Large p-group actions with $\frac{|G|}{g^2} \geq \frac{4}{(p^2-1)^2}$.

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Abstract

Let k be an algebraically closed field of characteristic $p > 0$ and C a connected nonsingular projective curve over k with genus $g \geq 2$. Let (C, G) be a *big action*, i.e. a pair (C, G) where G is a p-group of k-automorphisms of C such that $|G| > \frac{2\,p}{p-1}\,g$. This paper is twofold: we first give preliminary results to establish a classification of such actions. In particular, we study the finiteness of the quotients $\frac{|G|}{g^2}$ and $\frac{|G|}{g}$ when (C, G) runs over the big actions satisfying $\frac{|G|}{g^2} \geq M$, for a given positive real $M > 0$. As an application, we display the classification and the parametrization of such actions for $M := \frac{4}{(p^2-1)^2}$.

1 Introduction.

This paper continues the work begun in [MR08] and [Ro09], namely the study of G-actions on connected nonsingular projective curves of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$, when G is a p-group of k-automorphims such $|G| \geq \frac{2p}{p-1}\,g$. One of the aims of this set of papers is to pursue the classification of such actions as initiated in [LM05], namely to give the parametrization of the function field of the curve and to discuss the corresponding deformation space. This paper is twofold: we first give preliminary results towards the classification of such G-actions and, as an application, we display their classification when $|G| \geq \frac{4}{(p^2-1)^2}\,g^2$.

Setting. Let k be an algebraically closed field of positive characteristic $p \geq 0$ and C a connected nonsingular projective curve over k, with genus $g \geq 2$. In characteristic zero, the k-automorphism group $\mathrm{Aut}(C)$ of the curve C was proved by Hurwitz to be finite and of order at most $84(g-1)$. An open question concerns the classification of these automorphism groups for a given genus g. This classification has been partially achieved for the Hurwitz groups, that is the groups whose order reaches the Hurwitz bound $84(g-1)$ (cf. [Con90]), but also for *large* automorphism groups G, *large* meaning that the order of G is greater than $4\,(g-1)$ (cf.[Kul97]). Indeed, combined with the Hurwitz genus formula, this lower bound imposes restrictions on the genus of the quotient curve C/G, namely $g_{C/G} = 0$, on the number r of points of the quotient curve C/G ramified in C, namely $r \in \{3, 4\}$, and on the corresponding ramification indices (cf. [Kul97], [Br00]). Following the works of Kulkarni and Breuer, Magaard et alii ([MSSV02]) give the list of the *large* automorphism groups of compact Riemann surfaces of genus g up to $g = 10$.

In positive characteristic $p > 0$, the automorphism group $\mathrm{Aut}_k(C)$ is still finite. Nevertheless, for groups whose order is not prime to p, the Hurwitz linear bound is replaced by biquadratic ones (cf. [St73]). These bounds are optimal: so, in positive characteristic, the automorphism groups can be very large compared with the case of characteritic zero. This is due to the appearance of wild ramification. To rigidify the situation in characteristic $p > 0$ as has been done in characteristic zero, an idea is to focus on large automorphism p-groups, more specifically on p-subgroups of $\mathrm{Aut}_k(C)$ such that $|G| > \frac{2p}{p-1}\,g$. Following Lehr and Matignon ([LM05]), we call such a pair (C, G) a *big action.* Under this condition,the quotient curve C/G is isomorphic to the projective line $\mathbb{P}^1_k$ and only one point of C, say ∞, is ramified in C/G. In the study of big actions, a major role is played by the second lower ramification group G_2 of G at ∞. In [MR08], we prove that this group can be characterized in a purely algebraic way since it coincides with the derived group G' of G.

Motivation and purpose. If big actions are defined through the value taken by the quotient $\frac{|G|}{g}$, it turns out that the key criterion to classify them is the value of another quotient, namely $\frac{|G|}{g^2}$. Indeed, the quotient $\frac{|G|}{g^2}$ has, to some extent, a sieve effect among big actions. If (C, G) is a big action, we

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first deduce from [Na87] (Theorem 1) that $\frac{|G|}{g^2} \leq \frac{4p}{(p-1)^2}$. Lehr and Matignon then prove that the big actions such that $\frac{|G|}{g^2} \geq \frac{4}{(p-1)^2}$ correspond to the p-cyclic étale covers of the affine line parametrized by an Artin-Schreier equation: $W^p - W = X\,S(X) + c\,X \in k[X]$, where $S(X)$ runs over the additive polynomials of $k[X]$. In [MR08], we show that the big actions satisfying $\frac{|G|}{g^2} \geq \frac{4}{(p^2-1)^2}$ are étale covers of the affine line with Galois group $G' \simeq (\mathbb{Z}/p\mathbb{Z})^n$ with $n \leq 3$. The study of big actions with a p-elementary abelian G' is precisely the topic of [Ro09] where we generalize the structure theorem obtained in the p-cyclic case. In this paper, we deduce from this the parametrization of the functions f_i's for big actions satisfying $\frac{|G|}{g^2} \geq \frac{4}{(p^2-1)^2}$. In particular cases, we shall discuss the space of parameters of the families obtained in this way. This work should be compared with the results of Pries ([Pr05]) and Kontogeorgis ([Kon07]) who give an upper bound for the dimension of the deformation space of such G-actions when G is a p-group.

Outline ot the paper. In Section 2, we recall generalities about big actions, among others the ramification conditions imposed on the cover $C \to C/G$ by the choice of the lower bound $|G| > \frac{2p}{p-1}\,g$. In Section 4, we prove that, for a given $M > 0$, the order of G' only takes a finite number of values when (C, G) runs over the set of big actions satisfying $\frac{|G|}{g^2} \geq M$ (Lemma 4.1). When exploring similar finiteness results for g and $|G|$, we are lead to a purely group-theoretic discussion on the Frattini subgroup $\mathrm{Fratt}(G')$ of G' and the commutator subgroup $[G', G]$ of G' and G. Both of these groups play a major role in the study of big actions. Indeed, $\mathrm{Fratt}(G')$ is trivial if and only if G' is p-elementary abelian whereas $[G', G]$ is trivial if and only if G' is included in the center of G. Note that these two cases have been investigated in [Ro09]. The main results of Section 2 are gathered in the following

Theorem:

1. Let (C, G) be a big action. Then $\mathrm{Fratt}(G') \subset [G', G]$ (Lemma 4.5).
2. Fix $M > 0$ a positive real. Let (C, G) run over the set of big actions satisfying $\frac{|G|}{g^2} \geq M$.
   (a) If $\mathrm{Fratt}\,(G') \subsetneq [G', G]$, then g and $\frac{|G|}{g}$ take a finite number of values (Section 4.3).
   (b) If $\mathrm{Fratt}\,(G') = [G', G]$,
      i. $\frac{|G|}{g}$ is no more bounded (Remark 4.8).
      ii. if $\mathrm{Fratt}\,(G') = [G', G] = \{e\}$, $\frac{|G|}{g^2}$ takes a finite number of values (Section 4.4.2).
      iii. if $\mathrm{Fratt}\,(G') = [G', G] \neq \{e\}$ and if $p > 2$, G' is non-abelian (Section 4.4.3).

Note that we do not know yet examples of big actions with a non-abelian G'. This finiteness result also justifies the choice of the quotient $\frac{|G|}{g^2}$ as a criterion to classify big actions with an abelian G'.

Another central question raised in this paper is the link between the subgroups G of $\mathrm{Aut}_k(C)$ such that (C, G) is a big action and a p-Sylow subgroup of $\mathrm{Aut}_k(C)$ containing G (Section 3). Among other things, we prove that they have the same derived subgroup. This, together with the fact that the order of G' takes a finite number of values for big actions satisfying $\frac{|G|}{g^2} \geq M$, implies, on the one hand, that the order of G' is a key criterion to classify big actions and, on the other hand, that we can concentrate on the p-Sylow subgroups of $\mathrm{Aut}_k(C)$. As an application, we finally display in Section 5 the classification and the parametrization of big actions (C, G) satisfying $\frac{|G|}{g^2} \geq M$ for $M := \frac{4}{(p^2-1)^2}$. As in the theorem above, we shall distinguish the cases $[G', G] = \mathrm{Fratt}(G')(= \{e\})$ and $[G', G] \supsetneq \mathrm{Fratt}(G')(= \{e\})$.

Notation and preliminary remarks. Throughout this paper, k denotes an algebraically closed field of characteristic $p > 0$, C is a connected nonsingular projective curve over k with genus g and $\mathrm{Aut}_k(C)$ means its k-automorphism group.

We denote by F the Frobenius endomorphism for a k-algebra. Then $\wp$ means the Frobenius operator minus identity. We denote by $k\{F\}$ the k-subspace of $k[X]$ generated by the polynomials $F^i(X)$, with $i \in \mathbb{N}$. It is a ring under the composition. Moreover, for all α in k, $F\,\alpha = \alpha^p\,F$. The elements of $k\{F\}$ are the additive polynomials, i.e. the polynomials $P(X)$ of $k[X]$ such that for all α and β in k, $P(\alpha + \beta) = P(\alpha) + P(\beta)$.

Let G be a finite group. A G-action (also called a G-curve) is a pair (C, ϕ) where C is a curve as above and ϕ an injective morphism $G \hookrightarrow \mathrm{Aut}_k(C)$. Following [BeRo08], we say that two G-actions (C, ϕ) and (C', ϕ') are equivalent if and only if there exists a G-isomorphism f: $C \to C'$, i.e. a k-isomorphism f such that $\forall\, g \in G$, $f\,\phi(g) = \phi'(g)\,f$. In particular, f induces a k-isomorphism $\overline{f}$: $C/\phi(G) \to C'/\phi'(G)$. In what follows, we omit ϕ and confuse G with $\phi(G)$.

2 The setting: generalities on big actions.

2.1 Definitions.

Definition 2.1. *Let C be a connected nonsingular projective curve over k with genus $g \geq 1$. Let G be a subgroup of $\mathrm{Aut}_k(C)$. We say that the pair (C, G) is a big action if G is a finite p-group such that*

$$\frac{|G|}{g} > \frac{2\,p}{p-1}. \tag{1}$$

We also introduce definitions related to the classification.

Definition 2.2. *Let C be a connected nonsingular projective curve over k with genus $g \geq 1$. Let G be a subgroup of $\mathrm{Aut}_k(C)$. Fix $M > 0$ a positive real.*

1. *We say that (C, G) satisfies condition $\mathcal{C}_M$ if (C, G) is a big action with $\frac{|G|}{g^2} \geq M$.*
2. *We call $\mathcal{S}_M$ the set of big actions satisfying condition $\mathcal{C}_M$, i.e. such that $\frac{|G|}{g^2} \geq M$.*
3. *If (C, G) satisfies condition $\mathcal{C}_M$ for $M = \frac{4}{(p^2-1)^2}$, we say that (C, G) satisfies condition $(*)$.*

Remark 2.3. *There exists big actions satisfying condition $\mathcal{C}_M$ if and only if $M \leq \frac{4\,p}{(p-1)^2}$ (see [St73]).*

2.2 The ramification conditions and the embedding problem.

Combined with the bound (1), the Hurwitz and the Deuring-Shafarevitch formulas applied to $\pi : C \to C/G$ impose some restrictions on the ramification of π.

Proposition 2.4. *([LM05], [MR08] Section 2). Let (C, G) be a big action with genus $g \geq 2$.*

1. *The Hasse-Witt invariant of C vanishes. Moreover, the ramification locus (resp. branch locus) of the cover $\pi : C \to C/G$ is reduced to one point, say ∞ (resp. $\pi(\infty)$.)*
2. *For all $i \geq -1$, call G_i the i-th lower ramification group of G at ∞ (cf. [St93] Definition III 8.5). Then $G = G_{-1} = G_0 = G_1$. Moreover, the two quotient curves C/G and C/G_2 are isomorphic to the projective line $\mathbb{P}^1_k$. In particular, G_2 is nontrivial.*
3. *The second ramification group G_2 is strictly included in G. Then the quotient group G/G_2 acts as a group of translations of the affine line $C/G_2 - \{\infty\} = \mathrm{Spec}\, k[X]$, through $X \to X + y$, where y runs over a subgroup V of k. Thus, we obtain the exact sequence:*

$$0 \longrightarrow G_2 \longrightarrow G = G_1 \overset{\pi}{\longrightarrow} V \simeq (\mathbb{Z}/\,p\,\mathbb{Z})^v \longrightarrow 0,$$

where

$$\pi : \begin{cases} G \to V \\ g \to g(X) - X. \end{cases}$$

4. *Let H be a normal subgroup of G such that $H \subsetneq G_2$. Then $(C/H, G/H)$ is a big action with second ramification group $(G/H)_2 = G_2/H$.*

Remark 2.5. *Let C be a connected nonsingular projective curve over k with genus $g \geq 2$. In [GK07], Giuletti and Korchmáros give a classification of the groups $G \subset \mathrm{Aut}_k(C)$ such that there exists a point $P \in C$ with*

$$|G_{P,1}| > \frac{p}{p-1}\, g, \tag{2}$$

where $G_{P,1}$ means the wild inertia subgroup of G at P. Note that the lower bound given in the definition of a big action *is slightly stronger than the bound in* (2). *The reason for this choice is that the bound* (1) *induces an* embedding problem *as expressed in Proposition 2.4.3.*

More precisely, the strict inequality in (1) *is necessary to get the strict inclusion $G_2 \subsetneq G$. For instance, put $p > 2$ and consider $C : W^p - W = X^2$ together with $G = \langle \sigma \rangle$ where $\sigma(X) = X$ and $\sigma(W) = W + 1$. Then $G_2 = G_1 = G \simeq \mathbb{Z}/p\mathbb{Z}$ and $g = \frac{p-1}{2}p$. So $\frac{|G|}{g} = \frac{2\,p}{p-1}$.*

In [MR08], we also give a purely algebraic characterization of G_2.

Proposition 2.6. *([MR08] Theorem 2.7.4). Let (C, G) be a big action. Then $G_2 = G'$, where G' means the commutator subgroup of G.*

2.3 The special case of big actions with a p-elementary abelian G'.

The special case of big actions with a p-elementary abelian G' is the main object of [Ro09].

