SEMICLASSICAL RESONANCES AND TRACE FORMULAE FOR NON-SEMI-BOUNDED HAMILTONIANS

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1. Introduction

The formulas relating the scattering resonances and the trace of some functions of the perturbed and unperturbed operators play an important role in the scattering theory and in the analysis of the distributions of resonances. For compactly supported perturbations of the Laplacian in odd dimension a formula connecting the trace of the wave group to the resonances was proved by Lax-Phillips [24] and with successive extension by Bardos-Guillot-Ralston [3], Melrose [26], Sjöstrand-Zworski [39], Zworski [45].

Recently a substantial progress has been given in the analysis of the Schrödinger operator with long-range perturbations going to 0 as $|x| \longrightarrow +\infty$ and the works around the trace formulae generated many results on the upper and lower bounds of resonances, the Breit-Wigner approximation and the Weyl-type asymptotics of the spectral shift function (see [36], [37], [28], [29], [6], [7], [9], [4], [5] and the references given there). The approach developed in these works cannot be applied directly to non semi-bounded Hamiltonians as Stark Hamiltonians like $P_2(h) = -h^2\Delta + V(x) + x_1$ since the symbol $|\xi|^2 + x_1 + V(x)$ does not converge to $|\xi|^2$ as $|x| \to +\infty$ and the operator $P_2(h)$ is not elliptic.

We generalize for non semi-bounded Schrödinger type operators the result of [6] proving a representation of the derivative of the spectral shift function $\xi(\lambda, h)$ related to the semi-classical resonances. We obtain the same result for Stark Hamiltonians $P_2(h)$. Also we examine the resonances of the two-dimensional Schrödinger operator $P_1(B,\beta) = (D_x - By)^2 + D_y^2 + \beta x + V(x,y)$, $B > 0, \beta > 0$, with constant magnetic and electric fields. We define the resonances of $P_1(B,\beta)$ and the spectral shift function $\xi(\lambda)$, related to $P_1(B,\beta)$ and $P_0(B,\beta) = P_1(B,\beta) - V(x,y)$, without any restriction on B and β. For strong magnetic fields ($B \to \infty$) we obtain a representation of the derivative of $\xi(\lambda)$, a trace formula for $\mathrm{tr}(f(P_1(B,\beta) - f(P_0(B,\beta)))$ and an upper bound for the number of the resonances lying in $\{z \in \mathbb{C} : |\Re z - (2n-1)B| \leq \alpha B, \operatorname{Im} z \geq \mu \operatorname{Im}\theta\}$, $0 < \alpha < 1$, $0 < \mu < 1$, $\operatorname{Im}\theta < 0$. Moreover, for $B \to \infty$ we examine the free resonances domains and show that the resonances are included in the neighborhoods $\{z \in \mathbb{C} : |\Re z - (2n-1)B| \leq C_0\}$, where $(2n-1)B$ are the Landau levels and $C_0 > 0$ is a constant independent on B and $n \in \mathbb{N}^*$.

2. Long range perturbations

Consider two self-adjoint operators $L_j = L_j(h)$, $j = 1, 2$, in $L^2(\mathbb{R}^n)$ and assume that

$$L_j u = \sum_{|\nu| \leq 2} a_{j,\nu}(x;h)(hD_x)^\nu u, \ u \in C_0^\infty(\mathbb{R}^n)$$

with $a_{j,\nu}(x;h) = a_{j,\nu}(x)$ independent of h for $|\nu| = 2$ and $a_{j,\nu} \in C_b^\infty(\mathbb{R}^n)$ uniformly bounded with respect to h. Assume that there exists $C > 0$ such that

$$l_{j,0}(x,\xi) = \sum_{|\nu|=2} a_{j,\nu}(x)\xi^\nu \geq C|\xi|^2, \tag{2.1}$$

$$\sum_{|\nu|\leq 2} a_{j,\nu}(x;h)\xi^\nu \longrightarrow |\xi|^2,\ \ |x| \longrightarrow \infty \tag{2.2}$$

and suppose that

$$\Big| a_{1,\nu}(x;h) - a_{2,\nu}(x;h)\Big| \leq \mathcal{O}(1)\langle x\rangle^{-n-\epsilon_1},\ \epsilon_1 > 0,\ |\nu| \leq 2 \tag{2.3}$$

uniformly with respect to h.

