SPECTRAL SHIFT FUNCTION AND RESONANCES FOR NON SEMI-BOUNDED AND STARK HAMILTONIANS

MOUEZ DIMASSI AND VESSELIN PETKOV

Abstract. We generalize for non semi-bounded Schrödinger type operators the result of [8] proving a representation of the derivative of the spectral shift function $\xi(\lambda, h)$ related to the semi-classical resonances. For Stark Hamiltonians $P_2(h) = -h^2\Delta + \beta x_1 + V(x)$, $\beta > 0$, we obtain the same result as well as a local trace formula. We establish an upper bound $\mathcal{O}(h^{-n})$ for the number of the resonances in a compact domain $\Omega \subset \mathbb{C}_-$ and we obtain a Weyl-type asymptotics of $\xi(\lambda, h)$ for $V \in C^\infty(\mathbb{R}^n)$ with $\mathrm{supp}_{x_1} V \subset [R, +\infty[$. Finally, we establish the existence of resonances in every h-independent complex neighborhood of E_0 if E_0 is an analytic singularity of a suitable measure related to V.

Résumé. On généralise pour des opérateurs de Schrödinger non semi-bornés le résultat de [8] en obtenant une représentation de la dérivée de la fonction de décalage spectral $\xi(\lambda, h)$ associée aux résonances semi-classiques. Pour des hamiltoniens de Stark $P_2(h) = -h^2\Delta + \beta x_1 + V(x)$, $\beta > 0$, on obtient le même résultat et aussi une formule de trace locale. On démontre une borne supérieure $\mathcal{O}(h^{-n})$ pour le nombre de résonances dans un domaine compact $\Omega \subset \mathbb{C}_-$ et on établit une asymptotique de Weyl pour $\xi(\lambda, h)$ dans le cas quand $V \in C^\infty(\mathbb{R}^n)$ a la propriété $\mathrm{supp}_{x_1} V \subset [R, +\infty[$. Finalement, on démontre qu'il existe des résonances dans chaque h-indépendant voisinage complexe de E_0 si E_0 est une singularité analytique d'une mesure convenable associée à V.

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1. Introduction

The main problem examined in this work is the relation between the spectral shift function $\xi(\lambda, h)$ and the resonances for the semiclassical Stark hamiltonian

$$P_2(h) = -h^2\Delta + \beta x_1 + V(x),$$

where $V(x) \in C^\infty(\mathbb{R}^n)$ is a real-valued potential decreasing as $|x| \longrightarrow +\infty$, $h > 0$ and $\beta > 0$. The spectrum of $P_2(h)$ is absolutely continuous and coincides with $\mathbb{R}$ (see [1], [15]). Without loss of generality, throughout the paper we suppose that $\beta = 1$ and we consider $P_2(h)$ as a perturbation of the operator $P_1(h) = -h^2\Delta + x_1$.

The case $h = 1$ has been studied by many authors (see [1], [14], [15], [17], [41], [31], [32], [34], [42]) and the scattering theory has been developed (see e.g. [1], [41], [31]). The problem of resonances has been examined mainly for $\beta \searrow 0$ and only the existence of resonances close to a negative eigenvalue E_0 of the operator $-\Delta + V(x)$ has been treated (see for instance [33], [34], [42], [22]).

Recently a substantial progress has been given in the analysis of the Schrödinger operator with long-range perturbations going to 0 as $|x| \longrightarrow +\infty$ and the works around the trace formulae generated many results on the upper and lower bounds of resonances, the Breit-Wigner approximation and the Weyl-type asymptotics of the spectral shift function (see [39], [36], [37], [26], [27], [2], [3], [8], [9] and the references given there). The approach developed in these works cannot be applied directly to Stark hamiltonians like $P_2(h)$, since the symbol $|\xi|^2 + x_1 + V(x)$ does not converge to $|\xi|^2$ as $|x| \to +\infty$ and the operator $P_1(h)$ is not elliptic. The spectral shift function (SSF) $\xi(\lambda)$ associated to $P_1(1)$ and $P_2(1)$ has been studied by Robert and Wang [32] for short-range perturbations $V(x)$ and by Korotaev and Pushinitski [23] for perturbations $V(x)$ with compact support. Nevertheless, there are no works treating the link between the derivative of SSF and the resonances via a local trace formula in the spirit of Sjöstrand [36] (see also [27], [8]). Moreover, there are no results on the upper bounds of the (semiclassical) resonances z lying in a compact domain $\Omega \subset \mathbb{C}$ as well as the works treating the lower bounds of the number of the resonances produced by an analytic singularity.

In this paper we deal with all these problems studying trapping perturbations of $P_1(h)$ in the semiclassical setup without the assumption $\beta \searrow 0$. We are inspired by the ideas and tools in [36], [37], [26], [27], [8]. In Section 3 we generalize in the semiclassical "black box" setup, introduced in [36], the result of [8] for non semi-bounded operators without any assumption on the spectrum of the operators $L_j(h)$. The novelty in our proof is Lemma 1 based on a complex analysis argument related to the behavior of the functions $\sigma_{\pm}(z)$ in $\mathbb{C}_{\pm}$. The rest of the proof follows with some modifications that of Theorem 1 in [8].

