Well balanced ALE :
simple (lazy man's) time dependent mesh adaptation for balance laws

L. Arpaia and M. Ricchiuto

Inria BSO - Team CARDAMOM

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Mathematical setting

Model equations

Seek approximate solutions of

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = \mathcal{S}(u,\, g(\vec{x})) \tag{1}$$

Remarks

Equation (1) assumed to admit non-trivial steady equilibria characterized by

$$\eta(u, g) = \eta_0 = \text{const}$$

Mathematical setting

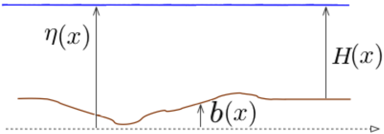
Model equations

Seek approximate solutions of

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = \mathcal{S}(u,\, g(\vec{x})) \tag{1}$$

Example

Shallow water flow : $\eta_0 = H(x) + b(x)$



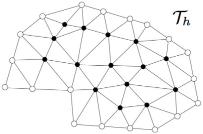
Mathematical setting

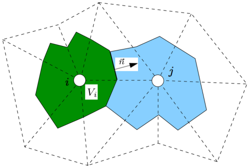
Model equations

Seek approximate solutions of

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = \mathcal{S}(u,\, g(\vec{x})) \qquad (1)$$

on an unstructured mesh $\mathcal{T}_h$.





Discrete equation

$$V_i u_i^{n+1} - V_i u_i^n + \Delta t \oint_{\partial V_i} \widehat{F}(u^n) \cdot \vec{n} = \Delta t\, \Sigma_i(u^n, g)$$

Mathematical setting

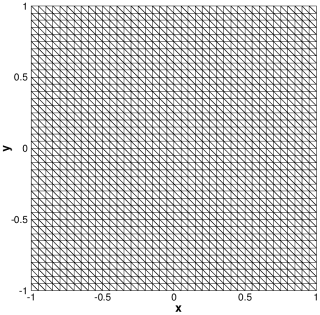
Model equation

Seek approximate solutions of

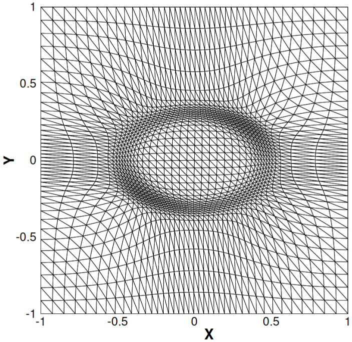
$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = \mathcal{S}(u,\, g(\vec{x})) \qquad (1)$$

on a time dependent unstructured mesh $\mathcal{T}_h(t)$.

$\mathcal{T}_h(t = t_1)$



$\mathcal{T}_h(t = t_2)$



Mathematical setting

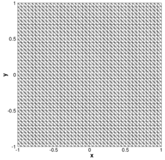
Building blocks

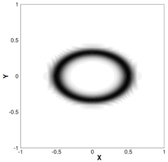
1. Discrete model for $\mathcal{T}_h(t)$: Time dependent mesh adaptation
2. Steady equilibria on moving meshes : Well balanced ALE
3. Coupling strategy : projection and evolution or ALE ?

1. Time dependent mesh adaptation

- Alauzet et al *JCP* 222, 2007 :
  re-mesh and adapt to all solutions in a given time slab
- Guardone et al *JCP* 230, 2011 :
  continuous deformation model for re-mesh, ALE projection (variable topology)
- Alauzet *Eng.w.Computers* 30, 2014 :
  continuous deformation model for re-mesh, ALE projection (variable topology)
- Tang and Tang SINUM 41, 2003 (conservation laws+adaptation) :
  continuous deformation with fixed mesh topology : constant data structure
- Baker *et al.* 2005 (compressible flow+moving bodies) :
  elastic deformation with fixed mesh topology : constant data structure
- etc. etc

Mesh adaptation by continuous deformation





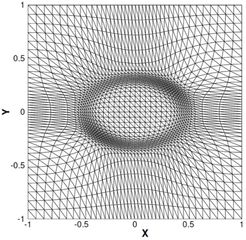
HANDBOOK OF

GRID GENERATION,

Thompson, Soni,

and Weatherill Eds,

CRC Press, 1998



Time dependent mesh adaptation by continuous deformation

Elliptic mesh movement

Given the mesh in the reference frame $\vec{X} = (X_1,\, X_2)$, seek $\vec{x} = \vec{x}(\vec{X})$ such that

$$\nabla_{\vec{X}} \cdot \big(\omega(\nabla_{\vec{x}} u)\, \nabla_{\vec{X}} \vec{x}\big) = \text{bc.s}$$

- Elliptic non-linear system of equations for the mapped (new) point positions $\vec{x}$
- Nonlinear monitor $\omega = \omega(\nabla_{\vec{x}} u)$:

$$\omega(\nabla_{\vec{x}} u_h) = \sqrt{1 + \alpha \nabla u^*}\,, \quad \nabla u^* = \min\left(1,\, \frac{\|\nabla_{\vec{x}} u_h\|^2}{\beta^2 \max_i \|\nabla_{\vec{x}} u_i\|^2}\right)$$

Elliptic mesh movement

In terms of displacements $\vec{\delta} = \vec{x} - \vec{X}$ and force $\vec{F} = -\mathbf{I}_2 \cdot \nabla_X \omega$

$$\nabla_X \cdot \left(\omega(\nabla_{\vec{x}} u_h) \, \nabla_{\vec{X}} \vec{\delta} \right) = \vec{F} + \text{bc.s}$$

- Elliptic non-linear system of equations for displacements $\vec{\delta}$
- Nonlinear monitor $\omega = \omega(\nabla_{\vec{x}} u)$ controlling stiffness and force

$$\omega(\nabla_{\vec{x}} u_h) = \sqrt{1 + \alpha \nabla u^*}\,, \quad \nabla u^* = \min\left(1, \frac{\|\nabla_{\vec{x}} u_h\|^2}{\beta^2 \max_i \|\nabla_{\vec{x}} u_i\|^2}\right)$$

Elliptic mesh movement : in practice

1. Elliptic PDE discretized on the reference mesh $\vec{X}$ with P^1 Galerkin FEM :

$$\sum_j \kappa_{ij}(\vec{\delta})\vec{\delta}_j = f_i(\vec{\delta}) \quad \forall\, i$$

with $\kappa_{ij}(\delta)$ the FEM stiffness matrix

2. Solution algorithm : relaxed Newton-Jacobi iterations

$$\vec{\delta}_i^{k+1} = \vec{\delta}_i^{k} - \frac{\sum_{j\neq i} \kappa_{ij}^k\, \vec{\delta}_j^k - f_i}{\kappa_{ii}^k}$$

$$\vec{x}_{k+1} = \vec{x}_k + \mu\, \vec{\delta}^{k+1}$$

Remarks

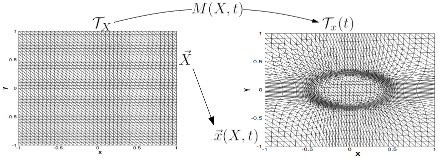
- At each iteration the FEM stiffness matrix κ_{ij}^{k} depends on $\nabla_{\vec{x}_k} u_h$ via ω
- At each iteration we need to compute $u_h(\vec{x}_k)$, the projection of the function u on the mesh $\vec{x}_k$

$k_{\max}$ iterations - $k_{\max}$ projections

2. Well balanced schemes on moving meshes

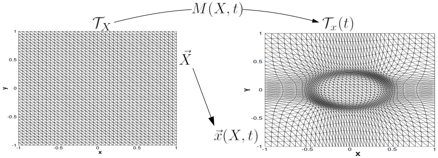
- ref ????