Next we assume that there exist $\theta_0 \in]0, \frac{\pi}{2}[$, $\epsilon > 0$ and $R_1 > R_0$ so that the coefficients $a_{j,\nu}(x;h)$ of L_j can be extended holomorphically in x to

$$\Gamma = \{r\omega:\ \omega \in \mathbb{C}^n,\ \mathrm{dist}\,(\omega, S^{n-1}) < \epsilon, r \in \mathbb{C}, r \in e^{i[0,\theta_0]}]R_1, +\infty[\}$$

and (2.2), (2.3) extend to Γ. The *spectral shift function* $\xi(\lambda, h)$ is a distribution in $\mathcal{D}'(\mathbb{R})$ such that for $f(\lambda) \in C_0^\infty(\mathbb{R})$ we have

$$< \xi'(\lambda,h), f(\lambda) >= \mathrm{tr}\Big(f(L_2) - f(L_1)\Big).$$

We define the resonances $w \in \overline{\mathbb{C}}_-$ by the complex scaling method as the eigenvalues of the complex scaling operators $L_{j,\theta}$, $j = 1,2$ (see [38], [36], [37]). Denote by $\mathrm{Res}\ L_j(h)$, $j = 1,2$, the set of resonances of $L_j(h)$ and introduce the notation $[a_j]_{j=1}^2 = a_2 - a_1$.

Theorem 1. *Under the above assumptions let*

$$\Omega \subset\subset e^{]-2\theta,2\theta[}]0, +\infty[,\ 0 < \theta \leq \theta_0 < \pi/2$$

be an open simply connected set and let $W \subset\subset \Omega$ be an open simply connected relatively compact set which is symmetric with respect to $\mathbb{R}$. Assume that $J = \Omega \cap \mathbb{R}^+$, $I = \Omega \cap \mathbb{R}^+$ are intervals. Then for $\lambda \in I$ we have

$$\xi'(\lambda,h) = \frac{1}{\pi}\mathrm{Im}\, r(\lambda,h) + \Big[\sum_{w\in \mathrm{Res}\ L_j\cap\Omega,\ \mathrm{Im}\ w\neq 0} \frac{-\mathrm{Im}\ w}{\pi|\lambda - w|^2} + \sum_{w\in\mathrm{Res}\ L_j\cap J}\delta(\lambda - w)\Big]_{j=1}^2,$$

where $r(z,h) = g_+(z,h) - \overline{g_+(\overline{z},h)}$, $g_+(z,h)$ is a function holomorphic in Ω and $g_+(z,h)$ satisfies the estimate

$$|g_+(z,h)| \leq C(W)h^{-n},\ \ z \in W.$$

Remark. In the case of "black box" long-range perturbations, Theorem 1 is proved in [6] under the assumption that $L_j(h)$ are semi-bounded from below. This assumption were removed in [11]. The novelty in our approach is the proof of formula (3.5) based on a complex analysis argument related to the behavior of the functions $\sigma_\pm(z)$ in $\mathbb{C}_\pm$ (see Section 3).

3. Stark Hamiltonians

The Schrödinger operator describing the particles in a homogeneous electric field can be written in the form

$$P_1(h) = -h^2\Delta + \beta x_1,$$

where $\beta > 0$, $h > 0$ and $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$. The perturbation to the homogeneous electric field has the form

$$P_2(h) = -h^2\Delta + \beta x_1 + V(x), \tag{3.1}$$

where $V(x)$ is a real-valued $C^\infty(\mathbb{R}^n)$ function. We assume that

$$|\partial^\alpha V(x)| \le C_\alpha \langle x_1\rangle^{-s_1}\langle x'\rangle^{-s_2}, \ \forall\alpha \tag{3.2}$$

for $s_1 > \frac{n+1}{2}$ and $s_2 > n-1$, where $\langle x\rangle = (1+|x|^2)^{1/2}$.

The assumption (3.1) insures that the operator $f(P_2(h)) - f(P_1(h))$ is trace class for every $f \in C_0^\infty(\mathbb{R})$. We denote by $\xi(\lambda, h) \in \mathcal{D}'(\mathbb{R})$ the spectral shift function defined by

$$\langle \xi'(\lambda,h), f(\lambda)\rangle = \mathrm{tr}\Big(f(P_2(h)) - f(P_1(h))\Big).$$

The case $h = 1$ has been studied by many authors (see [2], [16], [17], [18], [21], [44], [33], [34], [35], [40]) and the scattering theory has been developed (see e.g. [2], [44], [33]). The problem of resonances has been examined mainly for $\beta \searrow 0$ and only the resonances close to a negative eigenvalue E_0 of $-\Delta + V(x)$ have been treated (see for instance [35], [40], [22], [21]).

In the following we suppose for simplicity that $\beta = 1$. To define the resonances, we assume that V admits a holomorphic extension in the x_1-variable into the region

$$\Gamma_{\delta_0,R} := \{z \in \mathbb{C} : \ \Re z < R, |\mathrm{Im}\, z| \le \delta_0\},$$

for some $\delta_0 > 0$ and $R > 0$. We also assume that (3.2) remains true on $\Gamma_{\delta_0,\,R}$ and

$$|\partial^\alpha V(x_1, x')| \le C_\alpha \langle |\Re x_1|\rangle^{-s_1}\langle x'\rangle^{-s_2}, \ \forall\alpha. \tag{3.3}$$

Let $\chi_0 \in C^\infty(\mathbb{R})$ be such that $\chi_0(t) = t$ for $t \le -\epsilon < 0$ and $\chi_0(t) = 0$ for $t \ge 0$. Set $v(t) = 1 - e^{\chi_0(t-R_0)}$, where $R_0 < R$ and for $\theta \in \mathbb{R}$ define $\Phi_\theta(x) = (x_1 + \theta v(x_1), x')$. We denote by $J_\theta(x) = \det\,[D\Phi_\theta(x)] = 1 + \theta v'(x_1)$ the Jacobian of $\Phi_\theta(x)$. Then, for $|\theta|$ small, $U(\theta)$ defined by $U(\theta)f(x) = J_\theta^{1/2}(x) f(\Phi_\theta(x))$ is unitary on $L^2(\mathbb{R}^n)$.