In Section 4 we introduce the SSF $\xi(\lambda, h)$ for Stark hamiltonians. Our purpose is to obtain a representation of $\xi'(\lambda, h)$ having the form involved in Theorem 1. The resonances of $P_2(h)$ are introduced in Section 5 by the method of analytic distortion applied in the classical case ($h = 1$) as $\beta \searrow 0$ in [42], [22] (see also [14], [17], [33], [34]). We study the resonances in a domain

$$\Omega = \Omega_{\theta,\alpha} \subset \{z \in \mathbb{C} : \operatorname{Re} z \leq R - 3\epsilon,\ \operatorname{Im} z \geq \alpha(1 - e^{-\epsilon}) \operatorname{Im} \theta\},$$

where $\operatorname{Im}\theta < 0$, $0 < \alpha < 1$, $\epsilon > 0$ and R are given by the analytic distortion. The resonances are determined as the eigenvalues of the distorted operator $P_{2,\theta}(h)$ in Ω. Following the approach of [37], we construct an operator $\widehat{P}_{2,\theta}(h)$ so that

$$P_{2,\theta}(h) - \widehat{P}_{2,\theta}(h) = K \ \text{ has rank } \mathcal{O}(h^{-n})$$

and $\|(\widehat{P}_{2,\theta}(h) - z)^{-1}\| = \mathcal{O}(1)$, uniformly for $z \in \overline{\Omega}$. This makes it possible to apply the argument of Theorem 1 and to exploit the estimates of $\det\Big(I + K(z - \widehat{P}_2)^{-1}\Big)$ given in [37]. The main result concerning the Stark operator $P_2(h)$ is Theorem 2 (Section 6). Moreover, we establish an upper bound

$$\#\{z \in \operatorname{Res} P_2(h),\ z \in \Omega\} \leq C(\Omega) h^{-n}$$

which seems to be the first result of this type for Stark hamiltonians (Proposition 2). In Section 7 we obtain a local trace formula in the spirit of Sjöstrand [37] (see also [27], [8]). We apply this trace formula to show the existence of $\mathcal{O}(h^{-n})$ resonances (Theorem 7) in a h-independent neighborhood of an analytic singularity of suitable measure related to V (see [35], [11] for similar results) . For this purpose we need an asymptotic expansion of the trace

$$\operatorname{tr}[f(P_2(h)) - f(P_1(h))] \sim \sum_{j=0}^{\infty} a_j h^{j-n},\ h \searrow 0 \tag{1.1}$$

for $f \in C_0^\infty(\mathbb{R})$.

In the case, where the operators $P_i(h),\ i = 1,2$ are elliptic, the asymptotics like (1.1) are well known (see [10], [30], [20] and the references cited there). On the other hand, the approach developed in [10], [30] and [20] cannot be applied directly to the case of non-elliptic operators. For potentials V satisfying

$$\operatorname{supp}_{x_1} V \subset [R, +\infty[, \tag{1.2}$$

our strategy in Section 7 will be to show that

$$\operatorname{tr}\Big(\Big[f(P_j(h))\Big]_{j=1}^{2}\Big) = -\operatorname{tr}\Big[(\partial_{x_1} V) f(\tilde{P}_2(h))\Big] + \mathcal{O}(h^\infty), \tag{1.3}$$

where $\tilde{P}_2(h)$ is an elliptic operator. This makes it possible to apply the results for elliptic operators. In Section 7, we consider also the case where f depends on the semi-classical parameter h and we obtain an asymptotic expansion (Theorem 5). To our best knowledge, such complete asymptotics in powers of h for the Stark hamiltonians are new. Combining the representation of $\xi'(\lambda, h)$ in Theorem 2 with Theorem 5, we are able for the class of potentials satisfying (1.2) to establish a Weyl-type formula for the spectral shift function with remainder $\mathcal{O}(h^{1-n})$. For this purpose we exploit the sum of harmonic measures related to the Breit-Wigner factors following the approach in [8], [12]. We notice that all previous results on Weyl-type asymptotics for Stark hamiltonians have been obtained in the classical case ($h = 1$) (see [32], [23]). We will discuss the case of short range potentials elsewhere.

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2. Preliminaries

We start with the abstract "black box" scattering assumptions introduced in [39], [36] and [37]. The operators $L_j(h) = L_j, j = 1,2,\ 0 < h \leq h_0$, are defined in domains $\mathcal{D}_j \subset \mathcal{H}_j$ of a complex Hilbert space $\mathcal{H}_j$ with an orthogonal decomposition

$$\mathcal{H}_j = \mathcal{H}_{R_0,j} \oplus L^2(\mathbb{R}^n \setminus B(0,R_0)),\ B(0,R_0) = \{x \in \mathbb{R}^n : |x| \leq R_0\},\ \ R_0 > 0,\ \ n \geq 2.$$

Below $h > 0$ is a small parameter and we suppose the assumptions satisfied for $j = 1,2$. We suppose that $\mathcal{D}_j$ satisfies

$$\mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)} \mathcal{D}_j = H^2(\mathbb{R}^n \setminus B(0,R_0)), \tag{2.1}$$

uniformly with respect to h in the sense of [36]. More precisely, equip $H^2(\mathbb{R}^n \setminus B(0,R_0))$ with the norm $\| < hD >^2 u\|_{L^2},\ < hD >^2 = 1 + (hD)^2$, and equip $\mathcal{D}_j$ with the norm $\|(L_j + i)u\|_{\mathcal{H}_j}$. Then we require that $\mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)} : \mathcal{D}_j \longrightarrow H^2(\mathbb{R}^n \setminus B(0,R_0))$ is uniformly bounded with respect to h and this map has a uniformly bounded right inverse.

Assume that

$$\mathbb{1}_{B(0,R_0)}(L_j + i)^{-1} \text{is compact} \tag{2.2}$$

and

$$(L_j u)|_{\mathbb{R}^n \setminus \overline{B(0,R_0)}} = Q_j\Big(u|_{\mathbb{R}^n \setminus \overline{B(0,R_0)}}\Big), \tag{2.3}$$

where Q_j is a formally self-adjoint differential operator

$$Q_j u = \sum_{|\nu|\leq 2} a_{j,\nu}(x;h)(hD_x)^{\nu} u, \tag{2.4}$$

with $a_{j,\nu}(x;h) = a_{j,\nu}(x)$ independent of h for $|\nu| = 2$ and $a_{j,\nu} \in C_b^{\infty}(\mathbb{R}^n)$ uniformly bounded with respect to h.

We assume also the following properties:

There exists $C > 0$ such that

$$l_{j,0}(x,\xi) = \sum_{|\nu|=2} a_{j,\nu}(x)\xi^{\nu} \geq C|\xi|^2, \tag{2.5}$$

$$\sum_{|\nu|\leq 2} a_{j,\nu}(x;h)\xi^{\nu} \longrightarrow |\xi|^2, \ |x| \longrightarrow \infty \tag{2.6}$$

uniformly with respect to h.

