Global flux dG-SEM for systems of balance laws with a discretely well balanced entropy correction

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Acknowledgements

Joint work with

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Johannes Gutenberg-University, Mainz (Germany)

Credit to them for the good stuff in the talk
blame me for the rest

Setting: balance laws 1

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \nabla \cdot F(U) = S(U; \varphi(x)), \tag{1}$$

Typical examples

- Shallow water equations with topography/friction/Coriolis/etc
- Euler equations with gravity
- Shallow Water/Euler in pseudo-1D form (section variation)
- etc.

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A multi-D generalization is possible, and there will be 2D examples.

But for simplicity the discussion is done for the 1D case

Setting: balance laws 2

We seek solutions of the (hyperbolic) system of balance laws

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Property 1. Non-trivial steady states.

Setting: balance laws 2

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = 0,$$

Property 1. Non-trivial steady states.

For the homogeneous case:

- $U = U_0$ constant in space and time is an exact solution
- Fundamental consequence:
  consistency condition at the discrete level which is the exactness wrt constant U
- Polynomial approximation explicitly embed this condition in their construction

Setting: balance laws 2

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$$\partial_t U + \partial_x F(U) = S(U; \varphi(x)), \tag{1}$$

Property 1. Non-trivial steady states.

Balance law case:

- $U = U_0$ constant is rarely an exact solution
- Fundamental consequence:
  exactness wrt constant U is not an adequate *consistency condition* at the discrete level
- Using (only) this condition in the construction of discrete approximations may lead to large errors

Setting: balance laws 3

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x)), \tag{1}$$

Consistency conditions 1: (steady) invariant states

In some cases, one can establish other "simple" ***invariants*** (cf. later shallow water):

$$\partial_t u + \partial_x (u^2/2) + u \partial_x \varphi(x) = 0$$

A constant (in space and time) value $V =: u + \varphi(x) = V_0$ is a relevant consistency condition.

Indeed we can rewrite the PDE as

$$\partial_t u + u \partial_x V = 0 \quad \text{or} \quad \partial_t V + u \partial_x V = 0$$

Setting: balance laws 4

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Consistency conditions 2: (steady) integral relations

In other cases, invariants can emerge from *exact integral relations*:

$$\partial_t u + \partial_x (u^2/2) + \varphi(x) u = 0 \quad \Rightarrow \text{steady ODE: } \partial_x u + \varphi(x) = 0$$

A constant value $V =: u - u_0 + \int_{x_0}^{x} \varphi(s) ds$ is a relevant consistency condition.

Indeed, we can rewrite the PDE as

$$\partial_t u + u \partial_x V = 0 \quad \text{or} \quad \partial_t V + u \partial_x V = 0$$

Setting: balance laws 5

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x)), \tag{1}$$

Consistency conditions 3: global fluxes

More generally, we can consider the pseudo-conservative form of the balance law

$$\partial_t U + \partial_x F + \partial_x R = 0 \ , \tag{1}$$

having introduced the *source integral*

$$R(U, x) - R_0 := -\int_{x_0}^{x} S(U, \varphi) ds \tag{2}$$

A constant value in space of the *global flux* $G =: F + R$ is a relevant consistency condition.

The value of the global flux is only known a priori if the analytical form of a primitive of S is available.

Setting: balance laws 6

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x)), \tag{1}$$

Property 2. Entropy balance.

Setting: balance laws 6

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x)), \quad (1)$$

Property 2. Entropy balance.

System (1) is endowed with an auxiliary constraint

$$\partial_t \eta + \partial_x F_\eta(U) \leq S_\eta(U; \varphi(x)), \quad (4)$$

With:

$\eta = \eta(U)$ a mathematical entropy,
$F_\eta(U)$ the entropy flux,
$S_\eta(U; \varphi(x))$ a dissipation/production term.

Setting: balance laws 6

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For smooth solutions the inequality becomes an equality giving an auxiliary entropy balance law.

Continuous setting

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x)), \tag{1}$$

- Possible genelization of the notion of consistency wrt constants (in space) :
  1. Steady invariants
  2. Steady integral relations
  3. Global fluxes
  4. other declinations (continuous or dicrete level)...
- Well balanced scheme: discrete approximation embedding one (or more) of these notions

Remark: consistency and entropy conservation

All of the above relate to the main PDE.

Exact consistency with constant entropy flux, *viz entropy conservation* comes as an extra contraint. A well balanced approach may or may not satisfy this constraint.

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Example: shallow water equations

Example: shallow water equations 1

2D Cartesian shallow water equations.

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + p(h) \\ huv \end{pmatrix} + \partial_y \begin{pmatrix} hv \\ huv \\ hv^2 + p(h) \end{pmatrix} = -h \begin{pmatrix} 0 \\ \partial_x \varphi + c_f u + \omega v \\ \partial_y \varphi + c_f v - \omega u \end{pmatrix}, \qquad (5)$$

Notation.

h water depth

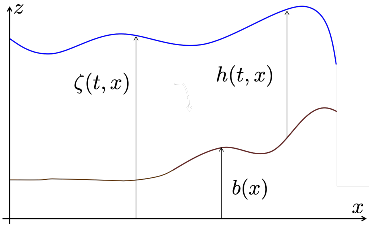
$\vec{v} = (u, v)$ horizontal velocity

$p = gh^2/2$ hydrostatic pressure (g gravity acceleration)

$\varphi = gb$ gravitational potential ($b(x, y)$ bottom topography)

$c_f = c_f(h, \vec{v})$ friction coefficient

ω Coriolis coefficient



Example: shallow water equations 2

1D rotating shallow water equations[1].

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + p(h) \\ huv \end{pmatrix} = -h \begin{pmatrix} 0 \\ \partial_x \varphi + c_f u + \omega v \\ c_f v - \omega u \end{pmatrix}, \quad (6)$$

Notation.

h water depth

$\vec{v} = (u, v)$ horizontal velocity

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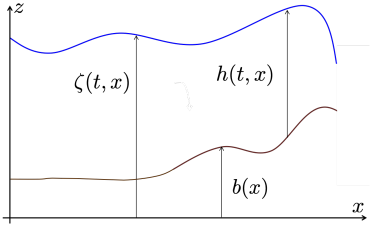


[1] Castro et al, SISC 31, 2008

Example: shallow water equations 3

1D rotating shallow water equations.

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + p(h) \\ huv \end{pmatrix} = -h \begin{pmatrix} 0 \\ \partial_x \varphi + c_f u + \omega v \\ c_f v - \omega u \end{pmatrix}, \tag{6}$$

Non-trivial steady states.

Example 1: constant energy moving equilibrium (no friction, no Coriolis).

$$hu = q_0\,, \quad v = 0\,, \quad g(h+b) + k = E_0$$

Example 2: lake at rest with Coriolis perturbation (no friction).

$$u = 0\,, \quad v = V(x)\,, \quad g(h+b) + \omega \int_x V = Z_0$$

Example 3: moving equilibrium with friction and slope variations[2]

$$hu = q_0\,, \quad v = 0\,, \quad g(h+b) + k + \int_{x_0}^{x} c_f u\, ds = E_0$$

[2] See e.g. (Michel-Dansac et al. *J.Comput.Phys.* 335, 2017) for examples

Example: shallow water equations 4

Shallow water equations.

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + p(h) \\ huv \end{pmatrix} = -h \begin{pmatrix} 0 \\ \partial_x \varphi + c_f u + \omega v \\ c_f v - \omega u \end{pmatrix}, \quad (6)$$

Entropy balance.

$$\partial_t \eta + \partial_x F_\eta = -\mathcal{D}_f \quad (7)$$

with *total entropy/energy* η and entropy flux F_η:

$$\eta = p(h) + hk + h\varphi = p(h) + hk + ghb\ , \quad F_\eta = hu\,(gh + k + \varphi) = hu\,(g\zeta + k) \quad (8)$$

In absence of friction, and for smooth solution total entropy/energy is conserved

This talk

Focus of this work

- Discretely well balanced method agnostic of the equilibrium
- Use idea of global flux approximation within dGSEM formulation
- Characterize the notion of discrete equilibrium associated to this formulation
- Investigate the compatibility of global flux consistency with entropy/energy conservation

This talk

Numerics: WB reconstruction, global fluxes, modified Riemann problem

Numerics: WB reconstruction, global fluxes, modified Riemann problem

We focus on techniques which can be related to some "special quadrature" of the source.

Other points of view are possible:

- hydrostatic reconstructions and generalizations
(Audusse et al *SISC* 25 2004; Castro et al., *Math.Mod.Meth.Appl.Sci.* 5, 2007)
- Reference solutions
(Klingenbert et al *SISC* 41, 2019; Castro & Pares *J.Sci.Comp.* 82, 2020)

There are of course relations among all these approaches...

Numerics: WB reconstruction, global fluxes, modified Riemann problem 1

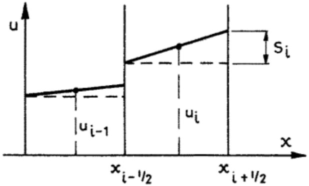
Once upon a time ...

P.L. Roe. Upwind differencing schemes for hyperbolic conservation laws with source terms. In Claude Carasso, Denis Serre, and Pierre-Arnaud Raviart, editors, *Nonlinear Hyperbolic Problems*, pages 41–51, Berlin, Heidelberg, 1987. Springer Berlin Heidelberg.

Numerics: WB reconstruction, global fluxes, modified Riemann problem 1

Once upon a time ...

$$\partial_t u + a\partial_x u = q\,, \quad a > 0$$



- Integrate in space and time
- upwind fluxes
- integrate the source term along characteristics

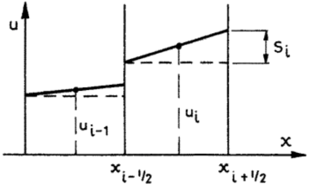
$$u_i^{n+1} = u_i^n - \nu(u_i^n - u_{i-1}^n) + \tfrac{1}{2}\nu(1-\nu)S_{i-1} - \tfrac{1}{2}\nu(1-\nu)S_i + \left[(1-\tfrac{1}{2}\nu)q_i + \tfrac{1}{2}\nu q_{i-1}\right]\Delta t$$

$$\nu = a\Delta t/\Delta x$$

Numerics: WB reconstruction, global fluxes, modified Riemann problem 1

Once upon a time ...

$$\partial_t u + a\partial_x u = q\,, \quad a > 0$$



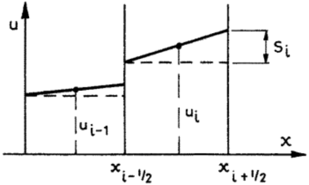
Choice of the slope...

problem (non-linear systems) several authors [6,7,8] have felt the attraction of considering data which is in piecewise equilibrium. That is, the data is projected into a representation such that the steady flow equations are satisfied within each cell. In our simple model equation, that means choosing

$$S_i = \frac{q_i \Delta x}{a} = \frac{q_i \Delta t}{\nu}$$

Numerics: WB reconstruction, global fluxes, modified Riemann problem 1

Once upon a time ...



$$\partial_t u + a\partial_x u = q\,, \quad a > 0$$

Upwind difference/source splitting:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\phi_{i-\frac{1}{2}}$$

we can measure the extent to which they are out of equilibrium (with each other now, now internally) by the quantity

$$\phi_{i-\frac{1}{2}} = a(u_i - u_{i-1}) - \tfrac{1}{2}\Delta x(q_{i-1} + q_i)$$

Numerics: WB reconstruction, global fluxes, modified Riemann problem 1

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U;\varphi(x)), \tag{1}$$

Well balanced upwind (difference splitting) schemes

$$\Delta x \frac{dU_i}{dt} + (A^- A^{-1})^{\text{Roe}}_{i+1/2}(F_{i+1} - F_i - \Delta x\, S_{i+1/2}) + (A^+ A^{-1})^{\text{Roe}}_{i-1/2}(F_i - F_{i-1} - \Delta x\, S_{i-1/2}) = 0$$

- Bermudez & Vazquez, *Computers & Fluids* 8, 1994; Vazquez-Cendon, *J.Comput.Phys.* 148, 1999
- Parés & Castro, *M²AN* 38, 2004; Parés, *SINUM* 44, 2006
- Castro et al., *Math.Mod.Meth.Appl.Sci. 5, 2007*

Numerics: WB reconstruction, global fluxes, modified Riemann problem 2

General formalism: residual distribution and global fluxes

$$\Delta x \frac{dU_i}{dt} + (A^- A^{-1})^{\text{Roe}}_{i+1/2}(F_{i+1} - F_i - \Delta x\, S_{i+1/2}) + (A^+ A^{-1})^{\text{Roe}}_{i-1/2}(F_i - F_{i-1} - \Delta x\, S_{i-1/2}) = 0$$