Well balanced ALE



Well balanced ALE = Well balanced + ALE

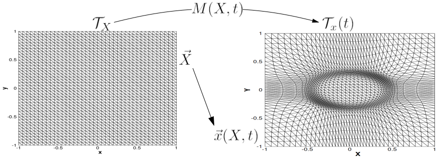
Well balanced ALE



Well balanced discretizations on fixed meshes

- Bermúdez and M.E. Vázquez, Computers and Fluids 23, 1994
- Greenberg and Leroux, SINUM 33, 1996
- Hubbard and Garcia-Navaro, J.Comput.Phys 165, 2000
- etc etc etc

Well balanced ALE



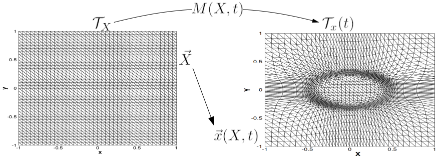
Well balanced discretizations on fixed meshes

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = \mathcal{S}(u,\, g(\vec{x}))$$

Assumed to admit non-trivial steady equilibria $\nabla \cdot \boldsymbol{\mathcal{F}}(u) = \mathcal{S}(u,\, g(\vec{x}))$ characterized by

$$\eta(u,g) = \eta_0 = \text{const}$$

Well balanced ALE



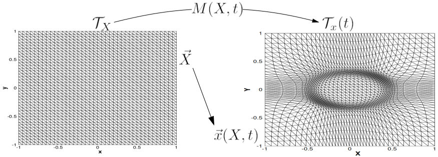
Well balanced discretizations on fixed meshes

$$V_i u_i^{n+1} - V_i u_i^n + \Delta t \oint_{\partial V_i} \widehat{F}(u^n)\cdot\vec{n} = \Delta t\, \Sigma_i(u^n, g(\vec{x}))$$

Is well-balanced if

$$\eta_i(u^0, g) = \eta_0 = \text{const} \quad \Longrightarrow \quad \begin{cases} \eta_i(u^n, g) = \eta_0 \\ u_i^{n+1} = u_i^n = u_i^0 \end{cases} \quad \forall\, n > 0$$

Well balanced ALE



Well balanced discretizations on fixed meshes

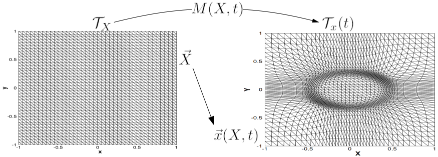
- Compatibility :

$$\oint_{\partial V_i} \widehat{F}(u^n) = \Sigma_i(u^n, g(\vec{x})) \quad \Longleftrightarrow \quad \eta_i(u^n, g) = \eta_0 \; \forall \, n > 0$$

- Exact discrete analog of $\nabla \cdot \boldsymbol{\mathcal{F}}(u) = \mathcal{S}(u,\ g(\vec{x}))$
- General strategies to satisfy this contraint : research topic[1]

[1] There exist N well balanced schemes... with N very very large..

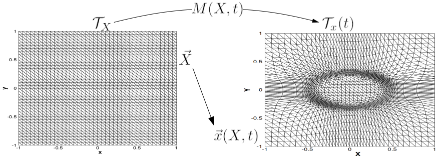
Well balanced ALE



ALE recap for $\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = 0$

- Farahat et al IJNMF 21 1995 ;
- Lesoinne and Farahat, CMAME 134, 1996 ;
- Farahat et al *JCP* 174 2001
- etc.

Well balanced ALE



ALE recap for $\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = 0$

Definitions :

Deformation speed

$$\sigma = \frac{d\vec{x}}{dt}$$

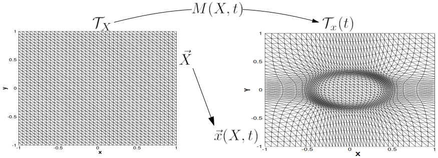
Deformation Jacobian

$$J = \det \frac{\partial \vec{x}}{\partial \vec{X}}$$

Volume :

$$V(t) = \int_{V(t)} d\vec{x} = \int_{V(t=0)} J \, d\vec{X}$$

Well balanced ALE



ALE recap for $\partial_t u + \nabla\cdot\boldsymbol{\mathcal{F}}(u) = 0$

Main results :

- Geometric Conservation Law (GCL, evolution of volume) :

$$\partial_t J\big|_{\vec{X}} = J\nabla_{\vec{x}}\cdot\sigma$$

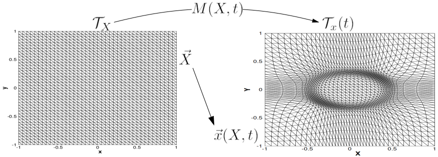
- Conservation law in ALE form (ALE-CL) :

$$\partial_t (Ju)\big|_{\vec{X}} + J\nabla_{\vec{x}}\cdot(\boldsymbol{\mathcal{F}}(u) - \sigma u) = 0$$

Fundamental relation

ALE-CL reduces to GCL for constant u !!!!

Well balanced ALE



ALE recap for $\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = 0$

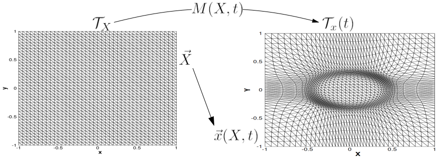
Discretization of ALE-CL, e.g. explicit FV on cell V_i :

$$V_i^{n+1} u_i^{n+1} - V_i^n u_i^n + \int_{t^n}^{t^{n+1}} \int_{\partial V_i(t)} \left(\widehat{F}(u^n) - \widehat{\sigma u}^n \right) \cdot \vec{n}(t) = 0$$

- $\widehat{F}(u)$ and $\widehat{\sigma u}$ FV numerical fluxes consistent with $\boldsymbol{\mathcal{F}}(u)$ and σu
- Discrete point diplacement speed

$$\sigma_i = \frac{\vec{x}_i^{n+1} - \vec{x}_i^n}{\Delta t} = \frac{\vec{\delta}_i}{\Delta t}$$

Well balanced ALE



ALE recap for $\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = 0$

Discretization of ALE-CL, e.g. explicit FV on cell V_i :

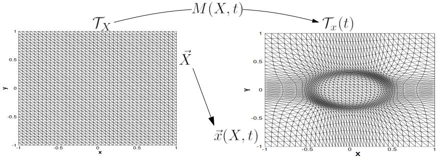
$$V_i^{n+1} u_i^{n+1} - V_i^n u_i^n + \int_{t^n}^{t^{n+1}} \int_{\partial V_i(t)} \left(\widehat{F}(u^n) - \widehat{\sigma u}^n \right) \cdot \vec{n}(t) = 0$$

Fundamental relation : Discrete-GCL

To be consistent with a constant state, for $u = u_0$, the scheme MUST reduce to the identity

$$u_0 \left(V_i^{n+1} - V_i^n - \int_{t^n}^{t^{n+1}} \int_{\partial V_i(t)} \widehat{\sigma} \cdot \vec{n}(t) \right) = 0$$

Well balanced ALE



ALE recap for $\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = 0$

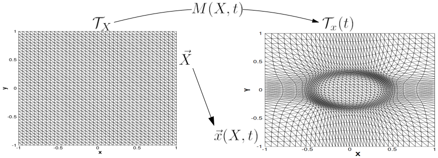
- Discrete-GCL is the compatibility :

$$V_i^{n+1} - V_i^n = \int_{t^n}^{t^{n+1}} \int_{\partial V_i(t)} \widehat{\sigma} \cdot \vec{n}(t) \qquad \Longleftrightarrow \qquad u_i^n = u_0 \;\forall\, n > 0$$

- Exact discrete analog of $\partial_t J \big|_{\vec{X}} = J \nabla_{\vec{x}} \cdot \sigma$
- General strategies to satisfy this contraint : research topic[2]

[2] There exist N ways to get the DGCL.. with N not so large ..