There exists $\overline{n} > n$ such that we have

$$\Big| a_{1,\nu}(x;h) - a_{2,\nu}(x;h) \Big| \leq \mathcal{O}(1)\langle x\rangle^{-\overline{n}} \tag{2.7}$$

uniformly with respect to h. This assumption will guarantee that for every $f \in C_0^{\infty}(\mathbb{R})$ the operator $f(L_1) - f(L_2)$ is "trace class near infinity".

There exist $\theta_0 \in]0, \frac{\pi}{2}[$, $\epsilon > 0$ and $R_1 > R_0$ so that the coefficients $a_{j,\nu}(x;h)$ of Q_j can be extended holomorphically in x to

$$\Gamma = \{r\omega;\ \omega \in \mathbb{C}^n,\ \text{dist}\,(\omega, S^{n-1}) < \epsilon,\ r \in \mathbb{C}, r \in e^{i[0,\theta_0]}]R_1, +\infty[\} \tag{2.8}$$

and (2.6), (2.7) extend to Γ.

Let $R > R_0$, $T_{\tilde{R}} = (\mathbb{R}/\tilde{R}\mathbb{Z})^n$, $\tilde{R} > 2R$. Set

$$\mathcal{H}_j^{\#} = \mathcal{H}_{R_0,j} \oplus L^2(T_{\tilde{R}} \setminus B(0,R_0))$$

and consider a differential operator

$$Q_j^{\#} = \sum_{|\nu|\leq 2} a_{j,\nu}^{\#}(x;h)(hD)^{\nu}$$

on $T_{\tilde{R}}$ with $a_{j,\nu}^{\#}(x;h) = a_{j,\nu}(x;h)$ for $|x| \leq R$ satisfying (2.3), (2.4), (2.5) with $\mathbb{R}^n$ replaced by $T_{\tilde{R}}$. Consider a self-adjoint operator $L_j^{\#} : \mathcal{H}_j^{\#} \longrightarrow \mathcal{H}_j^{\#}$ defined by

$$L_j^{\#} u = L_j \varphi u + Q_j^{\#}(1-\varphi)u,\ u \in \mathcal{D}_j^{\#},$$

with domain

$$\mathcal{D}_j^{\#} = \{u \in \mathcal{H}_j^{\#} :\ \varphi u \in \mathcal{D}_j,\ (1-\varphi)u \in H^2\},$$

where $\varphi \in C_0^{\infty}(B(0,R);[0,1])$ is equal to 1 near $\overline{B(0,R_0)}$.

Denote by $N(L_j^{\#}, [-\lambda,\lambda])$ the number of eigenvalues of $L_j^{\#}$ in the interval $[-\lambda,\lambda]$. Then we assume that

$$N(L_j^{\#}, [-\lambda,\lambda]) = \mathcal{O}\Big(\Big(\frac{\lambda}{h^2}\Big)^{n_j^{\#}/2}\Big),\ n_j^{\#} \geq n,\ \lambda \geq 1. \tag{2.9}$$

Given $f \in C_0^\infty(\mathbb{R})$, independent on h, and $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on $\overline{B(0, R_0)}$, we can define $\mathrm{tr}_{\mathrm{bb}}[f(L_j)]_{j=1}^2$, as in [36], [37], by the equality

$$\mathrm{tr}_{\mathrm{bb}}\Big(f(L_2) - f(L_1)\Big) = [\mathrm{tr}(\chi f(L_j)\chi + \chi f(L_j)(1-\chi) + (1-\chi)f(L_j)\chi)]_{j=1}^2$$
$$+\mathrm{tr}[(1-\chi)f(L_j)(1-\chi)]_{j=1}^2 .$$

Following [36], [37], we define the resonances $w \in \overline{\mathbb{C}}_-$ by the complex scaling method as the eigenvalues of the complex scaling operators $L_{j,\theta}$, $j = 1, 2$. We denote by $\mathrm{Res}\, L_j(h)$, $j = 1, 2$, the set of resonances and set $n^\# = \max\{n_1^\#, n_2^\#\}$. In the paper $[a_j]_{j=1}^2$ means $a_2 - a_1$ and C denotes a positive constant which may change from line to line.

The spectral shift function $\xi(\lambda, h) \in \mathcal{D}'(\mathbb{R})$ related to L_1, L_2 is defined by

$$\langle \xi'(\lambda, h), f(\lambda)\rangle = \mathrm{tr}_{\mathrm{bb}}\Big(f(L_2) - f(L_1)\Big),\ \ f \in C_0^\infty(\mathbb{R})$$

and our result concerning the derivative of $\xi(\lambda, h)$ is the following.

Theorem 1. *Assume that $L_j(h)$, $j = 1, 2$, satisfy assumptions* (2.1) – (2.9). *Let*

$$\Omega \subset\subset e^{i]-2\theta_0, 2\theta_0[}]0, +\infty[,\ 0 < \theta_0 < \pi/2,$$

be an open simply connected set and let $W \subset\subset \Omega$ be an open simply connected and relatively compact set which is symmetric with respect to $\mathbb{R}$. Assume that $J = \Omega \cap \mathbb{R}^+$, $I = W \cap \mathbb{R}^+$ are intervals. Then for $\lambda \in I$ we have the representation

$$\xi'(\lambda, h) = \frac{1}{\pi}\mathrm{Im}\, r(\lambda, h) + \Big[\sum_{\substack{w\in \mathrm{Res}\, L_j \cap\Omega,\\ \mathrm{Im}\, w\neq 0}} \frac{-\mathrm{Im}\, w}{\pi|\lambda - w|^2} + \sum_{w \in \mathrm{Res}\, L_j\cap J} \delta(\lambda - w)\Big]_{j=1}^2, \tag{2.10}$$

where $r(z, h) = g_+(z, h) - \overline{g_+(\overline{z}, h)}$, $g_+(z, h)$ is a function holomorphic in Ω and $g_+(z, h)$ satisfies the estimate

$$|g_+(z, h)| \le C(W)h^{-n^\#},\ \ z \in W \tag{2.11}$$

with $C(W) > 0$ independent on $h \in]0, h_0]$.

