



11^{ème} Biennale de la
Société des Mathématiques
Appliquées et Industrielles

**Bore propagation in channels with sloping banks:
numerical and asymptotic analysis**

M. Ricciuto

Centre Inria de l'Université de Bordeaux, Team CARDAMOM

<https://team.inria.fr/cardamom/>

Inria

du 22 au 26
Mai 2023



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**Bore propagation in channels with sloping banks:
numerical and asymptotic analysis**

Joint work with:

P. Bonneton (EPOC), **R. Chassagne** (LEGI/IRSTEA),

A.G. Filippini (BRGM/R3C), **M. Kazolea** (Inria/CARDAMOM)



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Intro

Bore propagation in channels with sloping banks:
numerical and asymptotic analysis





Saint Pardon, Dordogne river

<https://vimeo.com/106090912>, Jean-Marc Chauvet, Septembre 2014



Severn River - England



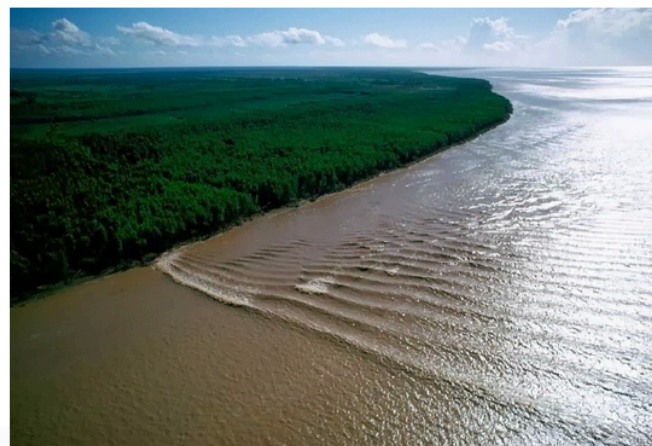
Severn River - England



Gironde - France



Qiantang River - China



Amazon River - Brazil



Kampar River - Sumatra

Tidal bores bear striking similarity to tsunami bores, and bores generated in laboratory experiments



**Naka river at Hitachinaka city, Japan
2011 Tohoku Tsunami**



**Sunaoshi River in Tagajo city, Japan
7.4 earthquake 21/11/2016**



**Sendai bay, Japan
2011 Tohoku Tsunami**

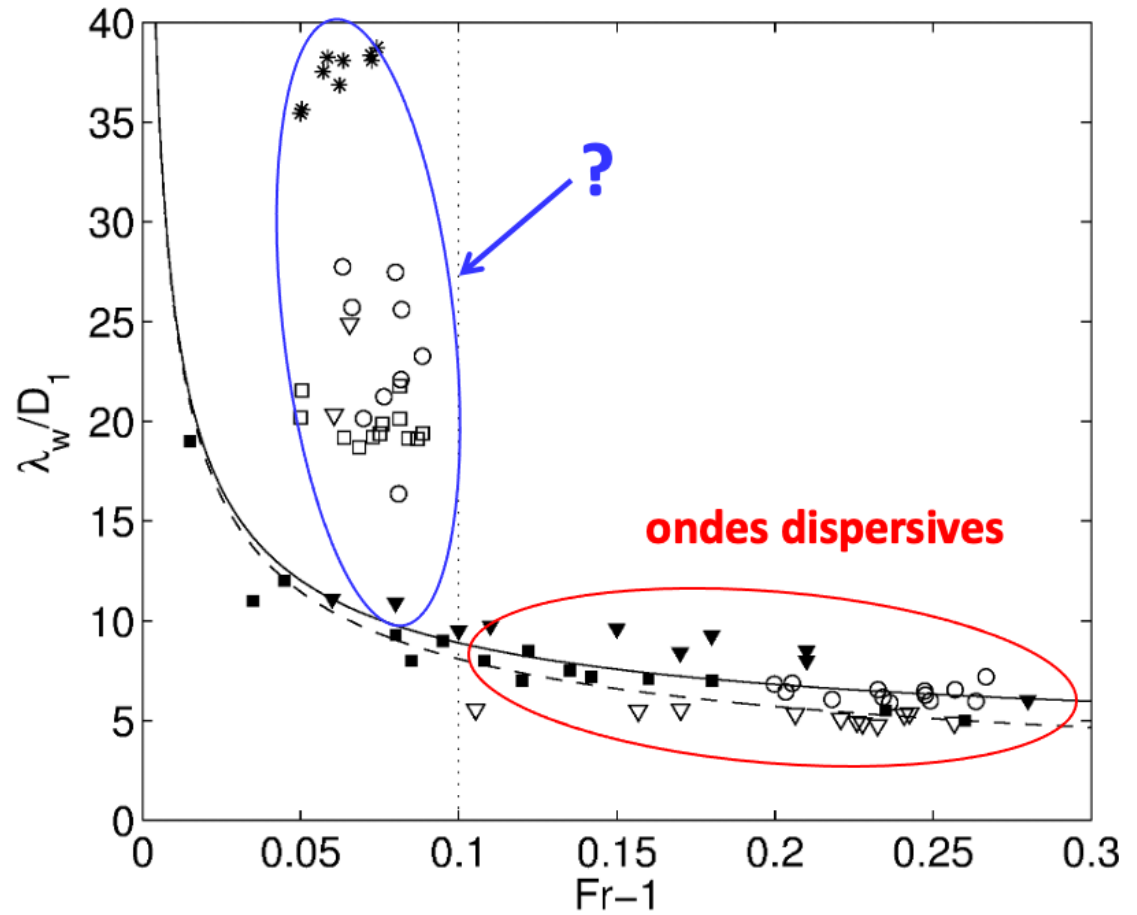
Low Fr transition in Seine and Gironde: the unseen Mascaret

3 field campaigns :
a unique long-term high-frequency database

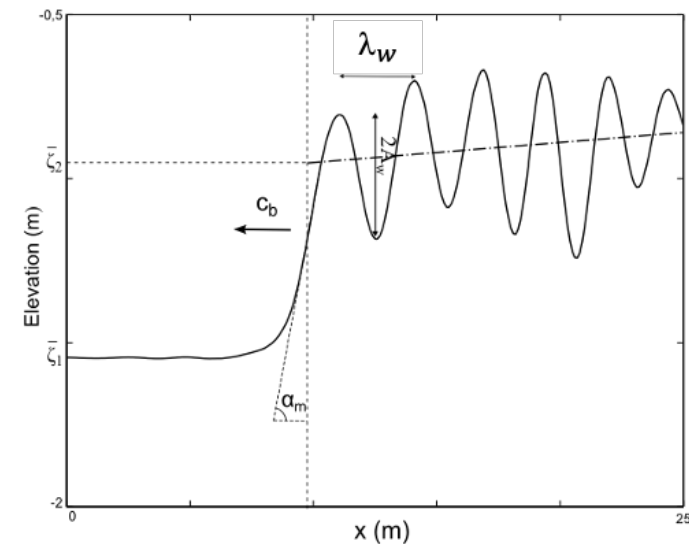


Bonneton et al, *Comptes Rendus Geoscience*, 2012

Bonneton et al, *J. Geophysical Research - Oceans*, 2015



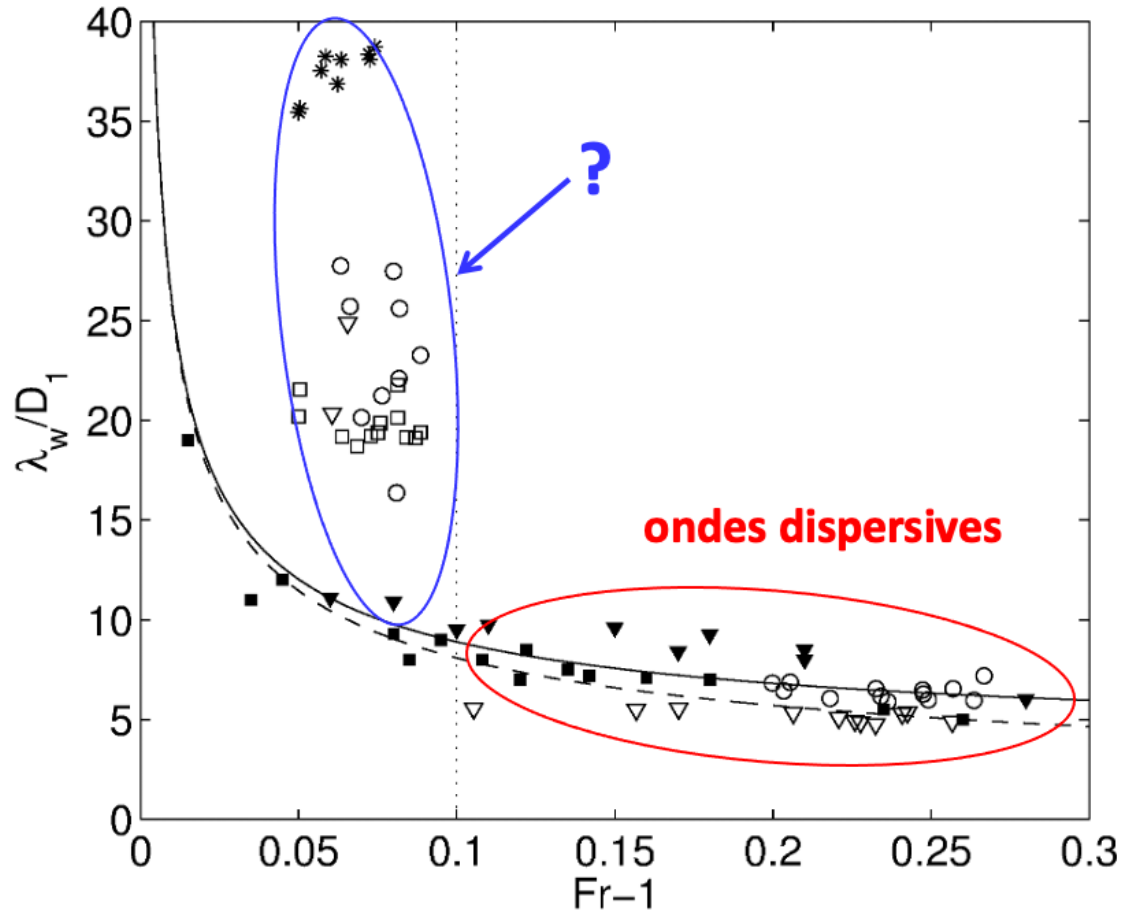
Transition for $Fr=F_T$ ($F_T \approx 1.1$)



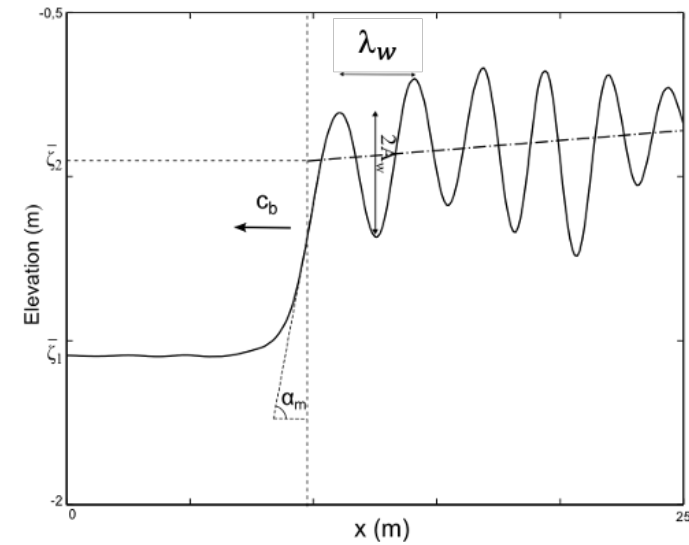
Ressaut de marée (n.lle terminologie **Bonneton et al,**)

- not visible naked eyes
- mechanism not known

*Comptes Rendus
Geoscience, 2012*



Transition for $Fr = F_T$ ($F_T \approx 1.1$)

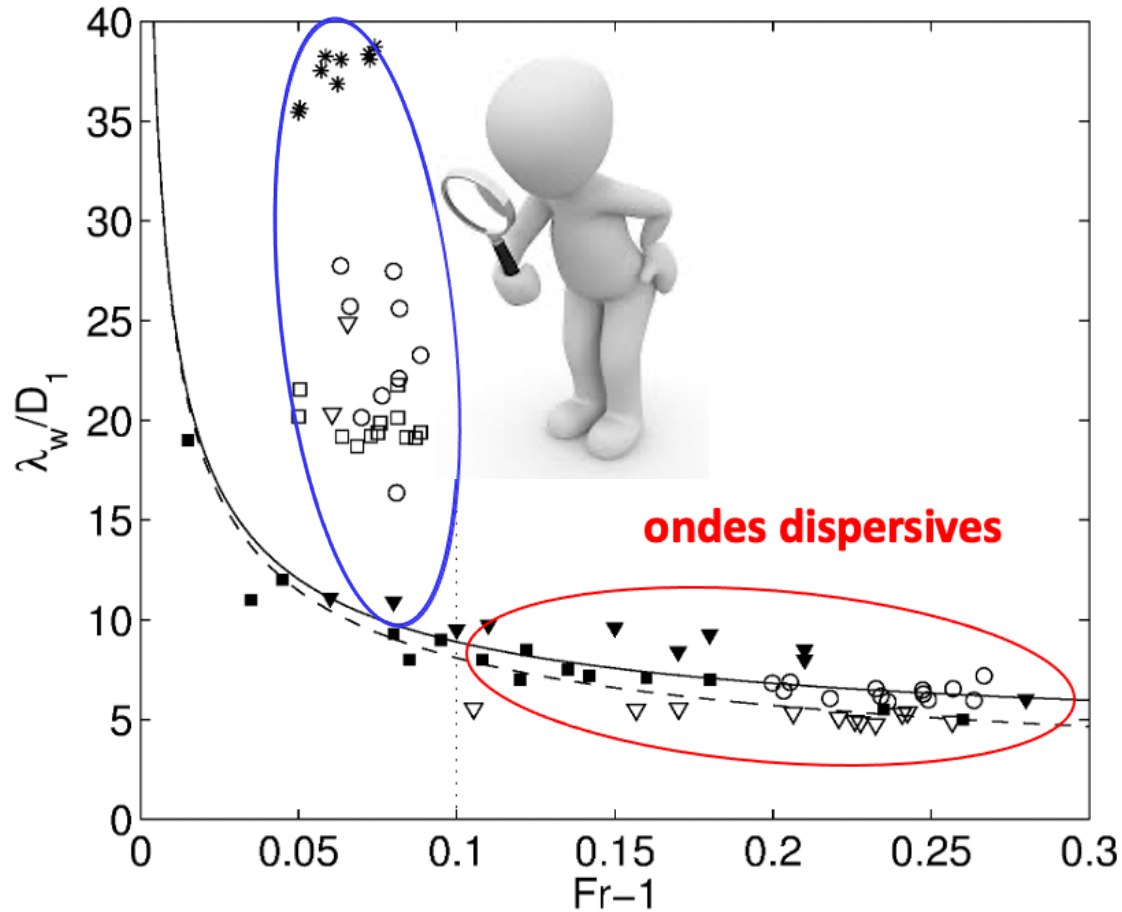


Common undular tidal bore (mascaret):
Favre wave (cf later)

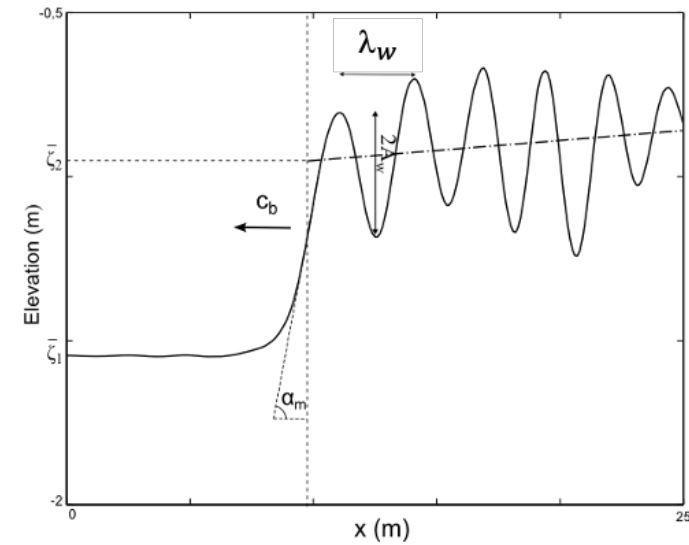
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Transition for $Fr = F_T$ ($F_T \approx 1.1$)



Common undular tidal bore (mascaret):
Favre wave (cf later)

1.

1.a Undular bores in rectangular channels : non-linearity vs dispersion

1.b Favre experiments in trapezoidal channels: low Froude transition

2.

2.a Modelling: asymptotic weakly nonlinear dispersive models

2.b Numerical approximation in multi-D

2.c Simulation results

3.

3.a Main ansatz

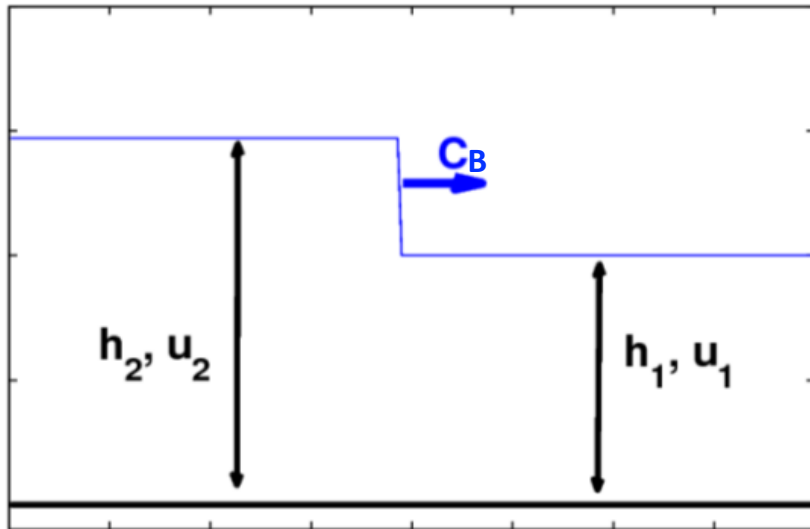
3.b Asymptotic analysis

3.c Physical validation

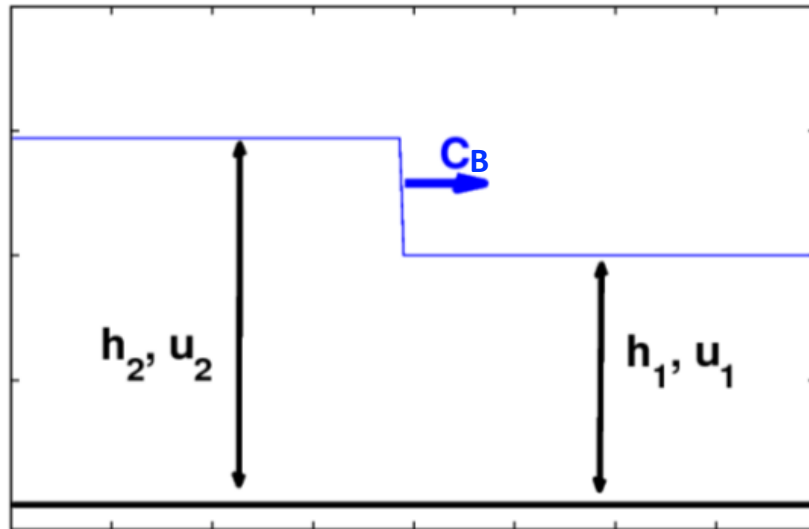
4. Conclusion/perspectives

Nonlinearity vs dispersion

Bore: positive surge or hydraulic jump in translation



Bore: positive surge or hydraulic jump in translation



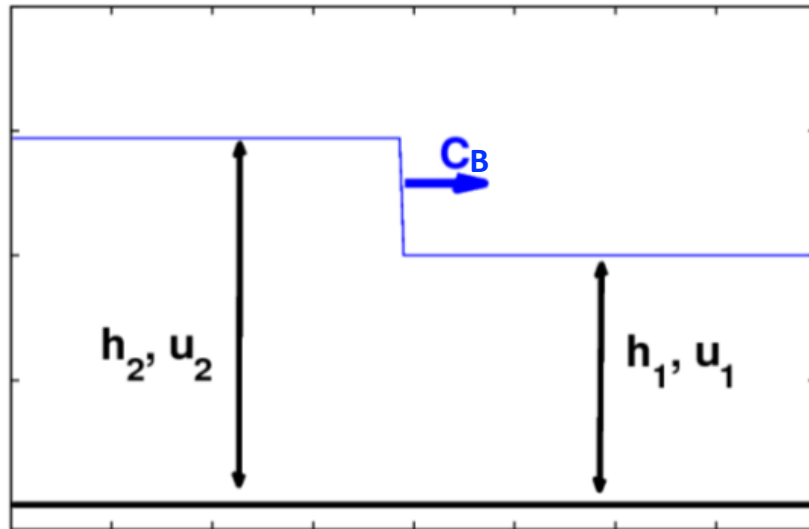
Shallow water equations (hydrostatic, shallow limit)

- 1.
- 2.
- 3.
- 4.

$$\partial_t h + \partial_x (hu) = 0$$

$$\partial_t (hu) + \partial_x (hu^2 + gh^2/2) = 0$$

Bore: positive surge or hydraulic jump in translation



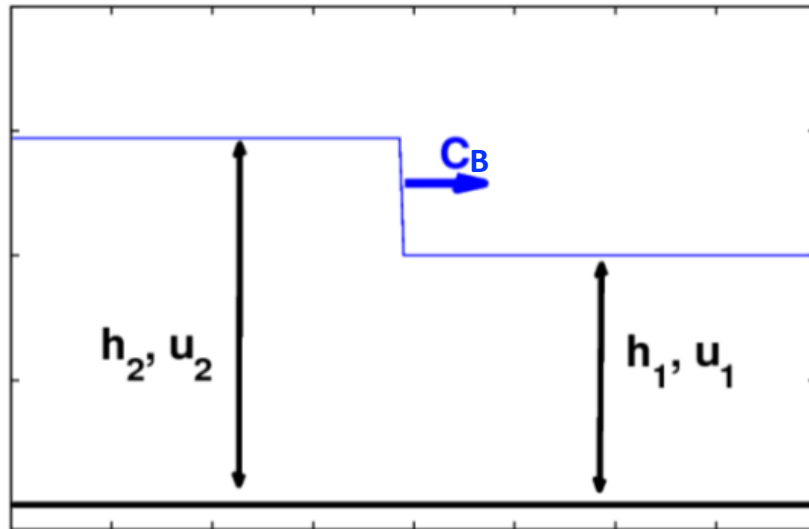
Shallow water equations (hydrostatic, shallow limit)

1. hyperbolic
- 2.
- 3.
- 4.

$$\partial_t \begin{pmatrix} h \\ hu \end{pmatrix} + A \partial_x \begin{pmatrix} h \\ hu \end{pmatrix} = 0$$

$$A = R \text{diag}(u - c, u + c) R^{-1}, \quad c = \sqrt{gh}$$

Bore: positive surge or hydraulic jump in translation



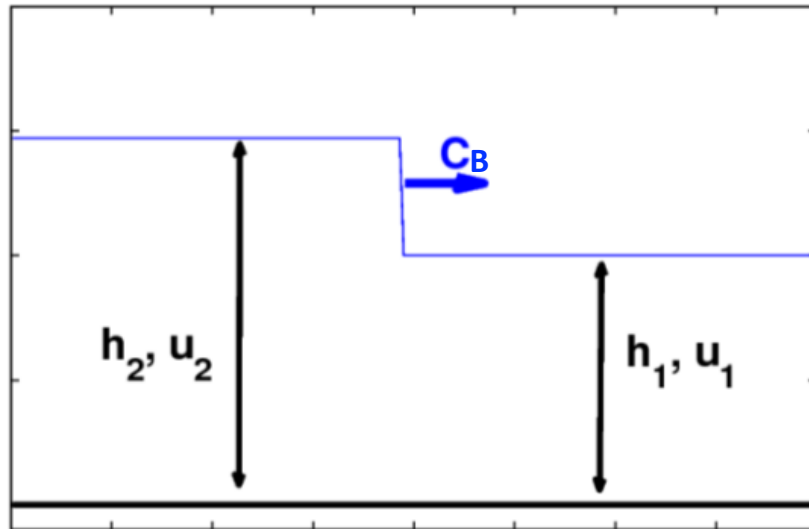
Shallow water equations (hydrostatic, shallow limit)

1. hyperbolic
2. mathematical entropy (energy)
- 3.
- 4.

$$\underbrace{\partial_t \left(gh^2/2 + hu^2/2 \right)}_E + \partial_x \underbrace{\left[hu(gh + u^2/2) \right]}_{F_E} \leq 0$$

= 0 only for fully continuous flows/solutions (cf. later)

Bore: positive surge or hydraulic jump in translation



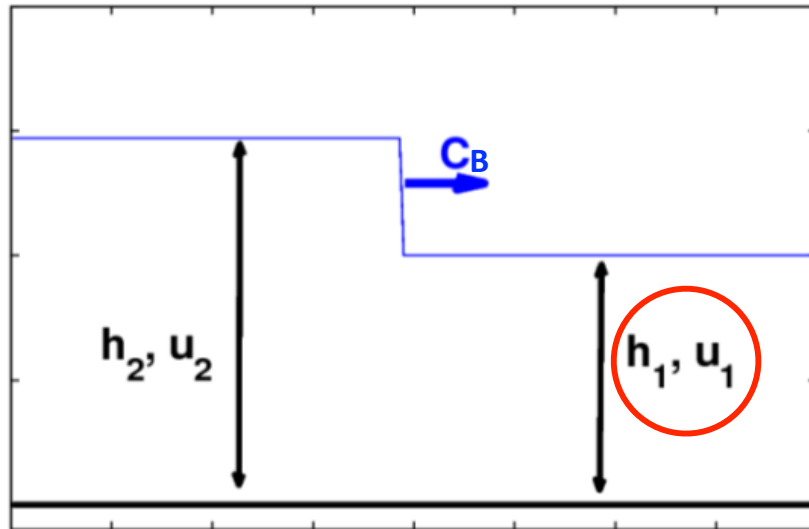
- Shallow water equations (hydrostatic, shallow limit)
1. hyperbolic
 2. mathematical entropy (energy)
 3. discontinuities (bores): Rankine-Hugoniot conditions
 - 4.

Conservation (Mass and momentum)

$$C_B(h_2 - h_1) = (h_2u_2 - h_1u_1)$$

$$C_B(h_2u_2 - h_1u_1) = (h_2u_2^2 - h_1u_1^2 + gh_2^2/2 - gh_1^2/2)$$

Bore: positive surge or hydraulic jump in translation



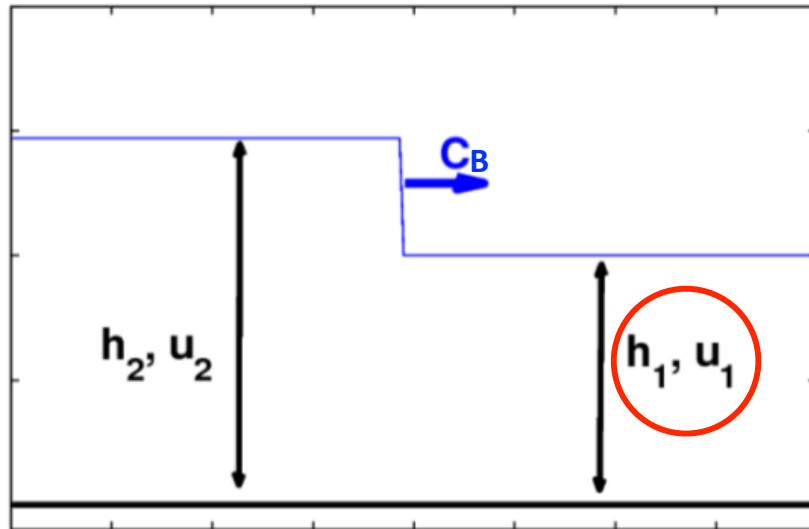
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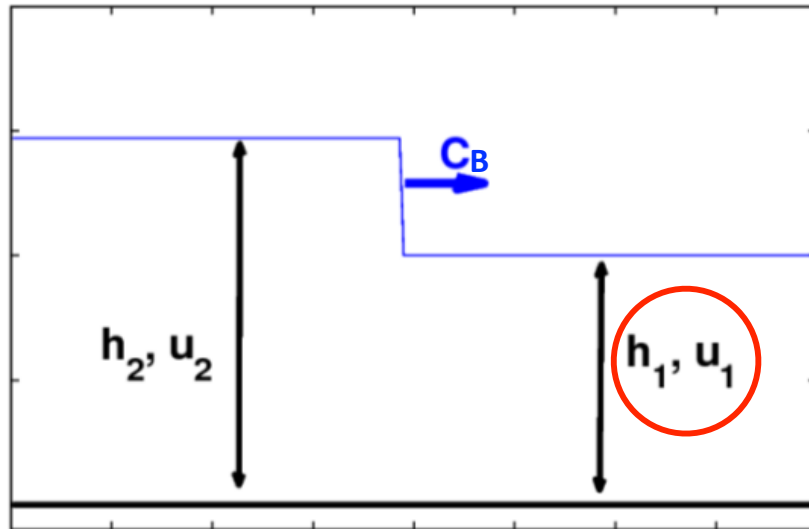
Conservation (Mass and momentum)

$$Fr := \frac{u_1 - C_B}{\sqrt{gh_1}} \Rightarrow C_B \equiv Fr$$

$$\frac{h_2}{h_1} = \frac{\sqrt{1 + 8Fr^2} - 1}{2} \Rightarrow \frac{h_2}{h_1} \equiv Fr$$

Bore strength

Bore: positive surge or hydraulic jump in translation



- Shallow water equations (hydrostatic, shallow limit)
1. hyperbolic
 2. mathematical entropy (energy)
 3. discontinuities (bores): Rankine-Hugoniot conditions
 - 4.

