On the numerical treatment of some dispersive PDEs for free surface hydrodynamics

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Tulane University

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Thanks to ....

- A. Filippini (PhD student, Inria BSO - CARDAMOM), P. Bacigaluppi (now PhD student Zürich University)
- M. Kazolea (research scientist, Inria BSO - CARDAMOM)
- P. Bonneton, D. Lannes, F. Marche

Main content ....

1. MR and A. Filippini, J.Comput.Phys. 271, 2014
2. A. Filippini, M. Kazolea and MR, J.Comput.Phys., 2016 (in press, available online)
3. M. Ricchiuto, A. Filippini and M. Kazolea, in preparation

Context 1/5

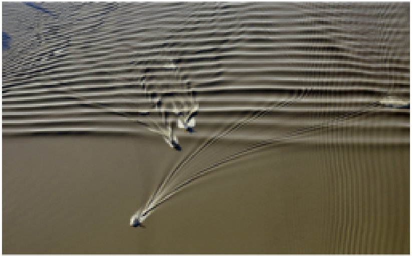
Aim: efficient simulation of nonlinear, dispersive water waves in the near-shore.

- Understanding local wave direction, height, strength
- Influence of local bathymetry and climate
- Erosion process, sediment transport, management of local activities (tourism, oyster cultures, surf competitions ...)

Context 2/5

Aim: efficient simulation of nonlinear, dispersive water waves in the near-shore.

- Understanding local wave direction, height, strength
- Propagation and impact of tsunami (left : Sumatra 2004)
- Propagation and inundation due to tidal bores (right : Garonne river)

Context 3/5

Aim: efficient simulation of nonlinear, dispersive water waves in the near-shore.

- Understanding local wave direction, height, strength
- Propagation and impact of tsunami (left : Tohoku, Naka river 2011)
- Propagation and inundation due to tidal bores (right : Garonne river)

Context 4/5

Aim: efficient simulation of nonlinear, dispersive water waves in the near-shore.

- the best description: 3D incompressible Euler or NS equations
- large scales : depth-averaged approximation and 2D restriction

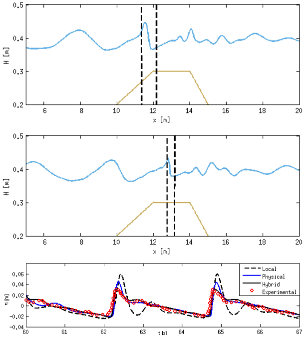
Context 5/5

Near shore hydrodynamics

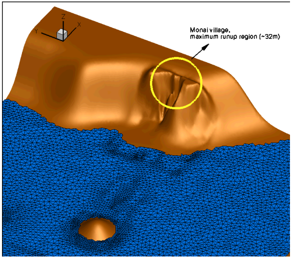
Bonneton,Chazel,Lannes,Marche,Tissier - Delis,Kazolea,Synolakis - Kirby,Grilli,et al (FUNWAVE-TVD) - Smit,Zijlema et al (SWASH) - Ricchiuto et al - etc.

(Ribbed channel clip)

Propagation: nonlinear dispersive model



Breaking: local coupling with "dissipative" model



Runup/flooding : hydrostatic model

Context in short

What are the main ingredients ?

1. Time dependent
2. Wave propagation : small dissipation and **dispersion** error
3. Steep fronts (bores) and dry states : non-oscillatory/positivity preserving
4. Hyperbolic component of the PDEs : need "upwinding"
5. Complex topographies, moving fronts: unstructured adaptive meshes
6. BCs issue: sponge layers and wave generation layers .. possibly combined with local mesh coarsening/refinement

Same as compressible flow or aeroacoustics ??

CONTENTS

An example : solitary wave propagation over a shelf

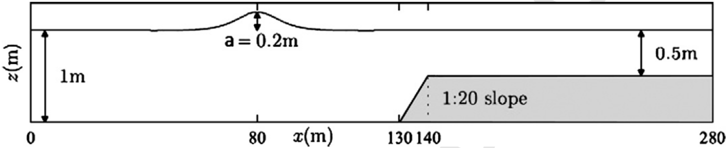
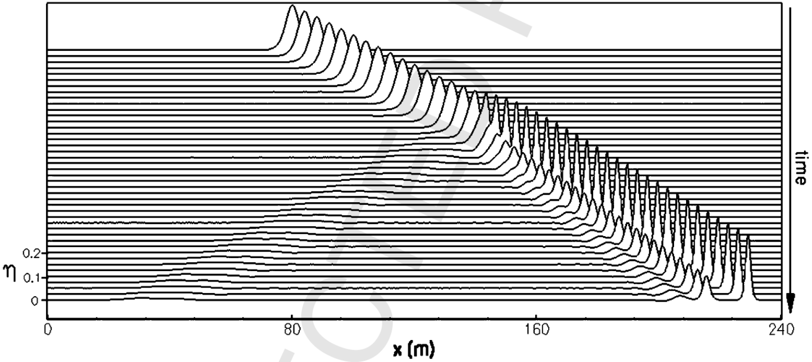


Fig. 12. Sketch of the submerged shelf test.

An example : solitary wave propagation over a shelf



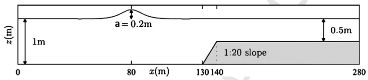
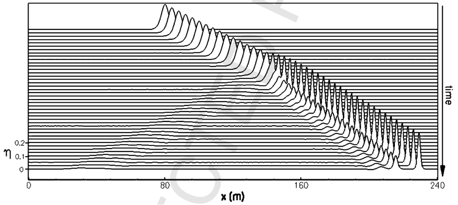


Fig. 12. Sketch of the submerged shelf test.

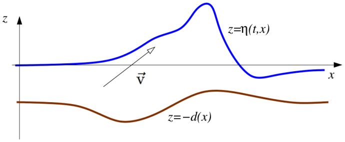


Dispersive wave propagation

- Initial propagation : undisturbed
- Shoaling (transformation) : wave getting higher and steeper
- Dispersion (transformation) : new frequencies are separated→ wave groups
- Propagation : each "group" propagates at its own speed

Physics and models 4/13

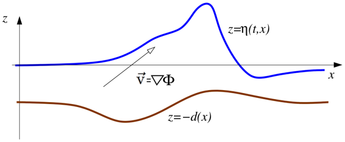
$$
\begin{aligned}
\nabla \cdot \vec{v} &= 0 \\
u_t + u u_x + w u_z + p_x &= 0 \\
w_t + u w_x + w w_z + p_z &= -g \\
p &= 0 \quad \text{in } z = \eta \\
\color{red}{\eta_t + u \eta_x} &\color{red}{= w} \quad \text{in } z = \eta \\
u d_x &= w \quad \text{in } z = -d
\end{aligned}
$$



Dispersive models : Airy theory

Incompressible Euler eq.s in terms of velocity $\vec{v} = (u, w)$ $(\rho = 1)$

$$\Delta\Phi = 0$$
$$\Phi_{tx} + \Phi_x\Phi_{xx} + \Phi_z\Phi_{xz} + p_x = 0$$
$$\Phi_{tz} + \Phi_x\Phi_{xz} + \Phi_z\Phi_{zz} + p_z = -g$$
$$p = 0 \quad \text{in } z = \eta$$
$$\eta_t + \Phi_x\eta_x = \Phi_z \quad \text{in } z = \eta$$
$$\Phi_x d_x = \Phi_z \quad \text{in } z = -d$$



Dispersive models : Airy theory

Incompressible Euler eq.s with assumption of irrotational flow in terms of velocity potential $\vec{v} = \nabla\Phi$ ($\rho = 1$)

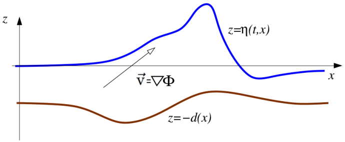
$$\Delta\Phi = 0$$

$$\nabla\left(\Phi_t + \frac{1}{2}\Phi_x^2 + \frac{1}{2}\Phi_z^2 + p + g\,z\right) = 0$$

$$p = 0 \quad \text{in } z = \eta$$

$$\eta_t + \Phi_x\eta_x = \Phi_z \quad \text{in } z = \eta$$

$$\Phi_x d_x = \Phi_z \quad \text{in } z = -d$$



Dispersive models : Airy theory

Bernoulli's theorem

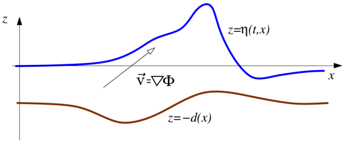
$$\Delta\Phi = 0$$

$$\Phi_t + \frac{1}{2}\Phi_x^2 + \frac{1}{2}\Phi_z^2 + p + g\,z = f(t) = 0$$

$$p = 0 \quad \text{in } z = \eta$$

$$\eta_t + \Phi_x \eta_x = \Phi_z \quad \text{in } z = \eta$$

$$\Phi_x d_x = \Phi_z \quad \text{in } z = -d$$



Dispersive models : Airy theory

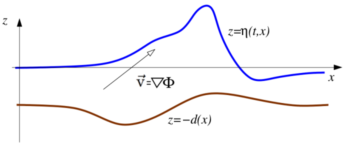
Bernoulli's theorem

$$\Delta\Phi = 0$$

$$\Phi_t + \frac{1}{2}\Phi_x^2 + \frac{1}{2}\Phi_z^2 + g\eta = 0 \quad \text{in } z = \eta$$

$$\eta_t + \Phi_x \eta_x = \Phi_z \quad \text{in } z = \eta$$

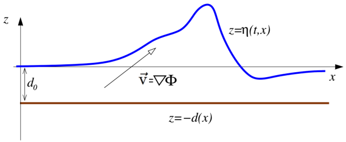
$$\Phi_x d_x = \Phi_z \quad \text{in } z = -d$$