Well balanced ALE



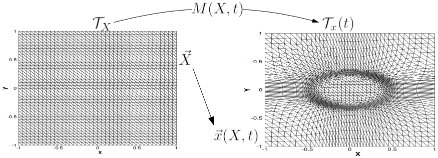
ALE for a balance law

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = \mathcal{S}(u,\, g(\vec{x}))$$

admitting a steady state characterized by

$$\eta(u,g) = \eta_0 = \text{const} \Rightarrow \nabla \cdot \boldsymbol{\mathcal{F}} = \mathcal{S}(u,\, g(\vec{x}))$$

Well balanced ALE



ALE for a balance law

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = \mathcal{S}(u,\, g(\vec{x}))$$

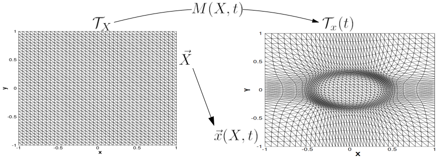
Straightforward application of ALE theory

$$\partial_t (Ju)\big|_{\vec{X}} + J\nabla_{\vec{x}} \cdot (\boldsymbol{\mathcal{F}}(u) - \sigma u) = J\mathcal{S}(u,\, g(\vec{x}))$$

plus the GCL

$$\partial_t J\big|_{\vec{X}} = J\nabla_{\vec{x}} \cdot \sigma$$

Well balanced ALE



ALE for a balance law

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = \mathcal{S}(u,\, g(\vec{x}))$$

Straightforward application of ALE theory

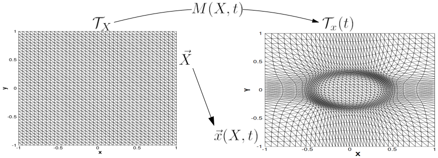
$$\partial_t (Ju)\big|_{\vec{X}} + J\nabla_{\vec{x}} \cdot (\boldsymbol{\mathcal{F}}(u) - \sigma u) = J\mathcal{S}(u,\, g(\vec{x}))$$

plus the GCL

$$\partial_t J\big|_{\vec{X}} = J\nabla_{\vec{x}} \cdot \sigma$$

Take now $\eta(u,g) = \eta_0 =$ const $\Rightarrow \nabla \cdot \boldsymbol{\mathcal{F}} = \mathcal{S}$ and combine these two relations

Well balanced ALE



ALE for a balance law

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = \mathcal{S}(u,\, g(\vec{x}))$$

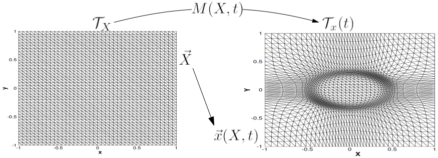
Straightforward application of ALE theory

If we take $\eta(u,g) = \eta_0 = \text{const} \Rightarrow \nabla \cdot \boldsymbol{\mathcal{F}} = \mathcal{S}$ and using both relations above

$$J\partial_t u\big|_{\vec{X}} - J\sigma \cdot \nabla_{\vec{x}} u = 0$$

is this true ?

Well balanced ALE



ALE for a balance law

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = \mathcal{S}(u,\, g(\vec{x}))$$

Straightforward application of ALE theory

Yes (!!) since in the moving frame and for $\eta(u, g) = \eta_0 =$ const :

$$\partial_t g\big|_{\vec{X}} = \sigma \cdot \nabla_{\vec{x}} g$$

and

$$0 = \partial_t \eta\big|_{\vec{X}} - \sigma \cdot \nabla_{\vec{x}} \eta = \partial_u \eta(\partial_t u\big|_{\vec{X}} - \sigma \cdot \nabla_{\vec{x}} u) + \partial_g \eta(\underbrace{\partial_t g\big|_{\vec{X}} - \sigma \cdot \nabla_{\vec{x}} g}_{=0})$$

Well balanced ALE

Standard ALE :

$$\partial_t (Ju)\big|_{\vec{X}} + J\nabla_{\vec{x}} \cdot (\boldsymbol{\mathcal{F}}(u) - \sigma u) = J\mathcal{S}(u,\, g(\vec{x}))$$

Problematic equilibrium on moving meshes

Discretize the standard ALE and set $\eta = u + F(g(\vec{x})) = \eta_0 =$ const

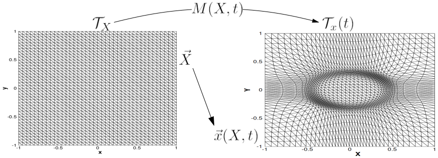
$$J(\partial_t u\big|_{\vec{X}} - \sigma \cdot \nabla_{\vec{x}} u) + u\,\underbrace{(\partial_t J\big|_{\vec{X}} - \nabla_{\vec{x}} \cdot \sigma)}_{\text{DGCL}} + J\,\overbrace{(\nabla_{\vec{x}} \boldsymbol{\mathcal{F}} - \mathcal{S})}^{\text{Well Balanced}} = 0$$

A scheme which *verifies the DGCL*, and which is *well balanced* on fixed meshes, will not be on moving meshes. The error is related to the discretization of the term

$$\partial_t u\big|_{\vec{X}} - \sigma \cdot \nabla_{\vec{x}} u$$

embedded in the discrete equations ....

Well balanced ALE



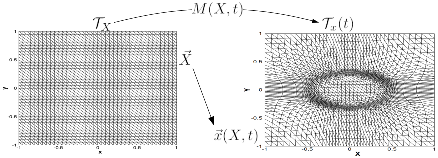
A particular case

Assume that the steady balance is described by the invariant

$$\eta(u,g) = u + F(g) \Rightarrow \partial\eta = \partial u + F'(g)\partial g$$

Modified ALE form

Well balanced ALE



A particular case

Assume that the steady balance is described by the invariant

$$\eta(u,g) = u + F(g) \Rightarrow \partial\eta = \partial u + F'(g)\partial g$$

Modified ALE form

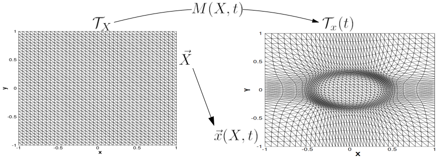
Start from the "straightforward" ALE form

$$\partial_t(Ju)\big|_{\vec{X}} + J\nabla_{\vec{x}}\cdot(\boldsymbol{\mathcal{F}}(u) - \sigma u) = J\mathcal{S}(u,\, g(\vec{x}))$$

and add the following quantities (both equal to zero) :

$$F(g)\underbrace{\left(\partial_t J\big|_{\vec{X}} - J\nabla_{\vec{x}}\cdot\sigma\right)}_{\text{GCL}} = 0 \quad \text{and} \quad F'(g)\underbrace{\left(J\partial_t g\big|_{\vec{X}} - J\sigma\cdot\nabla_{\vec{x}} g\right)}_{\text{Local time variation in moving frame}} = 0$$