Numerics: WB reconstruction, global fluxes, modified Riemann problem 2

General formalism: residual distribution and global fluxes

$$\Delta x \frac{dU_i}{dt} + (A^- A^{-1})^{\text{Roe}}_{i+1/2} \overbrace{(F_{i+1} - F_i - \Delta x\, S_{i+1/2})}^{\Phi^{i+1/2}}$$
$$+ (A^+ A^{-1})^{\text{Roe}}_{i-1/2} \underbrace{(F_i - F_{i-1} - \Delta x\, S_{i-1/2})}_{\Phi^{i-1/2}} = 0$$

Numerics: WB reconstruction, global fluxes, modified Riemann problem 2

General formalism: residual distribution and global fluxes

$$\Delta x \frac{dU_i}{dt} + B_i^{i+1/2}\Phi^{i+1/2} + B_i^{i-1/2}\Phi^{i-1/2} = 0$$

$$\Phi^{i-1/2} := \int_{x_{i-1}}^{x_i} (\partial_x F - S)$$

$$B_i^{i+1/2} + B_{i+1}^{i+1/2} = \mathrm{Id}$$

- Abgrall & Ricchiuto, `arXiv:2109.08491`, 2021; Abgrall & Ricchiuto, *ECM* 2nd Edition, 2017
- Ricchiuto, *J.Comput.Phys.* 280, 2015
- Chou & Shu, *J.Comput.Phys.* 214, 2006; J. Lin et al., *J.Sci.Comp.* 79, 2019

Numerics: WB reconstruction, global fluxes, modified Riemann problem 2

General formalism: residual distribution and global fluxes

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$$B_i^{i+1/2} + B_{i+1}^{i+1/2} = \mathrm{Id}$$

Steady state/well balanced conditions

1. $B_i^{i\pm 1/2}$ uniformly bounded
2. for data at equilibrium $\Phi^{i\pm 1/2} = 0$

Numerics: WB reconstruction, global fluxes, modified Riemann problem 2

General formalism: residual distribution and global fluxes

Trick. Set now

$$G_0 = F_0 \ , \quad G_i = G_{i-1} + \int_{x_{i-1}}^{x_i} (\partial_x F - S)$$

The flux splitting/RD prototype can equivalently be written in global flux form

$$\Delta x \frac{dU_i}{dt} = - B_i^{i+1/2}(G_{i+1} - G_i) - B_i^{i-1/2}(G_i - G_{i-1})$$

$$= - (\hat{G}_{i+1/2} - \hat{G}_{i-1/2})$$

with

$$\hat{G}_{i+1/2} = B_i^{i+1/2}(G_{i+1} - G_i) - G_i = G_{i+1} - B_{i+1}^{i+1/2}(G_{i+1} - G_i)$$

Numerics: WB reconstruction, global fluxes, modified Riemann problem 2

General formalism: residual distribution and global fluxes

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$$= - (\hat{G}_{i+1/2} - \hat{G}_{i-1/2})$$

Consistency/well balanced conditions

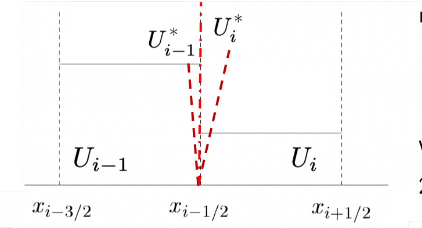
1. $G_i = G_0$ forall i, for data at equilibrium
2. $\hat{G}_{i+1/2} = G_0$ forall i, for data at equilibrium

Numerics: WB reconstruction, global fluxes, modified Riemann problem 3

Global flux consistency via full well balanced Riemann solver

Numerics: WB reconstruction, global fluxes, modified Riemann problem 3

Global flux consistency via full well balanced Riemann solver



Two intermediate states RP

For the shallow water equations, consider the Godunov method with numerical flux

$$\hat{F} = \frac{F_L + F_R}{2} - \frac{1}{2}\sum_k |\lambda_k| [\![U]\!]_k$$

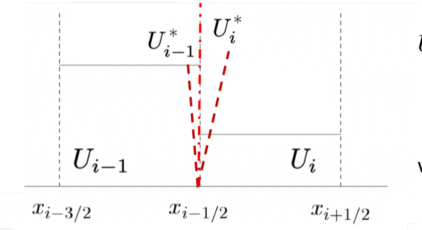
with λ_k the waves of the approximate Riemann solver,

2 physical waves plus a 0-wave :

$$\lambda_0 = u - \sqrt{gh}\,, \quad \lambda_1 = 0\,, \quad \lambda_2 = u + \sqrt{gh}$$

Numerics: WB reconstruction, global fluxes, modified Riemann problem 3

Global flux consistency via full well balanced Riemann solver



Two intermediate states RP

The space-time discrete FV prototype

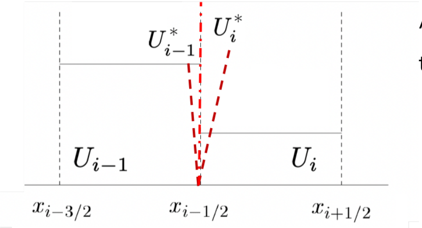
$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}(\hat{F}_{i+1/2} - \hat{F}_{i-1/2}) + \frac{\Delta t}{2}(\hat{S}_{i+1/2} + \hat{S}_{i+1/2})$$
$$= U_i^n - \frac{\Delta t}{\Delta x}(\Phi_i^{i+1/2} + \Phi_i^{i-1/2})$$

where simple manipulations show

$$\Phi_i^{i-1/2} = \frac{1}{2}\left(F_i - F_{i-1} - \sum_k |\lambda_k| [\![U]\!]_k - \Delta x \hat{S}_{i-1/2}\right)$$

Numerics: WB reconstruction, global fluxes, modified Riemann problem 3

Global flux consistency via full well balanced Riemann solver



Two intermediate states RP

A steady 0-wave introduces 2 intermediate states:

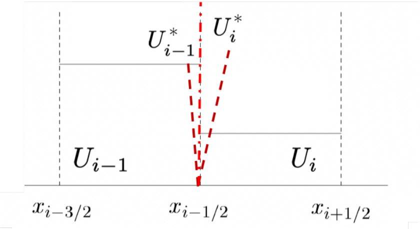
to obtain them we can impose 4 conditions

- 2 conditions: $F_i - F_{i-1} - \Delta x \hat{S} = \sum_k \lambda_k [\![U]\!]$ (space-time integration)
- 2 conditions: steady state relations across 0-wave, e.g.

$$[\![hu]\!]_0 = 0\ ,\quad [\![g\zeta + k]\!]_0 = 0$$

Numerics: WB reconstruction, global fluxes, modified Riemann problem 3

Global flux consistency via full well balanced Riemann solver



Two intermediate states RP

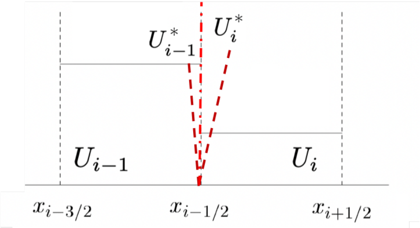
- Steady condition $U_j = U^*_j \Rightarrow$ algebraic eq. defining $\hat{S}$
- The intermediate states U^*_j depend on $\hat{S}$
- Automatically verify $\Phi_i^{i\pm1/2} = 0$ for steady initial data

For data at equilibrium the schemes verify

$$\Phi_i^{i-1/2} = \frac{1}{2}\left(F_i - F_{i-1} - \Delta x \hat{S}_{i-1/2}\right)$$
$$\approx \frac{1}{2}\int_{-\Delta x/2}^{\Delta x/2} (\partial_x F - S) = \frac{1}{2}\int_{-\Delta x/2}^{\Delta x/2} \partial_x G = 0$$

Numerics: WB reconstruction, global fluxes, modified Riemann problem 3

Global flux consistency via full well balanced Riemann solver



Two intermediate states RP

- Greenberg & Leroux, *SINUM* 33, 1996
- Gosse, *Comput.Math.Appli.* 39, 2000
- Gallouet et al., *Computers & Fluids* 32, 2003
- Berthon & Chalons *Math.Comp.* 85, 2016; Michel-Dansac et al. *J.Comput.Phys.* 154, 2017
- etc

Numerics: WB reconstruction, global fluxes, modified Riemann problem 4

Direct global flux reconstruction and discrete equilibria

Numerics: WB reconstruction, global fluxes, modified Riemann problem 4

Direct global flux reconstruction and discrete equilibria

1. Consider a consistent source quadrature which is not WB: $\Delta x \hat{S} \neq \Delta F$ along exact equilibria
2. Define local approximations of the global flux as:

$$R_0 := 0\,, \quad G_0 = F(U_{s0})$$

$$R_{i+1/2} := R_{i-1/2} + \int_{x_{i-1}}^{x_i} S(U,\varphi)\,, \quad G_i := F(U_i) + (R_{i+1/2} + R_{i-1/2})/2$$

3. Given (G_j, R_j) assume you can invert the relation $F(U_j) = G_j - R_j \Rightarrow U_j$

Numerics: WB reconstruction, global fluxes, modified Riemann problem 4

Direct global flux reconstruction and discrete equilibria

Definition (Discrete steady state). *The discrete steady state* $U(x)$ *of the global flux method is defined from*

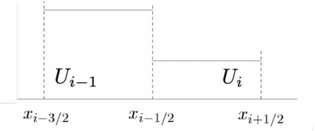
$$F(U(x)) + R(U(x)) = G_0 = F(U_{s0}) \tag{9}$$

Equation (9) defines a nonlinear algebraic system which can be solved iteratively:

1. $F(U_1) + R(U_1, \varphi)/2 = G_0$
2. $F(U_2) + R(U_2, \varphi)/2 = G_0 - R(U_1, \varphi)/2$
3. etc

Numerics: WB reconstruction, global fluxes, modified Riemann problem 4

Direct global flux reconstruction and discrete equilibria

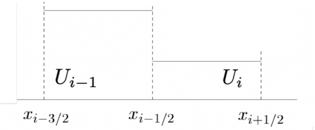


Global flux FV scheme: At each interface $i \pm 1/2$

1. Reconstruction:
   cell global flux polynomials $G_i(x)$ are built from cell averages using std. methods

Numerics: WB reconstruction, global fluxes, modified Riemann problem 4

Direct global flux reconstruction and discrete equilibria



Global flux FV scheme: At each interface $i \pm 1/2$

2. Solution recovery: interface values $U^{\pm}_{i+1/2}$ are recovered from

$$F(U^{-}_{i+1/2}) = G_i(x_{i+1/2}) - R_{i+1/2}\ , \quad F(U^{+}_{i+1/2}) = G_{i+1}(x_{i+1/2}) - R_{i+1/2}$$

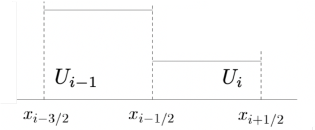
Note that

$$G_i(x_{i+1/2}) = G_{i+1}(x_{i+1/2}) \Rightarrow F(U^{-}_{i+1/2}) = F(U^{+}_{i+1/2})$$

In this case, the algebraic solver inverting $F(U) = G - R$ gives no interface jumps.

Numerics: WB reconstruction, global fluxes, modified Riemann problem 4

Direct global flux reconstruction and discrete equilibria



Global flux FV scheme: At each interface $i \pm 1/2$

3. Approximate Riemann problem :

$$\hat{G}_{i+1/2} = \gamma G_i(x_{i+1/2}) + (1-\gamma)G_{i+1}(x_{i+1/2}) + D(U^+_{i+1/2} - U^-_{i+1/2})$$

with D some dissipation matrix.

Numerics: WB reconstruction, global fluxes, modified Riemann problem 4

Direct global flux reconstruction and discrete equilibria

Definition (Discrete steady state). *The discrete steady state $U(x)$ of the global flux method is defined from*

$$F(U(x)) + R(U(x)) = G_0 = F(U_{s0}) \qquad (9)$$

Global flux FV scheme:

$$\Delta x \frac{dU_i}{dt} + \hat{G}_{i+1/2} - \hat{G}_{i-1/2} = 0$$

Proposition (Discrete well balanced). *On a given mesh the global flux FV scheme preserves exactly constant global flux states, and the associated discrete steady state $U(x)$ computed from (9), with an expected error wrt the exact steady state $\|U(x) - U_s(x)\|$ with U_s the exact steady equilibrium.*

Numerics: WB reconstruction, global fluxes, modified Riemann problem 4

Direct global flux reconstruction and discrete equilibria

- Chertock et al., *J.Comput.Phys.* 358, 2018
- Cheng et al., *J.Sci.Comp.* 80, 2019
- Chertock et al, *J.Sci.Comp.* 90, 2022

Next steps...

