Smooth curves having a large automorphism p-group in characteristic $p > 0$.

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Notation.

- k is an algebraically closed field of characteristic $p \geq 0$.
- C/k is a connected nonsingular projective curve with genus $g \geq 2$.
- $\mathrm{Aut}_k(C)$ is the full k-automorphism group of C.

The case of characteristic 0.

- In characteristic 0:

$$\boxed{|\mathrm{Aut}_k(C)| \leq 84\,(g-1)} \qquad \text{(Hurwitz bound-1892)}$$

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- Problem: classification of automorphism groups for a given genus.
- Partial answers:
  - for *Hurwitz groups*, i.e. $|\mathrm{Aut}_k(C)| = 84\,(g-1)$.
  - more generally, for *large groups* $G \subset \mathrm{Aut}_k(C)$, i.e.

$$|G| \geq 4\,(g-1) \quad \text{[Kulkarni 1997] [Breuer 2000]}$$

By the Hurwitz formula, $g_{C/G} = 0$ and $C \to C/G$ ramified in 3 or 4 points.

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In characteristic $p > 0$:

- $\mathrm{Aut}_k(C)$ is still finite [Schmid 1938].
- But the bound is **biquadratic**:

$$\boxed{|\mathrm{Aut}_k(C)| \leq 16g^4} \quad \text{[Stichtenoth-1973]}$$

except for $C : W^q + W = X^{1+q}$, $q = p^n \geq 3$.

This is due to **wild ramification**.

Definition of a big action.

From now on, $\mathrm{char}(k) = p > 0$.

Definition

Let C/k be a connected nonsingular projective curve with genus $g \geq 2$.

Let G be a subgroup of $\mathrm{Aut}_k(C)$.

A pair (C, G) is called a ***big action*** *if:*

- *G is a p-group.*
- $$\boxed{|G| > \frac{2p}{p-1} g.}$$

Ramification conditions.

Let (C, G) be a big action.
Apply the Hurwitz and Deuring-Shafarevitch formulas to $C \to C/G$.

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Apply the Hurwitz and Deuring-Shafarevitch formulas to $C \to C/G$.

- Then $h = 0$, where h is the Hasse-Witt invariant of C.
- Only one point $\infty \in C$ is ramified and even totally ramified.
- Let G_i be the i-th lower ramification group of G at ∞.
  Then $G = G_{-1} = G_0 = G_1$.
- $C/G \simeq \mathbb{P}^1_k$ and $C/G_2 \simeq \mathbb{P}^1_k$. In particular, $G_2 \neq \{e\}$.

Choice of the bound: the ”embedding problem”.

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- $|G| > \frac{\boxed{2}p}{p-1} g$ is necessary for $G_2 \boxed{\subsetneq} G_1 = G$.

  (see $W^p - W = X^2$, $p > 2$, with $G = \langle \sigma \rangle$, $\sigma(X) = X$ and $\sigma(W) = W + 1$).

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  (see $W^p - W = X^2$, $p > 2$, with $G = \langle \sigma \rangle$, $\sigma(X) = X$ and $\sigma(W) = W + 1$).

- So $G/G_2 \simeq \{X \to X + y, y \in V\}$
  with $C/G_2 - \{\infty\} = \operatorname{Spec} k[X]$ and $V \subset k$ an $\mathbb{F}_p$-vector space with:

$$0 \longrightarrow G_2 \longrightarrow G \xrightarrow{\pi} V \simeq (\mathbb{Z}/p\mathbb{Z})^v \longrightarrow 0,$$

where

$$\pi : \begin{cases} G \to V \\ g \to g(X) - X. \end{cases}$$

Examples related to curves with many rational points.

Notation: $q := p^e$, $e \in \mathbb{N}^*$, $K := \mathbb{F}_q(X)$, K^{alg} fixed, $m \in \mathbb{N}$.

