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I-BLaws I-BLaws SWEs I-WB GFI-dGSEI dGSEM GFIux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG I-EC EC-GF-dG WBRes 3 Global flux dG-SEM for systems of balance laws with a discretely well balanced entropy correction

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# Acknowledgements

Joint work with

#### EC-GFdGSEM

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I-BLaws SWEs I-WB GF-dGSEM dGSEM dGFbx WBRes 1 UB2d 2D GFdG WBRes 2 EC-GF-dG L-EC EC-GF-dG WBRes 3 EC-GF-dG

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Philipp Öffner Johannes Gutenberg-University, Mainz (Germany)

Credit to them for the good stuff in the talk blame me for the rest



# Setting: balance laws 1

(1)

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + 
abla \cdot F(U) = S(U; arphi(x)),$$

### Typical examples

- Shallow water equations with topography/friction/Coriolis/etc
- Euler equations with gravity
- Shallow Water/Euler in pseudo-1D form (section variation)
- etc.



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### I-BLaws

I-WB Gf-dGSEM dGSEM GFlux WBRes 1 WB2d

#### 2D Gf-dG WBRes 2

EC-Gf-dG I-EC EC-Gf-dG

End

# Setting: balance laws 1

(1)

We seek solutions of the (hyperbolic) system of balance laws

 $\partial_t U + \partial_x F(U) = S(U; \varphi(x)),$ 

### Typical examples

- Shallow water equations with topography/friction/Coriolis/etc
- Euler equations with gravity
- Shallow Water/Euler in pseudo-1D form (section variation)

• etc.

A multi-D generalization is possible, and there will be 2D examples. But for simplicity the discussion is done for the 1D case



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#### Intro I-BLaws

SWEs I-WB Gf-dGSEN

GFlux

WBRes

M/P24

2D.GE

WBRes 2 EC-Gf-dG

EC-Gf-dG WBRes 3

End

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#### ntro

I-BLaws SWEs I-WB GF-dGSEM GFlac WBRes 1 WB2d 2D GFdG WBRes 2 EC-GF-dG I-EC EC-GF-dG WBRes 3

# Setting: balance laws 2

(1)

### We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; arphi(x)),$$

### **Property 1. Non-trivial steady states.**



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#### ntro

# Setting: balance laws 2

We seek solutions of the (hyperbolic) system of balance laws

 $\partial_t U + \partial_x F(U) = 0,$ 

### Property 1. Non-trivial steady states.

For the homogeneous case:

- $U = U_0$  constant in space and time is an exact solution
- Fundamental consequence:
  - consistency condition at the discrete level which is the exactness wrt constant  $\boldsymbol{U}$
- Polynomial approximation explicitly embed this condition in their construction



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# Setting: balance laws 2

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; arphi(x)),$$
 (1)

### Property 1. Non-trivial steady states.

Balance law case:

•  $U = U_0$  constant is rarely an exact solution

### • Fundamental consequence: exactness wrt constant U is not an adequate *consistency condition* at the discrete level

• Using (only) this condition in the construction of discrete approximations may lead to large errors



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# Setting: balance laws 3

### We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x)),$$
 (1)

### **Consistency conditions 1: (steady) invariant states**

In some cases, one can establish other "simple" *invariants* (cf. later shallow water):

$$\partial_t u + \partial_x (u^2/2) + u \partial_x arphi(x) = 0$$

A constant (in space and time) value  $V=:u+\varphi(x)=V_0$  is a relevant consistency condition. Indeed we can rewrite the PDE as

$$\partial_t u + u \partial_x V = 0$$
 or  $\partial_t V + u \partial_x V = 0$ 





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# Setting: balance laws 4

We seek solutions of the (hyperbolic) system of balance laws

 $\partial_t U + \partial_x F(U) = S(U; \varphi(x)),$  (1)

Consistency conditions 2: (steady) integral relations

In other cases, invariants can emerge from exact integral relations:

$$\partial_t u + \partial_x (u^2/2) + arphi(x) u = 0 \quad \Rightarrow \mathsf{steady ODE:} \ \partial_x u + arphi(x) = 0$$

A constant value  $V=:u-u_0+\int\limits_{x_0}^x arphi(s)ds$  is a relevant consistency condition.

Indeed, we can rewrite the PDE as

$$\partial_t u + u \partial_x V = 0$$
 or  $\partial_t V + u \partial_x V = 0$ 





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# Setting: balance laws 5

### We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x)),$$
 (1)

### **Consistency conditions 3: global fluxes**

More generally, we can consider the pseudo-conservative form of the balance law

$$\partial_t U + \partial_x F + \partial_x R = 0$$
, (1)

having introduced the source integral

$$R(U,x) - R_0 := -\int_{x_0}^x S(U,\varphi) ds$$
<sup>(2)</sup>

A constant value in space of the global flux G =: F + R is a relevant consistency condition.

The value of the global flux is only known a priori if the analytical form of a primitive of S is available.

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#### ntro

I-BLaws SWEs I-WB GF-dGSEM dGSEM GFlux WBRes 1

2D Gf-d WBRes

EC-Gf-dG I-EC EC-Gf-dG WBRes 3

End

# Setting: balance laws 6

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x)),$$
 (1)

### **Property 2. Entropy balance.**



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# Setting: balance laws 6

(4)

### We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x)),$$
 (1)

### Property 2. Entropy balance.

System (1) is endowed with an auxiliary constraint

$$\partial_t \eta + \partial_x F_\eta(U) \leq S_\eta(U; arphi(x)),$$

With:

 $\eta = \eta(U)$  a mathematical entropy,  $F_n(U)$  the entropy flux,  $S_n(U;\varphi(x))$  a dissipation/production term.





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I-BLaws SWEs I-WB GF-dGSEM dGSEM GFitx WBRes 1 20 GF-dG WBRes 2 EC-GF-dG I-EC EC-GF-dG WBRes 3 EC-GF-dG

# Setting: balance laws 6

(4)

### We seek solutions of the (hyperbolic) system of balance laws

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With:

 $\eta = \eta(U)$  a mathematical entropy,  $F_{\eta}(U)$  the entropy flux,  $S_{\eta}(U; \varphi(x))$  a dissipation/production term.

For smooth solutions the inequality becomes an equality giving an auxiliary entropy balance law.



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### Intro

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x)),$$
 (1)

**Continuous setting** 

• Possible genelization of the notion of consistency wrt constants (in space) :

- 1 Steady invariants
- 2 Steady integral relations
- 3 Global fluxes
- 4 other declinations (continuous or dicrete level)...
- Well balanced scheme: discrete approximation embedding one (or more) of these notions

### Remark: consistency and entropy conservation

All of the above relate to the main PDE.

Exact consistency with constant entropy flux, *viz entropy conservation* comes as an extra contraint. A well balanced approach may or may not satisfy this constraint.



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### Intro

SWEs I-WB GF-dGSEM dGSEM dGSEM 2D GF-dG 2D GF-dG WBRes 2 EC-GF-dG I-EC EC-GF-dG WBRes 3

# **Continuous setting**

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x)),$$
 (1)

• Possible genelization of the notion of consistency wrt constants (in space) :

- 1 Steady invariants
- 2 Steady integral relations
- 3 Global fluxes
- 4 other declinations (continuous or dicrete level)...
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All of the above relate to the main PDE.

Exact consistency with constant entropy flux, *viz entropy conservation* comes as an extra contraint. A well balanced approach may or may not satisfy this constraint.



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# Example: shallow water equations

SWEs



# Example: shallow water equations 1

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# 2D Cartesian shallow water equations.

$$\partial_t \left(egin{array}{c} h \ hu \ hv \end{array}
ight) + \partial_x \left(egin{array}{c} hu \ hu^2 + p(h) \ huv \end{array}
ight) + \partial_y \left(egin{array}{c} hv \ huv \ hv^2 + p(h) \end{array}
ight) = -h \left(egin{array}{c} 0 \ \partial_x arphi + c_f u + \omega v \ \partial_y arphi + c_f v - \omega u \end{array}
ight)$$

### Notation.

h water depth

 $\vec{v} = (u, v)$  horizontal velocity  $p = gh^2/2$  hydrostatic pressure (g gravity acceleration)  $\varphi = gb$  gravitational potential (b(x, y) bottom topography)  $c_f = c_f(h, \vec{v})$  friction coefficient  $\omega$  Coriolis coefficient



(5)



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# 1D rotating shallow water equations<sup>1</sup>.

$$\partial_t \left( egin{array}{c} h \\ hu \\ hv \end{array} 
ight) + \partial_x \left( egin{array}{c} hu \\ hu^2 + p(h) \\ huv \end{array} 
ight) = -h \left( egin{array}{c} 0 \\ \partial_x \varphi + c_f u + \omega v \\ c_f v - \omega u \end{array} 
ight),$$
 (6)

### Notation.

h water depth

 $\vec{v} = (u, v)$  horizontal velocity  $p = gh^2/2$  hydrostatic pressure (g gravity acceleration)  $\varphi = gb$  gravitational potential (b(x, y) bottom topography)  $c_f = c_f(h, \vec{v})$  friction coefficient  $\omega$  Coriolis coefficient





# Example: shallow water equations 2

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# Example: shallow water equations 3

### 1D rotating shallow water equations.

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + p(h) \\ huv \end{pmatrix} = -h \begin{pmatrix} 0 \\ \partial_x \varphi + c_f u + \omega v \\ c_f v - \omega u \end{pmatrix},$$
 (6)

### Non-trivial steady states.

Example 1: constant energy moving equilibrium (no friction, no Coriolis).

$$hu=q_0\,,\;\;v=0\,,\;\;g(h+b)+k=E_0$$

Example 2: lake at rest with Coriolis perturbation (no friction).

$$u=0\,,\;\;v=V(x)\,,\;\;g(h+b)+\omega\int_xV=Z_0$$

Example 3: moving equilibrium with friction and slope variations<sup>2</sup>

$$hu=q_0\,,\;\;v=0\,,\;\;g(h+b)+k+\int_{x_0}^x c_f u\,ds=E_0$$

<sup>2</sup>See e.g. (Michel-Dansac et al. J.Comput.Phys. 335, 2017) for examples

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#### I-BLaws I-BLaws SWEs I-WB GF-dGSEM dGSEM dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG I-EC EC-GF-dG WBRes 3 F=-1

# Example: shallow water equations 4

### Shallow water equations.

$$\partial_t \begin{pmatrix} h\\ hu\\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu\\ hu^2 + p(h)\\ huv \end{pmatrix} = -h \begin{pmatrix} 0\\ \partial_x \varphi + c_f u + \omega v\\ c_f v - \omega u \end{pmatrix},$$
(6)

### Entropy balance.

$$\partial_t \eta + \partial_x F_\eta = -\mathcal{D}_f$$
 (7)

with total entropy/energy  $\eta$  and entropy flux  $F_{\eta}$ :

$$\eta = p(h) + hk + h\varphi = p(h) + hk + ghb, \quad F_{\eta} = hu \left(gh + k + \varphi\right) = hu \left(g\zeta + k\right)$$
(8)

In absence of friction, and for smooth solution total entropy/energy is conserved



### Focus of this work

- Discretely well balanced method agnostic of the equilibrium
- Use idea of global flux approximation within dGSEM formulation
- Characterize the notion of discrete equilibrium associated to this formulation
- Investigate the compatibility of global flux consistency with entropy/energy conservation

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EC-GF-

dGSEM Ricchiuto

SWEs

EC-GF-

dGSEM

SWEs

## 1 Introduction

Balance laws, consistency, well balanced, conservation Shallow water equations Global flux related well balanced techniques: incomplete taxonomy

## 2 Global Flux Collocated Discontinuous Galerkin

dGSEM basics Global flux assembly

# **3** Numerical results: batch 1

# 4 A simple extension to 2D systems

Global flux dGSEM in 2D Numerical results: batch 2

## **5** Embedding entropy conservation

Introduction: discrete entropy and DGSEM Issue with entropy correction with Gf-dG Numerical results: batch 3

# 6 Conclusion and perspectives

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Gf-dGSI

GFlux

WBRes 1

WB2d

2D Gf-dC

EC-Gf-dG I-EC EC-Gf-dG

End



# Numerics: WB reconstruction, global fluxes, modified Riemann problem

I-WB

# Numerics: WB reconstruction, global fluxes, modified Riemann problem

We focus on techniques which can be related to some "special quadrature" of the source.

Other points of view are possible:

- hydrostatic reconstructions and generalizations (Audusse et al SISC 25 2004; Castro et al., Math.Mod.Meth.Appl.Sci. 5, 2007)
- Reference solutions (Klingenbert et al SISC 41, 2019; Castro & Pares J.Sci.Comp. 82, 2020)

There are of course relations among all these approaches...



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Intro I-BLaws SWEs I-WB

Gf-dGSE dGSEM GFlux

WBRes 1

WB2d 2D Gf-dG WBRes 2

EC-Gt-dC I-EC EC-Gf-dG WBRes 3

End

# Numerics: WB reconstruction, global fluxes, modified Riemann problem 1

Once upon a time ...