Dispersive models : Airy theory

Nonlinear wave equations : 1st order PDEs plus an elliptic equation

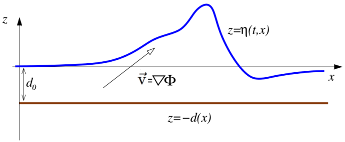
$$\begin{aligned} \Delta\Phi &= 0 \\ \Phi_t + g\eta &= 0 \quad \text{in } z = \eta \\ \eta_t &= \Phi_z \quad \text{in } z = \eta \\ 0 &= \Phi_z \quad \text{in } z = -d \end{aligned}$$



DISPERSIVE MODELS : AIRY THEORY

Linearized wave equations : $|\Phi_x|, |\Phi_z| \ll 1, |\eta|, |\eta_x|, |\eta_z| \ll 1$

$$\begin{aligned} \Delta\Phi &= 0 \\ \Phi_t + g\eta &= 0 \quad \text{in } z = \eta \\ \eta_t &= \Phi_z \quad \text{in } z = \eta \\ 0 &= \Phi_z \quad \text{in } z = -d \end{aligned}$$



Dispersive models : Airy theory

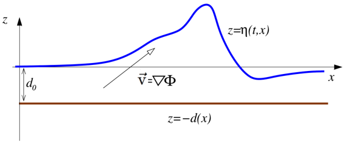
Propagating solutions of linearized equations on flat bathymetries :

$$\eta = a\sin(\mathsf{k}(x - Ct)) \quad \Phi = B(z)\cos(\mathsf{k}(x - Ct))$$

where

$$kC = \omega \quad \text{phase}$$

$$
\begin{aligned}
\Delta\Phi &= 0 \\
\Phi_t + g\eta &= 0 \quad \text{in } z = \eta \\
\eta_t &= \Phi_z \quad \text{in } z = \eta \\
0 &= \Phi_z \quad \text{in } z = -d
\end{aligned}
$$



DISPERSIVE MODELS : AIRY THEORY

Propagating solutions of linearized equations on flat bathymetries ($C_0^2 = gd_0$) :

$$\eta = a\sin(\mathsf{k}(x - Ct)) \quad \Phi = B(z)\cos(\mathsf{k}(x - Ct))$$

$$C^2 = \underbrace{C_0^2}_{\text{Linearized Shallow-Water}} \frac{\tanh(\mathsf{k}d_0)}{\mathsf{k}d_0} = \text{function of k !!!!!}$$

Dispersive models

1. Wave dispersion : even for linear equations
2. Wave dispersion : even for flat bathymetry
3. Wave dispersion : mainly a 3D effect (coupling vertical flow/surface deformation)
4. Surface wave propagation : wave heights and positions...

the BC in the wave equations !

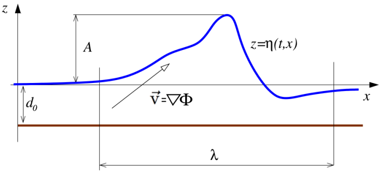
The "lazy" man's approach

DISPERSIVE MODELS : DEPTH AVERAGING (BOUSSINESQ 1872)

1. Depth averaging : reduce problem from 3D to 2D (just the BCs!)
2. Depth averaging : unknowns

$$\eta(t,x) \quad \text{and} \quad \overline{u}(t,x) = \frac{1}{d+\eta}\int_{-d}^{\eta} u(t,x,z)dz$$

3. Depth averaging : allow propagation over large domains
4. Wave dispersion : explicit presence of dispersive terms in equations !!!



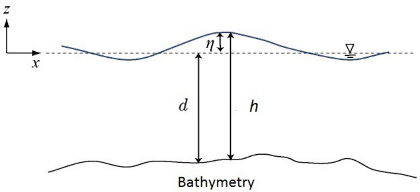
DISPERSIVE MODELS : DEPTH AVERAGING + ASYMPTOTIC ANALYSIS

1. Double asymptotic expansion
2. Small parameters (nonlinearity and dispersion)

$$\epsilon = \frac{A}{d_0}, \quad \mu = \frac{d_0}{\lambda}$$

3. Several family of models, different properties

Peregrine equations



System variables:

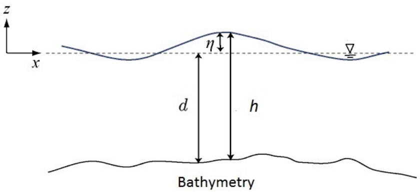
- η: free surface water level
- d: depth at still water
- h: water column height
- q: volume flux ($q = hu$)

$$\begin{cases} \partial_t \eta + \partial_x q = 0 \\ \partial_t w + \partial_x (uq) + gh\partial_x \eta = 0 \\ q - \dfrac{d^2}{3}\partial_{xx} q - \dfrac{1}{3} d\partial_x d\partial_x q = w \end{cases}$$

- Hyperbolic NonLinear Shallow Water (NLSW) equations
- Peregrine Boussinesq term

[1] Peregrine *J.Fluid.Mech.* 1967, Abbott et al. *Coast. Eng.* 1978

Green-Naghdi equations



System variables:

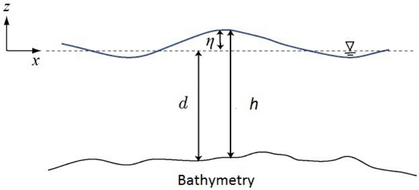
- η: free surface water level
- d: depth at still water
- h: water column height
- q: volume flux ($q = hu$)

$$
\begin{cases}
\partial_t \eta + \partial_x q = 0 \\
\partial_t q + \partial_x (uq) + gh\partial_x \eta = \phi \\
\phi - \alpha \mathcal{T}(\phi) = \mathcal{T}(gh\eta_x) - h\mathcal{Q}(u)
\end{cases}
$$

- Hyperbolic NonLinear Shallow Water (NLSW) equations
- GN non-hydrostatic source

[2] Green and Naghdi *J.Fluid.Mech.* 1976, Lannes and Bonneton *Phys. Fluids* 2009, Filippini et al *J.Comput.Phys.* 2015

Green-Naghdi equations: non-linear operators



System variables:

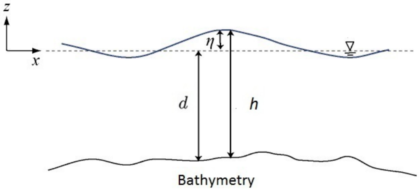
- η: free surface water level
- d: depth at still water
- h: water column height
- q: volume flux ($q = hu$)

$$\begin{cases} \mathcal{T}(\cdot) = S_1^*\left(hS_1\left(\frac{(\cdot)}{h}\right)\right) + S_2^*\left(hS_2\left(\frac{(\cdot)}{h}\right)\right) \\ S_1(\cdot) = \frac{h}{\sqrt{3}}\,(\cdot)_x - \frac{\sqrt{3}}{2} d_x\,(\cdot) \\ S_2(\cdot) = \frac{1}{2} d_x\,(\cdot) \end{cases}$$

Elliptic sub-system : coercive operator (the * denotes adjoint operators)

[2] Alvarez-Samaniego and Lannes *Indiana U. Math. J.* 2008, Filippini et al *J.Comput.Phys.* 2016

Green-Naghdi equations: non-linear operators



System variables:

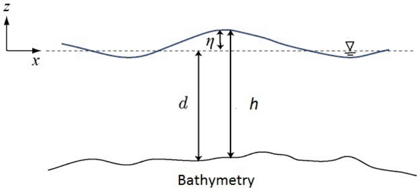
- η: free surface water level
- d: depth at still water
- h: water column height
- q: volume flux ($q = hu$)

$$\mathcal{T}(\cdot) = \partial_x (h^3 \partial_x \frac{(\cdot)}{h}) + (d_x h_x + \frac{1}{2} h d_{xx} - (d_x)^2)(\cdot)$$

