# On nonlinear shoaling properties of enhanced Boussinesq models

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**Abstract** In this paper, we investigate the nonlinear properties of Boussinesq models. In particular, we consider the wave shoaling obtained in physical regimes which go from linear to weakly nonlinear, to the wave breaking limit. For a given asymptotic accuracy in terms of dispersion and nonlinearity, we consider two families of models: the first depending on derivatives of the velocity, the second on derivatives of the volume flux. Ww show that, while linear dispersion and linear shoaling characteristics are strongly dependent on the type of dispersive terms introduced, when approaching the nonlinear regime the only influencing factor is whether the model is in amplitude-velocity of amplitude-flux form. We investigate these two alternative formulations of several known models, and propose a new model with a compact differential form, and the same linear characteristics of the model of Nwogu. The nonlinear shoaling properties of the models are investigated numerically showing that inside one given family, all the models have almost identical behaviour.

## 1 Introduction

This paper deals with Boussinesq-Type (BT) models for wave propagation and transformation. In the near shore region one has to deal with both nonlinear and dispersive effects which make the task of accurate modelling very difficult. Accounting for genuinely nonlinear effects is a research topics of high priority [4]. The simplest depth averaged model, the nonlinear shallow water equations system (NLSW), while capable of describing the energy dissipation in breaking regions [14], does not account for wave dispersion. In this work, we consider weakly nonlinear and disper-

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sive BT models obtained by adding linear differential terms to the NLSW system. These terms account for non hydrostatic effects, they improve the linear frequency dispersion, however, they do not include any dissipative effects. These effects can be recovered by locally reverting to the NLSW equations in properly detected regions, or by explicitly adding eddy viscosity terms [4,9,14]. In this case, the BT equations are required to accurately predict wave shapes and amplitudes to allow the breaking process to be triggered at the right time and place. In particular, they should provide an accurate description of the shoaling process. Differently from linear dispersion and shoaling, which can be studied analytically [5], nonlinear shoaling can only be investigated numerically. There exist several types of BT models with different linear dispersion and shoaling properties. For a given linear dispersion relation, and within the same asymptotics in terms of the nonlinearity $\varepsilon = a/d$ and dispersion $\sigma = d/\lambda$ parameters ($a$ the wave amplitude, $d$ the mean water level, $\lambda$ the wavelength), one can find at least two different models. These two systems of PDEs differ in the fact that the dispersive terms contain either derivatives of the velocity, or of the volume flux; thus we refer to them as to models in *wave amplitude-velocity* or *wave amplitude-volume flux* form.

The aim of this paper is to assess the nonlinear shoaling properties of BT models which have the same linear properties, but different non-linear PDE structure. We consider in our analysis the models of Peregrine (P) [12], Beji-Nadaoka (BN) [3], Madsen-Sørensen (MS) [10] and Nwogu (N) [11]. For all of them, we manipulate the differential equations adding terms which, keeping the same asymptotic accuracy, allow to obtain a *wave amplitude-velocity* model, for those systems written in *wave amplitude-volume flux* form (e.g the MS model), and vice versa for models in *wave amplitude-velocity* form (e.g the P, BN, N models). We show that, when approaching the nonlinear regime, only the nonlinear structure of the PDE has an influence on the shoaling effects.

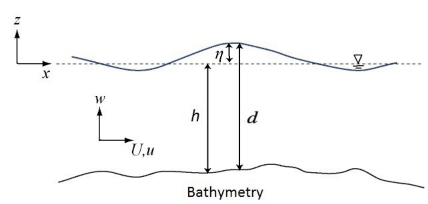



**Fig. 1** Sketch of the free surface flow problem, main parameters description.

## 2 Presentation of the models

The most common BT model is perhaps the one of Peregrine [12]. Starting from the incompressible Euler equations (see Fig. 1), considering asymptotic expansions in terms of the nonlinearity and dispersion parameters, $\varepsilon$ and $\sigma$, and depth averaging the resulting expressions, one can show that for $\varepsilon = \mathcal{O}(\sigma^2)$

$$\begin{cases} \eta_t + [(h+\varepsilon\eta)u]_x = 0 \\ u_t + \varepsilon u u_x + \eta_x + \sigma^2 \left( \frac{h^2}{6} u_{xxt} - \frac{h}{2}[hu]_{xxt} \right) = \mathscr{O}(\varepsilon\sigma^2, \sigma^4) \end{cases} \tag{1}$$