Coming up

1. Global flux formulation in the DGSEM context
2. Characterization of discrete equilibria
3. Compatibility between global flux and entropy conservation
4. Verification of well balanced for various sources in 1D and 2D

WB DG methods (not using global fluxes)

Y. Xing, Ohio State
M. Dumbser, U. Trento
E. Gaburro, (before U. Trento, now Inria)
M. Castro, (Edanya group in Malaga)
G. Gassner (Cologne U.), A. Winters (Linkoping U.) and co.
F. Giraldo (Navy),
B. Bonev, J. Hstehaven (EPFL),

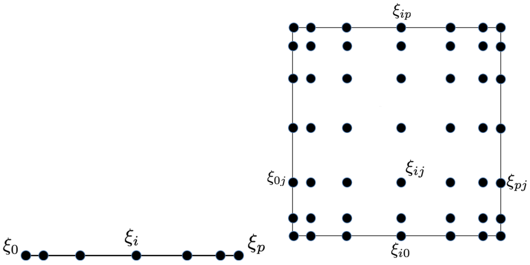
many many others...

dGSEM for conservation laws

dGSEM for conservation laws 1

Main notation

- Reference element $\xi \in [0, 1]$
- $x(\xi)$ linear map $K \mapsto [0, 1]$, here: $|K| = \mathrm{h}^d$
- $\{\phi_i(\xi)\}_{i=0,p}$ degree p Lagrange bases
- $\{\xi_i\}_{i=0,p}$ $p+1$ Gauss-Lobatto (GL) points
- Set $U_\mathrm{h} = \sum_{i=0}^{p} \phi_i(x(\xi))U_i$
- 2D extension by tensor products



dGSEM for conservation laws 2

DGSEM variational form: conservation laws

Consider for the moment the approximation of solutions of

$$\partial_t U + \partial_x F(U) = 0$$

On an element K, start from the dG approximation arising from the variational form

$$|K| \int_0^1 \varphi_i(\xi) \partial_t U_\mathrm{h} - \int_0^1 \partial_\xi \varphi_i(\xi) F_\mathrm{h} + (\varphi_i \hat{F}_\mathrm{h}(U_\mathrm{h}, U_\mathrm{h}^+))_{\xi=1} - (\varphi_i \hat{F}_\mathrm{h}(U_\mathrm{h}, U_\mathrm{h}^+))_{\xi=0} = 0$$

dGSEM for conservation laws 2

DGSEM variational form: conservation laws

Consider for the moment the approximation of solutions of

$$\partial_t U + \partial_x F(U) = 0$$

dGSEM: quadrature based on the same GL nodes used for the polynomial expansion[3]

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} - \widetilde{D}_x^T \mathbf{F} + \mathcal{M}^{-1}\mathcal{B}\widehat{\mathbf{F}} = 0$$

- $\mathcal{M} = \mathrm{diag}(\{w_i\}_{i=0,p})$ with $w_i = \mathrm{h}\phi_i(\xi_i)$ the quadrature weights
- $\widetilde{D}_x = \mathcal{M}D_x\mathcal{M}^{-1}$ with $(D_x)_{ij} = \partial_\xi \phi_i(\xi_j)$
- $\mathcal{B} = \mathrm{diag}(-1,\ldots,1)$ the matrix sampling boundary values
- $\mathbf{U}$, $\mathbf{F}$, $\widehat{\mathbf{F}}$ arrays of solution/flux/num. flux values

[3] Kopriva & Gassner *J.Sci.Comp.* 44, 2010 ; Hesthaven & Warburton, Springer 2008

dGSEM for conservation laws 3

SBP property and fluctuation form

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} - \widetilde{D}_x^T\mathbf{F} + \mathcal{M}^{-1}\mathcal{B}\widehat{\mathbf{F}} = 0$$

The semi-discrete dGSEM equations can be equivalently written as[4]

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} + \widetilde{D}_x\mathbf{F} + \mathcal{M}^{-1}\mathcal{B}(\widehat{\mathbf{F}} - \mathbf{F})$$

[4] Kopriva & Gassner *J.Sci.Comp.* 44, 2010; Gassner et al. *J.Comput.Phys.* 327, 2016

dGSEM for conservation laws 3

SBP property and fluctuation/RD form

In other words, the dGSEM can be written in fluctuation/strong form as

$$w_i \frac{dU_i}{dt} + \Phi_i + \Psi_i^L + \Psi_i^R = 0$$

with

$$\Phi_i := \int_K \phi_i \partial_x F_{\mathrm{h}}$$

$$\Psi_i^L := [\phi_i(\hat{F}_{\mathrm{h}} - F_{\mathrm{h}})]_{\xi=0}\ , \quad \Psi_i^R := [\phi_i(\hat{F}_{\mathrm{h}} - F_{\mathrm{h}})]_{\xi=1}$$

This form well suited to see that (trivially) $F_{\mathrm{h}} = F_0$ is an exact discrete steady state.

Gf-dGSEM: balance laws and global flux

Gf-dGSEM: balance laws and global flux 1

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x)), \tag{1}$$

by locally recasting it in the pseudo-conservative form

$$\partial_t U + \partial_x G(U; \varphi(x)) = 0$$

with

$$\begin{aligned} G(U; \varphi(x)) =& F(U) + R(U; \varphi(x)) \\ R(U; \varphi(x)) =& R_0 - \int_{x_0}^{x} S(U; \varphi(s)) ds \end{aligned}$$

Gf-dGSEM: balance laws and global flux 1

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1. from F to G: source integral assembly
2. discrete well balanced: definition and understanding

Gf-dGSEM: balance laws and global flux 2

Gf-dGSEM: global flux assembly in 1d

General definition

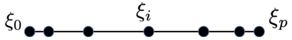
$$R(x) = R_0 - \int_{x_0}^{x} S(U;\varphi(s))ds$$

Gf-dGSEM: balance laws and global flux 2

Gf-dGSEM: global flux assembly in 1d

Evaluation of $\{R_i\}_{i=0,p}$, space-marching method: $\forall j \geq 0$

1. Set $R_0 = R^-$
2. $R_i = R_{i-1} + \int_{x_{i-1}}^{x_i} S(U;\varphi(s))ds$



In all $\{K_j\}_{j\geq 0}$ we set

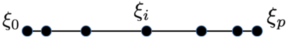
$$R_{\mathrm{h}} = \sum_{i=0,p} \varphi_i(x(\xi))R_i$$

Gf-dGSEM: balance laws and global flux 2

Gf-dGSEM: global flux assembly in 1d

Evaluation of $\{R_i\}_{i=0,p}$, space-marching method: $\forall j \geq 0$

1. Set $R_0 = R^-$
2. $R_i = R_{i-1} - h \sum_{l=0,p} \int_{\xi_{i-1}}^{\xi_i} \varphi_l(\xi) S_l ds$



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Gf-dGSEM: balance laws and global flux 2

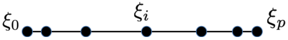
Gf-dGSEM: global flux assembly in 1d

Evaluation of $\{R_i\}_{i=0,p}$, space-marching method: $\forall j \geq 0$

1. Set $R_0^{K_j} = R^-$
2. $R_i = R_{i-1} - h \sum_{l=0,p} \int_{\xi_{i-1}}^{\xi_i} \varphi_l(\xi) S_l ds$

Over an element we have

$$\mathbf{R} = \mathbf{R}^- - \mathcal{I}\mathbf{S}$$



In all $\{K_j\}_{j\geq 0}$ we set

$$R_\mathrm{h} = \sum_{i=0,p} \varphi_i(x(\xi)) R_i$$

Remark. *The matrix $\mathcal{I}$ is the integration tableau of the $p+1$ stages RK-LobattoIIIA ODE solver*[5]

[5] A. Prothero & A. Robinson, Math.Comp. 28, 1974

Gf-dGSEM: balance laws and global flux 3

Gf-dGSEM: discrete well balanced

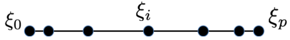
Gf-dGSEM: balance laws and global flux 3

Gf-dGSEM: discrete well balanced

Definition (Discrete steady state). *The discrete steady state U^* is the polynomial approximation arising from the solution of the elemental systems of nonlinear algebraic equations obtained as*

$$F(U_i^*) - \sum_{j=0,p} \mathcal{I}_{ij} S(U_j^*, \varphi(x_0 + c_j \mathrm{h})) = G_0 - R_0$$

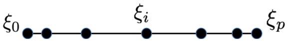
with G_0 a global (over the mesh) constant flux state, $R_0 = R^-$ the elemental initial value of R, and with $\mathcal{I}_{ij}$ and c_j the entries of the integration tableau of the p+1 stages implicit RK LobattoIIIA ODE integrator.



Gf-dGSEM: balance laws and global flux 3

Gf-dGSEM: discrete well balanced

Proposition (Discrete steady state). *Let* $R_0^{K_0} = 0$, *and* $\forall\, K_j$ *let* $R^- = R_p^{K_{j-1}}$. *For smooth enough data,* U^* *is an approximation of order* h^{2p} *to a continuous exact steady state* U.

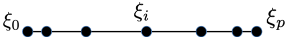


Gf-dGSEM: balance laws and global flux 3

Gf-dGSEM: discrete well balanced

Proposition (Discrete steady state). *Let* $R_0^{K_0} = 0$, *and* $\forall\, K_j$ *let* $R^- = R_p^{K_{j-1}}$. *For smooth enough data,* U^* *is an approximation of order* h^{2p} *to a continuous exact steady state* U..

Proof. The integration strategy reduces to the ODE integration with the A-stable implicit LobattoIIIA RK scheme of order $2p$[6] applied to $\partial_x F - S = 0$



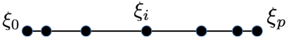
[6] A. Prothero & A. Robinson, Math.Comp. 28, 1974

Gf-dGSEM: balance laws and global flux 3

Gf-dGSEM: discrete well balanced

$$\mathbf{F}(\mathbf{U}) - \mathcal{I}\mathbf{S}(\mathbf{U};\varphi) = \mathbf{G}_0 - \mathbf{R}_0$$

$$\mathbf{R}_0^{K_j} = \mathbf{R}_p^{K_{j-1}}$$



In all $\{K_j\}_{j\geq 0}$ we set

$$R_{\mathrm{h}} = \sum_{i=0,p} \varphi_i(x(\xi))R_i$$

- The nonlinear equation is solved element by element via Newton iterations for $\{U_i\}_{i=0,p}$
- Nonlinear solver <u>only needed for the exact/IC</u> as data at quadrature points is directly evolved
- At equilibrium at interfaces $\{F(U_p) + R_p\}_{K_{j-1}} = \{F(U_0) + R_0\}_{K_j} \Longrightarrow$ no jumps of U
- For smooth solutions $U^* - U^{\text{ex}} = \mathcal{O}(\mathrm{h}^{2p}) \Rightarrow$ potential for superconvergence for $p \geq 2$

Gf-dGSEM: balance laws and global flux 4

Gf-dGSEM: variational form and numerical fluxes

Gf-dGSEM: balance laws and global flux 4

Gf-dGSEM: full discretization, well balanced, and numerical fluxes

The Gf-dGSEM can be written in fluctuation form

$$w_i \frac{dU_i}{dt} + \Phi_i + \Psi_i^L + \Psi_i^R = 0$$

with, setting $G_\mathrm{h} = \sum_{l=0,p} \phi_l(x(\xi)) G_l$, with $G_l = F(U_l) + R_l$

$$\Phi_i := \int_K \phi_i \partial_x G_\mathrm{h} \ , \quad \begin{cases} \Psi_i^L := [\phi_i(\hat{G}_\mathrm{h} - G_\mathrm{h})]_{\xi=0} \\ \Psi_i^R := [\phi_i(\hat{G}_\mathrm{h} - G_\mathrm{h})]_{\xi=1} \end{cases}$$

or equivalently in semi-discrete matrix form

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} + \widetilde{D}_x \mathbf{G} + \mathcal{M}^{-1}\mathcal{B}(\widehat{\mathbf{G}} - \mathbf{G}) = 0$$

with $\mathbf{G} = \mathbf{F} + \mathbf{R}$

Gf-dGSEM: balance laws and global flux 4

Gf-dGSEM: full discretization, well balanced, and numerical fluxes

The Gf-dGSEM can be written in fluctuation form

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$$\Phi_i := \int_K \phi_i \partial_x G_{\rm h} \ , \quad \begin{cases} \Psi_i^L := [\phi_i(\hat{G}_{\rm h} - G_{\rm h})]_{\xi=0} \\ \Psi_i^R := [\phi_i(\hat{G}_{\rm h} - G_{\rm h})]_{\xi=1} \end{cases}$$

Definition. (Consistent numerical global flux). *A numerical global flux* $\hat{G} = \hat{G}(U^+, \varphi^+; U^-, \varphi^-)$ *is said to be consistent if*

$$G(U^+, \varphi^+) = G(U^-, \varphi^-) = G_0 \quad \Rightarrow \quad \hat{G}(U^+, \varphi^+; U^-, \varphi^-) = G_0$$