Definition (Lauter 1999, Auer 1999)

We define $K^m \subset K^{alg}$ as the largest abelian extension L of K

- *with conductor $\leq m\infty$*
- *such that every place in $S := \{(X-y), y \in \mathbb{F}_q\}$ splits completely in L.*

Let $C_m/\mathbb{F}_q$ be the nonsingular projective curve with function field K^m.

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Remark

$$\mathrm{Gal}\,(K^m/K) \simeq \frac{1+Z\mathbb{F}_q[[Z]]}{\langle 1+Z^m\mathbb{F}_q[[Z]], 1-yZ, y \in \mathbb{F}_q\rangle}, \quad \textit{with } Z = X^{-1}.$$

$\mathrm{Gal}\,(K^m/K)$ *has exponent* 1 *or* $p \Leftrightarrow m < m_2 := p^{\lceil e/2 \rceil + 1} + p + 1$. *(Lauter)*

Big actions with G_2 abelian of arbitrary large exponent.

Proposition (Matignon-Rocher 2008)

- $\{X \to X+y, y \in \mathbb{F}_q\}$ *extends to a p-group* $G_m \subset \mathrm{Aut}_{\mathbb{F}_q}(C_m)$ *with*

$$0 \longrightarrow \mathrm{Gal}\,(K^m/K) \longrightarrow G_m \longrightarrow \mathbb{F}_q \longrightarrow 0.$$

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- *If* $e \geq 6$, (C_{m_2}, G_{m_2}) *is a big action with* $G_2 = \mathrm{Gal}(K^{m_2}/K)$ *abelian of exponent* p^2.

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Remark

Same method to construct G_2 *abelian of arbitrary large exponent.*

Link with algebraic curves with many rational points.

Remark

Let N_m be the number of $\mathbb{F}_q$-rational points on C_m. Then

$$N_m := |C_m(\mathbb{F}_q)| = 1 + q\,|\mathrm{Gal}\,(K^m/K)| = 1 + |G_m|.$$

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Let N_m be the number of $\mathbb{F}_q$-rational points on C_m. Then

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Hence,

$$\boxed{\frac{|G_m|}{g_{C_m}} \sim \frac{N_m}{g_{C_m}}.}$$

Construction of a "minimal subextension".

Problem:

- Construct a subextension of K^{m_2} such that $G_2 \simeq (\mathbb{Z}/p^2\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})^t$ with $t \geq 0$ minimal.

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Difficulty: The *"embedding problem"*.

- In the previous case, the stability under translation by $\mathbb{F}_q$ is due to the maximality and uniqueness.
- How to reduce to a system of equations that remains globally stable?

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  More generally, there is no big action with $G_2 \simeq \mathbb{Z}/p^n\mathbb{Z}$ except for $n = 1$.

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Answer:

- In [M-R], construction of a ”minimal subextension” with $t = O(\log_p(g))$.

Big actions with $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$ [Rocher 2009].

Notation. Let (C, G) be a big action with $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, $n \geq 1$. Let $L := k(C)$ and $k(X) := L^{G_2}$. Then

$$\boxed{L/k(X) : W_i^p - W_i = g_i(X) \in k[X], \quad 1 \leq i \leq n.}$$

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Definition

Let

$$\boxed{A := \frac{\wp(L) \cap k[X]}{\wp(k[X])} := < \overline{g_1(X)}, \ldots, \overline{g_n(X)} > \qquad \text{with} \quad \wp := F - id.}$$

A is the $\mathbb{F}_p$-vector subspace of $k[X]$ dual to G_2 with respect to the Artin-Schreier pairing:

$$\begin{cases} G_2 \times A \to \mathbb{Z}/p\mathbb{Z} \\ (g, \overline{\wp w}) \to g(w) - w \end{cases}$$

Action of V on A.