We refer to the Peregrine (P) model as the dimensional version of (1). Denoting the NLSW flux by $F^{SW} = uq + gd^2/2$, for the P model the volume flux $q = du$ verifies

$$q_t + F^{SW}_x - gdh_x + dP_t = 0\,,\;\; P = P(u) = \frac{h^2}{6} u_{xx} - \frac{h}{2}[hu]_{xx}\,, \tag{2}$$

Assuming $h_t = 0$, and since in non-dimensional form $d = h + \varepsilon\eta$, we have

$$(h+\varepsilon\eta)\sigma^2 \left( \frac{h^2}{6} u_{xxt} - \frac{h}{2}(hu)_{xxt} \right) = \sigma^2 \left( \frac{h^3}{6} \left(\frac{q}{h}\right)_{xxt} - \frac{h^2}{2} q_{xxt} \right) + \mathscr{O}(\varepsilon\sigma^2, \sigma^4)\,.$$

Thus, in the same asymptotics, we can replace (2) by

$$Q_t + F^{SW}_x - gdh_x = 0\,,\;\; Q = Q(q) = q + \frac{h^3}{6} \left(\frac{q}{h}\right)_{xx} - \frac{h^2}{2} q_{xx} \tag{3}$$

leading to the model presented in Abbott (A) [1]. Even if the P and A models are identical in the linear limit, they are substantially different in the nonlinear case. In particular, these are not the same PDEs written in terms of different unknowns; they actually include different differential terms. The difference between these terms allows to express the dispersive operators in terms of $q$ instead of in terms of $u$; in such sense the P and A model represent respectively the *amplitude-velocity* and the *amplitude-flux* form of the same linear dispersion relation.

As for the P and the A systems, two set of PDEs exist for a given couple linear dispersion relation-linear shoaling parameter. All dispersion enhanced BT models admit a *amplitude-velocity* form, and an *amplitude-flux* equivalent. Details on the derivation are given e.g. in [6] and in the second volume of [5], and are left out due to space limitations. Here we apply this theory to four linear dispersion relations, corresponding to the P model above and to the enhanced models of BN, MS and N. In total 8 models are considered. Three of them are new variants of existing models, so we speak of the *amplitude-flux* form of the BN model as the Beji-Nadaoka-Abbott (BNA) model, of the *amplitude-velocity* form of the MS model as the Madsen-Sørensen-Peregrine (MSP) model, and of the *amplitude-flux* form of the N model as Nwogu-Abbott (NA) model. They can be generally recast as

$$\begin{cases} h_t + \mathbf{K}_x = 0 \\ \mathbf{Q}_t + F^{SW}_x - ghd_x + h\mathbf{P}_t + g\mathbf{R} + u\boldsymbol{\Psi} = 0 \end{cases}. \tag{4}$$

For the definition of the differential operators $\mathbf{K}$, $\mathbf{Q}$, $\mathbf{P}$, $\mathbf{R}$ and $\boldsymbol{\Psi}$ for each specific model we refer to [6] and [3], [10], [11]. Here we observe that for models P, BN, MSP and N, written in *wave amplitude-velocity* form, $\mathbf{Q} = q$ and $\mathbf{P}$ is structurally similar to P(u) in (2); instead for models A, BNA, MS and NA, written in *wave* *amplitude-volume flux* form, $\mathbf{Q}$ is an elliptic operator structurally similar to Q(q) in (3), and $\mathbf{P}=0$. Only for the BN and MS models $\mathbf{R}\neq 0$, in particular $\mathbf{R}$ has the structure of $P(\eta)$; likewise, $\boldsymbol{\Psi}\neq 0$ only for the N model, assuming the structure of P(u). Finally, $\mathbf{K}\neq q$ only for the N model, with $\mathbf{K}=q+dP(u)$ in the *wave amplitude-velocity* case, while $\mathbf{K}=Q(q)$ in the *wave amplitude-volume flux* case. For each model, we will consider the standard dispersion/shoaling optimised values of the constants. Note that, among the 8 BT models taken into account, two of them present the following very simple structure:

$$\begin{cases} h_t+\mathbf{K}_x=0 \\ \mathbf{Q}_t+F_x^{SW}-ghd_x=0 \end{cases} \tag{5}$$

The first is the A system defined by (3) and $\mathbf{K}=q$, the other one is the NA model taking in (5) $\mathbf{K}=q+A_1h^2q_{xx}+A_2h^3(q/h)_{xx}$ and $\mathbf{Q}=q+B_1h^2q_{xx}+B_2h^3(q/h)_{xx}$ [6]. This system is the only enhanced BT model we know of sharing a compact structure very close to that of the NLSW equations.