We have

$$P_{1,\theta}(h) := U(\theta)P_1(h)U(\theta)^{-1} = -h^2\nabla\Big(a_\theta(x)\nabla\Big) + x_1 + \theta v(x_1) + h^2 g_\theta(x),$$

$$P_{2,\theta}(h) := U(\theta)P_2(h)U(\theta)^{-1} = P_{1,\theta}(h) + V(\Phi_\theta(x)),$$

where $a_\theta(x) = (a_{\theta,i,j}(x))_{i,j}$ is the diagonal matrix given by

$$a_{\theta,1,1}(x) = (1 + \theta v'(x_1))^{-2}, a_{\theta,j,j}(x) = 1, j \ne 1.$$

By the analytic assumption, $P_{j,\theta}(h)$ admits a holomorphic extension in θ into a complex disk $D(0,\theta_0) \subset \mathbb{C}$ with radius $\theta_0 \le \delta_0$. Consider an open simply connected relatively compact domain

$$\Omega_\theta \subset\subset \{z \in \mathbb{C} : \ \Re z \le R_0 - 3\epsilon, \ \mathrm{Im}\, z \ge \alpha(1 - e^{-\epsilon})\mathrm{Im}\,\theta\},$$

where $\mathrm{Im}\,\theta < 0$, $0 < \alpha < 1$, $\epsilon > 0$.

Lemma 1. *There exist $\theta_0 > 0,\ h_0 > 0$ small enough such that for $\theta \in D(0, \theta_0)$ with $\operatorname{Im}\theta < 0,\ h \in]0, h_0]$ we have*

$$\sigma(P_{1,\theta}) \cap \Omega_\theta = \emptyset.$$

Moreover, the operator $(P_{2,\theta}(h) - z)$ is a Fredholm one with index 0 for all $z \in \Omega_\theta$.

Let $\theta \in D(0, \theta_0)$ and $\operatorname{Im}\theta < 0$. We say that $z \in \mathbb{C}$ is a *resonance* of $P_{2,\theta}(h)$ if

$$\dim \operatorname{Ker}\,(P_{2,\theta}(h) - z) > 0.$$

The resonances depend on h but they are independent on $\theta \in D(0, \theta_0)$ with $\operatorname{Im}\theta < 0$. Moreover, there are no resonances with $\operatorname{Im} z \geq 0$.

3.1. Representation of $\xi'(\lambda, h)$ for Stark Hamiltonians

Let $\Omega = \Omega_\theta$ be the domain introduced above and let W be an open relatively compact subset of Ω. Assume that W and Ω are symmetric with respect to $\mathbb{R}$ and independent of h and suppose that $J = \Omega \cap \mathbb{R}$, $I = W \cap \mathbb{R}$ are intervals.

Theorem 2. *Assume* (3.3) *with $s_1 > \frac{n+1}{2}$ and $s_2 > n - 1$. Then $\xi'(\lambda, h)$ is real analytic in I and for $\lambda \in I$ we have the representation*

$$\xi'(\lambda, h) = \frac{1}{\pi}\operatorname{Im} r(\lambda, h) + \sum_{z \in \operatorname{Res}(P_2(h)) \cap \Omega} \frac{-\operatorname{Im}\omega}{\pi|\lambda - z|^2},$$

where $r(z, h)$ is a function holomorphic in Ω and

$$|r(z, h)| \leq C(W) h^{-n}, z \in W \tag{3.4}$$

with $C(W) > 0$ independent on $h \in]0, h_0[$.

