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Resolvent estimates and local energy decay for hyperbolic equations

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Abstract. We examine the cut-off resolvent $R_\chi(\lambda) = \chi(-\Delta_D - \lambda^2)^{-1}\chi$, where Δ_D is the Laplacian with Dirichlet boundary condition and $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 in a neighborhood of the obstacle K. We show that if $R_\chi(\lambda)$ has no poles for $\operatorname{Im}\lambda \geq -\delta$, $\delta > 0$, then $\|R_\chi(\lambda)\|_{L^2\to L^2} \leq C|\lambda|^{n-2}$, $\lambda \in \mathbb{R}$, $|\lambda| \geq C_0$. This estimate implies a local energy decay. We study the spectrum of the Lax-Phillips semigroup $Z(t)$ for trapping obstacles having at least one trapped ray.

1. Introduction

Let $K \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with C^∞ boundary ∂K and connected complement $\Omega = \mathbb{R}^n \setminus \overline{K}$. Such K is called an *obstacle* in $\mathbb{R}^n$. We consider the Dirichlet problem for the wave equation

$$\begin{cases} (\partial_t^2 - \Delta_x)u = 0 \text{ in } \mathbb{R}\times\Omega, \\ u = 0 \text{ on } \mathbb{R}\times\partial K, \\ u(0,x) = f_0(x),\ \partial_t u(0,x) = f_1(x). \end{cases} \tag{1}$$

Let $K \subset B_a = \{x \in \mathbb{R}^n : |x| \leq a\}$ and for $m \geq 0$ set

$$p_m(t) = \sup\Big[\frac{\|\nabla_x u\|_{L^2(B_a\cap\Omega)} + \|\partial_t u\|_{L^2(B_a\cap\Omega)}}{\|\nabla_x f_0\|_{H^m(B_a\cap\Omega)} + \|f_1\|_{H^m(B_1\cap\Omega)}},$$

$$(0,0) \neq (f_0, f_1) \in C_0^\infty(\Omega)\times C_0^\infty(\Omega),\ \operatorname{supp} f_i \subset B_a\}\Big].$$

For $\operatorname{Im} \lambda > 0$ consider the cut-off resolvent $R_\chi(\lambda) = \chi R(\lambda)\chi : L^2(\Omega) \longrightarrow L^2(\Omega)$, where $R(\lambda) = (-\Delta_D - \lambda^2)^{-1}$, $\chi \in C_0^\infty(B_{a+1})$, $\chi = 1$ on B_a and Δ_D is the Dirichlet Laplacian with domain $D(\Delta_D) = H_0^2(\Omega)$.

The following result of Vodev generalized the classical one of Morawetz for $n \geq 3$ odd.

Theorem 1 ([20]). *The following conditions are equivalents:*
(a) $\lim_{t\to+\infty} p_0(t) = 0$,
(b) *There exist* $C_0 > 0$, $C_1 > 0$ *so that*

$$\|\lambda R_\chi(\lambda)\| \leq C_1, \lambda \in \mathbb{R},\ |\lambda| \geq C_0,$$

(c) *There exist constants* $C > 0$, $\gamma > 0$ *so that*

$$p_0(t) \leq \begin{cases} Ce^{-\gamma t},\ n \text{ odd}, \\ Ct^{-n}, n \text{ even}. \end{cases}$$

It is known that (b) holds if the obstacle K is non-trapping, that is the singularities of the solution of the Dirichlet problem with initial data with compact support leave any compact $\omega \subset \mathbb{R}^n$ for $t \geq t(\omega)$ (see for instance [4] for more details). For trapping obstacles without any condition on the geometry of K we have the following

Theorem 2 ([3]). *We have the estimate*

$$\|R_\chi(\lambda)\| \leq Ce^{C|\lambda|},\ \lambda \in \mathbb{R},\ |\lambda| \geq C_0$$

and for every integer $m > 1$ *we have*

$$p_m(t) \leq \frac{C_m}{(\log t)^m},\ t > 1. \tag{2}$$

The cut-off resolvent $R_\chi(\lambda)$ has a meromorphic continuation in $\mathbb{C}$ for n odd and in $\mathbb{C}' = \{z \in \mathbb{C} : z \neq -i\mu,\ \mu \in \mathbb{R}\}$ for n even ([10], [19]). There are many examples when we have a domain

$$\{z \in \mathbb{C} :\ -\delta \leq \operatorname{Im} z \leq 0\},\ \delta > 0$$

without poles (resonances) of $R_\chi(\lambda)$ (cf. for example [7]). In this talk we some results showing that in this case we have a polynomial bound of the cut-off resolvent $R_\chi(\lambda)$ on $\mathbb{R}$ and a better local energy decay than (2). Our main result is the following

Theorem 3. *Assume that the cut-off resolvent* $R_\chi(\lambda)$ *has no poles for* $\operatorname{Im} \lambda \geq -\delta$, $\delta > 0$. *Then*

$$\|R_\chi(\lambda)\|_{L^2(\Omega)\to L^2(\Omega)} \leq C|\lambda|^{n-2},\ \lambda \in \mathbb{R},\ |\lambda| \geq C_0. \tag{3}$$