Recall 2.7. *([Ro09]) Let (C, G) be a big action such that $G' \simeq (\mathbb{Z}/p\mathbb{Z})^n$, $n \geq 1$.*

1. *We denote by L be the function field of the curve C and by $k(X) := L^{G'}$ the subfield of L fixed by G'. Then the extension $L/k(X)$ can be parametrized by n Artin-Schreier equations: $W_i^p - W_i = f_i(X) \in k[X]$ with $1 \leq i \leq n$, where the functions f_i satisfy some specific properties:*
   (a) *For all $i \in \{1, \dots, n\}$, each function f_i is assumed to be reduced mod $\wp(k[X])$*
   (b) *Moreover, $m_1 \leq m_2 \leq \dots \leq m_n$, where $m_i := \deg f_i$.*
   (c) *$\forall\ (\lambda_1, \dots \lambda_n) \in \mathbb{F}_p^n$ not all zeros,*
   $$\deg\,(\sum_{i=1}^{n} \lambda_i\, f_i(X)) = \max_{i \in \{1,\dots,n\}} \{\deg \lambda_i\, f_i(X)\}.$$
   Such a basis is called an adapted basis *(cf. [Ro09] Definition 2.7.4). Then the genus of the curve C is given by the following formula (cf. [Ro09] Corollary 2.11):*
   $$g = \frac{p-1}{2} \sum_{i=1}^{n} p^{i-1}\,(m_i - 1).$$

2. *Consider the $\mathbb{F}_p$-vector subspace of $k[X]$ generated by the classes of $\{f_1(X), \dots, f_n(X)\}$ modulo $\wp(k[X])$*
   $$A := \frac{\wp(L) \cap k[X]}{\wp(k[X])}.$$
   As an $\mathbb{F}_p$-vector space, A is isomorphic to G' with respect to the Artin-Schreier pairing (see [Ro09] Remark 2.3 or [Bou83] Chapter IX, Exercise 19).

3. *The group V (as defined in Proposition 2.4.3) acts on A by translation. More precisely, for all y in V, one can define an automorphism $\rho(y)$ of A as follows:*
   $$\rho(y) : \begin{cases} A \to A \\ \overline{f(X)} \to \overline{f(X+y)} \end{cases}$$
   where $\overline{f(X)}$ means the class in A of $f(X) \in k[X]$. For all y in V, the matrix of the automorphism $\rho(y)$ in the adapted basis fixed above is an upper triangular matrix of $\mathrm{GL}_n(\mathbb{F}_p)$ with identity on the diagonal, namely
   $$L(y) := \begin{pmatrix} 1 & \ell_{1,2}(y) & \ell_{1,3}(y) & \dots & \ell_{1,n}(y) \\ 0 & 1 & \ell_{2,3}(y) & \dots & \ell_{2,n}(y) \\ 0 & 0 & \dots & \dots & \ell_{i,n}(y) \\ 0 & 0 & 0 & 1 & \ell_{n-1,n}(y) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_n(\mathbb{F}_p),$$
   where, for all i in $\{1, \dots, n-1\}$, $\ell_{i,i+1}$ is a linear form from V to $\mathbb{F}_p$ (see [Ro09] Section 2.5). In other words,
   $$\forall\, y \in V,\ f_1(X+y) - f_1(X) = 0 \qquad \mathrm{mod}\ \ \wp(k[X]).$$
   $$\forall\, i \in \{2, \dots, n\},\ \forall\, y \in V,\ f_i(X+y) - f_i(X) = \sum_{j=1}^{i-1} \ell_{j,i}(y)\, f_j(X) \qquad \mathrm{mod}\ \ \wp(k[X]). \qquad (3)$$

4. *For all $t \geq 1$, we denote by Σ_t the k-vector subspace of $k[X]$ generated by 1 and the products of at most t additive polynomials of $k[X]$ (see [Ro09] Definition 3.1). Then for all i in $\{1, \dots, n\}$, f_i lies in Σ_{i+1} (see [Ro09] Theorem 3.14).*

3 A study on p-Sylow subgroups of $\mathrm{Aut}_k(C)$ inducing big actions.

Notation. In this section, we fix a connected nonsingular projective curve C over k with genus $g \geq 2$. We denote by $A := \mathrm{Aut}_k(C)$ the k-automorphism group of the curve C and by $S(A)_p$ any p-Sylow subgroup of A. For any point $P \in C$ and any $i \geq -1$, we denote by $A_{P,i}$ the i-th ramification group of A at P in lower notation.

Remark 3.1. *Assume that there exists a subgroup $G \subset A$ such that (C, G) is a big action.*

1. *Then for every p-Sylow subgroup $S(A)_p$ of A, $(C, S(A)_p)$ is a big action.*
2. *Moreover, A has a unique p-Sylow subgroup except in the three following cases (cf. [Han92] and [GK07]):*

   (a) *The Hermitian curve*
   $$C_H: \quad W^q + W = X^{1+q}$$
   with $p \geq 2$, $q = p^s$, $s \geq 1$. Then $g = \frac{1}{2}(q^2 - q)$ and $A \simeq \mathrm{PGU}_3(\mathbb{F}_{q^2})$ (cf. [Leo96]). It follows that $|A| = q^3(q^2-1)(q^3+1)$, so $\frac{|S(A)_p|}{g} = \frac{2q^2}{q-1} > \frac{2p}{p-1}$. Thus, $(C_H, S(A)_p)$ is a big action with $G' \simeq (\mathbb{Z}/p\mathbb{Z})^s$. Since $\frac{|S(A)_p|}{g^2} = \frac{4q}{(q-1)^2}$, this big action satisfies condition $(*)$ (see Definition 2.2) if and only if $1 \leq s \leq 3$.

   (b) *The Deligne-Lusztig curve arising from the Suzuki group*
   $$C_S: \quad W^q + W = X^{q_0}(X^q + X)$$
   with $p = 2$, $q_0 = 2^s$, $s \geq 1$ and $q = 2^{2s+1}$. In this case, $g = q_0(q-1)$ and $A \simeq Sz(q)$ is the Suzuki group. It follows that $|A| = q^2(q-1)(q^2+1)$, so $\frac{|S(A)_p|}{g} = \frac{q^2}{q_0(q-1)} > \frac{2p}{p-1}$. Thus, $(C_S, S(A)_p)$ is a big action with $G' \simeq (\mathbb{Z}/p\mathbb{Z})^{2s+1}$. Since $\frac{|S(A)_p|}{g^2} = \frac{q^2}{q_0^2(q-1)^2} < \frac{4}{(p^2-1)^2}$ for all $s \geq 1$, this big action never satisfies condition $(*)$.

   (c) *The Deligne-Lusztig curve arising from the Ree group*
   $$C_R: \quad W_1^q - W_1 = X^{q_0}(X^q - X) \quad \text{and} \quad W_2^q - W_2 = X^{2q_0}(X^q - X)$$
   with $p = 3$, $q_0 = 3^s$, $s \geq 1$ and $q = 3^{2s+1}$. Then $g = \frac{3}{2} q_0 (q-1)(q+q_0+1)$ and $A \simeq Ree(q)$ is the Ree group. It follows that $|A| = q^3(q-1)(q^3+1)$, so $\frac{|S(A)_p|}{g} = \frac{2q^3}{3q_0(q-1)(q+q_0+1)} > \frac{2p}{p-1}$. Thus, $(C_R, S(A)_p))$ is a big action with $G' \simeq (\mathbb{Z}/p\mathbb{Z})^{2(2s+1)}$. Since $\frac{|S(A)_p|}{g^2} = \frac{4q^3}{9q_0^2(q-1)^2(q+q_0+1)^2} < \frac{4}{(p^2-1)^2}$ for all $s \geq 1$, this big action never satisfies condition $(*)$.

   In each of these three cases, the group A is simple, so it has more than one p-Sylow subgroups.

We now highlight the link between the groups G such that (C, G) is a big action and the p-Sylow subgroup(s) of A containing G.

Proposition 3.2. *1. Let G be a subgroup of A such that (C, G) is a big action.*

(a) *There exists a point of C, say ∞, such that G is included in $A_{\infty,1}$. Then $(C, A_{\infty,1})$ is a big action with $A_{\infty,2} = G_2$, i.e. $(A_{\infty,1})' = G'$. Thus, we obtain the following diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{\infty,2} & \longrightarrow & A_{\infty,1} & \xrightarrow{\pi} & W \subset k & \longrightarrow & 0 \\ & & || & & \cup & & \cup & & \\ 0 & \longrightarrow & G_2 & \longrightarrow & G = G_1 & \xrightarrow{\pi} & V & \longrightarrow & 0. \end{array}$$

In particular, $G = \pi^{-1}(V)$ where V is an $\mathbb{F}_p$-vector subspace of W.

(b) *$A_{\infty,1}$ is a p-Sylow subgroup of A. Moreover, except in the three special cases mentionned in Remark 3.1, $A_{\infty,1}$ is the unique p-Sylow subgroup of A.*

(c) *Let M be a positive real such that (C, G) satisfies condition $\mathcal{C}_M$. Then $(C, A_{\infty,1})$ also satifies $\mathcal{G}_M$.*

2. *Conversely, let ∞ be a point of the curve C such that $(C, A_{\infty,1})$ is a big action. Consider V an $\mathbb{F}_p$-vector space of W defined as above and put $G := \pi^{-1}(V)$.*
   (a) *Then (C, G) is a big action if and only if*
   $$|W| \geq |V| > \frac{2\,p}{p-1}\,\frac{g}{|A_{\infty,2}|}.$$
   (b) *Let $M > 0$ such that $(C, A_{\infty,1})$ satisfies condition $\mathcal{C}_M$. Then (C, G) satisfies condition $\mathcal{C}_M$ if and only if*
   $$|W| \geq |V| \geq M\,\frac{g^2}{|A_{\infty,2}|}.$$

Proof: The first assertion (1.a) derives from [LM05] (Proposition 8.5) and [MR08] (Corollary 2.11). The second point (1.b) comes from [MR08] (Remark 2.12) together with Remark 3.1. The other claims are obtained via calculation. □

Remark 3.3. *Except in the three special cases mentionned in Remark 3.1, the point ∞ of C defined in Proposition 3.2 is uniquely determined. This means that if P is a point of C such that $(C, A_{P,1})$ is a big action, then $P = \infty$.*

As a conclusion, if (C, G) is a big action (satisfying condition $\mathcal{C}_M$) and if $A_{\infty,1}$ is a (or the) p-Sylow subgroup of A containing G, then $(C, A_{\infty,1})$ is also a big action (satisfying condition $\mathcal{C}_M$) with the same derived subgroup. So, in our attempt to classify the big actions (C, G) satisfying condition $\mathcal{C}_M$, we focus on the derived subgroup G' of G.

4 Finiteness results for big actions satisfying condition $\mathcal{C}_M$.

4.1 An upper bound on $|G'|$.

Lemma 4.1. *Let $M > 0$ be a positive real and (C, G) a big action satisfying condition $\mathcal{C}_M$. Then the order of G' is bounded as follows:*
$$p \leq |G'| \leq \frac{4\,p}{(p-1)^2}\,\frac{2 + M + 2\,\sqrt{1+M}}{M^2}.$$
Thus, $|G'|$ takes a finite number of values when (C, G) runs over $\mathcal{S}_M$.

Proof: We first recall that $G' = G_2$ is a non-trivial p-group (see Section 2). Now write the lower ramification filtration of G at ∞:
$$G = G_0 = G_1 \supsetneq G_2 = \ldots = G_{i_0} \supsetneq G_{i_0+1} = \ldots$$
Put $|G_2/G_{i_0+1}| = p^m$, with $m \geq 1$, and $\mathcal{B}_m := \frac{4}{M}\,\frac{|G_2/G_{i_0+1}|}{(|G_2/G_{i_0+1}|-1)^2} = \frac{4}{M}\,\frac{p^m}{(p^m-1)^2}$. By [LM05] (Theorem 8.6), $M \leq \frac{|G|}{g^2}$ implies $1 < |G_2| \leq \frac{4}{M}\,\frac{|G_2/G_{i_0+1}|^2}{(|G_2/G_{i_0+1}|-1)^2} = p^m\,\mathcal{B}_m$. From $|G_2| = p^m |G_{i_0+1}|$, we infer $1 \leq |G_{i_0+1}| \leq \mathcal{B}_m$. Since $(\mathcal{B}_m)_{m\geq 1}$ is a decreasing sequence which tends to 0 as m grows large, we conclude that m is bounded. More precisely, $m < m_0$ where m_0 is the smallest integer such that $\mathcal{B}_{m_0} < 1$. Since $M \leq \frac{4\,p}{(p-1)^2} \leq 8$ (see Remark 2.3), computation shows that $\mathcal{B}_m < 1 \Leftrightarrow p^m > \phi(M) := \frac{2+M+2\,\sqrt{1+M}}{M}$. Since $(\mathcal{B}_m)_{m\geq 1}$ is decreasing,
$$|G_2| \leq p^m\,\mathcal{B}_m \leq \phi(M)\,\mathcal{B}_1 = \frac{\phi(M)}{M}\,\frac{4\,p}{(p-1)^2}.$$
The claim follows. □

Next we deduce that, for big actions (C, G) satisfying condition $\mathcal{C}_M$, an upper bound on $|V|$ induces an upper bound on the genus g of C.

Corollary 4.2. *Let $M > 0$ be a positive real and (C, G) a big action satisfying condition $\mathcal{C}_M$. Then*
$$g < |G'|\,|V|\,\frac{p-1}{2\,p} \leq \frac{2}{p-1}\,\frac{2 + M + 2\sqrt{1+M}}{M^2}\,|V|.$$

This raises the following problem. If (C, G) is a big action satisfying condition $\mathcal{C}_M$, in which cases is $|V|$ (and then g) bounded from above? In other words, in which cases does the quotient $\frac{|G|}{g}$ take a finite number of values when (C, G) runs over $\mathcal{S}_M$? This question leads to a purely group-theoretic discussion on the inclusion $\mathrm{Fratt}(G') \subset [G', G]$, where $\mathrm{Fratt}(G')$ means the Frattini subgroup of G' and $[G', G]$ denotes the commutator subgroup of G' and G.

4.2 Preliminaries to a group-theoretic discussion.

We now introduce two groups $\mathrm{Fratt}(G')$ and $[G', G]$, both of them playing a major role in the study of big actions. Indeed, $\mathrm{Fratt}(G')$ is trivial if and only if G' is p-elementary abelian whereas $[G', G]$ is trivial if and only if G' is included in the center of G. These two cases have been investigated in [Ro09].