Main steps of the proof of Theorem 2

• Following the approach of Sjöstrand [37], we construct an operator $\widehat{P}_{2,\theta}(h) : \mathcal{D} \to L^2(\mathbb{R}^n)$ with the following properties:

$$K = \widehat{P}_{2,\theta}(h) - P_{2,\theta}(h) \text{ has rank } \mathcal{O}(h^{-n}),$$

$$(\widehat{P}_{2,\theta}(h) - z)^{-1} = \mathcal{O}(1) : L^2(\mathbb{R}^n) \to \mathcal{D},$$

uniformly on $z \in \overline{\Omega}$. Let $m > n/2$ and define for $\pm\operatorname{Im} z > 0$ the functions

$$\sigma_\pm(z) = (z^2 + 1)^m \times \operatorname{tr}\Big[(P_j(h) - i)^{-m}(P_j(h) + i)^{-m}(z - P_j(h))^{-1}\Big]_{j=1}^{2}.$$

• We prove that for $f \in C_0^\infty(\mathbb{R})$ we have

$$\langle \xi', f \rangle = \lim_{\epsilon \searrow 0} \frac{i}{2\pi} \int f(\lambda)\Big[\sigma_+(\lambda + i\epsilon) - \sigma_-(\lambda - i\epsilon)\Big] d\lambda. \tag{3.5}$$

Proposition 1. *There exists a function $a_+(z, h)$ holomorphic in Ω such that for $z \in \Omega \cap \operatorname{Im} z > 0$ we have*

$$\sigma_+(z) = \operatorname{tr}\Big((P_2 - z)^{-1} K (\widehat{P}_2 - z)^{-1}\Big) + a_+(z, h).$$

Moreover, $|a_+(z, h)| \leq C(\Omega) h^{-n},\ z \in \Omega.$

Next, to obtain a meromorphic continuation of $\sigma_+(z)$ through the real axis, it suffices to do this for the trace involving K. Setting $\tilde{K} = K(z - \widehat{P}_2)^{-1}$, we get the representation

$$-\mathrm{tr}\Big((P_2 - z)^{-1} K(\widehat{P}_2 - z)^{-1}\Big) = \mathrm{tr}\Big((1 + \tilde{K}(z))^{-1} \partial_z \tilde{K}(z)\Big) = \partial_z \log\det(1 + \tilde{K}(z)),$$

and the resonances of P_2 are precisely the zeros of the function $z \to \det(1 + \tilde{K}(z))$.

Now, Theorem 2 is a simple consequence of (3.5), Proposition 1 and the above equality. For the details we refer to [37], [6] and [11].

3.2 Applications

Using Theorem 2 and repeating the proof in [29], [6], we obtain the following local trace formula in the spirit of Sjöstrand [36], [37]

Theorem 3. *Assume that $P_j(h)$, $j = 1, 2$, satisfy the assumptions of Sections 3.1, and let Ω be as in Theorem 2. Suppose that f is holomorphic on a neighborhood of Ω and that $\psi \in C_0^\infty(\mathbb{R})$ satisfies*

$$\psi(\lambda) = \begin{cases} 0, & d(I, \lambda) > 2\epsilon, \\ 1, & d(I, \lambda) < \epsilon, \end{cases}$$

where $\epsilon > 0$ is sufficiently small. Then

$$\mathrm{tr}\Big[(\psi f)(P_j(h))\Big]_{j=1}^{2} = \sum_{z \in \mathrm{Res}\, P_2(h) \cap \Omega} f(z) + E_{\Omega, f, \psi}(h) \tag{3.6}$$

with

$$|E_{\Omega,f,\psi}(h)| \le M(\psi, \Omega) \sup\, \{|f(z)| \ : \ 0 \le d(\Omega, z) \le 2\epsilon\, , \ \mathrm{Im}\, z \le 0\} h^{-n}\, .$$

For the application of the above results to the Weyl-type asymptotics, we need the following weak asymptotics :

Theorem 4. *Assume that V satisfies (3.2) with $s_1 > \frac{n+1}{2}$ and $s_2 > n - 1$ and suppose that* $\mathrm{supp}\, V \subset \{x \in \mathbb{R}^n : \ x_1 > R\}$ *for some $R \in \mathbb{R}$. Then for $f \in C_0^\infty(\mathbb{R})$ we have*

$$\mathrm{tr}(f(P_2(h)) - f(P_1(h))) \sim \sum_{j=0}^{\infty} a_j h^{j-n}, \ h \searrow 0,$$

with

$$a_0 = -(2\pi)^{-n} \int_{\mathbb{R}^{2n}} (\partial_{x_1} V(x)) f(|\xi|^2 + x_1 + V(x)) dx d\xi.$$

Theorem 5. *In addition to the assumptions of Theorem 4 suppose that $p_2(x, \xi) = |\xi|^2 + V(x) + x_1$ is not critical on $\{(x, \xi) : \ p_2(x, \xi) = \tau\}$ for all $\tau \in [E_0, E_1]$, $E_0 < E_1$. Then there exist $C_0 > 0$ and h_0 small enough such that for $\theta \in C_0^\infty(]-\frac{1}{C_0}, \frac{1}{C_0}[;\, \mathbb{R})$, $\theta = 1$ in a neighborhood of 0, $f \in C_0^\infty(]E_0, E_1[)$ and $h \in]0, h_0]$ we have for $\forall m \in \mathbb{N}$, $\forall N \in \mathbb{N}$*

$$\mathrm{tr}\Big(\Big[\hat{\theta}_h(\tau - P_j(h)) f(P_j(h))\Big]_{j=1}^{2}\Big) = (2\pi h)^{-n} \Big(f(\tau) \sum_{j=0}^{N-1} \gamma_j(\tau) h^j + \mathcal{O}(h^N \langle\tau\rangle^{-m})\Big)$$

uniformly with respect to $\tau \in \mathbb{R}$. Here

$$\hat{\theta}_h(\tau) = (2\pi h)^{-1} \int e^{i\tau t/h} \theta(t) dt$$

is the semi-classical inverse Fourier transform of $\theta \in C_0^\infty(\mathbb{R})$.