Remark 1. Notice that if for some $M \geq 0$ we have the estimate

$$\|R_\chi(\lambda)\|_{L^2(\Omega)\to L^2(\Omega)} \leq C_1|\lambda|^M,\ \operatorname{Im}\lambda \geq -\delta,\ |\operatorname{Re}\lambda| \geq C_0,$$

then a result of N. Burq [5] says that

$$\|R_\chi(\lambda)\|_{L^2(\Omega)\to L^2(\Omega)} \leq C_2\frac{\log(2+|\lambda|^2)}{1+|\lambda|},\ \lambda \in \mathbb{R},\ |\lambda| \geq C_0.$$

In particular, a such estimate holds for two strictly convex disjoint obstacles and under some conditions for several strictly convex disjoint obstacles ([7]).

Remark 2. For the semiclassical Schrödinger operators $-h^2\Delta + V(x)$ in the case of dimension 1 a polynomial bound $\mathcal{O}(h^{-M})$ of the cut-off resolvent in

$$W = \{z \in \mathbb{C} : 0 < a_0 \leq \operatorname{Re} z \leq a_1,\ \operatorname{Im} z \geq -a_2 h, a_i > 0, i = 0,1,2\}$$

has been obtained in [2], provided that we have no resonances in W. It is natural to conjecture that under the condition of Theorem 3, the cut-off resolvent $R_\chi(\lambda)$ is bounded uniformly on $\mathbb{R}$ for any dimension $n \geq 3$.

2. Estimates of $R_\chi(\lambda)$

It is useful to transform the problem to a semi-classical one. Setting $\lambda = \frac{\sqrt{z}}{h}$, $0 < h \leq 1$, we have

$$(-\Delta_D - \lambda^2)^{-1} = h^2(-h^2\Delta_D - z)^{-1}$$

and we will study the operator $\chi(P(h)-z)^{-1}\chi$ with $P(h) = -h^2\Delta_D, h > 0$, in the domain

$$\mathcal{D}_{c_1} = \{z \in \mathbb{C} : 0 < a_0 \leq |\operatorname{Re} z| \leq a_1, -c_1 h \leq \operatorname{Im} z \leq c_2, a_i > 0, c_i, i = 0,1\}.$$

We will work in the "black box" setup ([15], [17]). For this purpose define $\mathcal{H}_a = L^2(\Omega \cap B_a)$ and set

$$\mathcal{L} = \mathcal{H}_a \oplus L^2(\mathbb{R}^n \setminus B_a).$$

We consider $P(h)$ as an operator $P(h) : \mathcal{L} \longrightarrow \mathcal{L}$ with domain $\mathcal{D}(P) \subset \mathcal{L}$ and the hypothesis in [15], [17] for a "black box" framework are satisfied. In particular, setting

$$\mathcal{H}^\sharp = \mathcal{H}_a \oplus L^2(\mathbb{T}^n_a \setminus B_1),\ \mathbb{T}^n_a = \mathbb{R}^n/(a\mathbb{Z}^n),$$

we introduce $P^\sharp(h)$ by replacing $-h^2\Delta_D$ by $-h^2\Delta_{\mathbb{T}^n_a}$. The operator $P^\sharp(h)$ has a discrete spectrum and we denote by $N(P^\sharp(h),\lambda)$ the number of eigenvalues of $P^\sharp(h)$ in $[-\lambda,\lambda]$. Then we we have

$$N(P^\sharp(h),\lambda) = \mathcal{O}\Big(\Big(\frac{\lambda}{h^2}\Big)^{n/2}\Big), \text{ for } \lambda \geq 1.$$

This follows from the Weyl asymptotic for the counting function for eigenvalues of $P^\sharp(h)$.

In the following for simplicity we will write P instead of $P(h)$. We will examine the resolvent of the complex dilated operator $P_\theta(h)$ defined as follows. Introduce a function $f_\theta(t) : \mathbb{R}^+ \longrightarrow \mathbb{C}$ having the properties:

$$f_\theta(t) = t \text{ for } t \leq a+1,$$
$$f_\theta(t) = e^{i\theta}t, \ t \gg 1,$$
$$0 \leq \arg f_\theta(t) \leq \theta, \ \partial_t f_\theta(t) \neq 0,$$
$$\arg f_\theta(t) \leq \arg \partial_t f_\theta(t) \leq \arg f_\theta(t) + \epsilon$$

with small $\epsilon > 0$. Let $\mu_\theta(t\omega) = f_\theta(t)\omega, \ t = |x| \in \mathbb{R}^+, \ \omega \in S^{n-1}$ and set $\Gamma_\theta = \mu_\theta(\mathbb{R}^n)$. Let $\Psi \in C_0^\infty(B_{a+1})$ be equal to 1 near B_a. As in [15], [16], we introduce the dilated operator P_θ

$$P_\theta u = P(\Psi u) - \Delta_{\Gamma_\theta}(1-\Psi)u$$

with domain

$$D_\theta = \{u \in L^2(\Gamma_\theta) : \Psi u \in D(P), \ (1-\Psi)u \in H^2(\Gamma_\theta)\},$$