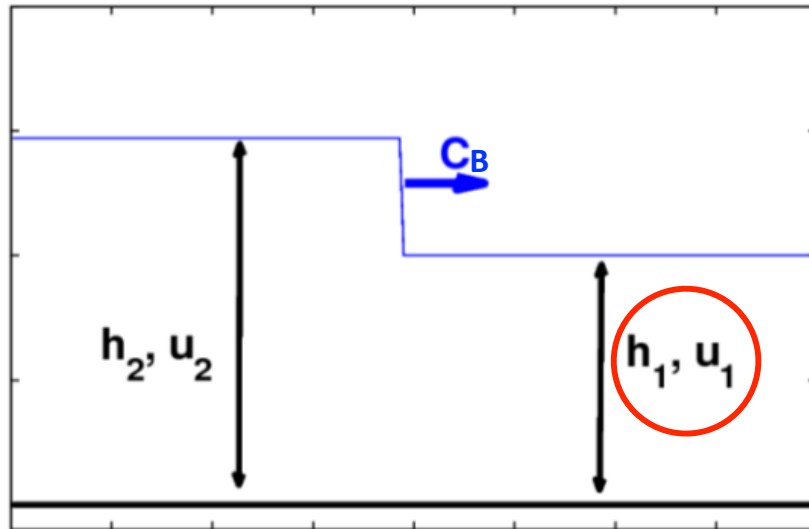
**Conservation (Mass and momentum)
and
Dissipation (Energy/entropy)**

$$C_B(h_2 - h_1) = (h_2u_2 - h_1u_1)$$

$$C_B(h_2u_2 - h_1u_1) = (h_2u_2^2 - h_1u_1^2 + gh_2^2/2 - gh_1^2/2)$$

$$D_B := C_B(E_2 - E_1) - (F_{E2} - F_{E1}) = -\frac{g}{4} \sqrt{\frac{g\bar{h}}{h_1h_2}} (h_2 - h_1)^3 < 0$$

Bore: positive surge or hydraulic jump in translation

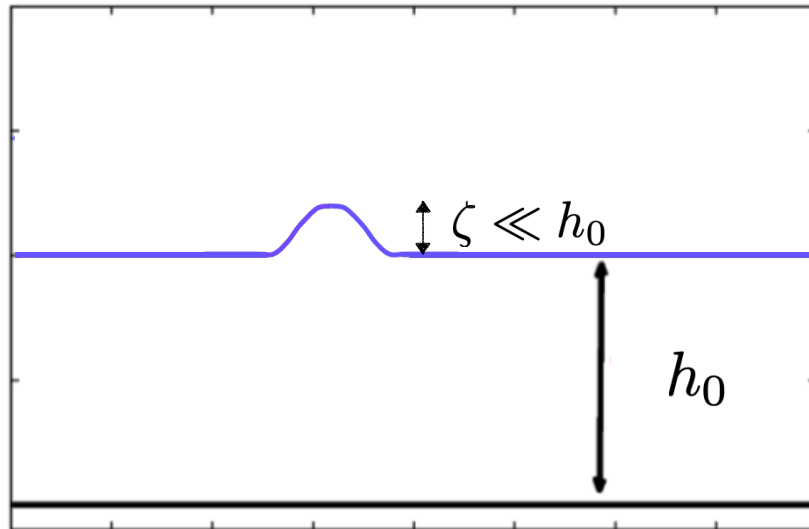


- Shallow water equations (hydrostatic, shallow limit)
1. hyperbolic
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 3. discontinuities (bores): Rankine-Hugoniot conditions
 4. linear propagation characteristics

$$\partial_t h + \partial_x(hu) = 0$$

$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2) = 0$$

Bore: positive surge or hydraulic jump in translation



- Shallow water equations (hydrostatic, shallow limit)
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Linearized eq.s

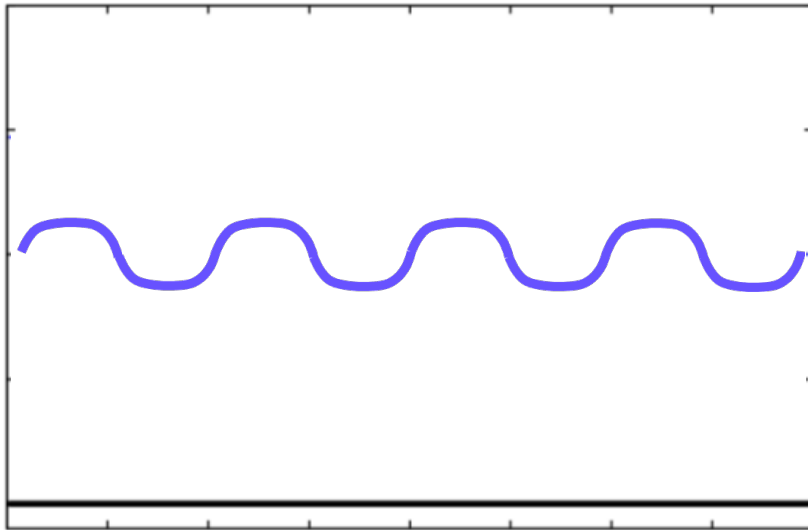
$$h = h_0 + \zeta$$

$$\zeta \ll h_0$$

$$\partial_t \zeta + h_0 \partial_x u = 0$$

$$\partial_t u + g \partial_x \zeta = 0$$

Bore: positive surge or hydraulic jump in translation



- Shallow water equations (hydrostatic, shallow limit)
1. hyperbolic
 2. mathematical entropy (energy)
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 4. linear propagation characteristics

Linearized eq.s

Fourier mode

$$\zeta = \zeta^* e^{i(\kappa x + \nu t)}$$

$$u = u^* e^{i(\kappa x + \nu t)}$$

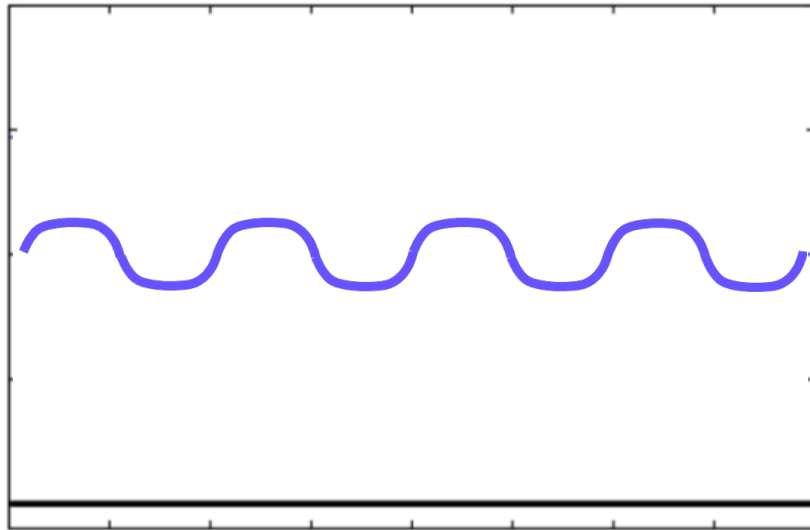
$$\nu = \omega + i\sigma$$

$$\partial_t \zeta + h_0 \partial_x u = 0$$

$$\partial_t u + g \partial_x \zeta = 0$$

$$\kappa = \frac{2\pi}{\lambda} \quad \text{wavenumber}$$

Bore: positive surge or hydraulic jump in translation



- Shallow water equations (hydrostatic, shallow limit)
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Linearized eq.s

Fourier mode

$$\zeta = \zeta^* e^{i\kappa(x \pm c_0 t)}$$

$$u = u^* e^{i\kappa(x \pm c_0 t)}$$

a) No damping

b) Constant celerity c_0

$$\omega = \pm \kappa c_0, \quad \sigma = 0$$

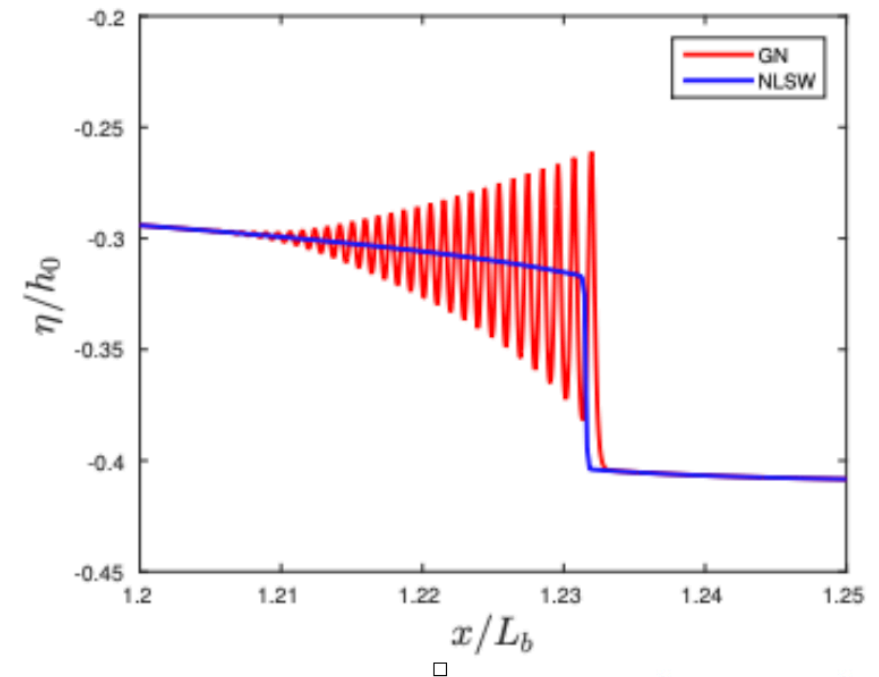
$$c_0 = \sqrt{gh_0}$$

No dispersion ...

In fluid dynamics, **dispersion** of water waves generally refers to frequency **dispersion**, which means that **waves** of different wavelengths travel at different phase speeds. Water **waves**, in this context, are **waves** propagating on the water surface, with gravity and surface tension as the restoring forces.

[Dispersion \(water waves\) - Wikipedia](https://en.wikipedia.org/wiki/Dispersion_(water_waves))

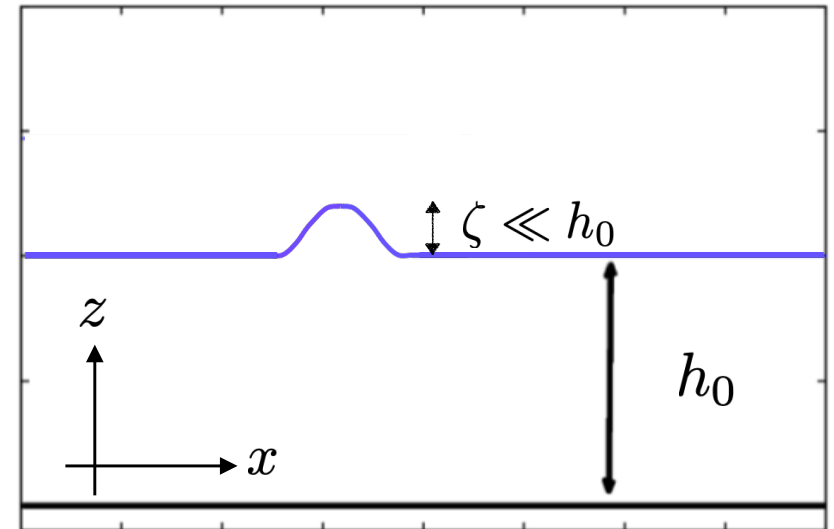
[https://en.wikipedia.org/wiki/Dispersion_\(water_waves\)](https://en.wikipedia.org/wiki/Dispersion_(water_waves))



$$\zeta = \zeta^* e^{i\kappa(x - c(\kappa)t)}$$

Euler equations (Airy theory)

$$\begin{aligned} \nabla \cdot \vec{v} &= 0 \\ \partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} + \nabla \tilde{p} &= -g \vec{1}_z \\ \partial_t \zeta + v_x \partial_x \zeta &= v_z & z = \zeta \\ \tilde{p} &= 0 & z = \zeta \\ v_z &= 0 & z = -h_0 \end{aligned}$$

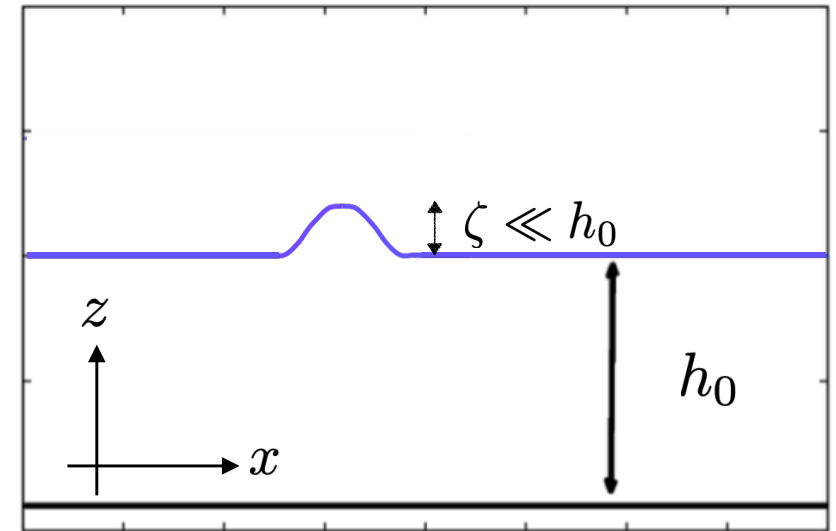


$$\zeta = \zeta^* e^{i\kappa(x - c(\kappa)t)}$$

Euler equations (Airy theory)

$$\begin{aligned} \nabla \times \vec{v} &= 0 \\ \vec{v} &= \nabla \Phi \end{aligned}$$

$$\begin{aligned} \Delta \Phi &= 0 & z = \zeta \\ \partial_t \Phi + \frac{1}{2} \|\nabla \Phi\|^2 + g\zeta &= 0 & z = \zeta \\ \partial_t \zeta + \partial_x \Phi \partial_x \zeta &= \partial_z \Phi & z = \zeta \\ \partial_z \Phi &= 0 & z = -h_0 \end{aligned}$$



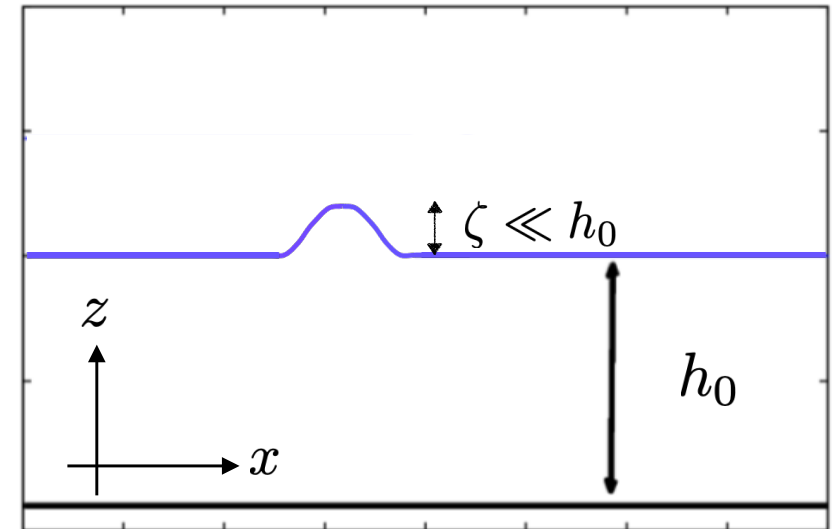
Non-linear potential equations

$$\zeta = \zeta^* e^{i\kappa(x - c(\kappa)t)}$$

Euler equations (Airy theory)

$$\begin{array}{l} \zeta \\ \|\nabla\zeta\| \\ \|\nabla\Phi\| \\ \text{etc} \end{array} \ll 1$$

$$\begin{array}{ll} \Delta\Phi = 0 & \\ \partial_t\Phi + g\zeta = 0 & z = \zeta \\ \partial_t\zeta = \partial_z\Phi & z = \zeta \\ \partial_z\Phi = 0 & z = -h_0 \end{array}$$



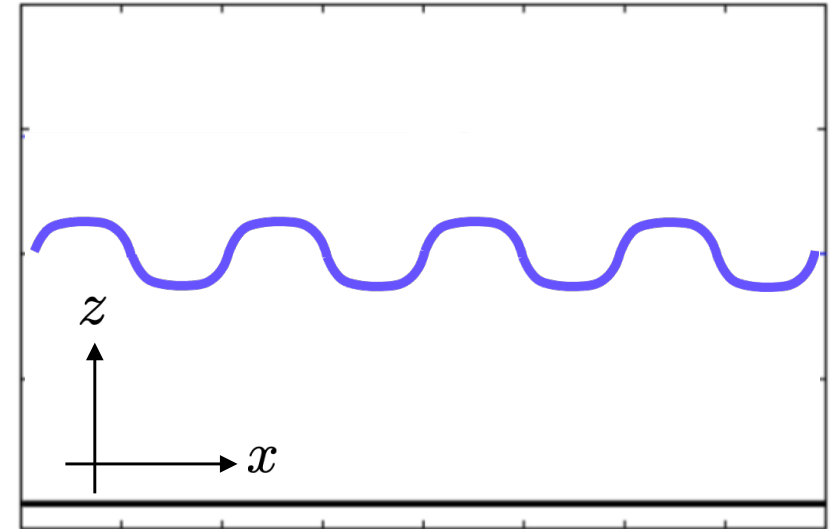
Linear potential equations

$$\zeta = \zeta^* e^{i\kappa(x - c(\kappa)t)}$$

Euler equations (Airy theory)

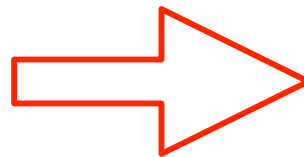
$$\left[\begin{array}{l} \zeta \\ \|\nabla\zeta\| \\ \|\nabla\Phi\| \\ \text{etc} \end{array} \right] \ll 1$$

$$\begin{aligned} \Delta\Phi &= 0 \\ \partial_t\Phi + g\zeta &= 0 & z = \zeta \\ \partial_t\zeta &= \partial_z\Phi & z = \zeta \\ \partial_z\Phi &= 0 & z = -h_0 \end{aligned}$$



Linear potential equations

$$\begin{aligned} \Phi &= B(z)\Phi^* e^{i(kx + \nu t)} \\ \zeta &= \zeta^* e^{i(kx + \nu t)} \\ \zeta^* &\ll 1 \end{aligned}$$



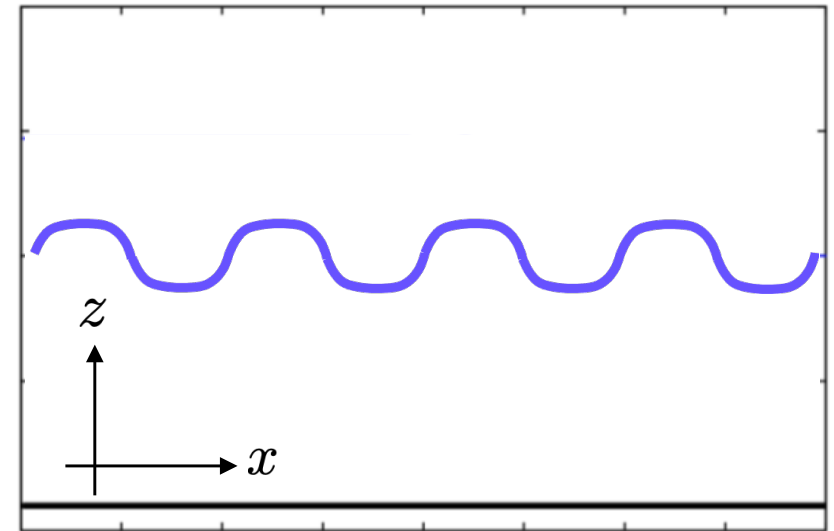
$$c^2(\kappa) = gh_0 \frac{\tanh(\kappa h_0)}{\kappa h_0}$$

$$\zeta = \zeta^* e^{i\kappa(x - c(\kappa)t)}$$

Euler equations (Airy theory)

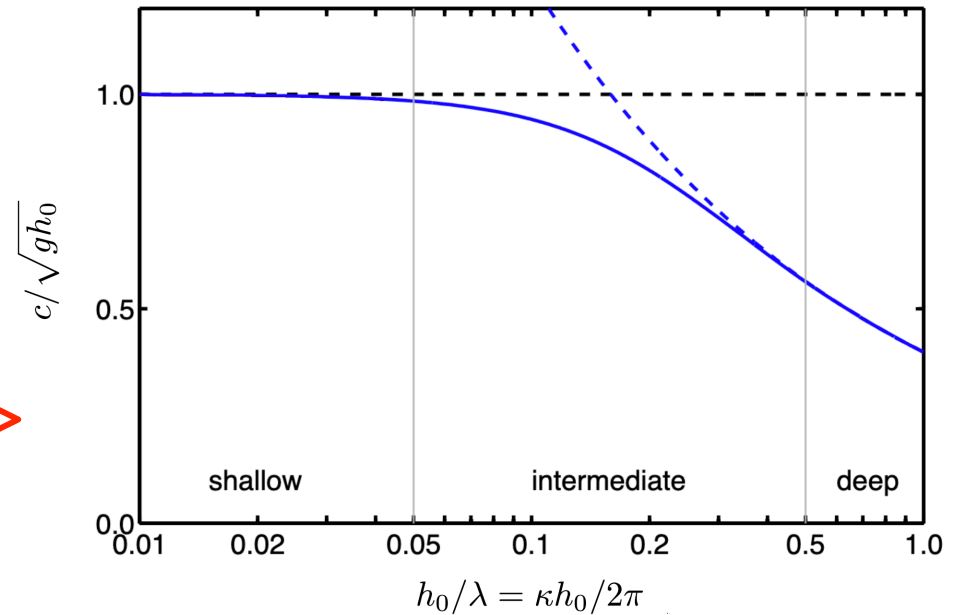
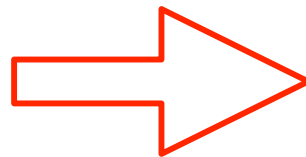
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$$\begin{aligned} \Delta\Phi &= 0 \\ \partial_t\Phi + g\zeta &= 0 & z = \zeta \\ \partial_t\zeta &= \partial_z\Phi & z = \zeta \\ \partial_z\Phi &= 0 & z = -h_0 \end{aligned}$$



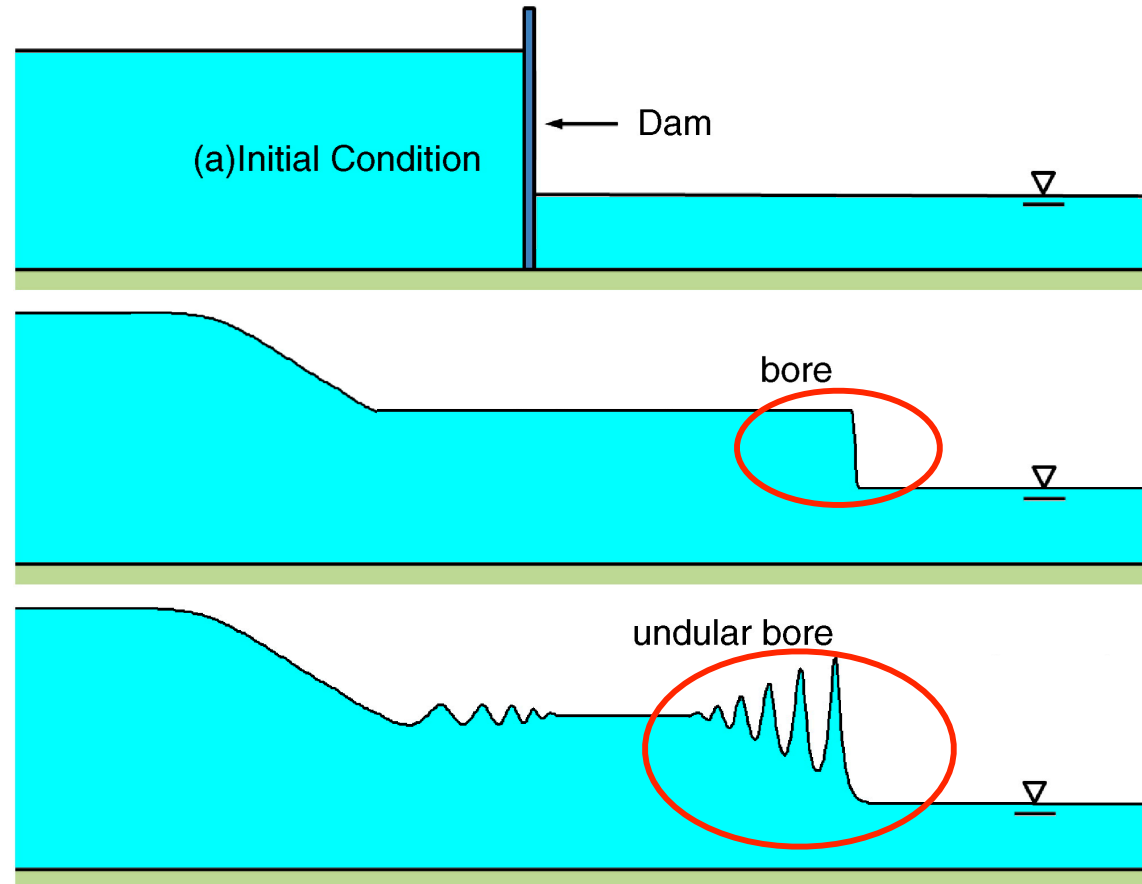
Linear potential equations

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$$\zeta = \zeta^* e^{i\kappa(x - c(\kappa)t)}$$

Bores in rectangular channels (1D/no banks)



Bores in rectangular channels (1D/no banks)

□ Mathematical and physical theories:

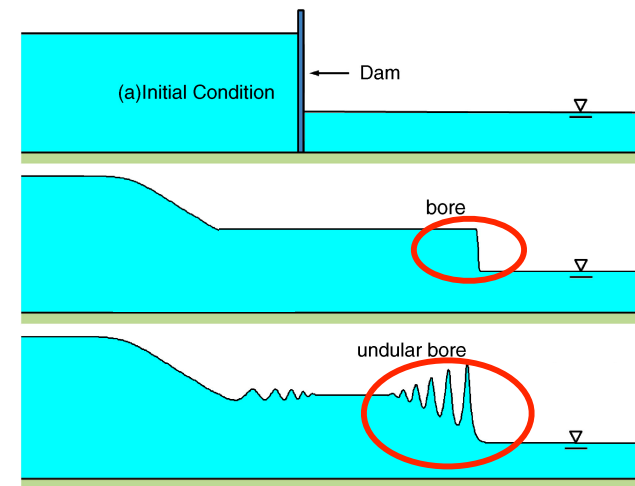
Rayleigh 1914, Lemoine 1948, Benjamin & Lighthill 1954, Serre 1954, Johnson 1970, Gurevich & Pitaevskii 1973, El et al. 2006, Congy et al 2021, and many others

□ Laboratory experiments:

Favre 1935, Sandover and Zienkiewics 1957, Bennet & Cunge 1971, Treske 1994, Chanson 1996 & 2009, Soares Frazao and Zech 2002, Simon 2013, Furgerot 2014, David et al. 2014, and many others ...

□ Numerical simulations:

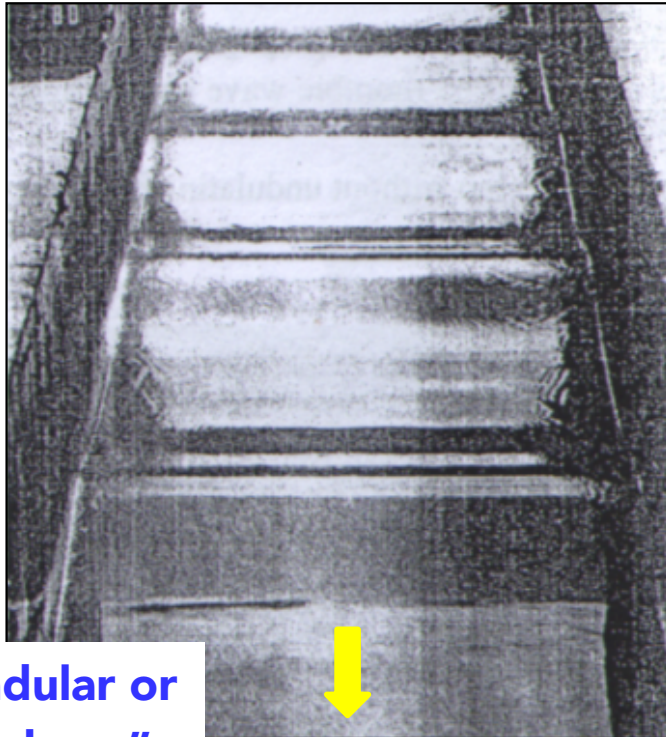
Peregrine 1966, Wei et al. 1995, Soares Frazao and Zech 2002, Lubin et al. 2010, Pan & Lu 2011, Tissier et al. 2011, Simon 2013, Filippini et al. 2019, and many others ...



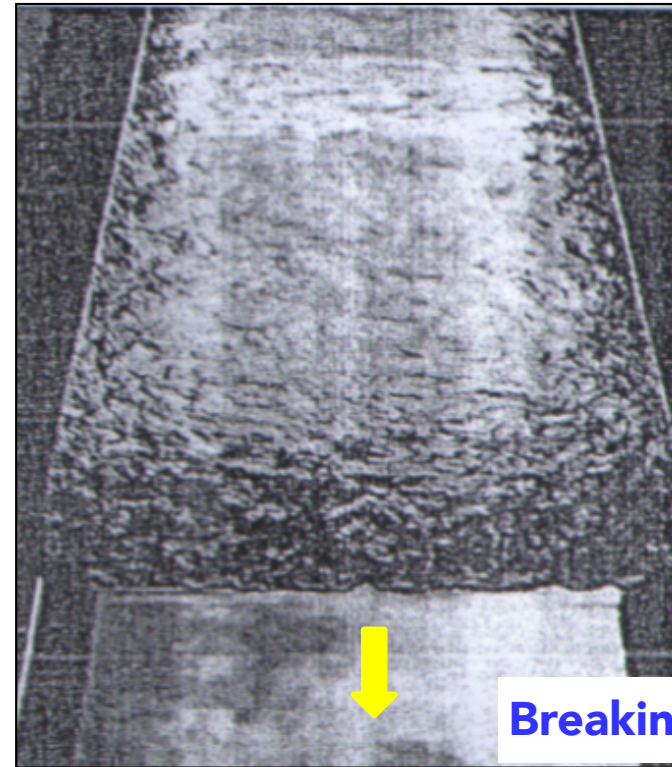
Experiments in rectangular channels (no banks)

Favre, *Dunod*, 1935

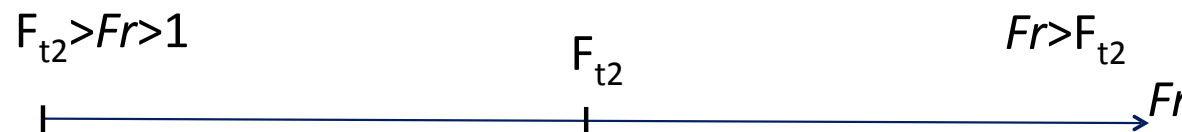
Treske, *J. Hydraulic Research*, 1994



classical undular or "dispersive bore" or "Favre wave"



Breaking bore



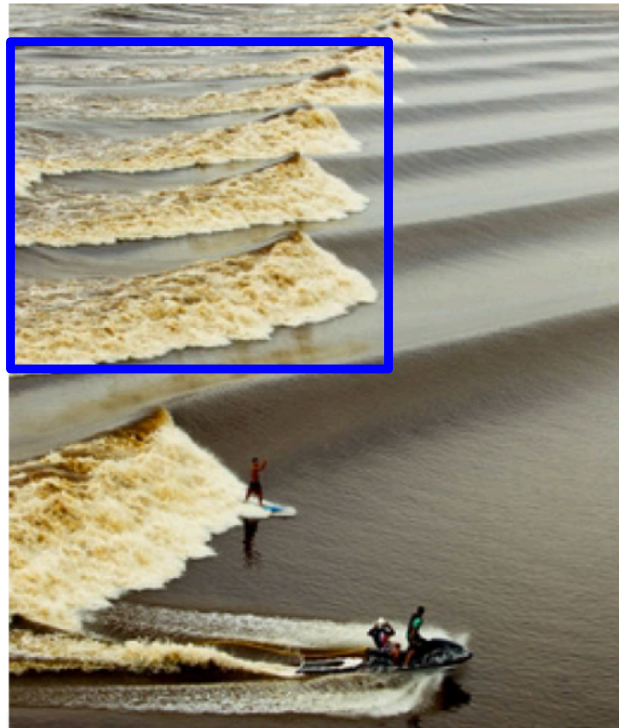


channels (no banks)