Elliptic sub-system : coercive operator

[2] Alvarez-Samaniego and Lannes *Indiana U. Math. J.* 2008, Filippini et al *J.Comput.Phys.* 2016

Green-Naghdi equations: non-linear operators



System variables:

- η: free surface water level
- d: depth at still water
- h: water column height
- q: volume flux ($q = hu$)

$$\mathcal{Q}(\cdot) = 2hh_x(\cdot)_x^2 + \frac{4}{3}h^2(\cdot)_x(\cdot)_{xx} - d_x h(\cdot)_x^2 - d_{xx}h(\cdot)(\cdot)_x - \left[d_{xx}h_x + \frac{1}{2}hd_{xxx} - d_x d_{xx}\right](\cdot)^2$$

Elliptic sub-system : nonlinear forcing

[3] Lannes and Bonneton *Phys. Fluids* 2009, Filippini et al *J.Comput.Phys.* 2016

Not so lazy after all

Complex systems of nonlinear dispersive PDEs

Asymptotic accuracy

Peregrine Weakly non-linear and weakly dispersive. We assume $\epsilon \approx \mu^2$ and terms of order ϵ^2, $\epsilon\mu^2$, and μ^4 are neglected ;

Green-Naghdi Fully nonlinear, weakly dispersive. We assume $\epsilon \approx 1$ and terms of order μ^4 are neglected.

Complex systems of nonlinear dispersive PDEs

State of the art "operational" models

FUNWAVE Fully nonlinear dispersive model
+ hybrid approach (FUNWAVE TVD)[1]

MIKE21 Weakly nonlinear dispersive model
+ eddy viscosity for wave breaking[2]

[1] University of Delaware, J.T. Kirby and co-workers

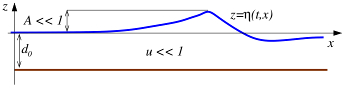
[2] Distributed by DHI Group, models by Tech. University of Denmark, P.A. Madsen-H. Schaffer and co-workers

HOW GOOD ARE THESE MODELS ?

1. Asymptotic approximations, error of order $\mathcal{O}(\epsilon^2, \epsilon\mu^2, \mu^4)$, or $\mathcal{O}(\mu^4)$ for GN
2. Irrotational potential flow: no breaking waves (rotational)
3. Asymptotic error in μ means waves may have the wrong phase relation (w.r.t. Euler eq.s)
4. Dispersion coeff.s α determined to minimize phase error (w.r.t. Euler eq.s)

Nonlinear Peregrine equations :

$$\begin{cases} \partial_t \eta + \partial_x q = 0 \\ \partial_t q - \frac{1}{3} d_0^2 \partial_{xxt} q + \partial_x (uq) + gh\partial_x \eta = 0 \end{cases}$$



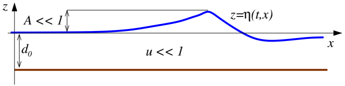
Dispersion analysis : continuous case

Linearized Peregrine equations (neglect quadratic/HO terms : ηu, η^2, u^2, etc)

$$\begin{cases} \partial_t \eta + d_0 \partial_x u = 0 \\ \partial_t u - \frac{1}{3} d_0^2 \partial_{xxt} u + g\partial_x \eta = 0 \end{cases}$$

Linearized Peregrine equations :

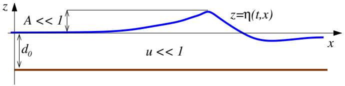
$$\begin{cases} \partial_t \eta + d_0 \partial_x u = 0 \\ \partial_t u - \frac{1}{3} d_0^2 \partial_{xxt} u + g \partial_x \eta = 0 \end{cases}$$



Dispersion analysis : continuous case

Linearized Peregrine equations :

$$\begin{cases} \partial_t \eta + d_0 \partial_x u = 0 \\ \partial_t u - \frac{1}{3} d_0^2 \partial_{xxt} u + g \partial_x \eta = 0 \end{cases}$$



Dispersion analysis : continuous case

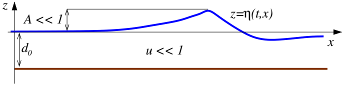
Look for solutions of the form

$$\eta = \eta_0 e^{\nu t + \mathrm{i}\, \mathrm{k} x}\,, \quad u = u_0 e^{\nu t + \mathrm{i}\, \mathrm{k} x}$$

with $\nu = \xi + \mathrm{i}\,\omega$ (ξ dissipation rate, $\omega = \mathrm{k}C$ the phase shift)

Linearized Peregrine equations :

$$\begin{cases} \partial_t \eta + d_0 \partial_x u = 0 \\ \partial_t u - \frac{1}{3} d_0^2 \partial_{xxt} u + g \partial_x \eta = 0 \end{cases}$$



Dispersion analysis : continuous case

Linear dispersion relations

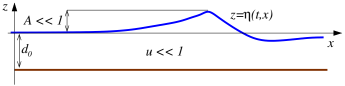
$$\nu \eta_0 + \mathrm{i}\,\mathsf{k} d_0 u_0 = 0$$

$$\nu(1 + B(\mathsf{k} d_0)^2) u_0 + \mathrm{i}\,\mathsf{k} g \eta_0 = 0$$

LINEAR DISPERSION ANALYSIS 6/11

Linearized Peregrine equations :

$$\begin{cases} \partial_t \eta + d_0 \partial_x u = 0 \\ \partial_t u - \dfrac{1}{3} d_0^2 \partial_{xxt} u + g \partial_x \eta = 0 \end{cases}$$



DISPERSION ANALYSIS : CONTINUOUS CASE

Complex eigenvalue problem for $\nu = \xi + \mathrm{i}\,\omega$ (ξ dissipation, $\omega = \mathrm{k}C$ phase)

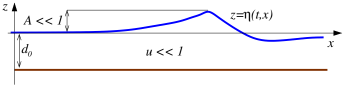
$$\begin{pmatrix} \nu & \mathrm{i}\,\mathrm{k} d_0 \\ \mathrm{i}\,\mathrm{k} g & \nu(1 + B(\mathrm{k} d_0)^2) \end{pmatrix} \begin{pmatrix} \eta_0 \\ u_0 \end{pmatrix} = 0$$

$$\Longleftrightarrow$$

$$\nu^2 (1 + B(\mathrm{k} d_0)^2) - (\mathrm{i}\,\mathrm{k} g d_0)^2 = 0$$

Linearized Peregrine equations :

$$\begin{cases} \partial_t \eta + d_0 \partial_x u = 0 \\ \partial_t u - \frac{1}{3} d_0^2 \partial_{xxt} u + g \partial_x \eta = 0 \end{cases}$$



DISPERSION ANALYSIS : CONTINUOUS CASE

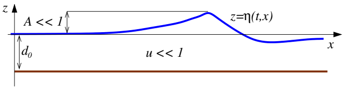
Result (set $C_0^2 = g d_0^2$)

$$\xi = 0$$

$$\omega^2 = \underbrace{(kC_0)^2}_{\text{Linearized Shallow Water}} \frac{1}{1 + \frac{(\mathrm{k} d_0)^2}{3}}$$

Linearized GN equations :

$$\begin{cases} \partial_t \eta + d_0 \partial_x u = 0 \\ \partial_t u - \dfrac{\alpha}{3} d_0^2 \partial_{xxt} u \\ \qquad - \dfrac{\alpha - 1}{3} g d_0^2 \partial_{xxx} \eta + g \partial_x \eta = 0 \end{cases}$$



Dispersion analysis : continuous case

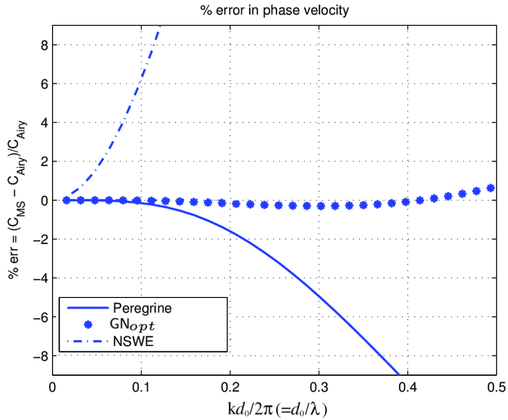
Result (set $C_0^2 = g d_0^2$)

$$\xi = 0$$

$$\omega^2 = \underbrace{(kC_0)^2}_{\text{Linearized Shallow Water}} \frac{1 + \dfrac{\alpha - 1}{3} (\mathsf{k} d_0)^2}{1 + \dfrac{\alpha}{3} (\mathsf{k} d_0)^2}$$

Coeff. α chosen by minimizing error w.r.t. Airy theory ($\alpha = 1 \rightarrow$ Peregrine !).