3. Representation of the derivative of the spectral shift function for non semi-bounded operators

Let $L_j, j = 1, 2$ be two operators satisfying the assumptions (2.1) - (2.9). Given $0 < \theta \le \theta_0 < \frac{\pi}{2}$, we choose $0 < \kappa < 1$ so that $2\theta < \kappa\pi$. Consider the functions

$$\sigma_\pm(z) = (z^2 + 1 - 2z\cos(\kappa\pi))^m\, \mathrm{tr}_{\mathrm{bb}}\Big[(L_j - e^{i\kappa\pi})^{-m}(L_j - e^{-i\kappa\pi})^{-m}(z - L_j)^{-1}\Big]_1^2,\ \ \pm\mathrm{Im}\, z > 0,$$

where $m > n/2$ is an integer. We will show below that $\sigma_\pm(z)$ is well defined and it is clear that we have

$$\sigma_+(z) = \overline{\sigma_-(\bar{z})},\ \mathrm{Im}\, z > 0.$$

Let $\Omega \subset\subset e^{i]-2\theta, 2\theta[}]0, +\infty[$ be a simply connected open relatively compact set such that $\Omega\cap\mathbb{R}^+ = J$ is an interval. The spectrum of $L_{j,\theta}$ outside of $e^{-2i\theta}[0, +\infty[$ consists of the negative eigenvalues of L_j and the eigenvalues in $e^{-i[0,2\theta[}]0, +\infty[$ (see [36]). We may choose $z_0 = e^{-i\kappa\pi}$, $0 < \kappa < 1$, so that z_0 and $\bar{z}_0$ are away from sp (L_j) and sp $(L_{j,\theta})$, $j = 1, 2$. Given a positive number $\delta > 0$, a trivial modification of the proof of Proposition 4.1 of Sjöstrand [37], yields that for all $z \in \Omega \cap \{z : \operatorname{Im} z \geq \delta\}$ we have

$$\operatorname{tr}_{\rm bb}\Big[(L_j - z)^{-1}(L_j - z_0)^{-m}(L_j - \bar{z}_0)^{-m}\Big]_{j=1}^2 \tag{3.1}$$
$$= \operatorname{tr}_{\rm bb}\Big[(L_{j,\theta} - z)^{-1}(L_{j,\theta} - z_0)^{-m}(L_{j,\theta} - \bar{z}_0)^{-m}\Big]_{j=1}^2,$$

where in the definition of the complex scaling operators $L_{j,\theta}$ the parameter ϵ_0 is chosen small enough.

Below we assume δ and θ fixed and we will drop in the notations L_j the index j writing $L_.$ when the properties are satisfied for both operators $L_j,\ j = 1, 2$. Following [37], Section 4, there exists an operator $\hat{L}_{.,\theta} : \mathcal{D}_. \longrightarrow \mathcal{H}_.$ so that

$$K_{.,\theta} = \hat{L}_{.,\theta} - L_{.,\theta} \text{ has rank } \mathcal{O}(h^{-n^\#})$$

and for all $N,\ M \in \mathbb{N}$ we have

$$K_{.,\theta} = \mathcal{O}(1) : \mathcal{D}(L_.^N) \longrightarrow \mathcal{D}(L_.^M).$$

Secondly, $K_{.,\theta}$ is compactly supported, that is if $\chi \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 on $B(0, R)$ for $R \geq R_0$ large enough, we have $K_{.,\theta} = \chi K_{.,\theta}\chi$ and, finally, for every $N \in \mathbb{N}$ we have

$$(\hat{L}_{.,\theta} - z)^{-1} = \mathcal{O}(1) : \mathcal{D}(L_.^N) \longrightarrow \mathcal{D}(L_.^{N+1}),$$

uniformly for $z \in \overline{\Omega}$. These properties imply for $z \in \Omega \cap \{\operatorname{Im} z > 0\}$ the representation

$$(L_{.,\theta} - z)^{-1} = (\hat{L}_{.,\theta} - z)^{-1} + (L_{.,\theta} - z)^{-1}K_{.,\theta}(\hat{L}_{.,\theta} - z)^{-1}. \tag{3.2}$$

The contributions related to the resolvent $(\hat{L}_{.,\theta} - z)^{-1}$ are examined in the following.

Proposition 1. *There exists a function $a_+(z, h)$ holomorphic in Ω such that for $z \in \Omega \cap \{\operatorname{Im} z > 0\}$ we have*

$$\sigma_+(z) = \operatorname{tr}\Big[(L_{j,\theta} - z)^{-1}K_{j,\theta}(\hat{L}_{j,\theta} - z)^{-1}\Big]_{j=1}^2 + a_+(z, h). \tag{3.3}$$

Moreover,

$$|a_+(z, h)| \leq C(\Omega)h^{-n^\#},\ \ z \in \Omega \tag{3.4}$$

with a constant $C(\Omega)$ independent on $h \in]0, h_0]$.

Proof. The proof is a modification of that of Proposition 2 in [8] and for the sake of completeness we will expose the main steps. According to (3.2), for $z \in \Omega \cap \{\operatorname{Im} z \geq \delta\}$ we have

$$\sigma_+(z) = \Big((z - z_0)(z - \bar{z}_0)\Big)^m \operatorname{tr}_{\rm bb}\Big[(\hat{L}_{j,\theta} - z)^{-1}(L_{j,\theta} - z_0)^{-m}(L_{j,\theta} - \bar{z}_0)^{-m}\Big]_{j=1}^2$$
$$+\Big((z - z_0)(z - \bar{z}_0)\Big)^m\Big[\operatorname{tr}\Big((L_{j,\theta} - z)^{-1}K_{j,\theta}(\hat{L}_{j,\theta} - z)^{-1}(L_{j,\theta} - z_0)^{-m}(L_{j,\theta} - \bar{z}_0)^{-m}\Big)\Big]_{j=1}^2 = A(z) + B(z).$$