Gf-dGSEM: balance laws and global flux 4

Gf-dGSEM: full discretization, well balanced, and numerical fluxes

The Gf-dGSEM can be written in fluctuation form

$$w_i \frac{dU_i}{dt} + \Phi_i + \Psi_i^L + \Psi_i^R = 0$$

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$$\Phi_i := \int_K \phi_i \partial_x G_{\mathrm{h}} \, , \quad \begin{cases} \Psi_i^L := [\phi_i(\hat{G}_{\mathrm{h}} - G_{\mathrm{h}})]_{\xi=0} \\ \Psi_i^R := [\phi_i(\hat{G}_{\mathrm{h}} - G_{\mathrm{h}})]_{\xi=1} \end{cases}$$

Example:

$$\hat{G}(U^+, \varphi^+; U^-, \varphi^-) = \alpha G(U^+, \varphi^+) + (\mathrm{Id} - \alpha) G(U^-, \varphi^-) - \mathcal{D}(U^+ - U^-)$$

Gf-dGSEM: balance laws and global flux 4

Gf-dGSEM: full discretization, well balanced, and numerical fluxes

The Gf-dGSEM can be written in fluctuation form

$$w_i \frac{dU_i}{dt} + \Phi_i + \Psi_i^L + \Psi_i^R = 0$$

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Proposition. (Gf-dGSEM and discrete well balanced). *The Gf-dGSEM scheme with a consistent numerical global flux is discretely well balanced, in the sense that (equivalently)*

1. *it preserves exactly the discrete equilibrium U^* associated to the quadrature defining G*
2. *It has a super-convergent behaviour of order h^{2p} wrt exact smooth steady states*

Gf-dGSEM: balance laws and global flux 5

Gf-dGSEM recap

On and element K_j we have

1. $w_i \dfrac{dU_i}{dt} + \int\limits_K \phi_i \partial_x G_\mathrm{h} + [\phi_i(\hat{G}_\mathrm{h} - G_\mathrm{h})]_{\xi=0} + [\phi_i(\hat{G}_\mathrm{h} - G_\mathrm{h})]_{\xi=1} = 0$
2. $G_\mathrm{h} = \sum_{i=0,p} \varphi_i(x(\xi))(F(U_i) + R_i)$
3. $R_0 = R_p^{K_{j-1}}$ and $R_i = R_0 - \sum_{l=0,p} \mathcal{I}_{il} S_l$

Remains to define the nodal values S_l

Gf-dGSEM: balance laws and global flux 6

Gf-dGSEM: well balanced fluxes and exact lake at rest

Proposition (Exact lake at rest preservation[7].) *For the shallow water equations the choice*

$$S_l = g\zeta_l \partial_x b_{\mathrm{h}}(x(\xi_l)) - \partial_x p_{\mathrm{h}}(b)(x(\xi_l)) + \omega h_l v_l + c_f h_l u_l$$

allows exact preservation of the analytical lake at rest $h_j + b_j = \zeta_0$, $hu = hv = 0$.

[7] Generalization of approach by Xing & Shu, *J.Comput.Phys.* 208, 2005

Gf-dGSEM: balance laws and global flux 6

Gf-dGSEM: well balanced fluxes and exact lake at rest

Proposition (Exact lake at rest preservation[7].) *For the shallow water equations the choice*

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allows exact preservation of the analytical lake at rest $h_j + b_j = \zeta_0$, $hu = hv = 0$.

Proof. consequence of approximation properties:

$$g\zeta_0 \int_0^{\xi_j} \phi_l(\xi)(\mathrm{h}^{-1} \sum_{i=0,p} \partial_\xi \phi_i(\xi_l) b_i) = g\zeta_0 \int_0^{\xi_j} \overbrace{\mathrm{h}^{-1} \partial_\xi \phi_i(\xi_l) b_i}^{\partial_x b_{\mathrm{h}}} = g\zeta_0 (b_j - b_0)$$

[7] Generalization of approach by Xing & Shu, *J.Comput.Phys.* 208, 2005

Gf-dGSEM: balance laws and global flux 6

Gf-dGSEM: well balanced fluxes and exact lake at rest

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allows exact preservation of the analytical lake at rest $h_j + b_j = \zeta_0$, $hu = hv = 0$.

Proof. consequence of approximation properties:

$$R_j - R_0 = g\zeta_0(b_j - b_0) - (p(b_j) - p(b_0)) \overbrace{=}^{h_j = \zeta_0 - b_j} -(p(h_j) - p(h_0)) = -(F_j - F_0)$$

so $G_j - G_0 = 0$ for the lake at rest state.

[7] Generalization of approach by Xing & Shu, *J.Comput.Phys.* 208, 2005

Numerical examples in 1d

1. Verification of super-convergence property
2. Perturbation of steady states
3. moving and non-moving equilibria (no ad hoc scheme modification)

Numerical examples in 1d

1. Verification of super-convergence property
2. Perturbation of steady states
3. moving and non-moving equilibria (no ad hoc scheme modification)

- Time integration: SSP-RK(p).
- (Non WB) dGSEM:

$$w_i \frac{dU_i}{dt} + \int_K \phi_i \partial_x F_{\rm h} + \int_K \phi_i S(U_{\rm h};\varphi) + [\phi_i(\hat{F}_{\rm h} - F_{\rm h})]_{\xi=0} + [\phi_i(\hat{F}_{\rm h} - F_{\rm h})]_{\xi=1} = 0$$

Numerical examples in 1d

1D rotating shallow water equations[8].

$$
\begin{aligned}
\partial_t h + \partial_x (hu) &= 0 \\
\partial_t (hu) + \partial_x (hu^2 + p(h)) + gh\partial_x b - \omega hv &= 0 \\
\partial_t (hv) + \partial_x (huv) + \omega hu &= 0
\end{aligned} \tag{10}
$$

With as usual $p(h) = gh^2/2$

[8] Castro et al, *SISC* 31, 2008

Numerical examples in 1d

Lake at rest

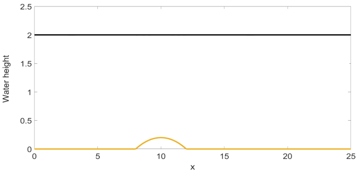
$$q = hu = q_0 = 0$$
$$q_y = hv = 0$$

and

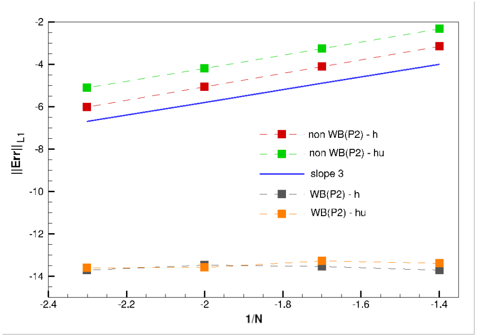
$$E = g(h+b) + k = E_0 = gh_0$$

or

$$K = hu^2 + p + \int_x gh\partial_x b = F_0 = p_0$$



$$b(x) = (0.2 - 0.05(x-10)^2)^+$$



unperturbed IC evolved up to $T = 5$

Numerical examples in 1d

Moving equilibria: sub-critical flow

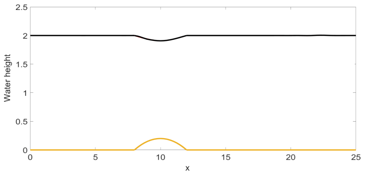
$$q = hu = q_0 = 4.42$$
$$q_y = hv = 0$$

and

$$E = g(h+b) + k = E_0 = 22.06$$

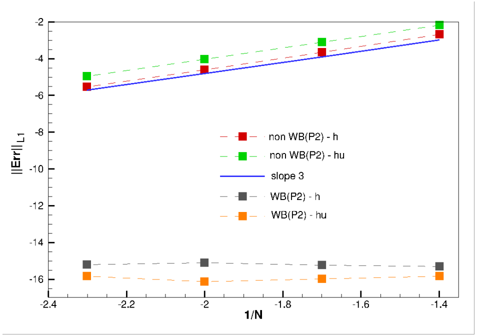
or

$$K = hu^2 + p + \int_x gh\partial_x b = F_0 = 29.41$$



$$b(x) = (0.2 - 0.05(x-10)^2)^+$$

Exact preservation of global flux/U^*



unperturbed IC evolved up to $T = 5$

Numerical examples in 1d

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$$q = hu = q_0 = 4.42$$
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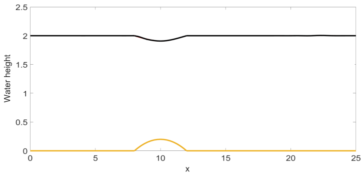
and

$$E = g(h+b) + k = E_0 = 22.06$$

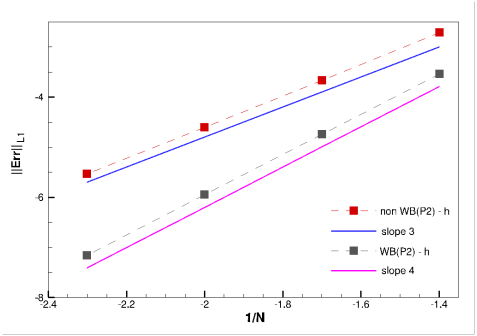
or

$$K = hu^2 + p + \int_x gh\partial_x b = F_0 = 29.41$$



$$b(x) = (0.2 - 0.05(x-10)^2)^+$$

(Super-)Convergence : $h^* - h^{ex}$



unperturbed IC evolved up to $T = 5$

Numerical examples in 1d

Moving equilibria: sub-critical flow

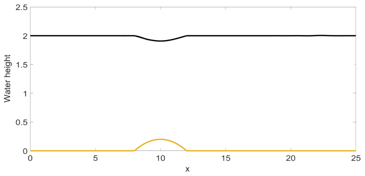
$$q = hu = q_0 = 4.42$$
$$q_y = hv = 0$$

and

$$E = g(h+b) + k = E_0 = 22.06$$

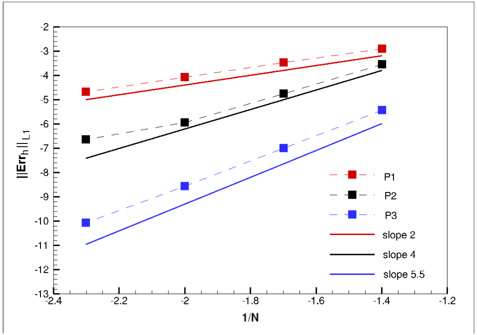
or

$$K = hu^2 + p + \int_x gh\partial_x b = F_0 = 29.41$$



$$b(x) = (0.2 - 0.05(x-10)^2)^+$$

(Super-)Convergence : $h^* - h^{\text{ex}}$



unperturbed IC evolved up to $T = 5$

Numerical examples in 1d

Moving equilibria: sub-critical flow

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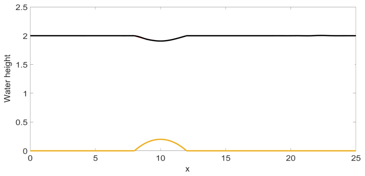
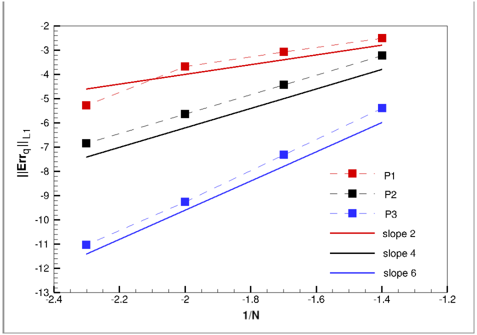


$$b(x) = (0.2 - 0.05(x-10)^2)^+$$

(Super-)Convergence : $hu^* - hu^{\text{ex}}$



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Moving equilibria: sub-critical flow

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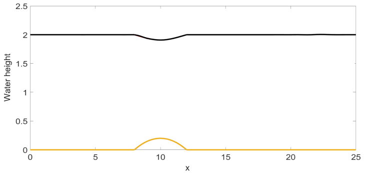
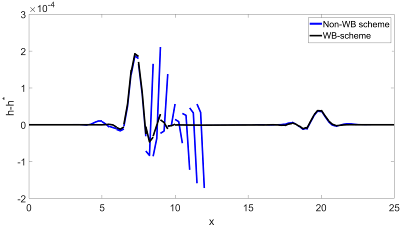


$$b(x) = (0.2 - 0.05(x-10)^2)^+$$

Small perturbation: global flux vs nonWB



$h = h_0 + 10^{-3}e^{-100(x-10)^2}$ evolved up to $T = 1.5$

Numerical examples in 1d

Moving equilibria: super-critical flow

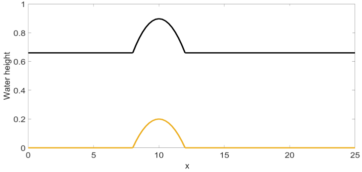
$$q = hu = q_0 = 4.42$$
$$q_y = hv = 0$$

and

$$E = g(h+b) + k = E_0 = 28.9$$

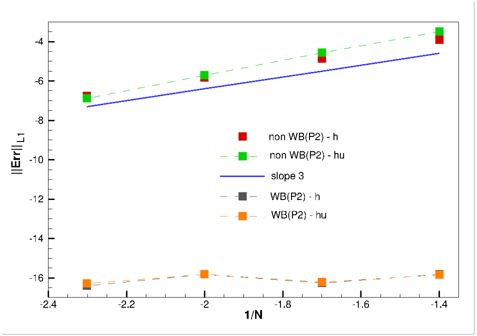
or

$$K = hu^2 + p + \int_x gh\partial_x b = F_0 = 31.74$$



$$b(x) = (0.2 - 0.05(x-10)^2)^+$$

Exact preservation of global flux/U^*



unperturbed IC evolved up to $T = 5$

Numerical examples in 1d

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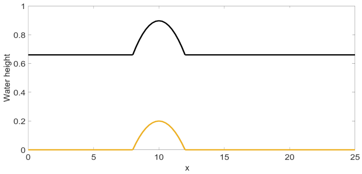