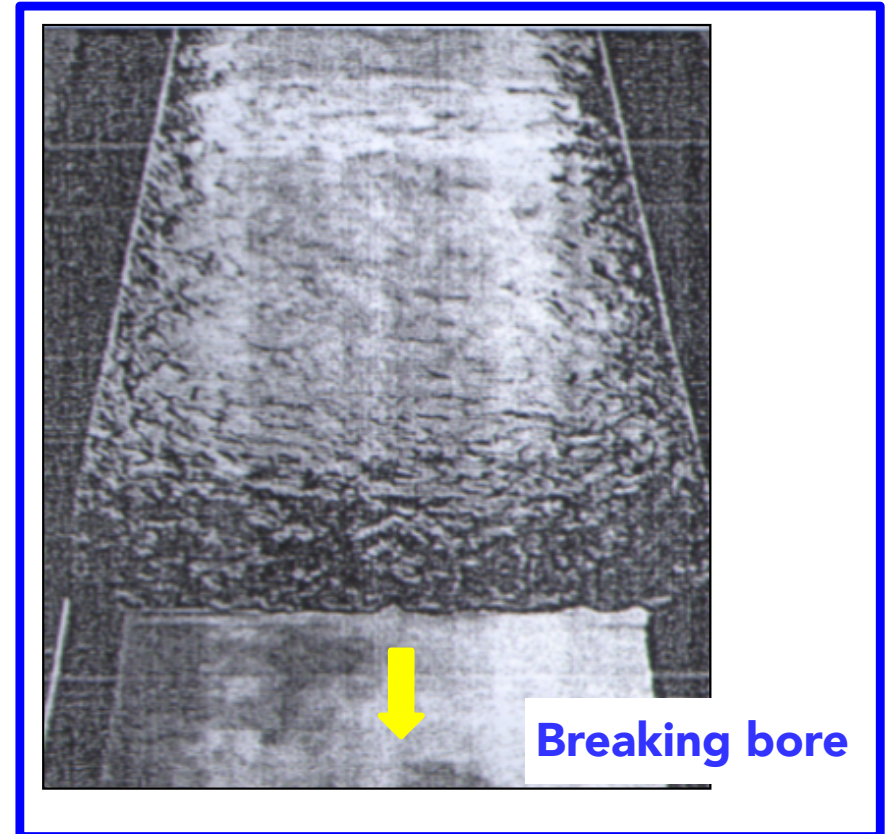
Favre, *Dunod*, 1935

Treske, *J. Hydraulic Research*, 1994

Qiantang River - China



Kampar River - Sumatra



$$F_{t2}$$

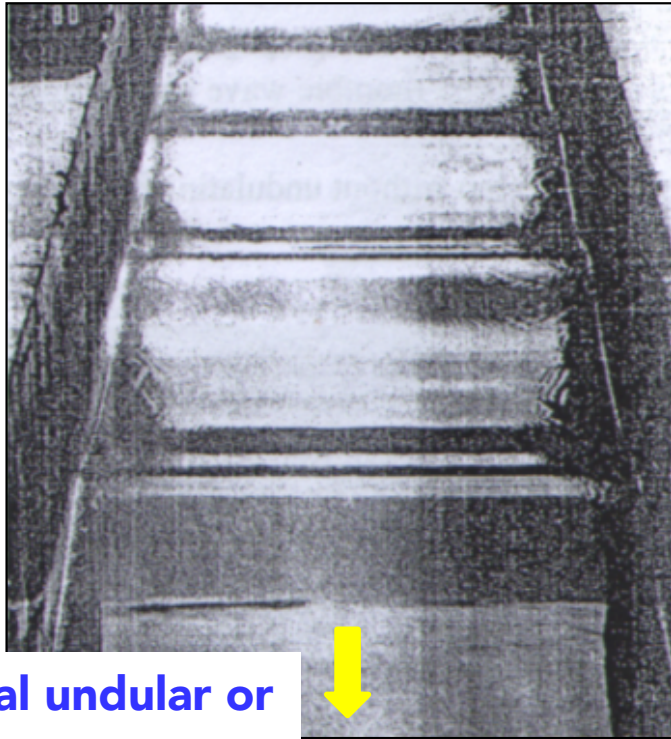
$$Fr > F_{t2}$$

Fr

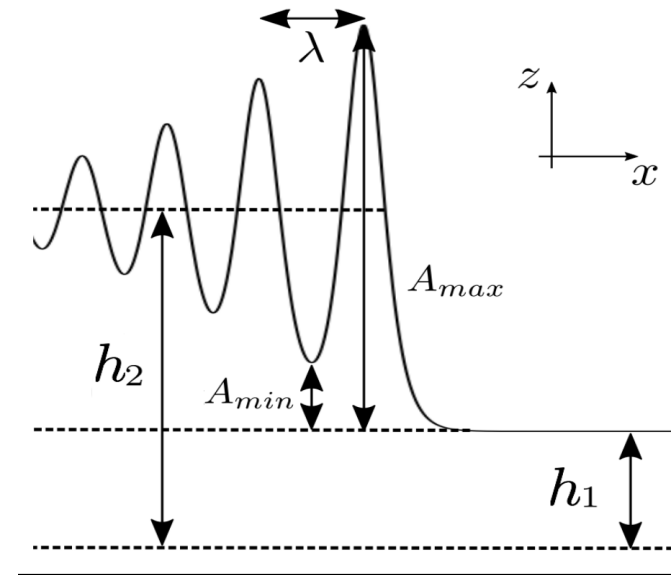
Experiments in rectangular channels (no banks)

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Treske, *J. Hydraulic Research*, 1994



classical undular or
"dispersive bore"
or "Favre wave"



$$1 < Fr < F_{t2}$$

$$F_{t2}$$

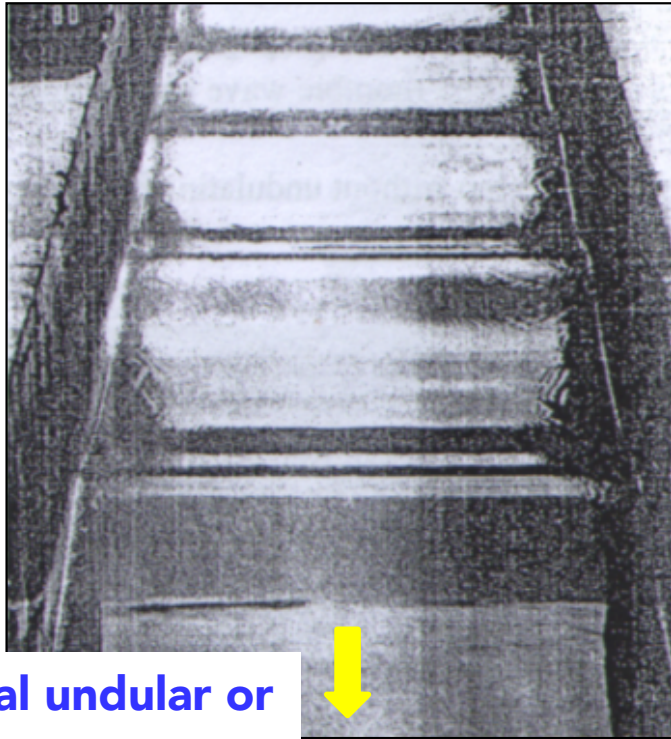
$$Fr > F_{t2}$$

$$Fr$$

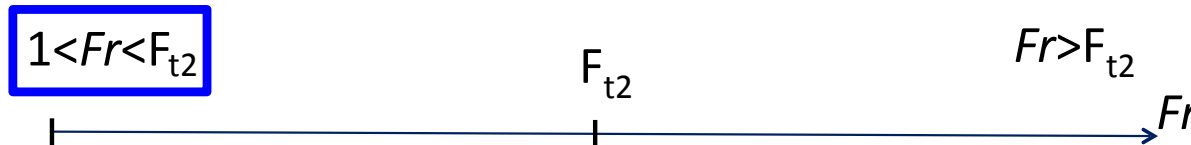
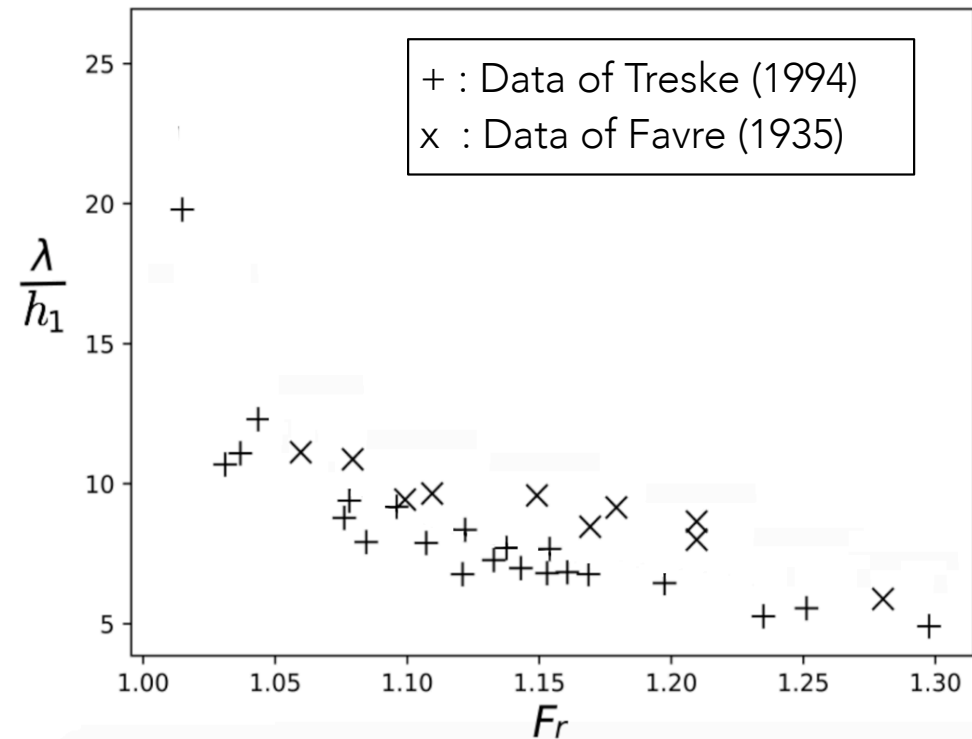
Experiments in rectangular channels (no banks)

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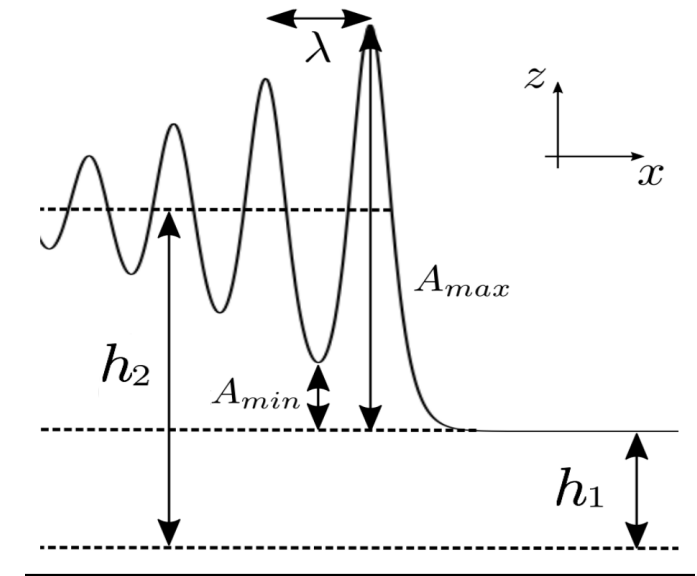
Some physical insight: Lemoine analogy

Lemoine, La Houille Blanche, 1948

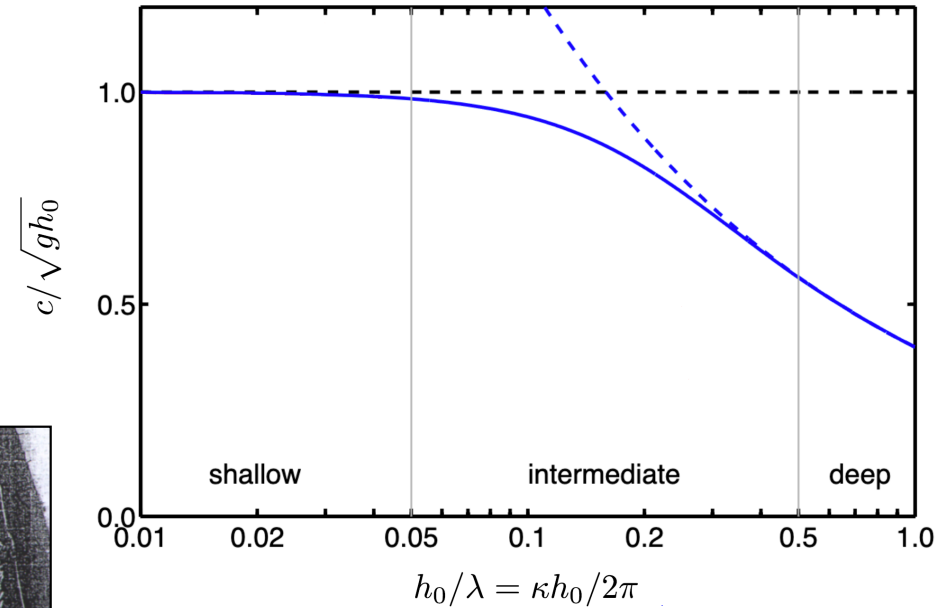
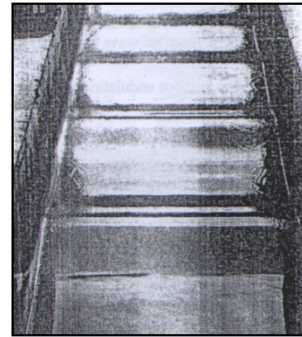
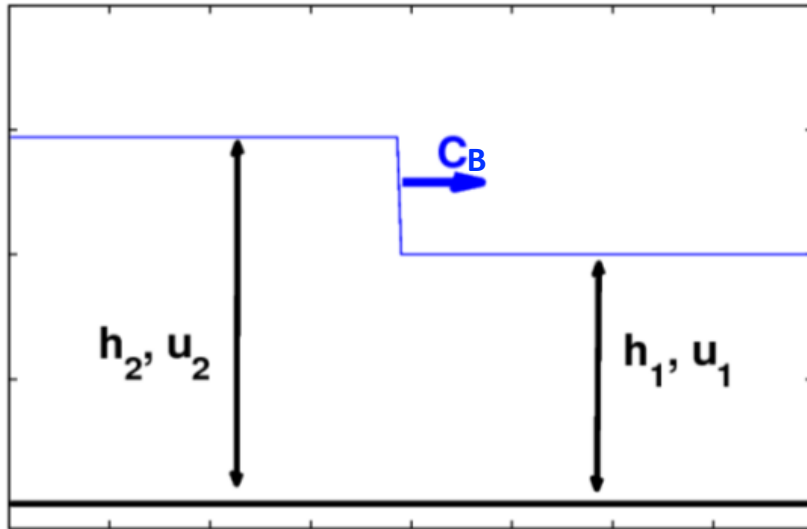


Some physical insight: Lemoine analogy

Lemoine, La Houille Blanche, 1948



1. Secondary waves conserve mass/momentum
2. The undular front moves at the speed corresponding to conservation
3. The energy normally dissipated goes into the secondary waves



Bore:

given the upstream/downstream conditions and using the jump conditions:

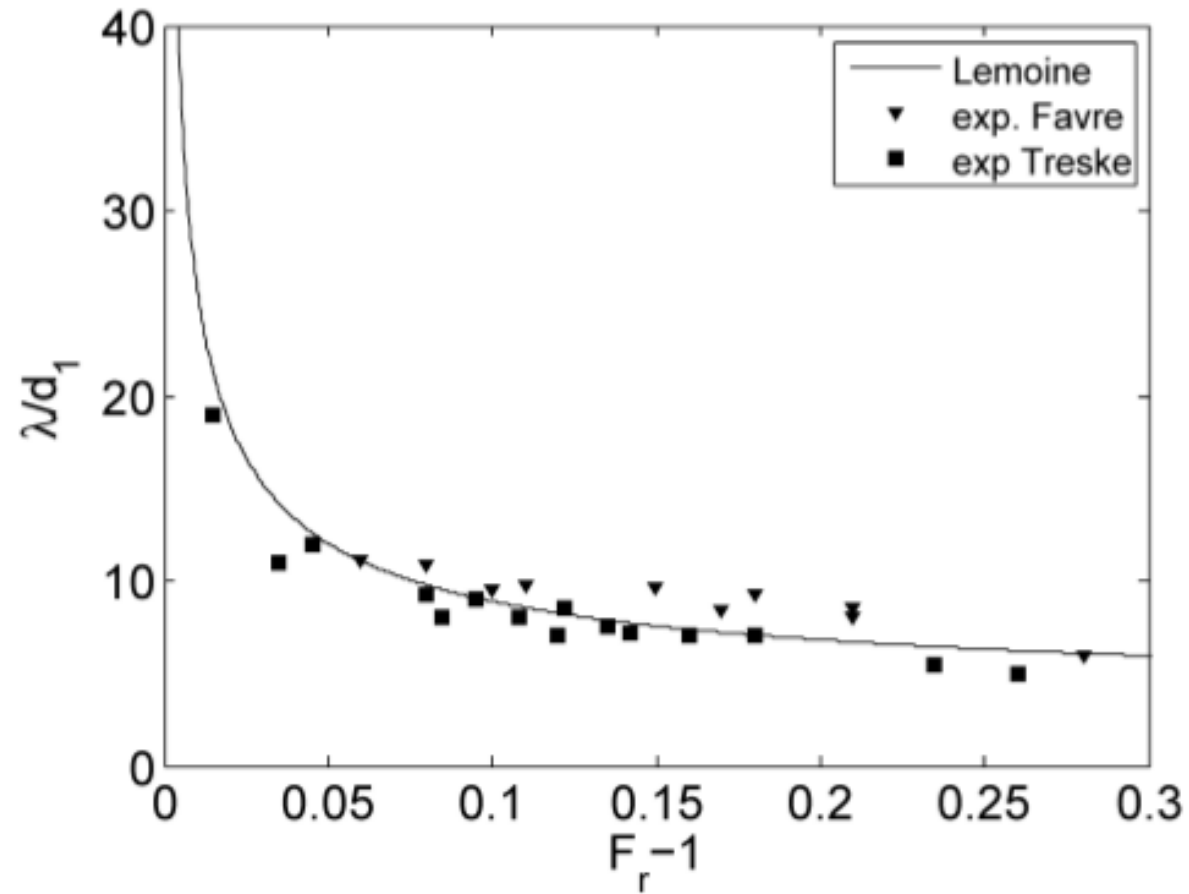
$$C_B = C_B(Fr) = C_B(h_2/h_1)$$

$$C = C(\lambda) = \sqrt{\frac{g\lambda}{2\pi} \tanh(2\pi h/\lambda)}$$

$$C_B(Fr) = C(\lambda) \implies \lambda(Fr)$$

Water waves:

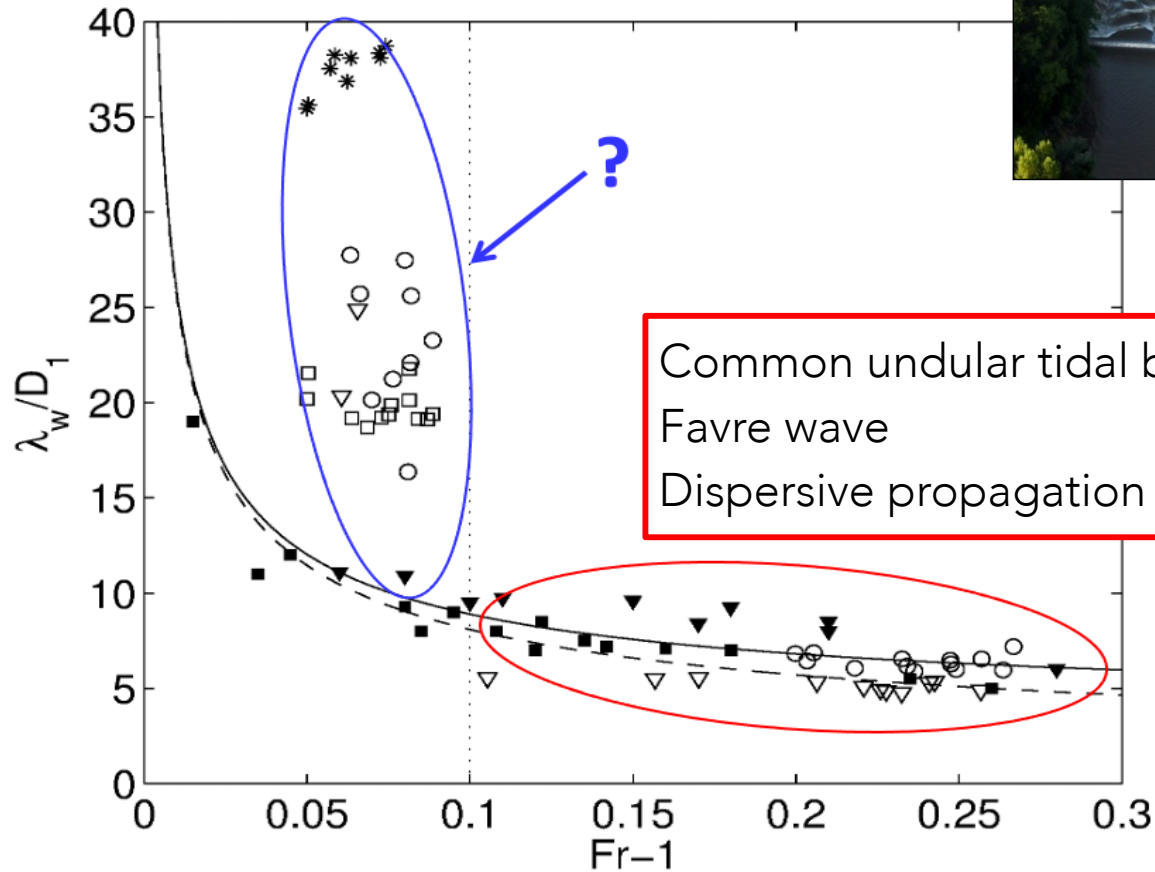
dispersion relation based on the linearized Euler equations (Airy theory)



$$\lambda(Fr)$$

Bonneton et al, *Comptes Rendus Geoscience*, 2012

Bonneton et al, *J. Geophysical Research - Oceans*, 2015

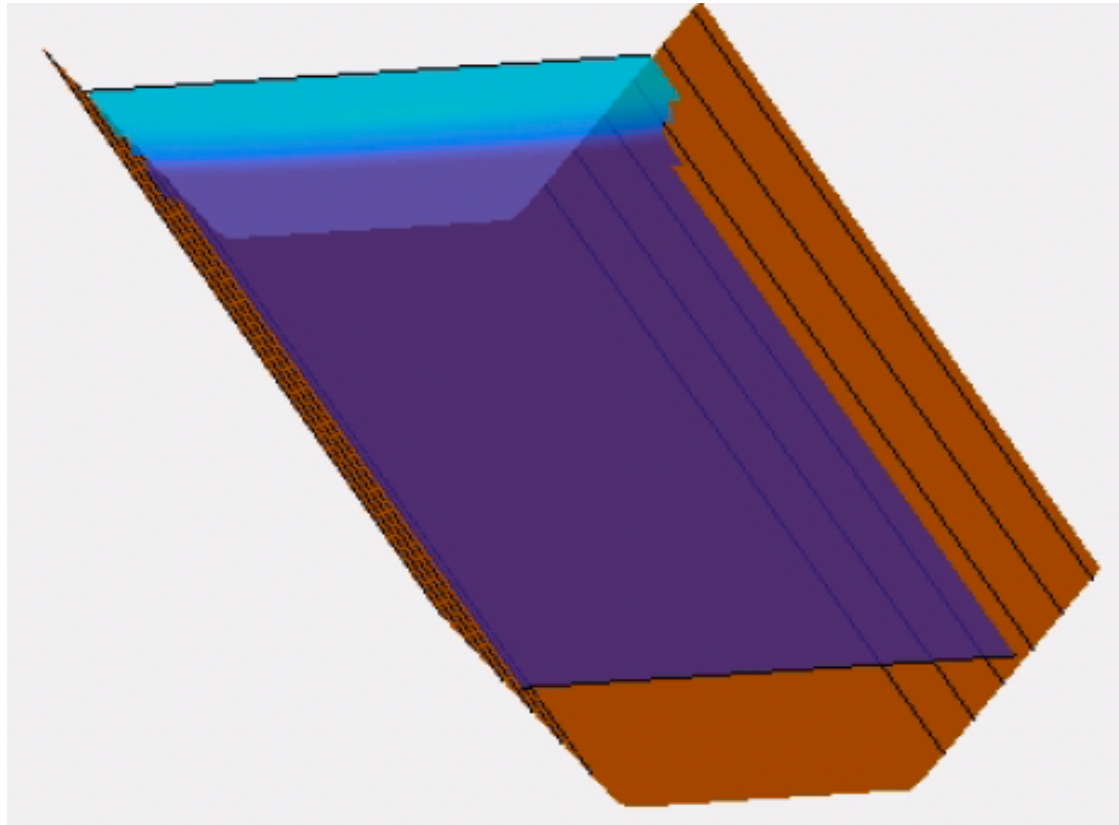


$$\lambda(Fr)$$

Favre experiments and low Froude transition

Experiments in trapezoidal channels

Treske, *J. Hydraulic Research*, 1994



Experiments in trapezoidal channels

Treske, *J. Hydraulic Research*, 1994



Fig. 8. Undular bore at Froude ~ 1.04.

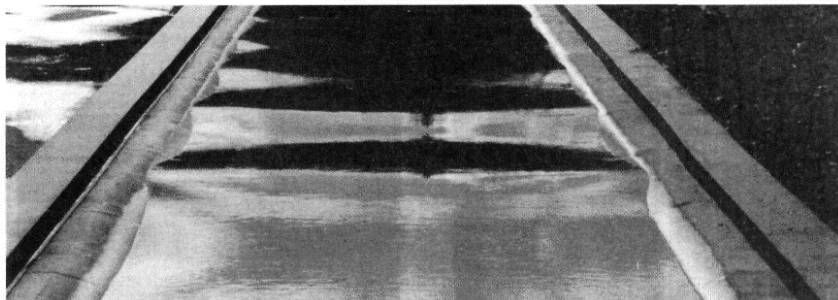


Fig. 9. Undular bore at Froude ~ 1.06.

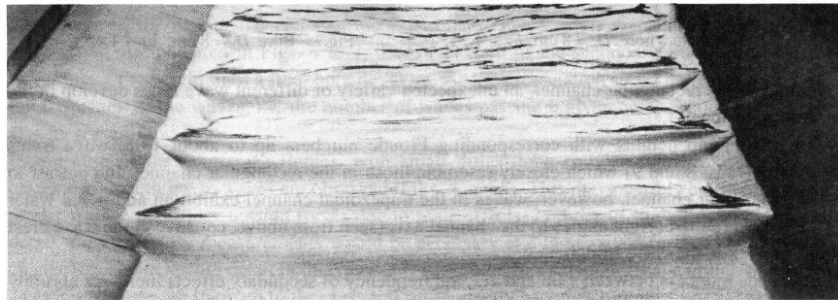


Fig. 10. Undular bore at Froude ~ 1.10.

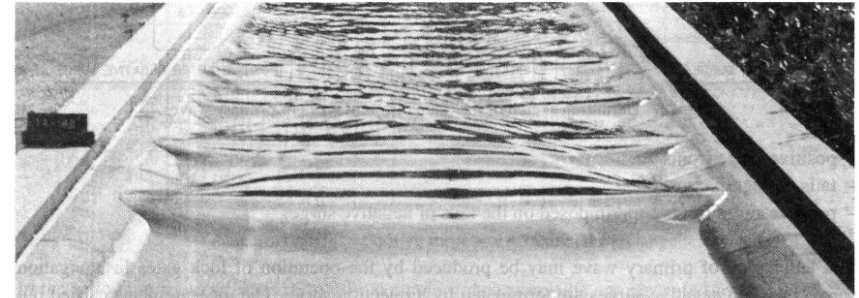


Fig. 11. Undular bore at Froude ~ 1.12.

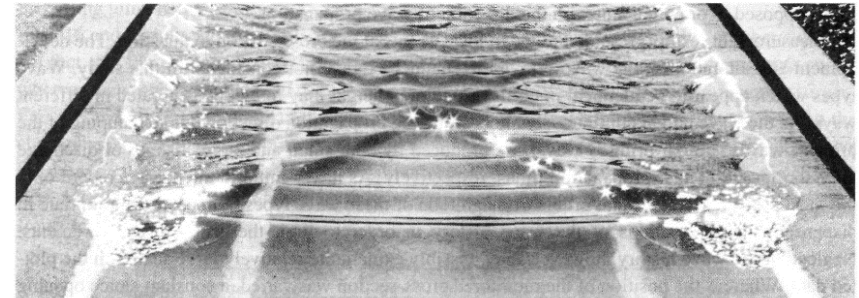


Fig. 12. Undular bore at Froude ~ 1.24.

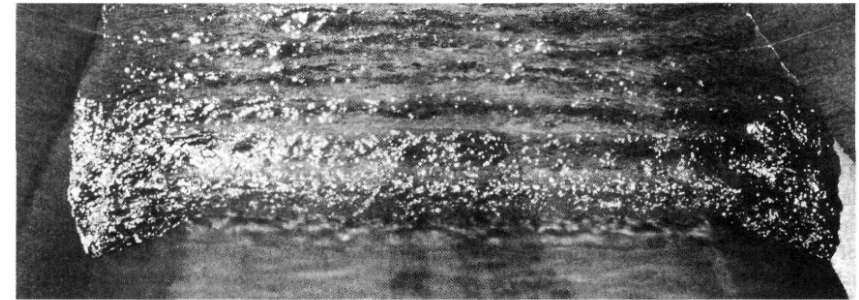
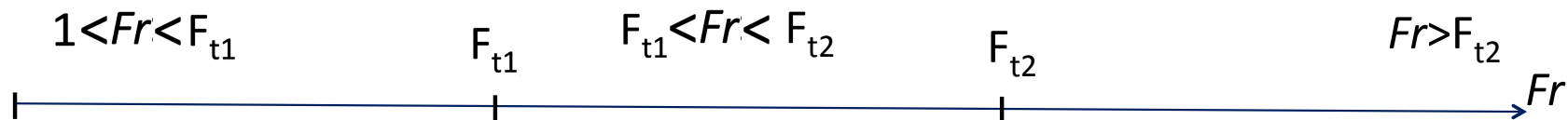


Fig. 13. Bore at Froude ~ 1.35.

Fr



Experiments in trapezoidal channels

Treske, *J. Hydraulic Research*, 1994



Fig. 8. Undular bore at Froude ~ 1.04.

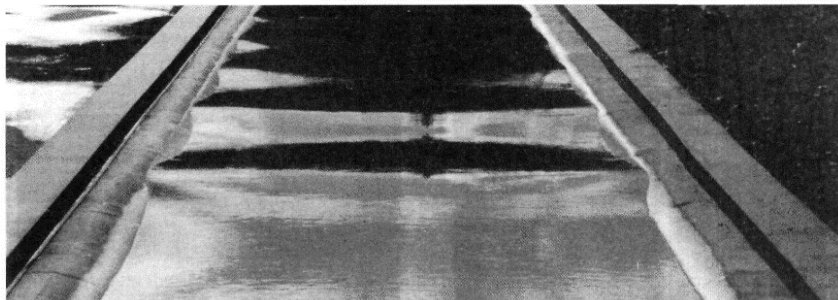


Fig. 9. Undular bore at Froude ~ 1.06.

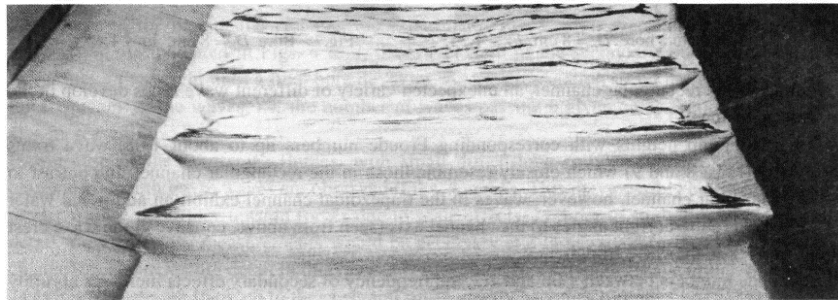


Fig. 10. Undular bore at Froude ~ 1.10.

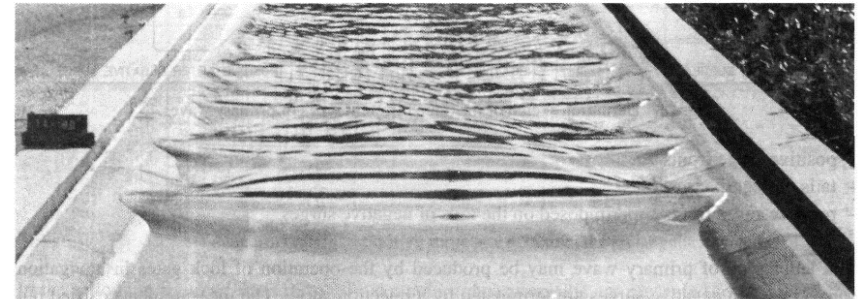


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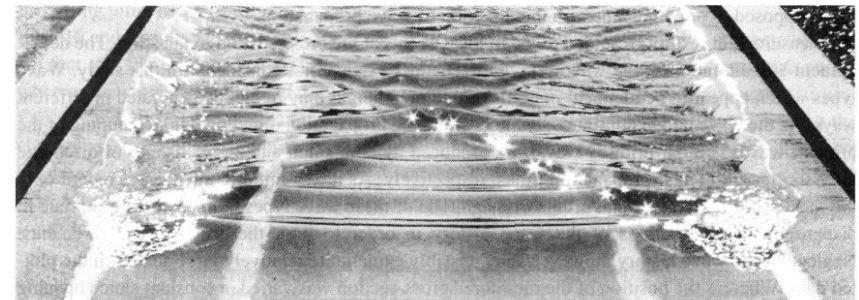


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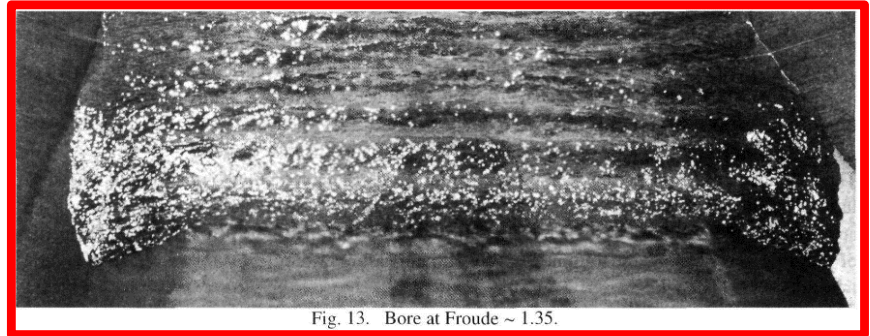
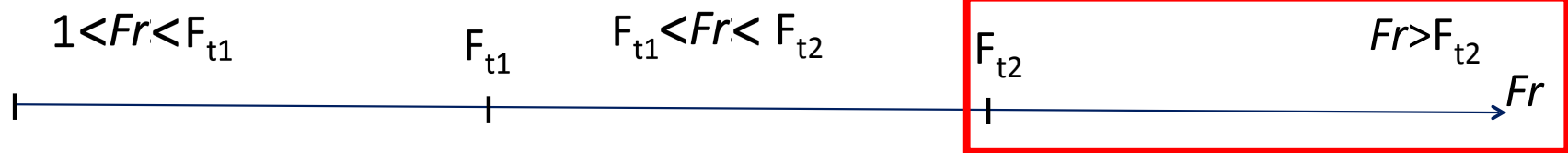


Fig. 13. Bore at Froude ~ 1.35.