In the case where the operators $P_i(h)$, $i = 1, 2$, are elliptic, Theorem 4 and Theorem 5 are well known (see [10], [19], [32] and the references given there). On the other hand, the approach developed in these papers cannot be applied directly to the case of non-elliptic operators. For potentials V satisfying $\mathrm{supp}_{x_1} V \subset [R, +\infty[$, we show that

$$\mathrm{tr}(f(P_2(h)) - f(P_1(h))) = -\mathrm{tr}\Big((\partial_{x_1} V) f(\tilde{P}_2(h))\Big) + \mathcal{O}(h^\infty),$$

where $\tilde{P}_2(h)$ is an elliptic operator. Consequently, the above theorems follows from the results for elliptic operators.

Theorem 6. *Assume the assumptions of Theorem 5 fulfilled and suppose that* $E_1 < \delta_1 = \inf\{x_1 \in \mathbb{R} : x_1 \in \mathrm{supp}_{x_1} V\}$. *Then there exists* $h_0 > 0$ *small enough such that for* $h \in]0, h_0]$ *we have*

$$\xi(\lambda, h) = (2\pi h)^{-n} c_0(\lambda) + \mathcal{O}(h^{-n+1}), \tag{3.7}$$

uniformly on $\lambda \in [E_0, E_1]$, *where*

$$c_0(\lambda) = -\frac{1}{n}\omega_n \int_{\mathbb{R}^n} \partial_{x_1} V(x)(\lambda - V(x) - x_1)_+^{n/2} dx$$

with $\omega_n = \mathrm{vol}\, S^{n-1}$. *Moreover, if*

$$\mathrm{Res}(P_2(h)) \cap \Big([E_0, E_1] - i[0, Nh\ln(1/h)]\Big) = \emptyset, \;\; h \in]0, h_N], \; \forall N \in \mathbb{N},$$

then we have

$$\xi'(\lambda, h) \sim \sum_{j=0}^{\infty} \gamma_j(\lambda) h^{j-n}, \;\; h \searrow 0$$

with $\gamma_0(\lambda) = c_0'(\lambda)$.

In the analysis of the counting function of eigenvalues, asymptotics like (3.7) are a simple consequence of Theorem 4, Theorem 5 and some Tauberian arguments. For the SSF, the main difficulty to establish (3.7) is that, in general, we do not know if $\xi(\lambda, h)$ is monotone with respect to λ and we cannot apply Tauberian theorems. To overcome this difficulty, we use the representation formula given in Theorem 2. In fact, Theorem 2 allows us to consider the integrals of the sum of the harmonic measures $\omega_{\mathbb{C}_-}(z)$ related to the resonances z, $\mathrm{Im}\, z < 0$, as a monotonic function and to apply a Tauberian argument as for the counting function of eigenvalues (see [27], [6]). Here the harmonic measures $\omega_{\mathbb{C}_-}(z)$ have the form

$$\omega_{\mathbb{C}_-}(z)(E) = -\frac{1}{\pi}\int_E \frac{\mathrm{Im}\, z}{|t - z|^2}\, dt, \;\; E \subset \mathbb{R}.$$

For the term involving $r(z, h)$ we use the estimate (3.4).

4. Stark Hamiltonian with strong magnetic field

The two-dimensional Schrödinger operator with electric and homogeneous magnetic fields can be written in the form

$$P_1(B, \beta) = P_0(B, \beta) + V(x, y), \quad P_0(B, \beta) = (D_x - By)^2 + D_y^2 + \beta x, \quad D_\nu = -i\frac{\partial}{\partial \nu},$$

where B and β are proportional to the strength of the homogeneous magnetic and electric fields. We assume that V satisfies (3.3) in $\Gamma_{\delta_0,+\infty} = \{z \in \mathbb{C} : |\mathrm{Im}\, z| \le \delta_0\}$, where now $x_1 = x$ and $x' = y$.

The essential spectrum of $P_1(B,0)$ and $P_0(B,0)$ are the same and it is well known that the spectrum of the operator $P_0(B,0)$ is given by

$$\cup_{n=1}^{\infty}\{(2n-1)B\}.$$

The numbers $\lambda_n = (2n-1)B$, $n \in \mathbb{N}^*$, called Landau levels, are the eigenvalues of infinite multiplicity (see [1]). Outside the Landau levels we have discrete eigenvalues caused by the potential V. The presence of electric field creates resonances which will be characterized as the eigenvalues of a distorted operator $P_1(B,\theta)$, $\operatorname{Im}\theta < 0$.