Lemma 4.3. *Let (C, G) be a big action. If the derived group G' is included in the center $Z(G)$ of G, then G' is p-elementary abelian.*

Proof: The hypothesis first requires $G' = G_2$ to be abelian. Now, assume that G_2 has exponent strictly greater than p. Then there exists a surjective map $\phi : G_2 \to \mathbb{Z}/p^2\mathbb{Z}$. So $H := \mathrm{Ker}\phi \subsetneq G_2 \subset Z(G)$ is a normal subgroup of G. It follows from Proposition 2.4 .4 that the pair $(C/H, G/H)$ is a big action with second ramification group $(G/H)_2 \simeq \mathbb{Z}/p^2\mathbb{Z}$. This contradicts [MR08] (Theorem 5.1). $\square$

Lemma 4.4. *Let (C, G) be a big action. Let $[G', G]$ be the commutator subgroup of G' and G.*

1. *Then the following assertions are equivalent.*
   (a) *The group $[G', G]$ is trivial*
   (b) *The derived group G' is included in the center $Z(G)$.*
   (c) *The function field of the curve C is parametrized by n Artin-Schreier equations:*
   $$\forall\, i \in \{1, \ldots, n\}, \quad W_i^p - W_i = f_i(X) = X\, S_i(X) + c_i\, X \in k[X],$$
   where S_i is an additive polynomial of $k[X]$ with degree $s_i \geq 1$ in F and $s_1 \leq s_2 \ldots \leq s_n$. Moreover, $V \subset \cap_{1 \leq i \leq n} Z(\mathrm{Ad}_{f_i})$ where Ad_{f_i} denotes the palindromic polynomial *related to f_i as defined in [Ro09] (Proposition 2.16). In particular, each Ad_{f_i} is an additive polynomial of degree p^{2s_i}.*
2. *The group $[G', G]$ is strictly included in G'.*
3. *Put $H := [G', G]$. The pair $(C/H, G/H)$ is a big action. Moreover, its second ramification group $(G/H)_2 = (G/H)' = G_2/H \subset Z(G/H)$ is p-elementary abelian.*

Proof:

1. The equivalence between the two first statements is clear. Now, if the second assertion is true, we deduce from Lemma 4.3 that G' is p-elementary abelian. Then the parametrization of the function field comes from [Ro09] (Proposition 2.16). The converse also derives from [Ro09] (Proposition 2.16).
2. Since G' is normal in G, then $[G', G] \subset G'$. Assume that $G' = [G', G]$. Then the lower central series of G is stationnary, which contradicts the fact that the p-group G is nilpotent (see e.g. [Su86] Chap.4). So $[G', G] \subsetneq G'$.
3. Since $H \subsetneq G' = G_2$ is normal in G, it follows from Proposition 2.4 .4 that the pair $(C/H, G/H)$ is a big action with second ramification group $(G/H)_2 = G_2/H$. From $H = [G_2, G]$, we gather that $G_2/H = (G/H)_2 \subset Z(G/H)$. Therefore, we deduce from Lemma 4.3 that $(G/H)_2$ is p-elementary abelian. $\square$

Lemma 4.5. *Let (C, G) be a big action. Let $\mathrm{Fratt}(G')$ be the Frattini subgroup of G'.*

1. *Then $\mathrm{Fratt}(G')$ is trivial if and only if G' is an elementary abelian p-group.*
2. *We have the following inclusions $\mathrm{Fratt}(G') \subset [G', G] \subsetneq G'$.*
3. *Put $F := \mathrm{Fratt}(G')$. The pair $(C/F, G/F)$ is a big action. Moreover, its second ramification group $(G/F)_2 = G_2/F = G'/F$ is p-elementary abelian.*
4. *Let $M > 0$ be a positive real. If (C, G) satisfies condition $\mathcal{C}_M$, then $(C/F, G/F)$ also satisfies condition $\mathcal{C}_M$.*

Proof:

1. Since G' is a p-group, $\mathrm{Fratt}(G') = (G')'(G')^p$, where $(G')'$ means the derived subgroup of G' and $(G')^p$ the subgroup generated by the p powers of elements of G' (cf. [LGM02] Proposition 1.2.4). This proves that if G' is p-elementary abelian, then $\mathrm{Fratt}(G')$ is trivial. The converse derives from the fact that $G'/\mathrm{Fratt}(G')$ is p-elementary abelian (cf. [LGM02] Proposition 1.2.4).

2. Using Lemma 4.4, the only inclusion that has to be shown is $\mathrm{Fratt}(G') = [G', G'](G')^p \subset [G', G]$. On the one hand, $[G', G'] \subset [G', G]$. On the other hand, by Lemma 4.5, $G'/[G', G]$ is p-elementary abelian. It follows that $(G')^p \subset [G', G]$.

3. Since $F \subsetneq G' = G_2$ is normal in G, we deduce from Proposition 2.4 .4 that the pair $(C/F, G/F)$ is a big action with second ramification group $(G/F)_2 = G_2/F = G'/F$. Moreover, since G_2 is a p-group, G_2/F is an elementary abelian p-group (see above).

4. This derives from [LM05] (Proposition 8.5 (ii)). □

This leads to discuss according to whether $\mathrm{Fratt}(G') \subsetneq [G', G]$ or $\mathrm{Fratt}(G') = [G', G]$. We shall start with the special case $\{e\} = \mathrm{Fratt}(G') \subsetneq [G', G]$, i.e. G' is p-elementary abelian and $G' \not\subset Z(G)$.

4.3 Case: $\mathrm{Fratt}(G') \subsetneq [G', G]$.

Proposition 4.6. *Let $M > 0$ be a positive real and (C, G) a big action satisfying condition $\mathcal{C}_M$. Suppose that $\{e\} = \mathrm{Fratt}(G') \subsetneq [G', G]$. Then $|V|$ and g are bounded as follows:*

$$|V| \leq \frac{4}{M} \frac{|G'|}{(p-1)^2} \leq \frac{16\, p}{(p-1)^4} \frac{2 + M + 2\sqrt{1+M}}{M^3} \tag{4}$$

and

$$\frac{p-1}{2} |V| \leq g < \frac{32\, p}{(p-1)^5} \frac{(2 + M + 2\sqrt{1+M})^2}{M^5}. \tag{5}$$

Thus, under these conditions, g, $|V|$ and the quotient $\frac{|G|}{g}$ take a finite number of values when (C, G) runs over $\mathcal{S}_M$.

Proof: Write $G' = G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, with $n \geq 1$. Since $G' \not\subset Z(G)$, Lemma 4.4 ensures the existence of a smaller integer $j_0 \geq 1$ such that $f_{j_0+1}(X)$ cannot be written as $c\,X + X S(X)$, with S an additive polynomial. If $j_0 \geq 2$, it follows that, for all y in V, the coefficients of the matrix $L(y)$ as defined in Recall 2.7.3 satisfy $\ell_{i,j}(y) = 0$ for all $2 \leq j \leq j_0$ and $1 \leq i \leq j-1$. Moreover, the matricial multiplication proves that, for all i in $\{1, \ldots, j_0\}$, the functions ℓ_{i,j_0+1} are linear forms from V to $\mathbb{F}_p$. Because of the form of $f_{j_0+1}(X)$, the maps ℓ_{i,j_0+1}'s cannot be all zero. Put $U := \bigcap_{1 \leq i \leq j_0} \mathrm{Ker}\, \ell_{i,j_0+1}$. Thus, U is the intersection of at most j_0 hyperplanes of V. As a consequence, $\dim_k U \geq \dim_k V - j_0$, which gives

$$|U| \geq \frac{|V|}{p^{j_0}}. \tag{6}$$

Let $C_{f_{j_0+1}}$ be the curve parametrized by $W^p - W = f_{j_0+1}(X)$. It defines an étale cover of the affine line with group $\Gamma_0 \simeq \mathbb{Z}/p\mathbb{Z}$. Since, for all y in U, $f_{i_0+1}(X+y) = f_{i_0+1}(X) \bmod \wp(k[X])$, the group of translations of the affine line: $\{X \rightarrow X + y,\ y \in U\}$ can be extended to a p-group of automorphisms of the curve $C_{f_{j_0+1}}$, say Γ, with the following exact sequence

$$0 \longrightarrow \Gamma_0 \simeq \mathbb{Z}/p\,\mathbb{Z} \longrightarrow \Gamma \longrightarrow U \longrightarrow 0.$$

The pair $(C_{f_{j_0+1}}, \Gamma)$ is not a big action. Otherwise, its second ramification group would be p-cyclic, which contradicts the form of the function $f_{j_0+1}(X)$, as compared with [MR08] (Proposition 2.5). Thus, $\frac{|\Gamma|}{g_{C_{f_{j_0+1}}}} = \frac{2\,p}{p-1} \frac{|U|}{(m_{j_0+1}-1)} \leq \frac{2\,p}{p-1}$. Using (6), we obtain $\frac{|V|}{p^{j_0}} \leq |U| \leq (m_{j_0+1} - 1)$. Combined with the formula given in Recall 2.7, this inequality yields a lower bound on the genus, namely

$$g = \frac{p-1}{2} \sum_{i=1}^{n} p^{i-1} (m_i - 1) \geq \frac{p-1}{2} p^{j_0} (m_{j_0+1} - 1) \geq \frac{p-1}{2} |V|.$$

It follows that $M \leq \frac{|G|}{g^2} = \frac{|G'|\,|V|}{g^2} \leq \frac{4\,|G'|}{(p-1)^2\,|V|}$. Using Lemma 4.1, we gather inequality (4). Inequality (5) then derives from Corollary 4.2. □

The following corollary generalizes the finiteness result of Proposition 4.6 to all big actions satisfying condition $\mathcal{C}_M$ with $\mathrm{Fratt}(G') \subsetneq [G', G]$.

Corollary 4.7. *Let $M > 0$ be a positive real and (C, G) a big action satisfying condition $\mathcal{C}_M$. Suppose that* $\mathrm{Fratt}(G') \subsetneq [G', G]$. *Then $|V|$ and g are bounded by the same bounds as in Proposition 4.6. So the quotient $\frac{|G|}{g}$ takes a finite number of values when (C, G) runs over $\mathcal{S}_M$.*

Proof: Put $F := \mathrm{Fratt}(G')$. Lemma 4.5 asserts that the pair $(C/F, G/F)$ is a big action satisfying condition $\mathcal{C}_M$ whose second ramification group $(G/F)_2 = G_2/F$ is p-elementary abelian. Note that the $\mathbb{F}_p$-vector space V associated with the big action $(C/F, G/F)$ is the same as the one related to the initial big action (C, G). From $F \subsetneq [G_2, G]$, we gather $\{e\} \subsetneq [G_2/F, G/F]$, which implies $(G/F)_2 = (G/F)' \not\subset Z(G/F)$. We deduce that $|V|$ is bounded from above as in Proposition 4.6. Then the bound on g derives from Corollary 4.2. □

4.4 Case: $\mathrm{Fratt}(G') = [G', G]$.

4.4.1 A preliminary remark.

It now remains to investigate the case where $\mathrm{Fratt}(G') = [G', G]$. In particular, this equality is satisfied when G' is included in the center of G and so is p-elementary abelian (cf. Lemma 4.3). The finiteness result on g obtained in the preceding section is no more true in this case, as illustrated by the remark below.

Remark 4.8. *Let $s \geq 1$ be an integer. Following [MR08] (Proposition 2.5), consider the big action (C, G) associated with the curve $C : W^p - W = X\,S(X)$ where S is an additive polynomial of $k[X]$ with degree p^s. In this case, $g = \frac{p-1}{2}\,p^s$, $V = Z(\mathrm{Ad}_f) \simeq (\mathbb{Z}/p\mathbb{Z})^{2s}$ and $G' = G_2 \simeq \mathbb{Z}/p\mathbb{Z} \subset Z(G)$. It follows that $\frac{|G|}{g^2} = \frac{4\,p}{(p-1)^2}$. So, for all $M \leq \frac{4\,p}{(p-1)^2}$, (C, G) satisfies condition $\mathcal{C}_M$ with $\{e\} =$* $\mathrm{Fratt}(G') = [G', G]$. *Nevertheless $g = \frac{p-1}{2}\,p^s$ grows arbitrary large with s.*

Therefore, in this case, the quotient $\frac{|G|}{g}$ is no more bounded when (C, G) runs over $\mathcal{S}_M$. This leads to question the finiteness of another quotient, namely $\frac{|G|}{g^2}$.

4.4.2 Case: $\mathrm{Fratt}(G') = [G', G] = \{e\}$.

Proposition 4.9. *Let $M > 0$ be a positive real and (C, G) a big action satisfying condition $\mathcal{C}_M$. Assume that* $[G', G] = \mathrm{Fratt}(G') = \{e\}$. *Then the quotient $\frac{|G|}{g^2}$ takes a finite number of values when (C, G) runs over $\mathcal{S}_M$.*

Proof: Write $G' = G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, with $n \geq 1$ and keep the notations of Lemma 4.4. First of all, Lemma 4.1 implies that p^n can only take a finite number of values. Moreover, as recalled in Lemma 4.4, $V \subset \bigcap_{i=1}^n Z(\mathrm{Ad}_{f_i})$. Since Ad_{f_1} has degree p^{2s_1}, $|G| = |G'||V| \leq p^{n+2\,s_1}$. We compute the genus by means of Recall 2.7.1:

$$g = \frac{p-1}{2} \sum_{i=1}^{n} p^{i-1}\,(m_i - 1) = \frac{p-1}{2}\,p^{s_1}\,(\sum_{i=1}^{n} p^{i-1} p^{s_i - s_1}).$$

It follows that

$$0 < M \leq \frac{|G|}{g^2} \leq \frac{4\,p^n}{(p-1)^2 (\sum_{i=1}^n p^{i-1} p^{s_i - s_1})^2},$$

which implies

$$(\sum_{i=1}^{n} p^{i-1} p^{s_i - s_1})^2 \leq \frac{4\,p^n}{M\,(p-1)^2}.$$

Since p^n is bounded from above (Lemma 4.1), the set $\{s_i - s_1, i \in [1, n]\} \subset \mathbb{N}$ is also bounded, and then finite. Thus, the quotient

$$\frac{g^2}{p^{2s_1}} = \frac{(p-1)^2}{4}\,(\sum_{i=1}^{n} p^{i-1} p^{s_i - s_1})^2$$

takes a finite number of values. Moreover, from $M \leq \frac{|G|}{g^2} = \frac{|V|\,p^n}{g^2}$, we infer that $\frac{1}{|V|} \leq \frac{p^n}{M\,g^2}$, which implies

$$1 \leq \frac{p^{2s_1}}{|V|} \leq \frac{p^{2\,s_1}\,p^n}{M\,g^2} = \frac{4\,p^n}{M\,(p-1)^2\,(\sum_{i=1}^n p^{i-1} p^{s_i - s_1})^2} \leq \frac{4\,p^n}{M\,(p-1)^2}.$$

It follows that the set $\{\frac{p^{2s_1}}{|V|}\} \subset \mathbb{N}$ is bounded, and then finite, as well as the set $\{\frac{|V|}{p^{2s_1}}\}$. Therefore, the quotient $\frac{|G|}{g^2} = p^n \frac{|V|}{p^{2s_1}} \frac{p^{2s_1}}{g^2}$ can only take a finite number of values. □

The last case to explore is $\mathrm{Fratt}(G') = [G', G] \neq \{e\}$. As shown in the next section, this case can only occur when G' is non-abelian. Note that we do not know yet examples of big actions with a non-abelian G'.