V acts on A by translation:

$$\phi : \begin{cases} V \longrightarrow \mathrm{Aut}\,(A) \simeq \mathrm{GL}_n(\mathbb{F}_p) \\ y \longrightarrow \phi(y) \end{cases}$$

with

$$\phi(y) : \begin{cases} A \longrightarrow A \\ \overline{f(X)} \longrightarrow \overline{f(X+y)} \end{cases}$$

Remark

$\mathrm{Im}\,\phi$ *is a unipotent subgroup of* $\mathrm{GL}_n(\mathbb{F}_p)$.

An adapted basis for A.

We construct a basis $\{\overline{f_1(X)}, \dots, \overline{f_n(X)}\}$ in which the matrix of $\phi(y)$ reads:

$$\Phi(y) := \begin{pmatrix} 1 & \ell_{1,2}(y) & \ell_{1,3}(y) & \dots & \ell_{1,n}(y) \\ 0 & 1 & \ell_{2,3}(y) & \dots & \ell_{2,n}(y) \\ 0 & 0 & \dots & \dots & \ell_{i,n}(y) \\ 0 & 0 & 0 & 1 & \ell_{n-1,n}(y) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_n(\mathbb{F}_p) \quad \forall y \in V$$

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In other words,

$$\boxed{\forall i \in \{1, \ldots, n\},\ \forall y \in V,\ f_i(X+y) - f_i(X) = \sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X) \qquad \text{mod } \wp(k[X])}.$$

i.e. stability of the system of equations under translation by V.

Case $n = 1$, i.e. $G_2 \simeq \mathbb{Z}/p\mathbb{Z}$.

Theorem (Lehr-Matignon 2005)

Let (C, G) be a big action such that $G_2 \simeq \mathbb{Z}/p\mathbb{Z}$.

- *Then*

$$\boxed{C \sim C_S : W^p - W = X\,S(X) \in k[X]}$$

with $S(X) = (a_0\, id + a_1\, F + \ldots + F^s)\,(X)$ ***additive*** *of degree* p^s, $s \geq 1$.

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  *with $S(X) = (a_0\, id + a_1\, F + \ldots + F^s)\,(X)$ **additive** of degree p^s, $s \geq 1$.*

- *Consider the **palindromic polynomial** of S [Elkies]:*

$$\boxed{\mathrm{Ad}_S := F^s \sum_{j=0}^{s} (a_j\, F^j + F^{-j}\, a_j) \quad \textit{and} \quad Z(\mathrm{Ad}_S) \simeq (\mathbb{Z}/p\mathbb{Z})^{2s}.}$$

Case $n = 1$, i.e. $G_2 \simeq \mathbb{Z}/p\mathbb{Z}$.

Theorem

- *Let* $A_{\infty,1}$ *be the wild inertia subgroup of* $\mathrm{Aut}_k(C)$ *at* ∞. *Then*

$$\boxed{\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & A_{\infty,1} & \xrightarrow{\pi} & Z(\mathrm{Ad}_S) & \longrightarrow & 0 \\ & & \| & & \bigcup & & \bigcup & & \\ 0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & G & \xrightarrow{\pi} & V & \longrightarrow & 0 \end{array}}$$

For $p \geq 3$, *unique* ***extraspecial group*** *of exponent* p *and order* p^{2s+1}.

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Theorem

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  For $p \geq 3$*, unique* ***extraspecial group*** *of exponent* p *and order* p^{2s+1}.

- *Conversely, if*

$$C_S : W^p - W = X\,S(X) \in k[X]$$

  with $S(X)$ *additive of degree* p^s*,* $s \geq 1$*,* $(C_S, A_{\infty,1})$ *is a big action with* $G_2 \simeq \mathbb{Z}/p\mathbb{Z}$.

A ring filtration of $k[X]$ related to the additive polynomials.

Problem: Generalize the case $n = 1$.

Definition (Rocher 2009)

Let $t \geq 1$. We define Σ_t as the k-vector subspace of $k[X]$ generated by 1 and the products of at most t additive polynomials.