Dispersion relations : models overview



[3] For GN $\alpha_{opt} \approx 1.159$

Next step : continuous to discrete

- Influence of the scheme
- dissipation for given mesh size
- dispersion error for given mesh size

NEXT STEP : CONTINUOUS TO DISCRETE

Model problem : AD equation

$$\partial_t u - \alpha \partial_{txx} u + a \partial_x u = 0$$

Admits solutions of the form $u = u_0 e^{\nu t + \mathrm{i}\,\mathrm{k} x}$ with $\nu = \xi + \mathrm{i}\,\omega$ and

$$\xi = 0 \qquad \text{no dumping}$$

$$\omega = - \underbrace{\mathrm{k}a}_{\text{pure advection}} \frac{1}{1 + \alpha \mathrm{k}^2} \qquad \text{dispersion}$$

Motivation

- High ratio accuracy/stencil
- High ratio 1/(cost $\times$ error)
- Potential for unstructured grids and adaptation
- Potential for higher order

Additional remarks

For time dependent problems always need to invert a “mass matrix”

- For hyperbolics often seen as a flaw (can’t simply do RK ODE integration)
- Here matrix inversion cannot be avoided[1]
- The PDEs are very complex. How to simplify and possibly reduce costs ? (e.g. matrix sizes)

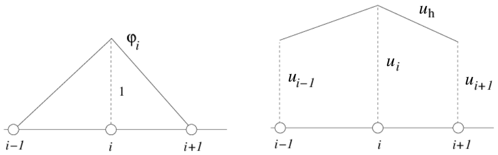
[1] presence of u_{xxt} terms, elliptic component hidden..

P^1 FEM AND $\partial_t u - \alpha \partial_{txx} u + a \partial_x u = 0$

- Consider a tessellation of the domain composed of non-overlapping elements (segments)
- Unknowns at nodes: $\{u_i(t)\}_{i \geq 1}$
- P^1 piecewise linear continuous approximation

$$u_{\mathrm{h}}(t,x) = \sum_{i \geq 1} u_i(t) \varphi_i(x) = \sum_K \sum_{j \in K} u_j(t) \varphi_j(x)$$

- φ_i are standard continuous piecewise linear finite element basis functions;



Continuous Galerkin (cG) for $\partial_t u - \alpha \partial_{txx} u + a \partial_x u = 0$

$$\int_{\Omega_h} \varphi_i \partial_t w_h - \int_{\Omega_h} a u_h \, \partial_x \varphi_i = 0$$

$$\int_{\Omega_h} \partial_x u \partial_x \varphi_i + \int_{\Omega_h} \varphi_i u_h = \int_{\Omega_h} \varphi_i w_h$$

1. A first order PDE : $\partial_t w + a u_x = 0$
2. plus an elliptic equation : $-\alpha u_{xx} + u = w$

Continuous Galerkin (cG) for $\partial_t u - \alpha \partial_{txx} u + a \partial_x u = 0$

cG SCHEME in FD form (uniform Δx)

$$\frac{\Delta x}{6}\left(\frac{du_{i-1}}{dt} + 4\frac{du_i}{dt} + \frac{du_{i+1}}{dt}\right) - \frac{\alpha}{\Delta x}\left(\frac{du_{i-1}}{dt} - 2\frac{du_i}{dt} + \frac{du_{i+1}}{dt}\right) + \frac{a}{2}(u_{i+1} - u_{i-1}) = 0$$

Continuous Galerkin (cG) for $\partial_t u - \alpha \partial_{txx} u + a \partial_x u = 0$

cG SCHEME in FD form (uniform Δx)

$$\frac{\Delta x}{6}\left(\frac{du_{i-1}}{dt} + 4\frac{du_i}{dt} + \frac{du_{i+1}}{dt}\right) - \frac{\alpha}{\Delta x}\left(\frac{du_{i-1}}{dt} - 2\frac{du_i}{dt} + \frac{du_{i+1}}{dt}\right) + \frac{a}{2}(u_{i+1} - u_{i-1}) = 0$$

FD2 scheme (same cost) :

$$\Delta x \frac{du_i}{dt} - \frac{\alpha}{\Delta x}\left(\frac{du_{i-1}}{dt} - 2\frac{du_i}{dt} + \frac{du_{i+1}}{dt}\right) + \frac{a}{2}(u_{i+1} - u_{i-1}) = 0$$

CONTINUOUS GALERKIN (cG) FOR $\partial_t u - \alpha\partial_{txx} u + a\partial_x u = 0$

cG SCHEME in FD form (uniform Δx)

$$\frac{\Delta x}{6}\left(\frac{du_{i-1}}{dt} + 4\frac{du_i}{dt} + \frac{du_{i+1}}{dt}\right) - \frac{\alpha}{\Delta x}\left(\frac{du_{i-1}}{dt} - 2\frac{du_i}{dt} + \frac{du_{i+1}}{dt}\right) + \frac{a}{2}(u_{i+1} - u_{i-1}) = 0$$

FD4 scheme (more expensive) :

$$\Delta x\frac{du_i}{dt} - \frac{\alpha}{12\Delta x}\left(-\frac{du_{i-2}}{dt} + 16\frac{du_{i-1}}{dt} - 30\frac{du_i}{dt} + 16\frac{du_{i+1}}{dt} - \frac{du_{i+2}}{dt}\right) + \frac{a}{12}(u_{i-2} - 8u_{i-1} + 8u_{i+1} - u_{i+2}) = 0$$

An upwind scheme

WHY !????!

To handle with one code the dispersive AND the hyperbolic limits ...

An upwind schemes : SUPG

Streamline Upwind Petrov-Galerkin[8] :

$$\text{Galerkin} + \sum_{\mathsf{K}\in\Omega_h} \int_K a\partial_x \varphi_i^{\mathsf{K}} \tau_{\mathsf{K}} r^K = 0$$

with τ_{K} a stabilization parameter, and with r^K the local residual

$$r^K = (\partial_t w_{\mathrm{h}} + a\partial_x u_{\mathrm{h}})_K$$

[8] see e.g. (Hughes, Scovazzi and Tezduyar, J.Sci.Comp. 43 2010) and references therein

A magic trick

Taking

$$\tau_{\mathsf{K}} = \frac{1}{\sum\limits_{j \in \mathsf{K}} |a \partial_x \varphi_j^K|}$$

We obtain for $a > 0$ (in FD form)

$$\frac{\Delta x}{6}\left(\frac{5}{2}\frac{dw_{i-1}}{dt} + 4\frac{dw_i}{dt} - \frac{1}{2}\frac{dw_{i+1}}{dt}\right) + a(u_i - u_{i-1}) = 0$$

$$\Delta x\, u_i - \frac{\alpha}{\Delta x}(u_{i-1} - 2u_i + u_{i+1}) = \Delta x\, w_i$$

The blue part is the first order upwind scheme !

In summary

- FD2 and cG schemes: same cost and same underlying elliptic discretization
- FD4 scheme: more accurate and more expensive
- SUPG: cost of FD4 with same underlying elliptic equation of FD2 and cG

Time continuous error analysis

We want to

1. Characterize the truncation errors (uniform Δx) for the different schemes (TE analysis)
2. Characterize the dispersion error (uniform Δx) for the different schemes (DE analysis)
3. Get some recipe to generalize to the systems we have seen

Time continuous TE analysis

Brute force ...

Time continuous TE analysis

Our problem

$$\partial_t u - \alpha \partial_{txx} u + a \partial_x u = 0$$

Truncation errors

$$\mathrm{TE}_{\mathrm{FD2}} = \frac{\Delta x^2}{6} \partial_{xx} (u_t - \frac{\alpha}{2} \partial_{txx} u) + \mathcal{O}(\Delta x^4)$$

$$\mathrm{TE}_{\mathrm{cG}} = \alpha \frac{\Delta x^2}{12} \partial_{xxxxt} u + \mathcal{O}(\Delta x^4)$$

$$\mathrm{TE}_{\mathrm{FD4}} = \frac{\Delta x^4}{30} \partial_{xxxx} (u_t + \frac{2}{3} \alpha \partial_{xxt} u) + \mathcal{O}(\Delta x^6)$$

$$\mathrm{TE}_{\mathrm{SUPG}} = \alpha \frac{\Delta x^2}{12} \partial_{xxxxt} u + \mathcal{O}(\Delta x^3)$$

Time continuous TE analysis

- For $\alpha = 0$ (pure advection) both cG and SUPG are more than second order accurate (resp. fourth order and third order)

- For $\alpha \neq 0$ their dispersive leading term contains the same derivative of FD4 : spurious numerical dispersion on same high frequencies as FD4