Fr



Experiments in trapezoidal channels

Treske, *J. Hydraulic Research*, 1994

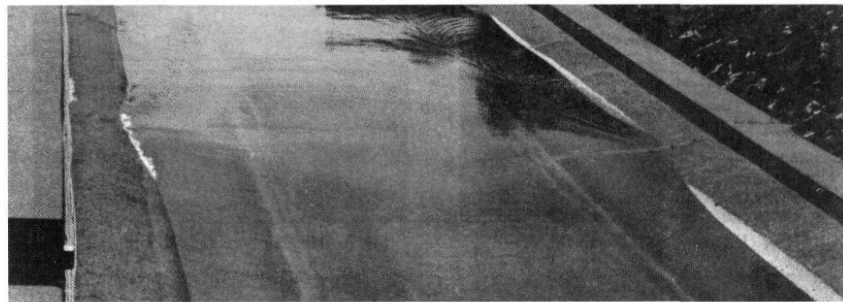


Fig. 8. Undular bore at Froude ~ 1.04.

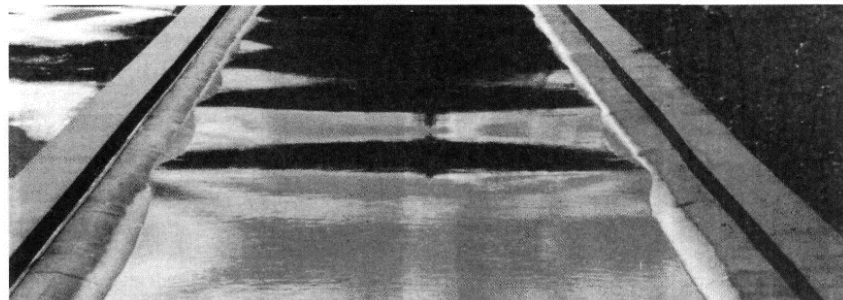


Fig. 9. Undular bore at Froude ~ 1.06.

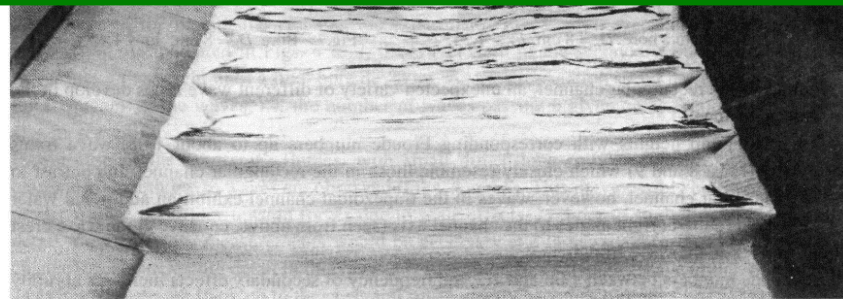


Fig. 10. Undular bore at Froude ~ 1.10.

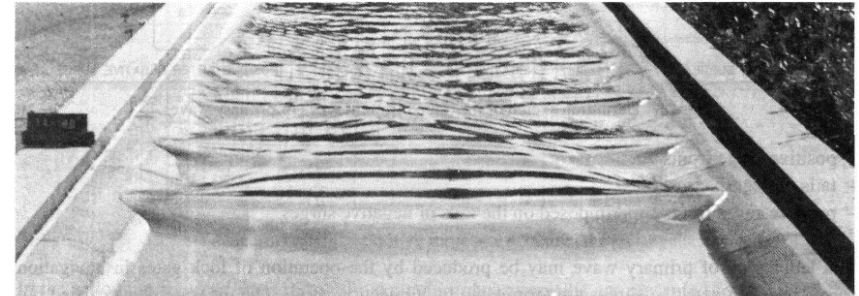


Fig. 11. Undular bore at Froude ~ 1.12.

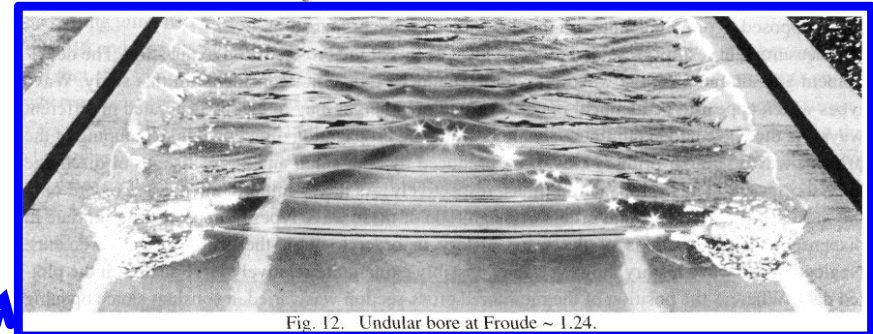


Fig. 12. Undular bore at Froude ~ 1.24.

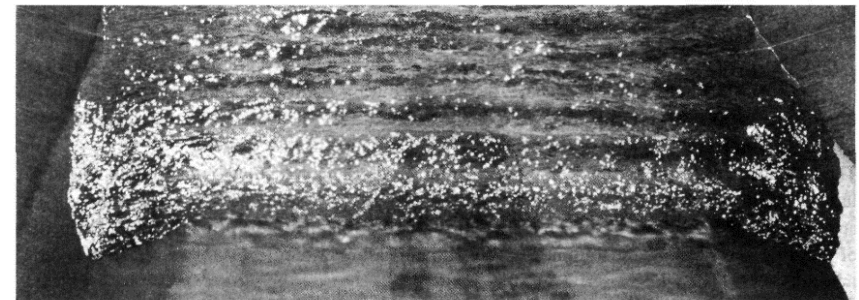


Fig. 13. Bore at Froude ~ 1.35.

Fr



$$1 < Fr < F_{t1}$$

F_{t1}

$$F_{t1} < Fr < F_{t2}$$

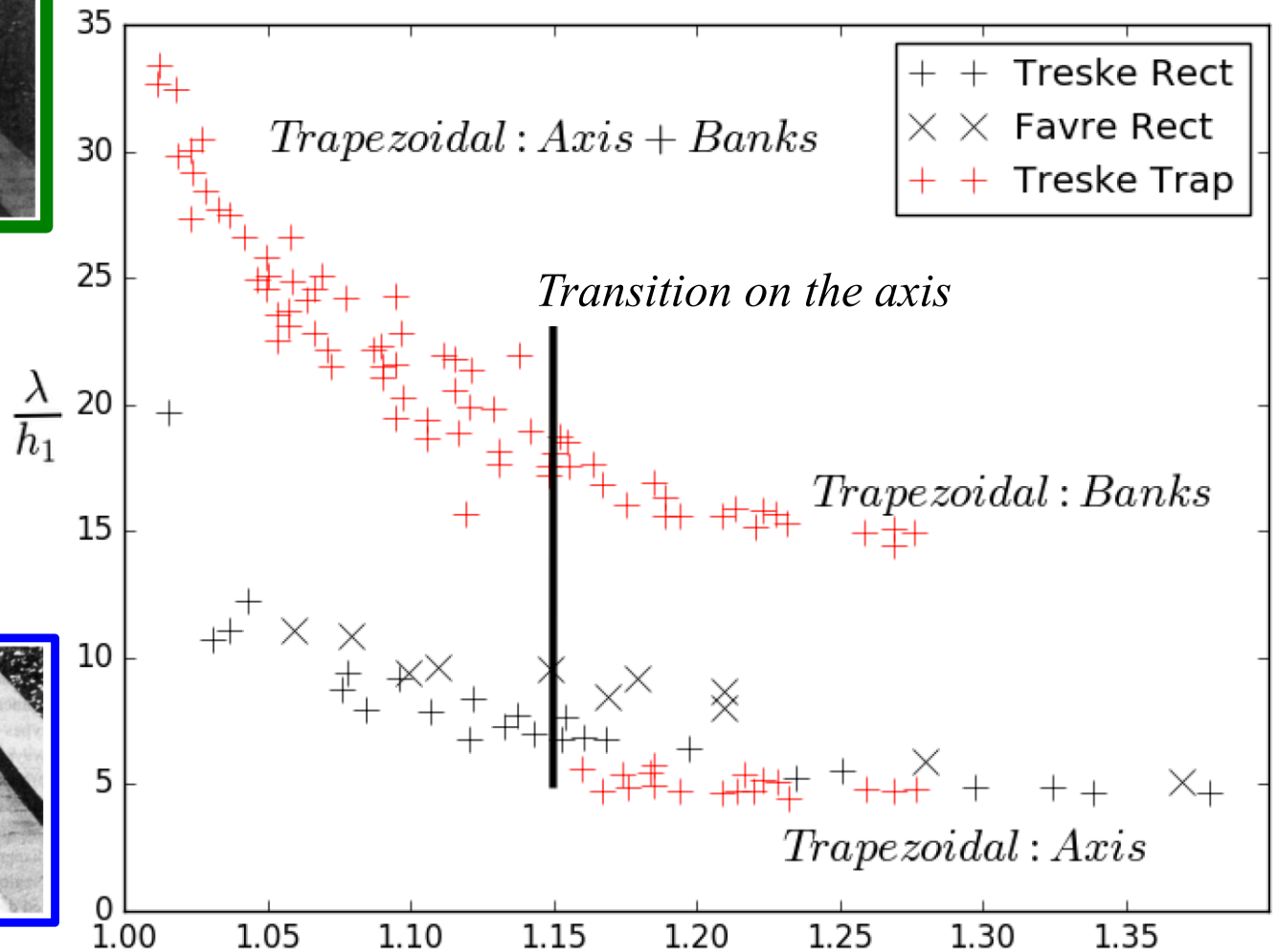
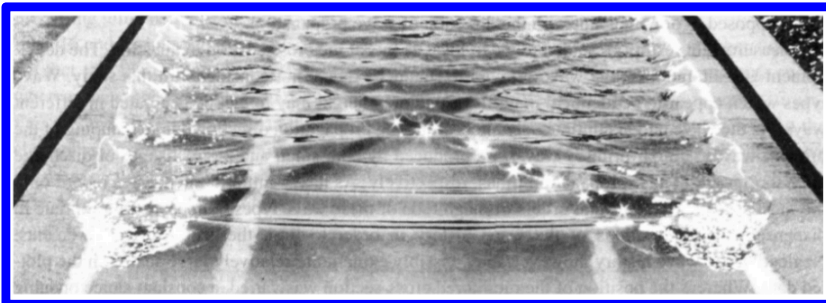
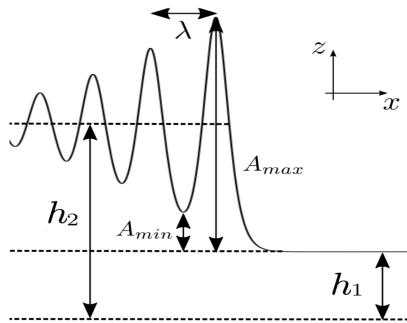
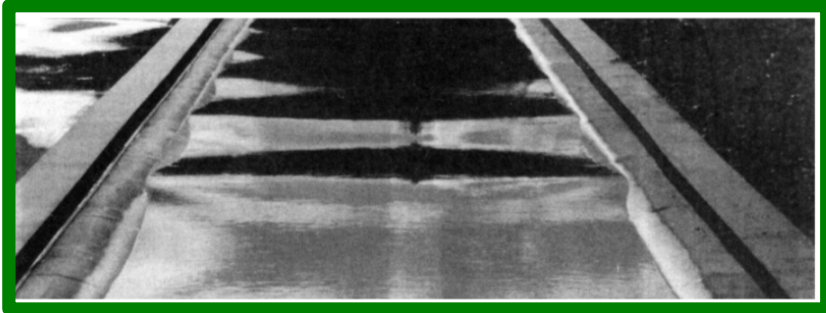
F_{t2}

$Fr > F_{t2}$

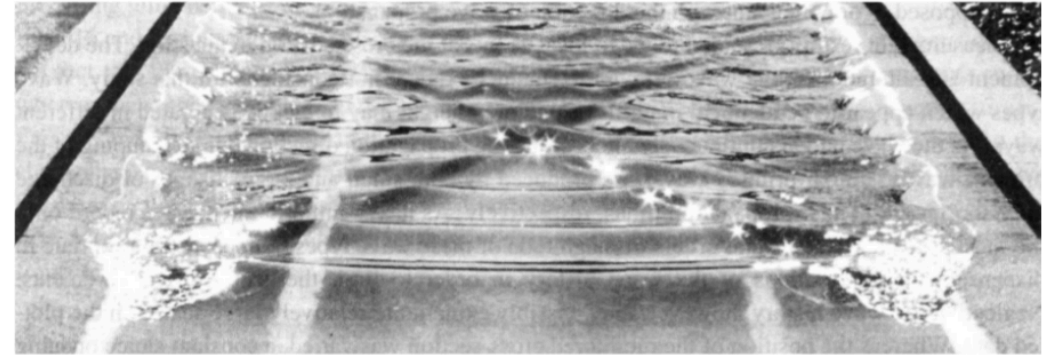
Fr

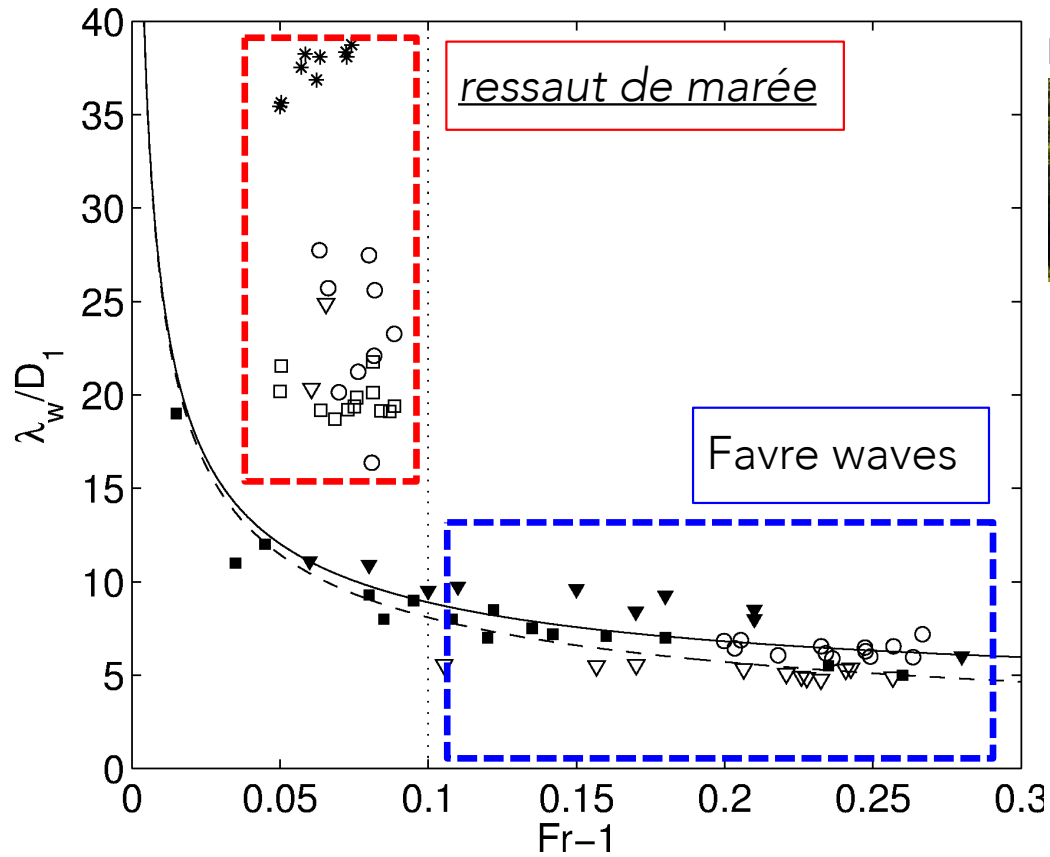
Experiments in trapezoidal channels

Treske, *J. Hydraulic Research*, 1994

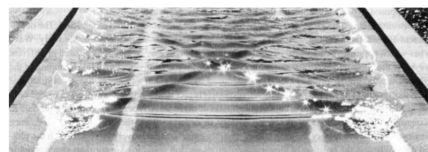


Striking similarities between the low Fr transition observed in field and laboratory experiments

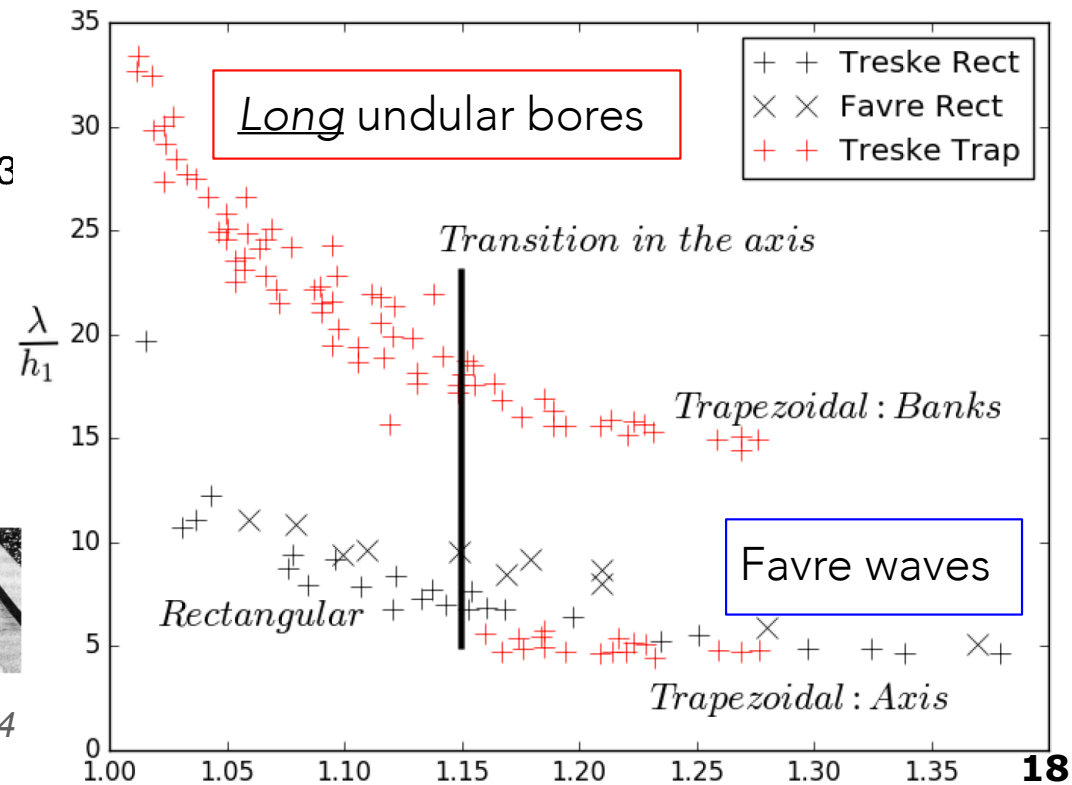


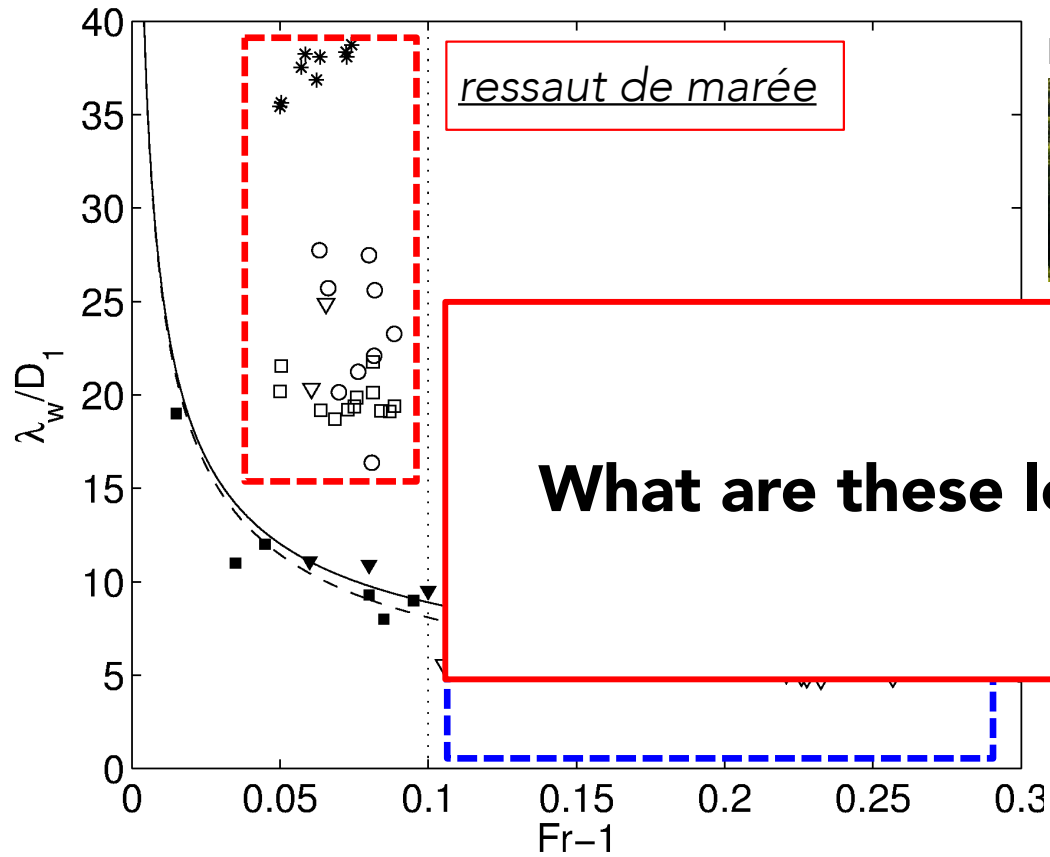


Bonneton et al, *J. Geophysical Research - Oceans*, 2015



Treske, *J. Hydraulic Research*, 1994

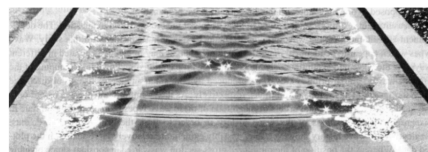




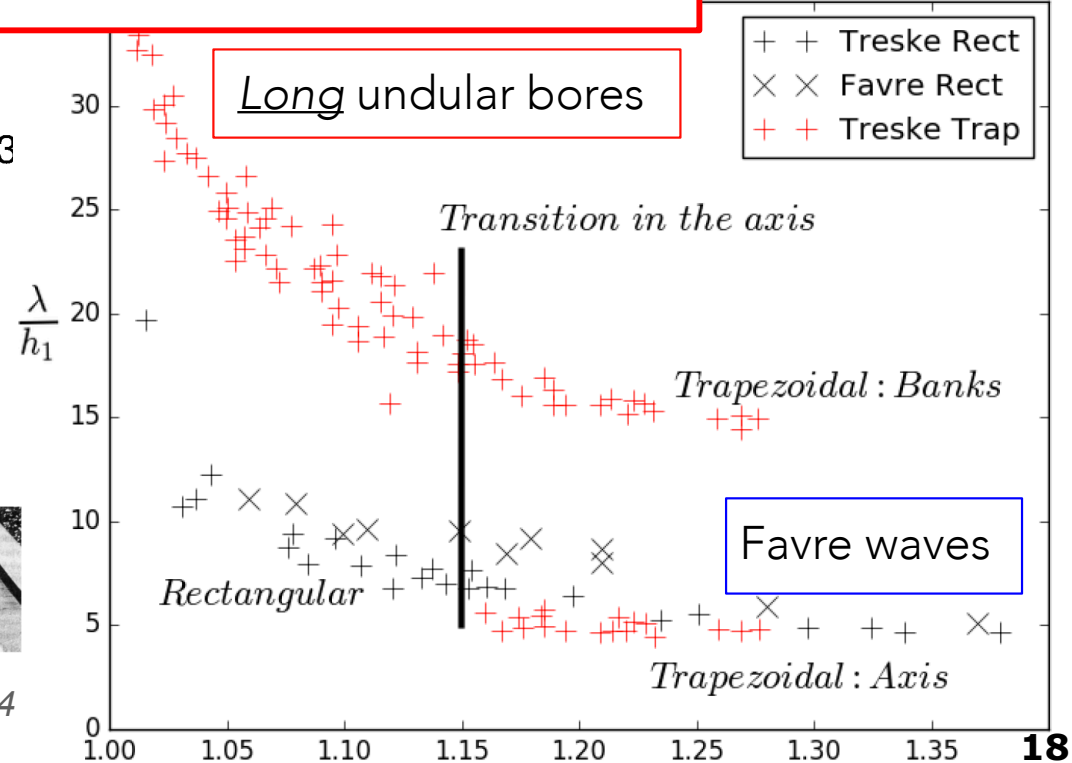
Bonneton et al, *J. Geophysical Research - Oceans*, 2015



What are these low Froude waves ?

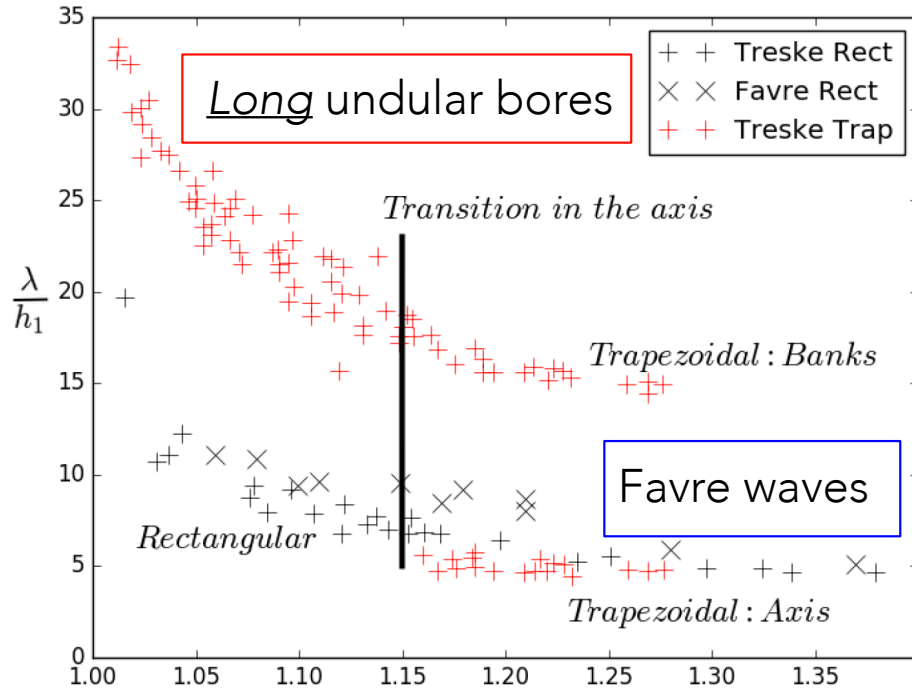


Treske, *J. Hydraulic Research*, 1994



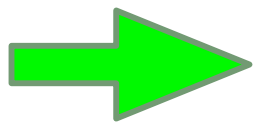
Dispersive wave models





- Complex free surface dynamics (2D)
- Variable flow parameters: Fr
- Variable geometrical parameters: channel geom.

Using full 3D models: overkill



Approximate 2d models



Fig. 8. Undular bore at Froude ~ 1.04.

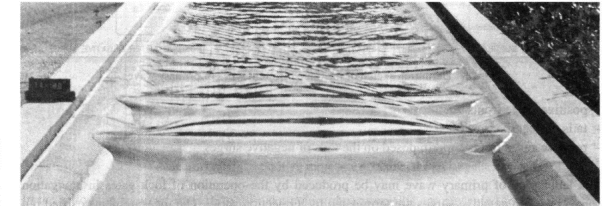


Fig. 11. Undular bore at Froude ~ 1.12.

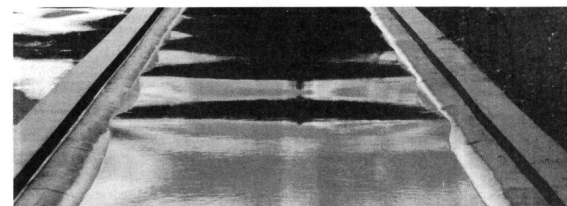


Fig. 9. Undular bore at Froude ~ 1.06.

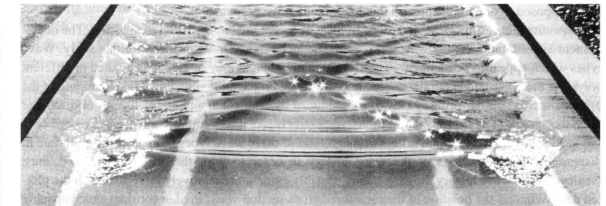


Fig. 12. Undular bore at Froude ~ 1.24.

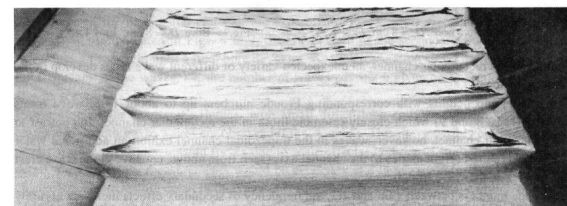


Fig. 10. Undular bore at Froude ~ 1.10.

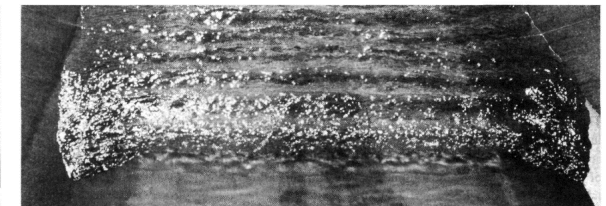
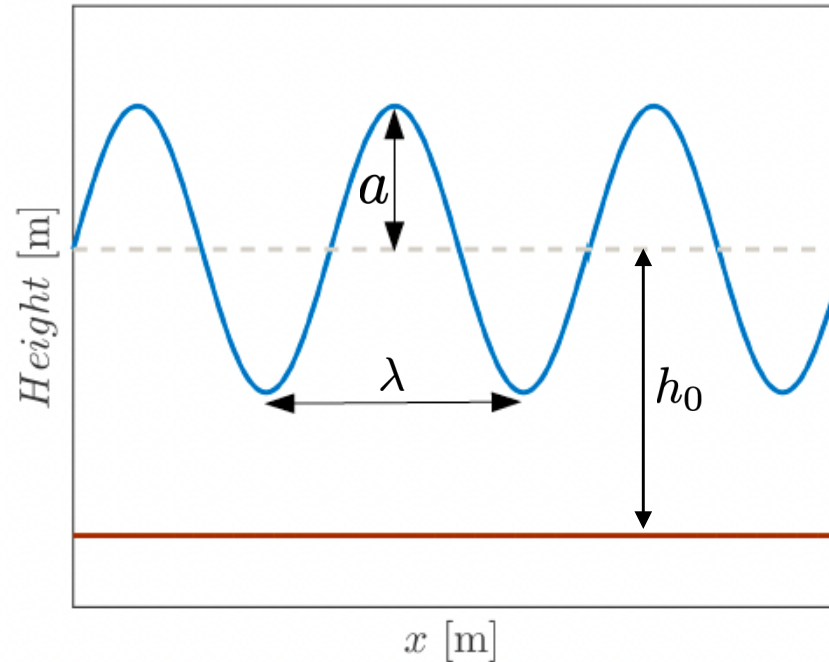


Fig. 13. Bore at Froude ~ 1.35.