4.4.3 Case: $\mathrm{Fratt}(G') = [G', G] \neq \{e\}$.

Theorem 4.10. *Assume that $p > 2$. Let (C, G) be a big action with $\mathrm{Fratt}(G') = [G', G] \neq \{e\}$. Then G' is non-abelian.*

We deduce the following

Corollary 4.11. *Assume that $p > 2$. Let $M > 0$ be a positive real. Then $\frac{|G|}{g^2}$ only takes a finite number of values when (C, G) runs over $\mathcal{S}_M$.*

Remark 4.12. *Theorem 4.10 is no more true for $p = 2$. A counterexample is given by [MR08] (Proposition 6.10). Indeed, keeping the notations of [MR08] (Proposition 6.10), take $q = p^e$ with $p = 2$, $e = 2s - 1$ and $s \geq 2$. Put $K = \mathbb{F}_q(X)$. Let $L := \mathbb{F}_q(X, W_1, V_1, W_2)$ be the extension of K parametrized by*

$$W_1^{2^{2s-1}} - W_1 = X^{2^{s-1}}(X^{2^{2s-1}} - X) \qquad V_1^{2^{2s-1}} - V_1 = X^{2^{s-2}}(X^{2^{2s-1}} - X)$$

$$[W_1, W_2]^2 - [W_1, W_2] = [X^{1+2^s}, 0] - [X^{1+2^{s-1}}, 0].$$

Let G be the p-group of $\mathbb{F}_q$-automorphisms of L constructed as in [MR08] (Proposition 6.10.3). Then the formula established for g_L in [MR08] (Proposition 6.10.4) shows that the pair (C, G) is a big action as soon as $s \geq 4$. In this case, $G' = G_2 \simeq \mathbb{Z}/2^2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{6s-4}$ (cf. [MR08] Proposition 6.10.2). In particular, $\mathrm{Fratt}(G') \neq \{e\}$. Moreover, since the functions $X^{2^{s-1}}(X^{2^{2s-1}} - X)$ and $X^{2^{s-2}}(X^{2^{2s-1}} - X)$ are products of two additive polynomials, it will be shown in the proof below (cf. point 6) that $[G', G] = \mathrm{Fratt}(G') \neq \{e\}$.

Proof of Theorem 4.10:

1. *Preliminary remarks: the link with Theorem 5.1 in [MR08].*
   (a) We first show that Theorem 4.10 implies Theorem 5.1 in [MR08]. We recall that the latter states that there is no big action (C, G) with $G_2 (= G')$ cyclic of exponent strictly greater than p. So assume that there exists one and show that Theorem 4.10 leads to a contradiction. If such a big action (C, G) exists, $G' = G_2$ is abelian and $\mathrm{Fratt}(G') = (G')^p \neq \{e\}$. To apply Theorem 4.10 and then find a contradiction, it remains to show that $F := \mathrm{Fratt}(G') = [G', G]$. From Lemma 4.5, we infer that $(C/F, G/F)$ is a big action whose second ramification group $(G/F)_2 = G_2/F$ is cyclic of order p as described in [MR08] (Proposition 2.5). Then $(G/F)_2 = (G/F)' \subset Z(G/F)$ (use Lemma 4.4). It follows from Lemma 4.4 that $[G'/F, G/F] = [(G/F)', G/F] = \{e\}$. Since $F \subset G'$, this induces $\{e\} \neq F = [G', G]$. Then Theorem 4.10 does not allow G' to be abelian.

   (b) The object of Theorem 4.10 is to prove that there exists no big action (C, G) with G' abelian of exponent strictly greater than p, such that $\mathrm{Fratt}(G') = [G', G]$. We have just underlined that this theorem is a refinement of Theorem 5.1 in [MR08]. That is the reason why the proof of Theorem 4.10 follows to some extent the same canvas as the proof of Theorem 5.1 in [MR08]. Nevertheless, to refine the arguments, we use the formalism related to the ring filtration $(\Sigma_i)_{i \geq 1}$ of $k[X]$ related to the additive polynomials as introduced in Recall 2.7.4.

   Now assume that there exists a big action (C, G) with G' abelian of exponent strictly greater than p such that $\mathrm{Fratt}(G') = [G', G]$.

2. *One can first suppose that $G' \simeq \mathbb{Z}/p^2\mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})^r$, with $r \geq 1$.*
   Indeed, write $G'/(G')^{p^2} \simeq (\mathbb{Z}/p^2\mathbb{Z})^a \times (\mathbb{Z}/p\mathbb{Z})^b$. By assumption, $a \geq 1$. Using [Su82] (Chapter 2, Theorem 19), one can find an index p-subgroup H of $(G')^p$, normal in G', such that $(G')^{p^2} \subset$

$H \subsetneq (G')^p \subsetneq G' = G_2$. Then we deduce from Proposition 2.4.4 that $(C/H, G/H)$ is a big action with second ramification group $(G/H)' = (G/H)_2 = G_2/H \simeq (\mathbb{Z}/p^2\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})^{a+b-1}$. It remains to show that $\mathrm{Fratt}((G/H)') = [(G/H)', G/H]$. Since G' and $(G/H)'$ are abelian, $\mathrm{Fratt}(G') = (G')^p$ and $\mathrm{Fratt}((G/H)') = ((G/H)')^p$. Since $H \subset (G')^p$ and since H is normal in G', we gather that $\mathrm{Fratt}((G/H)') = ((G/H)')^p = (G')^p/H = \mathrm{Fratt}(G')/H$. Finally we deduce from $\mathrm{Fratt}(G') = [G', G]$ that $\mathrm{Fratt}((G/H)') = \mathrm{Fratt}(G')/H = [(G/H)', G/H]$.

3. *Parametrization of the function field.*
We denote by $L := k(C)$ the function field of C and by $k(X) := L^{G'}$ the subfield of L fixed by G'. Following Artin-Schreier-Witt theory (also see [MR08], proof of Theorem 5.1, point 2), we introduce the $W_2(\mathbb{F}_p)$-module

$$A := \frac{\wp(W_2(L)) \cap W_2(k[X])}{\wp(W_2(k[X]))},$$

where $W_2(L)$ means the ring of Witt vectors of length 2 with coordinates in L and $\wp = F - id$. Using (cf. [Bou83] Chapter IX, Exercise 19), one can prove that A is isomorphic to the dual of G' with respect to the Artin-Schreier-Witt pairing

$$\begin{cases} G' \times A \longrightarrow W_2(\mathbb{F}_p) \\ (g, \overline{\wp x}) \longrightarrow [g, \overline{\wp x}\rangle := gx - x, \end{cases}$$

where $g \in G' \subset \mathrm{Aut}_k(L)$, $x \in L$ satisfies $\wp x \in k[X]$ and $\overline{\wp x}$ denotes the class of $\wp x$ mod $\wp(k[X])$.

Moreover, as a $\mathbb{Z}$-module, A is generated by the classes modulo $\wp(W_2(k[X]))$ of $(f_0(X), g_0(X))$ and $\{(0, f_i(X))\}_{1 \leq i \leq r}$. In other words, $L = k(X, W_i, V_0)_{0 \leq i \leq r}$ is parametrized by the following system of Artin-Schreier-Witt equations

$$\wp([W_0, V_0]) = [f_0(X), g_0(X)] \in W_2(k[X])$$

and

$$\forall\, i \in \{1, \ldots, r\}, \quad \wp(W_i) = f_i(X) \in k[X].$$

An exercise left to the reader shows that one can choose $g_0(X)$ and each $f_i(X)$, for $0 \leq i \leq r$, reduced modulo $\wp(k[X])$.

4. *We prove that $f_0 \in \Sigma_2$.*
As a $\mathbb{Z}$-module, pA is generated by the class of $(0, f_0(X))$ in A. By the Artin-Schreier-Witt pairing, pA corresponds to the kernel $G'[p]$ of the map

$$\begin{cases} G' \to G' \\ g \to g^p. \end{cases}$$

Thus, $G'[p] \subsetneq G' = G_2$ is a normal subgroup of G. Then it follows from Proposition 2.4.4 that the pair $(C/G'[p], G/G'[p])$ is a big action parametrized by $W^p - W = f_0(X)$ and with second ramification group $G'/G'[p] \simeq \mathbb{Z}/p\mathbb{Z}$. By [MR08] (Proposition 2.5) $f_0(X) = X\, S(X) + c\, X \in k[X]$, where S is an additive polynomial of $k\{F\}$ with degree $s \geq 1$ in F. So $f_0 \in \Sigma_2$.

5. *The embedding problem.*
Let V be the $\mathbb{F}_p$-vector subpsace of k defined as in Proposition 2.4. For any $y \in V$, the classes modulo $\wp(W_2(k[X]))$ of $(f_0(X+y), g_0(X+y))$ and $\{(0, f_i(X+y))\}_{1 \leq i \leq r}$ induces a new generating system of A. As in [MR08] (proof of Theorem 5.1, point 3), this is expressed by the equation

$$\forall\, y \in V, \quad (f_0(X+y), g_0(X+y)) = (f_0(X), g_0(X) + \sum_{i=0}^{r} \ell_i(y)\, f_i(X)) \quad \mathrm{mod}\ \wp(W_2(k[X])) \quad (7)$$

where, for all i in $\{0, \ldots, r\}$, ℓ_i is a linear form from V to $\mathbb{F}_p$. On the second coordinate, (7) reads

$$\forall\, y \in V, \quad \Delta_y(g_0) := g_0(X+y) - g_0(X) = \sum_{i=0}^{r} \ell_i(y)\, f_i(X) + \mathrm{coc}(y, X) \qquad \mathrm{mod}\ \wp(k[X]) \quad (8)$$

where $\mathrm{coc}(y,X)$ corresponds to the cocycle due to the addition in the ring of Witt vectors. More precisely,

$$\mathrm{coc}(y,X)=\sum_{i=1}^{p-1}\frac{(-1)^i}{i}\,y^{p-i}\,X^{i+p^{s+1}}+\text{lower degree terms in X} \tag{9}$$

For more details on calculation, we refer to [MR08] (proof of Theorem 5.1, point 3 and Lemma 5.2).

6. *We prove that each f_i lies in Σ_2, for all i in $\{0,\dots,r\}$, if and only if* $\mathrm{Fratt}(G')=[G',G]$.
Put $F:=\mathrm{Fratt}(G')$. We deduce from Lemma 4.5 that $(C/F,G/F)$ is a big action whose second ramification group $(G/F)'=(G/F)_2=G_2/F$ is p-elementary abelian. The function field of the curve C/F is then parametrized by the Artin-Schreier equation

$$\forall\, i\in\{0,\dots,r\},\quad \wp(W_i)=f_i(X)\in k[X].$$

Since $F\subset[G',G]$ (cf. Lemma 4.5),

$$F=[G',G]=[G_2,G]\Leftrightarrow\{e\}=[G_2/F,G/F]=[(G/F)',G/F]\Leftrightarrow(G/F)'\subset Z(G/F).$$

By Lemma 4.4, this occurs if and only if for all i in $\{0,\dots,r\}$, $f_i(X)=X\,S_i(X)+c_i\,X\in\Sigma_2$.

7. *We prove that g_0 does not belong to Σ_p.*
We first notice that the right-hand side of (8) does not belong to Σ_{p-1}. Indeed, the monomial $X^{p-1+p^{s+1}}\in\Sigma_p-\Sigma_{p-1}$ occurs once in $\mathrm{coc}(y,X)$ but not in $\sum_{i=0}^r\ell_i(y)\,f_i(X)$ which lies in $\Sigma_2\subset\Sigma_{p-1}$ for $p\geq 3$. Now, assume that $g_0\in\Sigma_p$. Then by [Ro09] (Lemma 3.9), the left-hand side of (8), namely $\Delta_y(g_0)$, lies in Σ_{p-1}, hence a contradiction.
Therefore, one can define an integer a such that X^a is the monomial of $g_0(X)$ with highest degree among those that do not belong to Σ_p. Note that since g_0 is reduced mod $\wp(k[X])$, then $a\neq 0$ modulo p.

8. *We prove that $a-1\geq p-1+p^{s+1}$.*
We have already seen that the monomial $X^{p-1+p^{s+1}}$ occurs in the right hand side of (8). In the left-hand side of (8), $X^{p-1+p^{s+1}}$ is produced by monomials X^b of $g_0(X)$ with $b>p-1+p^{s+1}$. If $b>a$, $X^b\in\Sigma_p$, so $\Delta_y(X^b)\in\Sigma_{p-1}$, which is not the case of $X^{p-1+p^{s+1}}$. It follows that $X^{p-1+p^{s+1}}$ comes from monomials X^b with $a\geq b>p-1+p^{s+1}$. Hence the expected inequality.

9. *We prove that p divides $a-1$.*
Assume that p does not divide $a-1$. In this case, the monomial X^{a-1} is reduced modulo $\wp(k[X])$ and (8) reads as follows:

$$\forall\, y\in V,\quad c_a(g_0)\,a\,y\,X^{a-1}+S_{p-1}(X)+R_{a-2}(X)=\mathrm{coc}(y,X)+\sum_{i=0}^r\ell_i(y)\,f_i(X)\mod\wp\,(k[X]),$$

where $c_a(g_0)\neq 0$ denotes the coefficient of X^a in g_0, $S_{p-1}(X)$ is the polynomial in Σ_{p-1} produced by the monomials X^b of g_0 with $b>a$ and $R_{a-2}(X)$ is the polynomial of $k[X]$ with degree lower than $a-2$ produced by the monomials X^b of g_0 with $b\leq a$.
We first notice that X^{a-1} does not occur in $S_{p-1}(X)$. Otherwise, $X^{a-1}\in\Sigma_{p-1}$ and $X^a=X^{a-1}\,X\in\Sigma_p$, hence a contradiction. Likewise, X^{a-1} does not occur in $\sum_{i=0}^r\ell_i(y)\,f_i(X)\in\Sigma_2$. Otherwise, $X^a=X^{a-1}\,X\in\Sigma_3\subset\Sigma_p$ since $p\geq 3$. It follows that X^{a-1} occurs in $\mathrm{coc}(y,X)$, which requires $a-1\leq\deg\mathrm{coc}(y,X)=p-1+p^{s+1}$. Then the previous point implies $a-1=p-1+p^{s+1}$, which contradicts $a\neq 0$ modulo p.
Thus, p divides $a-1$. So we can write $a=1+\lambda\,p^t$, with $t>0$, λ prime to p and $\lambda\geq 2$ because of the definition of a. We also define $j_0:=a-p^t=1+(\lambda-1)\,p^t$.