P.L. Roe. Upwind differencing schemes for hyperbolic conservation laws with source terms. In Claude Carasso, Denis Serre, and Pierre-Arnaud Raviart, editors, *Nonlinear Hyperbolic Problems*, pages 41–51, Berlin, Heidelberg, 1987. Springer Berlin Heidelberg.



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# Numerics: WB reconstruction, global fluxes, modified Riemann problem 1

### Once upon a time ...

$$\partial_t u + a \partial_x u = q\,, \quad a>0$$

- Integrate in space and time
- upwind fluxes
- integrate the source term along characteristics

$$u_{i}^{n+1} = u_{i}^{n} - v (u_{i}^{n} - u_{i-1}^{n}) + \frac{1}{2}v(1 - v)S_{i-1} - \frac{1}{2}v(1 - v)S_{i} + [(1 - \frac{1}{2}v)q_{i} + \frac{1}{2}vq_{i-1}]\Delta t$$

$$u = a \Delta t / \Delta x$$





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#### II-BLaws SWEs SWEs I-WB GF-dGSEM dGSEM dGSEM dGSEM 2D GFdG 2D GFdG 2D GFdG WBRes 2 EC-GF-dG LF-C EC-GF-dG WBRes 3

Numerics: WB reconstruction, global fluxes, modified Riemann problem 1

### Once upon a time ...

$$\partial_t u + a \partial_x u = q\,, \quad a>0$$

Choice of the slope ...



problem (non-linear systems) several authors [6,7,8] have felt the attraction of considering data which is in piecewise equilibrium. That is, the data is projected into a representation such that the steady flow equations are satisfied within each cell. In our simple model equation, that means choosing

$$S_i = \frac{q_i \Delta x}{a} = \frac{q_i \Delta t}{v}$$



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# Numerics: WB reconstruction, global fluxes, modified Riemann problem $\mathbf{1}$

### Once upon a time ...



### Upwind difference/source splitting:

 $u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} \phi_{i-\frac{1}{2}}$ 

we can measure the extent to which they are out of equilibrium (with each other now, now internally) by the quantity  $\phi_{i-\frac{1}{2}} = a(u_i - u_{i-1}) - \frac{1}{2}\Delta x(q_{i-1} + q_i)$ 





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# Numerics: WB reconstruction, global fluxes, modified Riemann problem 1

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x)),$$
 (1)

Well balanced upwind (difference splitting) schemes

$$\Delta x \frac{dU_i}{dt} + (A^- A^{-1})_{i+1/2}^{\text{Roe}} (F_{i+1} - F_i - \Delta x S_{i+1/2}) + (A^+ A^{-1})_{i-1/2}^{\text{Roe}} (F_i - F_{i-1} - \Delta x S_{i-1/2}) = 0$$

• Bermudez & Vazquez, Computers & Fluids 8, 1994; Vazquez-Cendon, J.Comput.Phys. 148, 1999

- Parés & Castro, *M*<sup>2</sup>*AN* 38, 2004; Parés, *SINUM* 44, 2006
- Castro et al., Math.Mod.Meth.Appl.Sci. 5, 2007



Intro I-BLaws SWEs I-WB Gf-dGSEl dGSEM GFlux WBRes 1 WB2d 2D Gf-dG

EC-Gf-dG I-EC EC-Gf-dG WBRes 3

End

# Numerics: WB reconstruction, global fluxes, modified Riemann problem 2

### General formalism: residual distribution and global fluxes

$$\begin{split} \Delta x \frac{dU_i}{dt} + (A^- A^{-1})^{\mathsf{Roe}}_{i+1/2}(F_{i+1} - F_i - \Delta x \, S_{i+1/2}) \\ &+ (A^+ A^{-1})^{\mathsf{Roe}}_{i-1/2}(F_i - F_{i-1} - \Delta x \, S_{i-1/2}) = 0 \end{split}$$



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WB2d

2D GT-dG

EC-Gf-dG I-EC EC-Gf-dG WBRes 3

End

# Numerics: WB reconstruction, global fluxes, modified Riemann problem 2

### General formalism: residual distribution and global fluxes

$$\Delta x \frac{dU_i}{dt} + (A^- A^{-1})_{i+1/2}^{\mathsf{Roe}} \underbrace{(F_{i+1} - F_i - \Delta x S_{i+1/2})}_{+ (A^+ A^{-1})_{i-1/2}^{\mathsf{Roe}} \underbrace{(F_i - F_{i-1} - \Delta x S_{i-1/2})}_{\Phi^{i-1/2}} = 0$$



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I-BLaws SWEs I-WB GF-dGSEM dGSEM dGSEM GFlux WBRes 1 2D.GF-dG WBRes 2 EC-GF-dG HEC EC-GF-dG WBRes 3 Numerics: WB reconstruction, global fluxes, modified Riemann problem 2

# General formalism: residual distribution and global fluxes

$$egin{aligned} &\Delta x rac{dU_i}{dt} + B_i^{i+1/2} \Phi^{i+1/2} + B_i^{i-1/2} \Phi^{i-1/2} = 0 \ &\Phi^{i-1/2} := \int_{x_{i-1}}^{x_i} (\partial_x F - S) \ &B_i^{i+1/2} + B_{i+1}^{i+1/2} = \mathrm{Id} \end{aligned}$$

- Abgrall & Ricchiuto, arXiv: 2109.08491, 2021; Abgrall & Ricchiuto, ECM 2nd Edition, 2017
- Ricchiuto, J.Comput.Phys. 280, 2015
- Chou & Shu, J.Comput.Phys. 214, 2006; J. Lin et al., J.Sci.Comp. 79, 2019



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I-BLaws SWEs I-WB GF-dGSEM dGSEM dGSEM 2D GF-bz 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG WBRes 3 Numerics: WB reconstruction, global fluxes, modified Riemann problem 2

# General formalism: residual distribution and global fluxes

$$\begin{split} \Delta x \frac{dU_i}{dt} + B_i^{i+1/2} \Phi^{i+1/2} + B_i^{i-1/2} \Phi^{i-1/2} &= 0\\ \Phi^{i-1/2} &:= \int_{x_{i-1}}^{x_i} (\partial_x F - S)\\ B_i^{i+1/2} + B_{i+1}^{i+1/2} &= \mathrm{Id} \end{split}$$

Steady state/well balanced conditions

 ${f 1} \; B_i^{i\pm 1/2}$  uniformly bounded

2 for data at equilibrium 
$$\Phi^{i\pm 1/2}=0$$



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End

# Numerics: WB reconstruction, global fluxes, modified Riemann problem 2

# General formalism: residual distribution and global fluxes

Trick. Set now

$$G_0 = F_0 \;, \quad G_i = G_{i-1} + \int_{x_{i-1}}^{x_i} (\partial_x F - S) \;.$$

The flux splitting/RD prototype can equivalently be written in global flux form

$$\Delta x \frac{dU_i}{dt} = -B_i^{i+1/2} (G_{i+1} - G_i) - B_i^{i-1/2} (G_i - G_{i-1})$$
$$= -(\hat{G}_{i+1/2} - \hat{G}_{i-1/2})$$

with

$$\hat{G}_{i+1/2} = B_i^{i+1/2}(G_{i+1} - G_i) - G_i = G_{i+1} - B_{i+1}^{i+1/2}(G_{i+1} - G_i)$$



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I-BLaws SWEs SWEs I-WB GF-dGSEM dGSEM GFlax WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG I-EC EC-GF-dG WBRes 3 EC-GF- Numerics: WB reconstruction, global fluxes, modified Riemann problem 2

# General formalism: residual distribution and global fluxes

Trick. Set now

$$G_0 = F_0 \;, \quad G_i = G_{i-1} + \int_{x_{i-1}}^{x_i} (\partial_x F - S)$$

The flux splitting/RD prototype can equivalently be written in global flux form

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$$= -(\hat{G}_{i+1/2} - \hat{G}_{i-1/2})$$

Consistency/well balanced conditions

 $\label{eq:Gi} \begin{tabular}{ll} \begin{tabular}{ll} G_i = G_0 \end{tabular} \end{tabular} \end{tabular} \begin{tabular}{ll} G_i = G_0 \end{tabular} \end{tabular} \end{tabular} \end{tabular} \end{tabular} \begin{tabular}{ll} \begin{tabular}{ll} G_i = G_0 \end{tabular} \end{tabular} \end{tabular} \end{tabular} \end{tabular} \end{tabular}$ 

2 
$$\hat{G}_{i+1/2} = G_0$$
 forall  $i$ , for data at equilibrium



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I-BLaws J-BLaws SWEs I-WB GF-dGSEM dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG I-EC EC-GF-dG WBRes 3

Global flux consistency via full well balanced Riemann solver



# Numerics: WB reconstruction, global fluxes, modified Riemann problem 3
I-WB



Two intermediate states RP

# Numerics: WB reconstruction, global fluxes, modified Riemann problem 3

# Global flux consistency via full well balanced Riemann solver

For the shallow water equations, consider the Godunov method with numerical flux

$$\hat{F} = \frac{F_L + F_R}{2} - \frac{1}{2}\sum_k |\lambda_k| \llbracket U \rrbracket_k$$

with  $\lambda_k$  the waves of the approximate Riemann solver, 2 physical waves plus a 0-wave :

$$\lambda_0=u-\sqrt{gh}\,,\;\;\lambda_1=0\,,\;\;\lambda_2=u+\sqrt{gh}$$



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EC-Gf-dG I-EC EC-Gf-dG WBRes 3

End

# 

Two intermediate states RP

# Numerics: WB reconstruction, global fluxes, modified Riemann problem 3

# Global flux consistency via full well balanced Riemann solver

The space-time discrete FV prototype  $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} (\hat{F}_{i+1/2} - \hat{F}_{i-1/2}) + \frac{\Delta t}{2} (\hat{S}_{i+1/2} + \hat{S}_{i+1/2})$   $= -n - \frac{\Delta t}{2} (\hat{F}_{i+1/2} - \hat{F}_{i-1/2})$ 

 $=U_i^n-rac{\Delta t}{\Delta x}(\Phi_i^{i+1/2}+\Phi_i^{i-1/2})$ 

where simple manipulations show

$$\Phi_{i}^{i-1/2} = \frac{1}{2} \left( F_{i} - F_{i-1} - \sum_{k} |\lambda_{k}| \llbracket U \rrbracket_{k} - \Delta x \hat{S}_{i-1/2} \right)$$



# I-WB

 $U_{i-1}$  $U_i$  $x_{i-3/2}$  $x_{i-1/2}$  $x_{i+1/2}$ 

# Numerics: WB reconstruction, global fluxes, modified Riemann problem 3

# Global flux consistency via full well balanced Riemann solver

A steady 0-wave introduces 2 intermediate states: to obtain them we can impose 4 conditions

- 2 conditions:  $F_i F_{i-1} \Delta x \hat{S} = \sum_k \lambda_k \llbracket U \rrbracket$ (space-time integration)
- 2 conditions: steady state relations across 0-wave, e.g.

 $\llbracket hu 
rbracket_0 = 0 \;, \; \ \llbracket g \zeta + k 
rbracket_0 = 0$ 



Two intermediate states RP

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# I-WB



Two intermediate states RP

# Numerics: WB reconstruction, global fluxes, modified Riemann problem 3

# Global flux consistency via full well balanced Riemann solver

Φ

• Steady condition  $U_i = U_i^* \Rightarrow$  algebraic eq. defining  $\hat{S}$ 

• The intermediate states  $U_i^*$  depend on  $\hat{S}$ 

• Automatically verify  $\Phi_i^{i\pm 1/2}=0$  for steady initial data

For data at equilibrium the schemes verify

$$egin{aligned} &\hat{F}_{i}^{i-1/2} =& rac{1}{2} \left( F_{i} - F_{i-1} - \Delta x \hat{S}_{i-1/2} 
ight) \ &pprox rac{1}{2} \int_{-\Delta x/2}^{\Delta x/2} (\partial_{x} F - S) = rac{1}{2} \int_{-\Delta x/2}^{\Delta x/2} \partial_{x} G = 0 \end{aligned}$$



# I-WB

 $U^*$  $U_{i-1}$  $U_i$  $x_{i-3/2}$  $x_{i-1/2}$  $x_{i+1/2}$ 

Two intermediate states RP

# Numerics: WB reconstruction, global fluxes, modified Riemann problem 3

# Global flux consistency via full well balanced Riemann solver



- Greenberg & Leroux, SINUM 33, 1996
- Gosse, Comput.Math.Appli. 39, 2000
- Gallouet et al., Computers & Fluids 32, 2003 ۲
- Berthon & Chalons Math.Comp. 85, 2016: • Michel-Dansac et al. J.Comput.Phys. 154, 2017
- etc



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I-BLaws SWEs I-WB Gf-dGSEM dGSEM GFlux WBRes 1 WB2d 2D Gf-dG WBRes 2 EC-Gf-dG I-EC EC-Gf-dG WBRes 3

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# Numerics: WB reconstruction, global fluxes, modified Riemann problem 4