## 3 Numerical experiments : shoaling tests

The numerical tests discussed hereafter have been repeated with two different discretizations and on several meshes, to ensure scheme and mesh independent results. The scheme used are the finite difference scheme proposed by Wei and Kirby, which discretizes the shallow water terms using fourth-order formulas and the dispersive terms to second order accuracy [15], and a $P^1$ continuous finite element method based on a standard Galerkin solution of the elliptic sub-problems defining $\mathbf{K}$, $\mathbf{Q}$, $\mathbf{P}$, etc, plus a Galerkin projection for the first order PDEs (4) or (5). This procedure has been recently used in [13] for the MS equations, and shown analytically and numerically to have accuracy close to a fourth order finite difference scheme, and to that of the scheme of [15]. As in [13] high order implicit time integration is used to allow the choice of the time step based on physical arguments. In all the tests the two discretizations have given virtually indistinguishable results. Due to space limitations, we will not show this comparison, but only report the main findings.

### *3.1 Linear shoaling test*

As discussed in section §2, we consider BT equations which, in couples, reduce to the same four linearized systems. In particular the P together with the A model, the BN together with the BNA model, the MSP together with the MS model and the N together with the NA model degenerate to the same linear systems. We refer the interested reader to the references given in section §2. These systems, thus, should manifest the same linear dispersion and linear shoaling behaviour.

To verify that our implementation correctly reproduces the linear shoaling, and in particular that indeed different models collapse onto one another for small amplitude waves, we perform an experiment proposed in [10] : a monochromatic wave of amplitude $a = 0.05$m and period $T = 4$s, propagating over a depth $h_0 = 13$m, and shoaling over a slope of 1 : 50 starting 50m from the inlet. These data give a nonlinearity parameter $\varepsilon \in [0.0038, 0.25]$, which is in the linear range. The generation of the periodic signals is performed adding a source to the mass equation (see [13] for details on this aspect, and [10] for further details on this test).

The results of the test are summarized on Fig. 2 where we have reported the distribution of the wave amplitude for all the models. It can be observed that the implemented schemes well reproduce the theoretical linear behaviors expected for all models. In fact Fig. 2 shows that the schemes of N and NA give nearly identical results, as well as the models MSP and MS and the models BN and BNA do.

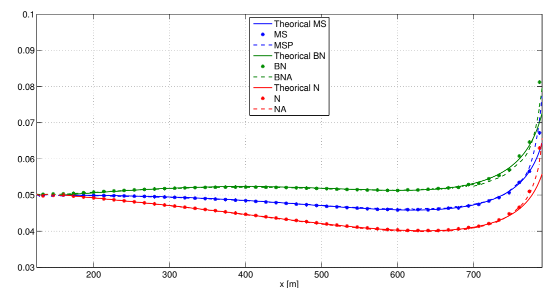



**Fig. 2** Linear shoaling of a periodic wave from deep to shallow water: envelope of the maximum elevations computed by the several models and comparison w.r.t. the theoretical results.

## 3.2 *Nonlinear shoaling test*

We compare now the nonlinear properties of the models on the shoaling test of Grilli et al. in [7] : a solitary wave of amplitude $A/h_0 = 0.2$ m propagating on a depth $h_0 = 0.44$ m, and shoaling on a slope of 1 : 35. In this test $\varepsilon = a_0/h \in [0.45, 1.7]$ which is in the nonlinear range $\varepsilon \geq 1$. A sketch of the test is given on the left of Fig. 3, also showing the position of the gauges where wave height measurements are available.