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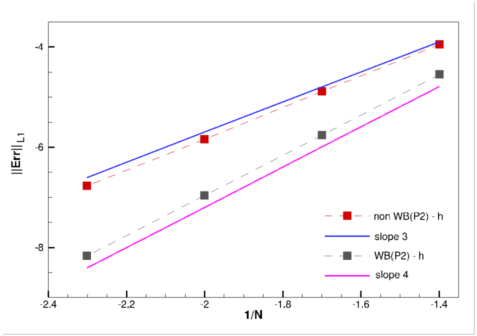
(Super-)Convergence : $h^* - h^{\text{ex}}$



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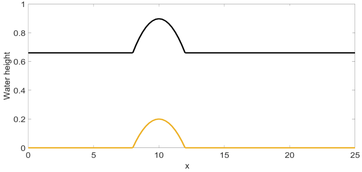

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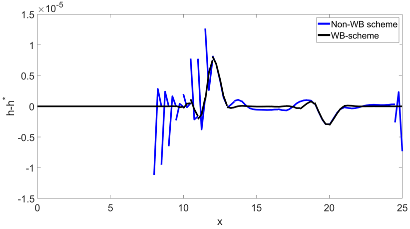
Small perturbation: global flux vs nonWB



$h = h_0 + 10^{-5} e^{-100(x-10)^2}$ evolved up to $T = 1.5$

Numerical examples in 1d

Algebraic source: pressure/Coriolis force equilibrium

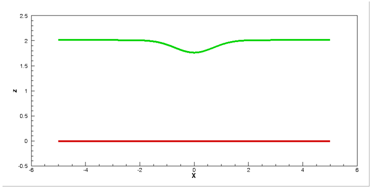
$$q = hu = q_0 = 0$$
$$q_y = hv = q_y(x)$$

and

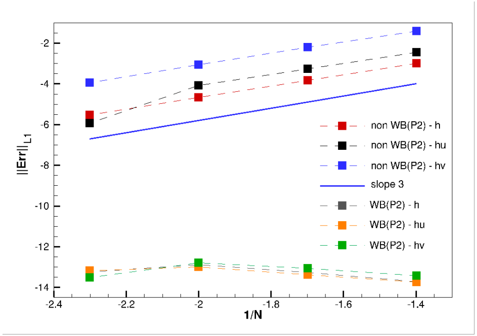
$$E = gh + k = E_0 = gh_0$$

or

$$K = p - \int_x \omega q_y(x) = p_0$$



Exact preservation of global flux/U^*



unperturbed IC evolved up to $T = 5$

Numerical examples in 1d

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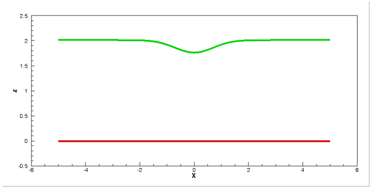
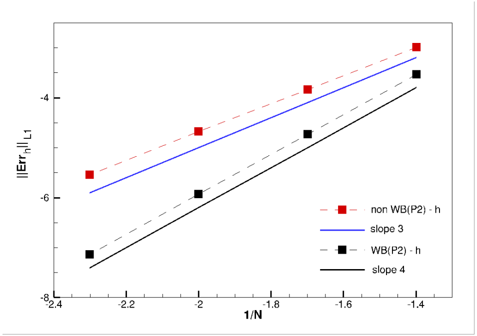


(Super-)Convergence : $h^* - h^{\text{ex}}$



unperturbed IC evolved up to $T = 5$

Numerical examples in 1d

Algebraic source: pressure/Coriolis force equilibrium

$$q = hu = q_0 = 0$$
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and

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or

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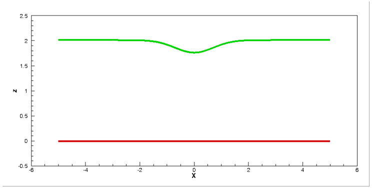
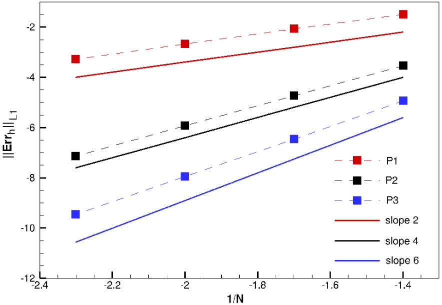


(Super-)Convergence : $h^* - h^{\mathrm{ex}}$



unperturbed IC evolved up to $T = 5$

Numerical examples in 1d

Algebraic source: pressure/Coriolis force equilibrium

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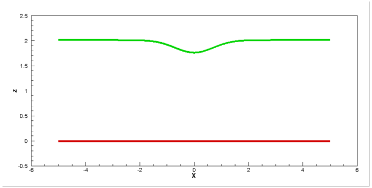
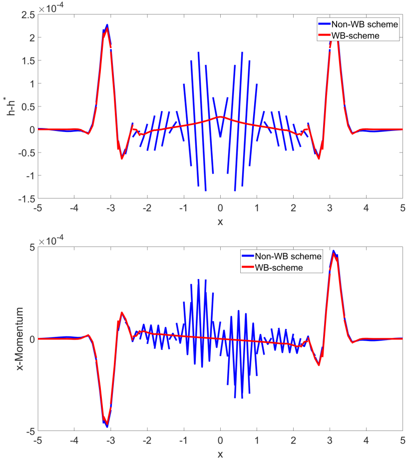


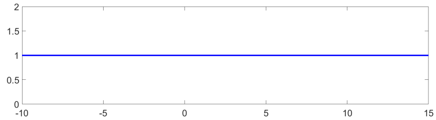
Small perturbation: global flux vs nonWB

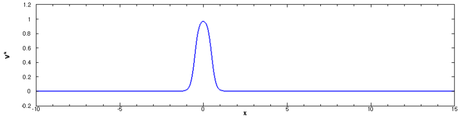


$h = h_0 + 10^{-3} e^{-100x^2}$ evolved up to $T = 1.5$

Numerical examples in 1d

Rotating shallow water: geostrophic adjustment[9]





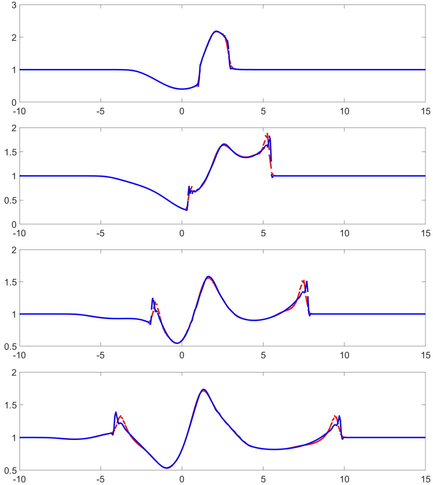
$$
\begin{aligned}
\partial_t h + \partial_x (hu) &= 0 \\
\partial_t (hu) + \partial_x (hu^2 + p(h)) + gh\partial_x b - \omega hv &= 0 \\
\partial_t (hv) + \partial_x (huv) + \omega hu &= 0
\end{aligned}
\tag{11}
$$

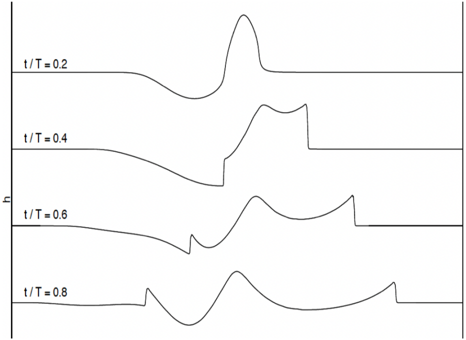
[9] Bouchut et al. *J.Fluid Mech.* 514, 2004 ; Castro et al. *SISC* 31, 2008

Numerical examples in 1d

Rotating shallow water: geostrophic adjustment[9]

Free surface evolution (Gf-dGSEM: $P1$ in red - $P2$ in blue - right pic from Castro et al.)





[9]Bouchut et al. *J.Fluid Mech.* 514, 2004 ; Castro et al. *SISC* 31, 2008

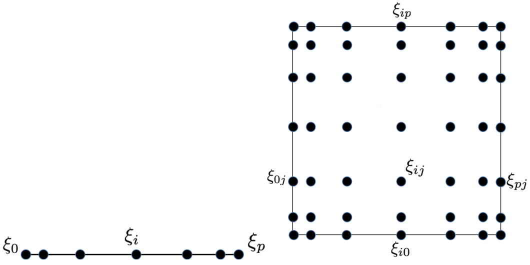
Gf-dGSEM in 2D 1

A 2D extension

Gf-dGSEM in 2D 1

Main notation

- Reference element $\xi \in [0,1]$
- $x(\xi)$ linear map $K \mapsto [0,1]$, here: $|K| = \mathrm{h}^d$
- $\{\phi_i(\xi)\}_{i=0,p}$ degree p Lagrange bases
- $\{\xi_i\}_{i=0,p}$ the $p+1$ Gauss-Lobatto (GL) points
- Set $U_\mathrm{h} = \sum_{i=0}^{p} \phi_i(x(\xi))U_i$
- 2D extension by tensor products



Gf-dGSEM in 2D 2

2D Cartesian shallow water equations.

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + p(h) \\ huv \end{pmatrix} + \partial_y \begin{pmatrix} hv \\ huv \\ hv^2 + p(h) \end{pmatrix} = -h \begin{pmatrix} 0 \\ \partial_x \varphi + c_f u + \omega v \\ \partial_y \varphi + c_f v - \omega u \end{pmatrix}, \qquad (5)$$

Notation.

h water depth

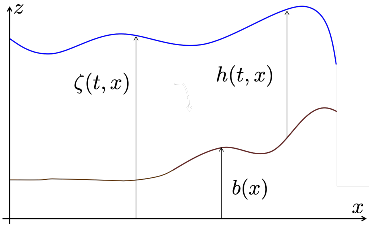
$\vec{v} = (u, v)$ horizontal velocity

$p = gh^2/2$ hydrostatic pressure (g gravity acceleration)

$\varphi = gb$ gravitational potential ($b(x, y)$ bottom topography)

$c_f = c_f(h, \vec{v})$ friction coefficient

ω Coriolis coefficient

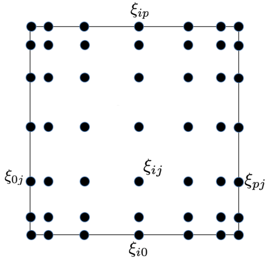


Gf-dGSEM in 2D 3

2D Cartesian shallow water equations.

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + p(h) + rx \\ huv \end{pmatrix} + \partial_y \begin{pmatrix} hv \\ huv \\ hv^2 + p(h) + ry \end{pmatrix} = 0, \tag{11}$$

Global flux/pressure formulation.



Gf-dGSEM in 2D 3

2D Cartesian shallow water equations.

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + p(h) + rx \\ huv \end{pmatrix} + \partial_y \begin{pmatrix} hv \\ huv \\ hv^2 + p(h) + ry \end{pmatrix} = 0, \qquad (12)$$

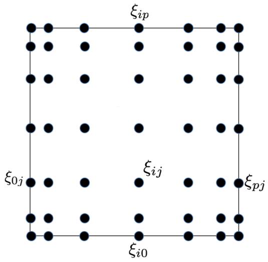
Global flux/pressure formulation.

Line based evaluation of (rx, ry): $\forall K_{lm}$ we set

$$\mathbf{rx}_j = \mathbf{rx}_j^- - \mathcal{I}\mathbf{Sx}_j$$

where

- $\mathbf{rx}_j$ and $\mathbf{Sx}_j$ contain the $\{rx_{ij}\}_{i=0,p}$ and $\{Sx_{ij}\}_{i=0,p}$ values
- $Sx := gh\partial_x b - \omega hv + c_f hu$ is the x-momentum source
- $\mathcal{I}$ is the implicit RK-LobattoIIIA tableau
- The IC is taken as $rx_{ij}^- = (rx_{pj})_{K_{l-1\,m}}$



Gf-dGSEM in 2D 3

2D Cartesian shallow water equations.