10. *We search for the coefficient of the monomial X^{j_0} in the left-hand side of* (8).
Since p does not divide j_0, the monomial X^{j_0} is reduced modulo $\wp(k[X])$. In the left-hand side of (8), namely $\Delta_y(g_0)$ modulo $\wp(k[X])$, the monomial X^{j_0} comes from the monomials X^b

of $g_0(X)$ with $b \geq j_0+1$. However, as seen above, the monomials X^b with $b > a$ produce in $\Delta_y(g_0)$ elements that belong to Σ_{p-1}, whereas $X^{j_0} \notin \Sigma_{p-1}$. Otherwise, $X^a = X^{j_0}\, X^{p^t} \in \Sigma_p$, which contradicts the definition of a. So we only have to consider the monomials X^b of $g_0(X)$ with $b \in \{j_0+1, \dots, a\}$. Then using the same arguments as those used in [MR08] (proof of Theorem 5.1, point 6), we conclude that the coefficient of X^{j_0} in the left-hand side of (8) is $T(y)$ where $T(Y)$ denotes a polynomial of $k[X]$ with degree p^t.

11. *We identify with the coefficient of X^{j_0} in the right-hand side of* (8) *and gather a contradiction.* As mentionned above, the monomial X^{j_0} does not occur in $\sum_{i=0}^{r} \ell_i(y)\, f_i(X) \in \Sigma_2 \subset \Sigma_{p-1}$, for $p \geq 3$. Assume that the monomial X^{j_0} appears in $\mathrm{coc}(y, X)$. This implies that $j_0 \leq p-1+p^{s+1}$. Using the same arguments as in [MR08] (proof of Theorem 5.1, point 7), we gather that $j_0 = 1 + (\lambda - 1)\, p^t = 1 + p^{s+1}$. Then X^{j_0} lies in Σ_2, which leads to the same contradiction as above. Therefore, the monomial X^{j_0} does not occur in the right-hand side of (8). Then $T(y) = 0$ for all y in V, which means that $|V| \leq p^t$. Call C_0 the curve whose function field is parametrized by $\wp([W_0, V_0]) = [f_0(X), g_0(X)]$. The same calculation as in [MR08] (proof of Theorem 5.1, point 7) shows that $g_{C_0} \geq p^{t+1}\,(p-1)$. Moreover, $g \geq p^r\, g_{C_0}$ (see e.g. [LM05] Proposition 8.5, formula (8)). Since $|G| = |G_2||V| \leq p^{2+r+t}$, it follows that $\frac{|G|}{g} = \frac{p}{p-1} < \frac{2\,p}{p-1}$, hence a contradiction. □

5 Classification of big actions such that $\frac{|G|}{g^2} \geq \frac{4}{(p^2-1)^2}$.

We now pursue the classification of big actions initiated by Lehr and Matignon. In [LM05], they characterize the big actions (C, G) satisfying $\frac{|G|}{g^2} \geq \frac{4}{(p-1)^2}$. In this section, we exhibit a parametrization for big actions (C, G) satisfying condition $(*)$, namely:

$$\frac{|G|}{g^2} \geq \frac{4}{(p^2-1)^2}.$$

As proved in [MR08] (Proposition 4.1 and Proposition 4.2), this condition requires G' (i.e. G_2) to be an elementary abelian p-group with order dividing p^3. Since G_2 cannot be trivial (cf. Proposition 2.4), this leaves three possibilities: $G_2 \simeq \mathbb{Z}/p\mathbb{Z}$ (Section 5.1), $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^2$ (Section 5.2), $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^3$ (Section 5.3). In each case, we shall distinguish the cases $[G', G] = \mathrm{Fratt}(G')(= \{e\})$ and $[G', G] \supsetneq \mathrm{Fratt}(G')(= \{e\})$.

Notation. Throughout this last part, the automorphim group $A := \mathrm{Aut}_k(C)$ and its wild inertia subgroups $A_{P,1}$ are defined as in Section 3. For any group G, $Z(G)$ means the center of G, G' its derived subgroup and $\mathrm{Fratt}(G)$ its Frattini subgroup.

Let (C, G) be a big action with $G' \simeq (\mathbb{Z}/p\mathbb{Z})^n$, $n \geq 1$. As seen in Recall 2.7.4, one can choose an adapted basis $\{f_1, \dots, f_n\}$ to parametrize the system of equations $W_i^p - W_i = f_i(X)$ of the Artin-Schreier extension $k(C)/k(C)^{G'}$ such that each f_i lies in Σ_{i+1}, i.e. is a linear combination over k of products of at most $i+1$ additive polynomials of $k[X]$. Let V be the $\mathbb{F}_p$-vector space defined in Proposition 2.4.3. For all $y \in V$, the matrix $L(y)$ and the maps $\ell_{i,j} : V \to \mathbb{F}_p$ are defined as in Recall 2.7.3. For all map ℓ, we write $\ell = 0$ if ℓ is identically zero and $\ell \neq 0$ otherwise.

When (C, G) is a big action with $G' \simeq (\mathbb{Z}/p\mathbb{Z})^n$ and $[G', G] = \{e\}$, we use the notations introduced in Lemma 4.4. In particular, each function f_i as mentionned above reads $f_i(X) = X\, S_i(X) + c_i\, X \in k[X]$, where S_i is an additive polynomial of $k[X]$ with degree $s_i \geq 1$ in F and $s_1 \leq s_2 \ldots \leq s_n$. Then we define the palindromic polynomial Ad_{f_i} as in [Ro09] (Proposition 2.16). Its set of zeroes denoted by $Z(\mathrm{Ad}_{f_i})$ is an $\mathbb{F}_p$-vector subspace of k.

5.1 Big actions satisfying condition $(*)$ with $G' \simeq \mathbb{Z}/p\mathbb{Z}$.

The following description derives from Remark 3.1, Proposition 3.2 and [LM05] (Theorem 1.1 I and 3.1).

Proposition 5.1. *We keep the notations defined above.*

1. *The pair (C, G) is a big action with $G' \simeq \mathbb{Z}/p\mathbb{Z}$ if and only if C is birational to a curve C_f parametrized by $W^p - W = f(X) = X\, S(X) \in k[X]$, where S is a (monic) additive polynomial with degree $s \geq 1$ in F.*

2. *In what follows, we assume that C is birational to a curve C_f as described in the first point.*

   (a) *If $s \geq 2$, $A_{\infty,1}$ is the unique p-Sylow subgroup of A.*

   (b) *If $s = 1$, there exists $r := p^3 + 1$ points of C: $P_0 := \infty, P_1, \ldots, P_r$ such that $(A_{P_i,1})_{0 \leq i \leq r}$ are the p-Sylow subgroups of A. In this case, for all i in $\{1, \ldots, r\}$, there exists $\sigma_i \in A$ such that $\sigma_i(P_i) = \infty$.*

   In both cases, $A_{\infty,1}$ is an extraspecial group (see [Su86] Definition 4.14) with exponent p (resp. p^2) if $p > 2$ (resp. $p = 2$) and order p^{2s+1}. Thus,

$$0 \longrightarrow Z(A_{\infty,1}) = (A_{\infty,1})' \simeq \mathbb{Z}/p\mathbb{Z} \longrightarrow A_{\infty,1} \xrightarrow{\pi} Z(\mathrm{Ad}_f) \simeq (\mathbb{Z}/p\mathbb{Z})^{2s} \longrightarrow 0.$$

   Furthermore, $(C, A_{\infty,1})$, and so each $(C, A_{P_i,1})$, with $1 \leq i \leq r$, are big actions satisfying condition $(*)$

3. *Let V be a vector subspace of $Z(\mathrm{Ad}_f)$ with dimension v over $\mathbb{F}_p$. Then $(C, \pi^{-1}(V))$ is also a big action satisfying condition $(*)$ if and only if*

$$\begin{array}{ll} \text{if } \ p \neq 2, & 2s \geq v \geq \max\{s+1, 2s-3\} \\ \text{if } \ p = 2, & 2s \geq v \geq \max\{s+1, 2s-4\} \end{array}$$

   This gives the different possibilities collected in the table below:

case	v	s	V	G
1	$2s$	$s \geq 1^\dagger$	$Z(\mathrm{Ad}_f)^\dagger$	$A_{\infty,1}^\dagger$
2	$2s-1$	$s \geq 2$	index p subgroup of $Z(\mathrm{Ad}_f)$	index p subgroup of $A_{\infty,1}$
3	$2s-2$	$s \geq 3$	index p^2 subgroup of $Z(\mathrm{Ad}_f)$	index p^2 subgroup of $A_{\infty,1}$
4	$2s-3$	$s \geq 4$	index p^3 subgroup of $Z(\mathrm{Ad}_f)$	index p^3 subgroup of $A_{\infty,1}$
5 (p=2)	$2s-4$	$s \geq 5$	index p^4 subgroup of $Z(\mathrm{Ad}_f)$	index p^4 subgroup of $A_{\infty,1}$

case	$\lvert G\rvert/g$	$\lvert G\rvert/g^2$
1	$\frac{2\,p}{p-1}\,p^s$	$\frac{4}{(p^2-1)^2}\,(p+1)^2\,p$
2	$\frac{2\,p}{p-1}\,p^{s-1}$	$\frac{4}{(p^2-1)^2}\,(p+1)^2$
3	$\frac{2\,p}{p-1}\,p^{s-2}$	$\frac{4}{(p^2-1)^2}\,\frac{(p+1)^2}{p}$
4	$\frac{2\,p}{p-1}\,p^{s-3}$	$\frac{4}{(p^2-1)^2}\,\frac{(p+1)^2}{p^2}$
5 (p=2)	$\frac{2\,p}{p-1}\,p^{s-4}$	$\frac{4}{(p^2-1)^2}\,\frac{(p+1)^2}{p^3}$

† *Note:* In the case $s = 1$, this result is true up to conjugation by σ_i as defined in Proposition 5.1.

Remark 5.2. *Note that, for $p > 2$, the solutions can be parametrized by s algebraically independent variables over $\mathbb{F}_p$, namely the s coefficients of S assumed monic after an homothety on the variable X. Note that $s \sim \log g$.*

5.2 Big actions satisfying condition $(*)$ with $G' \simeq (\mathbb{Z}/p\mathbb{Z})^2$.

5.2.1 Case $[G', G] = \mathrm{Fratt}(G') = \{e\}$.

We deduce from [Ro09] (Proposition 4.3.3), Remark 3.1 and Proposition 3.2, the following description. For a complete proof and details of calculation, see [Ro08].

Proposition 5.3. *Let (C, G) be a big action satisfying condition $(*)$ with $G' \simeq (\mathbb{Z}/p\mathbb{Z})^2$. Assume that $[G', G] = \{e\}$ and keep the notations defined at the beginning of Section 5.*

1. *The pair $(C, A_{\infty,1})$ is a big action satisfying condition $(*)$. Moreover, $A_{\infty,1}$ is a special group (see [Su86] Definition 4.14) with exponent p (resp. p^2) for $p > 2$ (resp. $p = 2$) and order $p^{2+2\,s_1}$. Thus,*

$$0 \longrightarrow Z(A_{\infty,1}) = (A_{\infty,1})' \simeq (\mathbb{Z}/p\mathbb{Z})^2 \longrightarrow A_{\infty,1} \xrightarrow{\pi} Z(\mathrm{Ad}_{f_1}) \simeq (\mathbb{Z}/p\mathbb{Z})^{2s_1} \longrightarrow 0.$$

2. *Moreover, $s_2 = s_1$ or $s_2 = s_1 + 1$.*

(a) *If* $s_2 = s_1$, $G = \pi^{-1}(V)$, *where* V *is a vector subspace of* $Z(\mathrm{Ad}_{f_1})$ *with dimension* v *over* $\mathbb{F}_p$ *such that* $2\,s_1 - 2 \leq v \leq 2\,s_1$. *Then* $A_{\infty,1}$ *is a* p*-Sylow subgroup of* A. *It is normal except if* C *is birational to the Hermitian curve* $W^q - W = X^{1+q}$ *with* $q = p^2$.

(b) *If* $s_2 = s_1 + 1$, $V = Z(\mathrm{Ad}_{f_1})$ *and* $G = A_{\infty,1}$ *is the unique* p*-Sylow subgroup of* A.

The different possibilities are listed in the table below:

s_1	s_2	v	V	G
$s \geq 2$	s	$2s$	$Z(\mathrm{Ad}_{f_1}) = Z(\mathrm{Ad}_{f_2})$	$A_{\infty,1}$
$s \geq 2$	s	$2s-1$	index p subgroup of $Z(\mathrm{Ad}_{f_1})$	index p subgroup of $A_{\infty,1}$
$s \geq 3$	s	$2s-2$	index p^2 subgroup of $Z(\mathrm{Ad}_{f_1})$	index p^2 subgroup of $A_{\infty,1}$
$s \geq 3$	$s+1$	$2s$	$Z(\mathrm{Ad}_{f_1})$	$A_{\infty,1}$

$\lvert G\rvert/g$	$\lvert G\rvert/g^2$
$\frac{2\,p}{p-1}\,\frac{p^{1+s}}{1+p}$	$\frac{4}{(p^2-1)^2}\,p^2$
$\frac{2\,p}{p-1}\,\frac{p^{s}}{1+p}$	$\frac{4}{(p^2-1)^2}\,p$
$\frac{2\,p}{p-1}\,\frac{p^{s-1}}{1+p}$	$\frac{4}{(p^2-1)^2}$
$\frac{2\,p}{p-1}\,\frac{p^{1+s}}{1+p}$	$\frac{4}{(p^2-1)^2}\,\frac{p^2(p+1)^2}{(1+p^2)^2}$

Since $V \subset \cap_{1\leq i\leq 2}\, Z(\mathrm{Ad}_{f_i})$ (see Lemma 4.4), the difficulty in going further in the parametrization of the functions f_i's lies in finding the GCD for the family of palindromic polynomials Ad_{f_i}. The simplest case is the one where all these palindromic polynomials are equal with $V = Z(\mathrm{Ad}_{f_i})$. This case has been fully described in [Ro09] (Section 4). In the other cases, one has to work in the Ore ring of Laurent polynomials $k\{F, F^{-1}\}$ (see [El99] Section 3 or [Go96], 1.6) and introduce two additive polynomials:

$$R := \prod_{y\in V} (X - y) \quad \text{and} \quad T := \gcd\{\mathrm{Ad}_{f_1}, \mathrm{Ad}_{f_2}\}.$$

Then, knowing that R divides T which divides each Ad_{f_i} and that we have information on their respective degree, one can discuss relationships between these polynomials (see [Ro08] for a few examples).