Well balanced ALE



A particular case

Assume that the steady balance is described by the invariant

$$\eta(u,g) = u + F(g) \Rightarrow \partial\eta = \partial u + F'(g)\partial g$$

Modified ALE form

WELL BALANCED ALE formulation

$$\partial_t(J\eta)\big|_{\vec{X}} + J\nabla_{\vec{x}}\cdot(\boldsymbol{\mathcal{F}}(u) - \sigma\eta) = J\mathcal{S}(u,\, g(\vec{x}))$$

Well balanced ALE

A particular case

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = \mathcal{S}(u,\, g(\vec{x}))$$

Assume that the steady balance is described by the invariant

$$\eta(u, g) = u + F(g) \Rightarrow \partial\eta = \partial u + F'(g)\partial g$$

Equilibrium on moving meshes

- WELL BALANCED ALE for $\eta = u + F(g(\vec{x})) = \eta_0$

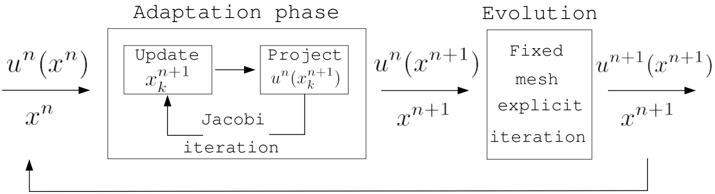
$$J \overbrace{\left(\partial_t \eta_0\big|_{\vec{X}} - \sigma \cdot \nabla_{\vec{x}} \eta_0\right)}^{\partial\eta_0 = 0} + \eta_0 \underbrace{\left(\partial_t J\big|_{\vec{X}} - \nabla_{\vec{x}} \cdot \sigma\right)}_{\text{DGCL}} + J \overbrace{\left(\nabla_{\vec{x}} \boldsymbol{\mathcal{F}} - \mathcal{S}\right)}^{\text{Well Balanced}} = 0$$

A scheme which is well balanced on fixed meshes will also be on moving meshes provided it verifies the DGCL

3. Adaptation-discretization coupling : ALE-remap vs ALE

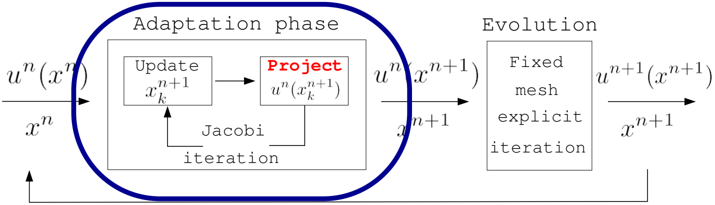
Tang and Tang, *SINUM 2003* - Xu et al. *J.Comput.Phys* 2013

DPE METHOD



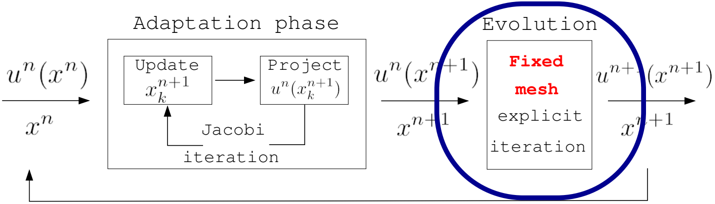
Deformation-Projection-Evolution

DPE METHOD



- To get $\vec{x}_i^{n+1}$: nonlinear elliptic deformation eq. solved with initial guess $\vec{x}_i^n$
- $k_{\max}$ Jacobi iterations are performed
- To compute $\omega(\nabla_{\vec{x}} u)$ we need to define a projection to get u^n onto each x_k^{n+1} (important bit)

DPE METHOD



The scheme is applied on the fixed mesh as if no adaptation was used at all

1. Conservation requires the projection step needs to be conservative
2. Second order of accuracy requires the projection step to be second order accurate
3. Monotonicity requires the projection step needs to be monotone

Cost of the projection step ?

ALE REMAP

FV scheme for $\partial_t (J\eta)\big|_{\vec{X}} + J\nabla_{\vec{x}} \cdot (\boldsymbol{\mathcal{F}} - \sigma\eta) = J\mathcal{S}$

$$V_i^{n+1}\eta_i^{n+1} - V_i^n \eta_i^n + \int_{t^n}^{t^{n+1}} \int_{\partial V_i(t)} \widehat{F}(u^n) \cdot \vec{n}(t) - \int_{t^n}^{t^{n+1}} \int_{\partial V_i(t)} \widehat{\sigma\eta}^n \cdot \vec{n}(t) = \int_{t^n}^{t^{n+1}} \Sigma_i^n$$

ALE remap

Use the fact that

$$\sigma = \frac{\vec{x}^{n+1} - \vec{x}^{n}}{\Delta t} = \frac{\vec{\delta}}{\Delta t}$$

and $\vec{\delta}$ given from the current mesh deformation step

DPE method

ALE remap

FV scheme for $\partial_t(J\eta)\big|_{\vec{X}} + J\nabla_{\vec{x}} \cdot (\boldsymbol{\mathcal{F}} - \sigma\eta) = J\mathcal{S}$

$$V_i^{n+1}\eta_i^{n+1} - V_i^n\eta_i^n + \int_{t^n}^{t^{n+1}} \int_{\partial V_i(t)} \widehat{F}(u^n) \cdot \vec{n} - \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \int_{\partial V_i(t)} \widehat{\delta\eta}^n \cdot \vec{n} = \int_{t^n}^{t^{n+1}} \Sigma_i^n$$

DPE METHOD

ALE REMAP

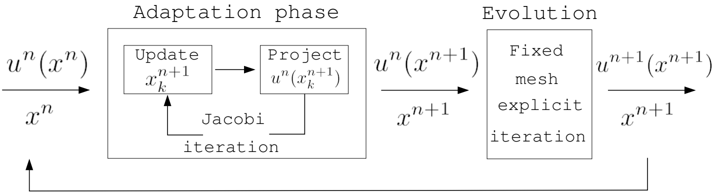
Let now $\Delta t \to 0$ and keep the displacement δ finite **to get the projection**

$$V_i^{n+1}\eta_i^{n+1} - V_i^n\eta_i^n + \int_{\partial V_i^n} \widehat{\delta\eta}^n \cdot \vec{n} = 0$$

1. Conservative high order and well balanced projection obtained from a conservative high order well balanced scheme
2. Same cost of discretisation of scalar advection equation
3. Repeated at each Jacobi iteration and for each variable : costly for high order with limiter
   (see next)

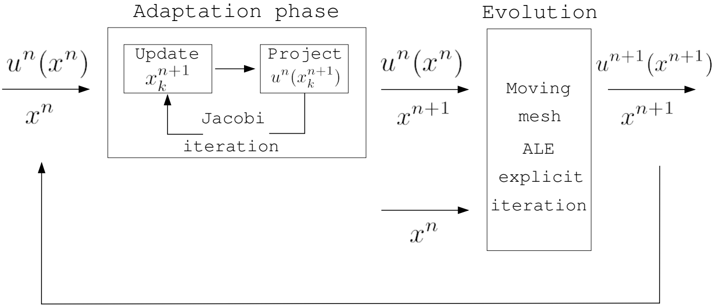
Can we do better ?