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + p(h) + rx \\ huv \end{pmatrix} + \partial_y \begin{pmatrix} hv \\ huv \\ hv^2 + p(h) + ry \end{pmatrix} = 0, \tag{13}$$

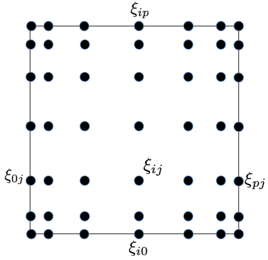
Global flux/pressure formulation.

Line based evaluation of (rx, ry): $\forall K_{lm}$ we set

$$\mathbf{ry}_i = \mathbf{ry}_i^- - \mathcal{I}\mathbf{Sy}_i$$

where

- $\mathbf{ry}_i$ and $\mathbf{Sy}_i$ contain the $\{ry_{ij}\}_{i=0,p}$ and $\{Sy_{ij}\}_{i=0,p}$ values
- $Sy := gh\partial_y b + \omega hu + c_f hv$ is the x-momentum source
- $\mathcal{I}$ is the implicit RK-LobattoIIIA tableau
- The IC is taken as $ry_{ij}^- = (ry_{ip})_{K_{l\,m-1}}$



Gf-DGSEM in 2D 4

$$\frac{dU_{ij}}{dt} + \widetilde{D}_x \mathbf{Gx} + \mathcal{M}^{-1}\mathcal{B}(\widehat{\mathbf{Gx}} - \mathbf{Gx}) + \widetilde{D}_y \mathbf{Gy} + \mathcal{M}^{-1}\mathcal{B}(\widehat{\mathbf{Gy}} - \mathbf{Gy}) = 0$$

Well balanced direction-wise[10]

Proposition. (Discrete well balanced along x- and y-) *The line Gf-dGSEM with consistent numerical global flux is discretely well balanced along the x- and y- directions, in the sense that (equivalently)*

- *it preserves exactly 1d discrete equilibrium U^* associated to the quadrature defining Gx (or Gy)*
- *It has a super-convergent h^{2p} behaviour wrt exact smooth 1d steady states in the x or y direction*

[10] cf. e.g. (Michel-Dansac et al. *Computers & Fluids* 230, 2021) for similar result

Numerical examples in 2d

Moving equilibria: sub-critical flow

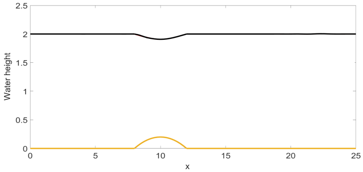
$$qx = hu = q_0$$
$$qy = hv = 0$$

and

$$E = g(h+b) + k = E_0$$

or

$$K = hu^2 + p + \int_x gh\partial_x b = F_0$$



Numerical examples in 2d

Moving equilibria: sub-critical flow

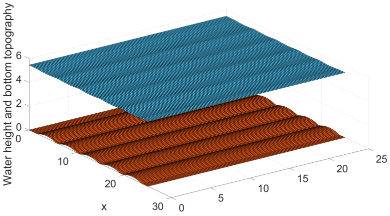
$$
\begin{aligned}
qx =& hu = q_0 \\
qy =& hv = 0
\end{aligned}
$$

and

$$
E = g(h+b) + k = E_0
$$

or

$$
K = hu^2 + p + \int_x gh\partial_x b = F_0
$$



Numerical examples in 2d

Moving equilibria: sub-critical flow

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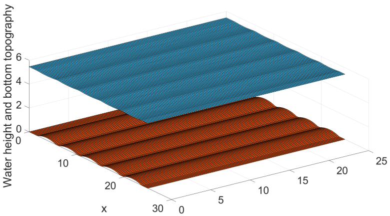
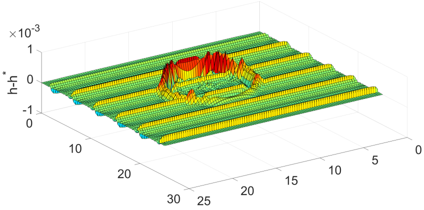
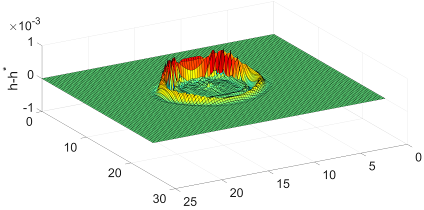


Small perturbation: global flux vs nonWB



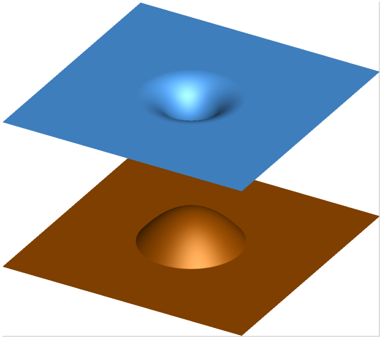


$h = h^* + 0.05e^{-100(r-r^*)^2}$ evolved up to $T = 2$

Numerical examples in 2d

Steady vortex with bathymetry and Coriolis forces

Modification of the vortex used e.g. in[11]



[11] Audusse et al. *J.Comput.Phys.* 228, 2009 - Chertock et al, *Numerische Mathematik* 128, 2018

Numerical examples in 2d

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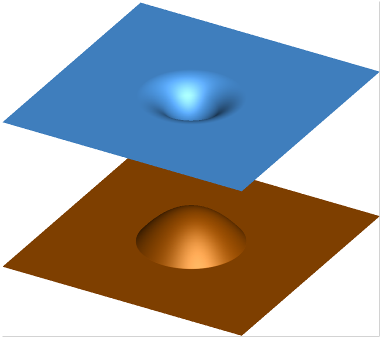

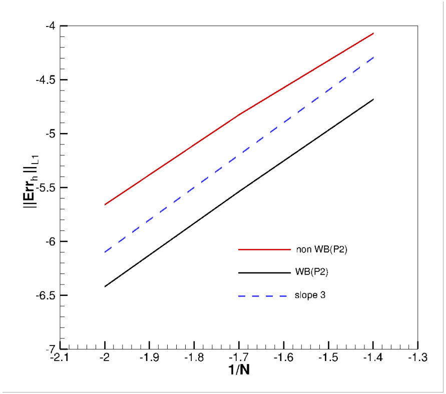

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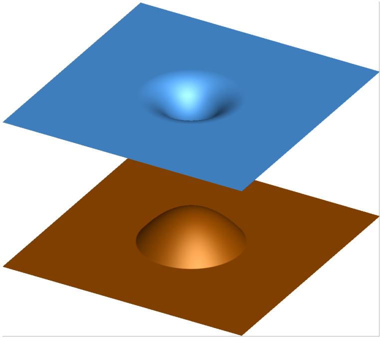
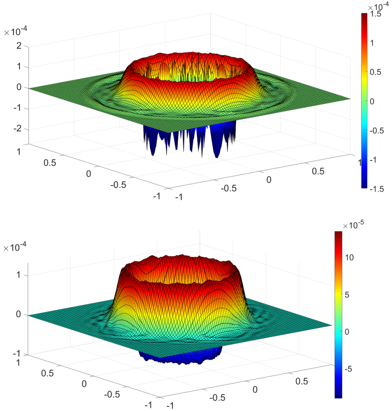


Perturbation $h = h^* + 10^{-3} e^{-100 r^2}$

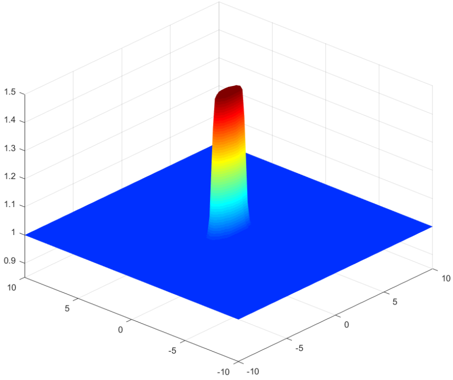


[11] Audusse et al. *J.Comput.Phys.* 228, 2009 - Chertock et al, *Numerische Mathematik* 128, 2018

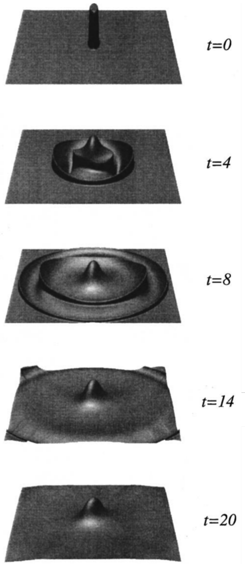
Numerical examples in 2d

2D geostropyc adjustiment[12]

Shallow water + Coriolis, non-symmetric initial free surface



Initial

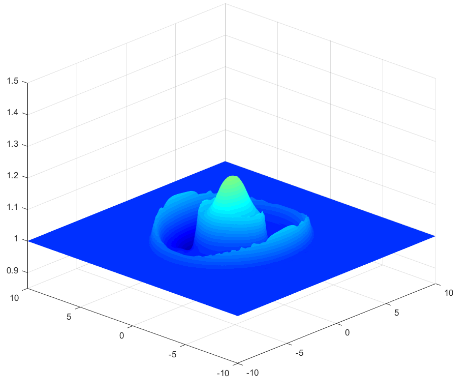


[12] Kuo & Polvani *Phys.Fl.* 12, 2000 - Castro et al *SISC* 31, 2008

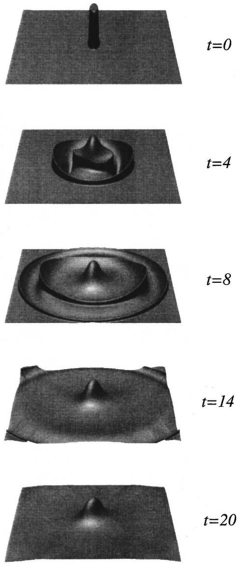
Numerical examples in 2d

2D geostropyc adjustiment[12]

Shallow water + Coriolis, non-symmetric initial free surface



Gf-dGSEM(P2), T=4

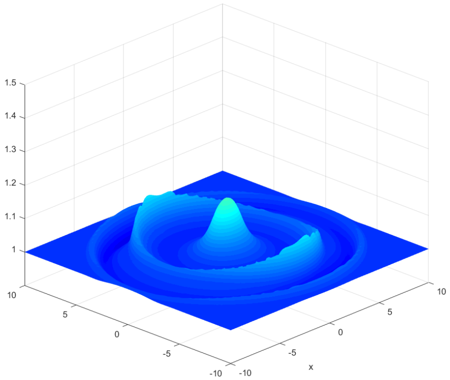


[12] Kuo & Polvani *Phys.Fl.* 12, 2000 - Castro et al *SISC* 31, 2008

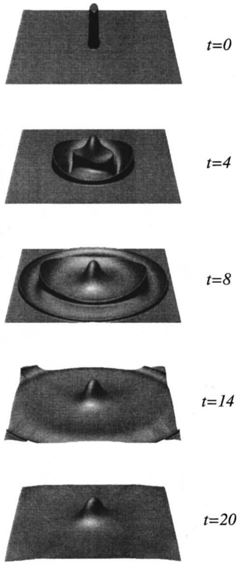
Numerical examples in 2d

2D geostropyc adjustment[12]

Shallow water + Coriolis, non-symmetric initial free surface



Gf-dGSEM(P2), T=8

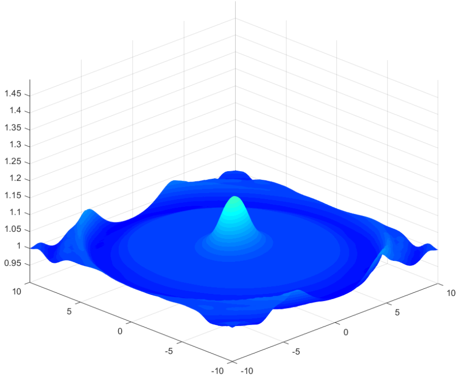


[12] Kuo & Polvani *Phys.Fl.* 12, 2000 - Castro et al *SISC* 31, 2008

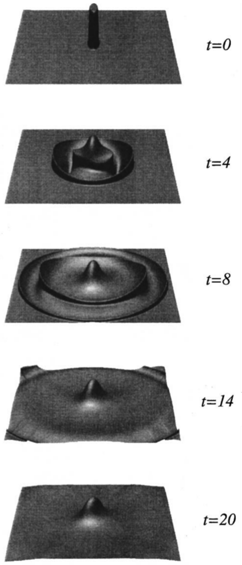
Numerical examples in 2d

2D geostropyc adjustiment[12]

Shallow water + Coriolis, non-symmetric initial free surface



Gf-dGSEM(P2), T=12

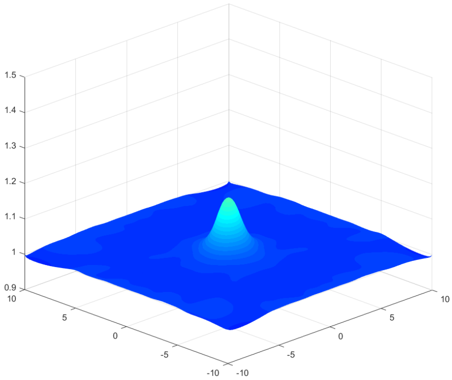


[12]Kuo & Polvani *Phys.Fl.* 12, 2000 - Castro et al *SISC* 31, 2008

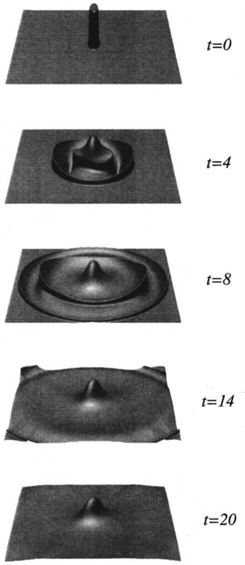
Numerical examples in 2d

2D geostropyc adjustiment[12]

Shallow water + Coriolis, non-symmetric initial free surface



Gf-dGSEM(P2), T=20



[12]Kuo & Polvani *Phys.Fl.* 12, 2000 - Castro et al *SISC* 31, 2008

Entropy conservation and global fluxes

Entropy conservation and global fluxes

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x)), \quad (1)$$

- Possible genelization of the notion of consistency wrt constants (in space) :
  1. Steady invariants
  2. Steady integral relations
  3. Global fluxes
  4. other declinations (continuous or dicrete level)...
- Well balanced scheme: discrete approximation embedding one (or more) of these notions

Remark: consistency and entropy conservation

All of the above relate to the main PDE.