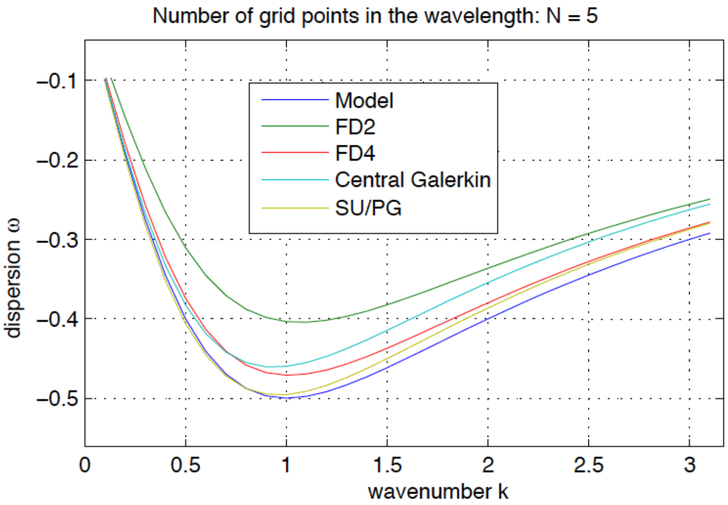
- FD2: dispersive leading error of the same type as the physical one. Numerical dispersion on the same wave numbers as the physical one

Time continuous DE analysis

As in the continuous case :

1. Set $u_i = u_0 e^{\nu t + \mathrm{i}\,\mathrm{k} x}$
2. Replace in the FD form of the scheme ($\partial_t u_i = \nu u_i$, $\partial_x u_i = \mathrm{i}\,\mathrm{k} u$, $u_{i+1} = e^{\mathrm{i}\,\mathrm{k}\Delta x} u_i$etc.)
3. Solve complex algebraic equation for $\nu = \xi + \mathrm{i}\,\omega$
4. $\omega = \omega(\mathrm{k}, \Delta x)$

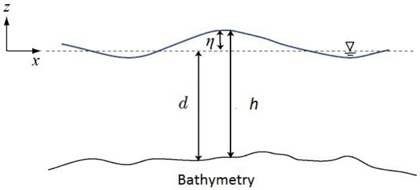
Time continuous DE analysis



Test 1. Repeat the exercise for Peregrine's model

Test 1. Repeat the exercise for Peregrine's model

Model equations (reminder)



System variables:

- η: free surface water level
- d: depth at still water
- h: water column height
- q: volume flux ($q = hu$)

$$\begin{cases} \partial_t \eta + \partial_x q = 0 \\ \partial_t w + \partial_x (uq) + gh\partial_x \eta = 0 \\ q - \dfrac{d^2}{3}\partial_{xx} q - \dfrac{1}{3} d\partial_x d\partial_x q = w \end{cases}$$

Test 1. Repeat the exercise for Peregrine's model

cG formulation

$$
\begin{cases}
\int\limits_{\Omega_h} \varphi_i \partial_t \eta_h - \int\limits_{\Omega_h} q_h \partial_x \varphi_i = 0 \\
\int\limits_{\Omega_h} \varphi_i \partial_t w_h - \int\limits_{\Omega_h} \left(u_h q_h + \frac{1}{2} g (h_h)^2 \right) \partial_x \varphi_i - \int\limits_{\Omega_h} \varphi_i g h_h \partial_x d_h = 0 \\
\int\limits_{\Omega_h} \varphi_i q_h + \frac{1}{3} \int\limits_{\Omega_h} \partial_x \varphi_i (d_h)^2 \partial_x q_h + \frac{1}{3} \int\limits_{\Omega_h} \varphi_i d_h \partial_x d_h \partial_x q_h = \int\limits_{\Omega_h} \varphi_i w_h
\end{cases}
$$

Test 1. Repeat the exercise for Peregrine's model

SUPG formulation

Set

$$R_i^{\mathsf{cG}} = \begin{pmatrix} \int\limits_{\Omega_\mathrm{h}} \varphi_i \partial_t \eta_\mathrm{h} - \int\limits_{\Omega_\mathrm{h}} q_\mathrm{h} \partial_x \varphi_i \\ \\ \int\limits_{\Omega_\mathrm{h}} \varphi_i \partial_t w_\mathrm{h} - \int\limits_{\Omega_\mathrm{h}} \left(u_\mathrm{h} q_\mathrm{h} + \frac{1}{2} g (h_\mathrm{h})^2 \right) \partial_x \varphi_i - \int\limits_{\Omega_\mathrm{h}} \varphi_i g h_\mathrm{h} \partial_x d_\mathrm{h} \end{pmatrix}$$

Test 1. Repeat the exercise for Peregrine's model

SUPG formulation

$$\begin{cases} R_i^{\mathsf{cG}} + \sum_{\mathsf{K}\in\Omega_h} \int_K A\partial_x \varphi_i^{\mathsf{K}}\, \tau_{\mathsf{K}}\, r^K = 0 \\ \\ \int_{\Omega_\mathrm{h}} \varphi_i q_\mathrm{h} + \frac{1}{3}\int_{\Omega_\mathrm{h}} \partial_x\varphi_i (d_\mathrm{h})^2 \partial_x q_\mathrm{h} + \frac{1}{3}\int_{\Omega_\mathrm{h}} \varphi_i d_\mathrm{h} \partial_x d_\mathrm{h} \partial_x q_\mathrm{h} = \int_{\Omega_\mathrm{h}} \varphi_i w_\mathrm{h} \end{cases}$$

- With A the Shallow-Water Jacobian matrix
- With r^K the residual

$$r^K = \begin{pmatrix} \partial_t \eta_\mathrm{h} + \partial_x q_\mathrm{h} \\ \\ \partial_t w_\mathrm{h} + \partial_x(u_\mathrm{h} q_\mathrm{h}) + g h_\mathrm{h} \partial_x \eta_\mathrm{h} \end{pmatrix}$$

Test 1. Repeat the exercise for Peregrine's model

SUPG and Roe's method

As in the scalar case, setting

$$\tau_{\mathsf{K}} = (\sum_{j \in \mathsf{K}} |A \partial_x \varphi_j^K|)^{-1}$$

we recover

- Roe's flux difference splitting for the flux part
- Well known well balanced (upwind) approximation of the bathymetry
- plus a non-symmetric mass matrix coupling $\partial_t \eta_{\mathrm{h}}$ and $\partial_t w_{\mathrm{h}}$

TE analysis for (linearized) Peregrine

FD2 scheme.

$$\mathrm{TE}^{\eta}_{\mathsf{FD2}} = \frac{d_0 \Delta x^2}{6} \partial_{xxx} u + \mathcal{O}(\Delta x^4)$$

$$\mathrm{TE}^{u}_{\mathsf{FD2}} = \frac{\Delta x^2}{6} \partial_{xxx} \left(-\frac{d_0^2}{6} \partial_{xt} u + g\,\eta \right) + \mathcal{O}(\Delta x^4)$$

cG scheme.

$$\mathrm{TE}^{\eta}_{\mathsf{cG}} = \frac{\Delta x^4}{24} \partial_{xxxx} \left(\frac{1}{3} \partial_t \eta + \frac{d_0}{5} \partial_x u \right) + \mathcal{O}(\Delta x^6)$$

$$\mathrm{TE}^{u}_{\mathsf{cG}} = \frac{\Delta x^2}{36} d_0^2 \partial_{xxxxt} \partial_t u + \mathcal{O}(\Delta x^4)$$

FD4 scheme.

$$\mathrm{TE}^{\eta}_{\mathsf{FD4}} = \frac{d_0 \Delta x^4}{30} \partial_{xxxxx} u + \mathcal{O}(\Delta x^6)$$

$$\mathrm{TE}^{u}_{\mathsf{FD4}} = \frac{\Delta x^4}{15} \partial_{xxxx} \left(-\frac{2}{27} d_0^2 \partial_t u + \frac{1}{2} g \partial_x \eta \right) + \mathcal{O}(\Delta x^6)$$

SUPG scheme.