$D(P)$ being the domain of P. Here $-\Delta_{\Gamma_\theta}$ is the dilated Laplacian corresponding to the change $\mathbb{R}^n \ni x \longrightarrow f_\theta(t)\omega \in \mathbb{C}^n$ and we refer to [15], [16], [17] for more details. Next set $\theta = c_1 h$ so that in the domain

$$\Omega_\theta = \{z \in \mathbb{C} : |z-\omega| \leq \theta, \ -\theta \leq \operatorname{Im} z \leq a_2\theta\} \subset \mathcal{D}_{c_1}, \ a_2 \gg 1$$

there are no eigenvalues of P_θ. Note that the eigenvalues of P_θ coincide with their multiplicities with the resonances of P ([15], [16], [17]). From [8], [11] the counting function of the eigenvalues of $P^\#(h)$ satisfies

$$N(P^\#, [\lambda-h, \lambda+h]) = \mathcal{O}(h^{1-n}),$$

for $\lambda \in [a_0, a_1]$. Then, following a construction, given by one of the authors [1], we may find a finite rank operator L so that

$$(P_\theta + \theta L - z)^{-1} = \mathcal{O}(\theta^{-1}), \ \forall z \in \Omega_\theta, \tag{4}$$

$$(P_\theta - z)^{-1} = \mathcal{O}(\theta^{-1}),\ z \in \Omega_\theta^+ = \Omega_\theta \cap \{\operatorname{Im} z \geq \epsilon\theta\},\ 0 < \epsilon < a_2, \quad (5)$$
$$\|L\|_{D(P_\theta) \to L^2(\Gamma_\theta)} = \mathcal{O}(1),\ \text{ rank } L = \kappa \leq C_0 h^{-n+1}$$

with a constant $C_0 > 0$ independent on h. This construction generalises that of Sjöstrand [16] with a finite rank operator K_0, $\text{rank} K_0 = Ch^{-n}$. Now consider the Grushin problem

$$\begin{cases} (P_\theta - z)u + R_-(z)u_- = v, \\ R_+(z)u = v_+, \end{cases} \tag{6}$$

where $u \in D(P_\theta), v \in L^2(\Gamma_\theta)$, while $u_-, v_+ \in \mathbb{C}^k$. Given an orthonormal basis $(e_1, ..., e_\kappa)$ in Image L^*, the operators $R_\pm$ have the form

$$R_+ u = (u, e_j)_{j=1,\dots,\kappa},$$

$$R_- u_- = \sum_{j=1}^{\kappa} u_{-,j}(P_\theta + \theta L - z)e_j,\ u_- = (u_{-,1}, ..., u_{-,\kappa}),$$

where $(,)$ is the scalar product in $L^2(\Gamma_\theta)$.

Following the results in Section 6, [16], the problem (6) is invertible and the inverse operator is given by

$$\begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-,+}(z) \end{pmatrix}.$$

To estimate the operators $E(z)$, $E_-(z)$, $E_+(z)$, $E_{-,+}(z)$, consider an orthonormal basis

$$(e_{\kappa+1}, ..., e_m, ...)$$

in $\Big(\text{Image } L^*\Big)^\perp = \operatorname{Ker} L$. Let

$$u = \sum_{j=1}^{\kappa} u_j e_j + \sum_{j=\kappa+1}^{\infty} u_j e_j = u' + u''.$$

From (6) we get

$$(P_\theta - z)(u' + u'') + (P_\theta + \theta L - z)(\sum_{j=1}^{\kappa} u_{-,j} e_j) = v,$$

$$(u, e_j) = u_j = v_{+,j},\ j = 1, ..., \kappa.$$

This implies

$$(P_\theta + \theta L - z)(u'' + \sum_{j=1}^{\kappa} u_{-,j} e_j) = v - (P_\theta - z)u'$$

$$= v - (P_\theta - z) \sum_{j=1}^{\kappa} u_j e_j,$$

hence

$$\Big(u'' + \sum_{j=1}^{\kappa} u_{-,j} e_j \Big) = \Big(P_\theta + \theta L - z \Big)^{-1} v$$

$$- \Big(P_\theta + \theta L - z \Big)^{-1} (P_\theta - z) \sum_{j=1}^{\kappa} v_{+,j} e_j = A + B.$$

The estimate (4) leads to $A = \mathcal{O}(\theta^{-1}) \|v\|$. Using once more (4), we get

$$B = \Big(-I + (P_\theta + \theta L - z)^{-1} \theta L \Big) \sum_{j=1}^{\kappa} v_{+,j} e_j = \mathcal{O}(1) \|v_+\|_{\mathbb{C}^\kappa}.$$