DPE METHOD



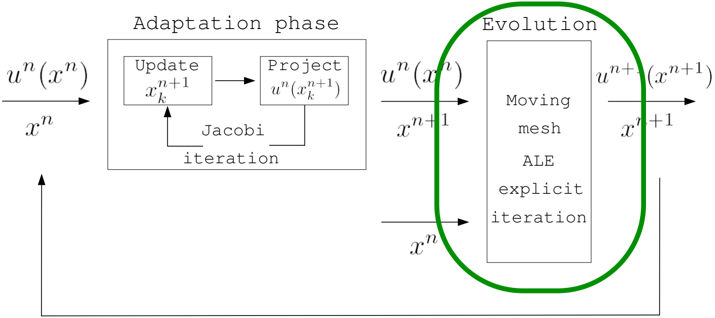
Deformation-Projection-Evolution

DALE METHOD



Deformation-ALE evolution

DALE METHOD



The ALE evolution guarantees that the overall algorithm is

1. Conservative
2. Second order accurate
3. Monotone

The projection step can be simplified considerably...

Numerical examples : schemes implemented

Finite volume

- Std well balanced Roe scheme (Bermudez-Vazquez, *Computers & Fluids* 1994)
- Muscl reconstruction with van Albada limiter
- Second order SSP Runge Kutta integration
- ALE formulation following e.g. (Farahat et al *JCP* 174, 2001)

Residual distribution

- Second order, positivity preserving, well balanced approach proposed in (Ricchiuto *J.Comput.Phys.* 2015)
- ALE extension proposed in (Arpaia, Ricchiuto, Abgrall *J.Sci.Comp.* 2014) for compressible gas dyn.

Scalar balance law mimicking the SW equations

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = \vec{a}(u) \cdot \nabla g(\vec{x})$$

For $\vec{a}(u) = \partial_u \boldsymbol{\mathcal{F}}$ we have a simple steady state invariant :

$$\eta = u + g(\vec{x})$$

Example 1 : linear transport with source

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = \vec{a} \cdot \nabla g(\vec{x})$$

with

$$\vec{\boldsymbol{\mathcal{F}}} = \vec{a}u, \quad g = 0.8e^{-50(x-0.5)^2-5(y-0.9)^2)}, \quad \text{and } \vec{a}(\vec{x}) = (0,1)$$

with initial solution ($r^2 = (x-0.5)^2 + (y-0.5)^2$)

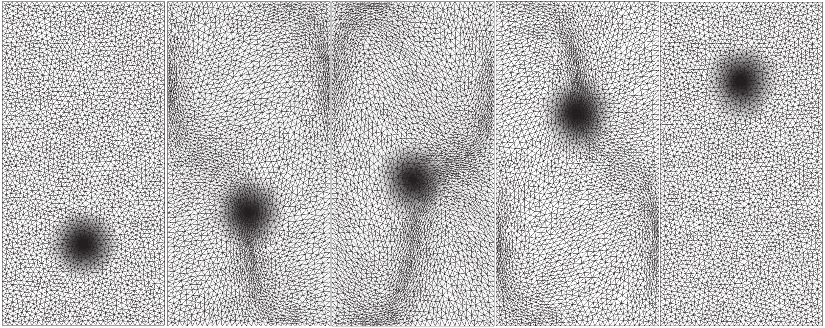
$$\eta = 1 + \psi(x,y), \quad \psi = \begin{cases} \cos^2(2\pi r) & \text{if } r < 1/4 \\ 0 & \text{otherwise} \end{cases}$$

solved on $[0,1] \times [0,2]$ superimposing the time dependent mapping

$$\begin{cases} x = X + 0.1\sin(2\pi X)\sin(\pi Y)\sin(2\pi t) \\ y = Y + 0.2\sin(2\pi X)\sin(\pi Y)\sin(4\pi t) \end{cases}$$

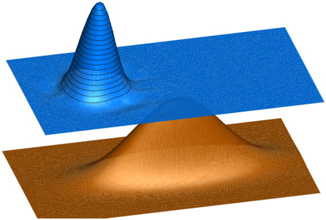
Example 1 : linear transport with source

Mesh movement ($t = 0$, 0.2, 0.4, 0.6, 1)

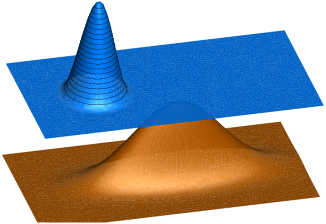


Example 1 : linear transport with source

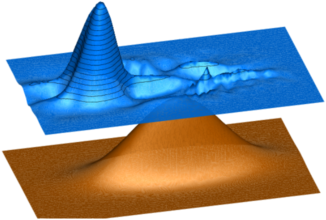
Results with linear second order RD scheme



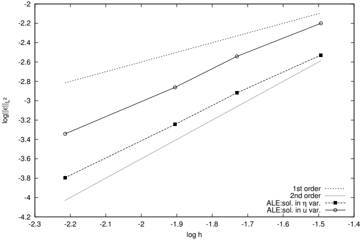
Well balanced ALE $t = 1$



Exact $t = 1$



Standard ALE $t = 1$



Grid convergence

Example 2 : rigid body rotation with source

$$\partial_t u + \nabla \cdot \vec{\boldsymbol{\mathcal{F}}} = \vec{a} \cdot \nabla g(\vec{x})$$

with

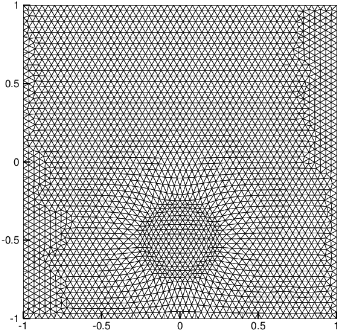
$$\vec{\boldsymbol{\mathcal{F}}} = \vec{a}(\vec{x})u, \quad g = 0.6e^{-5(x^2+y^2)}, \quad \text{and } \vec{a}(\vec{x}) = (y, -x)$$

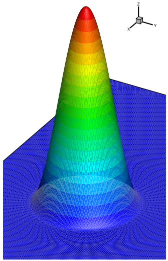
with initial solution ($r^2 = (x+0.5)^2 + y^2$)

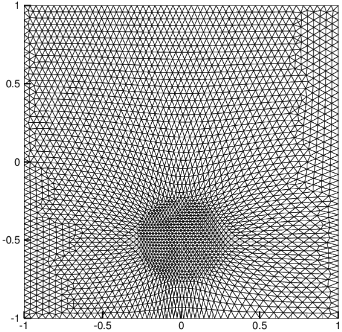
$$\eta = 1 + \psi(x,y), \quad \psi = \begin{cases} \cos^2(2\pi r) & \text{if } r < 1/4 \\ 0 & \text{otherwise} \end{cases}$$

solved on $[-1,1]^2$ testing both the DPE and DALE approaches.