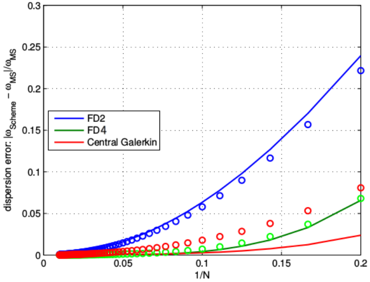
$$\mathrm{TE}^{\eta}_{\mathsf{SUPG}} = \frac{C_0 \Delta x^3}{6g} \partial_{xxx} \left(\partial_t u - \frac{d_0^2}{2} \partial_{xxt} u + \frac{1}{2} g \partial_x \eta \right) + \mathcal{O}(\Delta x^4)$$

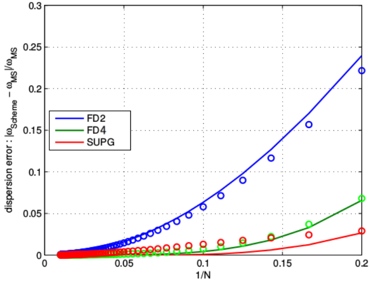
$$\mathrm{TE}^{u}_{\mathsf{SUPG}} = \frac{\Delta x^2}{36} d_0^2 \partial_{xxxxt} u + \mathcal{O}(\Delta x^3)$$

Time continuous TE analysis for Peregrine

- Without dispersion (Shallow Water) both cG and SUPG are more than second order accurate (resp. fourth order and third order)
- In the dispersive case, the dispersive leading term has the same derivative of FD4 : spurious numerical dispersion on same high frequencies as FD4
- FD2: dispersive leading error providing spurious numerical dispersion on the same wave numbers as the physical one

DE analysis : Boussinesq model - cG and SUPG





Solid line : k$d_0 = 0.5$ - Circles k$d_0 = 2.6$

N : points per wavelength

Our recipe

1. **Hyperbolic component**: suitable higher order (at least third) scheme

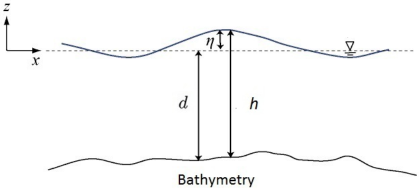
2. **Elliptic sub-problem**: suitable scheme, second order is enough to preserve the dispersion relation

Our recipe

Let's apply the recipe to the GN equations !

Let's apply the recipe to the GN equations

GN (reminder)



System variables:

- η: free surface water level
- d: depth at still water
- h: water column height
- q: volume flux ($q = hu$)

$$\begin{cases} \partial_t \eta + \partial_x q = 0 \\ \partial_t q + \partial_x (uq) + gh\partial_x \eta = \phi \\ \phi - \alpha \mathcal{T}(\phi) = \mathcal{T}(gh\eta_x) - h\mathcal{Q}(u) \end{cases}$$

Let's apply the recipe to the GN equations

GN non-linear operators (reminder)

$$\left\{\begin{aligned} \mathcal{T}(\cdot) &= S_1^*\left(hS_1\Big(\frac{(\cdot)}{h}\Big)\right) + S_2^*\left(hS_2\Big(\frac{(\cdot)}{h}\Big)\right) \\ \mathcal{Q}(\cdot) &= 2hh_x(\cdot)_x^2 + \frac{4}{3}h^2(\cdot)_x(\cdot)_{xx} - d_xh(\cdot)_x^2 - d_{xx}h(\cdot)(\cdot)_x \\ &\qquad\qquad - \left[d_{xx}h_x + \frac{1}{2}hd_{xxx} - d_xd_{xx}\right](\cdot)^2 \end{aligned}\right.$$

Let's apply the recipe to the GN equations

Equation for non-hydrostatic source ϕ

P^1 continuous Galerkin formulation :

$$\int_{\Omega_\mathrm{h}} \varphi_i \phi_\mathrm{h} + \int_{\Omega_\mathrm{h}} S_1(\varphi_i) h_\mathrm{h} S_1\left(\frac{\phi_\mathrm{h}}{h_\mathrm{h}}\right) + \int_{\Omega_\mathrm{h}} S_2(\varphi_i) h_\mathrm{h} S_2\left(\frac{\phi_\mathrm{h}}{h_\mathrm{h}}\right) = \mathrm{R}_i^\mathrm{h}(h_\mathrm{h}, u_\mathrm{h}, d_\mathrm{h})$$

where recall that

$$S_1(\cdot) = \frac{h}{\sqrt{3}}\,(\cdot)_x - \frac{\sqrt{3}}{2} d_x\,(\cdot)\,, \qquad S_2(\cdot) = \frac{1}{2} d_x\,(\cdot)$$

and with $\mathrm{R}_i^\mathrm{h}(h_\mathrm{h}, u_\mathrm{h}, d_\mathrm{h})$ obtained by recasting the forcing terms $h\mathcal{Q}(\cdot)$ and $\mathcal{T}(gh\eta_x)$ in variational form (omitted for brevity)

Let's apply the recipe to the GN equations

Equation for non-hydrostatic source ϕ

P^1 continuous Galerkin formulation :

$$\mathrm{T}(h_{\mathrm{h}}, u_{\mathrm{h}}, d_{\mathrm{h}})\Phi = \mathrm{R}^{\mathrm{h}}(h_{\mathrm{h}}, u_{\mathrm{h}}, d_{\mathrm{h}})$$

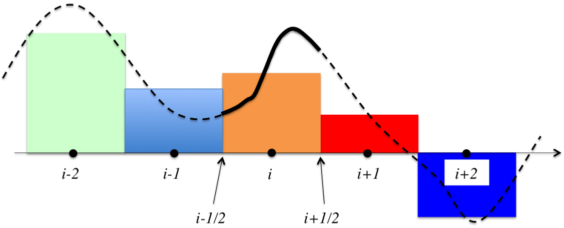
Linear system for Φ: the vector of nodal values of ϕ

Let's apply the recipe to the GN equations

Evolution of physical quantities: finite volume

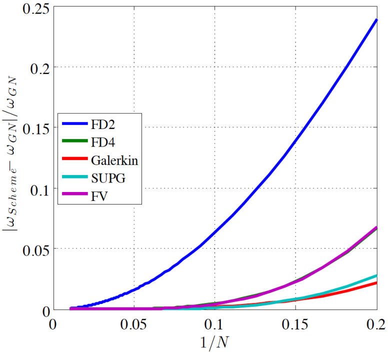
- parabolic (third order) reconstruction
- Roe's flux
- and well balanced treatment of topography[10]

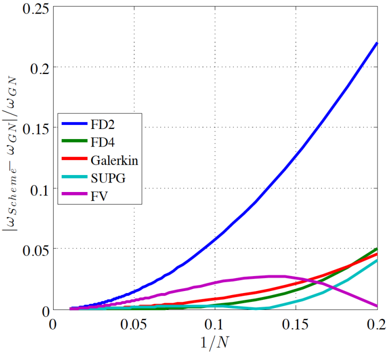
$$\Delta x_i \frac{d}{dt} \begin{pmatrix} \eta_i \\ q_i \end{pmatrix} + \widehat{F}_{i+1/2} - \widehat{F}_{i-1/2} + S_d = \int_{\Delta x_i} \phi_{\mathrm{h}}$$



[10](Bermudez-Vazquez, Computers&Fluids 1994)

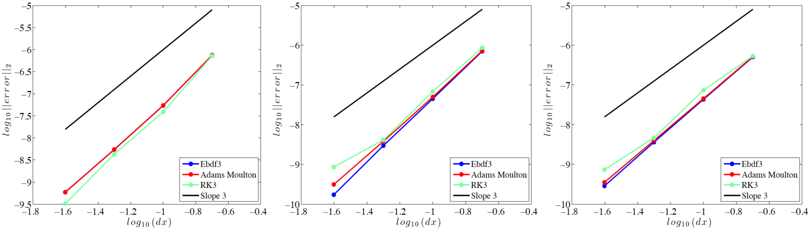
Time continuous dispersion error analysis





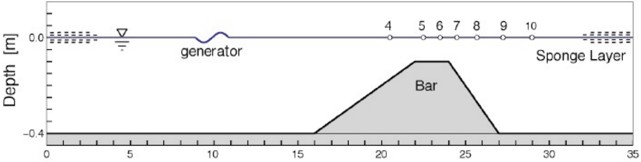
Left : k$d_0 = 0.5$ - Right k$d_0 = 2.6$

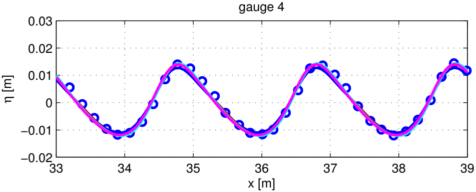
Solution error convergence



Hyperbolic phase :
Left: FV - Middle: SUPG - Right: cG

Weakly and fully nonlienar in 1D

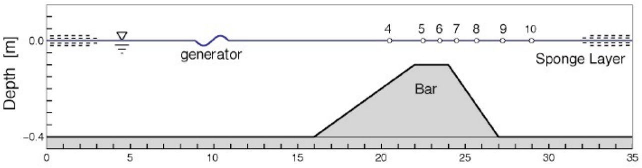


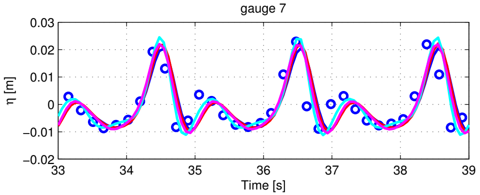


Experiments (○), FDWK (—), cG (—), SUPG (—), and URD (—)