Dimensionless parameters

- dispersion: $\mu = \frac{h_0}{\lambda} = \frac{\kappa h_0}{2\pi}$
- non-linearity: $\epsilon = \frac{a}{h_0}$

Physical hypotheses

Long waves : small μ

Weakly dispersive waves : $\mu^2 \ll 1$, μ^4 negligible

Weak/full non-linearity : $\epsilon = \mathcal{O}(\mu^2)$ and $\epsilon = \mathcal{O}(1)$ respectively

Principles: asymptotic expansion, depth averaging

Boussinesq, J.Math. Pures Appl., 1872
Dingemans, World Scientific, 1997
Lannes, AMS, 2013
Lannes, Nonlinearity, 2020

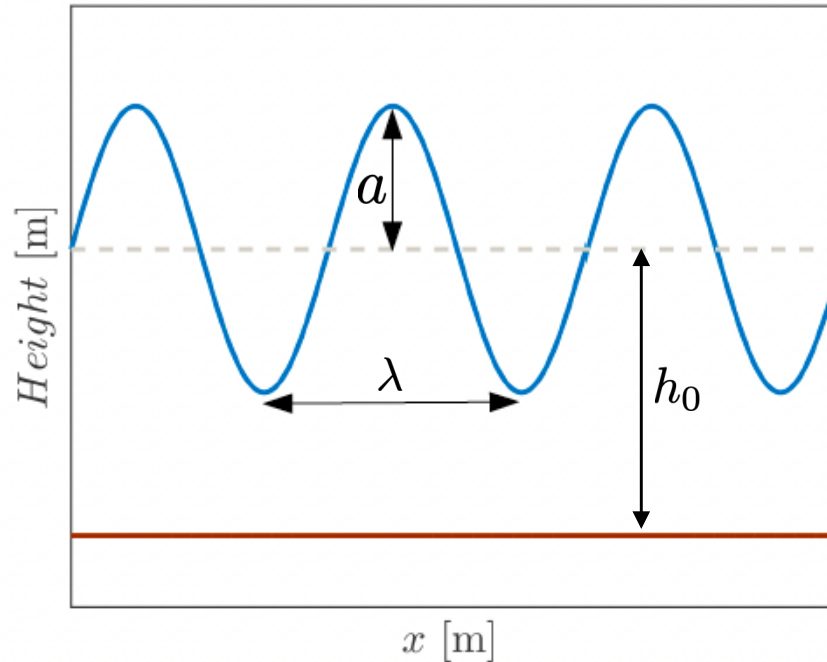
1. Starting point : nonlinear wave equations

$$\begin{aligned}\Delta\Phi &= 0 \\ \partial_t\Phi + \frac{1}{2}\|\nabla\Phi\|^2 + g\zeta &= 0 \\ \partial_t\zeta + \partial_x\Phi\partial_x\zeta &= \partial_z\Phi \\ \partial_z\Phi &= 0\end{aligned}$$

2. **Asymptotic dev.** wrt μ^2 : $\Phi = \Phi_0 + \mu^2\Phi_1 + \mu^4\Phi_2 + \dots$

3. **Depth averaging** : $\int_0^{h_0+\zeta} (\cdot) dz \quad \longrightarrow \quad h\vec{u} = \int_b^\zeta \vec{v} dz$

4. Retain appropriate order terms



Dimensionless parameters

- dispersion: $\mu = \frac{h_0}{\lambda} = \frac{\kappa h_0}{2\pi}$
- non-linearity: $\epsilon = \frac{a}{h_0}$

Physical hypotheses

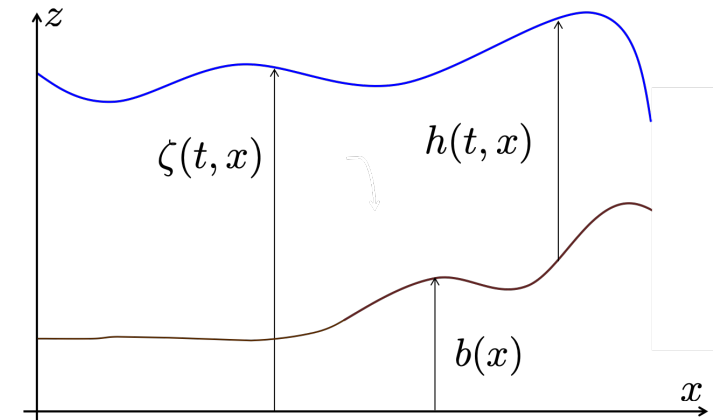
Long waves : small μ

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Zeroth order in μ

Shallow water/Saint Venant equations



$$\partial_t h + \partial_x(hu) = 0$$

$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2) + gh\partial_x b = 0$$

Bathymetry

Depth averaged velocity $h\vec{u} = \int_b^\zeta \vec{v} dz$

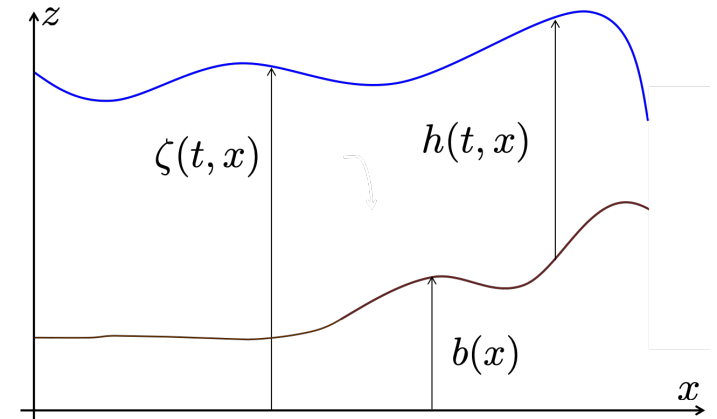
Weakly dispersive (μ^2) corrections

Weakly nonlinear: $\epsilon = \mathcal{O}(\mu^2)$

$$\partial_t h + \partial_x(hu) = 0$$

$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2) + gh\partial_x b = h\partial_t \left[\frac{d^2}{3} \partial_{xx} u + \frac{d}{3} \partial_x d \partial_x u \right]$$

$$d(x) = h_0 - b(x)$$



Peregrine, J.Fluid Mech., 1967

$$\partial_t h + \partial_x(hu) = 0$$

$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2) + gh\partial_x b = \mathcal{D}$$

$$\mathcal{D} = \partial_t \left[\beta d^2 \partial_{xx}(hu) + \frac{d}{3} \partial_x d \partial_x(hu) \right] + Bgd^2 [d\partial_{xxx}\zeta + 2\partial_x d \partial_{xx}\zeta]$$

Madsen & Sorensen, Coast.Eng., 1992

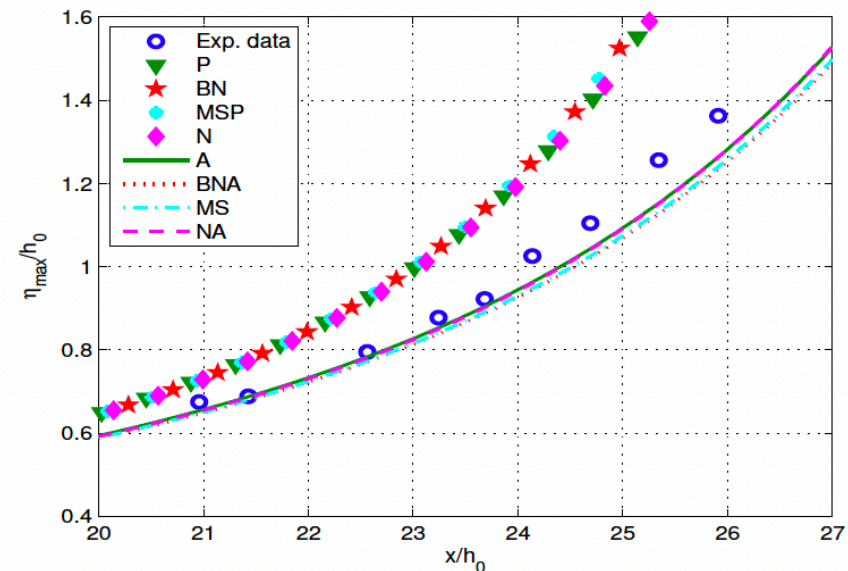
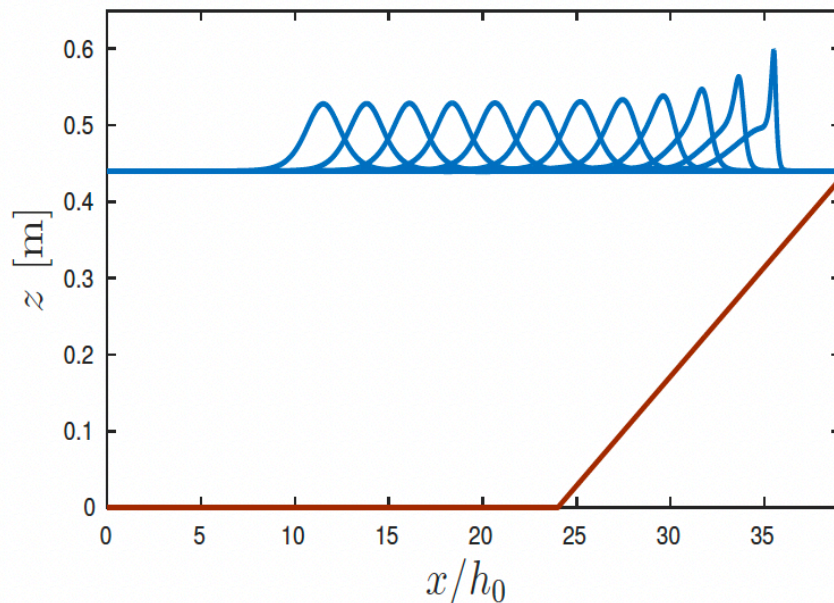
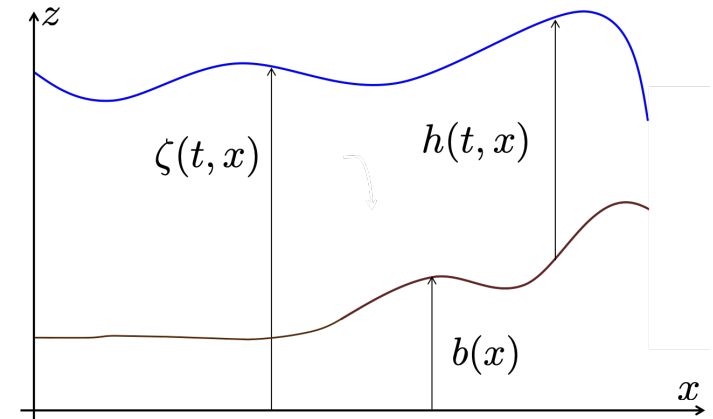
Weakly dispersive (μ^2) corrections

Weakly nonlinear: $\epsilon = \mathcal{O}(\mu^2)$

Many variations for a given asymptotic accuracy

* $\mu^2 d \equiv \mu^2 h$ as the difference is of order $\mu^2 \epsilon = \mu^4$

* $\mu^2 \partial_{xxt}(du) \equiv \mu^2 \partial_{xxt}(hu)$ as the difference is of order $\mu^2 \epsilon = \mu^4$



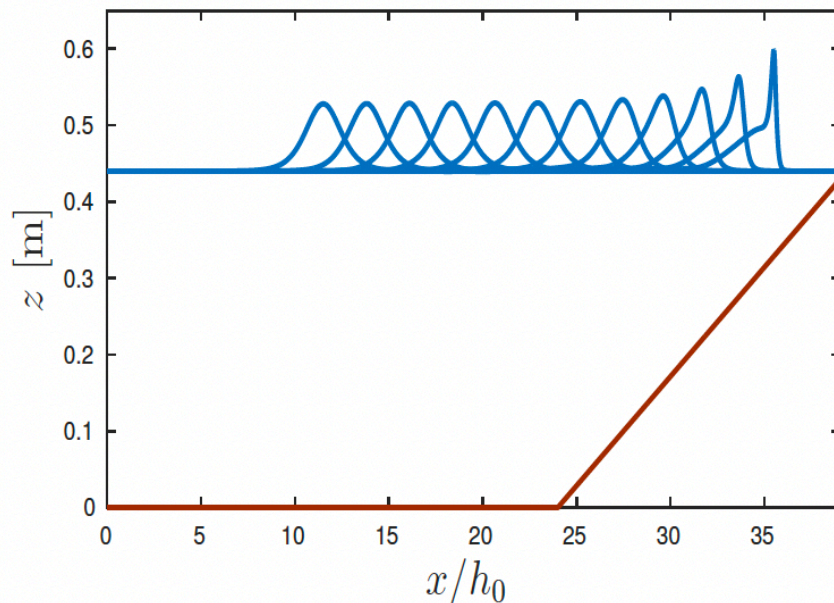
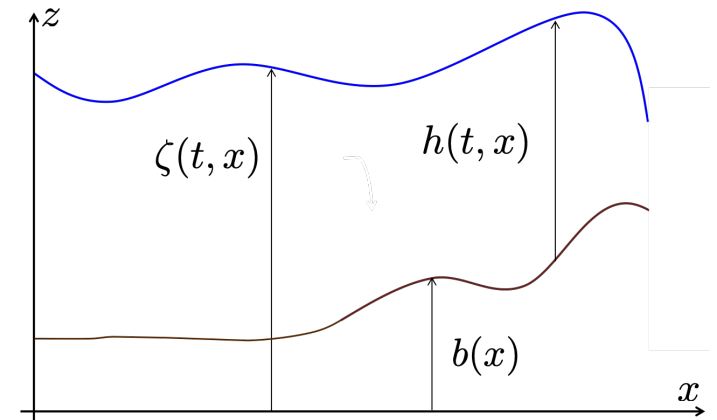
Weakly dispersive (μ^2) corrections

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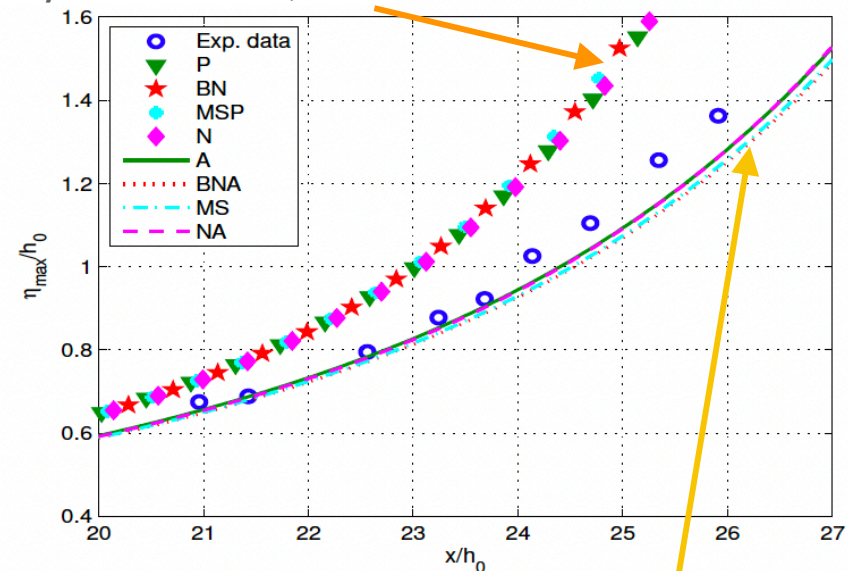
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Peregrine, J.Fluid Mech., 1967



Madsen & Sorensen, Coast.Eng., 1992

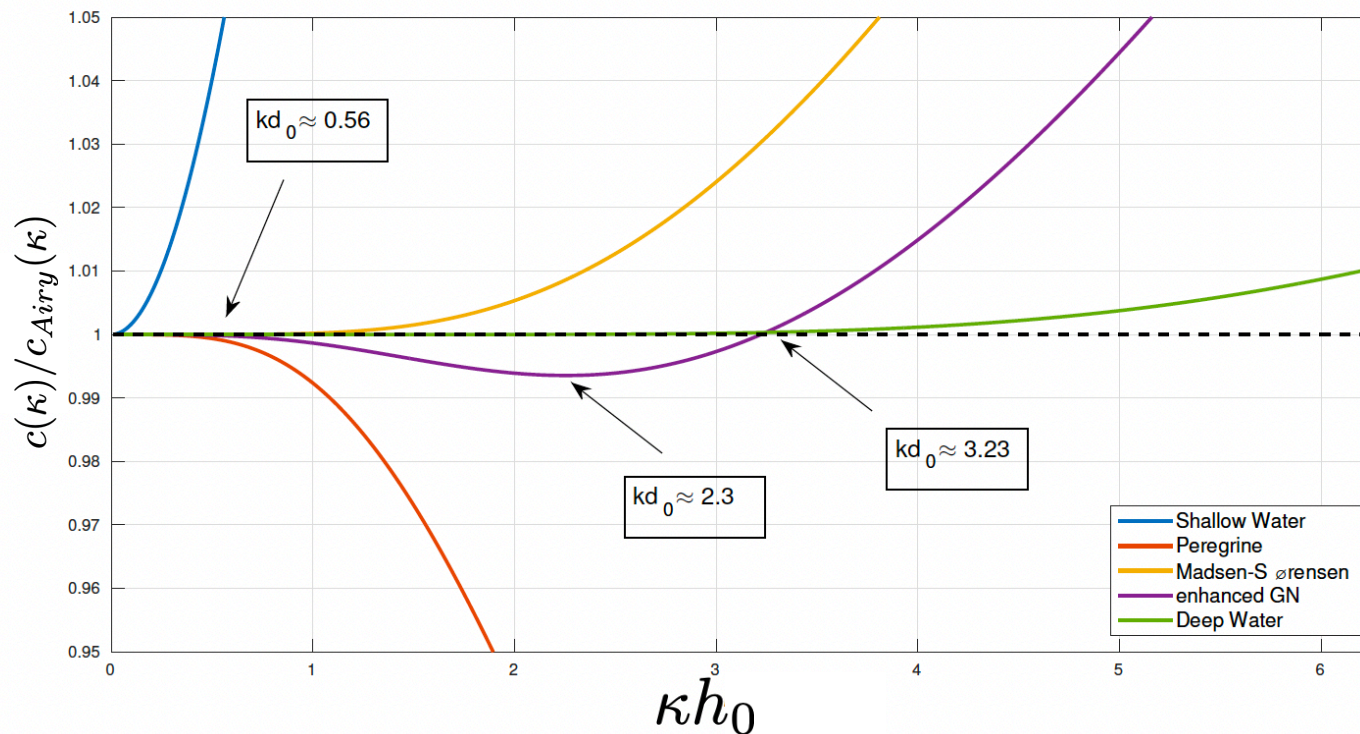
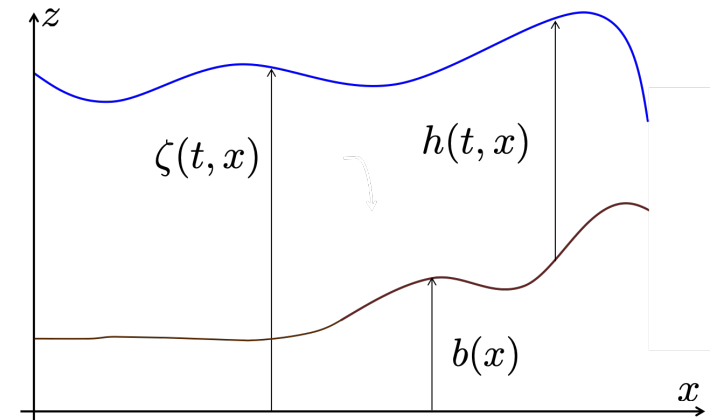
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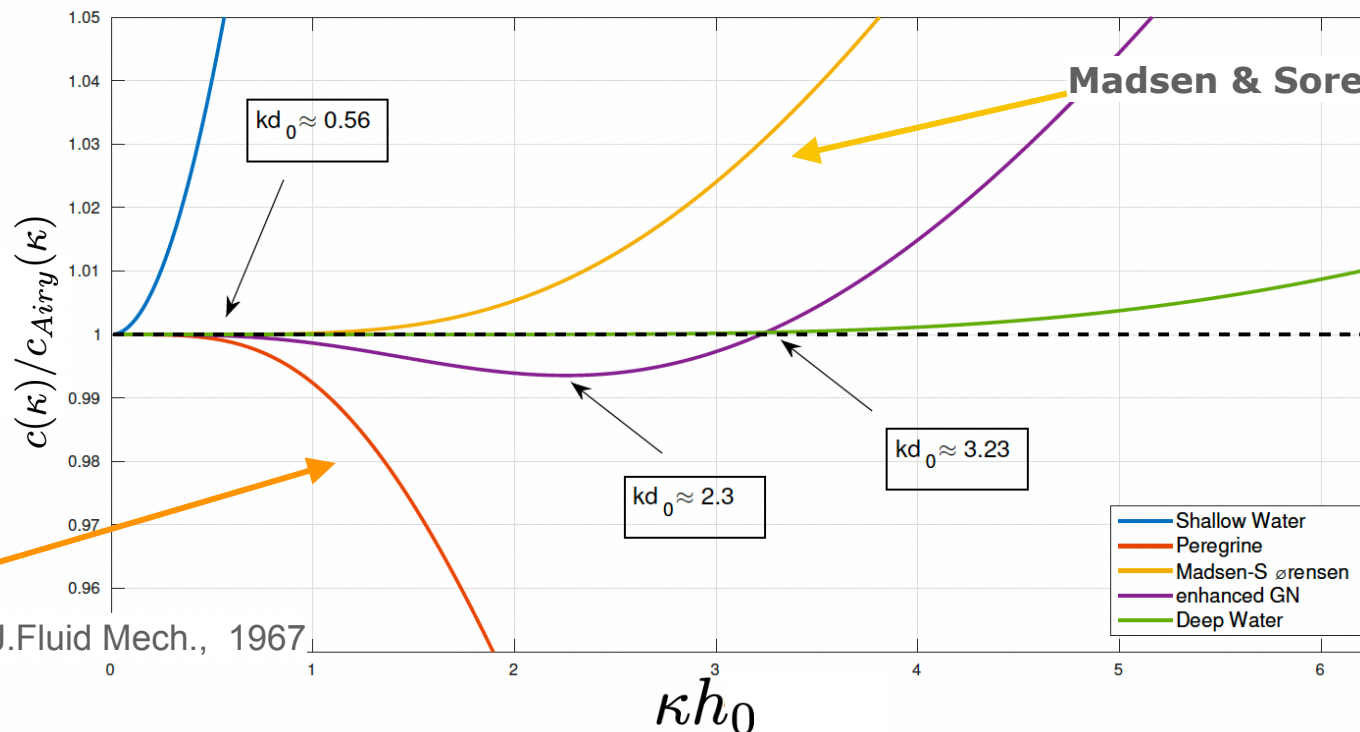
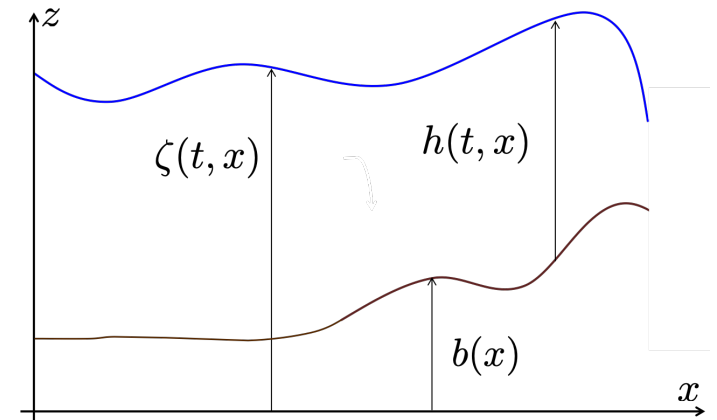
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Madsen & Sørensen, Coast.Eng., 1992

Peregrine, J.Fluid Mech., 1967

Weakly dispersive (μ^2) corrections

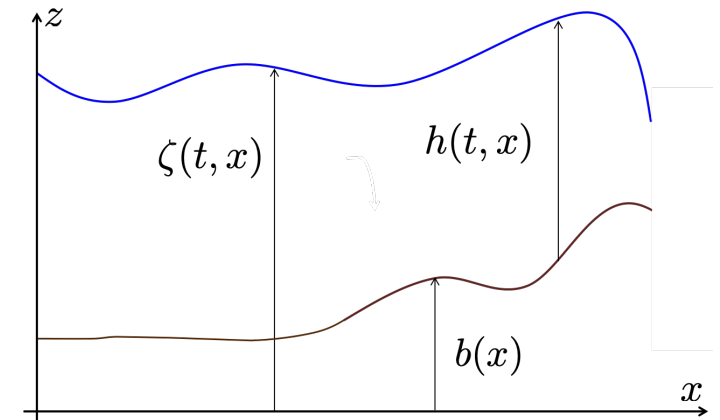
Fully nonlinear: $\epsilon = \mathcal{O}(1)$

$$\partial_t h + \partial_x(hu) = 0$$

$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2) + gh\partial_x b = \mathcal{D}$$

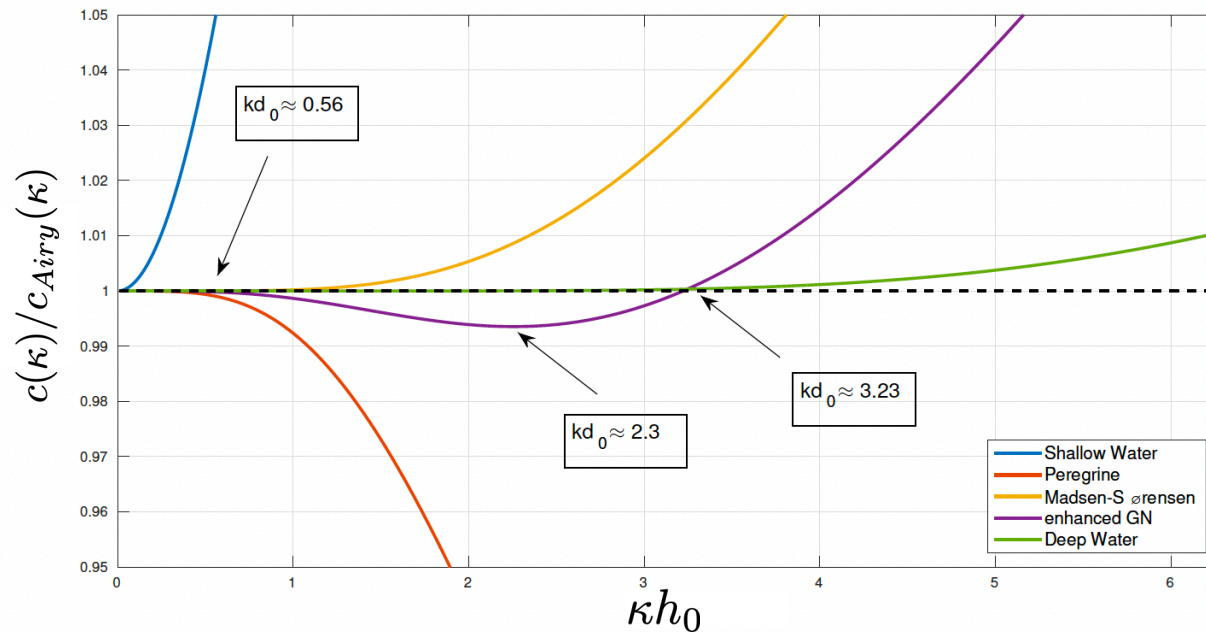
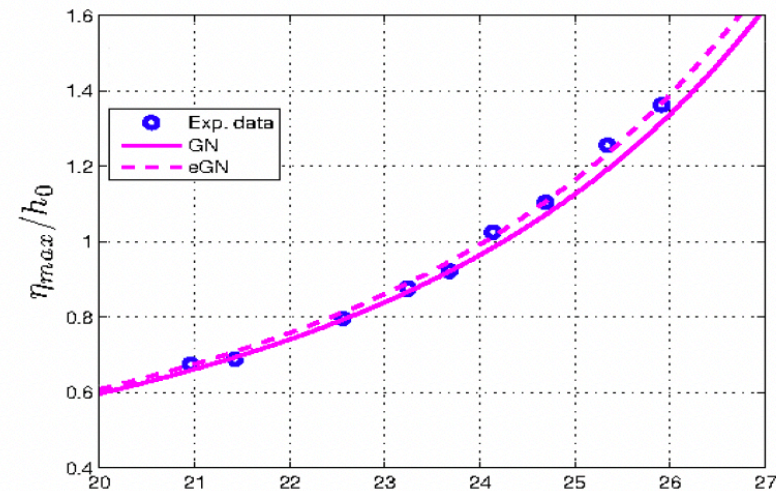
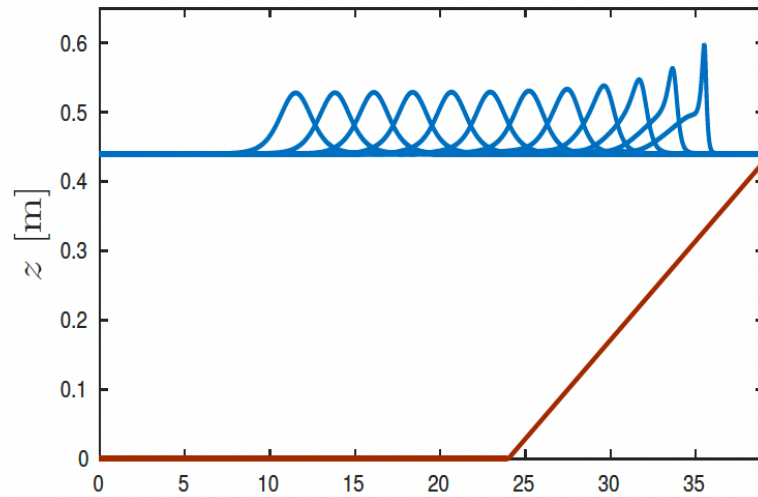
$$\mathcal{D} = \alpha \partial_x(h^2 \partial_x \dot{u}) + (\alpha - 1) \partial_x(h^2 \partial_{xx} \zeta) + \mathcal{Q}(u, b)$$

$$\begin{aligned} \mathcal{Q}(u, b) = & h \partial_x h^2 (\partial_x u)^2 + \frac{2}{3} h^3 \partial_x (\partial_x u)^2 \\ & + h^2 \partial_x b (\partial_x u)^2 + \frac{h}{2} \partial_{xx} b \partial_x u^2 + (\partial_x (h^2 \partial_{xx} b) + \partial_x (\partial_x b)^2) \frac{u^2}{2} \end{aligned}$$



Weakly dispersive (μ^2) corrections

Fully nonlinear: $\epsilon = \mathcal{O}(1)$



MultiD numerical approximation



Weakly nonlinear
Weakly dispersive

Fully nonlinear
Fully dispersive

Structured
Grids

- Madsen et al, 1992
- Nwogu, 1994
- Beji & Nadaoka, 1996
- etc. etc.

- Wei & Kirby, 1995
- Shi et al, 2012
- Lannes & Marche, 2015
- etc. etc.

Unstructured
Grids

- Walkley & Berzins, 2002
- Eskilsson & Sherwin, 2006
- Kazolea et al, 2012
- Ricchiuto & Filippini, 2014
- etc. etc.

- ☑ Filippini et al, 2017
- ☑ Duran & Marche, 2017
- ☑ Assiouene et al, 2020
- ☑ Busto et al, 2021
- ☑ etc. etc.

Certainly forgetting someone here ...