From the resolvent equation we obtain

$$\Big((z - z_0)(z - \bar{z}_0)\Big)^m(L_{j,\theta} - z_0)^{-m}(L_{j,\theta} - \bar{z}_0)^{-m}(L_{j,\theta} - z)^{-1} = (L_{j,\theta} - z)^{-1} - \sum_{k=1}^m (z - \bar{z}_0)^{k-1}(L_{j,\theta} - \bar{z}_0)^{-k}$$
$$-(z - \bar{z}_0)^m \sum_{k=1}^m (z - z_0)^{k-1}(L_{j,\theta} - \bar{z}_0)^{-m}(L_{j,\theta} - z_0)^{-k}.$$

To treat $B(z)$ we use the cyclicity of the trace and the above equality and conclude that $B(z)$ is equal to $\operatorname{tr}\Big[(L_{j,\theta}-z)^{-1}K_{j,\theta}(\hat{L}_{j,\theta}-z)^{-1}\Big]_{j=1}^{2}$ modulo a function holomorphic in Ω and bounded by $\mathcal{O}(h^{-n^{\#}})$.

Now we pass to the analysis of $A(z)$. Our purpose is to show that $A(z)$ is holomorphic in Ω and bounded by $\mathcal{O}(h^{-n^{\#}})$. By construction, $(\hat{L}_{j,\theta}-z)^{-1}$ is holomorphic on Ω and for any cut-off function $\chi \in C_0^{\infty}(\mathbb{R}^n)$, $\chi = 1$ on $\overline{B(0,R_0)}$ with $\operatorname{supp}\chi \subset B(0,R_1)$ the operators $\chi(L_{j,\theta}-\bar{z}_0)^{-m}$, $(L_{j,\theta}-\bar{z}_0)^{-m}\chi$ are trace class ones. This implies that the function $\operatorname{tr}\Big((\hat{L}_{j,\theta}-z)^{-1}(L_{j,\theta}-z_0)^{-m}(L_{j,\theta}-\bar{z}_0)^{-m}\chi\Big)$ is holomorphic in Ω. On the other hand,

$$\begin{aligned}(L_{j,\theta}-\bar{z}_0)^{-m}(L_{j,\theta}-z_0)^{-m}(\hat{L}_{j,\theta}-z)^{-1} - (\hat{L}_{j,\theta}-z)^{-1}(L_{j,\theta}-z_0)^{-m}(L_{j,\theta}-\bar{z}_0)^{-m} \\ = (L_{j,\theta}-\bar{z}_0)^{-m}(L_{j,\theta}-z_0)^{-m}(L_{j,\theta}-z)^{-1}K_{j,\theta}(\hat{L}_{j,\theta}-z)^{-1} \\ -(L_{j,\theta}-z)^{-1}K_{j,\theta}(\hat{L}_{j,\theta}-z)^{-1}(L_{j,\theta}-z_0)^{-m}(L_{j,\theta}-\bar{z}_0)^{-m}.\end{aligned} \tag{3.5}$$

Consequently, for $\operatorname{Im} z > 0$ if $\chi_1 \in C_0^{\infty}(\mathbb{R}^n)$ is a cut-off function such that $\chi_1 \prec \chi$, applying the cyclicity of the trace once more, we get

$$\operatorname{tr}\Big(\chi_1(\hat{L}_{j,\theta}-z)^{-1}(L_{j,\theta}-z_0)^{-m}(L_{j,\theta}-\bar{z}_0)^{-m}(1-\chi)\Big) = 0.$$

Here and below $\psi \prec \varphi$ means that $\varphi(x) = 1$ on the support of $\psi(x)$. Thus it remains to examine

$$\tau_+(z) = \operatorname{tr}\Big[(1-\chi_1)(\hat{L}_{j,\theta}-z)^{-1}(1-\chi)(L_{j,\theta}-z_0)^{-m}(L_{j,\theta}-\bar{z}_0)^{-m}(1-\chi)\Big]_{j=1}^{2}.$$

Consider the operator $Q_{\cdot,\theta} = Q_{\cdot}|_{\Gamma_\theta}$ and note that for $\psi \in C^{\infty}$ supported away from $B(0,R_1)$ we have $L_{\cdot,\theta}\psi = Q_{\cdot,\theta}\psi$. Repeating the construction of $\hat{L}_{\cdot,\theta}$ in Section 4, [37], we can find an operator $\hat{Q}_{\cdot,\theta}: H^2(\Gamma_\theta) \longrightarrow L^2(\Gamma_\theta)$ so that

$$\hat{Q}_{\cdot,\theta} - Q_{\cdot,\theta} \text{ has rank } \mathcal{O}(h^{-n}),$$

the operator $\hat{Q}_{\cdot,\theta} - Q_{\cdot,\theta}$ is compactly supported and for $z \in \overline{\Omega}$ we have

$$(\hat{Q}_{\cdot,\theta}-z)^{-1} = \mathcal{O}(1) : D(Q_{\cdot}^{N}) \longrightarrow D(Q_{\cdot}^{N+1}),\ \forall N \in \mathbb{N}.$$

Moreover, for $\psi \in C^{\infty}$ supported away from $B(0,R_1)$ we have $\hat{L}_{\cdot,\theta}\psi = \hat{Q}_{\cdot,\theta}\psi$ and for $\chi \in C_0^{\infty}(\Gamma_\theta)$ equal to 1 on a sufficiently large set, $z \in \Omega$ and $\chi_1 \prec \chi_0 \prec \chi$ we obtain

$$\begin{aligned}(\hat{L}_{\cdot,\theta}-z)^{-1}(1-\chi) = (1-\chi_0)(\hat{Q}_{\cdot,\theta}-z)^{-1}(1-\chi) \\ +(\hat{L}_{\cdot,\theta}-z)^{-1}[\hat{Q}_{\cdot,\theta},\chi_0](\hat{Q}_{\cdot,\theta}-z)^{-1}(1-\chi).\end{aligned}$$