Thus

$$\|u''\| + \|u_-\|_{\mathbb{C}^\kappa} = \mathcal{O}(\theta^{-1}) \|v\| + \mathcal{O}(1) \|v_+\|_{\mathbb{C}^\kappa}$$

and $\|u'\|_{\mathbb{C}^\kappa} = \mathcal{O}(1) \|v_+\|_{\mathbb{C}^\kappa}$. Consequently,

$$\|u\| + \|u\|_{\mathbb{C}^\kappa} = \mathcal{O}(\theta^{-1}) \|v\| + \mathcal{O}(1) \|v_+\|_{\mathbb{C}^\kappa}$$

and we get the estimates

$$\|E(z)\| = \mathcal{O}(\theta^{-1}), \ \|E_-(z)\| = \mathcal{O}(\theta^{-1}),$$

$$\|E_+(z)\| = \mathcal{O}(1), \ \|E_{-,+}(z)\| = \mathcal{O}(1),$$

where $E_{-,+}(z) : \mathbb{C}^\kappa \longrightarrow \mathbb{C}^\kappa$. Moreover, the resolvent $(P_\theta - z)^{-1}$ and $(E_{-,+}(z))^{-1}$ are related by the equality (see for instance, [16])

$$(P_\theta - z)^{-1} = E(z) - E_+(z) ((E_{-,+}(z))^{-1} E_-(z)$$

and the above estimates yield

$$\begin{aligned} \|(P_\theta - z)^{-1}\| \leq & \|E(z)\| + \|E_+(z)\| \|(E_{-,+}(z))^{-1}\| \|E_-(z)\| \\ = & \mathcal{O}(\theta^{-1}) (1 + \|(E_{-,+}(z))^{-1}\|). \end{aligned}$$

Obviously,

$$(E_{-,+})^{-1} = \frac{{}^t\text{comatrix}(E_{-,+})}{\det(E_{-,+})} = \frac{\mathcal{O}(e^{C\kappa})}{\det(E_{-,+})}$$

and the problem is reduced to obtain a lower bound of $D(z) = \det(E_{-,+})$.

Since $\|E_{-,+}(z)\|_{\mathbb{C}^\kappa\to\mathbb{C}^\kappa} = \mathcal{O}(1)$, we have $|D(z)| \leq e^{C\kappa}$, $z \in \Omega_\theta$. On the other hand, in Ω_θ^+ we get

$$(E_{-,+}(z))^{-1}u_+ = -R_+(z)(P_\theta - z)^{-1}R_-(z)u_+$$

$$= -R_+(z)\Big(I + (P_\theta - z)^{-1}\theta L\Big)\sum_{j=1}^{\kappa} u_{-,j}e_j = \mathcal{O}(1)$$

and the estimate (5) for $z \in \Omega_\theta^+$ yields

$$|D(z)| = |\det(E_{-,+}(z))| = \Big|\frac{1}{\det((E_{-,+}(z))^{-1})}\Big| \geq e^{-C_1\kappa},\ z \in \Omega_\theta^+.$$

Recall that P_θ has no eigenvalues in Ω_θ, hence $D(z)$ has no zeros in Ω_θ. This makes possible to introduce the positive harmonic function $G(z) = C\kappa - \log|D(z)| \geq 0,\ z \in \Omega_\theta$. We have in Ω_θ^+ the estimate $\log|D(z)| \geq -C_1\kappa$, so we can apply the Harnack inequality for positive harmonic functions. In fact, for every $M \subset\subset \Omega_\theta$ we have

$$\sup_{z\in M} G(z) \leq C_M \inf_{z\in M} G(z) \leq C_M \inf_{z\in M\cap\Omega_\theta^+} G(z).$$

Making a small decrease of Ω_θ, which means to replace c_1 by a constant $0 < c_3 < c_1$, we deduce

$$G(z) \leq C_2\kappa,\ \log|D(z)| \geq -C_3\kappa,\ z \in \Omega_\theta,\ \theta = c_2 h.$$

Next suppose that Ω_θ is defined by c_3 instead of c_1. Combining the above estimates with the fact that $\kappa = C_0 h^{-n+1}$, we conclude that

$$\|(P_\theta - z)^{-1}\| \leq C_5 e^{C_4 h^{-n+1}},\ z \in \Omega_\theta.$$

Moreover, the same estimate is uniform with respect to choice of ω in Ω_θ, provided ω runs over a compact interval in $\mathbb{R}^+$ so that P_θ has no eigenvalues in Ω_θ. Thus we obtain

$$\|(P_\theta - z)^{-1}\| \leq C_5 e^{C_5 h^{-n+1}},\ z \in \mathcal{D}_{c_2}.$$

The complex scaling was chosen so that $f_\theta(t) = 1$ for $t \leq a+1$. Since $\mathrm{supp}\chi \subset B_{a+1}$, it is easy to see that

$$\chi(P - z)^{-1}\chi = \chi(P_\theta - z)^{-1}\chi,$$

hence

$$\|\chi(-h^2\Delta - z)^{-1}\chi\|_{L^2(\Omega)\longrightarrow L^2(\Omega)} \leq C_6 e^{C_6 h^{-n+1}},\ z \in \mathcal{D}_{c_2}.$$