Enhanced Serre-Green-Naghdi equations in multiD

$$\partial_t h + \partial_x(hu) = 0$$

$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2) + gh\partial_x b = \mathcal{D}$$

$$\mathcal{D} = \alpha \partial_x(h^2 \partial_x \dot{u}) + (\alpha - 1) \partial_x(h^2 \partial_{xx} \zeta) + \mathcal{Q}(u, b)$$

$$\begin{aligned} \mathcal{Q}(u, b) = & h \partial_x h^2 (\partial_x u)^2 + \frac{2}{3} h^3 \partial_x (\partial_x u)^2 \\ & + h^2 \partial_x b (\partial_x u)^2 + \frac{h}{2} \partial_{xx} b \partial_x u^2 + (\partial_x(h^2 \partial_{xx} b) + \partial_x (\partial_x b)^2) \frac{u^2}{2} \end{aligned}$$

Enhanced Serre-Green-Naghdi equations in multiD

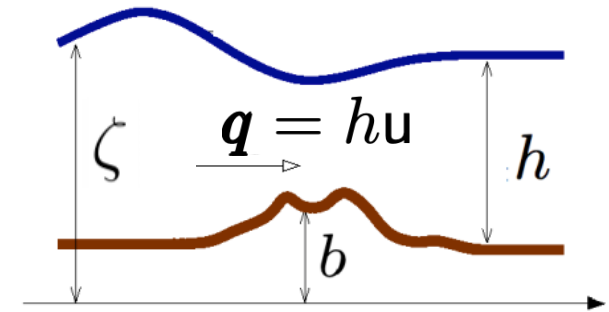
$$\partial_t h + \partial_x(hu) = 0$$

$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2) + gh\partial_x b = \varphi$$

$$\varphi - \alpha \partial_x(h\partial_x \varphi - \varphi \partial_x h) = -\partial_x(h^2 \partial_{xx} \zeta) + \mathcal{Q}$$

$$\begin{aligned} \mathcal{Q}(u, b) = & h\partial_x h^2 (\partial_x u)^2 + \frac{2}{3} h^3 \partial_x (\partial_x u)^2 \\ & + h^2 \partial_x b (\partial_x u)^2 + \frac{h}{2} \partial_{xx} b \partial_x u^2 + (\partial_x(h^2 \partial_{xx} b) + \partial_x(\partial_x b)^2) \frac{u^2}{2} \end{aligned}$$

Enhanced Serre-Green-Naghdi equations in multiD



$$\partial_t h + \nabla \cdot \mathbf{q} = 0$$

$$\partial_t \mathbf{q} + \nabla \cdot \left(\frac{\mathbf{q} \otimes \mathbf{q}}{h} \right) + gh \nabla \zeta - \boldsymbol{\varphi} = 0 \quad \text{Hyperbolic step}$$

$$\boldsymbol{\varphi} + \alpha \mathbb{T}_h(\boldsymbol{\varphi}) = \mathcal{R}(h, \mathbf{q}, b)$$

$$\mathcal{R}(h, \mathbf{q}, b) = \mathbb{T}_h(h \nabla \zeta) + \mathcal{Q} \left(\frac{\mathbf{q}}{h} \right) \quad \text{Elliptic step}$$

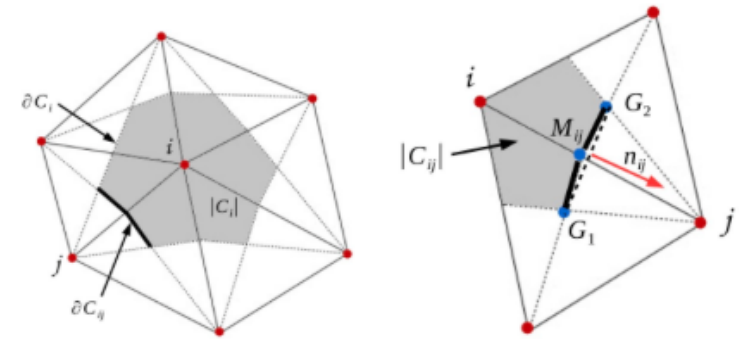
For constant bathymetry

$$\mathbb{T}_h(\boldsymbol{\varphi}) = -\nabla(h \nabla \cdot \boldsymbol{\varphi}) + \nabla(\boldsymbol{\varphi} \cdot \nabla h)$$

Enhanced Serre-Green-Naghdi solver

$$\partial_t h + \nabla \cdot \mathbf{q} = 0$$

$$\partial_t \mathbf{q} + \nabla \cdot \left(\frac{\mathbf{q} \otimes \mathbf{q}}{h} \right) + gh \nabla \zeta - \varphi = 0 \quad \text{Hyperbolic step}$$



Nodal Finite volume

Well-balanced Roe or HLL numerical fluxes/sources

Explicit high order time stepping (Runge-Kutta SSP3)

Compact nodal k-th derivative recovery via iterative corrected Green-Gauss*

Enhanced Serre-Green-Naghdi solver

$$\boldsymbol{\varphi} + \alpha \mathbb{T}_h(\boldsymbol{\varphi}) = \mathcal{R}(h, \mathbf{q}, b)$$

$$\mathcal{R}(h, \mathbf{q}, b) = \mathbb{T}_h(h \nabla \zeta) + \mathcal{Q} \left(\frac{\mathbf{q}}{h} \right) \quad \text{Elliptic step}$$

For constant bathymetry

$$\mathbb{T}_h(\boldsymbol{\varphi}) = -\nabla(h \nabla \cdot \boldsymbol{\varphi}) + \nabla(\boldsymbol{\varphi} \cdot \nabla h)$$

We solve it with standard H1 linear finite elements (not in H(div) ...) :

- Block SPD structure
- H1 is not the natural space
 - > spurious “curl modes” need stabilization/damping

Enhanced Serre-Green-Naghdi solver

On the stability of $\mathbf{I} - \text{grad div}$:

$H(\text{div})$ conforming FE space

Stability in H^1

Curl stabilization: $H^1 = H(\text{div}) + H(\text{curl})$ **Costabel**, J.Math.Anal.Appl. 1991

Bonnet-Ben Dhia et al, CRAS 2001, **Bonnet-Ben Dhia et al**, J. Comput. Appl. Math. 2007

Mixed form + stabilization, **Bonito et al**, M2NA 2016, **Chabassier & Duruflé** 2018

Laplacian stabilization

Mardal et al, SINUM 2002

Enhanced Serre-Green-Naghdi solver

On the stability of $I - \text{grad div}$:

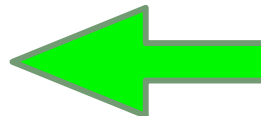
$$\varphi = (I + \alpha T_h)^{-1} \mathcal{R}(h^n, \mathbf{q}^n, b)$$

$$\frac{h^{n+1} - h^n}{\Delta t} + \nabla \cdot \hat{\mathbf{q}}^n = \nabla \cdot (D_h \widehat{\nabla} h^n)$$

$$\frac{\mathbf{q}^{n+1} - \mathbf{q}^n}{\Delta t} + \nabla \cdot \hat{\mathbf{F}}_q^n + \mathbf{S}_b^n - \varphi^n = \nabla \cdot (D_q \widehat{\nabla} \mathbf{q}^n)$$

○ Laplacian stabilization

Mardal et al, SINUM 2002

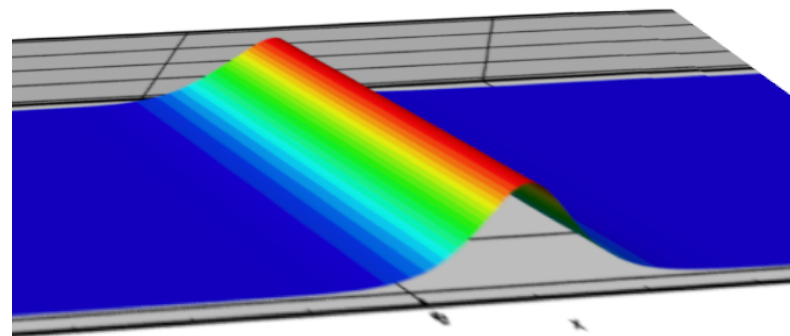
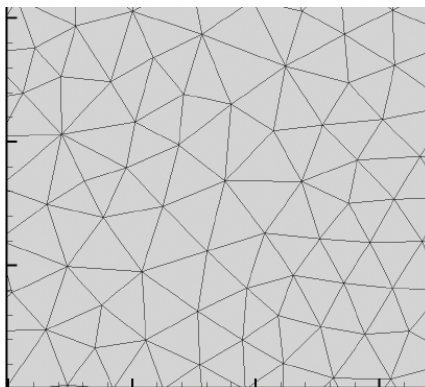
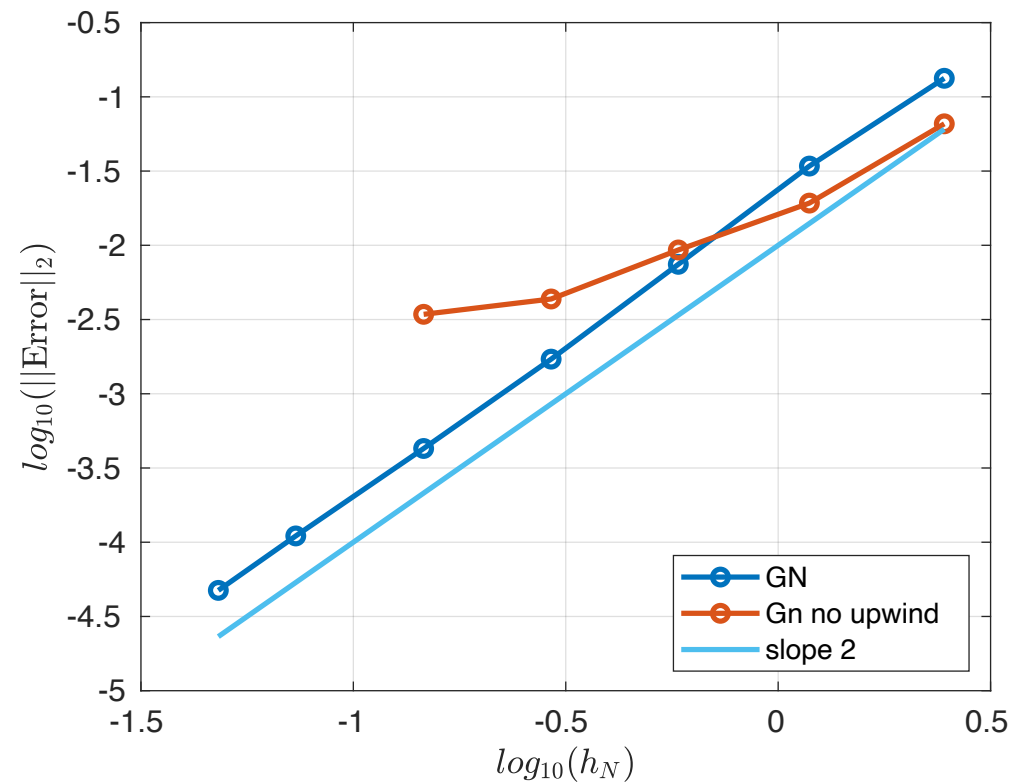
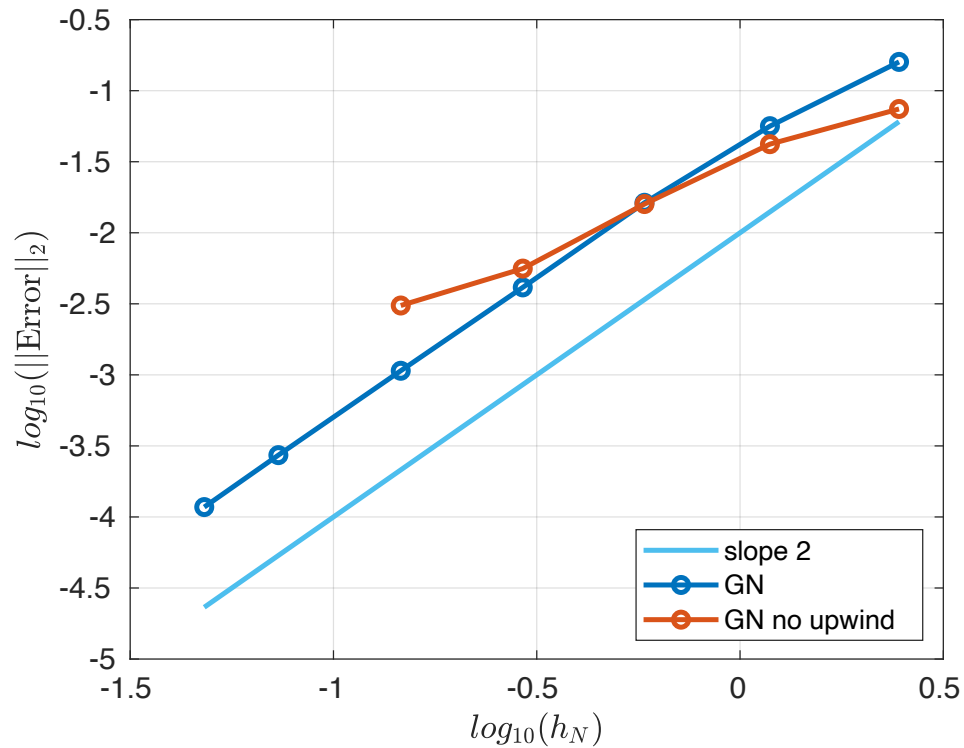


Embedded in FV numerical fluxes

Parabolic damping of spurious curl modes

Kazolea et al, Ocean Mod., 2023

Enhanced Serre-Green-Naghdi solver



Analytical solitary wave
 2d convergence
 Unstructured triangulations

Enhanced Serre-Green-Naghdi solver**The formal consistency of the method is**

$$\partial_t h + \nabla \cdot \mathbf{q} = \mathcal{O}(\Delta x^3)$$

$$(I + \alpha \mathbf{T}_h)(\partial_t \mathbf{q} + \nabla \cdot F_{\mathbf{q}} + \mathbf{S}_b) = \mathcal{R} + \mathcal{O}(\mu^2 \Delta x^2)$$

When using third order polynomial reconstruction and RK3

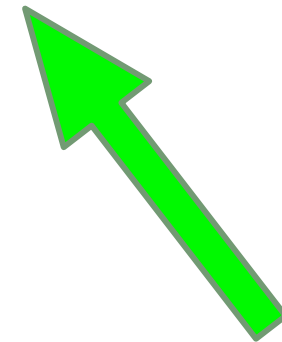
Enhanced Serre-Green-Naghdi solver

The formal consistency of the method is

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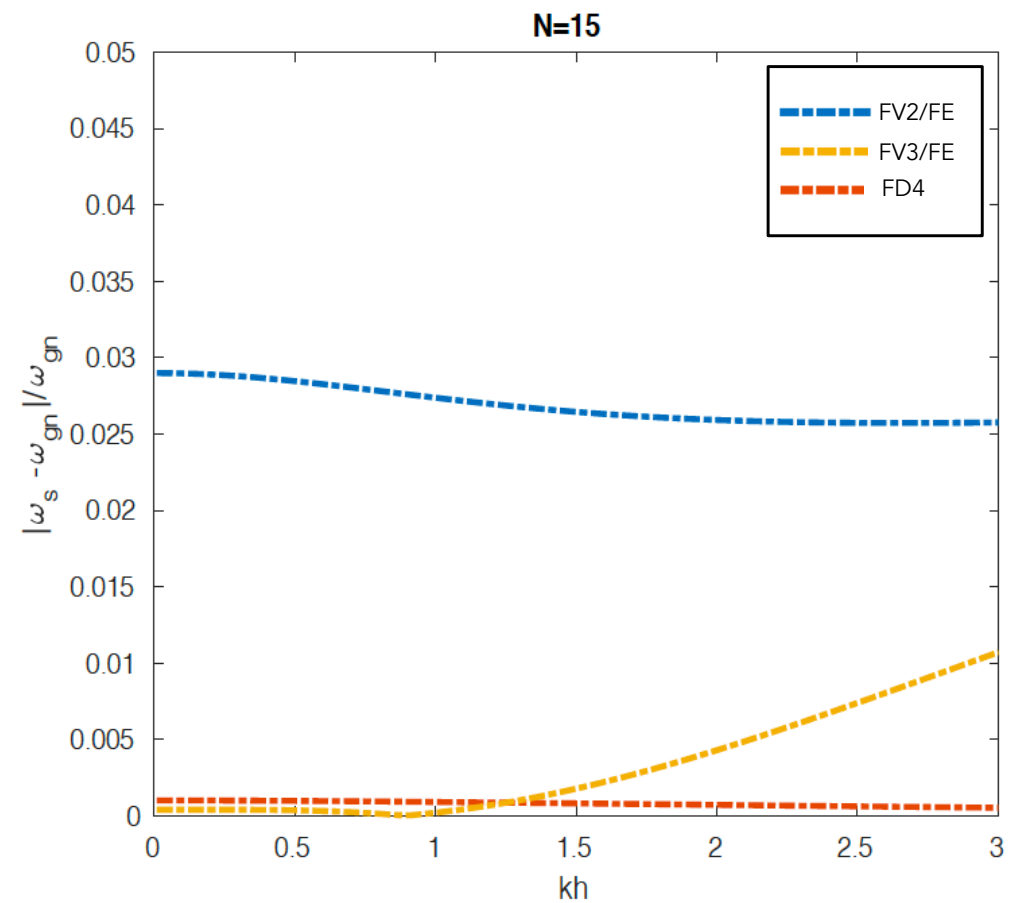
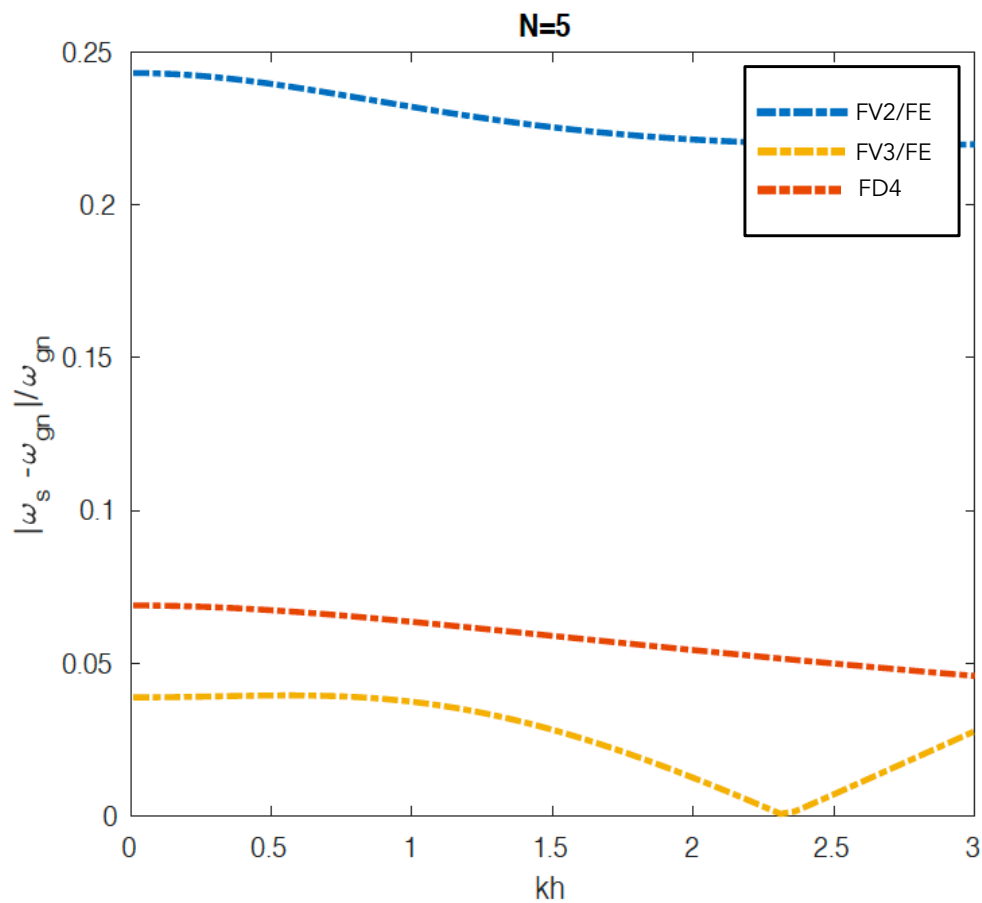
When using third order polynomial reconstruction and RK3



What is the effect of this ?

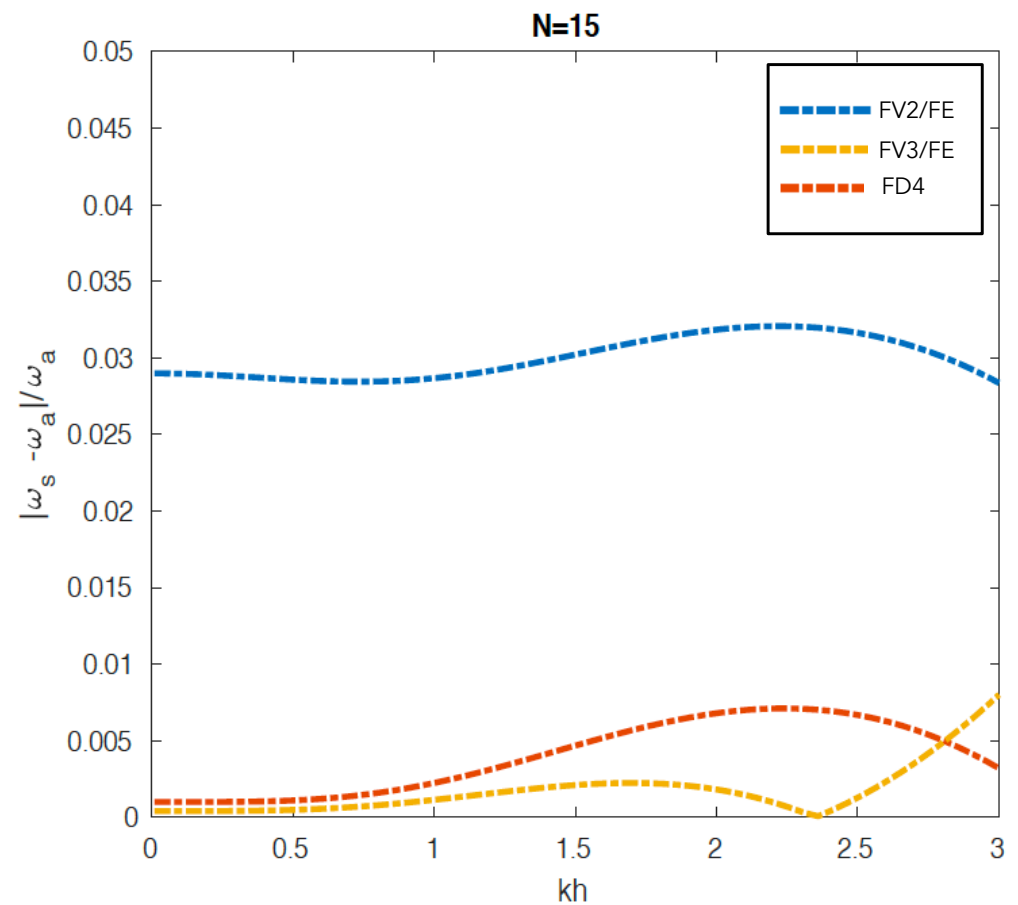
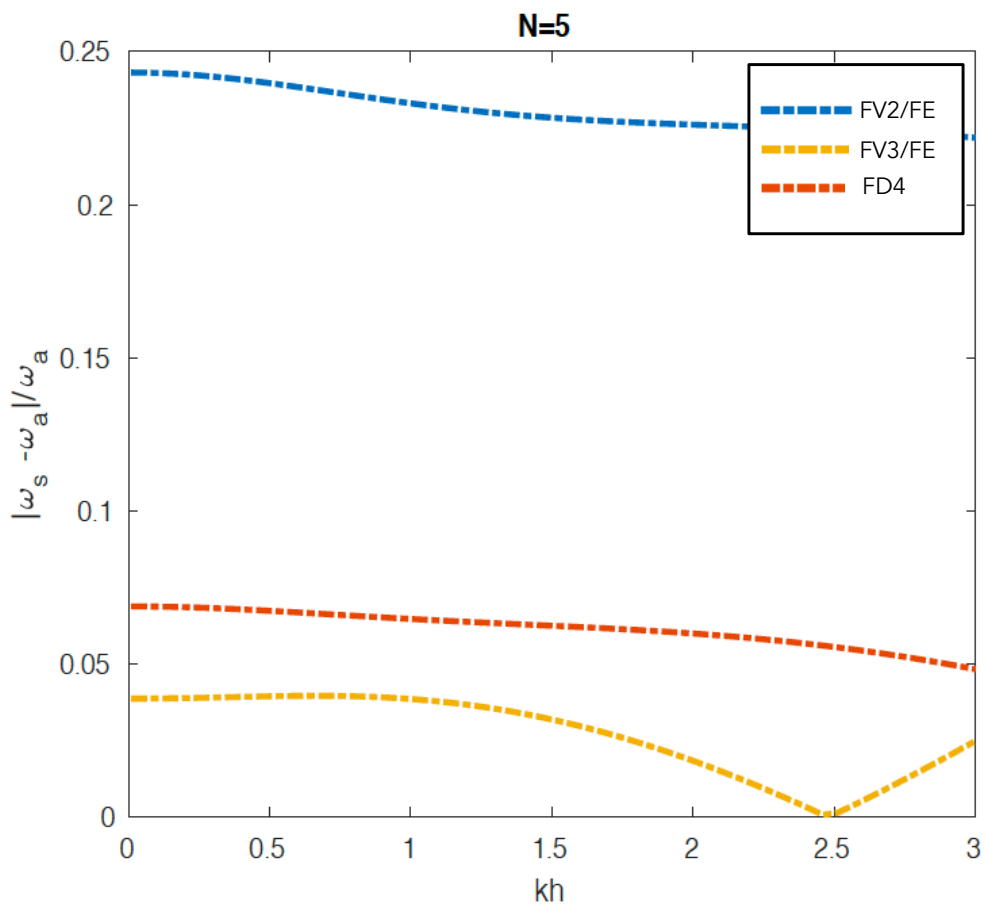
Enhanced Serre-Green-Naghdi solver

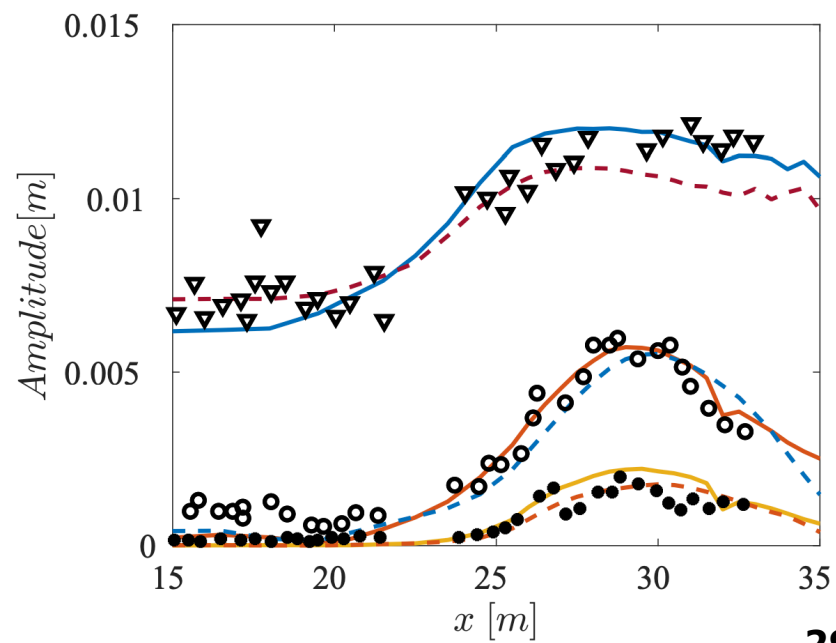
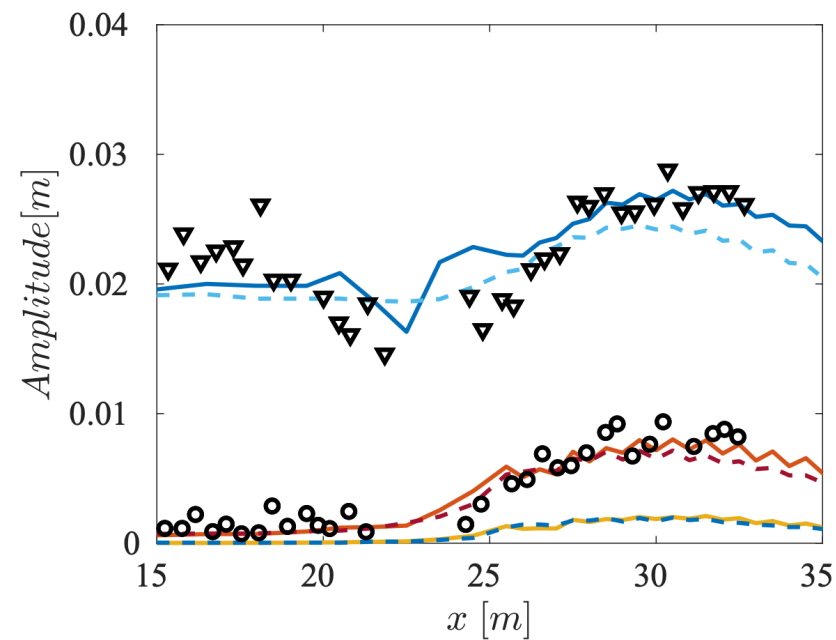
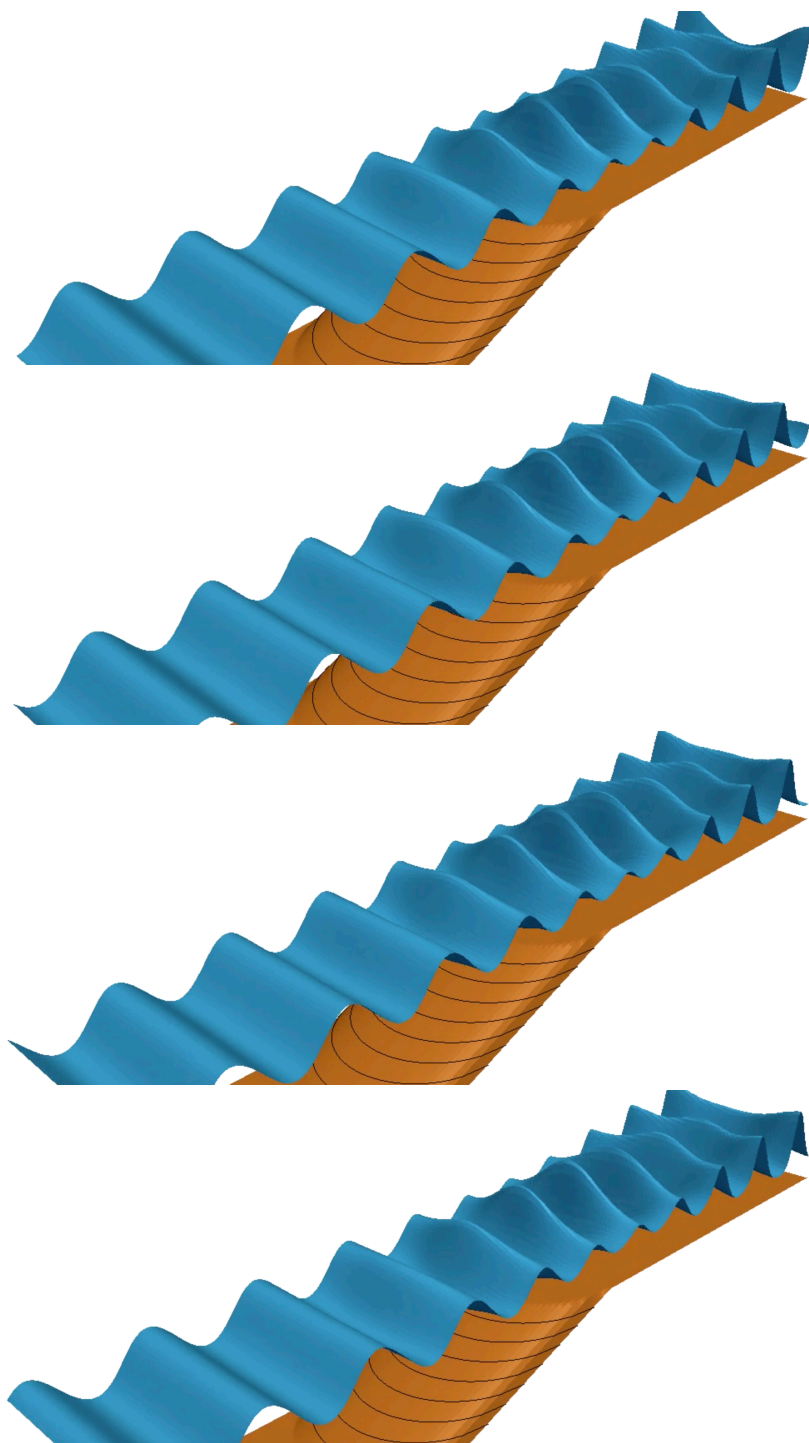
Dispersion error (time continuous): $c(\kappa)_{\text{num}} - c(\kappa)_{\text{GN}}$