5.2.2 Case $[G', G] \supsetneq \mathrm{Fratt}(G') = \{e\}$.

In this section, we first give the conditions satisfied by a big action (C, G) satisfying condition $(*)$ with $G' \simeq (\mathbb{Z}/p\mathbb{Z})^2$ and $[G', G] \neq \{e\}$ (Proposition 5.4). Then we study the converse, namely we indicate that these conditions induce such a big action (Remark 5.5). We finally discuss when two such G-actions are equivalent in the way defined in the introduction (Remark 5.6). Note that such a detailed study could be carried out in all cases mentionned in the next sections but would be too technical to display.

Proposition 5.4. *Let* (C, G) *be a big action satisfying condition* $(*)$ *with* $G' \simeq (\mathbb{Z}/p\mathbb{Z})^2$ *and* $[G', G] \neq \{e\}$. *We keep the notations defined at the beginning of Section 5.*

1. (a) *Then* $G = A_{\infty,1}$ *is the unique* p*-Sylow subgroup of* A.
   (b) *For all* i *in* $\{1, 2\}$, $f_i \in \Sigma_{i+1} - \Sigma_i$ *and* $\deg(f_i) = 1 + i\,p^s$, *with* $p \geq 3$ *and* $s \in \{1, 2\}$.
   (c) *Moreover,* $\dim_{\mathbb{F}_p}(V) = s + 1$. *More precisely,* $y \in V$ *if and only if* $\ell_{1,2}(y)^p - \ell_{1,2}(y) = 0$.
2. *There exists a coordinate* X *for the projective line* C/G_2 *such that the functions* f_i*'s are parametrized as follows:*

(a) If $s = 1$,

	$p>3$	$p=3$
f_1	$f_1(X)=X^{1+p}+a_2\,X^2$	$f_1(X)=X^4+a_2\,X^2$
V	$V=Z(\mathrm{Ad}_{f_1})=Z(X^{p^2}+2\,a_2^p\,X^p+X)$	$V=Z(\mathrm{Ad}_{f_1})=Z(X^9+2\,a_2^3\,X^3+X)$
f_2	$f_2(X)=b_{1+2\,p}\,X^{1+2\,p}+b_{2+p}\,X^{2+p}+b_3\,X^3+b_1\,X$	$f_2(X)=b_7\,X^7+b_5\,X^5+b_1\,X$
$b_{1+2\,p}$	$b_{1+2\,p}\in k^{\times}$	$b_7^{16}=1$
a_2	$2\,a_2^p=-b_{1+2\,p}^{-p}(b_{1+2\,p}^{p^2}+b_{1+2\,p})\Leftrightarrow b_{1+2\,p}\in V-\{0\}$	$2\,a_2^3=-b_7^6-b_7^{-2}\Leftrightarrow b_7\in V-\{0\}$
b_{2+p}	$b_{2+p}=-b_{1+2\,p}^p$	$b_5=-b_7^3$
b_3	$3\,b_3^p=b_{1+2\,p}^{-p}\,(b_{1+2\,p}^{2\,p^2}-b_{1+2\,p}^2)$	
b_1	$b_1\in k$	$b_1\in k$
$\ell_{1,2}$	$\ell_{1,2}(y)=2\,(b_{1+2\,p}\,y^p-b_{1+2\,p}^p\,y)$	$\ell_{1,2}(y)=2\,(b_7\,y^3-b_7^3\,y)$

Therefore, for $p>3$, the solutions are parametrized by 2 algebraically independent variables over $\mathbb{F}_p$, namely $b_{1+2\,p}\in k^{\times}$ and $b_1\in k$. For $p=3$, as the monomial X^3 can be reduced mod $\wp(k[X])$, the parameter $b_{1+2\,p}$ satisfies an additional algebraic relation: $b_7^{16}=1$. Then b_7 takes a finite number of values.

In both cases ($p=3$ or $p>3$),

$$\frac{|G|}{g}=\frac{2\,p}{p-1}\,\frac{p^2}{1+2\,p}\qquad \textit{and}\qquad \frac{|G|}{g^2}=\frac{4}{(p^2-1)^2}\,\frac{p^2\,(p+1)^2}{(1+2\,p)^2}.$$

(b) If $s=2$ and $p>3$,

f_1	$f_1(X)=X^{1+p^2}+a_{1+p}\,X^{1+p}+a_2\,X^2$
Ad_{f_1}	$X^{p^4}+a_{1+p}^{p^2}\,X^{p^3}+2\,a_2^{p^2}\,X^{p^2}+a_{1+p}^p\,X^p+X$
f_2	$f_2(X)=b_{1+2\,p^2}\,X^{1+2\,p^2}+b_{1+p+p^2}\,X^{1+p+p^2}+b_{2+p^2}\,X^{2+p^2}+b_{1+p^2}\,X^{1+p^2}+b_{1+2\,p}\,X^{1+2\,p}$ $+b_{2+p}X^{2+p}+b_{1+p}X^{1+p}+b_3\,X^3+b_2\,X^2+b_1\,X$
$b_{1+2\,p}$	$b_{1+2\,p}\in k^{\times}$
b_{2+p^2}	$b_{2+p^2}\in k^{\times}$
b_{1+p+p^2}	$b_{1+p+p^2}^p=-2\,b_{1+2\,p}^p\,(b_{2+p^2}^p\,b_{1+2\,p}^{-p^2}+b_{2+p^2}^{p-1})$
$\ell_{1,2}$	$\forall\,y\in V,\quad \ell_{1,2}(y)=2\,b_{1+2\,p}\,y^{p^2}+b_{1+p+p^2}\,y^p+2\,b_{2+p^2}\,y$
V	*V is an index p-subgroup of $Z(\mathrm{Ad}_{f_1})$* $V=Z(2\,b_{1+2\,p}^p\,X^{p^3}+(b_{1+p+p^2}^p-2\,b_{1+2\,p})\,X^{p^2}+(2b_{2+p^2}^p-b_{1+p+p^2})X^p-2b_{2+p^2}X)$
a_{1+p}	$a_{1+p}^{p^2}=-b_{1+2\,p}^{p-p^2}-b_{1+2\,p}^p\,b_{2+p^2}^{-1}-b_{2+p^2}^{p^2}\,b_{1+2\,p}^{-p^3}-b_{2+p^2}^{p^2-p}$
a_2	$2\,a_2^{p^2}=b_{2+p^2}^{p^2}\,b_{1+2\,p}^{-p^2}+b_{1+2\,p}\,b_{2+p^2}^{-1}+b_{2+p^2}^p\,b_{1+2\,p}^{p-2\,p^2}+2\,b_{2+p^2}^{p-1}\,b_{1+2\,p}^{p-p^2}+b_{1+2\,p}^p b_{2+p^2}^{p-2}$
$b_{1+2\,p}$	$b_{1+2\,p}^{p^2}=-b_{1+2\,p}^{2\,p-p^2}-b_{1+2\,p}^{2\,p}\,b_{2+p^2}^{-1}+b_{2+p^2}^{2\,p^2}\,b_{1+2\,p}^{p^2-2\,p^3}+2\,b_{2+p^2}^{2\,p^2-p}\,b_{1+2\,p}^{p^2-p^3}+b_{1+2\,p}^{p^2}\,b_{2+p^2}^{2\,p^2-2\,p}$
b_{2+p}	$b_{2+p}^{p^2}=b_{2+p^2}^p\,b_{1+2\,p}^{2\,p-2\,p^2}+2\,b_{2+p^2}^{p-1}\,b_{1+2\,p}^{2\,p-p^2}+b_{1+2\,p}^{2\,p}\,b_{2+p^2}^{p-2}-b_{2+p^2}^{2\,p^2}\,b_{1+2\,p}^{-p^3}-b_{2+p^2}^{2\,p^2-p}$
b_3	$3\,b_3^{p^2}=b_{2+p^2}^{2\,p^2}\,b_{1+2\,p}^{-p^2}-b_{2+p^2}^{2\,p}\,b_{1+2\,p}^{2\,p-3p^2}-3\,b_{2+p^2}^{2\,p-2}\,b_{1+2\,p}^{2\,p-p^2}-3\,b_{2+p^2}^{2\,p-1}\,b_{1+2\,p}^{2\,p-2\,p^2}-b_{1+2\,p}^{2\,p}b_{2+p^2}^{2\,p-3}+b_{1+2\,p}^2\,b_{2+p^2}^{-1}$
b_{1+p^2}	$b_{1+p^2}\in Z(b_{2+p^2}^{p^2}\,b_{1+2\,p}^{-p^3}\,X^{p^3}-(b_{2+p^2}^{p^2}\,b_{1+2\,p}^{-p^3}+b_{1+2\,p}^{p-p^2}+b_{2+p^2}^{p^2-p})\,X^{p^2}+$ $(b_{1+2\,p}^{p-p^2}+b_{1+2\,p}^p\,b_{2+p^2}^{-1}+b_{2+p^2}^{p^2-p})\,X^p-b_{1+2\,p}^p\,b_{2+p^2}^{-1}\,X)$
b_{1+p}	$b_{1+p}^{p^2}=-(b_{1+2\,p}^{p-p^2}+b_{2+p^2}^{p^2}\,b_{1+2\,p}^{-p^3}+b_{2+p^2}^{p^2-p})\,b_{1+p^2}^{p^2}-b_{1+2\,p}^p\,b_{2+p^2}^{-1}\,b_{1+p^2}$
b_2	$2\,b_2^{p^2}=(b_{2+p^2}^p\,b_{1+2\,p}^{p-2\,p^2}+b_{2+p^2}^{p-1}\,b_{1+2\,p}^{p-p^2-p}+b_{2+p^2}^{p^2}\,b_{1+2\,p}^{-p^2})b_{1+p^2}^{p^2}+$ $(b_{1+2\,p}^p b_{2+p^2}^{p-2}+b_{2+p^2}^{p-1}\,b_{1+2\,p}^{p-p^2}+b_{1+2\,p}\,b_{2+p^2}^{-1})\,b_{1+p^2}$
b_1	$b_1\in k$

Therefore, for $p>3$, the solutions can be parametrized by 3 algebraically independent variables over $\mathbb{F}_p$, namely $b_{1+2\,p^2}\in k^{\times}$, $b_{2+p^2}\in k^{\times}$ and $b_1\in k$. One also finds a fourth parameter b_{1+p^2} which runs over an $\mathbb{F}_p$-vector subspace of k, namely the set of zeroes of an additive separable polynomial whose coefficients are rational functions in $b_{1+2\,p^2}$ and b_{2+p^2}. So, for given $b_{1+2\,p}$ and b_{2+p^2}, the parameter b_{1+p^2} takes a finite number of values.

For $p=3$,

$f_1(X) =$	$X^{10} + a_4\, X^4 + a_2\, X^2$
$f_2(X) =$	$b_{19}\, X^{19} + b_{13}\, X^{13} + b_{11}\, X^{11} + b_{10}\, X^{10} + b_7\, X^7 + b_5\, X^5 +$ $b_4\, X^4 + b_2\, X^2 + b_1\, X$

with a_4, a_2, b_{13}, b_7, b_5, b_3 and b_2 satisfying the same relations as above. But, this time, the parameters b_{19} and b_{11} are linked through an algebraic relation, namely:

$$b_{11}^{18}\, b_{19}^{-9} - b_{11}^{6}\, b_{19}^{-21} - b_{19}^{6}\, b_{11}^{3} + b_{19}^{2}\, b_{11}^{-1} = 0.$$

In both cases ($p = 3$ or $p > 3$),

$$\frac{|G|}{g} = \frac{2\,p}{p-1}\,\frac{p^2}{1+2\,p} \qquad \textit{and} \qquad \frac{|G|}{g^2} = \frac{4}{(p^2-1)^2}\,\frac{p\,(p+1)^2}{(1+2\,p)^2}.$$

Remark 5.5. *One can show that the conditions obtained in Proposition 5.4.2 conversely give a big action (C, G) satisfying condition $(*)$ with $G' \simeq (\mathbb{Z}/p\mathbb{Z})^2$ and $[G', G] \neq \{e\}$.*

Indeed, take $(b_{1+2p}, b_1) \in (k^\times, k)$. Define $f_1(X)$, V, $f_2(X)$ and $\ell_{1,2}$ as in Proposition 5.4.2 in the case $p > 3$ and $s = 1$. Consider the curve C parametrized by $W_i^p - W_i = f_i(X)$, $1 \leq i \leq 2$. Build $\sigma := \sigma_{y,z_1,z_2} \in \mathrm{Aut}_k(C)$ such that

$$\begin{aligned} \sigma(X) &= X + y \\ \sigma(W_1) &= W_1 + P_y(X) + z_1 \\ \sigma(W_2) &= W_2 + \ell_{1,2}(y)\, W_1 + Q_y(X) + z_2, \end{aligned}$$

with

$$\begin{array}{lll} y \in V & & \\ z_1 \in k & \textit{such that} & f_1(y) = \wp(z_1) \\ z_2 \in k & \textit{such that} & f_2(y) = \wp(z_2) \\ P_y(X) \in k[X] & \textit{such that} & f_1(X+y) - f_1(X) - f_1(y) = \wp(P_y(X)), \quad P_y(0) = 0 \\ Q_y(X) \in k[X] & \textit{such that} & f_2(X+y) - f_2(X) - f_2(y) - \ell_{1,2}(y)\, f_1(X) = \wp(Q_y(X)), \quad Q_y(0) = 0. \end{array}$$

Let G be the group generated by the automorphisms σ_{y,z_1,z_2}. Since z_1 and z_2 are defined up to translation by $\mathbb{F}_p$, the order of G is at least $p^2\,|V|$. It follows that (C, G) is a big action satisfying condition $(*)$ with $G' \simeq (\mathbb{Z}/p\mathbb{Z})^2$ and $[G', G] \neq \{e\}$ as in Proposition 5.4. In particular, $G = A_{\infty,1}$.