Enhanced Boussinesq model

Numerical examples : submerged bar

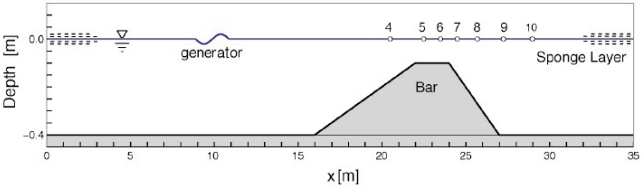


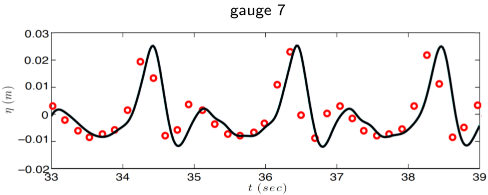


Experiments (o), FD4 (—), cG (—), SUPG (—), and FD2 (—)

Enhanced Boussinesq model

Numerical examples : submerged bar

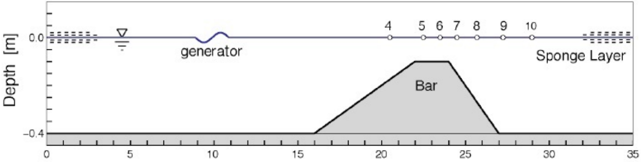


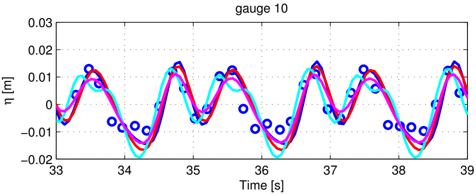


Hybrid FV-cG scheme

Enhanced GN model

Numerical examples : submerged bar

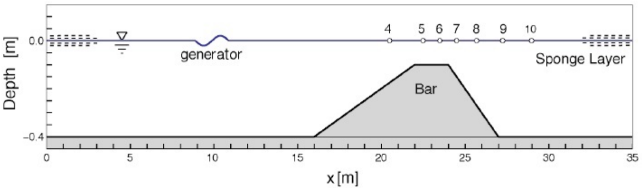


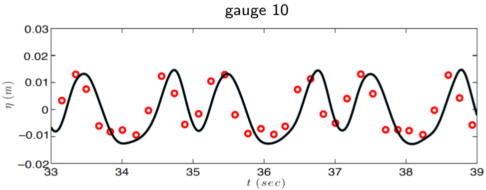


Experiments (o), FD4 (—), cG (—), SUPG (—), and FD2 (—)

Enhanced Boussinesq model

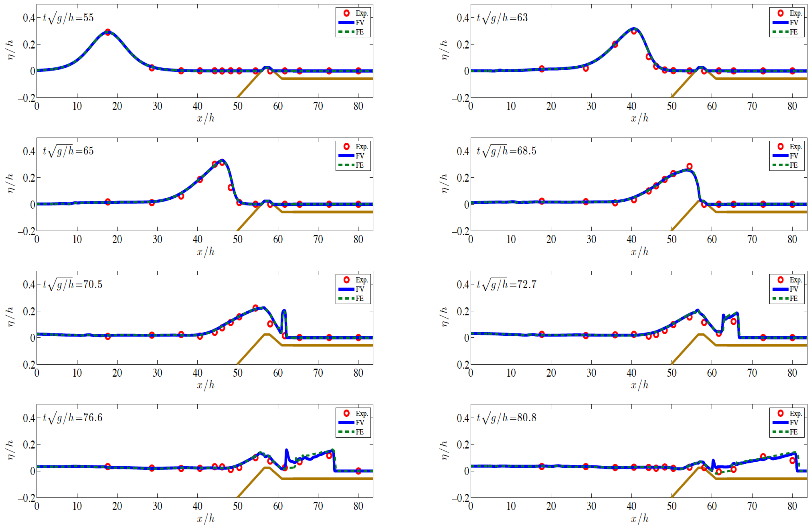
Numerical examples : submerged bar



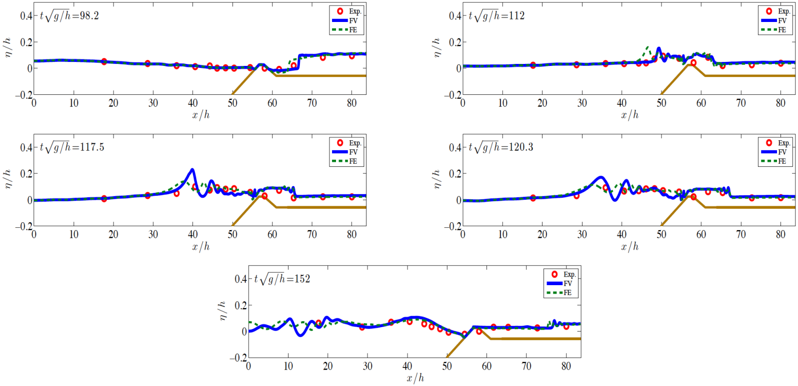


Hybrid FV-cG scheme
Enhanced GN model

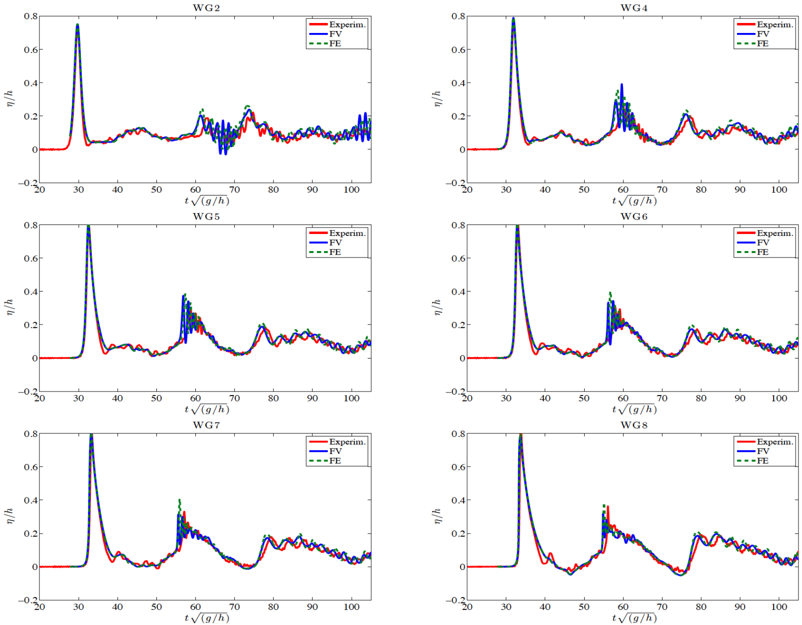
Numerical examples : reef overtopping



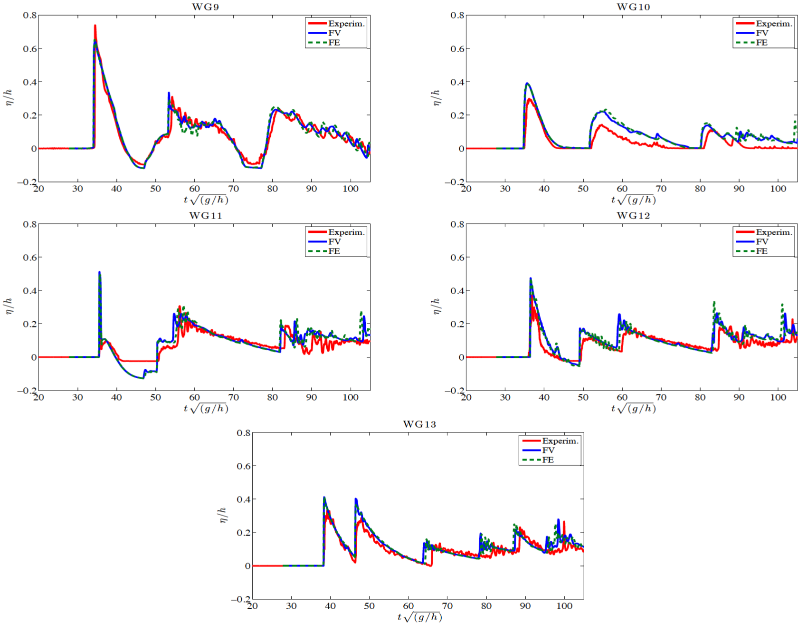
Numerical examples : reef overtopping



Numerical examples : reef overtopping



Numerical examples : reef overtopping



Remark: wave breaking

1. Detect breaking fronts
2. Set $\phi = 0$ to approximate breaking waves with bores
3. Purely algebraic modification !

$$\Delta x_i \frac{d}{dt} \left(\begin{array}{c} \eta_i \\ q_i \end{array} \right) + \widehat{F}_{i+1/2} - \widehat{F}_{i-1/2} + S_d = \int_{\Delta x_i} \phi_{\mathrm{h}}$$