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Lemma

- *Let $a \in \mathbb{N}$ with p-adic expansion: $a = a_0 + a_1 p + \ldots a_\ell p^\ell, 0 \leq a_i \leq p-1$.*
  Then $\boxed{X^a \in \Sigma_t \quad \Leftrightarrow \quad \mathrm{S}_p(a) := a_0 + a_1 + \ldots + a_\ell \leq t.}$
- *Let $f(X) \in k[X] - \{0\}$ such that $f(X) = \sum_{a \in \mathbb{N}} c_a(f) X^a$.*
  Then $\boxed{f \in \Sigma_t \quad \Leftrightarrow \quad \forall a \in \mathbb{N} \text{ with } c_a(f) \neq 0,\ \mathrm{S}_p(a) \leq t.}$

Parametrization of the big actions with $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$.

Theorem (Rocher 2009)

Let (C, G) be a big action such that $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, $n \geq 1$.
Then $\forall\, i \in \{1, \ldots, n\}, f_i \in \Sigma_{i+1}$.

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Remark

- *This generalizes the case $n = 1$ where $f(X) = X\,S(X) \in \Sigma_2$.*
- *For $n \geq 2$, the converse is no longer true.*
  Obstruction related to the embedding problem.

The special case: each $f_i \in \Sigma_{i+1} - \Sigma_i$.

Prelimary remark: Let (C, G) be a big action. Then $G_2 = D(G)$. [M-R]

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Definition (Rocher 2009)

For any group G, we define an ascending sequence of characteristic subgroups of G as follows:

$$\Lambda_0(G) = \{e\}$$

$$\forall i \geq 1, \frac{\Lambda_i(G)}{\Lambda_{i-1}(G)} = Z(\frac{G}{\Lambda_{i-1}(G)}) \cap D(\frac{G}{\Lambda_{i-1}(G)}).$$

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Theorem (Rocher 2009)

Let (C,G) be a big action such that $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, $n \geq 2$. Then the following assertions are equivalent:

- $\forall i \in \{1,\ldots,n\}, f_i \in \Sigma_{i+1} - \Sigma_i$.
- $\forall i \in \{1,\ldots,n\}, \frac{\Lambda_i(G)}{\Lambda_{i-1}(G)} \simeq \mathbb{Z}/p\mathbb{Z}$.

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- $\forall i \in \{1, \ldots, n\}$, $\frac{\Lambda_i(G)}{\Lambda_{i-1}(G)} \simeq \mathbb{Z}/p\mathbb{Z}$.

Then,

- $n \leq p - 1$.
- $\forall i \in \{1, \ldots, n\}$, $\deg(f_i) = 1 + ip^s$.
- $\dim_{\mathbb{F}_p} V = s + 1$.
- *The $\Lambda_i(G)$'s coincide with the upper ramification groups of G_2:*
$$\forall i \in \{0, \ldots, n\}, \quad \Lambda_{n-i}(G) = (G_2)^{v_i}.$$

Parametrization of the family for $p = 5$ and $s = 1$.

- One deduces an algorithmic method to parametrize the functions f_i's.
- We illustrate this method for $p = 5$ and $s = 1$, i.e. $\dim_{\mathbb{F}_p} V = 2$..

Parametrization of the family for $p = 5$ et $s = 1$.

- $\underline{n = 2}$ [Rocher 2009]

$$\boxed{\begin{array}{l} f_1(X) = X^6 + 2\,(b_{11}^{25} + b_{11})\, b_{11}^{-5}\, X^2 \\ \\ f_2(X) = b_{11}^5\, X^{11} + 4\, b_{11}^{25}\, X^7 + 2\,(b_{11}^{50} - b_{11}^2)\, b_{11}^{-5}\, X^3 + b_1\, X \end{array}}$$

where $b_{11} \in k^\times$ and $b_1 \in k$ are algebraically independent parameters.