# Direct global flux reconstruction and discrete equilibria

#### I-BLaws SWEs SWEs GF-dGSEM dGSEM dGSEM 2D GF-dG WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG WBRes 3 EFord

# Numerics: WB reconstruction, global fluxes, modified Riemann problem 4

# Direct global flux reconstruction and discrete equilibria

- 1 Consider a consistent source quadrature which is not WB:  $\Delta x \hat{S} \neq \Delta F$  along exact equilibria
- 2 Define local approximations of the global flux as:

$$R_0 := 0 \,, \quad G_0 = F(U_{s0})$$
 $R_{i+1/2} := R_{i-1/2} + \int_{x_{i-1}}^{x_i} S(U, arphi) \,, \quad G_i := F(U_i) + (R_{i+1/2} + R_{i-1/2})/2$ 

3 Given  $(G_j,R_j)$  assume you can invert the relation  $F(U_j)=G_j-R_j\Rightarrow U_j$ 



HILTO SWEs SWEs FWB GF-dGSEM dGSEM dGSEM dGSEM 2D GF-dG 2D GF-dG 2D GF-dG L-EC EC-GF-dG WBRes 3 F-C

# Numerics: WB reconstruction, global fluxes, modified Riemann problem 4

# Direct global flux reconstruction and discrete equilibria

**Definition** (Discrete steady state). The discrete steady state U(x) of the global flux method is defined from

$$F(U(x)) + R(U(x)) = G_0 = F(U_{s0})$$
(9)

Equation (9) defines a nonlinear algebraic system which can be solved iteratively:

1 
$$F(U_1) + R(U_1, \varphi)/2 = G_0$$
  
2  $F(U_2) + R(U_2, \varphi)/2 = G_0 - R(U_1, \varphi)/2$   
3 etc



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I-BEaves SWEs F-WB GF-dGSEM dGSEM dG

# Numerics: WB reconstruction, global fluxes, modified Riemann problem 4

Direct global flux reconstruction and discrete equilibria



# Global flux FV scheme: At each interface $i \pm 1/2$

1 Reconstruction:

cell global flux polynomials  $G_i(x)$  are built from cell averages using std. methods



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I-WB

# Numerics: WB reconstruction, global fluxes, modified Riemann problem 4

# Direct global flux reconstruction and discrete equilibria



# Global flux FV scheme: At each interface $i \pm 1/2$ 2 Solution recovery: interface values $U_{i+1/2}^{\pm}$ are recovered from

$$F(U_{i+1/2}^{-}) = G_i(x_{i+1/2}) - R_{i+1/2}, \quad F(U_{i+1/2}^{+}) = G_{i+1}(x_{i+1/2}) - R_{i+1/2}$$

Note that

$$G_i(x_{i+1/2}) = G_{i+1}(x_{i+1/2}) \Rightarrow F(U_{i+1/2}^-) = F(U_{i+1/2}^+)$$

In this case, the algebraic solver inverting F(U) = G - R gives no interface jumps.



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I-BLaws SWEs I-WB GF-dGSEM dGSEM dGSEM 2D GF-dG WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG I-EC EC-GF-dG WBRes 3

# Numerics: WB reconstruction, global fluxes, modified Riemann problem 4

# Direct global flux reconstruction and discrete equilibria



# Global flux FV scheme: At each interface $i \pm 1/2$

3 Approximate Riemann problem :

$$\hat{G}_{i+1/2} = \gamma G_i(x_{i+1/2}) + (1-\gamma)G_{i+1}(x_{i+1/2}) + D(U_{i+1/2}^+ - U_{i+1/2}^-)$$

with  $\boldsymbol{D}$  some dissipation matrix.



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# Numerics: WB reconstruction, global fluxes, modified Riemann problem 4

# Direct global flux reconstruction and discrete equilibria

**Definition** (Discrete steady state). The discrete steady state U(x) of the global flux method is defined from

$$F(U(x)) + R(U(x)) = G_0 = F(U_{s0})$$
(9)

### Global flux FV scheme:

$$\Delta x rac{dU_i}{dt} + \hat{G}_{i+1/2} - \hat{G}_{i-1/2} = 0 \; ,$$

**Proposition** (Discrete well balanced). On a given mesh the global flux FV scheme preserves exactly constant global flux states, and the associated discrete steady state U(x) computed from (9), with an expected error wrt the exact steady state  $||U(x) - U_s(x)||$  with  $U_s$  the exact steady equilibrium.



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# Numerics: WB reconstruction, global fluxes, modified Riemann problem 4

# Direct global flux reconstruction and discrete equilibria

- Chertock et al., J.Comput.Phys. 358, 2018
- Cheng et al., *J.Sci.Comp.* 80, 2019
- Chertock et al, J.Sci.Comp. 90, 2022



# Next steps...

#### EC-GFdGSEM

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I-BLaws SWEs F-WB GF-dGSEM dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG I-EC

# Coming up

- $\label{eq:global_state} \ensuremath{{\rm I}}\xspace$  Global flux formulation in the DGSEM context
- 2 Characterization of discrete equilibria
- $\ensuremath{\mathfrak{S}}$  Compatibility between global flux and entropy conservation
- ${\scriptstyle \textcircled{4}}$  Verification of well balanced for various sources in 1D and 2D

# WB DG methods (not using global fluxes)

- Y. Xing, Ohio State
- M. Dumbser, U. Trento
- E. Gaburro, (before U. Trento, now Inria)
- M. Castro, (Edanya group in Malaga)
- G. Gassner (Cologne U.), A. Winters (Linkoping U.) and co.
- F. Giraldo (Navy),
- B. Bonev, J. Hstehaven (EPFL),

many many others...



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#### Intro

SWE

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#### Gf-dGSE

dGSEM

GFlux

WBRes 1

WB2d

2D Gf-d

EC-Gf-dG

WBRes 3

End



# dGSEM for conservation laws

# dGSEM for conservation laws 1

#### EC-GFdGSEM

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Intro I-BLaws SWEs I-WB

dGSEM

# Main notation

- Reference element  $\xi \in [0,1]$
- $x(\xi)$  linear map  $K\mapsto [0,1]$ , here:  $|K|=\mathrm{h}^d$
- $\{\phi_i(\xi)\}_{i=0,p}$  degree p Lagrange bases
- $\{\xi_i\}_{i=0,p} \ p+1$  Gauss-Lobatto (GL) points
- Set  $U_{
  m h} = \sum_{i=0}^p \phi_i(x(\xi)) U_i$
- 2D extension by tensor products







#### Intro I-BLaws SWEs I-WB

#### Gf-dGSEM

dGSEM

WBRes 1

2D Gf-d

EC-Gf-dG I-EC EC-Gf-dG WBRes 3

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End

## DGSEM variational form: conservation laws

Consider for the moment the approximation of solutions of

$$\partial_t U + \partial_x F(U) = 0$$

On an element K, start from the dG approximation arising from the variational form

$$|K| \int_{0}^{1} \varphi_{i}(\xi) \partial_{t} U_{\rm h} - \int_{0}^{1} \partial_{\xi} \varphi_{i}(\xi) F_{\rm h} + (\varphi_{i} \hat{F}_{\rm h}(U_{\rm h}, U_{\rm h}^{+}))_{\xi=1} - (\varphi_{i} \hat{F}_{\rm h}(U_{\rm h}, U_{\rm h}^{+}))_{\xi=0} = 0$$

# dGSEM for conservation laws 2



#### Intro I-BLaws SWEs I-WB

#### dGSEM

GFlux WBRes 1 WB2d 2D Gf-dG WBRes 2 EC-Gf-dG I-EC EC-Gf-dG

End

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# DGSEM variational form: conservation laws

Consider for the moment the approximation of solutions of

 $\partial_t U + \partial_x F(U) = 0$ 

dGSEM: quadrature based on the same GL nodes used for the polynomial expansion<sup>3</sup>

$$rac{\mathrm{d} \mathbf{U}}{\mathrm{d} t} - \widetilde{D}_x^T \mathbf{F} + \mathcal{M}^{-1} \mathcal{B} \widehat{\mathbf{F}} = 0$$

- $\mathcal{M} = ext{diag}(\{w_i\}_{i=0,p})$  with  $w_i = ext{h}\phi_i(\xi_i)$  the quadrature weights
- $\widetilde{D}_x = \mathcal{M} D_x \mathcal{M}^{-1}$  with  $(D_x)_{ij} = \partial_{\xi} \phi_i(\xi_j)$
- $\mathcal{B} = \operatorname{diag}(-1,\ldots,1)$  the matrix sampling boundary values
- $\mathbf{U},\,\mathbf{F},\,\widehat{\mathbf{F}}$  arrays of solution/flux/num. flux values

<sup>3</sup>Kopriva & Gassner J.Sci.Comp. 44, 2010 ; Hesthaven & Warburton, Springer 2008

# dGSEM for conservation laws 2

# dGSEM for conservation laws 3

## SBP property and fluctuation form

$$rac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} - \widetilde{D}_x^T\mathbf{F} + \mathcal{M}^{-1}\mathcal{B}\widehat{\mathbf{F}} = 0$$

The semi-discrete dGSEM equations can be equivalently written as<sup>4</sup>

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} + \widetilde{D}_x\mathbf{F} + \mathcal{M}^{-1}\mathcal{B}(\widehat{\mathbf{F}} - \mathbf{F})$$

<sup>4</sup>Kopriva & Gassner J.Sci.Comp. 44, 2010; Gassner et al. J.Comput.Phys. 327, 2016

#### EC-GFdGSEM

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#### dGSEM

GFlux

WB2d

WBRes 2

EC-Gf-dC I-EC EC-Gf-dG

End

# dGSEM for conservation laws 3

#### EC-GFdGSEM

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# dGSEM

# SBP property and fluctuation/RD form

In other words, the dGSEM can be written in fluctuation/strong form as

$$w_irac{dU_i}{dt}+\Phi_i+\Psi^L_i+\Psi^R_i=0$$

#### with

$$\Phi_i := \int\limits_K \phi_i \partial_x F_{
m h}$$

$$\Psi_i^L := [\phi_i(\hat{F}_{\rm h} - F_{\rm h})]_{\xi=0} , \ \Psi_i^R := [\phi_i(\hat{F}_{\rm h} - F_{\rm h})]_{\xi=1}$$

This form well suited to see that (trivially)  $F_{
m h}=F_0$  is an exact discrete steady state.



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Intro

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GFlux

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2D Gf-c

WBRes 2

EC-Gf-d I-EC EC-Gf-dG

WBRes 3

End



# Gf-dGSEM: balance laws and global flux

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#### Intro

SWEs I-WB Gf-dGSEM dGSEM GFlux

WB2d 2D Gf-dG WBRes 2 EC-Gf-dC

with

I-EC EC-Gf-dG WBRes 3

End

# Gf-dGSEM: balance laws and global flux $\boldsymbol{1}$

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x)),$$
 (1

by locally recasting it in the pseudo-conservative form

 $\partial_t U + \partial_x G(U; arphi(x)) = 0$ 

$$egin{aligned} G(U;arphi(x)) =& F(U) + R(U;arphi(x)) \ R(U;arphi(x)) =& R_0 - \int_{x_0}^x S(U;arphi(s)) ds \end{aligned}$$



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GFlux

Gf-dGSEM: balance laws and global flux 1

(1)

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; arphi(x)),$$

by locally recasting it in the pseudo-conservative form

 $\partial_t U + \partial_x G(U; arphi(x)) = 0$ 

- 1 from F to G: source integral assembly
- 2 discrete well balanced: definition and understanding



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II-BLaws SWEs II-WB GF-dGSEM dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG II-EC EC-GF-dG WBRes 3

# Gf-dGSEM: global flux assembly in 1d

# General definition

$$R(x)=R_0-\int_{x_0}^x S(U;arphi(s))ds$$



# Gf-dGSEM: balance laws and global flux 2

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#### Intro I-BLaws SWEs

GFlux

# Gf-dGSEM: global flux assembly in 1d

Evaluation of  $\{R_i\}_{i=0,p}$ , space-marching method:  $\forall j \ge 0$ 1 Set  $R_0 = R^-$ 

2 
$$R_i=R_{i-1}+\int\limits_{x_{i-1}}^{x_i}S(U;arphi(s))ds$$

# Gf-dGSEM: balance laws and global flux 2



In all  $\{K_j\}_{j\geq 0}$  we set





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# Gf-dGSEM: balance laws and global flux 2

# Gf-dGSEM: global flux assembly in 1d

Evaluation of  $\{R_i\}_{i=0,p}$ , space-marching method:  $\forall j \geq 0$ 1) Set  $R_0 = R^-$ 

**2** 
$$R_i = R_{i-1} - h \sum_{l=0,p} \int_{\xi_{i-1}}^{\xi_i} \varphi_l(\xi) S_l ds$$



In all  $\{K_j\}_{j\geq 0}$  we set

 $R_{
m h} = \sum_{i=0,p} arphi_i(x(\xi)) R_i$ 



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#### I-BLaws SWEs I-WB GF-dGSEM dGSEM dGSEM dGSEM 20 GF-dG 20 GF-dG WBRes 2 EC-GF-dG EC-GF-dG WBRes 3 EC-GF-dG