Taking into account the scaling $\lambda = \frac{\sqrt{z}}{h}$, for $z \in \mathcal{D}_{c_2}$ we get

$$\operatorname{Re} z = h^2(\operatorname{Re}^2 \lambda - \operatorname{Im}^2 \lambda) \geq a_0, \ \operatorname{Im} z = 2h^2 \operatorname{Re} \lambda \operatorname{Im} \lambda \geq -c_2 h$$

which imply

$$\operatorname{Re} \lambda \geq \frac{a_0}{h} \geq a_0 > 0, \ \operatorname{Im} \lambda \geq -\frac{c_2}{2\sqrt{a_0}} = -a_2.$$

Consequently, we obtain

$$\|R_\chi(\lambda)\| \leq C_7 e^{C_7 |\lambda|^{n-1}}, \ \operatorname{Re} \lambda \geq a_0, \ \operatorname{Im} \lambda \geq -a_2. \tag{7}$$

In the same way we treat the domain $\operatorname{Re} \lambda \leq -a_0$, $\operatorname{Im} \lambda \geq -a_2$ and we get (7) for $|\operatorname{Re} \lambda| \geq a_0 > 0$.

3. Estimates on the real axis and decay of local energy

Proposition 1. *Let $f(z)$ be a holomorphic function in*

$$U_\alpha = \{z \in \mathbb{C} : \ \operatorname{Im} z \geq -\alpha\}, \ \alpha > 0,$$

such that

$$|f(z)| \leq C_0 e^{C|z|^m}, \ z \in U_\alpha, \ m \geq 1,$$

$$|f(z)| \leq \frac{C_1}{|z| \operatorname{Im} z}, \ \operatorname{Im} z > 0.$$

Then we have $|f(z)| \leq C_2(1+|z|)^{m-1}$, $z \in \mathbb{R}$.

Proof. Introduce the function $g(z) = e^{iAz^{m+1}} f(z)$, where $A > 0$ is sufficiently large. Consider the domain bounded by the curves:

$$\gamma_+ = \{z \in \mathbb{C} : \ \operatorname{Im} z = \frac{1}{|z|^m}, \ \operatorname{Re} z \geq 1\},$$

$$\gamma_- = \{z \in \mathbb{C} : \ \operatorname{Im} z = -\alpha, \ \operatorname{Re} z \geq 1\},$$

$$\gamma_0 = \{z \in \mathbb{C} : \ -\alpha \leq \operatorname{Im} z \leq \frac{1}{|z|^m}, \ \operatorname{Re} z = 1\}.$$

For $z \in \gamma_-$ and $\operatorname{Re} z \gg 1$ we have

$$|g(z)| \leq C_0 e^{C'(\operatorname{Re} z)^m} \exp\Big(-A\frac{(m+1)}{2}(\operatorname{Re} z)^m \operatorname{Im} z\Big) \leq C_3$$

taking $2C' - A(m+1)\alpha < 0$. On the curve γ_+ we obtain

$$|g(z)| \leq C_4 |z|^{m-1} \exp\Big((m+1)A(\operatorname{Re} z)^m \operatorname{Im} z\Big[1 + \mathcal{O}\Big(\frac{1}{|\operatorname{Re} z|}\Big)\Big]\Big)$$

$$\leq C_4|z|^{m-1}\exp\Big(\frac{B(\operatorname{Re} z)^m}{|z|^m}(m+1)\Big) \leq C_5|z|^{m-1}.$$

To obtain the estimate, we apply the Pragmen-Lindelöf theorem for the function $g(z)$ and deduce

$$|g(z) \leq C_6|z|^{m-1}$$

for

$$\operatorname{Re} z \geq 1,\ -\alpha \leq \operatorname{Im} z \leq \frac{1}{|z|^m}.$$

In particular, for $z \geq 1$ we get

$$|f(z)| \leq C_6|z|^{m-1}.$$

In a similar way we treat the case $z \leq -1$.

To apply Proposition 1, notice that the operator $-\Delta_D$ with Dirichlet boundary condition on $\partial\Omega$ is a self-adjoint positive operator and it is easy to see that

$$\|R(\lambda)\|_{L^2(\Omega)\to L^2(\Omega)} \leq \frac{C}{|z|\operatorname{Im} z},\ \operatorname{Im} z > 0.$$

Combining the estimate (7) and Proposition 1 with $m = n-1$, we obtain (3) and the proof of Theorem 3 is complete.