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Remark

- *The space of parameters is a Zariski open of the affine space* $\mathbb{A}_k^2$*: it is irreducible, hence only one possibility for G (up to isomorphism).*
- *Two curves* $C(b_{11}, b_1)$ *and* $C(b'_{11}, b'_1)$ *are isomorphic if and only if*

$$\boxed{\left(\frac{b'_{11}}{b_{11}}\right)^{24} = 1} \quad \textit{and} \quad \boxed{b'_1 = c\,\frac{b'_{11}}{b_{11}}\,b_1} \quad \textit{with} \quad c \in \mathbb{F}_5^\times .$$

- $n = 3$ [Rocher 2009]

$$f_1(X) = X^6 + 2\,(b_{11}^{25} + b_{11})\,b_{11}^{-5}\,X^2$$

$$f_2(X) = b_{11}^5\,X^{11} + 4\,b_{11}^{25}\,X^7 + 2\,(b_{11}^{50} - b_{11}^2)\,b_{11}^{-5}\,X^3 + 2\,(c_6 - c_6^5)\,b_{11}^{-5}\,X$$

$$\begin{aligned} f_3(X) = \;& 4\,b_{11}^{10}\,X^{16} + 4\,b_{11}^{30}\,X^{12} + c_{11}^5\,X^{11} + 4\,b_{11}^{50}\,X^8 + 4\,c_{11}^{25}\,X^7 \\ & + c_6^5\,X^6 + 4\,(b_{11}^{75} + b_{11}^3)\,b_{11}^{-5}\,X^4 \\ & + \{(b_{11}^{25} + b_{11})\,c_{11}\,b_{11}^{-5} + 2\,(b_{11}^{25} + b_{11})^2\,c_{11}^5\,b_{11}^{-10}\}\,X^3 \\ & + 2\,(c_6^5\,b_{11}^{25} + c_6\,b_{11})\,b_{11}^{-5}\,X^2 + c_1\,X \end{aligned}$$

where $b_{11} \in k^\times$, $c_6 \in k$ and $c_1 \in k$ are algebraically independent parameters and $c_{11} \in k$ satisfies

$$\boxed{c_{11}^{25} + 4\,(b_{11}^{25} + b_{11})\, b_{11}^{-5}\, c_{11}^{5} + c_{11} = 0.}$$

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Remark

The space of parameters is no more connected.
One finds two non-isomorphic models for the group G (MAGMA).

Application: towards a classification of big actions.

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- By [Stichtenoth, 1973], $\boxed{\frac{|G|}{g^2} \leq \frac{4p}{(p-1)^2}}$.
- If $\boxed{\frac{|G|}{g^2} \geq \frac{4}{(p-1)^2}}$, then $G_2 \simeq \mathbb{Z}/p\mathbb{Z}$. Description in [L-M].

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The criterion to classify big actions is $\frac{|G|}{g^2}$.

Let (C, G) be a big action.

- By [Stichtenoth, 1973], $\boxed{\frac{|G|}{g^2} \leq \frac{4p}{(p-1)^2}}$.
- If $\boxed{\frac{|G|}{g^2} \geq \frac{4}{(p-1)^2}}$, then $G_2 \simeq \mathbb{Z}/p\mathbb{Z}$. Description in [L-M].
- If $\boxed{\frac{|G|}{g^2} \geq \frac{4}{(p^2-1)^2}}$, then $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, $1 \leq n \leq 3$. Proof in [M-R].
  Classification and parametrization given in [Rocher 2].

Outline of my thesis, publications, prepublications.

- Chapter 1: General background on G-actions on curves.
  - Link with recent works on deformation by [Bertin-Mezard 00], [Cornelissen- Kato 03], [Pries 05,06,08], [Kontogeorgis 06,07], [Bertin-Romagny 08]...
- Chapter 2: *"Smooth curves having a large automorphism p-group in characteristic $p > 0$"* (joint with M. Matignon), Algebra and Number Theory (2008).
- Chapter 3: *"Large p-group actions with a p-elementary abelian derived group"*, Journal of Algebra (2009).
- Chapter 4: *"Large p-group actions with $\frac{|G|}{g^2} \geq \frac{4}{(p^2-1)^2}$."*