# Gf-dGSEM: balance laws and global flux 2

# Gf-dGSEM: global flux assembly in 1d

Evaluation of  $\{R_i\}_{i=0,p},$  space-marching method:  $\forall j\geq 0$  . 1 Set  $R_0^{K_j}=R^-$ 

2 
$$R_i = R_{i-1} - h \sum_{l=0,p} \int_{\xi_{i-1}}^{\xi_i} \varphi_l(\xi) S_l ds$$

Over an element we have

 $\mathbf{R}=\mathbf{R}^{-}-\mathcal{I}\mathbf{S}$ 



In all  $\{K_j\}_{j\geq 0}$  we set

 $R_{
m h} = \sum_{i=0,p} arphi_i(x(\xi)) R_i$ 

**Remark.** The matrix  $\mathcal I$  is the integration tableau of the p+1 stages RK-LobattoIIIA ODE solver<sup>5</sup>



<sup>5</sup>A. Prothero & A. Robinson, Math.Comp. 28, 1974

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I-BLaws SWEs I-WB Gf-dGSE

#### GFlux

WBRes 3

WB2d 2D Gf-dC

EC-Gf-dG I-EC EC-Gf-dG WBRes 3

End

# Gf-dGSEM: discrete well balanced

Gf-dGSEM: balance laws and global flux 3



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I-BLaws SWEs I-WB GF-dGSEM dGSEM GFlux WBRes 1 UB2d 2D GFdG WBRes 2 EC-GF-dG UBRes 2 EC-GF-dG WBRes 3 EC-BF-dG

# Gf-dGSEM: balance laws and global flux 3

### Gf-dGSEM: discrete well balanced

**Definition** (Discrete steady state). The discrete steady state  $U^*$  is the polynomial approximation arising from the solution of the elemental systems of nonlinear algebraic equations obtained as

$$F(U_i^*) - \sum_{j=0,p} \mathcal{I}_{ij}S(U_j^*, arphi(x_0+c_j\mathbf{h})) = G_0 - R_0$$

with  $G_0$  a global (over the mesh) constant flux state,  $R_0 = R^-$  the elemental initial value of R, and with  $\mathcal{I}_{ij}$  and  $c_j$  the entries of the integration tableau of the p+1 stages implicit RK LobattoIIIA ODE integrator.





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# Gf-dGSEM: balance laws and global flux 3

# Gf-dGSEM: discrete well balanced

**Proposition** (Discrete steady state). Let  $R_0^{K_0} = 0$ , and  $\forall K_j$  let  $R^- = R_p^{K_{j-1}}$ . For smooth enough data,  $U^*$  is an approximation of order  $h^{2p}$  to a continuous exact steady state U.





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#### I-BLaws SWEs I-WB GF-GGSEM dGSEM GFlux WBRes 1 2D GF-dG WBRes 2 2D GF-dG WBRes 2 2D GF-dG WBRes 3 EC-GF-dG

# Gf-dGSEM: balance laws and global flux 3

# Gf-dGSEM: discrete well balanced

**Proposition** (Discrete steady state). Let  $R_0^{K_0} = 0$ , and  $\forall K_j$  let  $R^- = R_p^{K_{j-1}}$ . For smooth enough data,  $U^*$  is an approximation of order  $h^{2p}$  to a continuous exact steady state U.

*Proof.* The integration strategy reduces to the ODE integration with the A-stable implicit LobattoIIIA RK scheme of order  $2p^6$  applied to  $\partial_x F - S = 0$ 



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<sup>6</sup>A. Prothero & A. Robinson, Math.Comp. 28, 1974

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I-BLaws SWEs I-WB GF-dGSEN dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG WBRes 3

End

## Gf-dGSEM: discrete well balanced

 $\mathbf{F}(\mathbf{U}) - \mathcal{I}\mathbf{S}(\mathbf{U}; arphi) = \mathbf{G}_0 - \mathbf{R}_0$  $\mathbf{R}_0^{K_j} = \mathbf{R}_p^{K_{j-1}}$ 



In all  $\{K_j\}_{j\geq 0}$  we set

$$R_{
m h} = \sum_{i=0,p} arphi_i(x(\xi)) R_i$$

- The nonlinear equation is solved element by element via Newton iterations for  $\{U_i\}_{i=0,p}$
- Nonlinear solver only needed for the exact/IC as data at quadrature points is directly evolved
- At equilibrium at interfaces  $\{F(U_p) + R_p\}_{K_{j-1}} = \{F(U_0) + R_0\}_{K_j} \Longrightarrow$  no jumps of U
- For smooth solutions  $U^* U^{\mathsf{ex}} = \mathcal{O}(\mathrm{h}^{2p}) \Rightarrow$  potential for superconvergence for  $p \geq 2$



# Gf-dGSEM: balance laws and global flux 3

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Intro I-BLaw SWFs

CE ACSE

dGSEN

GFlux

WBRes 1

WB2d

WBRes 2

EC-Gf-di I-EC EC-Gf-dG

WBRes 3

End



Gf-dGSEM: variational form and numerical fluxes



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# GFlux

Gf-dGSEM: balance laws and global flux 4

**Gf-dGSEM:** full discretization, well balanced, and numerical fluxes The Gf-dGSEM can be written in fluctuation form

$$w_i rac{dU_i}{dt} + \Phi_i + \Psi_i^L + \Psi_i^R = 0$$

with, setting 
$$G_{\mathrm{h}} = \sum_{l=0,p} \phi_l(x(\xi))G_l$$
, with  $G_l = F(U_l) + R_l$   
$$\Phi_i := \int_K \phi_i \partial_x G_{\mathrm{h}} , \quad \begin{cases} \Psi_i^L := [\phi_i(\hat{G}_{\mathrm{h}} - G_{\mathrm{h}})]_{\xi=0} \\ \Psi_i^R := [\phi_i(\hat{G}_{\mathrm{h}} - G_{\mathrm{h}})]_{\xi=1} \end{cases}$$

or equivalently in semi-discrete matrix form

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} + \widetilde{D}_x\mathbf{G} + \mathcal{M}^{-1}\mathcal{B}(\widehat{\mathbf{G}} - \mathbf{G}) = 0$$

with  $\mathbf{G}=\mathbf{F}+\mathbf{R}$ 





I-BLaws SWEs I-WB GF-dGSEM dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG EC-GF-dG EC-GF-dG EC-GF-dG

# Gf-dGSEM: balance laws and global flux 4

**Gf-dGSEM:** full discretization, well balanced, and numerical fluxes The Gf-dGSEM can be written in fluctuation form

$$w_i rac{dU_i}{dt} + \Phi_i + \Psi_i^L + \Psi_i^R = 0$$

with, setting 
$$G_{
m h} = \sum\limits_{l=0,p} \phi_l(x(\xi)) G_l$$
, with  $G_l = F(U_l) + R_l$ 

$$\Phi_i := \int\limits_K \phi_i \partial_x G_\mathrm{h} \;, \quad \left\{ egin{array}{c} \Psi_i^L := [\phi_i (G_\mathrm{h} - G_\mathrm{h})]_{\xi=0} \ \Psi_i^R := [\phi_i (\hat{G}_\mathrm{h} - G_\mathrm{h})]_{\xi=1} \end{array} 
ight.$$

**Definition.** (Consistent numerical global flux). A numerical global flux  $\hat{G} = \hat{G}(U^+, \varphi^+; U^-, \varphi^-)$  is said to be consistent if

$$G(U^+,\varphi^+) = G(U^-,\varphi^-) = G_0 \quad \Rightarrow \quad \hat{G}(U^+,\varphi^+;U^-,\varphi^-) = G_0$$



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# GFlux

Gf-dGSEM: full discretization, well balanced, and numerical fluxes The Gf-dGSEM can be written in fluctuation form

$$w_irac{dU_i}{dt}+\Phi_i+\Psi^L_i+\Psi^R_i=0$$

setting 
$$G_{\mathrm{h}} = \sum_{l=0,p} \phi_l(x(\xi))G_l$$
, with  $G_l = F(U_l) + R_l$   
$$\Phi_i := \int_K \phi_i \partial_x G_{\mathrm{h}} , \quad \begin{cases} \Psi_i^L := [\phi_i(\hat{G}_{\mathrm{h}} - G_{\mathrm{h}})]_{\xi=0} \\ \Psi_i^R := [\phi_i(\hat{G}_{\mathrm{h}} - G_{\mathrm{h}})]_{\xi=1} \end{cases}$$

### Example:

with.

$$\hat{G}(U^+,\varphi^+;U^-,\varphi^-) = \alpha G(U^+,\varphi^+) + (\mathrm{Id} - \alpha)G(U^-,\varphi^-) - \mathcal{D}(U^+ - U^-)$$



# Gf-dGSEM: balance laws and global flux 4


I-BLaws SWEs I-WB GF-dGSEM dGSEM dGSEM 2D GF-dG 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG WBRes 3 End Gf-dGSEM: balance laws and global flux 4

**Gf-dGSEM:** full discretization, well balanced, and numerical fluxes The Gf-dGSEM can be written in fluctuation form

$$w_irac{dU_i}{dt}+\Phi_i+\Psi^L_i+\Psi^R_i=0$$

with, setting 
$$G_{\mathrm{h}} = \sum_{l=0,p} \phi_l(x(\xi))G_l$$
, with  $G_l = F(U_l) + R_l$   
$$\Phi_i := \int_K \phi_i \partial_x G_{\mathrm{h}} , \quad \begin{cases} \Psi_i^L := [\phi_i(\hat{G}_{\mathrm{h}} - G_{\mathrm{h}})]_{\xi=0} \\ \Psi_i^R := [\phi_i(\hat{G}_{\mathrm{h}} - G_{\mathrm{h}})]_{\xi=1} \end{cases}$$

**Proposition.** (Gf-dGSEM and discrete well balanced). The Gf-dGSEM scheme with a consistent numerical global flux is discretely well balanced, in the sense that (equivalently)

- $\mathbf{1}$  it preserves exactly the discrete equilibrium  $U^*$  associated to the quadrature defining G
- 2 It has a super-convergent behaviour of order  $h^{2p}$  wrt exact smooth steady states



#### EC-GFdGSEM

#### Ricchiuto

I-BLaws SWEs SWEs I-WB GF-dGSEM dGSEM dGSEM VBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG WBRes 3 EC-GF-dG

# Gf-dGSEM: balance laws and global flux 5

# **Gf-dGSEM** recap

On and element  $K_j$  we have

1 
$$w_i \frac{dU_i}{dt} + \int_K \phi_i \partial_x G_h + [\phi_i (\hat{G}_h - G_h)]_{\xi=0} + [\phi_i (\hat{G}_h - G_h)]_{\xi=1} = 0$$

2 
$$G_{
m h} = \sum_{i=0,p} \varphi_i(x(\xi))(F(U_i) + R_i)$$

3 
$$R_0=R_p^{K_{j-1}}$$
 and  $R_i=R_0-\sum_{l=0,p}\mathcal{I}_{il}S_l$ 

Remains to define the nodal values  $S_l$ 



EC-GFdGSEM

#### Ricchiuto

I-BLaws SWEs I-WB GF-dGSEM dGSEM dGSEM 2D GF-dG WBRes 1 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG WBRes 3 End

# Gf-dGSEM: balance laws and global flux 6

## Gf-dGSEM: well balanced fluxes and exact lake at rest

Proposition (Exact lake at rest preservation<sup>7</sup>.) For the shallow water equations the choice

 $S_l = g \zeta_l \partial_x b_{
m h}(x(\xi_l)) - \partial_x p_{
m h}(b)(x(\xi_l)) + \omega h_l v_l + c_f h_l u_l$ 

allows exact preservation of the analytical lake at rest  $h_j + b_j = \zeta_0$ , hu = hv = 0.



<sup>&</sup>lt;sup>7</sup>Generalization of approach by Xing & Shu, J.Comput.Phys. 208, 2005



GFlux

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# Gf-dGSEM: balance laws and global flux 6

# Gf-dGSEM: well balanced fluxes and exact lake at rest

**Proposition** (Exact lake at rest preservation<sup>7</sup>.) For the shallow water equations the choice  $S_l = g\zeta_l \partial_x b_h(x(\xi_l)) - \partial_x p_h(b)(x(\xi_l)) + \omega h_l v_l + c_f h_l u_l$ allows exact preservation of the analytical lake at rest  $h_j + b_j = \zeta_0$ , hu = hv = 0.