We present the results on Fig. 3, 4, and 5, in terms of evolution of the wave maximum in space, and of temporal evolution of the wave height in the gauges. In both cases, the experimental data of [7] are reported as well. Looking at the figures we see clearly that only two main behaviors are observed. All the models derived in terms of *amplitude-velocity* (Fig.5) quickly over-shoal. The evolution of the peak height is quite independent of the linear dispersion relation of the models which all give almost the same curves. The same is true for the models derived in terms of *amplitude-volume flux* (Fig.4), which however give a shoaling height closer to the experimental ones.

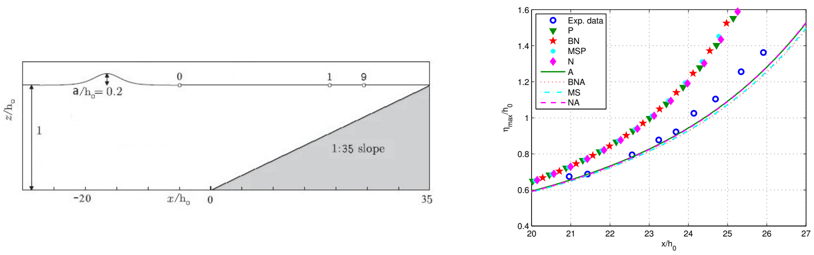



**Fig. 3** Shoaling of a solitary wave. Left: computational set-up. Right: comparison between computed and measured maximum value of the relative wave height.

**Fig. 4** Shoaling of a solitary wave up to breaking: comparison between computed and measured data for relative wave height at gauges 1, 3, 5, 7 and 9 and for the BT model of Abbott, Madsen and Sørensen, and Nwogu-Abbott.

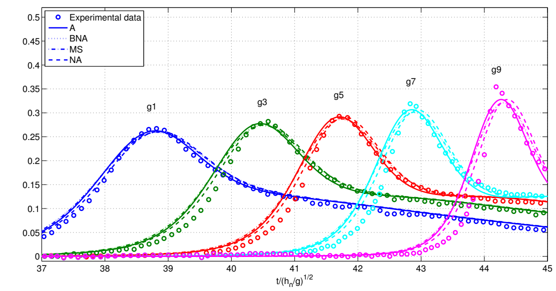
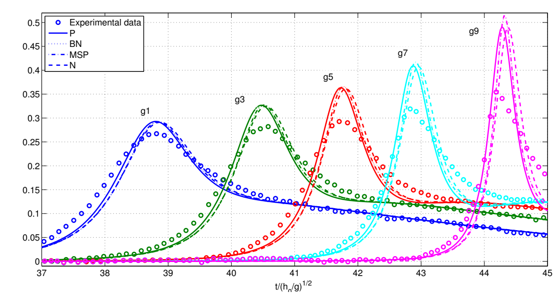



**Fig. 5** Shoaling of a solitary wave up to breaking: comparison between computed and measured data for relative wave height at gauges 1, 3, 5, 7 and 9 and for the BT model of Peregrine, Nwogu, and Beji and Nadaoka.




In particular, it is reported in [7] than the breaking point is gauge 9. We can see from Fig. 4, and 5 that models obtained in terms of velocity all have already given higher waves already in gauge 5, while the models obtained in terms of volume flux under-shoal and never reach this height. This means that very early breaking will be likely to occur if one uses one family of models, while late or no breaking will be observed with the others. Note that here no breaking modeling is considered, so velocity based models keep on shoaling giving very tall and steep waves in the last gauges. The same figures show that the steep front of the waves are generally better described by models with dispersive terms written in terms of the velocity, while the tail of the wave is much better approximated by models in volume flux form.

## 4 Conclusions and perspectives

We have discussed the nonlinear behaviour of weakly nonlinear Boussinesq models. Under the same asymptotic accuracy, a linear dispersion relation/shoaling coefficient define at least two models written either in terms of *velocity* or *volume flux* derivatives. This has allowed in this paper to reformulate existing models, propose new ones, and studied their linear and nonlinear properties. The results show that in the nonlinear case only the *amplitude-velocity* form or *amplitude-volume flux* form counts: models with different linear shoaling and dispersion relations give practically the same results.

Future works will involve the study of the coupling of these models with wave breaking criteria to assess the impact of these properties on the behavior of breaking models, and the extension of this study to genuinely nonlinear equations and to the set of equations recently developed in [2] and [8] which couples the dispersive properties of BT models and the dissipative features of the NLSW equations.

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