Example 2 : rigid body rotation with source





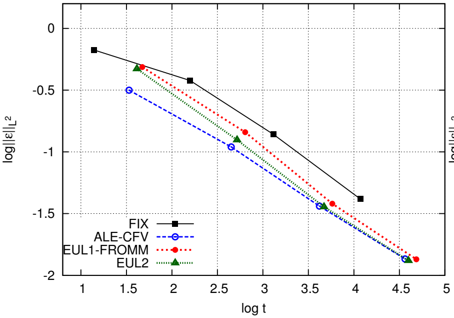


Initial

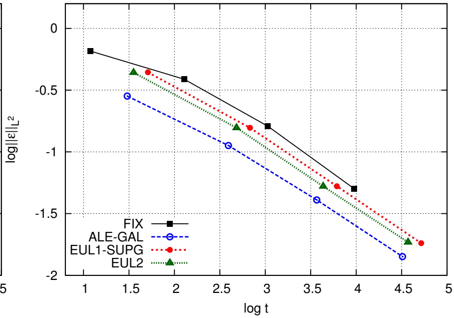
Ofter one rotation

Example 2 : rigid body rotation with source

Grid convergence : error vs CPU time



FV scheme



RD scheme

Example 3 : nonlinear balance law

$$\partial_t u + \nabla \cdot \boldsymbol{\mathcal{F}}(u) = \vec{a}(u) \cdot \nabla g(\vec{x})$$

with

$$\vec{\boldsymbol{\mathcal{F}}} = (u^2/2,\, u^2/2), \quad g = 0.6e^{-5(x^2+y^2)}, \quad \text{and } \vec{a}(u) = (u, u)$$

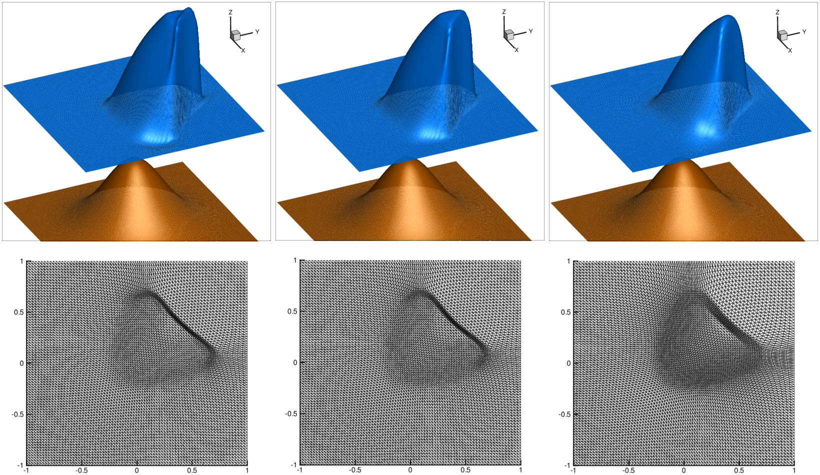
with initial solution ($r^2 = (x + 0.5)^2 + y^2$)

$$\eta = 1 + \psi(x, y), \quad \psi = \begin{cases} 1.4 & \text{if } \vec{x} \in [-0.9, -0.2]^2 \\ 0.8 & \text{otherwise} \end{cases}$$

solved on $[-1, 1]^2$ testing both the DPE and DALE approaches.

Example 3 : nonlinear balance law

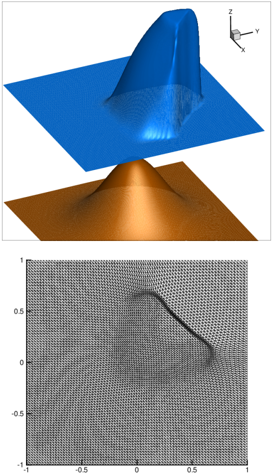
DPE results for FV



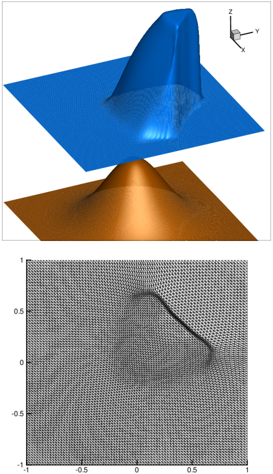
2nd order proj. | High order proj. (VL limiter) | 1st order proj.

Example 3 : nonlinear balance law

DALE results for FV



Simplified central 2nd order proj.



1st order proj.

CPU gain roughly 30% w.r.t DPE

Shallow water results

Standard form

Used in the DPE algorithm

$$\partial_t \begin{bmatrix} H \\ \vec{q} \end{bmatrix} + \nabla \cdot \begin{bmatrix} \vec{q} \\ \vec{u} \otimes \vec{q} + g \dfrac{H^2}{2} \end{bmatrix} + gH \begin{bmatrix} 0 \\ \nabla b \end{bmatrix} = 0$$

Well balanced ALE form

Used in the DALE algorithm

$$\partial_t \begin{bmatrix} J\eta \\ J\vec{q} \end{bmatrix} + J\nabla \cdot \begin{bmatrix} \vec{q} - \sigma\eta \\ \vec{u} \otimes \vec{q} + g \dfrac{H^2}{2} - \sigma \otimes \vec{q} \end{bmatrix} + J\,gH \begin{bmatrix} 0 \\ \nabla b \end{bmatrix} = 0$$

Shallow water results with RD

Perturbation over smooth bathymetry

Over the domain $[0,2] \times [0,1]$ take

$$b(x,y) = 0.8e^{-50(x-0.9)^2 - 5(y-0.5)^2}$$

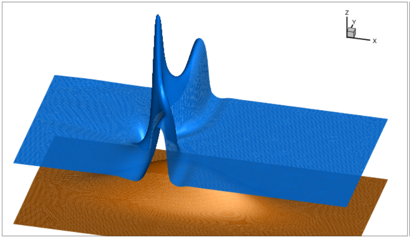
and set as initial solution still flow and free surface level

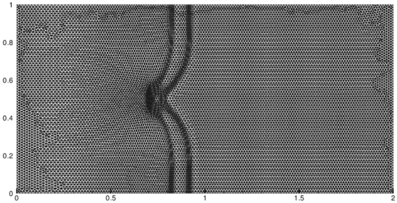
$$\eta = \begin{cases} 1.01 & \text{if } 0.05 \leq x \leq 0.15 \\ 1 & \text{otherwise} \end{cases}$$