# Proof. consequence of approximation properties:

$$g\zeta_0 \int_0^{\xi_j} \phi_l(\xi) (\mathrm{h}^{-1} \sum_{i=0,p} \partial_{\xi} \phi_i(\xi_l) b_i) = g\zeta_0 \int_0^{\xi_j} \underbrace{\mathrm{h}^{-1} \partial_{\xi} \phi_i(\xi_l) b_i}_{h^{-1} \partial_{\xi} \phi_i(\xi_l) b_i} = g\zeta_0 (b_j - b_0)$$

<sup>7</sup>Generalization of approach by Xing & Shu, J.Comput.Phys. 208, 2005



#### I-BLaws SWEs I-WB GF-dGSEM dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG WBRes 3 EC-GF-dG

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# Gf-dGSEM: balance laws and global flux 6

# Gf-dGSEM: well balanced fluxes and exact lake at rest

**Proposition** (Exact lake at rest preservation<sup>7</sup>.) For the shallow water equations the choice  $S_l = g\zeta_l \partial_x b_{\rm h}(x(\xi_l)) - \partial_x p_{\rm h}(b)(x(\xi_l)) + \omega h_l v_l + c_f h_l u_l$ 

allows exact preservation of the analytical lake at rest  $h_j + b_j = \zeta_0$ , hu = hv = 0.

*Proof.* consequence of approximation properties:

$$R_j - R_0 = g\zeta_0(b_j - b_0) - (p(b_j) - p(b_0)) \xrightarrow{h_j = \zeta_0 - b_j} -(p(h_j) - p(h_0)) = -(F_j - F_0)$$

so  $G_j - G_0 = 0$  for the lake at rest state.

<sup>&</sup>lt;sup>7</sup>Generalization of approach by Xing & Shu, J.Comput.Phys. 208, 2005

EC-GFdGSEM

#### Ricchiuto

- I-BLaws SWEs I-WB
- Gf-dGSEI
- GFlux
- WBRes 1
- WB2d 2D Gf-dG
- EC-Gf-dG I-EC EC-Gf-dG
- End



- Perturbation of steady states
- 3 moving and non-moving equilibria (no ad hoc scheme modification)



- 1 Verification of super-convergence property
- 2 Perturbation of steady states
- 3 moving and non-moving equilibria (no ad hoc scheme modification)
- Time integration: SSP-RK(p).
- (Non WB) dGSEM:

$$w_i \frac{dU_i}{dt} + \int_K \phi_i \partial_x F_{\rm h} + \int_K \phi_i S(U_{\rm h};\varphi) + [\phi_i (\hat{F}_{\rm h} - F_{\rm h})]_{\xi=0} + [\phi_i (\hat{F}_{\rm h} - F_{\rm h})]_{\xi=1} = 0$$



#### EC-GFdGSEM

#### Ricchiuto

#### SWEs I-WB GF-dGSEM dGSEM GFlux WBRes 1 WB2d

- 2D Gf-dG WBRes 2 EC-Gf-d0 I-EC
- EC-Gf-dG WBRes 3
- End

# 1D rotating shallow water equations<sup>8</sup>.

 $\partial$ W

. .

<sup>8</sup>Castro et al, *SISC* 31, 2008

$$egin{aligned} &\partial_t h + \partial_x (hu) = 0 \ &\partial_t (hu) + \partial_x (hu^2 + p(h)) + gh \partial_x b - \omega hv = 0 \ &\partial_t (hv) + \partial_x (huv) + \omega hu = 0 \end{aligned}$$

$$\iota)=0$$

ith as usual 
$$p(h)=gh^2/2$$

EC-GFdGSEM

#### Ricchiuto

WBRes 1

Ínría-

#### EC-GFdGSEM

#### Ricchiuto

WBRes 1

# Ínría-

# Numerical examples in 1d

#### Lake at rest

 $q = hu = q_0 = 0$  $q_y = hv = 0$ 

# and

 $E = g(h+b) + k = E_0 = gh_0$ 

or

$$K=hu^2+p+\int_x gh\partial_x b=F_0=p_0$$





unperturbed IC evolved up to T=5

WBRes 1

# Moving equilibria: sub-critical flow

 $q = hu = q_0 = 4.42$  $q_y = hv = 0$ 

#### and

 $E = q(h+b) + k = E_0 = 22.06$ 

#### or

$$K=hu^2+p+\int_x gh\partial_x b=F_0=29.41$$



Exact preservation of global flux/ $U^*$ 



unperturbed IC evolved up to T=5



# Numerical examples in 1d

WBRes 1

# Moving equilibria: sub-critical flow

 $q = hu = q_0 = 4.42$  $q_y = hv = 0$ 

#### and

 $E = q(h + b) + k = E_0 = 22.06$ 

#### or

$$K = hu^2 + p + \int_x gh\partial_x b = F_0 = 29.41$$



(Super-)Convergence :  $h^* - h^{ex}$ 



unperturbed IC evolved up to T=5

# Numerical examples in 1d

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WBRes 1

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# Moving equilibria: sub-critical flow

 $q = hu = q_0 = 4.42$  $q_y = hv = 0$ 

#### and

 $E = q(h + b) + k = E_0 = 22.06$ 

#### or

$$K = hu^2 + p + \int_x gh\partial_x b = F_0 = 29.41$$



(Super-)Convergence :  $h^* - h^{ex}$ 



unperturbed IC evolved up to T=5

# Numerical examples in 1d

WBRes 1

# Moving equilibria: sub-critical flow

 $q = hu = q_0 = 4.42$  $q_u = hv = 0$ 

#### and

 $E = g(h + b) + k = E_0 = 22.06$ 

#### or

$$K = hu^2 + p + \int_x gh\partial_x b = F_0 = 29.41$$





unperturbed IC evolved up to T=5



#### I-BLaws SWEs I-WB Gf-dGSEM dGSEM GFlux WBRes 1

2D Gf-dG WBRes 2 EC-Gf-dG

EC-Gf-dG WBRes 3

End

# Ínría\_

# Moving equilibria: sub-critical flow

 $q=hu=q_0=4.42$  $q_y=hv=0$ 

#### and

 $E = g(h + b) + k = E_0 = 22.06$ 

#### or

$$K = hu^2 + p + \int_x gh\partial_x b = F_0 = 29.41$$



#### Small perturbation: global flux vs nonWB



# Numerical examples in 1d

I-BLaws SWEs I-WB Gf-dGSEN dGSEM GFlux WBRes 1 WB2d

2D Gf-dG WBRes 2

I-EC EC-Gf-dG WBRes 3

End

# Moving equilibria: super-critical flow

 $q = hu = q_0 = 4.42$  $q_y = hv = 0$ 

#### and

 $E = g(h + b) + k = E_0 = 28.9$ 

#### or

$$K=hu^2+p+\int_x gh\partial_x b=F_0=31.74$$



Exact preservation of global  $flux/U^*$ 



unperturbed IC evolved up to  $T=5\,$ 

# Numerical examples in 1d

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I-BLaws SWEs I-WB Gf-dGSEM dGSEM GFlux WBRes 1

2D Gf-dG WBRes 2 EC-Gf-dG I-EC

EC-Gf-dG WBRes 3

End

# Moving equilibria: super-critical flow

 $q = hu = q_0 = 4.42$  $q_y = hv = 0$ 

#### and

 $E = g(h + b) + k = E_0 = 28.9$ 

#### or

$$K = hu^2 + p + \int_x^r gh\partial_x b = F_0 = 31.74$$



(Super-)Convergence :  $h^* - h^{
m ex}$ 



unperturbed IC evolved up to  $T=5\,$ 

# Ínría-

# Numerical examples in 1d

#### I-BLaws SWEs I-WB Gf-dGSEN dGSEM GFlux WBRes 1

WB2d 2D Gf-dG WBRes 2 EC-Gf-dG I-EC EC-Gf-dG

End

# Moving equilibria: super-critical flow

 $q = hu = q_0 = 4.42$  $q_y = hv = 0$ 

#### and

 $E = g(h + b) + k = E_0 = 28.9$ 

#### or

$$K = hu^2 + p + \int_x gh\partial_x b = F_0 = 31.74$$



Small perturbation: global flux vs nonWB



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# Numerical examples in 1d

WBRes 1

# Algebraic source: pressure/Coriolis force equilibrium

 $q = hu = q_0 = 0$  $q_y = hv = q_y(x)$ 

and

 $E = qh + k = E_0 = qh_0$ 

or

$$K = p - \int_x \omega q_y(x) = p_0$$



Exact preservation of global flux/ $U^*$ 



Numerical examples in 1d

unperturbed IC evolved up to T=5







#### I-BLaws SWEs I-WB Gf-dGSEM dGSEM GFlux WBRes 1

2D Gf-dG WBRes 2 EC-Gf-dC

I-EC EC-Gf-dG WBRes 3

End

# Algebraic source: pressure/Coriolis force equilibrium

 $egin{aligned} q =& hu = q_0 = 0 \ q_y =& hv = q_y(x) \end{aligned}$ 

and

 $E = gh + k = E_0 = gh_0$ 

or





(Super-)Convergence :  $h^* - h^{ex}$ 



unperturbed IC evolved up to  ${\cal T}=5$ 

Numerical examples in 1d

Ínría-

# WBRes 1

# Algebraic source: pressure/Coriolis force equilibrium

 $q = hu = q_0 = 0$  $q_u = hv = q_u(x)$ 

and

 $E = ah + k = E_0 = ah_0$ 

or





(Super-)Convergence :  $h^* - h^{ex}$ 



unperturbed IC evolved up to T=5

Ínría-

# Numerical examples in 1d

# I-BLaws SWEs I-WB Gf-dGSEM dGSEM GFlux WBRes 1

2D Gf-dG WBRes 2

EC-Gf-dC I-EC EC-Gf-dG

End

# Algebraic source: pressure/Coriolis force equilibrium

 $egin{aligned} q =& hu = q_0 = 0 \ q_y =& hv = q_y(x) \end{aligned}$ 

and

 $E=gh+k=E_0=gh_0$ 

or

$$K = p - \int_x \omega q_y(x) = p_0$$





Numerical examples in 1d

 $h=h_0+10^{-3}e^{-100x^2}$  evolved up to T=1.5



#### EC-GFdGSEM

#### Ricchiuto

#### I-BLaws SWEs I-WB

#### Gf-dGSEM

GFlux

#### WBRes 1

WB2d

2D Gf-dG WBRes 2

```
EC-Gf-dC
I-EC
EC-Gf-dG
WBRes 3
```

End

# **Rotating shallow water: geostrophic adjustment**<sup>9</sup>





$$\partial_t h + \partial_x (hu) = 0$$
  
 $\partial_t (hu) + \partial_x (hu^2 + p(h)) + gh \partial_x b - \omega hv = 0$  (11)  
 $\partial_t (hv) + \partial_x (huv) + \omega hu = 0$ 

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<sup>9</sup>Bouchut et al. J.Fluid Mech. 514, 2004 ; Castro et al. SISC 31, 2008

#### EC-GFdGSEM

#### Ricchiuto

#### Intro I-BLaws SWEs I-WB GF-dGSEM dGSEM GFlux WBRes 1 WB2d 2D GF-dG VBRes 2 EC-GF-dG I-EC EC-GF-dG

# Rotating shallow water: geostrophic adjustment<sup>9</sup>

Free surface evolution (Gf-dGSEM: P1 in red - P2 in blue - right pic from Castro et al.)



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<sup>9</sup>Bouchut et al. J.Fluid Mech. 514, 2004 ; Castro et al. SISC 31, 2008

EC-GFdGSEM

#### Ricchiuto

I-BLaws SWEs I-WB Gf-dGSE dGSEM GFlux WBRes

2D Gf-dG

A 2D extension



# ${\it Gf-dGSEM \ in \ 2D \ 1}$

# Gf-dGSEM in 2D 1

#### EC-GFdGSEM

#### Ricchiuto

FBLaws SWEs I-WB GF-dGSEM dGSEM GFlux WBRes 1 WBRes 1 WBRes 2 EC-GF-dG EC-GF-dG EC-GF-dG EC-GF-dG EC-GF-dG

# Main notation

- Reference element  $\xi \in [0, 1]$
- $x(\xi)$  linear map  $K\mapsto [0,1]$ , here:  $|K|=\mathrm{h}^d$
- $\{\phi_i(\xi)\}_{i=0,p}$  degree p Lagrange bases
- $\{\xi_i\}_{i=0,p}$  the p+1 Gauss-Lobatto (GL) points
- Set  $U_{
  m h} = \sum_{i=0}^p \phi_i(x(\xi)) U_i$
- 2D extension by tensor products





# Gf-dGSEM in 2D 2

(5)

#### EC-GFdGSEM

#### Ricchiuto

I-BLaws SWEs I-WB GF-dGSEM dGSEM dGSEM WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG WBRes 3 End

# 2D Cartesian shallow water equations.

$$\partial_t \left( egin{array}{c} h \ hu \ hv \end{array} 
ight) + \partial_x \left( egin{array}{c} hu \ hu^2 + p(h) \ huv \end{array} 
ight) + \partial_y \left( egin{array}{c} hv \ huv \ hv^2 + p(h) \end{array} 
ight) = -h \left( egin{array}{c} 0 \ \partial_x arphi + c_f u + \omega v \ \partial_y arphi + c_f v - \omega u \end{array} 
ight)$$

# Notation.

h water depth

 $\vec{v} = (u, v)$  horizontal velocity  $p = gh^2/2$  hydrostatic pressure (g gravity acceleration)  $\varphi = gb$  gravitational potential (b(x, y) bottom topography)  $c_f = c_f(h, \vec{v})$  friction coefficient  $\omega$  Coriolis coefficient