Theorem 3 makes possible to apply a result of G.Popov and G. Vodev (see Proposition 1.4 in [13]) in order to obtain the following

Theorem 4. *Under the hypothesis of Theorem* 3 *for every* $m > 0$ *and* $t > 1$ *we have for* n *odd the estimate*

$$p_m(t) \leq C(t^{-1}\log t)^{m/(n-1)},$$

while for n *even and* $t > 1$ *we have*

$$p_m(t) \leq \begin{cases} C(t^{-1}\log t)^{m/(n-1)}, \text{ for } 0 < m \leq n(n-1), \\ Ct^{-n} \text{ for } m > n(n-1). \end{cases}$$

The factor $m/(n-1)$ comes from the estimate of the resolvent of the generator G of the unitary group $U(t) = e^{itG}$ related to the problem (1). More precisely, we have

$$G = -i\begin{pmatrix} 0 & Id \\ \Delta_D & 0 \end{pmatrix}$$

with domain

$$D(G) = \{(u,v) : \ u \in H_0^2(\Omega), \ v \in H_D(\Omega)\} \subset \mathcal{H},$$

where $\mathcal{H} = \{(u,v) : \ u \in H_D(\Omega), \ v \in L^2(\Omega)\}$ and $H_D(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|\varphi\|_D^2 = \int_\Omega |\nabla \varphi|^2 dx.$$

For the resolvent $(G-\lambda)^{-1}$ we have the representation

$$(G-\lambda)^{-1} = \begin{pmatrix} \lambda R(\lambda) & -iR(\lambda) \\ -i\Delta_D R(\lambda) & \lambda R(\lambda) \end{pmatrix}. \tag{8}$$

Therefore (3) implies the estimates (see [20], [5])

$$\|\lambda R_\chi(\lambda)\|_{H_D \to H_D} \leq C|\lambda|^{n-1}, \ \|\chi \Delta_D R(\lambda)\chi\|_{H_D \to L^2} \leq C|\lambda|^{n-1},$$

$$\|R_\chi(\lambda)\|_{L^2 \to H_D} \leq C|\lambda|^{n-1}, \ \lambda \in \mathbb{R}, |\lambda| \geq C_0$$

and we obtain

$$\|\chi(G-\lambda)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} \leq C|\lambda|^{n-1}, \ \lambda \in \mathbb{R}, \ |\lambda| \geq C_0.$$

4. Spectre of the Lax-Phillips semigroup $Z(t)$

In this section we assume $n \geq 3$, n, odd and we examine the spectrum of the Lax-Phillips semigroup $Z^b(t) = P_+^b U(t) P_-^b$, $t \geq 0$, where $U(t)$ is the unitary group introduced in Section 3 and $P_\pm^a$ are the orthogonal projections on the orthogonal complements of the spaces

$$D_\pm^b = \{f \in \mathcal{H} : U_0(t)f = 0, \ |x| < \pm t + b\}, \ b > a.$$

Here $U_0(t)$ is the unitary group related to the Cauchy problem for the wave equation in $\mathbb{R}_t \times \mathbb{R}^n$ (see [10]). We choose $\chi \in C_0^\infty(\mathbb{R}^n)$ so that $\chi = 1$ for $|x| \leq a$, $\chi = 0$ for $|x| \geq b$. We fix $b > a$ with this property and note that $P_\pm^b \chi = \chi = \chi P_\pm^b$ and for simplicity we will write $Z(t)$ instead of $Z^b(t)$. Let B be the generator of $Z(t)$. Therefore,

$$\sigma(B) \subset \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$$

and the eigenvalues z_j of iB coincide with their multiplicities with the poles of $R_\chi(\lambda)$ (see [10]). The condition

$$\sup_{\lambda \in \mathbb{R}} \|\lambda R_\chi(\lambda)\|_{L^2(\Omega) \to L^2(\Omega)} = +\infty \tag{9}$$

implies

$$\sup_{\lambda\in\mathbb{R}} \|(B+i\lambda)^{-1}\|_{\mathcal{H}\to\mathcal{H}} = +\infty. \tag{10}$$

In fact, for $\operatorname{Re}\lambda > 0$ we have

$$\chi(iG-\lambda)^{-1}\chi = -\int_0^\infty e^{-\lambda t}\chi e^{itG}\chi dt$$

$$= -\int_0^\infty e^{-\lambda t}\chi Z(t)\chi dt = \chi(B-\lambda)^{-1}$$

and by analytic continuation for $\operatorname{Re}\lambda \geq 0$ we obtain

$$\chi(iG+i\lambda)^{-1}\chi, = \chi(B+i\lambda)^{-1}\chi,\ \forall\lambda\in\mathbb{R}$$

and we may exploit the representation (8). On the other hand, (9) means that the condition (b) of Theorem 1 is not satisfied, so we have not an uniform decay of the local energy. This holds for obstacles having at least one generalized non-degenerate trapping ray (see [14] and [12] for more details).