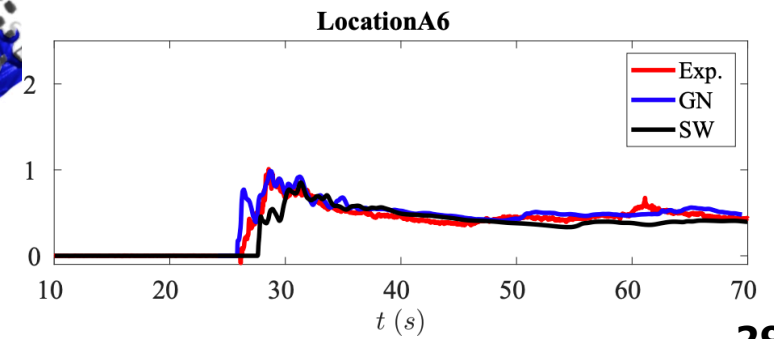
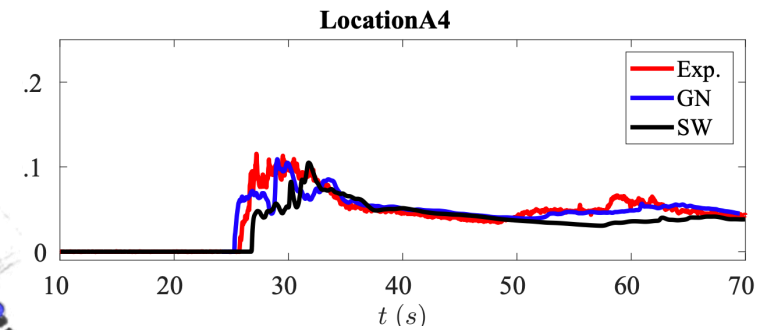
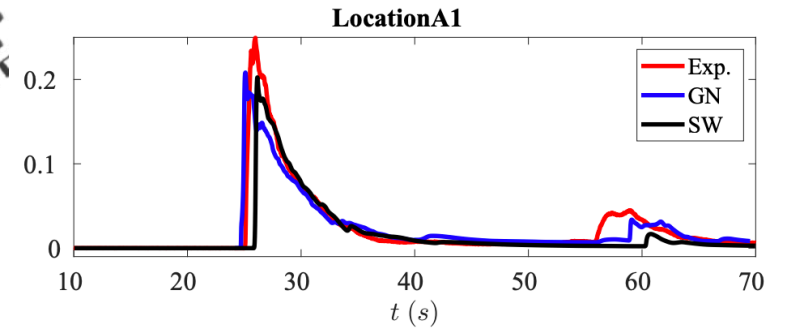
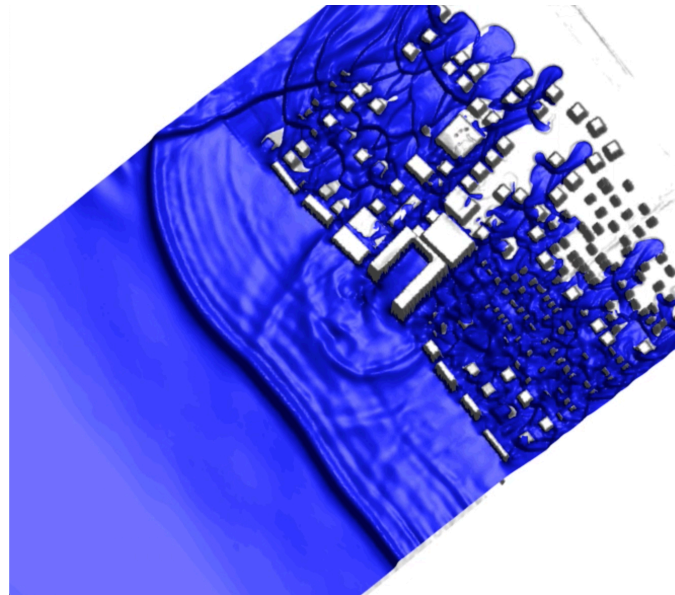
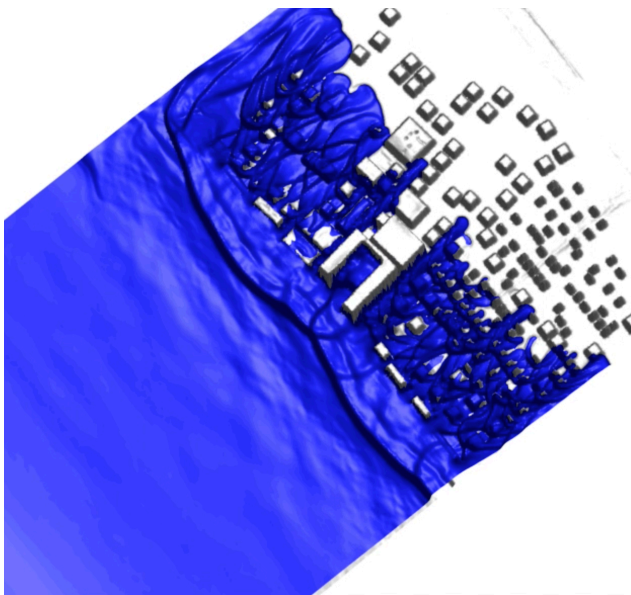
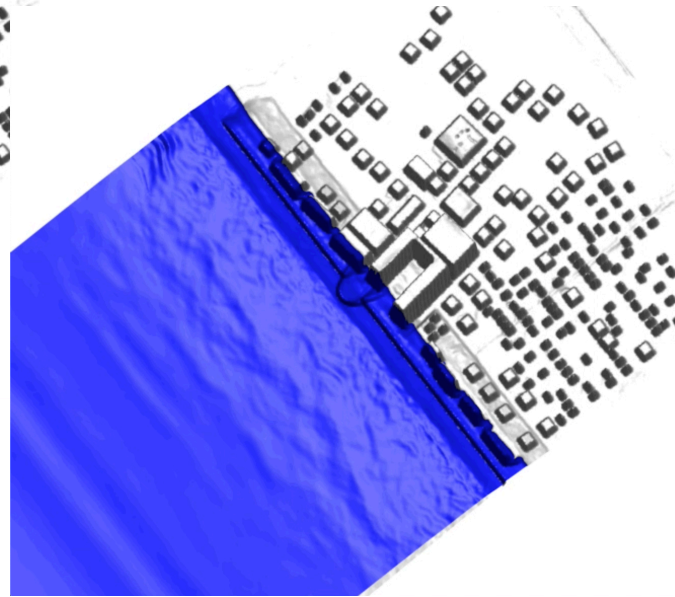
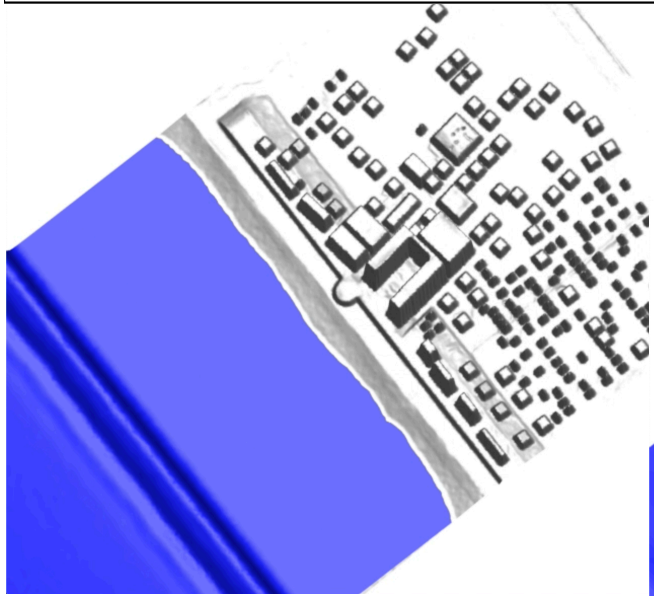


Enhanced Serre-Green-Naghdi solver

Dispersion error (time continuous): $c(\kappa)_{\text{num}} - c(\kappa)_{\text{Euler}}$

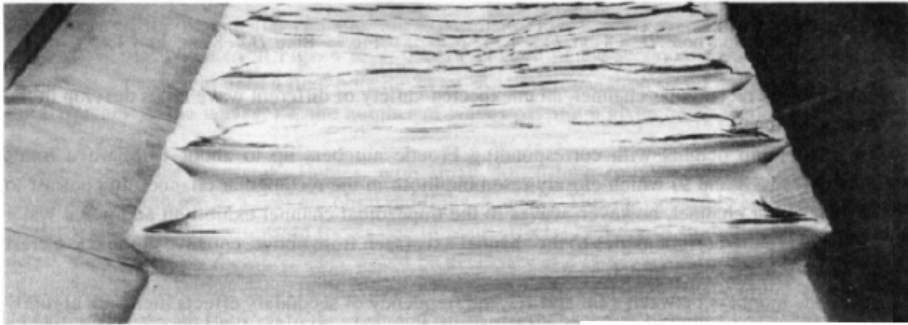




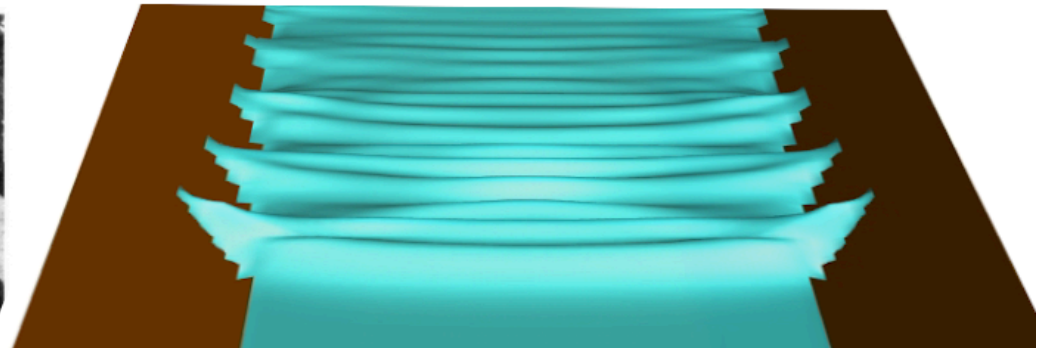
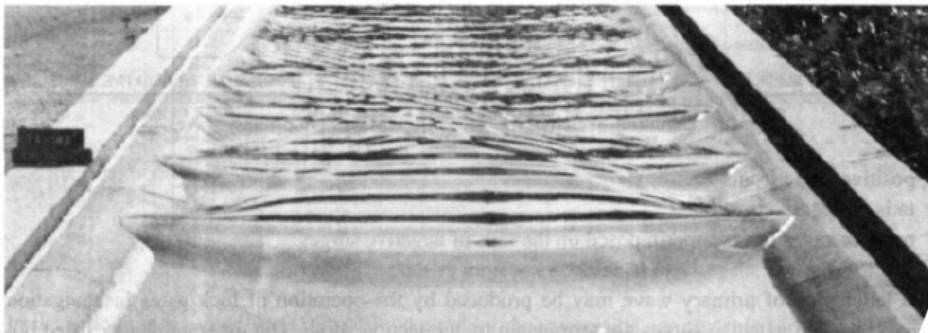


Undular bores simulations

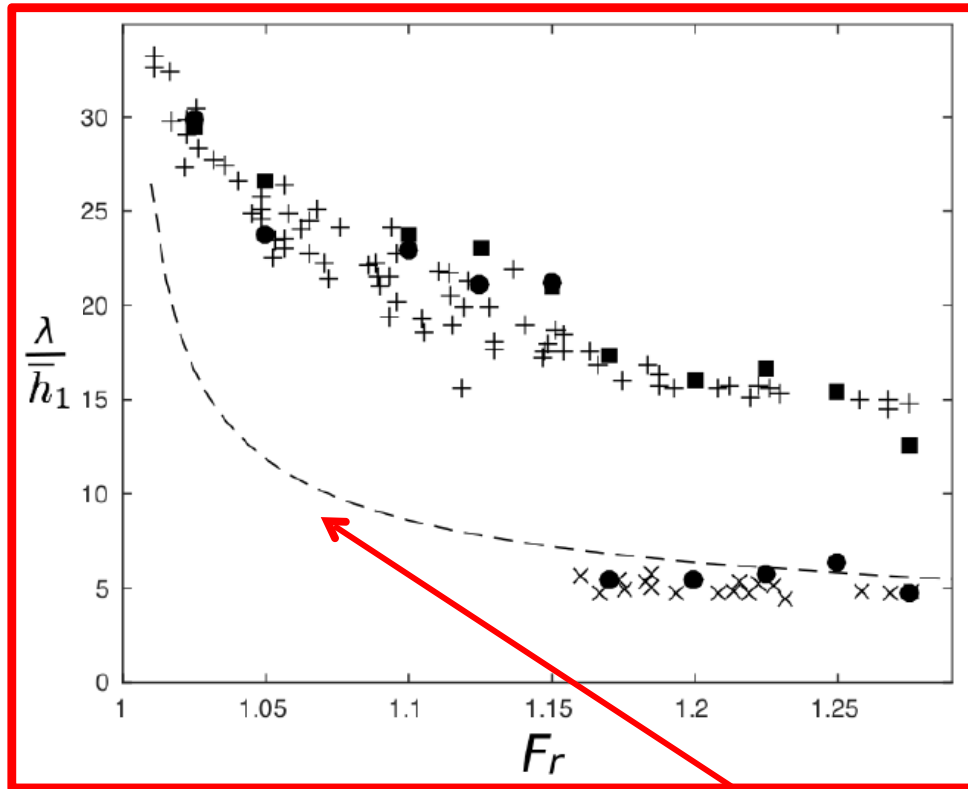
$Fr = 1.10$



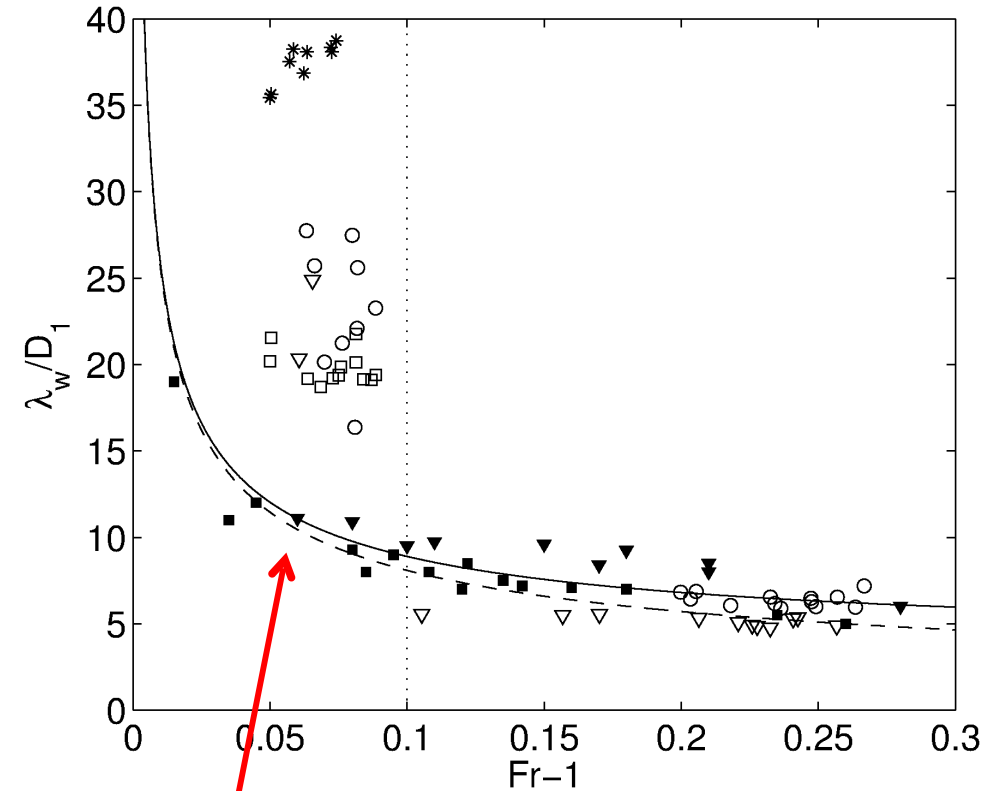
$Fr = 1.17$



Treske, *J. Hydraulic Research*, 1994



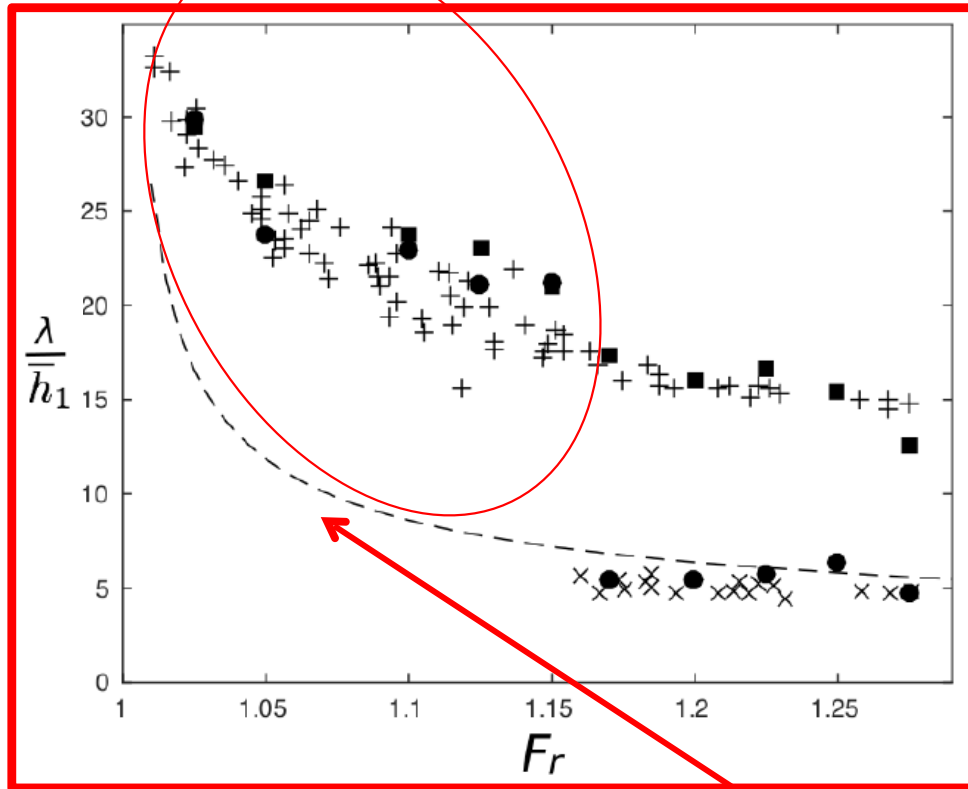
Bonneton et al, *J. Geophysical Research - Oceans*, 2015



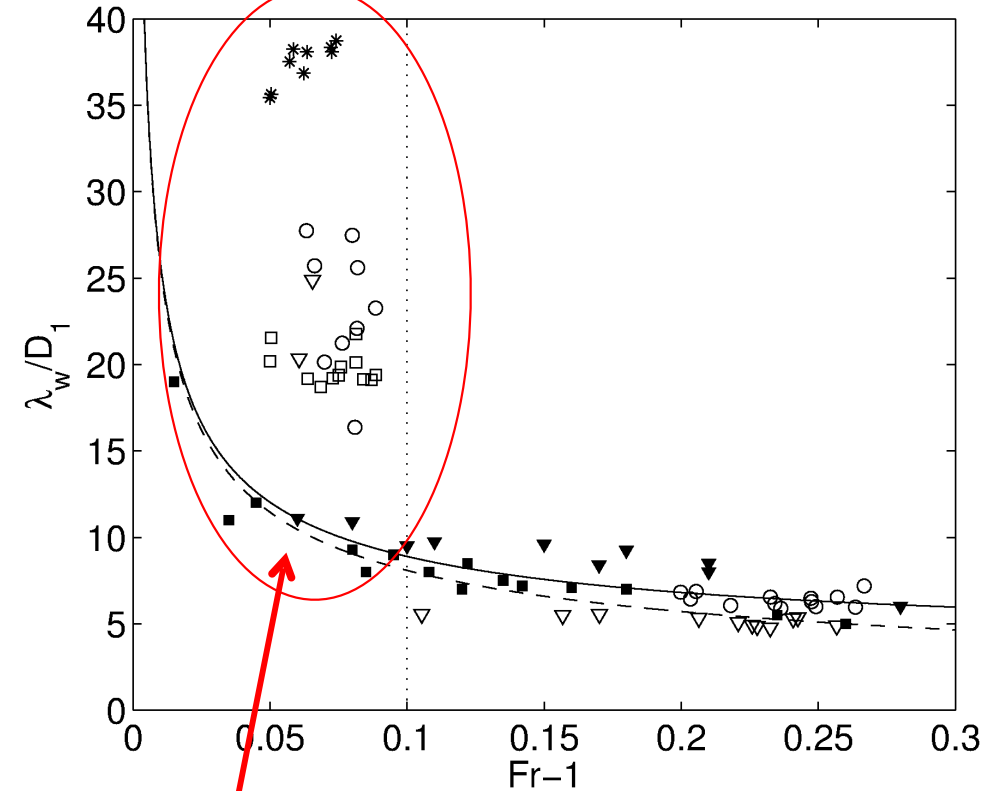
**Comparison with data:
transition from low Froude
to high Froude regime**

Lemoine theory for Favre waves

Treske, *J. Hydraulic Research*, 1994



Bonneton et al, *J. Geophysical Research - Oceans*, 2015



**Comparison with data:
transition from low Froude
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Lemoine theory for Favre waves

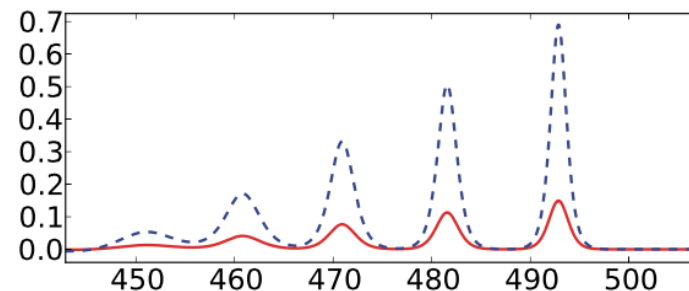
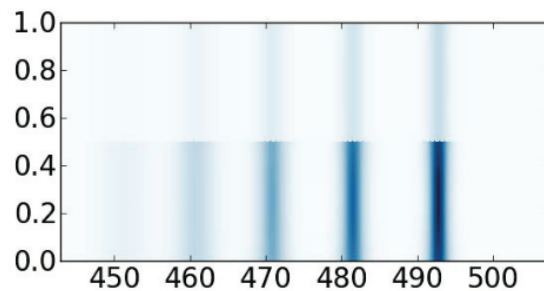
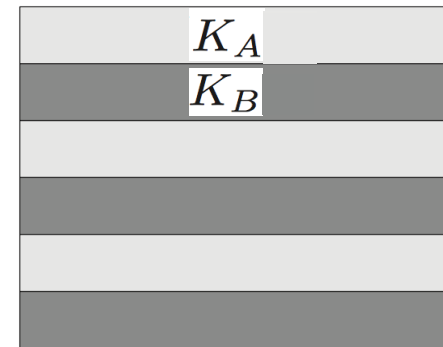
Asymptotic analysis

Several elements suggest that it may be an hydrostatic phenomenon:

- It is predominant on the banks (very shallow limit)
- It involves long(er) waves
- Dispersion in wave propagation in heterogenous media

Ketcheson & Quessada de Luna, Multiscale Mod. Simul., 2015

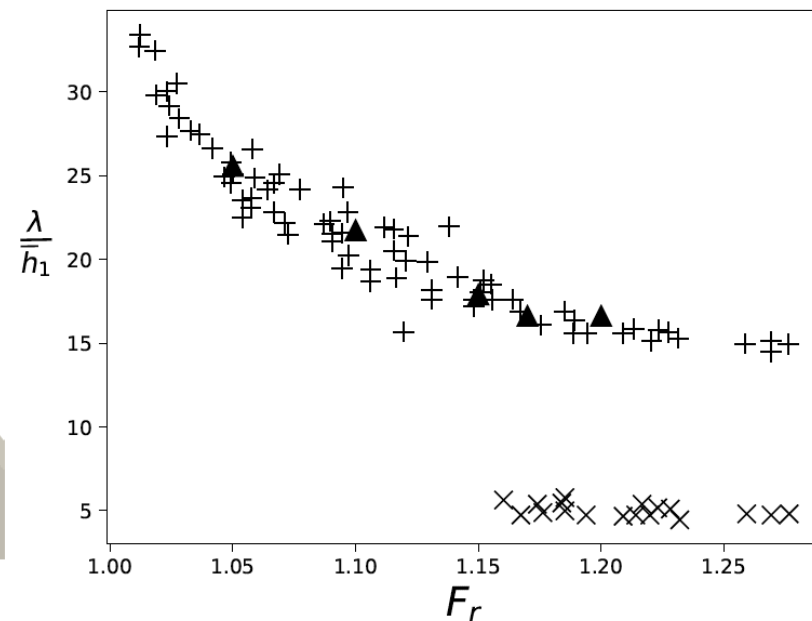
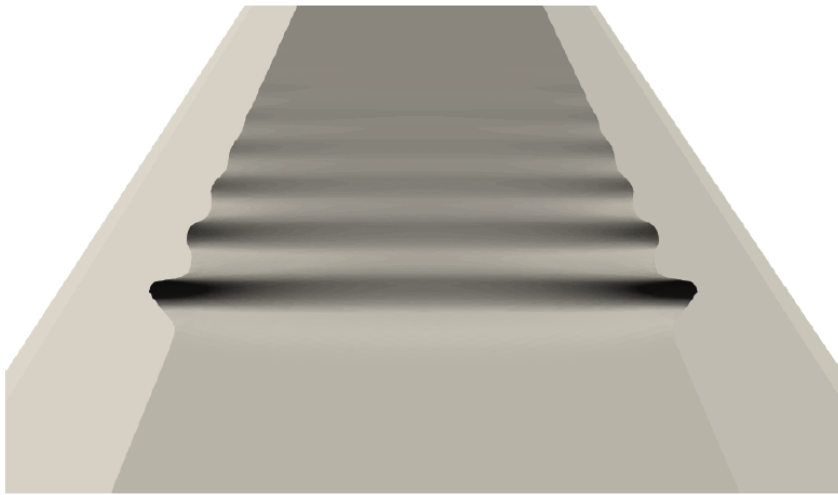
$$\epsilon_{tt} - \nabla \cdot \left(\frac{1}{\rho(\mathbf{x})} \nabla \sigma(\epsilon, \mathbf{x}) \right) = 0, \quad \sigma(\epsilon, \mathbf{x}) = \exp(K(\mathbf{x})\epsilon) - 1$$



Several elements suggest that it may be an hydrostatic phenomenon:

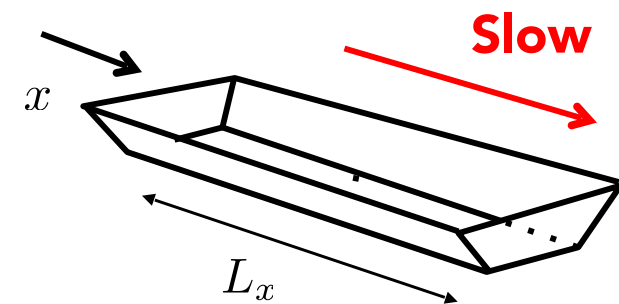
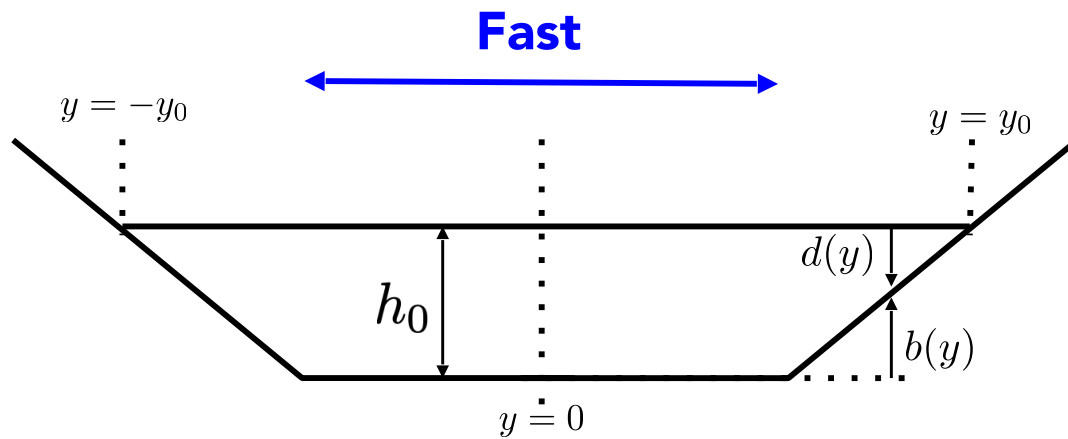
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Ketcheson & Quessada de Luna, Multiscale Mod. Simul., 2015



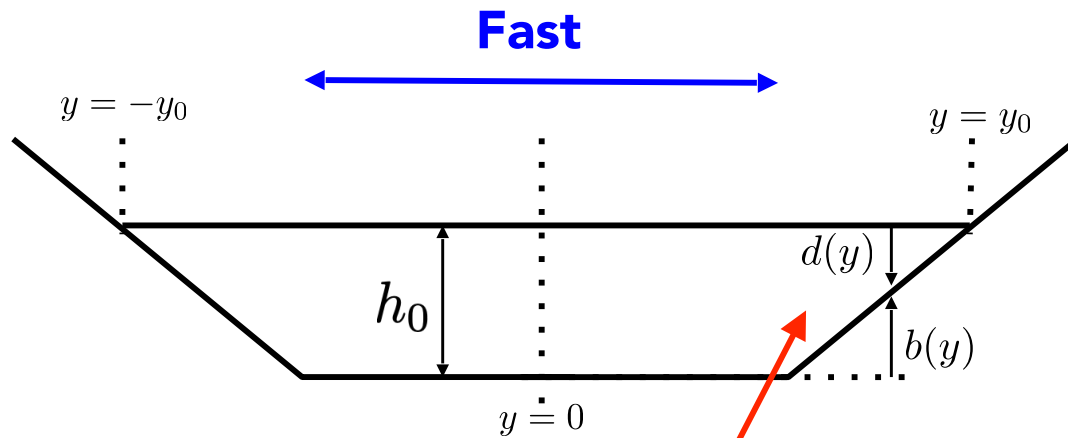
Shallow water simulations !