Shallow water results with RD

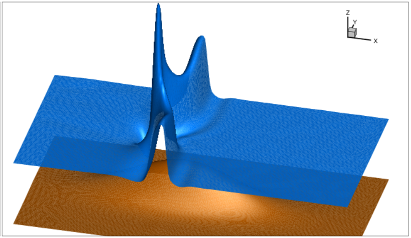
Perturbation over smooth bathymetry

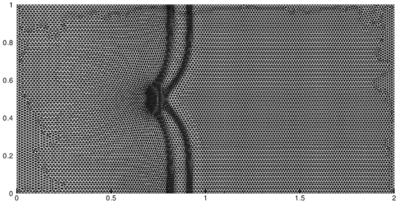
DPE with second order projection





DALE with centered projection

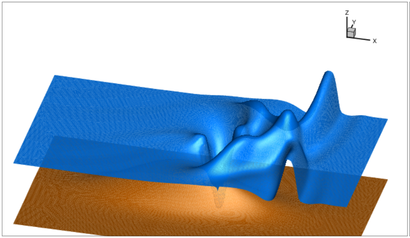




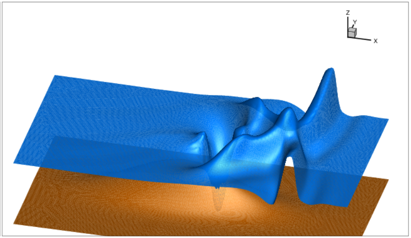
Shallow water results with RD

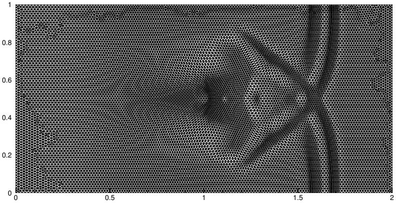
Perturbation over smooth bathymetry

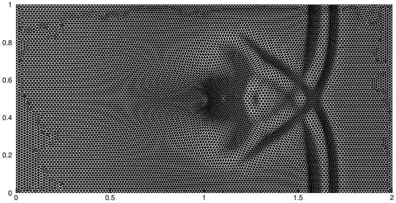
DPE with second order projection



DALE with centered projection



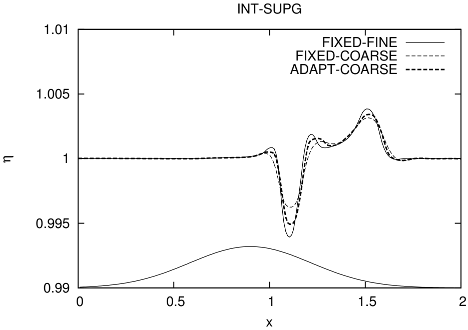




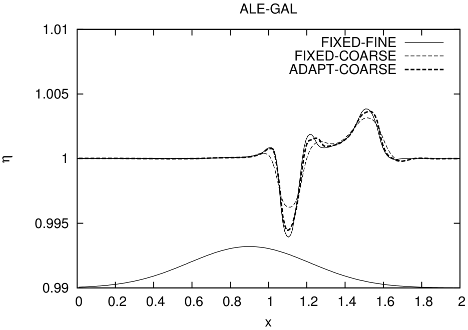
Shallow water results with RD

Perturbation over smooth bathymetry

DPE with second order projection



DALE with centered projection



CPU times :
Fixed fine : 843[s]
DPE : 346[s]
DALE : 260[s]

Shallow water results with RD

Dam break

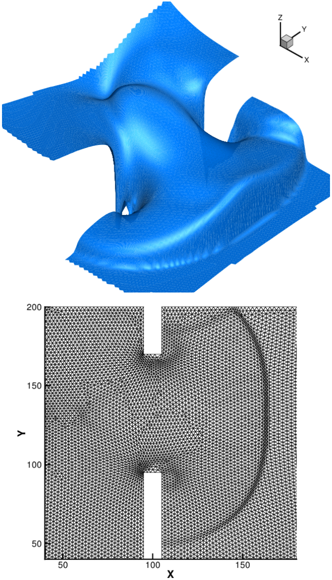
Initial solution involving still flow and

$$H_{\mathsf{left}} = 10[m] \quad \text{and} \quad H_{\mathsf{right}} = 5[m]$$

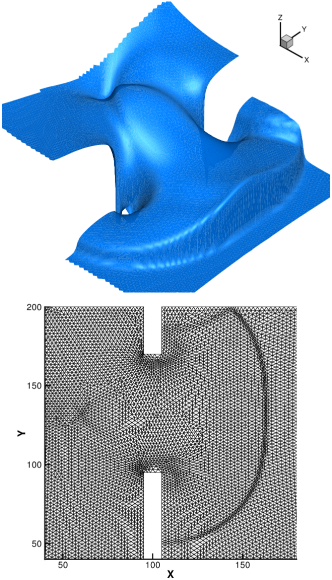
Shallow water results

Dam break

DPE with second order projection



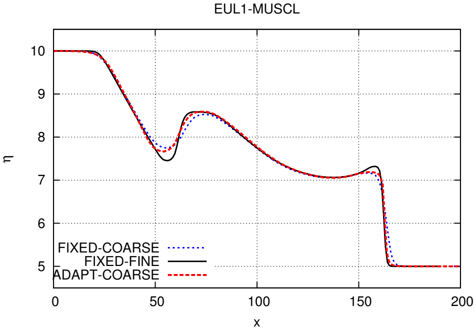
DALE with centered projection



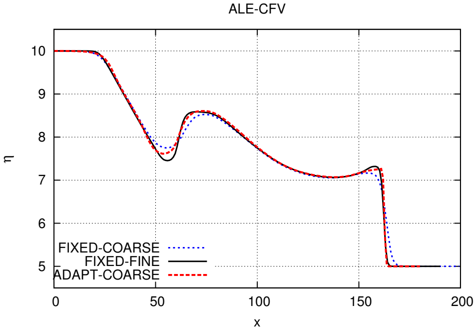
Shallow water results

Dam break : FV schemes

DPE with second order projection



DALE with centered projection

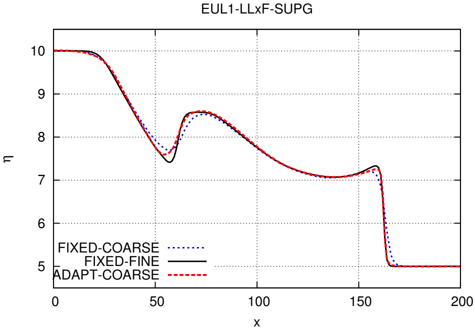


CPU times :
Fixed fine : 207[s]
DPE : 150[s] (Simplified DPE: 111[s])
DALE : 100[s]

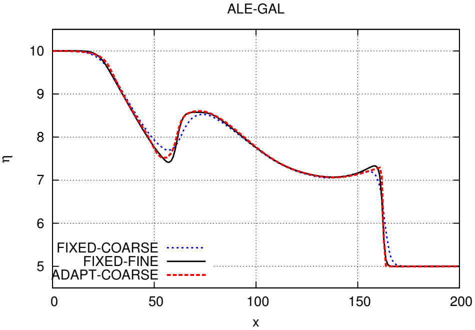
Shallow water results

Dam break : RD schemes

DPE with second order projection



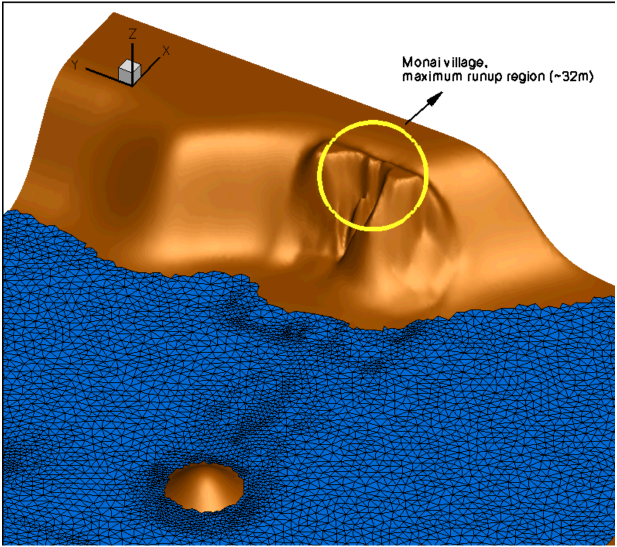
DALE with centered projection



CPU times :
Fixed fine : 185[s]
DPE : 179[s] (Simplified DPE: 98[s])
DALE : 77[s]