The spectral properties of the 2D Schrödinger operator $P_1(B,0)$ have been intensively studied in the last ten years. In the case of perturbations the Landau levels λ_n become accumulation points of the eigenvalues of $P_1(B,0)$ and the asymptotics of the function counting the number of the eigenvalues lying in a neighborhood of λ_n have been examined by many authors in different aspects (see [30], [19], [20], [31], [25] and the references given there). We would like to mention that it seems difficult to obtain a trace formula involving some summation over the eigenvalues close to a Landau level (see [23] for a result in this direction).

For the 2D Schrödinger operator with crossed magnetic and electric fields ($\beta \neq 0$) the situation completely changes and $\sigma_{\text{ess}}P_0(B,\beta) = \sigma_{\text{ess}}P_1(B,\beta) = \mathbb{R}$. For decreasing potentials the operator $P_1(B,\beta)$ can have embedded eigenvalues $\lambda \in \mathbb{R}$, but this question seems not sufficiently investigated. From physical point of view, it is expected that $V(x,y)$ creates resonances $z \in \mathbb{C}$, $\operatorname{Im} z \leq 0$, and it natural to define and to study the spectral shift function (SSF) $\xi(\lambda)$ related to $P_1(B,\beta)$ and $P_0(B,\beta)$. There are only few works treating magnetic Stark resonances. The case $B \to \infty$ was studied in [41], [42], while the case $\beta \to 0$ has been examined in [13], [14] (see also [40]). In these works the authors study mainly the resonances close to the eigenvalues of the non-perturbed operator $P_0(B,\beta)$. Moreover, in [41] the complex scaling and the definition of the resonances for $B \to \infty$ lead to some difficulties when we try to show that there are no resonances z with $\operatorname{Im} z > 0$ and this was an open problem in [41]. We can define SSF following the general setup [43] , but to our best knowledge the SSF for magnetic Stark Hamiltonians has been not investigated, as well as there are no trace formulae involving the resonances lying in a compact domain in $\mathbb{C}$.

4.1 Resonances for magnetic Stark Hamiltonians

From now on we assume that $\beta = 1$, and we write $P_j(B)$ instead of $P_j(B,\beta)$. Let $D(0,\theta_0)$ be the disk in $\mathbb{C}$ of center 0 and radius $\theta_0 > 0$. For $\theta \in D(0,\theta_0)$, $\theta_0 > 0$ small, we will use the dilatation $(x,y) \longrightarrow (x+\theta,y)$. More precisely, for $\theta \in \mathbb{R}$, consider the unitary operator

$$\mathcal{U}_\theta : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \quad f \to f(x+\theta,y).$$

It is clear that

$$\mathcal{U}_\theta^{-1} P_0(B) \mathcal{U}_\theta := \tilde{P}_0(B,\theta) = P_0(B) + \theta, \tag{4.1}$$

$$\mathcal{U}_\theta^{-1} P_1(B) \mathcal{U}_\theta = P_1(B,\theta) = P_0(B,\theta) + V(x+\theta,y). \tag{4.2}$$

Using the analytic assumption on V, we obtain

Lemma 2. *There exists $\theta_0 > 0$ such that the self-adjoint operator $P_1(B,\theta)$, defined for $\theta \in]-\theta_0,\theta_0[$, extends to an analytic type-A family of operators on $D(0,\theta_0)$ with the same domain $\mathcal{D}$ as that of $\tilde{P}_0(B,0)$. Moreover,*

$$\sigma_{\text{ess}}(P_1(B,\theta)) = \sigma_{\text{ess}}(P_0(B,\theta)) = \sigma(P_0(B,\theta)) = \{\lambda+\theta;\ \lambda \in \mathbb{R}\}.$$

Definition 1. *Let* $\operatorname{Im}\theta < 0$*. We say that* $z \in \Omega_\theta$ *is a resonance of* $P_1(B)$ *if*

$$\dim \operatorname{Ker}\ (P_1(B,\theta) - z) > 0.$$

As in section 3, we show that $P_1(B)$ has no resonances z with $\operatorname{Im} z > 0$, as well as, that the resonances in $\{z \in \mathbb{C} :\ \operatorname{Im} z > \operatorname{Im}\theta_2 > \operatorname{Im}\theta_1\}$ are independent of the choice of θ satisfying the condition $0 > \operatorname{Im}\theta_2 \geq \operatorname{Im}\theta \geq \operatorname{Im}\theta_1$. We define the multiplicity of a resonance z_0 by

$$m(z_0) = \operatorname{rank} \frac{1}{2\pi i} \int_{\gamma_\nu(z_0)} (z - \tilde{P}_1(B,\theta))^{-1} dz,$$

where $\gamma_\epsilon(z_0) = \{z = z_0 + \nu e^{i\varphi},\ 0 \leq \varphi < 2\pi\}$ and $\nu > 0$ is small enough. In the following we fix $\theta \in D(0,\theta_0)$ with $\operatorname{Im}\theta < 0$ and we denote the resonances of $P_1(B)$ by $\operatorname{Res} P_1(B)$.