# Gf-dGSEM in 2D 3

#### EC-GFdGSEM

#### Ricchiuto

I-BLaws SWEs I-WB GF-dGSEM dGSEM GFlux WBRes 1 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG EC-GF-dG 2D Cartesian shallow water equations.

$$\partial_t \left(egin{array}{c} h \ hu \ hv \end{array}
ight) + \partial_x \left(egin{array}{c} hu \ hu^2 + p(h) + rx \ huv \end{array}
ight) + \partial_y \left(egin{array}{c} hv \ huv \ hv^2 + p(h) + ry \end{array}
ight) = 0,$$
 (11)

# Global flux/pressure formulation.





#### EC-GFdGSEM

#### Ricchiuto

III. BLaws SWEs I-WB GF-dGSEM dGSEM dGSEM dGSEM 2D GF-dG WBRes 1 2D GF-dG WBRes 2 EC-GF-dG I-EC EC-GF-dG I-EC EC-GF-dG WBRes 3

# 2D Cartesian shallow water equations.

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + p(h) + rx \\ huv \end{pmatrix} + \partial_y \begin{pmatrix} hv \\ huv \\ hv^2 + p(h) + ry \end{pmatrix} = 0,$$
(12)

# **Global flux/pressure formulation.**

Line based evaluation of (rx, ry):  $\forall K_{lm}$  we set

$$\mathbf{r}\mathbf{x}_j = \mathbf{r}\mathbf{x}_j^- - \mathcal{I}\mathbf{S}\mathbf{x}_j$$

# where

- $\mathbf{rx}_j$  and  $\mathbf{Sx}_j$  contain the  $\{rx_{ij}\}_{i=0,p}$  and  $\{Sx_{ij}\}_{i=0,p}$  values
- $Sx:=gh\partial_x b-\omega hv+c_fhu$  is the x-momentum source
- ${\mathcal I}$  is the implicit RK-LobattoIIIA tableau
- The IC is taken as  $rx_{ij}^- = (rx_{pj})_{K_{l-1\,m}}$





# Gf-dGSEM in 2D 3

#### EC-GFdGSEM

#### Ricchiuto

III. BLaws SWEs I-WB GF-dGSEM dGSEM dGSEM dGSEM 2D GF-dG WBRes 1 2D GF-dG WBRes 2 EC-GF-dG I-EC EC-GF-dG I-EC EC-GF-dG WBRes 3

# 2D Cartesian shallow water equations.

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + p(h) + rx \\ huv \end{pmatrix} + \partial_y \begin{pmatrix} hv \\ huv \\ hv^2 + p(h) + ry \end{pmatrix} = 0,$$
(13)

# Global flux/pressure formulation.

Line based evaluation of (rx, ry):  $\forall K_{lm}$  we set

$$\mathbf{ry}_i = \mathbf{ry}_i^- - \mathcal{I}\mathbf{Sy}_i$$

#### where

- $\mathbf{ry}_i$  and  $\mathbf{Sy}_i$  contain the  $\{ry_{ij}\}_{i=0,p}$  and  $\{Sy_{ij}\}_{i=0,p}$  values
- $Sy:=gh\partial_y b+\omega hu+c_fhv$  is the x-momentum source
- $\mathcal I$  is the implicit RK-LobattoIIIA tableau
- The IC is taken as  $ry_{ij}^- = (ry_{ip})_{K_{l\,m-1}}$





# Gf-dGSEM in 2D 3

# Gf-DGSEM in 2D 4

$$egin{aligned} & rac{dU_{ij}}{dt} \!+\! \widetilde{D}_x \mathbf{G} \mathbf{x} + \mathcal{M}^{-1} \mathcal{B}(\widehat{\mathbf{G} \mathbf{x}} - \mathbf{G} \mathbf{x}) \ & + \widetilde{D}_y \mathbf{G} \mathbf{y} + \mathcal{M}^{-1} \mathcal{B}(\widehat{\mathbf{G} \mathbf{y}} - \mathbf{G} \mathbf{y}) = 0 \end{aligned}$$

# Well balanced direction-wise<sup>10</sup>

EC-GF-

dGSEM Ricchiuto

2D Gf-dG

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**Proposition.** (Discrete well balanced along x- and y-) The line Gf-dGSEM with consistent numerical global flux is discretely well balanced along the x- and y- directions, in the sense that (equivalently)

- it preserves exactly 1d discrete equilibrium  $U^*$  associated to the quadrature defining Gx (or Gy)
- It has a super-convergent h<sup>2p</sup> behaviour wrt exact smooth 1d steady states in the x or y direction

<sup>10</sup>cf. e.g. (Michel-Dansac et al. Computers & Fluids 230, 2021) for similar result

# WBRes 2

# Numerical examples in 2d

#### Moving equilibria: sub-critical flow

or

 $qx = hu = q_0$ qy = hv = 0

 $E = q(h+b) + k = E_0$ 

#### and





#### I-BLaws SWEs I-WB GF-dSEM dGSEM GFlux WBRes 1 2D GFdG 2D GFdG 2D GFdG UBRes 2 EC-GF-dG UBRes 3 EC-GF-dG

# Ínría-

# Moving equilibria: sub-critical flow

 $egin{array}{ll} qx=&hu=q_0\ qy=&hv=0 \end{array}$ 

#### and

 $E = g(h+b) + k = E_0$ 

#### or

$$K = hu^2 + p + \int_x gh\partial_x b = F_0$$



# Numerical examples in 2d

#### EC-GFdGSEM

#### Ricchiuto

I-BLaws SWEs I-WB GF-dGSEM dGSEM dGSEM WBRes 2 D GF-dG WBRes 2 EC-GF-dG I-EC EC-GF-dG WBRes 3 F-C

# Moving equilibria: sub-critical flow

qx	=hu=q	0
qy	=hv=0	

#### and

 $E=g(h+b)+k=E_0$ 

#### or

$$K = hu^2 + p + \int_x gh\partial_x b = F_0$$



# Numerical examples in 2d

# Small perturbation: global flux vs nonWB





#### EC-GFdGSEM

#### Ricchiuto

WBRes 2

# Steady vortex with bathymetry and Coriolis forces Modification of the vortex used e.g. $in^{11}$



Ínría-

<sup>11</sup>Audusse et al. J.Comput.Phys. 228, 2009 - Chertock et al, Numerische Mathematik 128, 2018

#### EC-GFdGSEM

#### Ricchiuto

I-BLaws SWEs I-WB GF-dGSEM dGSEM dGSEM 2D GF-dG 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG WBRes 3

# Steady vortex with bathymetry and Coriolis forces Modification of the vortex used e.g. in $^{11}\,$





<sup>11</sup>Audusse et al. J.Comput.Phys. 228, 2009 - Chertock et al, Numerische Mathematik 128, 2018



#### Ricchiuto

Intro I-BLaws SWEs SWEs I-WB GF-dGSEM dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG I-EC EC-GF-dG WBRes 3

# Steady vortex with bathymetry and Coriolis forces

Modification of the vortex used e.g.  $\ensuremath{\mathsf{in}^{11}}$ 







<sup>11</sup>Audusse et al. J.Comput.Phys. 228, 2009 - Chertock et al, Numerische Mathematik 128, 2018
### Ricchiuto

### I-BLaws J-BLaws SWEs I-WB GF-dGSEI dGSEM GFlux WBRes 1 WB2d 2D Gf-dG WBRes 2 EC-Gf-dG I-EC EC-Gf-dG WBRes 3

# **2D geostropyc adjustiment**<sup>12</sup>

Shallow water + Coriolis, non-symmetric initial free surface

1.5 1.4 1.3 12 0.9 10 -10 -10 Initial



<sup>12</sup>Kuo & Polvani *Phys.Fl.* 12, 2000 - Castro et al *SISC* 31, 2008

### Ricchiuto

### I-BLaws I-BLaws SWEs I-WB GF-dGSEI dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG I-EC EC-GF-dG WBRes 3

# **2D geostropyc adjustiment**<sup>12</sup>

Shallow water + Coriolis, non-symmetric initial free surface

1.5 1.4 1.3 1.2 0.9 10 -10 -10 Gf-dGSEM(P2), T=4



<sup>12</sup>Kuo & Polvani *Phys.Fl.* 12, 2000 - Castro et al *SISC* 31, 2008

# Numerical examples in 2d



### Ricchiuto

### I-BLaws I-BLaws SWEs I-WB GFI-dGSEI dGSEM GFIux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG U-EC EC-GF-dG WBRes 3

# **2D geostropyc adjustiment**<sup>12</sup>

Shallow water + Coriolis, non-symmetric initial free surface

14 1.2 10 -5 -10 -10 Gf-dGSEM(P2), T=8





<sup>12</sup>Kuo & Polvani *Phys.Fl.* 12, 2000 - Castro et al *SISC* 31, 2008

# Numerical examples in 2d

### Ricchiuto

### I-BLaws SWEs I-WB GF-dGSEM dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG I-EC EC-GF-dG WBRes 3

# **2D** geostropyc adjustiment<sup>12</sup>

# Shallow water + Coriolis, non-symmetric initial free surface







<sup>12</sup>Kuo & Polvani *Phys.Fl.* 12, 2000 - Castro et al *SISC* 31, 2008

### Ricchiuto

### I-BLaws J-BLaws SWEs I-WB GF-dGSEM dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG I-EC EC-GF-dG WBRes 3

# **2D geostropyc adjustiment**<sup>12</sup>

Shallow water + Coriolis, non-symmetric initial free surface

1.5 1.4 12 0.9 -5 -10 -10 Gf-dGSEM(P2), T=20

t=0t=4t=8t = 14t = 20



<sup>12</sup>Kuo & Polvani *Phys.Fl.* 12, 2000 - Castro et al *SISC* 31, 2008

# Numerical examples in 2d

### Ricchiuto

### Intro

SWI

I-WB

Gf-dGS

dGSEN

VVDRes

WB2d

2D Gf-d

EC-Gf-dG

I-EC EC-Gf-d

WBRes 3

End



# Entropy conservation and global fluxes

### Ricchiuto

### Intro

SWEs I-WB Gf-dGSEN

- ICCEN
- CElux
- WBRes 1
- 2D Gf
- WBRe
- EC-Gf-d0
- WBRes
- End

# Entropy conservation and global fluxes

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x)),$$
 (1)

- Possible genelization of the notion of consistency wrt constants (in space) :
  - 1 Steady invariants
  - 2 Steady integral relations
  - 3 Global fluxes
  - 4 other declinations (continuous or dicrete level)...
- Well balanced scheme: discrete approximation embedding one (or more) of these notions

### Remark: consistency and entropy conservation

All of the above relate to the main PDE.

Exact consistency with constant entropy flux, *viz entropy conservation* comes as an extra contraint. A well balanced approach may or may not satisfy this constraint.



# Entropy conservation and global fluxes

# Shallow water equations (no friction).

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + p(h) \\ huv \end{pmatrix} = -h \begin{pmatrix} 0 \\ \partial_x \varphi + \omega v \\ -\omega u \end{pmatrix},$$
(6)

# Entropy conservation.

where

I-EC EC-Gf-dG WBRes 3

EC-GF-

dGSEM Ricchiuto

End

$$\partial_t \eta + \partial_x F_\eta = 0$$

$$\eta = p(h) + hk + h\varphi$$
,  $F_{\eta} = hu (g\zeta + k) = hu E$  (14)



### Ricchiuto

I-BLaws SWEs I-WB GF-GGSEM GFbax WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG WBRes 3

# Entropy conservation and global fluxes

# Shallow water equations (no friction).

$$\partial_t \left(egin{array}{c} h\ hu\ hv\end{array}
ight) + \partial_x \left(egin{array}{c} hu\ hu^2 + p(h) + r\ huv + r\omega\end{array}
ight) = 0$$

### **Conservation and consistency/steady states.** Analytical steady state

Global flux consistent steady state :

 $hu=q_0 \qquad \qquad hu=q_0$ 

 $F_{\eta} = q_0 E_0 \Rightarrow E = E_0$   $K = hu^2 + p + r = K_0$ 

 $V_{\omega}=huv+r_{\omega}=V_0$ 

Mismatch between entropy consistent fluxes and global-flux consistency



# Entropy conservation and global fluxes

### EC-GFdGSEM

### Ricchiuto

### I-BLaws SWEs I-WB GF-dGSEM dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG FEC EC-GF-dG WBRes 3

# Conservation and consistency/steady states.

Initial data corresponding to  $U^*$ /global flux consistency



Mismatch between entropy consistent fluxes and global-flux consistency



### Ricchiuto

### I-BLaws SWEs I-WB GF-dGSEM dGSEM dGSEM 2D GF-dG WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG FEC EC-GF-dG WBRes 3

# Entropy conservation and global fluxes

$$\partial_t \eta + \partial_x F_\eta = 0$$

# Conservation and consistency/steady states.

- 1 How to write a scheme with some control on  $\partial_t \eta$  for Gf-dGSEM
- 2 How to concile exactness with constant G and  $\partial_t \eta = 0$  (in some sense)