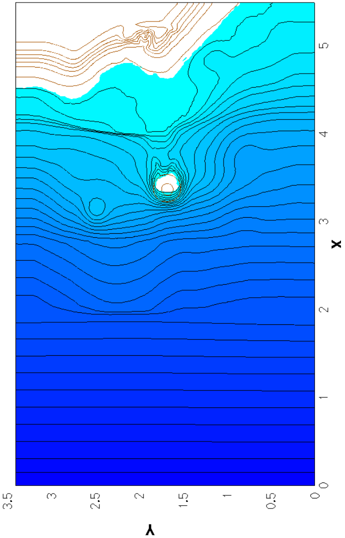
Shallow water results

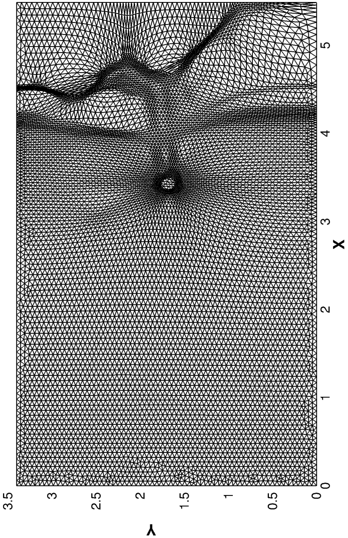
1993 Okushiri Tsunami (Monai Valley)



Shallow water results

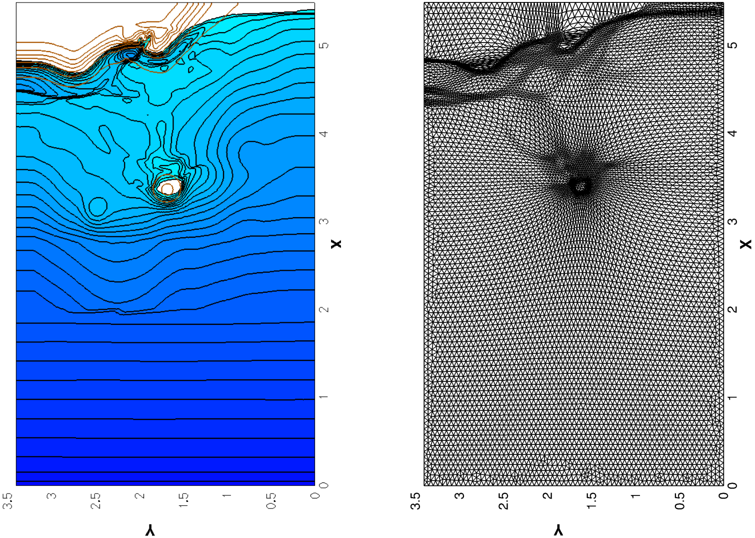
1993 Okushiri Tsunami (Monai Valley): RD





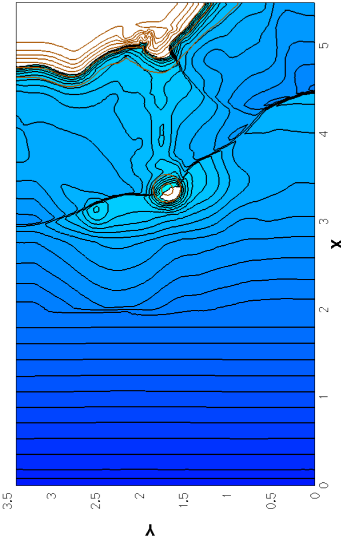
Shallow water results

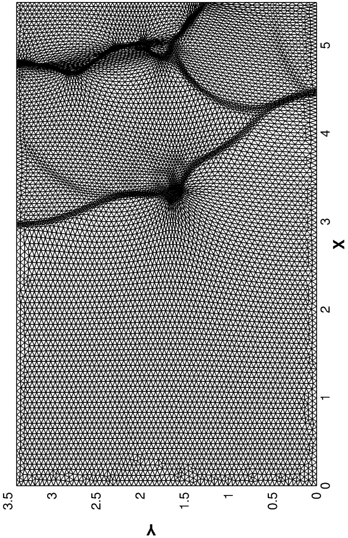
1993 Okushiri Tsunami (Monai Valley): RD



Shallow water results

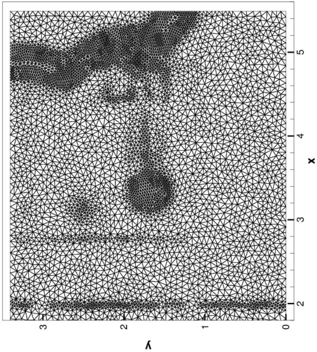
1993 Okushiri Tsunami (Monai Valley): RD



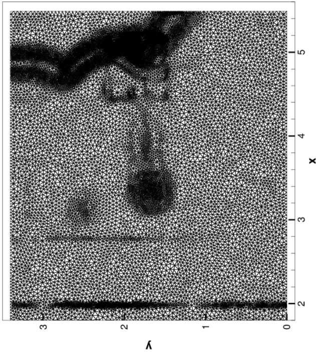


Shallow water results

1993 Okushiri Tsunami (Monai Valley): runup plot



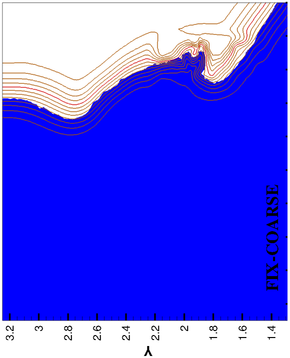
14k-Elements



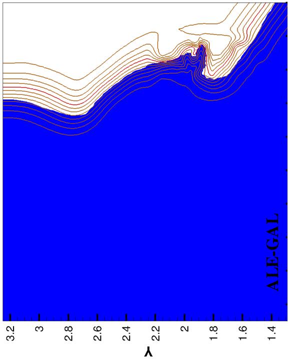
37k-Elements

Shallow water results

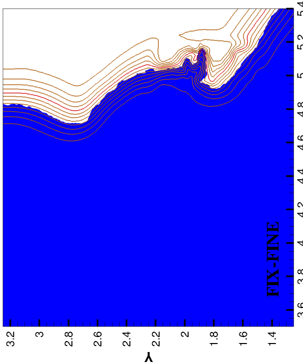
1993 Okushiri Tsunami (Monai Valley): runup plot



14k-Elements



16k-Elements



37k-Elements

Conclusions and perspectives

Summary

- Constant topology adaptation by deformation
- Well balanced ALE formulation
- Improved treatment by ALE evolution

Perspectives and work in progress

- 3D, elasticity, anisotropic formulations, coupling with re-meshing (with. C. Dobrzynski)
- Nonlinear mesh PDEs, high order meshes (with. C. Dobrzynski and R. Abgrall)
- Coupling with uncertainty quantification : adapt w.r.t. sensitivities (with P. Congedo)
- Multi-step schemes, implicit and local time stepping, etc etc etc