Battjes. This test has been first done by Dingemans to verify the Delft Hydraulics model HISWA, and then repeated by Beji and Battjes [7, 17]. To give an overview of the qualitative behaviour of the solution, wave profiles at different breaking instants are reported on Fig. 3 for the three tested breaking criteria. In the figure we report the wave profiles at the first breaking instance, at an intermediate time (same for all criteria), and at the last seen breaking instance for a given wave. The vertical lines delimit the breaking region in which the shallow water equations are used. The criterion of [10] provides the strongest and most regular breaking behaviour, with wave heights considerably decreasing along the plateau. The local implementation of the convective criterion proposed gives weaker breaking, and slightly higher waves. We have also observed numerically a more intermittent behaviour of the flag. Lastly, the criterion of [19] gives the weakest breaking, with wave heights only slightly decreasing.

These observations are confirmed by the temporal evolution of the wave height in four experimental gauges (respectively at the beginning and the end of the upward slope of the bar and in the middle and end of the plateau), reported in Fig. 4. The results obtained with the criterion of [10] show very good agreement with experiments, while the convective criterion is slightly worse in terms of wave heights. The non-linearity sensor of [19] fails to detect some wave breaking areas, at least on this level of mesh size. We mention that better results are obtained in [19] on much finer grids, and that the results of the convective criterion could be improved by increasing the value of the constant α in the definition of the free surface velocity (under investigation).

4.2 Run-up of a Periodic Wave

This test, known as the spilling breaking test of Hansen and Svendsen, involves the shoaling and breaking over a shore of a set of regular waves, and corresponds to the test 051041 described in [8]. A qualitative view of the wave profiles obtained is

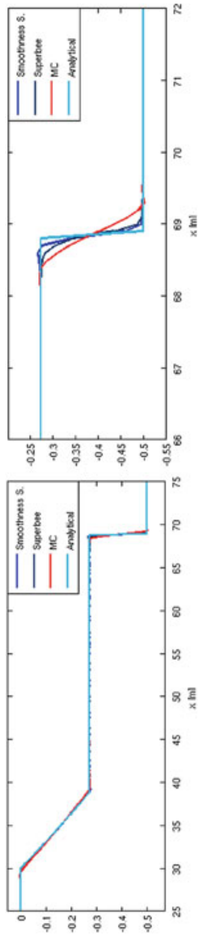

Fig. 2 Riemann problem: smoothness sensor versus Superbee and Monotonized Central limiters

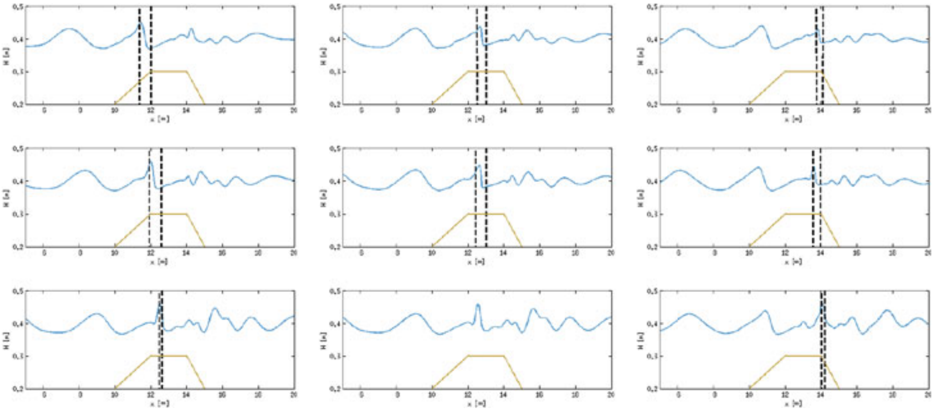

Fig. 3 Plunging breaking test. Wave profiles corresponding to: first (*left*), intermediate (*center*), and last breaking instants. Breaking criterion: Kazolea, Delis and Synolakis (*top*), convective (*middle*), Tonelli and Petti (*bottom*). The *vertical lines* delimit the breaking (shallow water) region

reported on Fig. 5, showing the effects of wave shoaling and wave breaking over a constant slope bathymetry. On Fig. 6, instead, we report a quantitative comparison of the time-average of the wave height and of the wave set-up along the shore with experimental data. The numerical results are those obtained with the criterion of [10], and with the convective criterion. We can see from the change in slope in the computed results that wave breaking is predicted too early by the criterion of Kazolea, Delis and Synolakis, while the wave heights are under-predicted in both cases. This, according to [20], might be due to the use of a weakly nonlinear Boussinesq model for propagation. For this test, the convective criterion gives a better prediction of the breaking position. The wave set-up is predicted very well by both models.