### EC-GFdGSEM

### Ricchiuto

I-BLaws SWEs I-WB GF-dGSEI dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG WBRes 3

### Entropy conservative dGSEM

- Gassner *SISC* 35, 2013
- Gassner et al Appl.Math.Comp. 272, 2016
- Chen J.Comput.Phys 362, 2017
- Wen et al *J.Sci.Comp.* 83, 2020
- Chen & Shu CSIAM Trans. Appl. Math. 1, 2020
- Renac J.Comput.Phys 382, 2019
- and many others



### Ricchiuto

### I-BLaws SWEs I-WB GF-dGSEN dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG EC-GF-dG EC-GF-dG

End

# Entropy conservative dGSEM

# Numerics: entropy conservation 2

$$\begin{split} \int_{K} \phi_i \partial_t h + \int_{K} \phi_i \partial_x q_{\mathbf{h}} + [\phi_i (\hat{q}_{\mathbf{h}}^* - q_{\mathbf{h}})]_{\xi=0} + [\phi_i (\hat{q}_{\mathbf{h}}^* - q_{\mathbf{h}})]_{\xi=1} &= 0 \\ \int_{K} \frac{\phi_i}{2} (\partial_t q_{\mathbf{h}} + h_{\mathbf{h}} \partial_t u_{\mathbf{h}}) + \int_{K} \frac{\phi_i}{2} (\partial_x (hu^2)_{\mathbf{h}} + q_{\mathbf{h}} \partial_x u_{\mathbf{h}}) \\ &+ \int_{K} \phi_i g h_{\mathbf{h}} \partial_x \zeta_{\mathbf{h}} + [\phi_i (\hat{f}_{\mathbf{h}}^* - f_{\mathbf{h}})]_{\xi=0} + [\phi_i (\hat{f}_{\mathbf{h}}^* - f_{\mathbf{h}})]_{\xi=1} = 0 \end{split}$$

# Main ingredients

- SBP property to work (indifferently) with the strong/weak form of the prob.
- 2 Skew-symmetric split form to enforce kinetic energy conservation
- 3 Entropy conservative fluxes  $\hat{F}^* = (q^*, f^*)$  to guarantee global entropy conservation:

$$[\![W]\!]^T \hat{F}^* = [\![\psi]\!] = [\![up(h)]\!]$$



### Ricchiuto

### I-BLaws SWEs I-WB GF-dGSEM dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG F-EC EC-GF-dG WBRes 3

# Entropy conservative dGSEM

$$\begin{split} \int\limits_{K} \phi_i \partial_t h + \int\limits_{K} \phi_i \underline{\partial_x q_{\rm h}} + [\phi_i (\underline{\hat{q}_{\rm h}^* - q_{\rm h}})]_{\xi=0} + [\phi_i (\underline{\hat{q}_{\rm h}^* - q_{\rm h}})]_{\xi=1} = 0 \\ \int\limits_{K} \frac{\phi_i}{2} (\partial_t q_{\rm h} + h_{\rm h} \partial_t u_{\rm h}) + \int\limits_{K} \frac{\phi_i}{2} (\partial_x (hu^2)_{\rm h} + q_{\rm h} \partial_x u_{\rm h}) \\ + \int\limits_{K} \phi_i g h_{\rm h} \partial_x \zeta_{\rm h} + [\phi_i (\hat{f}_{\rm h}^* - f_{\rm h})]_{\xi=0} + [\phi_i (\hat{f}_{\rm h}^* - f_{\rm h})]_{\xi=1} = 0 \end{split}$$

Numerics: entropy conservation 2

# Main ingredients

# ${\rm I\!I}$ SBP property to work (indifferently) with the strong/weak form of the prob.

- 2 Skew-symmetric split form to enforce kinetic energy conservation
- ${\mathfrak F}$  Entropy conservative fluxes  $\hat{F}^*=(q^*,f^*)$  to guarantee global entropy conservation:

$$\llbracket W \rrbracket^T \hat{F}^* = \llbracket \psi \rrbracket = \llbracket up(h) \rrbracket$$



### Ricchiuto

# I-BLaws SWEs I-WB GF-dGSEN dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG VBRes 3

# Entropy conservative dGSEM

$$\begin{split} \int\limits_{K} \phi_i \partial_t h + \int\limits_{K} \phi_i \partial_x q_{\mathbf{h}} + [\phi_i(\hat{q}_{\mathbf{h}}^* - q_{\mathbf{h}})]_{\xi=0} + [\phi_i(\hat{q}_{\mathbf{h}}^* - q_{\mathbf{h}})]_{\xi=1} &= 0\\ \int\limits_{K} \frac{\phi_i}{2} (\partial_t q_{\mathbf{h}} + h_{\mathbf{h}} \partial_t u_{\mathbf{h}}) + \int\limits_{K} \frac{\phi_i}{2} (\partial_x (hu^2)_{\mathbf{h}} + q_{\mathbf{h}} \partial_x u_{\mathbf{h}}) \\ &+ \int\limits_{K} \phi_i g h_{\mathbf{h}} \partial_x \zeta_{\mathbf{h}} + [\phi_i(\hat{f}_{\mathbf{h}}^* - f_{\mathbf{h}})]_{\xi=0} + [\phi_i(\hat{f}_{\mathbf{h}}^* - f_{\mathbf{h}})]_{\xi=1} = 0 \end{split}$$

Numerics: entropy conservation 2

### Main ingredients

- ${\rm 1}\!{\rm I}$  SBP property to work (indifferently) with the strong/weak form of the prob.
- 2 Skew-symmetric split form to enforce kinetic energy conservation
- ${\mathfrak F}$  Entropy conservative fluxes  $\hat{F}^*=(q^*,f^*)$  to guarantee global entropy conservation:

$$\llbracket W \rrbracket^T \hat{F}^* = \llbracket \psi \rrbracket = \llbracket up(h) \rrbracket$$



### Ricchiuto

# I-BLaws SWEs I-WB GF-dGSEN dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG VBRes 3

# Entropy conservative dGSEM

$$\begin{split} \int\limits_{K} \phi_i \partial_t h + \int\limits_{K} \phi_i \partial_x q_{\rm h} + [\phi_i (\underline{\hat{q}_{\rm h}^*} - q_{\rm h})]_{\xi=0} + [\phi_i (\underline{\hat{q}_{\rm h}^*} - q_{\rm h})]_{\xi=1} &= 0\\ \int\limits_{K} \frac{\phi_i}{2} (\partial_t q_{\rm h} + h_{\rm h} \partial_t u_{\rm h}) + \int\limits_{K} \frac{\phi_i}{2} (\partial_x (hu^2)_{\rm h} + q_{\rm h} \partial_x u_{\rm h}) \\ &+ \int\limits_{K} \phi_i g h_{\rm h} \partial_x \zeta_{\rm h} + [\phi_i (\underline{\hat{f}_{\rm h}^*} - f_{\rm h})]_{\xi=0} + [\phi_i (\underline{\hat{f}_{\rm h}^*} - f_{\rm h})]_{\xi=1} = 0 \end{split}$$

Numerics: entropy conservation 2

### Main ingredients

- **1** SBP property to work (indifferently) with the strong/weak form of the prob.
- 2 Skew-symmetric split form to enforce kinetic energy conservation
- 3 Entropy conservative fluxes  $\hat{F}^* = (q^*, f^*)$  to guarantee global entropy conservation:

$$\llbracket W \rrbracket^T \hat{F}^* = \llbracket \psi \rrbracket = \llbracket up(h) \rrbracket$$





### Ricchiuto

I-BLaws SWEs I-WB GF-dGSEN dGSEM GFlux WBRes 1 WB2d UWBRes 2 EC-GF-dG WBRes 3

# Entropy conservative dGSEM

- The PDE plays an important role in the process
- Not clear how to ensure properties 2. and 3. in the global flux formulation

$$egin{aligned} &\partial_t h + \partial_x q = 0 \ &\partial_t q + \partial_x (hu^2 + p + r) = 0 \end{aligned}$$

Main ingredients

- 1 SBP property to work (indifferently) with the strong/weak form of the prob.
- 2 Skew-symmetric split form to enforce kinetic energy conservation
- 3 Entropy conservative fluxes  $\hat{F}^* = (q^*, f^*)$  to guarantee global entropy conservation:

$$\llbracket W \rrbracket^T \hat{F}^* = \llbracket \psi \rrbracket = \llbracket up(h) \rrbracket$$

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# Numerics: entropy conservation 2

### Ricchiuto

### I-BLaws SWEs I-WB Gf-dGS

GFlux

WBRes 1

WB2d

WBRes 2

EC-Gf-dG I-EC EC-Gf-dG WBRes 3

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End

# Entropy correction technique<sup>13</sup>

<sup>13</sup>Abgrall J.Comput.Phys. 372, 2018 - Abgrall et al, J.Comput.Phys. 453 2022

# Numerics: entropy conservation 3

### Ricchiuto

### Intro I-BLav

I-WB

Gf-dGSE

GElux

WBRes 1

WB2d

2D Gf-d0

EC-Gf-dO I-EC

EC-Gf-dG WBRes 3

End

# Numerics: entropy conservation 3

# **Entropy correction technique**<sup>13</sup>

$$w_i rac{dU_i}{dt} + \Phi_i + \Psi^L_i + \Psi^R_i = 0$$

<sup>&</sup>lt;sup>13</sup>Abgrall J.Comput.Phys. 372, 2018 - Abgrall et al, J.Comput.Phys. 453 2022

# Entropy correction technique<sup>13</sup>

$$w_i rac{dU_i}{dt} + \Phi_i + \Psi_i^L + \Psi_i^R + lpha_K \int\limits_K \partial_x \phi_i A_0 \partial_x W_{
m h} = 0$$

- W are the entropy variables such that  $W^T\partial U=\partial\eta$
- $A_0 = \partial U / \partial W$  is the SPD entropy Hessian inverse



I-EC

EC-GF-

dGSEM Ricchiuto

<sup>&</sup>lt;sup>13</sup>Abgrall J.Comput.Phys. 372, 2018 - Abgrall et al, J.Comput.Phys. 453 2022

### EC-GFdGSEM Ricchiuto

I-BLaws SWEs I-WB GF-dGSEM dGSEM dGSEM gFlux WBRes 1 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG WBRes 3

# Entropy correction technique<sup>13</sup>

$$w_irac{dU_i}{dt}+\Phi_i+\Psi^L_i+\Psi^R_i+lpha_K\int\limits_K\partial_x\phi_iA_0\partial_xW_{
m h}=0$$

Multiply by  $W_i^T$ , and add up over  $i \in K$  and over all Ks:

$$\sum_{K}\sum_{i\in K}w_{i}W_{i}^{T}\frac{dU_{i}}{dt}+\sum_{K}\sum_{i\in K}W_{i}^{T}(\Phi_{i}+\Psi_{i}^{L}+\Psi_{i}^{R})+\sum_{K}\mathcal{D}_{K}=0$$



<sup>&</sup>lt;sup>13</sup>Abgrall J.Comput.Phys. 372, 2018 - Abgrall et al, J.Comput.Phys. 453 2022

### Ricchiuto

EC-GF-

dGSEM

I-BLaws SWEs I-WB GF-dGSEM dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG WBRes 3

# Entropy correction technique<sup>13</sup>

$$w_i rac{dU_i}{dt} + \Phi_i + \Psi^L_i + \Psi^R_i + lpha_K \int\limits_K \partial_x \phi_i A_0 \partial_x W_{
m h} = 0$$

Multiply by  $W_i^T$ , and add up over  $i \in K$  and over all Ks:

$$\sum_{K} \underbrace{\sum_{i \in K} w_i W_i^T \frac{dU_i}{dt}}_{= \int_K \partial_t \eta_h} + \sum_{K} \underbrace{\sum_{i \in K} W_i^T (\Phi_i + \Psi_i^L + \Psi_i^R)}_{:= \Phi_\eta^K} + \sum_{K} \mathcal{D}_K = 0$$

<sup>13</sup>Abgrall J.Comput.Phys. 372, 2018 - Abgrall et al, J.Comput.Phys. 453 2022



# Entropy correction technique<sup>13</sup>

$$w_i rac{dU_i}{dt} + \Phi_i + \Psi^L_i + \Psi^R_i + lpha_K \int\limits_K \partial_x \phi_i A_0 \partial_x W_{
m h} = 0$$

Multiply by  $W_i^T$ , and add up over  $i \in K$  and over all Ks:

$$egin{aligned} &\int_{\Omega}\partial_t\eta_{
m h} + \sum_K (\Phi^K_\eta + \mathcal{D}_K) = 0 \ &\mathcal{D}_K = &lpha_K \|\partial_x W\|^2_{L^2_{A,0}(K)} \end{aligned}$$

EC-GFdGSEM

### Ricchiuto

SWEs I-WB GF-dGSEM dGSEM dGSEM gFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG FC EC-GF-dG WBRes 3

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<sup>&</sup>lt;sup>13</sup>Abgrall J.Comput.Phys. 372, 2018 - Abgrall et al, J.Comput.Phys. 453 2022