Proposition 2. *Let* V *satisfy* (3.3) *with* $s_1 > 2$ *and* $s_2 > 1$ *and assume that*

$$1 + \partial_x V(x,y) > 0.$$

Then, there exists θ_0 , $\operatorname{Im}\theta_0 < 0$, *such that* $P_1(B)$ *has no resonances in* Ω_{θ_0}.

Proof. First, since $\partial_x V(x,y)$ tends to 0 when $|(x,y)|$ tends to infinity, it follows from our assumptions that

$$1 + \partial_x V(x,y) \geq \eta > 0,$$

uniformly on $(x,y) \in \mathbb{R}^2$. For u in the domain of $P_0(B)$ we have

$$-\operatorname{Im}\left((P_1(B,\theta) - z)u, u\right) = (\operatorname{Im} z - \operatorname{Im}\theta)\|u\|^2 - \operatorname{Im}\left(V(\cdot + \theta, \cdot)u, u\right).$$

Applying Taylor formula for the function $\theta \mapsto V(x+\theta, y)$, we obtain

$$\operatorname{Im} V(x+\theta, y) = \operatorname{Im}\theta\, \partial_x V(x + \Re\theta, y) + \mathcal{O}(|\operatorname{Im}\theta|^2).$$

Thus

$$-\operatorname{Im}\left((P_1(B,\theta) - z)u, u\right) = \operatorname{Im} z\|u\|^2 - \operatorname{Im}\theta\left((1 + \partial_x V(\cdot + \Re\theta, \cdot))u, u\right) + \mathcal{O}(|\operatorname{Im}\theta|^2)\|u\|^2.$$

Next, we choose $\operatorname{Im}\theta < 0$ small enough, and using the above inequality we get the proposition. □

4.2. Representation of the derivative of the spectral shift function for strong magnetic fields

In this section we will examine the case of strong magnetic field characterized by $B \to \infty$. For simplicity we assume $\theta \in i\mathbb{R}$. First, by using a symplectic change of variables (see [8], [19], [15]), there exists an unitary operator U such that

$$\tilde{P}_0(B,\theta) = U^{-1} P_0(B,\theta) U = B(D_y^2 + y^2) + x + \theta - \frac{1}{4B^2},$$

$$\tilde{P}_1(B,\theta) = U^{-1} P_1(B,\theta) U = \tilde{P}_0(B,\theta) + V^\omega,$$

where

$$V^\omega := V^\omega(x + \theta - B^{-1/2} D_y - \frac{1}{2B^2}, B^{-1/2} y + B^{-1} D_x).$$

Let φ_n be the n-th real normalized Hermite function given by

$$(D_y^2 + y^2)\varphi_n = (2n+1)\varphi_n,\ \|\varphi_n\| = 1,\ n \in \mathbb{N}.$$

Fix $0 < \alpha < 2$, $0 < \alpha_1$ and $0 < \mu < 1$. Set

$$\Omega_n = \{z \in \mathbb{C} : |\Re z - (2n+1)B| \leq \alpha B,\ \alpha_1 B \geq \operatorname{Im} z \geq \mu \operatorname{Im}\theta\}\,.$$

Let Π be the spectral projection corresponding to eigenspace generated by φ_n.

Proposition 3. *For $B \gg 1$ sufficiently large and $z \in \Omega_n$ the operator $\Big((I-\Pi)\tilde{P}_1(B,\theta)(I-\Pi) - z\Big)^{-1}(I-\Pi)$ is well defined and there exists a constant $\gamma > 0$, independent on B, such that*

$$\|\Big((I-\Pi)\tilde{P}_1(B,\theta) - z\Big)(I-\Pi)u\| \geq \gamma|\operatorname{Im}\theta|\|(I-\Pi)u\|,\ u \in \mathcal{D} \tag{4.3}$$

uniformly with respect to $z \in \Omega_n$.

The existence of double characteristics of the operator $(D_x - By)^2 + D_y^2$ which is not globally elliptic, combined with the Stark effects caused by x, lead to several difficulties. In particular, the proof of Proposition 3 is rather technical and too long and we refer to Proposition 3 in [12] for more details.