5 Conclusions and Perspectives

We have presented a one-dimensional finite element model for non-hydrostatic wave propagation, breaking, and run-up. The model combines a weakly non-linear Boussinesq model with the hyperbolic shallow water equations. The blending is obtained via a wave breaking criterion based on physical arguments. We propose an enhancement of the stabilized finite element method of [16] consisting in a discontinuity capturing technique relying on a nonlinear lumping of the mass matrix. This allows the local treatment of discontinuous shallow water flows, and wetting/drying fronts. When combined with the hybrid dispersive-hyperbolic modeling approach, this method allows to provide an accurate description of the wave transformation in the near shore region.

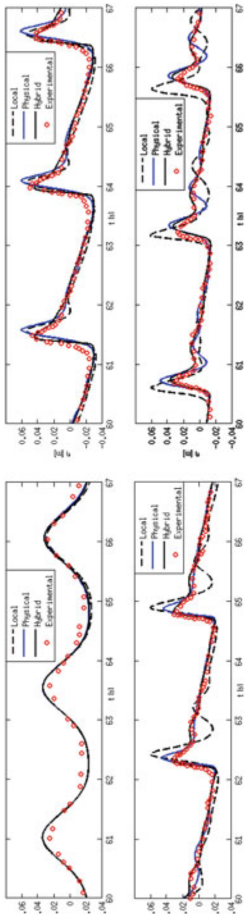

Fig. 4 Gauge data for the submerged bar test. *Top* gauge 1 (*left*) and 2 (*right*). *Bottom* gauge 3 (*left*) and 4 (*right*). Breaking criteria: Tonelli and Petti (local in the legend), Kazolea, Delis and Synolakis (hybrid in the legend), and convective (physical in the legend). Symbols: experiments

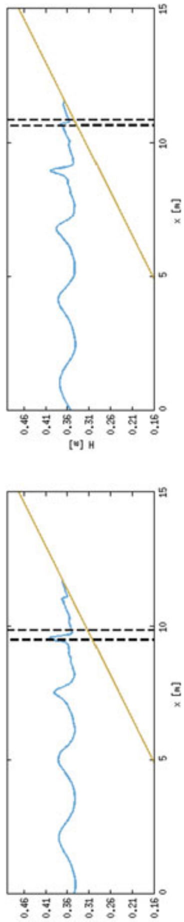

Fig. 5 Hansen and Svendsen test 051041. Snapshots of the wave profiles at different instants

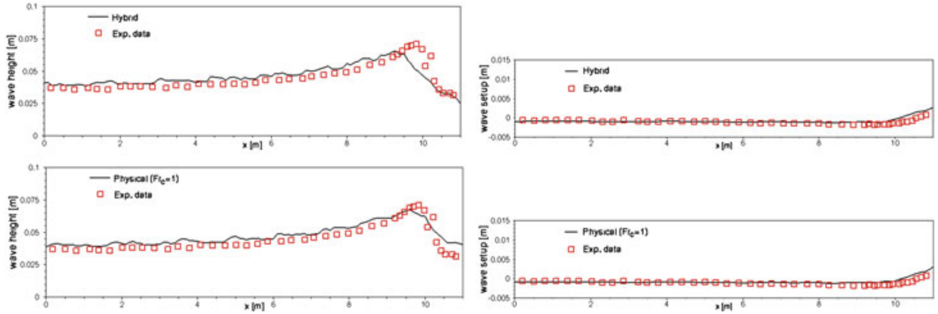

Fig. 6 Hansen and Svendsen test 051041. Wave height *(left)* and mean water level (set-up) *(right)* for Kazolea, Delis and Synolakis (*top row*, hybrid in the legend), and convective criterion (*bottom row*, physical in the legend)

The numerical results, while confirming the stability and robustness of the numerics proposed, also provide an initial validation for the different breaking criteria tested. Our implementation of the convective breaking criterion of [3] shows some promise, even though the criterion of Kazolea et al. gives similar, and sometimes better, results, with a much simpler implementation. The very simple criterion of [19] is not able to provide similar results.

The work planned for the future involves a more systematic study of the definition of the free surface velocity used on the convective criterion, the implementation of the model in two dimensions and on unstructured meshes, following [10, 16], and the use of fully non-linear dispersive models, as in [5, 11].

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