As above, we assume that $z_0 = e^{-i\kappa\pi}$ is chosen so that $z_0 \notin \operatorname{sp}(Q_j)$, $z_0 \notin \operatorname{sp}(Q_{j,\theta})$, $j = 1,2$. For simplicity of the notations below we omit the index θ. Repeating the argument of Section 4 in [8], we conclude that there exists a function $b(z,h)$, holomorphic in Ω, and bounded by $\mathcal{O}(h^{-n^{\#}})$, so that

$$\tau_+(z) = b(z,h) + \operatorname{tr}\Big[(1-\chi)(\hat{Q}_j-z)^{-1}(Q_j-z_0)^{-m}(Q_j-\bar{z}_0)^{-m}(1-\chi)\Big]_{j=1}^{2}. \tag{3.6}$$

We write

$$(\hat{Q}_2-z)^{-1}(Q_2-z_0)^{-m}(Q_2-\bar{z}_0)^{-m} - (\hat{Q}_1-z)^{-1}(Q_1-z_0)^{-m}(Q_1-\bar{z}_0)^{-m}$$

$$= (\hat{Q}_2 - z)^{-1}\Big[\Big((Q_2-z_0)^{-m} - (Q_1-z_0)^{-m}\Big)(Q_2-\bar{z}_0)^{-m} + (Q_1-z_0)^{-m}\Big((Q_2-\bar{z}_0)^{-m} - (Q_1-\bar{z}_0)^{-m}\Big)\Big]$$

$$\Big[(\hat{Q}_2 - z)^{-1} - (\hat{Q}_1 - z)^{-1}\Big](Q_1 - z_0)^{-m}(Q_1 - \bar{z}_0)^{-m} = I + II\,.$$

We treat these terms, as in [8], and we conclude that $\tau_+(z)$ is holomorphic in Ω and bounded by $\mathcal{O}(h^{-n^\#})$. To complete the proof we use an analytic continuation. $\square$

Next we will obtain a representation of the derivative $\xi'(\lambda, h)$. Let $f \in C_0^\infty(\mathbb{R}^+)$ and let $\tilde{f}(z) \in C_0^\infty(\mathbb{C})$ be an almost analytic extension of f. Set

$$g(z) = f(z)(z^2 + 1 - 2z\cos(\kappa\pi))^m.$$

Then

$$g(L_.) = -\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)(z^2+1-2z\cos(\kappa\pi))^m\ (z - L_.)^{-1}L(dz),$$

where $L(dz)$ denotes the Lebesgue measure on $\mathbb{C}$. Clearly,

$$f(L_.) = (L_. - e^{i\kappa\pi})^{-m}(L_. - e^{-i\kappa\pi})^{-m}g(L_.)$$

$$= -\frac{1}{\pi}\int \bar{\partial}_z\tilde{f}(z)(z^2+1-2z\cos(\kappa\pi))^m(L_. - e^{i\kappa\pi})^{-m}(L_. - e^{-i\kappa\pi})^{-m}(z-L_.)^{-1}L(dz)$$

which implies

$$\mathrm{tr_{bb}}\Big(f(L_2) - f(L_1)\Big) = -\frac{1}{\pi}\int \bar{\partial}_z\tilde{f}(z)(z^2+1-2z\cos(\kappa\pi))^m \tag{3.7}$$

$$\times\, \mathrm{tr_{bb}}\Big[(L_j - e^{i\kappa\pi})^{-m}(L_j - e^{-i\kappa\pi})^{-m}(z-L_j)^{-1}\Big]_1^2 L(dz).$$

We have $\sigma_\pm(z) = \mathcal{O}(h^{-n^\#}|\operatorname{Im} z|^{-2})$ and $\bar{\partial}_z\tilde{f} = \mathcal{O}(|\operatorname{Im} z|^2)$ so we may write the right hand side of (3.9) as

$$\langle \xi', f\rangle = \mathrm{tr_{bb}}\Big(f(L_2) - f(L_1)\Big)$$

$$= \lim_{\epsilon\searrow 0} -\frac{1}{\pi}\Big(\int_{\operatorname{Im} z>0}\bar{\partial}_z\tilde{f}(z)\sigma_+(z+i\epsilon)L(dz) + \int_{\operatorname{Im} z<0}\bar{\partial}_z\tilde{f}(z)\sigma_-(z-i\epsilon)L(dz)\Big).$$

According to Proposition 1, the function $\sigma_+(z+i\epsilon)$ (resp. $\sigma_-(z-i\epsilon)$) is holomorphic on $\{z\in\Omega:\ \operatorname{Im} z>0\}$ (resp. $\{z\in\Omega:\ \operatorname{Im} z<0\}$) and applying the Green formula we obtain the following

Lemma 1. *We have*

$$\langle\xi', f\rangle = \lim_{\epsilon\searrow 0}\frac{i}{2\pi}\int f(\lambda)\Big[\sigma_+(\lambda+i\epsilon) - \sigma_-(\lambda - i\epsilon)\Big]d\lambda$$

where the limit is taken in the sense of distributions.

Now Theorem 1 follows from the analysis of the singularities of $\sigma_+(z)$ for $\operatorname{Im} z \searrow 0$ which is a straightforward repetition of that in [8]. This competes the proof. $\square$

4. Spectral shift function for Stark Hamiltonians

The Schrödinger operator describing the particles in a homogeneous electric field can be written in the form

$$P_1(h) = -h^2\Delta + \beta x_1,$$

where $\beta > 0,\ h > 0$ and $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$.

For a perturbation to the homogeneous electric field, the corresponding Schrödinger operator has the form

$$P_2(h) = -h^2\Delta + \beta x_1 + V(x), \tag{4.1}$$

where $V(x)$ is a real-valued $C^\infty(\mathbb{R}^n)$ function. It is natural to assume that for $|x| \to \infty$ the potential $V(x)$ is small in comparison with βx_1. More precisely, we assume that

$$|\partial^\alpha V(x)| \leq C_\alpha \langle x_1 \rangle^{-s_1} \langle x' \rangle^{-s_2},\ \forall \alpha \tag{4.2}$$

for some positive constants $s_1 > 0,\ s_2 > 0$, where $\langle x \rangle = (1 + |x|^2)^{1/2}$.