Exact consistency with constant entropy flux, *viz entropy conservation* comes as an extra contraint. A well balanced approach may or may not satisfy this constraint.

Entropy conservation and global fluxes

Shallow water equations (no friction).

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + p(h) \\ huv \end{pmatrix} = -h \begin{pmatrix} 0 \\ \partial_x \varphi + \omega v \\ -\omega u \end{pmatrix}, \qquad (6)$$

Entropy conservation.

$$\partial_t \eta + \partial_x F_\eta = 0$$

where

$$\eta = p(h) + hk + h\varphi \, , \quad F_\eta = hu\,(g\zeta + k) = hu\,E \qquad (14)$$

Entropy conservation and global fluxes

Shallow water equations (no friction).

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + p(h) + r \\ huv + r_\omega \end{pmatrix} = 0$$

Conservation and consistency/steady states.

Analytical steady state

$$hu = q_0$$

$$F_\eta = q_0 E_0 \Rightarrow E = E_0$$

Global flux consistent steady state :

$$hu = q_0$$

$$K = hu^2 + p + r = K_0$$

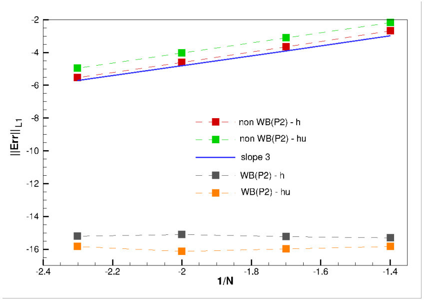
$$V_\omega = huv + r_\omega = V_0$$

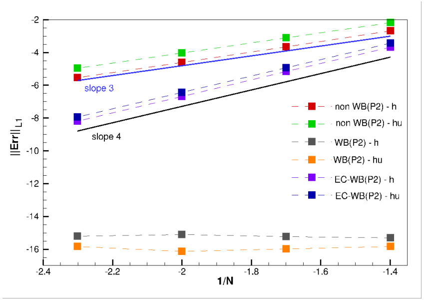
Mismatch between entropy consistent fluxes and global-flux consistency

Entropy conservation and global fluxes

Conservation and consistency/steady states.

Initial data corresponding to U^*/global flux consistency





Mismatch between entropy consistent fluxes and global-flux consistency

Entropy conservation and global fluxes

$$\partial_t \eta + \partial_x F_\eta = 0$$

Conservation and consistency/steady states.

1. How to write a scheme with some control on $\partial_t \eta$ for Gf-dGSEM
2. How to concile exactness with constant G and $\partial_t \eta = 0$ (in some sense)

Numerics: entropy conservation 1

Entropy conservative dGSEM

- Gassner *SISC* 35, 2013
- Gassner et al *Appl.Math.Comp.* 272, 2016
- Chen *J.Comput.Phys* 362, 2017
- Wen et al *J.Sci.Comp.* 83, 2020
- Chen & Shu *CSIAM Trans. Appl. Math.* 1, 2020
- Renac *J.Comput.Phys* 382, 2019
- and many others

Numerics: entropy conservation 2

Entropy conservative dGSEM

$$\int_K \phi_i \partial_t h + \int_K \phi_i \partial_x q_{\mathrm{h}} + [\phi_i(\hat{q}^*_{\mathrm{h}} - q_{\mathrm{h}})]_{\xi=0} + [\phi_i(\hat{q}^*_{\mathrm{h}} - q_{\mathrm{h}})]_{\xi=1} = 0$$

$$\int_K \frac{\phi_i}{2}(\partial_t q_{\mathrm{h}} + h_{\mathrm{h}}\partial_t u_{\mathrm{h}}) + \int_K \frac{\phi_i}{2}(\partial_x (hu^2)_{\mathrm{h}} + q_{\mathrm{h}}\partial_x u_{\mathrm{h}})$$

$$+ \int_K \phi_i g h_{\mathrm{h}} \partial_x \zeta_{\mathrm{h}} + [\phi_i(\hat{f}^*_{\mathrm{h}} - f_{\mathrm{h}})]_{\xi=0} + [\phi_i(\hat{f}^*_{\mathrm{h}} - f_{\mathrm{h}})]_{\xi=1} = 0$$

Main ingredients

1. SBP property to work (indifferently) with the strong/weak form of the prob.
2. Skew-symmetric split form to enforce kinetic energy conservation
3. Entropy conservative fluxes $\hat{F}^* = (q^*, f^*)$ to guarantee global entropy conservation:

$$[\![W]\!]^T \hat{F}^* = [\![\psi]\!] = [\![up(h)]\!]$$

Numerics: entropy conservation 2

Entropy conservative dGSEM

$$\int_K \phi_i \partial_t h + \int_K \phi_i \underline{\partial_x q_{\mathrm{h}}} + [\phi_i \underline{(\hat{q}^*_{\mathrm{h}} - q_{\mathrm{h}})}]_{\xi=0} + [\phi_i \underline{(\hat{q}^*_{\mathrm{h}} - q_{\mathrm{h}})}]_{\xi=1} = 0$$

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$$+ \int_K \phi_i g h_{\mathrm{h}} \partial_x \zeta_{\mathrm{h}} + [\phi_i (\hat{f}^*_{\mathrm{h}} - f_{\mathrm{h}})]_{\xi=0} + [\phi_i (\hat{f}^*_{\mathrm{h}} - f_{\mathrm{h}})]_{\xi=1} = 0$$

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$$\int_K \frac{\phi_i}{2}(\partial_t q_{\mathrm{h}} + h_{\mathrm{h}}\partial_t u_{\mathrm{h}}) + \int_K \underline{\frac{\phi_i}{2}(\partial_x (hu^2)_{\mathrm{h}} + q_{\mathrm{h}}\partial_x u_{\mathrm{h}})}$$

$$+ \int_K \phi_i g h_{\mathrm{h}} \partial_x \zeta_{\mathrm{h}} + [\phi_i(\hat{f}^*_{\mathrm{h}} - f_{\mathrm{h}})]_{\xi=0} + [\phi_i(\hat{f}^*_{\mathrm{h}} - f_{\mathrm{h}})]_{\xi=1} = 0$$

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Numerics: entropy conservation 2

Entropy conservative dGSEM

$$\int_K \phi_i \partial_t h + \int_K \phi_i \partial_x q_h + [\phi_i(\underline{\hat{q}_h^*} - q_h)]_{\xi=0} + [\phi_i(\underline{\hat{q}_h^*} - q_h)]_{\xi=1} = 0$$

$$\int_K \frac{\phi_i}{2}(\partial_t q_h + h_h \partial_t u_h) + \int_K \frac{\phi_i}{2}(\partial_x (hu^2)_h + q_h \partial_x u_h)$$
$$+ \int_K \phi_i g h_h \partial_x \zeta_h + [\phi_i(\underline{\hat{f}_h^*} - f_h)]_{\xi=0} + [\phi_i(\underline{\hat{f}_h^*} - f_h)]_{\xi=1} = 0$$

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$$[\![W]\!]^T \hat{F}^* = [\![\psi]\!] = [\![up(h)]\!]$$

Numerics: entropy conservation 2

Entropy conservative dGSEM

- The PDE plays an important role in the process
- Not clear how to ensure properties 2. and 3. in the global flux formulation

$$\partial_t h + \partial_x q = 0$$
$$\partial_t q + \partial_x (hu^2 + p + r) = 0$$

Main ingredients

1. SBP property to work (indifferently) with the strong/weak form of the prob.
2. Skew-symmetric split form to enforce kinetic energy conservation
3. Entropy conservative fluxes $\hat{F}^* = (q^*, f^*)$ to guarantee global entropy conservation:

$$[\![W]\!]^T \hat{F}^* = [\![\psi]\!] = [\![up(h)]\!]$$

Numerics: entropy conservation 3

Entropy correction technique[13]

[13] Abgrall *J.Comput.Phys.* 372, 2018 - Abgrall et al, *J.Comput.Phys.* 453 2022

Numerics: entropy conservation 3

Entropy correction technique[13]

$$w_i \frac{dU_i}{dt} + \Phi_i + \Psi_i^L + \Psi_i^R = 0$$

[13] Abgrall *J.Comput.Phys.* 372, 2018 - Abgrall et al, *J.Comput.Phys.* 453 2022

Numerics: entropy conservation 3

Entropy correction technique[13]

$$w_i \frac{dU_i}{dt} + \Phi_i + \Psi_i^L + \Psi_i^R + \alpha_K \int_K \partial_x \phi_i A_0 \partial_x W_\mathrm{h} = 0$$

- W are the entropy variables such that $W^T \partial U = \partial \eta$
- $A_0 = \partial U / \partial W$ is the SPD entropy Hessian inverse

[13] Abgrall *J.Comput.Phys.* 372, 2018 - Abgrall et al, *J.Comput.Phys.* 453 2022

Numerics: entropy conservation 3

Entropy correction technique[13]

$$w_i \frac{dU_i}{dt} + \Phi_i + \Psi_i^L + \Psi_i^R + \alpha_K \int\limits_K \partial_x \phi_i A_0 \partial_x W_{\mathrm{h}} = 0$$

Multiply by W_i^T, and add up over $i \in K$ and over all Ks:

$$\sum_K \sum_{i \in K} w_i W_i^T \frac{dU_i}{dt} + \sum_K \sum_{i \in K} W_i^T (\Phi_i + \Psi_i^L + \Psi_i^R) + \sum_K \mathcal{D}_K = 0$$

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Multiply by W_i^T, and add up over $i \in K$ and over all Ks:

$$\sum_K \underbrace{\sum_{i\in K} w_i W_i^T \frac{dU_i}{dt}}_{=\int_K \partial_t \eta_{\rm h}} + \sum_K \underbrace{\sum_{i\in K} W_i^T (\Phi_i + \Psi_i^L + \Psi_i^R)}_{:=\Phi_\eta^K} + \sum_K \mathcal{D}_K = 0$$

[13] Abrall *J.Comput.Phys.* 372, 2018 - Abgrall et al, *J.Comput.Phys.* 453 2022

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Multiply by W_i^T, and add up over $i \in K$ and over all Ks:

$$\int_\Omega \partial_t \eta_h + \sum_K (\Phi_\eta^K + \mathcal{D}_K) = 0$$

$$\mathcal{D}_K = \alpha_K \|\partial_x W\|^2_{L^2_{A_0}(K)}$$

[13] Abgrall *J.Comput.Phys.* 372, 2018 - Abgrall et al, *J.Comput.Phys.* 453 2022

Numerics: entropy conservation 3

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$$w_i \frac{dU_i}{dt} + \Phi_i + \Psi_i^L + \Psi_i^R + \alpha_K \int_K \partial_x \phi_i A_0 \partial_x W_\mathrm{h} = 0$$

Multiply by W_i^T, and add up over $i \in K$ and over all Ks:

$$\int_\Omega \partial_t \eta_\mathrm{h} + \sum_K (\Phi_\eta^K + \mathcal{D}_K) = 0$$