- Shallow water waves (hydrostatic, no dispersion terms)
- Linear waves
- Scale separation between transverse (fast) and longitudinal (slow) waves

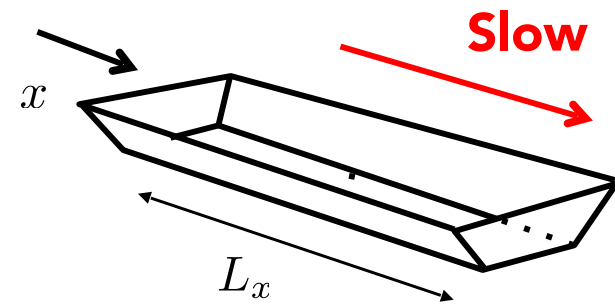


$$\tau_y \ll \tau_x$$

$$L_y = \tau_y \sqrt{gh_0} \ll L_x \quad \longrightarrow \quad \delta = \frac{L_y}{L_x} = \frac{\tau_y}{\tau_x} \ll 1$$



$$d = d(y)$$

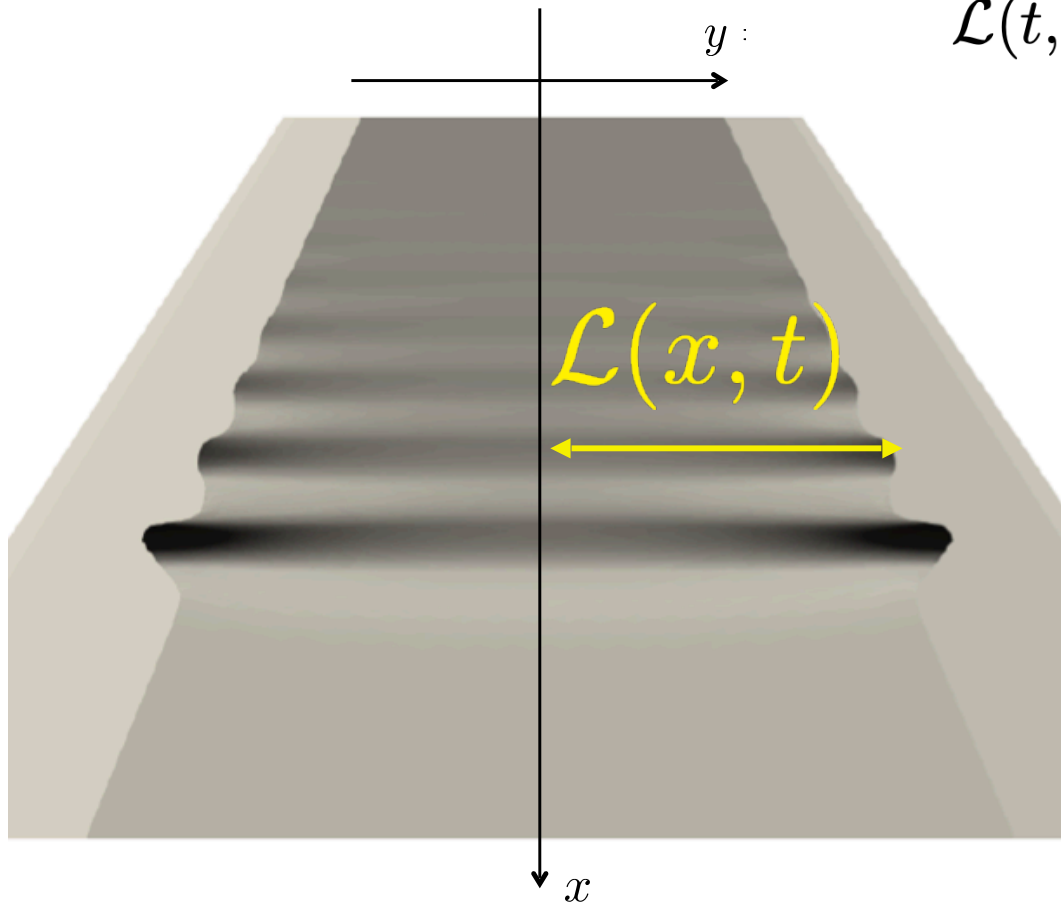


2D NLSW in dimensionless form

$$\partial_t \zeta + \partial_x((d + \epsilon \zeta)u) + \frac{1}{\delta} \partial_y((d + \epsilon \zeta)v) = 0$$

$$\partial_t u + \epsilon u \partial_x u + \frac{\epsilon}{\delta} v \partial_y u + \partial_x \zeta = 0$$

$$\partial_t v + \epsilon u \partial_x v + \frac{\epsilon}{\delta} v \partial_y v + \frac{1}{\delta} \partial_y \zeta = 0$$



$$\mathcal{L}(t, x) = y_0 + \frac{\epsilon}{\delta} \int_0^t v(s, x, y_{bank}(s, x)) ds$$

$$y = \pm \mathcal{L} \Rightarrow hv = 0$$

for banks and
straight walls

$$\overline{(\cdot)} := \frac{1}{2\mathcal{L}} \int_{-\mathcal{L}(t,x)}^{\mathcal{L}(t,x)} (\cdot)(t, x, y) dy$$

Linearized problem

$$\delta(\partial_t \zeta + d \partial_x u) + \partial_y (dv) = 0$$

$$\partial_t u + \partial_x \zeta = 0$$

$$\delta \partial_t v + \partial_y \zeta = 0$$

$$d = d(y)$$

With

$$\mathcal{L} = y_0 \Rightarrow y \in [-y_0, y_0]$$

$$y = y_0 \Rightarrow dv = 0$$

for banks and
straight walls

Linearized problem

$$\overline{(\cdot)} = \frac{1}{2y_0} \int_{-y_0}^{y_0} (\cdot) dy$$

$$\delta(\partial_t \zeta + d\partial_x u) + \partial_y(dv) = 0$$

$$\partial_t u + \partial_x \zeta = 0$$

$$\delta\partial_t v + \partial_y \zeta = 0$$

$$d = d(y)$$

$$y = y_0 \Rightarrow dv = 0$$

Linearized problem

$$\overline{(\cdot)} = \frac{1}{2y_0} \int_{-y_0}^{y_0} (\cdot) dy$$

$$\delta(\partial_t \zeta + d\partial_x u) + \partial_y(dv) = 0$$

$$\partial_t u + \partial_x \zeta = 0$$

$$\delta\partial_t v + \partial_y \zeta = 0$$

$$d = d(y)$$

$$y = y_0 \Rightarrow dv = 0$$

$$\overline{\zeta}_{tt} - \overline{(d\zeta)_{xx}} = 0$$

This is exact

$$\zeta = \sum_{j \geq 0} \delta^j \zeta_j, \quad u = \sum_{j \geq 0} \delta^j u_j, \quad v = \sum_{j \geq 0} \delta^j v_j$$

$$\delta(\partial_t \zeta + d\partial_x u) + \partial_y(dv) = 0$$

$$\partial_t u + \partial_x \zeta = 0$$

$$\delta\partial_t v + \partial_y \zeta = 0$$

$$\zeta = \sum_{j \geq 0} \delta^j \zeta_j, \quad u = \sum_{j \geq 0} \delta^j u_j, \quad v = \sum_{j \geq 0} \delta^j v_j$$

$$\partial_y(dv_{n+1}) = -(\partial_t \zeta_n + d\partial_x u_n) \quad \text{with BCs } dv = 0$$

$$\partial_t u_{n+1} = -\partial_x \zeta_{n+1}$$

$$\partial_y \zeta_{n+1} = -\partial_t v_n \implies \zeta_{n+1} = \bar{\zeta}_{n+1} + \boxed{Z_{n+1}} - \bar{Z}_{n+1}$$

arbitrary primitive of $-\partial_t v_n$

$$\zeta = \sum_{j \geq 0} \delta^j \zeta_j, \quad u = \sum_{j \geq 0} \delta^j u_j, \quad v = \sum_{j \geq 0} \delta^j v_j$$

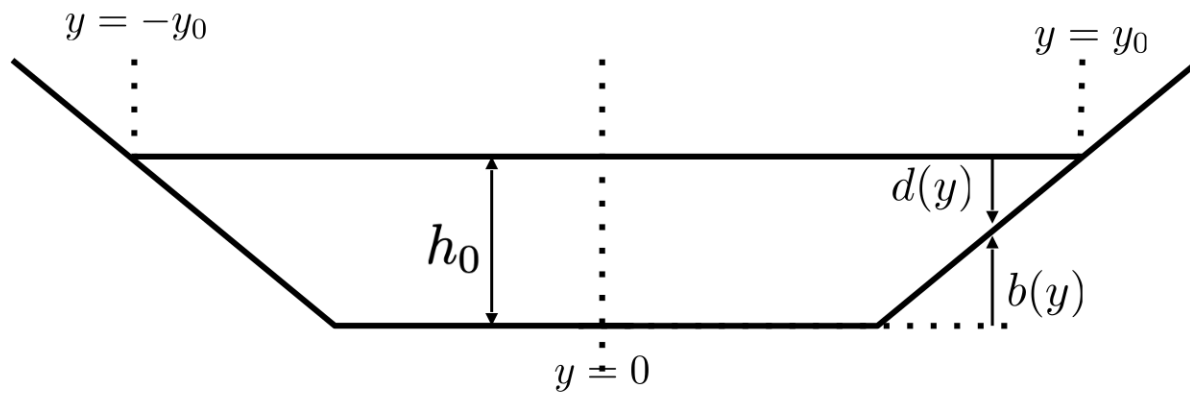
$$\partial_y(dv_{n+1}) = -(\partial_t \zeta_n + d\partial_x u_n) \quad \text{with BCs } dv = 0$$

$$\partial_t u_{n+1} = -\partial_x \zeta_{n+1}$$

$$\partial_y \zeta_{n+1} = -\partial_t v_n \implies \zeta_{n+1} = \bar{\zeta}_{n+1} + Z_{n+1} - \bar{Z}_{n+1}$$

With C.I.

$$\zeta_0 = \bar{\zeta}, \quad \partial_t u_0 = -\partial_x \zeta_0, \quad v_0 = 0$$



$$\zeta(x, y, t) = \bar{\zeta}(x, t) + \delta^2 (K(y) - \bar{K}) \bar{\zeta}_{xx} + \mathcal{O}(\delta^4)$$

$$K(y) := \int_{-y_0}^y \frac{h_0(y_0 + s) - D(s)}{d(s)} ds$$

$$D(y) := \int_{-y_0}^y d(s) ds$$

$$\bar{\zeta}_{tt} - \overline{(d\zeta)}_{xx} = 0$$



$$\bar{\zeta}_{tt} - c_0^2 \bar{\zeta}_{xx} - \chi c_0^2 \bar{\zeta}_{xxxx} = 0$$

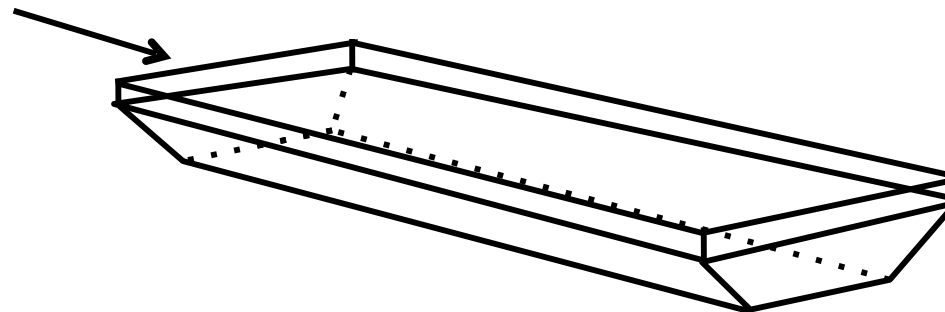
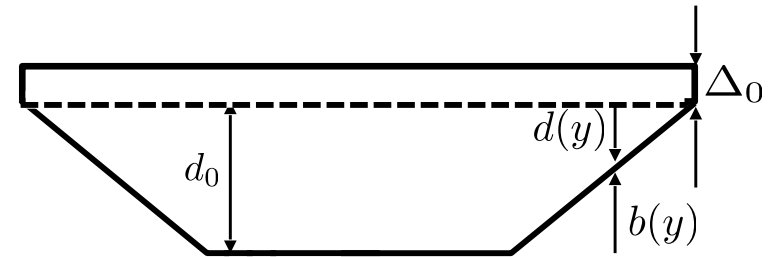
$$\chi := \overline{d(y)(K(y) - \bar{K})}$$

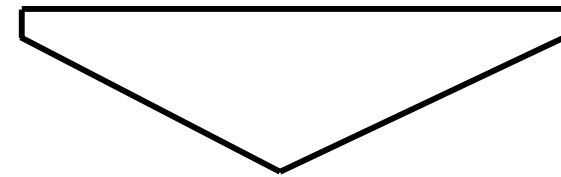
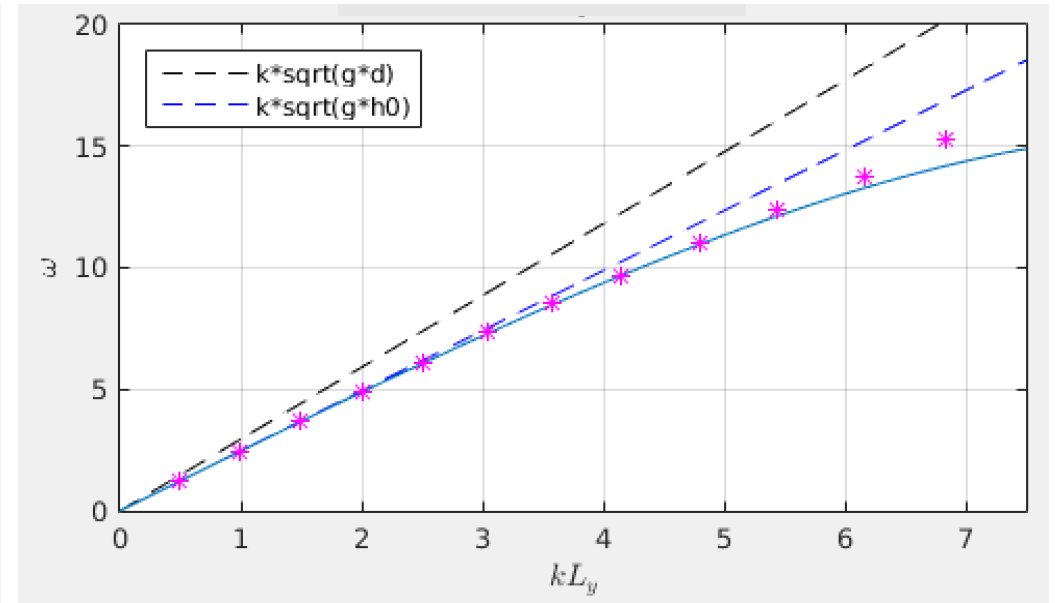
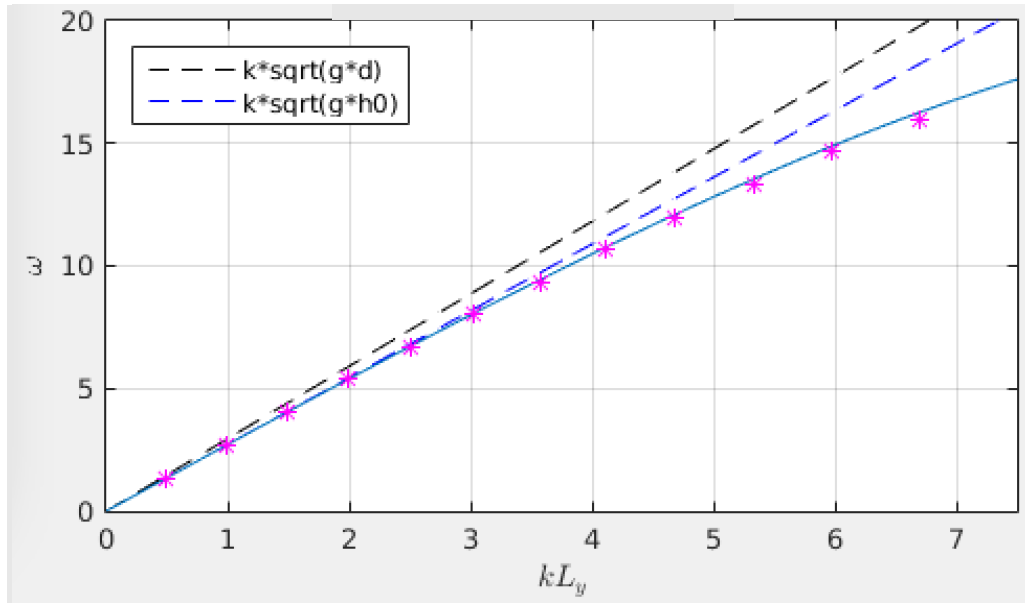
Dispersive behaviour due to diffraction within each section

$$\omega^2 = \kappa^2 c_0^2 (1 - \chi(\kappa y_0^2))$$

Shallow water simulations:

- 1- Linear periodic signal imposed at the inlet
- 2- The whole signal is (section-)averaged
- 3- Period and wavelength are measured downstream

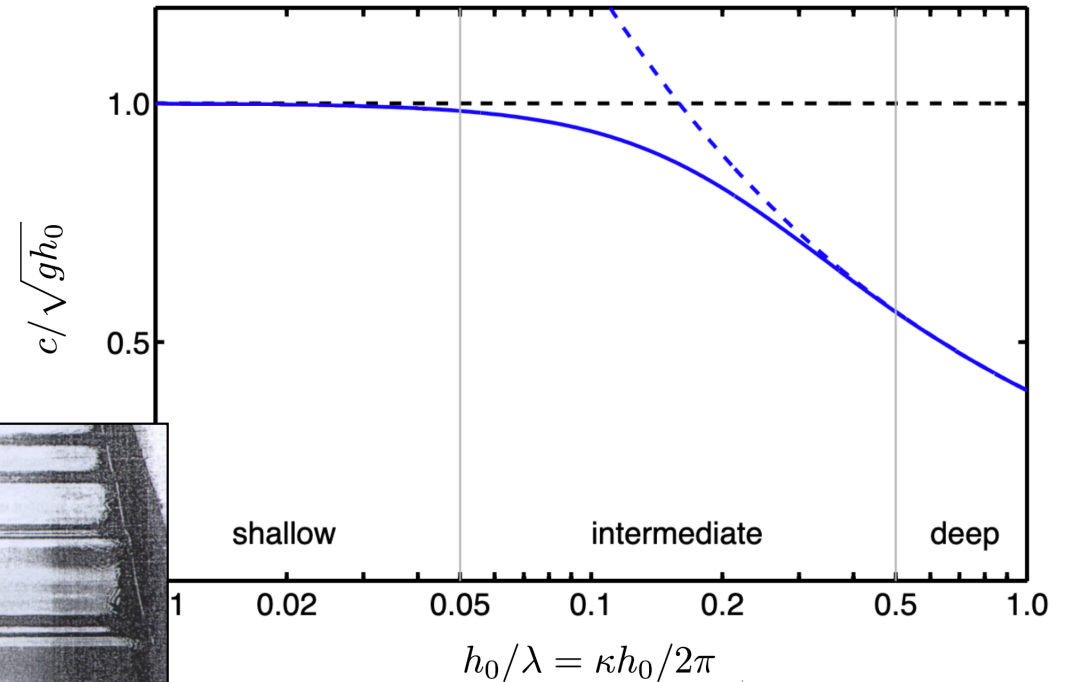
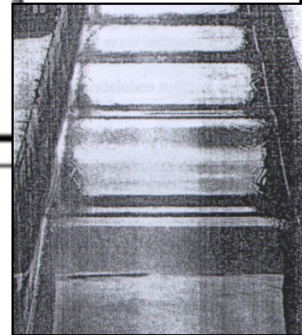
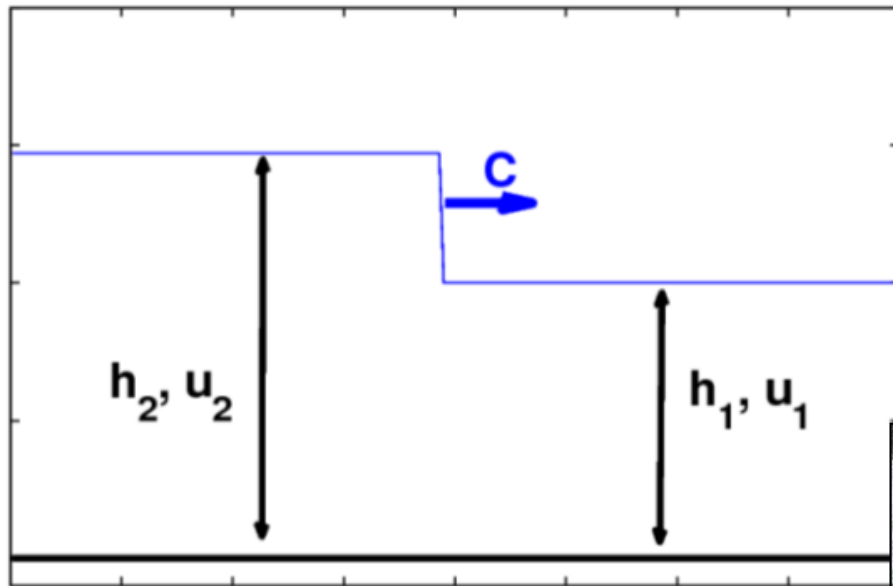




Asymptotic model



Simulations



Bore:

given the upstream/downstream conditions and using the jump conditions:

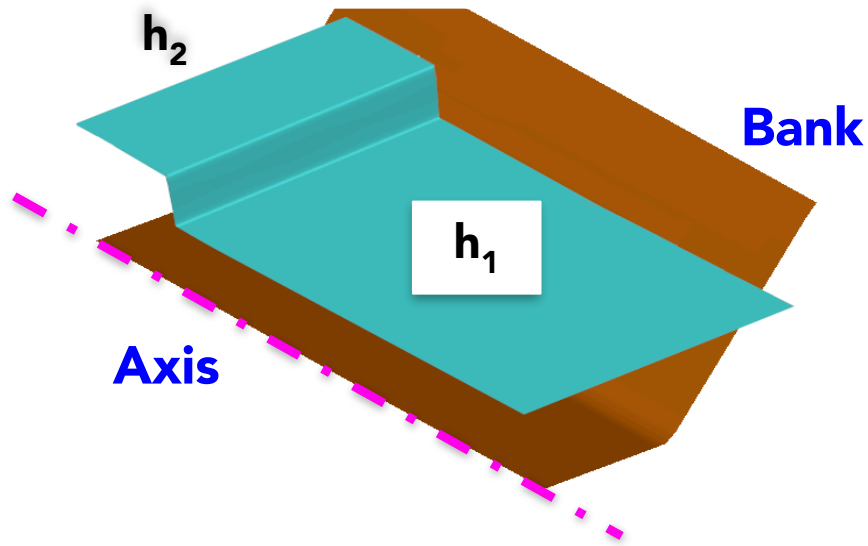
$$C = C(Fr) = C(h_2/h_1)$$

Water waves:

dispersion relation based on the linearized Euler equations (Airy theory)

$$C = C(\lambda) = \sqrt{\frac{g\lambda}{2\pi} \tanh(2\pi h/\lambda)}$$

$$C(Fr) = C(\lambda) \implies \lambda(Fr)$$



$$\bar{\zeta}_{tt} - \bar{\zeta}_{xx} - \delta^2 \chi \bar{\zeta}_{xxxx} = 0$$

$$\chi := \overline{d(y)(K(y) - \bar{K})}$$

Bore:

Jump conditions for section-averaged NLSW

Chanson, Elsevier, 2004

$$C = C(Fr)$$

Dispersive like-waves:

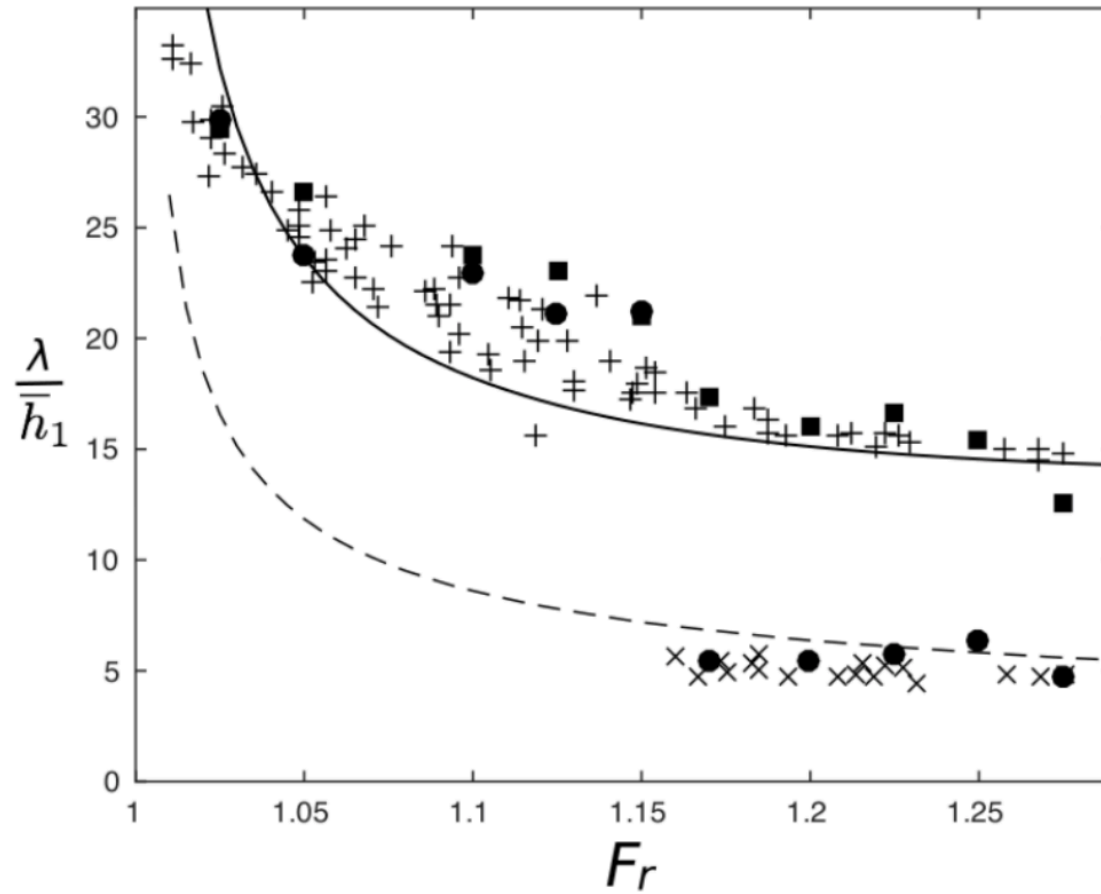
dispersion relation

$$C = C(\lambda)$$

$$\lambda = \lambda(Fr)$$

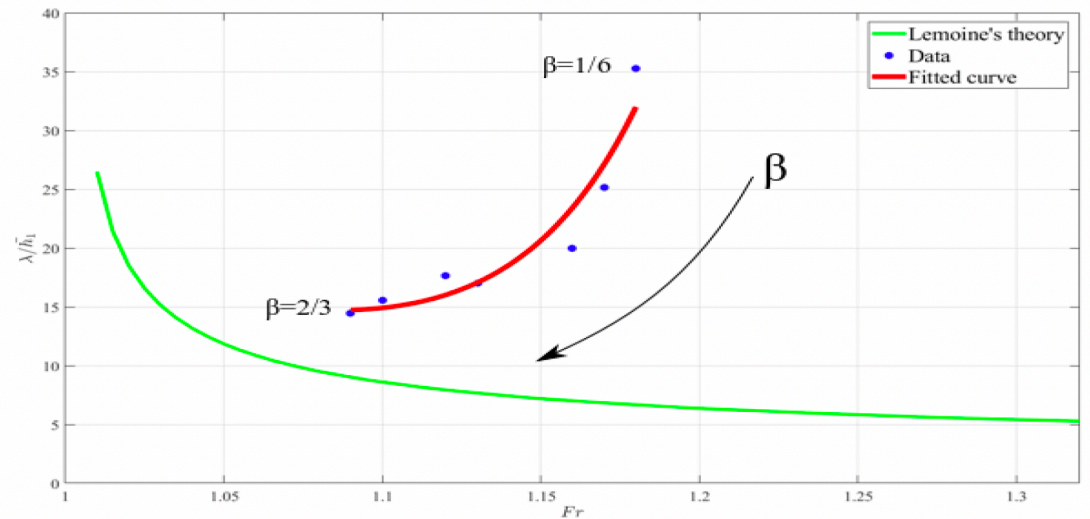
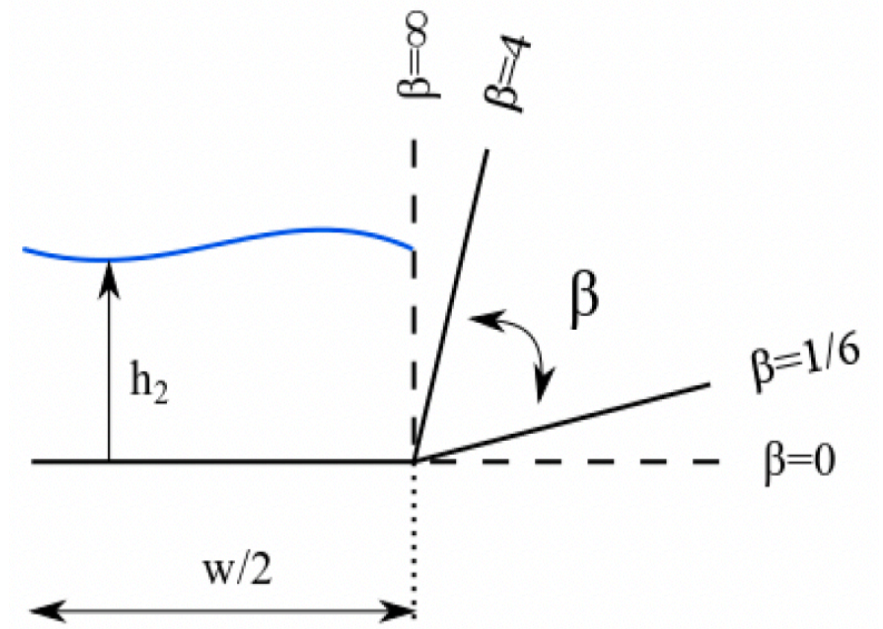
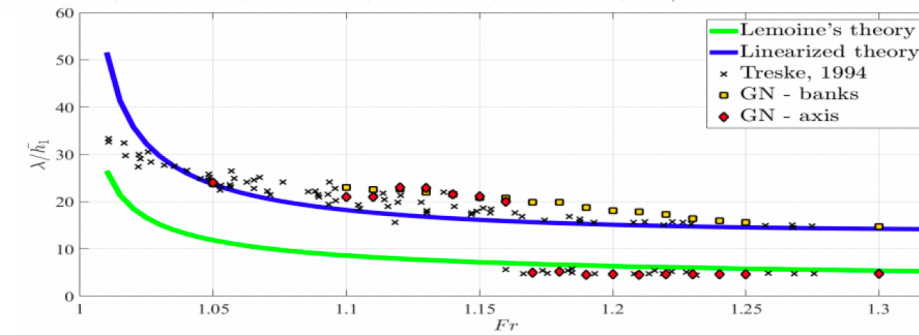
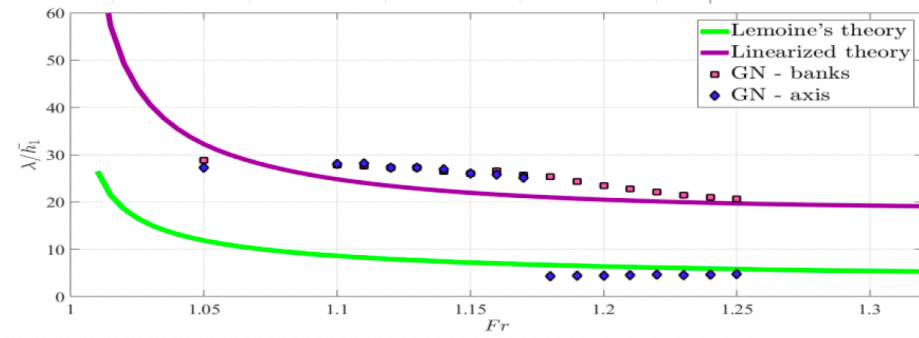
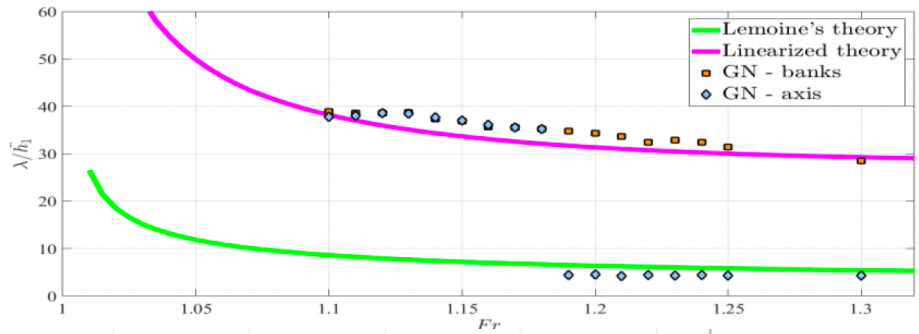
Serre-Green-Naghdi simulations

Compared to data and theory



Asymptotic analysis

Geometrical parameters



Ongoing foreseen work and open issues

Modelling

- Asymptotic analysis including :
 - some geometrical nonlinearity
 - “vertical” dispersion (on a Boussinesq model): connect the two dispersive behaviours
 - Nonlinear version of the above : 1D model for channels with some potential for applications (including both regimes)
- Why the transition : modulation equations (see work of El and Gavriluk) ?
- Coupled model with river bed morphology

Ongoing foreseen work and open issues

Numerics

- H(div) vs curl augmented approach vs current one for the elliptic pb
- Boundary conditions for the elliptic pb
- Further work on efficiency/accuracy of scheme (elliptic vs hyperbolic)
- Impact of numerical dissipation:
 - Use energy conservative fluxes (for SW)
 - Full energy conservation for GN
 - Dissipation vs solitary wave fission (cf work of EI, Physics D 2016)
- Physics based dissipation



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Related work

M. Ricchiuto and A.G. Filippini, Upwind residual discretization of enhanced Boussinesq equations for wave propagation over complex bathymetries, *J.Comput.Phys.* 271, 2014

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A.G. Filippini, M. Kazolea, and M. Ricchiuto, A flexible genuinely nonlinear approach for wave propagation, breaking and runup, *J.Comput.Phys.* 310, 2016

A.G. Filippini, M. Kazolea, and M. Ricchiuto, Hybrid finite-volume/element simulations of fully-nonlinear/weakly dispersive wave propagation breaking and runup on unstructured grids, SIAM-GS Conference, Sep 2017

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