There are two points of view of considering the hamiltonian (4.1). The first one, which is usually used for the study of resonances in Stark effect, is to examine $P_2(h)$ as a perturbation of $-h^2\Delta + V(x)$, (see [15], [17]). The second one, is to consider $P_2(h)$ as a perturbation of $P_1(h)$ just as in the scattering theory (see [1], [22], [42]). Here we will work with the later point of view with $\beta = 1$. Under assumption (4.2), it is well known that $P_2(h)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$, $\sigma(P_j(h)) = \mathbb{R},\ j = 1, 2$ and $P_j(h)$ have no eigenvalues [28], [14].

Lemma 2. *Let V satisfy (4.2) with $s_1 > \frac{n+1}{2}$ and $s_2 > n - 1$.*

i) For $k \in \mathbb{N}$ large enough the operator

$$(i - P_2(h))^{-k} - (i - P_1(h))^{-k}$$

is trace class one and its trace norm is $\mathcal{O}(h^{-n})$.

ii) Moreover, for $\operatorname{Im} z \neq 0$ we have

$$\|(i - P_2(h))^{-k}(z - P_2(h))^{-1} - (i - P_1(h))^{-k}(z - P_1(h))^{-1}\|_{\mathrm{tr}} = \mathcal{O}\Big(\frac{h^{-n}}{|\operatorname{Im} z|^2}\Big). \tag{4.3}$$

Proof. Taking $(k-1)$ derivatives in z in the resolvent identity

$$(z - P_2(h))^{-1} - (z - P_1(h))^{-1} = (z - P_2(h))^{-1} V (z - P_1(h))^{-1}$$

and setting $z = i$, we see that $(i - P_2(h))^{-k} - (i - P_1(h))^{-k}$ is a linear combination of terms

$$(i - P_2(h))^{-j} V (i - P_1(h))^{-(k+1-j)}$$

with $1 \leq j \leq k$. Hence, it suffices to prove that $(i - P_2(h))^{-l} V$ and $V(i - P_1(h))^{-l}$ are trace class ones for l large enough. On the other hand, by duality, we must show only that $V(i - P_j(h))^{-l}$ is trace class.

Since the operators $P_j(h), j = 1, 2$, are not elliptic, we cannot use the h-pseudodifferential calculus for the analysis of $(i - P_j(h))^{-l}$. To overcome this difficulty, we will decompose $V(i - P_j(h))^{-l}$ as a sum of three terms:

$$V(i - P_j(h))^{-l} = gV(i - P_j(h))^{-l} + (1 - g)V(1 - \gamma^w(x, hD_x))(i - P_j(h))^{-l}$$

$$+(1-g)V\gamma^w(x,hD_x)(i-P_j(h))^{-l} = F_{1,j}(h)+F_{2,j}(h)+F_{3,j}(h).$$

Here $g(x_1)\in C^\infty(\mathbb{R})$ with $g(x_1)=1$ for $x_1\geq -1$ and $g(x_1)=0$ for $x_1\leq -2$, and $\gamma(x,\xi)=\psi\Big(\frac{x_1+|\xi|^2}{\langle\xi\rangle}\Big)$, where $\psi(t)\in C_0^\infty([-2,2];\ [0,1])$, $\psi(t)=1$ on $[-1,1]$. We denote by $\gamma^w(x,hD_x)$ the h-pseudodifferential operator with symbol $\gamma(x,\xi)$.

On the support of g and $(1-\gamma)$, we can use the h-pseudodifferential calculus and prove that $F_{1,j}(h)$ (resp. $F_{2,j}(h)$) is an h-pseudodifferential operator with symbol in $S^0(m_1)$ (resp. $S^0(m_2)$), where

$$m_1(x,\xi)=\langle x_1\rangle^{-s_1}\langle x'\rangle^{-s_2}(1+|x_1|+|\xi|^2)^{-l},$$
$$m_2(x,\xi)=\langle x_1\rangle^{-s_1}\langle x'\rangle^{-s_2}\langle\xi\rangle^{-l}.$$

We denote by $S^0(m)$ the class of symbols

$$S^0(m)=\{a\in C^\infty(\mathbb{R}^{2n}):\ \partial_x^\alpha\partial_\xi^\beta a=\mathcal{O}_{\alpha,\beta}(m),\ \forall\alpha,\forall\beta\},$$

where m is an order function (see for instance, [10]). To justify this it is sufficient to observe that the operators $(P_j(h)-i)$ are elliptic for $x_1\in\operatorname{supp} g$ and $(x,\xi)\in\operatorname{supp}(1-\gamma)$ and to estimate the principal symbols of their inverse. Under our assumptions and for $l>n/2$ we have $m_i\in L^1(\mathbb{R}^{2n})$, $i=1,2$. Thus, it follows from Theorem 9.4 in [10] that $F_{i,j}(h)$, $i=1,2$, are trace class and $\|F_{i,j}(h)\|_{\mathrm{tr}}=\mathcal{O}(h^{-n})$.

Next, on the support of γ we have $|x_1+|\xi|^2|\leq 2\langle\xi\rangle$. So we can apply the h-pseudodifferentaial calculus to $V\gamma^w(x,hD_x)$ and we deduce

$$\|V\gamma\|_{\mathrm{tr}}\leq Ch^{-n}\int_{|x_1+|\xi|^2|\leq 2\langle\xi\rangle}\langle x_1\rangle^{-s_1}dx_1d\xi\int_{\mathbb{R}^{n-1}}\langle x'\rangle^{-s_2}dx'$$
$$\leq Ch^{-n}\Big(\int_{|x_1|<1,\ |\xi|\leq C}\langle x_1\rangle^{-s_1}dx_1d\xi+\int_1^\infty r^{n-2s_1}dr\Big)<+\infty$$

if $s_1>\frac{n+1}{2}$ and $s_2>n-1$. Consequently, $\|F_{3,j}(h)\|_{\mathrm{tr}}=\mathcal{O}(h^{-n})$ and this completes the proof of the first part of the lemma.