Proof of Proposition 5.4:

1. Since $\ell_{1,2} \neq 0$, the group G satisfies the third condition of [Ro09] (Proposition 5.2). Then, the equality $G = A_{\infty,1}$ derives from [Ro09] (Corollary 5.7). The unicity of the p-Sylow subgroup is explained in Remark 3.1. The second and third assertions come from [Ro09] (Theorem 5.6). Moreover, the description of V displayed in (c) is due to [Ro09] (Proposition 2.12.2). It remains to show that $s = 1$ or $s = 2$. Using the formula given in Recall 2.7.1, we compute $g = \frac{(p-1)}{2}\,(p^s + p\,(m_2 - 1)) = \frac{(p-1)}{2}\,p^s(1 + 2\,p)$. Since $|G| = p^{3+s}$, condition $(*)$ requires: $\frac{4}{(p^2-1)^2} \leq \frac{|G|}{g^2} = \frac{4}{(p^2-1)^2}\,\frac{(p+1)^2}{p^{s-3}\,(1+2\,p)^2}$. It follows that $3 - s > 0$, i.e. $1 \leq s \leq 2$.

2. We merely explain the case $s = 1$. One can find a coordinate X for the projective line C/G_2 such that $f_1(X) = X\,S_1(X) = X\,(X^p + a_2\,X)$ (cf. [Ro09] Corollary 2.15). Then, $\mathrm{Ad}_{f_1} = F^2 + 2\,a_2^p\,F + I$. From $V \subset Z(\mathrm{Ad}_{f_1})$ and $\dim_{\mathbb{F}_p} Z(\mathrm{Ad}_{f_1}) = 2 = s + 1 = \dim_{\mathbb{F}_p} V$, we deduce that $V = Z(\mathrm{Ad}_{f_1})$. Since $f_2 \in \Sigma_3 - \Sigma_2$ with $\deg f_2 = 1 + 2\,p^s$ and since the functions f_i's are supposed to be reduced mod $\wp(k[X])$, equation (3) reads:

   $$\forall\, y \in V, \qquad f_2(X+y) - f_2(X) = \ell_{1,2}(y)\, f_1(X) \qquad \mathrm{mod}\ \ \wp(k[X])$$

   $$\begin{array}{lll} \text{with} & f_1(X) = X^{1+p} + a_2\, X^2 & \\ \text{and} & f_2(X) = b_{1+2\,p}\, X^{1+2\,p} + b_{2+p}\, X^{2+p} + b_{1+p}\, X^{1+p} + b_3\, X^3 + b_2\, X^2 + b_1\, X & \text{for } p > 3 \\ \text{(resp.} & f_2(X) = b_{1+2\,p}\, X^{1+2\,p} + b_{2+p}\, X^{2+p} + b_{1+p}\, X^{1+p} + b_2\, X^2 + b_1\, X & \text{for } p = 3) \end{array}$$

   Then, calculation gives the relations gathered in the table. Note that we find $b_{1+p} \in \mathbb{F}_p$. Since we are working in the $\mathbb{F}_p$-space generated by $f_1(X)$ and $f_2(X)$, we can replace $f_2(X)$ with $f_2(X) - b_{1+p}\, f_1(X)$, hence the expected formula. We solve the case $s = 2$ in the same way. □

Remark 5.6. *Consider the big actions described in Proposition 5.4. To simplify the problem, we restrict to the case $p > 3$ and $s = 1$. A natural question is to wonder when two pairs of parameters (b_{1+2p}, b_1) and (b'_{1+2p}, b'_1) in $(k^\times, k)$ give two equivalent G-curves C and C' in the way defined in the introduction.*

Let $\{g_1(X), g_2(X)\}$ (resp. $\{h_1(X), h_2(X)\}$) be the specialization of the functions $\{f_1(X), f_2(X)\}$ corresponding to the parameters (b_{1+2p}, b_1) (resp. (b'_{1+2p}, b'_1)). Call f a G-isomorphism: $C \to C'$. As the ramification filtration is preserved by f, it induces a k-isomorphism $\overline{f}$: $C/G_i \to C'/G_i$, where the G_i's are the ramification groups of G at $\infty \in C$ (resp. at $f(\infty) \in C'$). Using [Ro09] (Theorem 5.8), this gives a k-isomorphism: $C_{g_1} \to C_{h_1}$ where $C_{g_1} : W^p - W = g_1(X)$ and C_{h_1} : $V^p - V = h_1(T)$. It follows from [LM05] (Proposition 3.3) that there exists $(a, b, c) \in (k^\times, k, \mathbb{F}_p^\times)$ such that $T = aX + b$ and $h_1(aX + b) = c\, g_1(X)$ modulo $\wp(k[X])$. Calculations show that $a \in \mathbb{F}_{p^2}^\times$ and $b \in Z(\mathrm{Ad}_{h_1})$ which coincides with V since $s = 1$.

Likewise, there exists $(\lambda, \mu) \in (\mathbb{F}_p^\times, \mathbb{F}_p)$ such that $h_2(aX + b) = h_2(aX) = \lambda\, g_2(X) + \mu\, g_1(X)$ modulo $\wp(k[X])$. Calculations show that this is equivalent to:

$$\frac{b'_{1+2p}}{b_{1+2p}} \in \mathbb{F}_{p^2}^\times \qquad \text{and} \qquad b_1'^p = c\, b_1^p\, \Big(\frac{b'_{1+2p}}{b_{1+2p}}\Big) \qquad \text{with} \qquad c \in \mathbb{F}_p^\times. \tag{10}$$

Conversely, two pairs of parameters (b_{1+2p}, b_1) and (b'_{1+2p}, b'_1) in $(k^\times, k)$ satisfying conditions (10) give two equivalent G-curves.

Remark 5.7. *One can now answer the second problem raised at the end of [Ro09] (Section 6). Indeed, one notices that the family obtained for $s = 2$ is larger than the one obtained after the additive base change: $X = Z^p + c\,Z$, $c \in k - \{0\}$ (see [MR08] Proposition 3.1) applied to the case $s = 1$. Indeed, such a base change does not produce any monomial Z^{1+p^2} in $f_2(Z)$.*

5.3 Big actions satisfying condition $(*)$ with $G' \simeq (\mathbb{Z}/p\mathbb{Z})^3$.

We begin with a preliminary remark which allows to use the results obtained in the preceding section.

Remark 5.8. *Let (C, G) be a big action satisfying condition $(*)$ with $G'(= G_2) \simeq (\mathbb{Z}/p\mathbb{Z})^3$. We keep the notations introduced at the beginning of Section 5.*

1. *Let $C_{1,2}$ be the curve parametrized by the two equations: $W_i^p - W_i = f_i(X)$, with $i \in \{1, 2\}$. Let $A_{1,2}$ be the $\mathbb{F}_p$-vector subspace of A generated by the classes of $f_1(X)$ and $f_2(X)$. Let $H_{1,2} \subsetneq G_2$ be the orthogonal of $A_{1,2}$ with respect to the Artin-Schreier pairing (see Recall 2.7.2). The pair $C_{1,2} = C/H_{1,2}$ and $G_{1,2} = G/H_{1,2}$. Then it follows from Proposition 2.4.4 and [LM05] (Proposition 8.5 (ii)) that the pair $(C_{1,2}, G_{1,2})$ is a big action sastisfying condition $(*)$ with second ramification group isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$.*

2. *Likewise, if $\ell_{2,3} = 0$, the two equations: $W_i^p - W_i = f_i(X)$, with $i \in \{1, 3\}$, also parametrize a big action satisfying condition $(*)$ with second ramification group isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$.*

3. *Similarly, if $\ell_{1,2} = \ell_{1,3} = 0$, the two equations: $W_i^p - W_i = f_i(X)$, with $i \in \{2, 3\}$, also parametrize a big action satisfying condition $(*)$ with second ramification group isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$.*

5.3.1 Case $[G', G] = \mathrm{Fratt}(G') = \{e\}$.

Proposition 5.9. *Let (C, G) be a big action satisfying condition $(*)$ with $G' \simeq (\mathbb{Z}/p\mathbb{Z})^3$ and $[G', G] = \{e\}$. We keep the notations defined at the beginning of Section 5.*

1. *Then $G = A_{\infty,1}$ is a special group of exponent p (resp. p^2) for $p > 2$ (resp. $p = 2$) and order $p^{3+2\,s_1}$. Thus,*

$$0 \longrightarrow Z(G) = G' \simeq (\mathbb{Z}/p\mathbb{Z})^3 \longrightarrow G \xrightarrow{\ \pi\ } Z(\mathrm{Ad}_{f_1}) = Z(\mathrm{Ad}_{f_2}) = Z(\mathrm{Ad}_{f_3}) \simeq (\mathbb{Z}/p\mathbb{Z})^{2s_1} \longrightarrow 0.$$

   Moreover, G is a p-Sylow subgroup of A. It is normal except when C is birational to the Hermitian curve: $W^q - W = X^{1+q}$, with $q = p^3$.

2. *There exists a coordinate X for the projective line C/G_2, $s \geq 2$, $d \geq 2$ dividing s, and γ_2, γ_3 in $\mathbb{F}_{p^d} - \mathbb{F}_p$ linearly independent over $\mathbb{F}_p$, $b_1 \in k$, $c_1 \in k$ such that:*

f_1	$f_1(X) = X\,S_1(X)$	*with*	$S_1(F) = \sum_{j=0}^{s/d} a_{jd}\,F^{jd} \in k\{F\} \quad a_s = 1$
f_2	$f_2(X) = X\,S_2(X) + b_1\,X$	*with*	$S_2 = \gamma_2\,S_1$
f_3	$f_3(X) = X\,S_3(X) + c_1\,X$	*with*	$S_3 = \gamma_3\,S_1$
V	$V = Z(\mathrm{Ad}_{f_1}) = Z(\mathrm{Ad}_{f_2}) = Z(\mathrm{Ad}_{f_3})$		

Therefore, the solutions can be parametrized by $s+4$ algebraically independent variables over $\mathbb{F}_p$, namely the s coefficients of S, $\gamma_2 \in \mathbb{F}_{p^d} - \mathbb{F}_p$, $\gamma_3 \in \mathbb{F}_{p^d} - \mathbb{F}_p$, $b_1 \in k$ and $c_1 \in k$.

Moreover,

$$\frac{|G|}{g} = \frac{2\,p}{p-1}\,\frac{p^s}{1+p+p^2} \quad \textit{and} \quad \frac{|G|}{g^2} = \frac{4}{(p^2-1)^2}\,\frac{p^3(p+1)^2}{(1+p+p^2)^2}.$$

Proof: The idea is to use Remark 5.8 to show that $\dim_{\mathbb{F}_p}(V) = 2\,s_3$. Then one can apply [Ro09] (Proposition 4.2, Proposition 4.3 and Corollary 4.5). For a complete proof, see [Ro08] .

5.3.2 Case $[G', G] \supsetneq \mathrm{Fratt}(G') = \{e\}$.

Lemma 5.10. *Let (C, G) be a big action satisfying condition $(*)$ with $G' \simeq (\mathbb{Z}/p\mathbb{Z})^3$ and $[G', G] \neq \{e\}$. We keep the notations introduced at the beginning of Section 5. Then one cannot have $\ell_{1,2} = \ell_{2,3} = 0$.*

Proof: the proof is based on calculation left to the reader. Otherwise, see [Ro08]. □

As a consequence, there are 3 cases to study:

$\ell_{1,2} \neq 0$ and $\ell_{2,3} = 0$ (cf. Proposition 5.11).

$\ell_{1,2} = 0$ or $\ell_{2,3} \neq 0$ (cf. Proposition 5.12).

$\ell_{1,2} \neq 0$ or $\ell_{2,3} \neq 0$ (cf. Proposition 5.13).

For a complete proof of each of these propositions, see [Ro08].

Proposition 5.11. *Let (C, G) be a big action satisfying condition $(*)$ with $G' \simeq (\mathbb{Z}/p\mathbb{Z})^3$ and $[G', G] \neq \{e\}$. We keep the notations defined at the beginning of Section 5. Assume that $\ell_{1,2} \neq 0$ and $\ell_{2,3} = 0$.*

1. *Then $p \geq 5$ and there exists a coordinate X for the projective line C/G_2 such that the functions f_i's can be parametrized as follows:*

f_1	$f_1(X) = X^{1+p} + a_2\,X^2$
V	$V = Z(\mathrm{Ad}_{f_1}) = Z(X^{p^2} + 2\,a_2^p\,X^p + X)$
f_2	$f_2(X) = b_{1+2\,p}\,X^{1+2\,p} + b_{2+p}\,X^{2+p} + b_3\,X^3 + b_1\,X$
$b_{1+2\,p}$	$b_{1+2\,p} \in k^\times$
a_2	$2\,a_2^p = -b_{1+2\,p}^{-p}\,(b_{1+2\,p} + b_{1+2\,p}^{p^2}) \Leftrightarrow b_{1+2\,p} \in V - \{0\}$
b_{2+p}	$b_{2+p} = -b_{1+2\,p}^p$
b_3	$3\,b_3^p = b_{1+2\,p}^{-p}\,(b_{1+2\,p}^{2\,p^2} - b_{1+2\,p}^2)$
b_1	$b_1 \in k$
$\ell_{1,2}$	$\ell_{1,2}(y) = 2\,(b_{1+2\,p}\,y^p - b_{1+2\,p}^p\,y)$
f_3	$f_3(X) = c_{1+2\,p}\,X^{1+2\,p} + c_{2+p}\,X^{2+p} + c_3\,X^3 + c_1\,X$
$c_{1+2\,p}$	$c_{1+2\,p} \in k^\times$ $c_{1+2\,p} \in V - \{0\}$, $c_{1+2\,p}$ *and* $b_{1+2\,p}$ $\mathbb{F}_p$*-independent*
c_{2+p}	$c_{2+p} = -c_{1+2\,p}^p$
c_3	$3\,c_3^p = -c_{1+2\,p}^{-p}\,(c_{1+2\,p}^{2\,p^2} + c_{1+2\,p}^2)$
c_1	$c_1 \in k$
$\ell_{1,3}$	$\ell_{1,3}(y) = 2\,(c_{1+2\,p}\,y^p - c_{1+2\,p}^p\,y)$
$\ell_{2,3}$	$\ell_{2,3}(y) = 0$

Therefore, the solutions are parametrized by 4 algebraically independent variables over $\mathbb{F}_p$, namely $b_{1+2\,p} \in k^\times$, $c_{1+2\,p} \in k^\times$, $b_1 \in k$ and $c_1 \in k$.
Moreover,

$$\frac{|G|}{g} = \frac{2\,p}{p-1}\,\frac{p^3}{1+2\,p+2\,p^2} \quad \textit{and} \quad \frac{|G|}{g^2} = \frac{4}{(p^2-1)^2}\,\frac{p^3(p+1)^2}{(1+2\,p+2\,p^2)^2}.$$