We now set

$$\Phi_\eta^K + \mathcal{D}_K = \Psi_\eta^K = \oint_{\partial_K} \hat{F}_\eta(U_\mathrm{h}) \cdot \hat{n} \quad \Rightarrow \quad \alpha_K = \frac{\Psi_\eta^K - \Phi_\eta^K}{\|\partial_x W\|^2_{L^2_{A_0}(K)}}$$

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Numerics: entropy conservation 3

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Multiply by W_i^T, and add up over $i \in K$ and over all Ks:

$$\int_\Omega \partial_t \eta_{\mathrm{h}} + \sum_K (\Phi_\eta^K + \mathcal{D}_K) = 0$$

In 1D

$$\Phi_\eta^K + \mathcal{D}_K = \Psi_\eta^K = \hat{F}\eta\big|_{\xi=1} - \hat{F}\eta\big|_{\xi=0} \quad \Rightarrow \quad \alpha_K = \frac{\Psi_\eta^K - \Phi_\eta^K}{\|\partial_x W\|^2_{L^2_{A_0}(K)}}$$

[13] Abgrall *J.Comput.Phys.* 372, 2018 - Abgrall et al, *J.Comput.Phys.* 453 2022

Numerics: entropy conservation 3

Entropy correction technique[13]

$$w_i \frac{dU_i}{dt} + \Phi_i + \Psi_i^L + \Psi_i^R + \alpha_K \int_K \partial_x \phi_i A_0 \partial_x W_h = 0$$

Proposition (Entropy conservative correction). *Let $\hat{F}_\eta$ be a consistent entropy flux, and*

$$\alpha_K := \frac{\Psi_\eta^K - \Phi_\eta^K}{\|\partial_x W\|^2_{L^2_{A_0}(K)}} \tag{14}$$

The resulting Gf-dGSEM semi-discretization

1. *verifies cell (and global) entropy conservation (time continuous)*
2. *verifies a $\mathcal{E} = \mathcal{O}(h^{p+1})$ consistency estimate*

[13] Abgrall *J.Comput.Phys.* 372, 2018 - Abgrall et al, *J.Comput.Phys.* 453 2022

Numerics: entropy conservation 3

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Entropy conservation and global fluxes

Shallow water equations (no friction).

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + p(h) + r \\ huv + r_\omega \end{pmatrix} = 0$$

Conservation and consistency/steady states.

Analytical steady state

$$hu = q_0$$

$$F_\eta = q_0 E_0 \Rightarrow E = E_0$$

Global flux consistent steady state :

$$hu = q_0$$

$$K = hu^2 + p + r = K_0$$

$$V_\omega = huv + r_\omega = V_0$$

Mismatch between entropy consistent fluxes and global-flux consistency

Entropy correction with global fluxes 1

Alternative definitions of a numerical entropy flux

Solution 1: approximation consistent with analytical entropy flux, e.g.

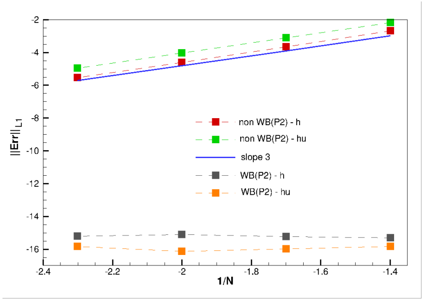
$$\hat{F}_\eta = \lambda F_\eta(U^-;\varphi^-) + (1-\lambda)F_\eta(U^+;\varphi^+)$$

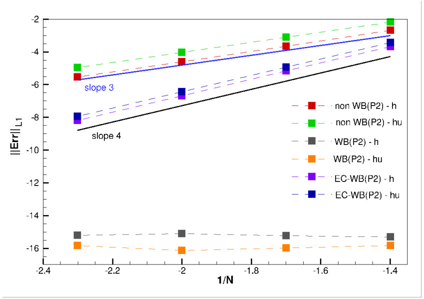
- globally entropy conservative (time continuous)
- exact for analytical steady data but not for constant global fluxes
- the correction term spoils the underlying consistency condtion

Entropy conservation and global fluxes

Conservation and consistency/steady states.

Initial data corresponding to U^*/global flux consistency





Mismatch between entropy consistent fluxes and global-flux consistency

Entropy correction with global fluxes 2

Alternative definitions of a numerical entropy flux

Solution 2: a global flux-consistent approximation.

First set $F^*_{\eta,0} = F_\eta(U_0)$ (left hand of the domain). Then $\forall \{K_j\}_{j\geq 1}$ do

1. Set $(F^*_\eta)_0 = F^*_\eta(U^-) + [\![F_\eta(U_\mathrm{h};\varphi)]\!] = F^*_\eta(x_p)_{K_{j-1}} + [\![F_\eta(U_\mathrm{h};\varphi)]\!]$
2. Compute: $F^*_\eta(x) = (F^*_\eta)_0 + \int_{x_0}^{x} W_\mathrm{h}^T \partial_x G_h$

Entropy correction with global fluxes 2

Alternative definitions of a numerical entropy flux

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2. Compute: $F^*_\eta(x) = (F^*_\eta)_0 + \int_{x_0}^{x} W_\mathrm{h}^T \partial_x G_h$

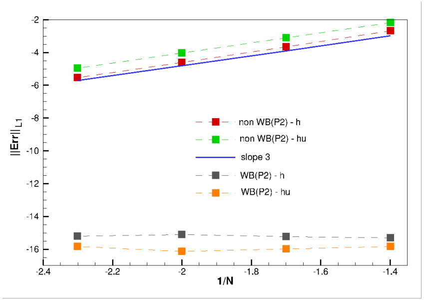
If we set $\hat{F}_\eta = \lambda(F^*_\eta)^- + (1-\lambda)(F^*_\eta)^+$, the Gf-dGSEM obtained is

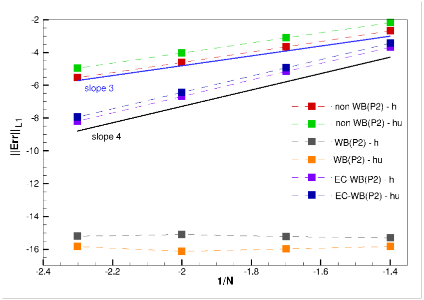
- globally entropy conservative (time continuous)
- compatible with constant global flux as long as φ is continuously approximated

Well balanced and entropy conservative results 1

Conservation and consistency/steady states.

Initial data corresponding to U^*/global flux consistency





Mismatch between entropy consistent fluxes and global-flux consistency

Well balanced and entropy conservative results 1

Conservation and consistency/steady states.

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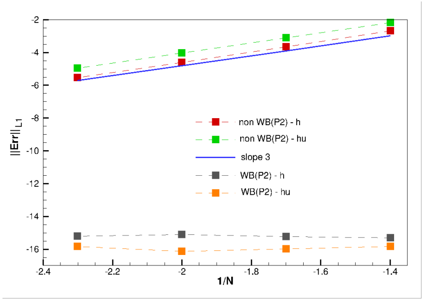
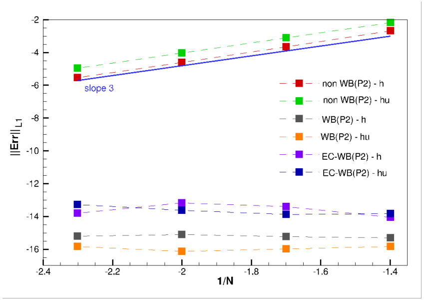
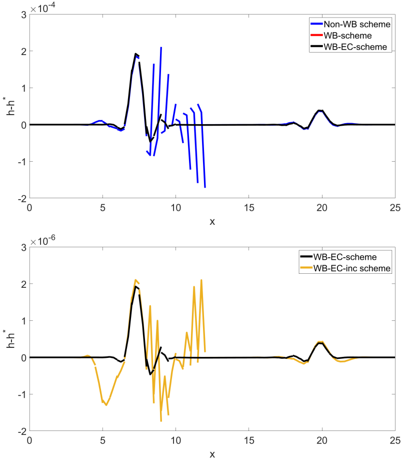


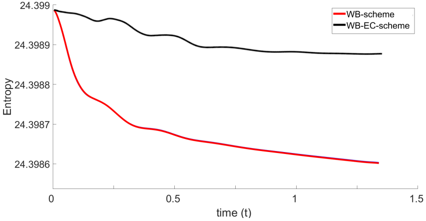


We can choose the EC correction depending on the initial data

Well balanced and entropy conservative results 2

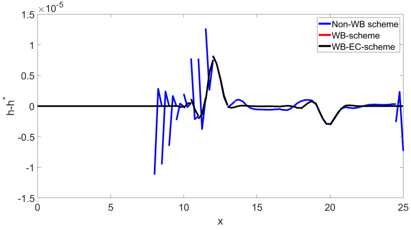
Moving steady state: sub-critical

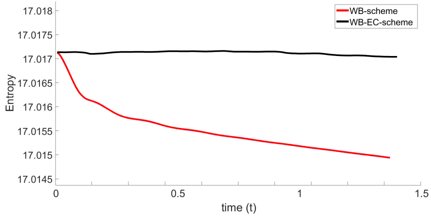


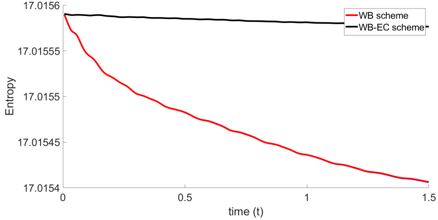


Well balanced and entropy conservative results 3

Moving steady state: super-critical







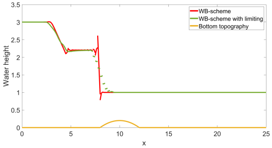
Summary

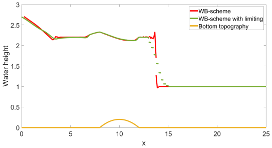
- Family of schemes discretely well balanced/super-convergent
- Agnostic of the particular equilibrium in 1D
- Relation between super-convergent behaviour and underlying ODE integrator
- Measurable net improvements for some 2D tests
- Correction allowing global (time continuous) entropy preservation compatible with both analytical or global flux initialization

Extensions future work

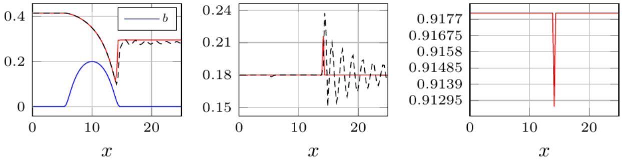
Nonlinear formulations

Troubled cell indicator/P0 switch:





Gf-WENO (with M. Ciallella & D. Torlo):



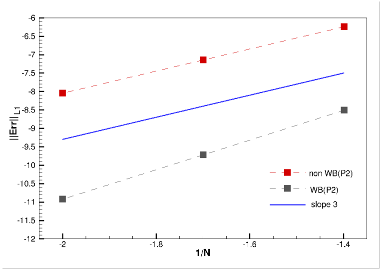
(a) z (b) q (c) K

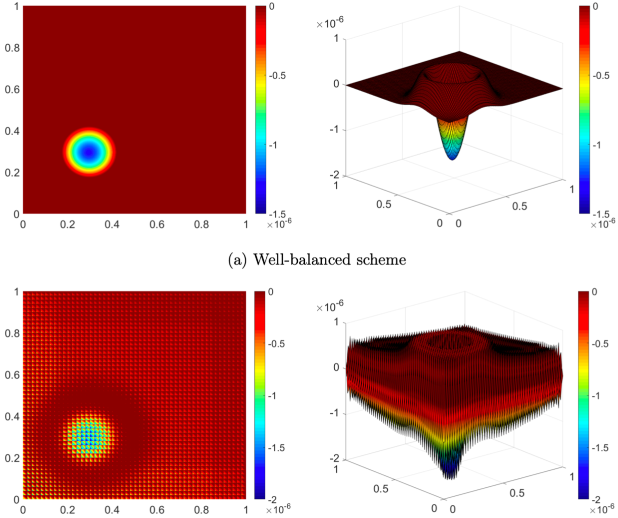
Transcritical flow: characteristic variables computed by GF-WENO5 (red) and WENO5 (black)

Extensions future work

Euler Equations with gravity

Isothermal eq. solution[14]





(a) Well-balanced scheme

(b) Non-well-balanced scheme

[14]See e.g. Chertock et a *J.Comput.Phys.* 358, 2018

Extensions future work

Ongoing/future work

- Well balanced cGSEM (with R. Abgrall, L. Micalizzi, S. Michel, & D. Torlo)
- Fully discrete entropy conservative ADER + relaxation (with E. Gaburro, P. Öffner & D. Torlo)
- Entropy controlled with a-posteriori limiter (with E. Gaburro, P. Öffner & D. Torlo)
- Sources depending on time derivatives: dispersive PDEs (with W. Barsukow and D. Torlo)
- Genuinely 2D : $G =$ const $\neq \nabla \cdot G = 0$ ...

.. Obrigado