Let us introduce the following operators

$$L_1(B,\theta) = (I-\Pi)\Big(B(D_y^2 + y^2) + x + \theta - \frac{1}{4B^2} + V^\omega\Big)(I-\Pi)\,,$$

$$L_2(B,\theta) = \Pi\Big(B(D_y^2 + y^2) + x + \theta - \frac{1}{4B^2}\Big)\Pi\,,$$

$$W^\omega = (I-\Pi)V^\omega\Pi + \Pi V^\omega(I-\Pi) + \Pi V^\omega\Pi\,.$$

It clear that

$$L_1(B,\theta) + L_2(B,\theta) - z + W^\omega = \tilde{P}_1(B,\theta) - z\,.$$

The operator $\tilde{L}(B,\theta) - z = L_1(B,\theta) + L_2(B,\theta) - z$ is invertible for $z \in \Omega_n$. In fact, we have

$$\|(\tilde{L}(B,\theta) - z)u\|^2 = \|(L_1(B,\theta) - z)(I-\Pi)u\|^2 + \|(L_2(B,\theta) - z)\Pi u\|^2\,.$$

For the first term at the right hand side we apply Proposition 3, while for the second one we estimate the imaginary part of $(L_2(B,\theta) - z)\Pi u, \Pi u)$. Thus for $z \in \Omega_n$ we obtain

$$\|(\tilde{L}(B,\theta) - z)u\|^2 \geq \gamma_1\|(I-\Pi)u\|^2 + \gamma_2\|\Pi u\|^2 \geq \gamma_3\|u\|^2,\ \gamma_j > 0,\ j = 1,2,3\,.$$

Since $[\Pi, V^\omega] = \mathcal{O}(B^{-1/2})$, for B large enough the operator

$$L(B,\theta) - z = \tilde{L}(B,\theta) + (I-\Pi)V^\omega\Pi + \Pi V^\omega(I-\Pi) - z$$

is invertible for $z \in \Omega_n$. On the other hand, by the h-pseudodifferential calculus (see for instance [10]) $K = \Pi V^\omega \Pi$ is an h- pseudodifferential operator in $L^2(\mathbb{R}^2)$, and

$$\|K\|_{\mathrm{tr}} \leq CB \tag{4.4}$$

with a constant $C > 0$, independent on B. Thus we have the following

Theorem 7. *Let B be sufficiently large. Then for $z \in \Omega_n$ we have*

$$z - \tilde{P}_1(B,\theta) = z - L(B,\theta) - K \tag{4.5}$$

and the operator $z - L(B,\theta)$ is invertible for $z \in \Omega_n$.

Denote by $\xi(\lambda, B)$ the spectral shift function related to operators $P_1(B)$, $P_0(B)$. Let $\Omega \subset \Omega_n$ and let W be an open relatively compact subset of Ω. Suppose that $J = \Omega \cap \mathbb{R}$, $I = W \cap \mathbb{R}$ are intervals. Now, repeating the proof of Theorem 2 and using the above theorem, we obtain the following

Theorem 8. *Let V satisfy (3.3) with $s_1 > 2$ and $s_2 > 1$. Then for B sufficiently large and $\lambda \in I$ we have the representation*

$$\xi'(\lambda, B) = \frac{1}{\pi}\mathrm{Im}\, r(\lambda, B) + \sum_{z \in \mathrm{Res}\,(P_1(B)) \cap \Omega,\ \mathrm{Im}\, z<0} \frac{-\mathrm{Im}\, z}{\pi|\lambda - z|^2} + \sum_{z \in \sigma_{pp}(P_1(B)) \cap J} \delta(\lambda - z), \tag{4.6}$$

where $r(z, B)$ is a function holomorphic in Ω and

$$|r(z, B)| \leq C(W)B,\ z \in W. \tag{4.7}$$

According to Proposition 3, the invertibility of $(\tilde{P}_1(B, \theta) - z)$ for $z \in \Omega_n$ is reduced to the invertibility of $\Pi(\tilde{P_1}(B, \theta) - z)\Pi + \mathcal{O}(B^{-1/2})$. Hence, considering a suitable Grushin problem, we can reduce the spectral study of $\tilde{P}_1(B, \theta)$ in Ω_n to the study of a h-pseudodifferential operator $E_{-+}(z, \theta, h)$. Moreover, the leading term of the symbol of $E_{-+}(z, \theta, h)$ can be explicitly calculated (see [12], section 6). In particular, by using the formula of $E_{-+}(z, \theta, h)$, we obtain the following

Proposition 4. *Let $0 < \mu < 1$, $n \in \mathbb{N}$ be fixed. Then there exists a constant $C_0 > 0$, independent on B, and B_n such that for $B \geq B_n$, the operator $P_1(B)$ has no resonances z lying in the domain*

$$\{z \in \mathbb{C} :\ C_0 \leq |\Re z - (2n-1)B| \leq B,\ \mathrm{Im}\, z \geq \mu\, \mathrm{Im}\, \theta\}.$$

Moreover, there are no resonances z with $\Re z < \alpha B$, $0 < \alpha < 1$.

Remark. Our results implies that the Landau levels λ_n are the only points that may play the role of attractors of resonances creating the gaps and free resonances regions. For fixed B it is proved in [13] that there are no resonances z of $P_1(B)$ with $|\Re z| \geq R_0(B) > 0$. In this direction the above proposition says that we have no resonances with negative real part.

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