# Entropy correction technique<sup>13</sup>

$$w_i rac{dU_i}{dt} + \Phi_i + \Psi^L_i + \Psi^R_i + lpha_K \int\limits_K \partial_x \phi_i A_0 \partial_x W_{
m h} = 0$$

Multiply by  $W_i^T$ , and add up over  $i \in K$  and over all Ks:

$$\int_\Omega \partial_t \eta_{
m h} + \sum_K (\Phi^K_\eta + \mathcal{D}_K) = 0$$

We now set

$$\Phi_{\eta}^{K} + \mathcal{D}_{K} = \Psi_{\eta}^{K} = \oint_{\partial_{K}} \hat{F}_{\eta}(U_{\rm h}) \cdot \hat{n} \quad \Rightarrow \quad \alpha_{K} = \frac{\Psi_{\eta}^{K} - \Phi_{\eta}^{K}}{\|\partial_{x}W\|_{L^{2}_{A_{0}}(K)}^{2}}$$

<sup>13</sup>Abgrall J.Comput.Phys. 372, 2018 - Abgrall et al, J.Comput.Phys. 453 2022



### EC-GFdGSEM

### Ricchiuto

I-WB Gf-dGSEM dGSEM GFlux WBRes 1 WB2d 2D Gf-dG WBRes 2 EC-Gf-dG I-EC

# Entropy correction technique<sup>13</sup>

$$w_i rac{dU_i}{dt} + \Phi_i + \Psi_i^L + \Psi_i^R + lpha_K \int\limits_K \partial_x \phi_i A_0 \partial_x W_{
m h} = 0$$

Multiply by  $W_i^T$ , and add up over  $i \in K$  and over all Ks:

$$\int_\Omega \partial_t \eta_{
m h} + \sum_K (\Phi^K_\eta + \mathcal{D}_K) = 0$$

In 1D

$$\Phi_{\eta}^{K} + \mathcal{D}_{K} = \Psi_{\eta}^{K} = \hat{F}_{\eta} \big|_{\xi=1} - \hat{F}_{\eta} \big|_{\xi=0} \quad \Rightarrow \quad \alpha_{K} = \frac{\Psi_{\eta}^{K} - \Phi_{\eta}^{K}}{\|\partial_{x}W\|_{L^{2}_{A_{0}}(K)}^{2}}$$

<sup>13</sup>Abgrall J.Comput.Phys. 372, 2018 - Abgrall et al, J.Comput.Phys. 453 2022



### EC-GFdGSEM

### Ricchiuto

SWES I-WB Gf-dGSEM dGSEM GFlux WBRes 1 WB2d 2D Gf-dG WBRes 2 EC-Gf-dG I-EC

# Entropy correction technique<sup>13</sup>

$$w_i rac{dU_i}{dt} + \Phi_i + \Psi_i^L + \Psi_i^R + lpha_K \int\limits_K \partial_x \phi_i A_0 \partial_x W_{
m h} = 0$$

**Proposition** (Entropy conservative correction). Let  $\hat{F}_{\eta}$  be a consistent entropy flux, and

$$\alpha_K := \frac{\Psi_\eta^K - \Phi_\eta^K}{\|\partial_x W\|_{L^2_{A_0}(K)}^2} \tag{14}$$

The resulting Gf-dGSEM semi-discretization

1 verifies cell (and global) entropy conservation (time continuous)

2 verifies a  $\mathcal{E} = \mathcal{O}(h^{p+1})$  consistency estimate

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### EC-GFdGSEM

### Ricchiuto

I-WB Gf-dGSEM dGSEM GFlux WBRes 1 WB2d 2D Gf-dG WBRes 2 EC-Gf-dG EC-Gf-dG EC-Gf-dG WBRes 3

<sup>&</sup>lt;sup>13</sup>Abgrall J.Comput.Phys. 372, 2018 - Abgrall et al, J.Comput.Phys. 453 2022

(14)

# Entropy correction technique<sup>13</sup>

$$w_i rac{dU_i}{dt} + \Phi_i + \Psi_i^L + \Psi_i^R + lpha_K \int\limits_K \partial_x \phi_i A_0 \partial_x W_{
m h} = 0$$

**Proposition** (Entropy conservative correction). Let  $\hat{F}_{\eta}$  be a consistent entropy flux, and

$$\alpha_K := \frac{\Psi_\eta^K - \Phi_\eta^K}{\|\partial_x W\|_{L^2_{A_0}(K)}^2}$$

The resulting Gf-dGSEM semi-discretization

- 1 verifies cell (and global) entropy conservation (time continuous)
- 2 verifies a  $\mathcal{E}=\mathcal{O}(h^{p+1})$  consistency estimate

### EC-GFdGSEM

### Ricchiuto

SWEs I-WB GF-dGSEM dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG EC-GF-dG WBRes 3 End

<sup>&</sup>lt;sup>13</sup>Abgrall J.Comput.Phys. 372, 2018 - Abgrall et al, J.Comput.Phys. 453 2022

# Entropy conservation and global fluxes

# Shallow water equations (no friction).

$$\partial_t \left(egin{array}{c} h \ hu \ hv \end{array}
ight) + \partial_x \left(egin{array}{c} hu \ hu^2 + p(h) + r \ huv + r_\omega \end{array}
ight) = 0$$

# **Conservation and consistency/steady states.** Analytical steady state

Global flux consistent steady state :

- $hu = q_0$   $F_n = q_0 E_0 \Rightarrow E = E_0$   $K = hu^2 + p + r = K_0$ 
  - $V_{\omega}=huv+r_{\omega}=V_0$

Mismatch between entropy consistent fluxes and global-flux consistency

### EC-GFdGSEM Ricchiuto

### I-BLaws SWEs I-WB GF-IdGSEM GF-IdGSEM UBRes 1 2D GF-0 2D GF-0 WBRes 2 EC-GF-0 EC-GF-0 WBRes 3

End



# Entropy correction with global fluxes 1

# Alternative definitions of a numerical entropy flux

Solution 1: approximation consistent with analytical entropy flux, e.g.

$$\hat{F}_\eta = \lambda F_\eta(U^-;arphi^-) + (1-\lambda)F_\eta(U^+;arphi^+)$$

- globally entropy conservative (time continuous)
- exact for analytical steady data but not for constant global fluxes
- the correction term spoils the underlying consistency condtion



EC-Gf-dG

EC-GF-

dGSEM Ricchiuto

# Entropy conservation and global fluxes

### EC-GFdGSEM

### Ricchiuto

### I-BLaws SWEs J-WB Gf-dGSEM dGSEM dGSEM dGSEM 2D GF-dG WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG WBRes 3 EC-GF-dG

# Conservation and consistency/steady states.

Initial data corresponding to  $U^*$ /global flux consistency



Mismatch between entropy consistent fluxes and global-flux consistency



### Ricchiuto

I-BLaws SWEs I-WB GF-dSGEM dGSEM GFlux WBRes 1 2D GFdG 2D GFdG WBRes 2 EC-GF-dG EC-GF-dG WBRes 3 EC-GF-dG

# Entropy correction with global fluxes 2

# **Alternative definitions of a numerical entropy flux** Solution 2: a global flux-consistent approximation.

First set  $F^*_{\eta,0} = F_\eta(U_0)$  (left hand of the domain). Then  $orall \{K_j\}_{j\geq 1}$  do

1 Set 
$$(F_{\eta}^{*})_{0} = F_{\eta}^{*}(U^{-}) + \llbracket F_{\eta}(U_{\mathrm{h}}; \varphi) 
rbracket = F_{\eta}^{*}(x_{p})_{K_{j-1}} + \llbracket F_{\eta}(U_{\mathrm{h}}; \varphi) 
rbracket$$

2 Compute: 
$$F^*_\eta(x)=(F^*_\eta)_0+\int_{x_0}^x W^T_{
m h}\partial_x G_h$$



### Ricchiuto

### I-BLaws SWEs I-WB GF-dGSEM dGSEM dGSEM 2D GF-dG 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG WBRes 2 EC-GF-dG

# Entropy correction with global fluxes 2

# Alternative definitions of a numerical entropy flux Solution 2: a global flux-consistent approximation.

First set  $F_{\eta,0}^* = F_{\eta}(U_0)$  (left hand of the domain). Then  $orall \{K_j\}_{j\geq 1}$  do

1 Set 
$$(F_{\eta}^{*})_{0} = F_{\eta}^{*}(U^{-}) + \llbracket F_{\eta}(U_{\mathrm{h}}; \varphi) 
rbracket = F_{\eta}^{*}(x_{p})_{K_{j-1}} + \llbracket F_{\eta}(U_{\mathrm{h}}; \varphi) 
rbracket$$

2 Compute: 
$$F^*_\eta(x) = (F^*_\eta)_0 + \int_{x_0}^x W^T_\mathrm{h} \partial_x G_h$$

If we set  $\hat{F}_\eta = \lambda (F^*_\eta)^- + (1-\lambda)(F^*_\eta)^+$ , the Gf-dGSEM obtained is

- globally entropy conservative (time continuous)
- compatible with constant global flux as long as  $\varphi$  is continuously approximated



# Well balanced and entropy conservative results 1

### EC-GFdGSEM Ricchiuto

# WBRes 3

### Conservation and consistency/steady states.

Initial data corresponding to  $U^*$ /global flux consistency



Mismatch between entropy consistent fluxes and global-flux consistency



# Well balanced and entropy conservative results 1

### Conservation and consistency/steady states.

Initial data corresponding to  $U^*$ /global flux consistency



We can choose the EC correction depending on the initial data



WBRes 3

EC-GF-

dGSEM Ricchiuto

# Well balanced and entropy conservative results 2

### EC-GFdGSEM

### Ricchiuto

I-BLaws SWEs I-WB GF-dGSEM dGSEM dGSEM dGSEM 2D GF-dG 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG WBRes 3

# <sup>3</sup> <sup>×10<sup>4</sup></sup> <sup>4</sup> <sup>WB-EC-ichame</sup> <sup>5</sup> <sup>1</sup> <sup>4</sup> <sup>WB-EC-ichame</sup> <sup>4</sup> <sup>WB-EC-ichame</sup> <sup>5</sup> <sup>10</sup> <sup>15</sup> <sup>20</sup> <sup>25</sup> <sup>3</sup> <sup>×10<sup>6</sup></sup>





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-2 <sup>L</sup>0

5

10

15

х

20

25

# Moving steady state: sub-critical

### Ricchiuto

I-BLaws SWEs I-WB GF-dGSEM dGSEM dGSEM dGSEM 2D GF-dG 2D GF-dG WBRes 2 EC-GF-dG I-EC EC-GF-dG

# Moving steady state: super-critical





# Well balanced and entropy conservative results 3

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# Summary

#### EC-GFdGSEM

#### Ricchiuto

I-BLaws SWEs I-WB GF-GGSEM dGSEM dGSEM 2D GF-L 2D GF-D 2D GF-D WBRes 2 EC-GF-dG EC-GF-dG WBRes 3

End



- Family of schemes discretely well balanced/super-convergent
- Agnostic of the particular equilibrium in 1D
- Relation between super-convergent behaviour and underlying ODE integrator
- Measurable net improvements for some 2D tests
- Correction allowing global (time continuous) entropy preservation compatible with both analytical or global flux initialization

# **Extensions future work**

#### EC-GFdGSEM

#### Ricchiuto

## I-BLaws SWEs I-WB GF-dGSEM GFlux WBRes 1 WB2d 2D GF-dG WBRes 2 EC-GF-dG EC-GF-dG WBRes 3

End

# Nonlinear formulations

Troubled cell indicator/P0 switch:



# Gf-WENO (with M. Ciallella & D. Torlo):





#### EC-GFdGSEM

#### Ricchiuto

# End

**Euler Equations with gravity** Isothermal eq. solution<sup>14</sup>





(b) Non-well-balanced scheme

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<sup>14</sup>See e.g. Chertock et a *J.Comput.Phys.* 358, 2018

**Extensions future work** 

# **Extensions future work**

#### EC-GFdGSEM

#### Ricchiuto

### I-BLaws SWEs I-WB GF-dGSEM dGSEM GFlux WBRes 1 WB2d 2D GF-dG VBRes 2 EC-GF-dG WBRes 3 EC-GF-dG

# **Ongoing/future work**

- Well balanced cGSEM (with R. Abgrall, L. Micalizzi, S. Michel, & D. Torlo)
- Fully discrete entropy conservative ADER + relaxation (with E. Gaburro, P. Öffner & D. Torlo)
- Entropy controlled with a-posteriori limiter (with E. Gaburro, P. Öffner & D. Torlo)
- Sources depending on time derivatives: dispersive PDEs (with W. Barsukow and D. Torlo)
- Genuinely 2D :  $G = \text{const} \neq \nabla \cdot G = 0 \dots$



EC-GFdGSEM

#### Ricchiuto

Intro

SWE

I-WB

Gf-dGSI

dGSEM

GFlux

WBRes

WB2d

2D Gf-d

EC-Gf-dC

I-EC EC-Gf-dG

WBRes 3

End

# .. Obrigado