2. *In this case, $G = A_{\infty,1}$ is the unique p-Sylow subgroup of A.*

Proposition 5.12. *Let (C, G) be a big action satisfying condition $(*)$ with $G' \simeq (\mathbb{Z}/p\mathbb{Z})^3$ and $[G', G] \neq \{e\}$. We keep the notations defined at the beginning of Section 5. Assume that $\ell_{1,2} = 0$ and $\ell_{2,3} \neq 0$.*

1. *Then $p \geq 5$ and there exists a coordinate X for the projective line C/G_2 such that the functions f_i's can be parametrized as follows:*

f_1	$f_1(X) = X^{1+p^2} + a_2\, X^2$
f_2	$f_2(X) = \gamma_2\, (X^{1+p^2} + a_2\, X^2) + b_1\, X$
b_1	$b_1 \in k$
γ_2	$\gamma_2 \in \mathbb{F}_{p^2} - \mathbb{F}_p$
V	$V = Z(\mathrm{Ad}_{f_1}) = Z(\mathrm{Ad}_{f_2}) = Z(X^{p^4} + 2\, a_2^{p^2}\, X^{p^2} + X)$

<u>*First case: $b_1 \neq 0$*</u>

f_3	$f_3(X) = c_{1+2\,p^2}\, X^{1+2\,p^2} + c_{2+p^2}\, X^{2+p^2} + c_{1+p^2}\, X^{1+p^2}$ $+ c_{1+p}\, X^{1+p} + c_3\, X^3 + c_2 X^2 + c_1\, X$
$c_{1+2\,p^2}$	$c_{1+2\,p^2} \in k^\times$
a_2	$2\, a_2^{p^2} = -c_{1+2\,p^2}^{-p^2}\, (c_{1+2\,p^2}^{p^4} + c_{1+2\,p^2}) \Leftrightarrow c_{1+2\,p^2} \in V - \{0\}$
c_{2+p^2}	$c_{2+p^2} = -c_{1+2\,p^2}^{p^2}$
c_3	$3\, c_3^{p^2} = -c_{1+2\,p^2}^{p^2}\, (3\, c_{1+2\,p^2}^{2\,p^4} + 4\, c_{1+2\,p^2}^{1+p^4} + c_{1+2\,p^2}^2)$
$e := c_{1+p^2} - c_{1+p^2}^{p^2}$	$e \in Z\,((c_{1+2\,p^2}^{p^7-p^3} + 1 + c_{1+2\,p^2}^{p-p^5} + c_{1+2\,p^2}^{p^7+p-p^5-p^3})\, X^{1+p^4}$ $-X^{1+p^2} - X^{p^2} - X - 1)$
b_1	$b_1^{p^5-p^4+p^3-p^2} = -e^{p^3-1}$
c_{1+p}	$c_{1+p}^{p+p^3} = -e^{1+p}$
c_2	$4\, c_2^{p^3\,(p-1)^2\,(p^2+1)} = c_{1+p^2}^{p^3\,(p^2+1)}$ $+(c_{1+2\,p^2}^{p^7-p^3} + 1 + c_{1+2\,p^2}^{p-p^5} + c_{1+2\,p^2}^{p^7+p-p^5-p^3})\,(c_{1+p^2} - c_{1+p^2}^{p^2})^{p^3+p^2+p-1}$
c_1	$c_1 \in k$

Therefore, the solutions can be parametrized by 3 algebraically independent variables over $\mathbb{F}_p$, namely $c_{1+2\,p^2} \in k^\times$, $c_1 \in k$ and $\gamma_2 \in \mathbb{F}_{p^2} - \mathbb{F}_p$. One also finds a fourth parameter $e := c_{1+p^2} - c_{1+p^2}^{p^2}$ which runs over the set of zeroes of a polynomial whose coefficients are rational functions in $c_{1+2\,p^2}$. So, for a given $c_{1+2\,p^2}$, the parameter e takes a finite number of values.

<u>*Second case: $b_1 = 0$*</u>

f_3	$f_3(X) = c_{1+2\,p^2}\, X^{1+2\,p^2} + c_{2+p^2}\, X^{2+p^2} + c_3\, X^3$
$c_{1+2\,p^2}$	$c_{1+2\,p^2} \in k^\times$
a_2	$2\, a_2^{p^2} = -c_{1+2\,p^2}^{-p^2}\, (c_{1+2\,p^2}^{p^4} + c_{1+2\,p^2}) \Leftrightarrow c_{1+2\,p^2} \in V - \{0\}$
c_{2+p^2}	$c_{2+p^2} = -c_{1+2\,p^2}^{p^2}$
c_3	$3\, c_3^{p^2} = -c_{1+2\,p^2}^{p^2}\, (3\, c_{1+2\,p^2}^{2\,p^4} + 4\, c_{1+2\,p^2}^{1+p^4} + c_{1+2\,p^2}^2)$
c_1	$c_1 \in k$

In this case, the solutions can be parametrized by 3 algebraically independent variables over $\mathbb{F}_p$, namely $c_{1+2\,p^2} \in k^\times$, $c_1 \in k$ and $\gamma_2 \in \mathbb{F}_{p^2} - \mathbb{F}_p$.

In both cases,

$$\frac{|G|}{g} = \frac{2\,p}{p-1}\, \frac{p^4}{1+p+2\,p^2} \quad \textit{and} \quad \frac{|G|}{g^2} = \frac{4}{(p^2-1)^2}\, \frac{p^3\,(p+1)^2}{(1+p+2\,p^2)^2}.$$

2. *Moreover, $G = A_{\infty,1}$ is the unique p-Sylow subgroup of A.*

The last case: $\ell_{1,2} \neq 0$ and $\ell_{2,3} \neq 0$, generalizes the results obtained in [Ro09] (Section 6.2).

Proposition 5.13. *Let (C, G) be a big action satisfying condition $(*)$ with $G' \simeq (\mathbb{Z}/p\mathbb{Z})^3$ and $[G', G] \neq \{e\}$. We keep the notations defined at the beginning of Section 5. Assume that $\ell_{1,2} \neq 0$ and $\ell_{2,3} \neq 0$.*

1. *Then $p \geq 11$ and there exists a coordinate X for the projective line C/G_2 such that the functions f_i's can be parametrized as follows:*

f_1	$f_1(X) = X^{1+p} + a_2\, X^2$
V	$V = Z(\mathrm{Ad}_{f_1}) = Z(X^{p^2} + 2\, a_2^p\, X^p + X)$
f_2	$f_2(X) = b_{1+2\,p}\, X^{1+2\,p} + b_{2+p}\, X^{2+p} + b_3\, X^3 + b_1\, X$
$b_{1+2\,p}$	$b_{1+2\,p} \in k^\times$
a_2	$2\, a_2^p = -b_{1+2\,p}^{-p}\, (b_{1+2\,p} + b_{1+2\,p}^{p^2}) \Leftrightarrow b_{1+2\,p} \in V - \{0\}$
b_{2+p}	$b_{2+p} = -b_{1+2\,p}^p$
b_3	$3\, b_3^p = b_{1+2\,p}^{-p}\, (b_{1+2\,p}^{2\,p^2} - b_{1+2\,p}^2)$
$\ell_{1,2}$	$\ell_{1,2}(y) = 2\, (b_{1+2\,p}\, y^p - b_{1+2\,p}^p\, y)$
f_3	$f_3(X) = c_{1+3\,p}\, X^{1+3\,p} + c_{2+2\,p}\, X^{2+2\,p} + c_{1+2\,p}\, X^{1+2\,p} + c_{3+p}\, X^{3+p}$ $+ c_{2+p}\, X^{2+p} + c_{1+p}\, X^{1+p} + c_4\, X^4 + c_3\, X^3 + c_2\, X^2 + c_1\, X$
$c_{1+3\,p}$	$3\, c_{1+3\,p} = 2\, b_{1+2\,p}^2$
$c_{2+2\,p}$	$c_{2+2\,p} = -b_{1+2\,p}^{1+p}$
c_{3+p}	$3\, c_{3+p} = 2\, b_{1+2\,p}^{2\,p}$
c_4	$6\, c_4^p = -b_{1+2\,p}^{-p}\, (b_{1+2\,p}^3 + b_{1+2\,p}^{3\,p^2})$
$c_{1+2\,p}$	$c_{1+2\,p} \in V$
c_{2+p}	$c_{2+p} = -c_{1+2\,p}^p$
c_3	$3\, c_3^p = b_{1+2\,p}^{-p}\, (b_{1+2\,p} + b_{1+2\,p}^{p^2})\, (c_{1+2\,p}^{p^2} - c_{1+2\,p})$
c_{1+p}	$c_{1+p} \in k$
b_1	$2\, b_1^p = b_{1+2\,p}^{-p}\, (c_{1+p}^p - c_{1+p})$
c_2	$2\, c_2^p = -b_{1+2\,p}^{-p}\, (c_{1+p}^p\, b_{1+2\,p}^{p^2} + c_{1+p}\, b_{1+2\,p})$
c_1	$c_1 \in k$
$\ell_{1,3}$	$\ell_{1,3}(y) = 2\, (c_{1+2\,p}\, y^p\, c_{1+p}^p - y) + 2\, b_{1+2\,p}^2\, y^{2\,p} - 4\, b_{1+2\,p}^{1+p}\, y^{1+p} + 2\, b_{1+2\,p}^{2\,p}\, y^2$ $= 2\, (c_{1+2\,p}\, y^p - c_{1+p}^p\, y) + \ell_{1,2}^2(y)/2$
$\ell_{2,3}$	$\ell_{2,3}(y) = 2\, (b_{1+2\,p}\, y^p - b_{1+2\,p}^p\, y)$

Therefore, the solutions can be parametrized by 3 algebraically independent variables over $\mathbb{F}_p$, namely $b_{1+2\,p} \in k^\times$, $c_{1+p} \in k$ and $c_1 \in k$. One also finds a fourth parameter $c_{1+2\,p}$ which runs over V. So, for a given $b_{1+2\,p}$, the parameter $c_{1+2\,p}$ takes a finite number of values. Moreover,

$$\frac{|G|}{g} = \frac{2\,p}{p-1}\, \frac{p^3}{1+2\,p+3\,p^2} \quad \textit{and} \quad \frac{|G|}{g^2} = \frac{4}{(p^2-1)^2}\, \frac{p^3\,(p+1)^2}{(1+2\,p+3\,p^2)^2}.$$

2. *$G = A_{\infty,1}$ is the unique p-Sylow subgroup of A. Furthermore, $Z(G)$ is cyclic of order p.*

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References

[BeRo08] J. Bertin, M. Romagny, *Champs d'Hurwitz*, (2008), disponible à `http://www.math.jussieu.fr/ ~ romagny/`

[Bou83] N. Bourbaki, *Algèbre commutative.* Eléments de Mathématiques, Masson, Paris, (1983)

[Br00] T. Breuer, *Characters and automorphism groups of compact Riemann surfaces.* London Mathematical Society Lecture Note Series, 280. Cambridge University Press, Cambridge, (2000).

[Con90] M. Conder, *Hurwitz groups: a brief survey.* Bull. Amer. Math. Soc. (N.S.) **23**, no. 2, (1990), 359-370.

[El99] N. Elkies, *Linearized algebra and finite groups of Lie type. I. Linear and symplectic groups* in *Applications of curves over finite fields*, (Seattle, WA, 1997), Contemporary Mathematics, vol 245, American Mathematical Society, Providence, RI, (1999).

[GK07] M. Giulietti, G. Korchmáros, *On large automorphism groups of algebraic curves in positive characteristic*, (2007), available at `http://front.math.ucdavis.edu/0706.2320`

[Go96] D. Goss, *Basic structures of function field arithmetic*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol.35, (Springer, Berlin, 1996)

[Han92] J. Hansen, *Deligne-Lusztig varieties and group codes.* Coding theory and algebraic geometry (Luminy, 1991), 63–81, Lecture Notes in Math., 1518, Springer, Berlin, (1992).

[Kon07] A. Kontogeorgis, *On the tangent space of the deformation functor of curves with automorphisms*, Algebra & Number Theory **1** no. 2, (2007), 119–161.

[Kul97] R. Kulkarni, *Riemann surfaces admitting large automorphism groups.* Contemp. Math., 201, Amer. Math. Soc., Providence, RI, (1997), 63–79.

[Leo96] H-W. Leopoldt, *Über die Automorphismengruppe des Fermatkörpers.* J. Number Theory **56**, no. 2, (1996), 256–282.

[LGM02] C.R. Leedham-Green, S. McKay, *The structure of groups of prime power order.* London Mathematical Society Monographs. New Series, 27. Oxford Science Publications. Oxford University Press, Oxford, (2002).

[LM05] C. Lehr, M. Matignon, *Automorphism groups for p-cyclic covers of the affine line.* Compositio Math. **141** , no. 5,(2005), 1213–1237.

[MR08] M. Matignon, M. Rocher, *Smooth curves having a large automorphism p-group in characteristic* $p > 0$, Algebra Number Theory 2, n° 8, (2008), 887–926

[MSSV02] K. Magaard, T. Shaska, S. Shpectorov, H. Völklein, *The locus of curves with prescribed automorphism group.* Communications in arithmetic fundamental groups (Kyoto, 1999/2001). Sūrikaisekikenkyūsho Kōkyūroku no. 1267 (2002), 112–141, available at `http://arxiv.org/abs/math/0205314`

[Na87] S. Nakajima, *p-ranks and automorphism groups of algebraic curves*, Trans. Amer. Math. Soc. **303**, no. 2, (1987), 595–607.

[Pr05] R. Pries, *Equiramified deformations of covers in positive characteristic*, (2005), avalaible at `http://arxiv.org/abs/math/0403056`

[Ro08] M. Rocher, *Large p-group actions with* $\frac{|G|}{g^2} \geq \frac{4}{(p^2-1)^2}$, (2008), available at `http://arxiv.org/abs/0804.3494`

[Ro09] M. Rocher, *Large p-group actions with a p-elementary abelian derived group.* Journal of Algebra **321** (2009), 704–740

[St73] H. Stichtenoth, *Über die Automorphismengruppe eines algebraischen Funktionkorpers von Primzahlcharakteristik I, II*, Arch. Math. (Basel) **24** (1973).

[St93] H. Stichtenoth, *Algebraic function fields and codes.* Universitext. Springer-Verlag, Berlin, (1993).

[Su82] M. Suzuki, *Group Theory I.* Grundlehren der Mathematischen Wissenschaften, 247. Springer-Verlag, Berlin-New York, (1982)

[Su86] M. Suzuki, *Group theory. II.* Grundlehren der Mathematischen Wissenschaften, 248. Springer-Verlag, New York, (1986)