In the following we assume the condition (9) satisfied. Suppose that there are only finite number of resonances in the domain

$$\{z\in\mathbb{C} : \operatorname{Im} z \geq -\delta\},\ \delta > 0.$$

Choose $0 \leq \alpha \leq \delta$ so that we have no resonances on the line $\{z \in \mathbb{C} : \operatorname{Im} z = -\alpha\}$, hence the resolvent $(B+\alpha+i\lambda)^{-1}$ exists for every $\lambda\in\mathbb{R}$. It is easy to see that

$$\sup_{\lambda\in\mathbb{R}} \|(B+\alpha+i\lambda)^{-1}\|_{\mathcal{H}\to\mathcal{H}} = +\infty. \tag{11}$$

Indeed, if the resolvent $(B+\alpha+i\lambda)^{-1}$ is uniformly bounded with respect to $\lambda\in\mathbb{R}$, the cut-off resolvent $\|\lambda R_\chi(-i\alpha+\lambda)\|_{L^2\to L^2}$ will be also bounded uniformly with respect to $\lambda\in\mathbb{R}$. Consider the domain

$$\{z\in\mathbb{C} : -\alpha \leq \operatorname{Im} z \leq c_0,\ |\operatorname{Re} z| \geq c_1,\ c_i > 0,\ i = 0,1\}$$

with sufficiently large c_1. For all z in this domain we have an estimate (see for example [18])

$$\|zR_\chi(z)\|_{L^2\to L^2} \leq Ce^{C|z|^n}$$

and an application of the Pragmen-Lindelöf theorem leads to a contradiction with (9). Next, assume that

$$e^{-\alpha-i\beta} \notin \sigma(e^B),\ \forall\beta\in\mathbb{R}.$$

Then $\|(e^{-\alpha-i\beta}-e^B)^{-1}\| \leq C_\alpha$, $\forall \beta \in \mathbb{R}$ and from the equality

$$I - e^{B+\alpha+i\beta} = -(B+\alpha+i\beta)\int_0^1 e^{t(B+\alpha+i\beta)}dt$$

we deduce

$$(B+\alpha+i\beta)^{-1} = -\int_0^1 e^{t(B+\alpha+i\beta)}dt(I-e^{B+\alpha+i\beta})^{-1}.$$

Consequently, the resolvent $(B+\alpha+i\beta)^{-1}$ is uniformly bounded with respect to $\beta \in \mathbb{R}$ and we obtain a contradiction with (11). This shows that there exists $\beta_0 \in \mathbb{R}$ so that

$$e^{-\alpha-i\beta_0} \in \sigma(e^B) \setminus e^{\sigma(B)}.$$

Now we are in position to apply the result in [9] saying that there exists a set $\mathcal{M}_\alpha \subset \mathbb{R}^+$ with Lebesgue measure zero so that for all $t \in]0,\infty[\setminus \mathcal{M}_\alpha$ we have

$$e^{t(-\alpha-i\beta_0)}e^{i\omega} \in \sigma(Z(t)) : \forall \omega \in \mathbb{R},$$

hence

$$e^{-\alpha t+i\omega} \in \sigma(Z(t)),\ \forall \omega \in \mathbb{R}.$$

Assume that for $\frac{p_n}{q_n} \in \mathbb{Q}$, $0 < \frac{p_n}{q_n} \leq \delta$ we have no resonances on the line

$$\{z \in \mathbb{C} :\ \operatorname{Im} z = -\frac{p_n}{q_n}\}.$$

The above argument implies the existence of a set $\mathcal{M}_n \subset \mathbb{R}^+$ with Lebesgue measure zero such that for $t \in]0,\infty[\setminus \mathcal{M}_n$ we have

$$e^{-t\frac{p_n}{q_n}+i\omega} \in \sigma(Z(t)).$$

The rationals are dense in $]0,\delta[$ and the spectrum $\sigma(Z(t))$ is closed. Thus for $t \in]0,\infty[\setminus\Big(\cup_{n\in\mathbb{N}}\mathcal{M}_n\Big)$ we get the relation

$$\{z = e^{-ty+i\omega} \in \sigma(Z(t)) :\ 0 \leq y \leq \delta,\ \omega \in \mathbb{R}\}.$$

Finally, we have the following

Theorem 5. *Suppose that we have a finite number of resonances z with* $\operatorname{Im} z \geq -\delta$, $\delta > 0$. *If the condition* (9) *holds, there exists a set $\mathcal{R} \subset \mathbb{R}^+$ with Lebesgue measure zero so that for all $t \in]0,\infty[\setminus\mathcal{R}$ we have*

$$\{z \in \mathbb{C} : e^{-t\delta} \leq |z| \leq 1\} \subset \sigma(Z(t)).$$

Next we will examine the singularities of the cut-off resolvent $\chi(U(t)-z)^{-1}\chi$ for $z \to z_0 \in \mathbb{S}^1$, $|z| > 1$. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a function such that $\psi(x) = 1$ for $|x| \le a+1$, $\psi(x) = 0$ for $|x| \ge a+2$. Introduce the operator

$$L_\psi(g,h) = \Big(0, \langle \nabla_x \psi, \nabla_x g\rangle + (\Delta\psi)g\Big).$$