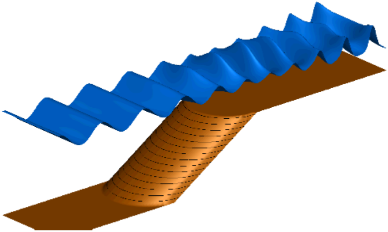
$$\begin{cases} \partial_t \eta + \nabla \cdot \vec{q} = 0 \\ \\ \partial_t q + \nabla \cdot (\vec{u} \otimes \vec{q}) + gH\nabla\eta + \vec{\psi} = 0 \end{cases}$$

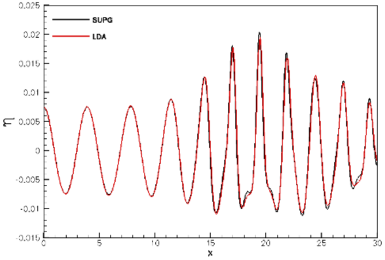
where $\vec{\psi} \equiv (\psi_x, \psi_y)$ are the dispersive terms of the model which can be written as

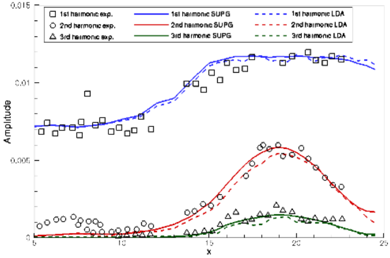
$$\begin{cases} \psi_x = -Bh^2\partial_{tx}\nabla \cdot \vec{q} - \dfrac{1}{6}h\partial_x h\partial_t \left(\nabla \cdot \vec{q} + \partial_x q_x\right) - \dfrac{1}{6}h\partial_y h\partial_{tx} q_y - \beta g h^2 \partial_x w^\eta \\ \\ \psi_y = -Bh^2\partial_{ty}\nabla \cdot \vec{q} - \dfrac{1}{6}h\partial_y h\partial_t \left(\nabla \cdot \vec{q} + \partial_y q_y\right) - \dfrac{1}{6}h\partial_x h\partial_{ty} q_x - \beta g h^2 \partial_y w^\eta \\ \\ w^\eta = \nabla \cdot (h\nabla\eta) \end{cases}$$

Schemes in 2D : similar stabilized approach with different upwinding strategies (cf. Ricchiuto-Filippini 2014)

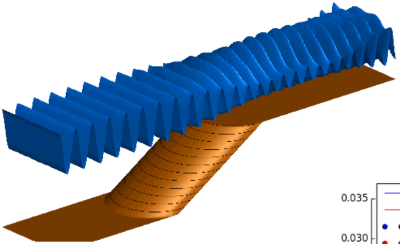
Numerical examples : circular shoal (hexagons) - $\Delta x = 0.1m$

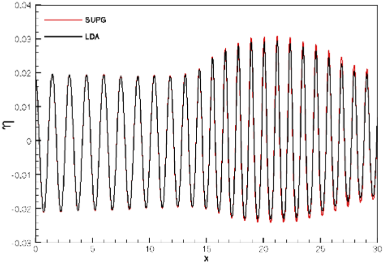


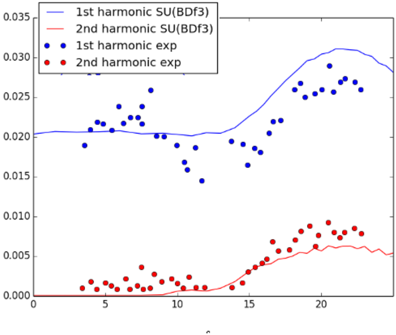




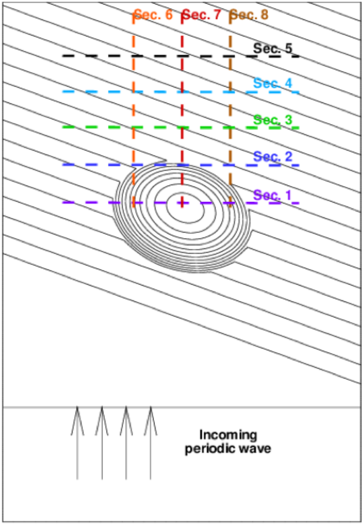
Numerical examples : circular shoal (hexagons) - $\Delta x = 0.1m$

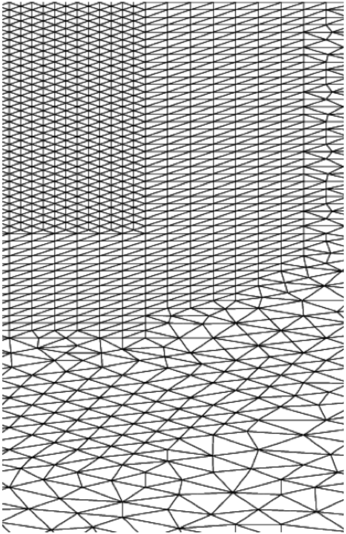




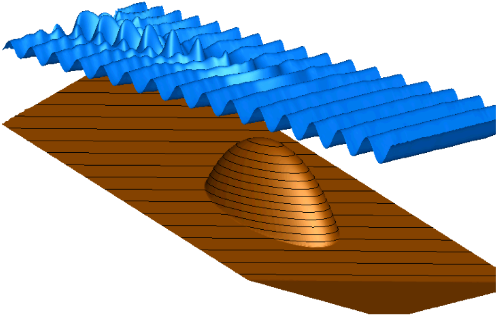


Numerical examples : elliptic shoal (unstructured)



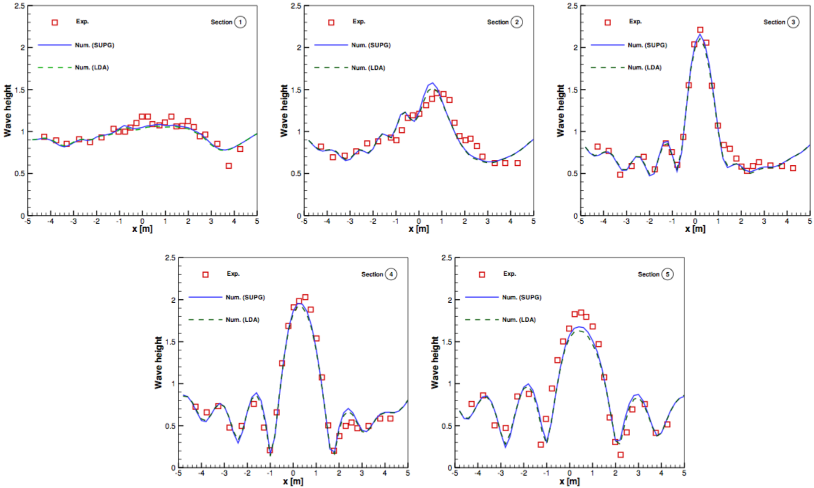


Numerical examples : elliptic shoal (unstructured)



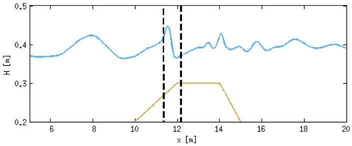
(Ribbed channel clip)

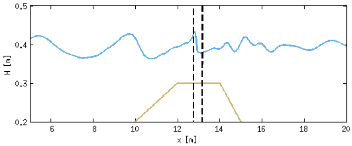
Numerical examples : elliptic shoal (unstructured)



The short of it ...

- Interaction between advective (hyperbolic) and higher order terms in depth averaged wave models
- High order on hyperbolic : at least third order required for good dispersion
- Elliptic component: can be treated with a second order method !
- Coupling with SW: your favorite scheme for SW + FEM for elliptic part
- Natural extension to 2D on unstructured meshes
- Wave breaking: evert to SW (cf. below)

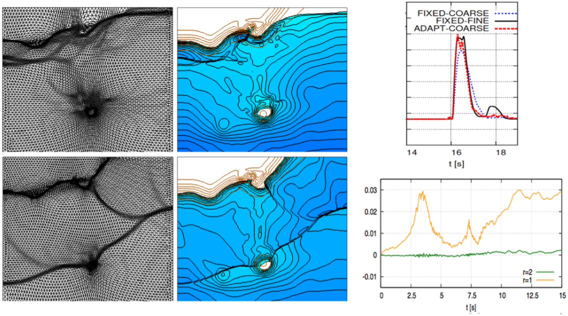




PERSPECTIVES

A few ongoing extensions

- 2D Green-Naghdi :
  couple FEM for the elliptic part with FV(SW), DG(SW), and RD(SW)
- Implicit or explicit time integration ... ?
- Evaluate PDE-based "wave breaking viscosity" (as in turbulence models)
- Time dependent : ALE based adaptation (cf. below)



... THY ...