In particular, we define $L_\psi(U(t)f)$ and $L_\psi(U_0(t)f)$ and will write simply $L_\psi U(t)$ and $L_\psi U_0(t)$. It is easy to see that we have the following equalities:

$$(1-\psi)U(t) = U_0(t)(1-\psi) + \int_0^t U_0(t)L_\psi U(t-s)ds,$$

$$U(t)(1-\psi) = (1-\psi)U_0(t) + \int_0^t U(t-s)L_\psi U_0(s)ds.$$

Applying these equalities, we get

$$U(t) = U(t)\psi + (1-\psi)U_0(t) + \int_0^t \psi U(t-s)L_\psi U_0(s)ds$$

$$+\int_0^t U_0(t-s)(1-\psi)L_\psi U_0(s)ds + \int_0^t\int_0^{t-s} U_0(\tau)L_\psi U(t-s-\tau)L_\psi U_0(s)dsd\tau$$

$$= \psi U(t)\psi + U_0(t)\psi(1-\psi) + (1-\psi)U_0(t) + \int_0^t \psi U(t-s)L_\psi U_0(s)ds$$

$$+\int_0^t U_0(s)L_\psi U(t-s)\psi ds + \int_0^t U_0(t-s)(1-\psi)L_\psi U_0(s)ds$$

$$+\int_0^t\int_0^{t-s} U_0(\tau)L_\psi U(t-s-\tau)L_\psi U_0(s)dsd\tau.$$

Now let $z \in \mathbb{C}$ be such that $|z| > 1$. Let $g \in C_0^\infty(B_{a+2})$ be a cut-off function equal to 1 on B_{a+1}. We choose the projectors $P_\pm^b = P_\pm$ so that

$$P_\pm\psi = \psi = \psi P_\pm,\ P_\pm g = g = gP_\pm.$$

Next we fix $b > 0$ and the projectors $P_\pm$ with these properties and note that $gL_\psi = L_\psi = L_\psi g$. Let $T_0 > 0$ be chosen so that $P_+U_0(t)P_- = 0$ for $t \ge T_0$. Given a $t > 0$, we have

$$(Z(t)-z)^{-1} = -\sum_{j=0}^\infty z^{-j-1}P_+U(jt)P_-$$

$$= P_+\psi(U(t)-z)^{-1}\psi P_- - \sum_{jt\le T_0} z^{-j-1}P_+U_0(jt)\psi(1-\psi)P_-$$

$$- \sum_{jt \leq T_0} z^{-j-1} P_+ (1-\psi) U_0(jt) P_-$$
$$- \int_0^{T_0} P_+ U_0(s) L_\psi (U(t)-z)^{-1} \Phi U(-s) \psi P_- ds$$
$$- \int_0^{T_0} P_+ \psi (U(t)-z)^{-1} \Phi U(-s) L_\psi U_0(s) P_- ds$$
$$- \sum_{jt \leq T_1} \int_0^{\min(jt,T_0)} z^{-j-1} P_+ U_0(jt) \Phi U_0(-s)(1-\psi) L_\psi U_0(s) P_- ds$$
$$- \int_0^{T_0} \int_0^{T_0} P_+ U_0(\tau) L_\psi (U(t)-z)^{-1} \Phi_1 U(-s-\tau) L_\psi U_0(s) P_- ds d\tau + G(z)$$

with a function $G(z)$ holomorphic for $z \neq 0$. Here Φ and Φ_1 are cut-off functions with compact support determined by the finite speed of propagation so that

$$(1-\Phi)U_0(-s)g = 0 \text{ for } 0 \leq s \leq T_0,$$
$$(1-\Phi_1)U(-t)g = 0 \text{ for } 0 \leq t \leq 2T_0.$$

Finally, $T_1 > 0$ is chosen so that $P_+ U(t) \Phi = 0$ for $t \geq T_1$. The terms in the above presentation of $(Z(t)-z)^{-1}$ given by finite sums are holomorphic functions with respect to z. Consequently, if

$$\lim_{z \to z_0, |z|>1} \|\Psi (U(t)-z)^{-1} \Psi\| < \infty$$

for $\Psi \in C_0^\infty(x \in \mathbb{R}^n : |x| \leq c+1)$ and equal to 1 for $|x| \leq c$ for some suitably large and fixed constant $c > 0$, we conclude that $(Z(t)-z)^{-1}$ is not singular at $z_0 \in \mathbb{S}^1$. Combining this argument with the fact under the condition (9) we have $\mathbb{S}^1 \subset \sigma(Z(t))$ for almost $t > 0$, we obtain the following

Theorem 6. *Assume the condition* (9) *fulfilled. Then for almost all* $t \in]0, \infty[$ *and all* $z_0 \in \mathbb{S}^1$ *we have*

$$\lim_{z \to z_0, |z|>1} \|\Psi (U(t)-z)^{-1} \Psi\| = +\infty.$$

This result is important for the analysis of the analytic continuation of the cut-off resolvent $U_\chi(z) = \chi(U(T,0)-z)^{-1}\chi$ of the monodromy operator $U(T,0)$ related to the propagator $U(t,s)$ for time-periodic perturbations of the wave equation. In particular,we conclude that for trapping periodically moving obstacles we have not a meromorphic continuation of $U_\chi(z)$ from $\{z \in \mathbb{C} : |z| \geq A \gg 1\}$ across the unit circle $\mathbb{S}$.

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