To obtain (4.3), it suffices to observe that the operator in the left hand side of (4.3) can be written in the form

$$\Big((i-P_2(h))^{-k}-(i-P_1(h))^{-k}\Big)(z-P_2(h))^{-1}+(i-P_1(h))^{-k}\Big((z-P_2(h))^{-1}-(z-P_1(h))^{-1}\Big)$$
$$=\Big((i-P_2(h))^{-k}-(i-P_1(h))^{-k}\Big)(z-P_2(h))^{-1}$$
$$-(z-P_1(h))^{-1}(i-P_1(h))^{-k}(P_2(h)-P_1(h))(z-P_2(h))^{-1}, \tag{4.4}$$

which together with the above estimates and the fact that $\|(z-P_j(h))^{-1}\|=\mathcal{O}(|\operatorname{Im} z|^{-1})$ yield (4.3). □

By using Helffer-Sjöstrand formula (see (3.9)) and the above lemma, we conclude that for $s_1>\frac{n+1}{2}$ and $s_2>n-1$, the operator $f(P_2(h))-f(P_1(h))$ is trace class for every $f\in C_0^\infty(\mathbb{R})$. We denote by $\xi'(\lambda,h)\in\mathcal{D}'(\mathbb{R})$ the spectral shift function related to the pair $(P_2(h),P_1(h))$ and defined by

$$\langle\xi'(\lambda,h),f(\lambda)\rangle=\mathrm{tr}\Big(f(P_2(h))-f(P_1(h))\Big).$$

5. Analytic distortion and resonances for Stark hamiltonians

To define the resonances, we will suppose that V admits a holomorphic extension in the x_1-variable into the region

$$\Gamma_{\delta_0,R} := \{z \in \mathbb{C} :\ \operatorname{Re} z < R, |\operatorname{Im} z| \leq \delta_0\}, \tag{5.1}$$

for some $\delta_0 > 0$ and $R > 0$. We also assume that (4.2) remains true on $\Gamma_{\delta_0,\,R}$ and

$$|\partial^\alpha V(x_1, x')| \leq C_\alpha \langle |\operatorname{Re} x_1|\rangle^{-s_1} \langle x'\rangle^{-s_2},\ \ \forall \alpha. \tag{5.2}$$

Let us recall the definition of the resonances for Stark hamiltonians by the method of analytic distortion (for more details we refer to [15], [17], [42]). Let $\chi_0 \in C^\infty(\mathbb{R})$ be such that $\chi_0(t) = t$ for $t \leq -\epsilon < 0$ and $\chi_0(t) = 0$ for $t \geq 0$. Set $v(t) = 1 - e^{\chi_0(t-R_0)}$, where $R_0 < R$ and for $\theta \in \mathbb{R}$ define

$$\Phi_\theta(x) = (x_1 + \theta v(x_1), x').$$

We denote by $J_\theta(x) = \det\,[D\Phi_\theta(x)] = 1 + \theta v'(x_1)$ the Jacobian of $\Phi_\theta(x)$. Then, for $|\theta|$ small, $U(\theta)$ defined by

$$U(\theta)f(x) = J_\theta^{1/2}(x) f(\Phi_\theta(x))$$

is unitary on $L^2(\mathbb{R}^n)$. A simple calculus shows that

$$P_{1,\theta}(h) := U(\theta)P_1(h)U(\theta)^{-1} = -h^2\nabla\Big(a_\theta(x)\nabla\Big) + x_1 + \theta v(x_1) + h^2 g_\theta(x),$$

$$P_{2,\theta}(h) := U(\theta)P_2(h)U(\theta)^{-1} = P_{1,\theta}(h) + V(\Phi_\theta(x)), \tag{5.3}$$

where $a_\theta(x) = (a_{\theta,i,j}(x))_{i,j}$ is the diagonal matrix given by

$$a_{\theta,1,1}(x) = (1 + \theta v'(x_1))^{-2}, a_{\theta,j,j}(x) = 1, j \neq 1,$$

and

$$g_\theta(x) = \frac{\theta}{2} v'''(x_1)\Big(a_{\theta,1,1}(x)\Big)^{3/2} - \frac{5}{4}\theta^2 v''(x_1)^2 \Big(a_{\theta,1,1}(x)\Big)^2.$$

By the analytic assumption, $P_{j,\theta}(h)$ admits a holomorphic extension in θ into a complex disk $D(0,\theta_0) \subset \mathbb{C}$ of center 0 and radius $\theta_0 \leq \delta_0$.

Below we set

$$G(x) := x_1 + V(x),\ \ b_\theta(x) := 1 - a_\theta(x).$$

The diagonal matrix $b_\theta(x) = (b_{\theta,i,j}(x))_{ij}$ has the form

$$b_{\theta,1,1}(x) = \Big(2\theta v'(x_1) + \theta^2 v'(x_1)^2\Big)(1 + \theta v'(x_1))^{-2} \text{ and } b_{\theta,j,j}(x) = 0 \text{ if } j > 1.$$

Clearly, from the definitions of $v(x_1)$ and Φ_θ, it follows that

$$\sup_{x\in\mathbb{R}^n} |b_{\theta,1,1}(x)G(x)| \leq C_0|\theta|,\ \ \ \sup_{x\in\mathbb{R}^n} |b_{\theta,1,1}(x)| \leq C_1|\theta| \tag{5.4}$$

and

$$\sup_{x\in\mathbb{R}^n} |G(\Phi_\theta(x)) - G(x)| \leq C_2|\theta|,\ \ \sup_{x\in\mathbb{R}^n} |g_\theta(x)| \leq C_3|\theta|, \tag{5.5}$$

where C_0, C_1, C_2, C_3 are independent on $\theta \in D(0,\theta_0)$.

Using (5.4) and the exponential decay properties of $v'(x_1)$, we obtain

$$\|h^2\nabla\Big(b_\theta\nabla\Big)u\|^2 \leq C_1|\theta|^2(\|P_1(h)u\|^2 + \|u\|^2 \leq C_0|\theta|^2\|(P_j(h) \pm i)u\|^2,\ j = 1,2 \tag{5.6}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$, where C_0 is independent on $h \in]0,1[$ and $\theta \in D(0,\theta_0)$.

We will use the notations $P_{\cdot}(h)$, $P_{\cdot,\theta}(h)$ for the operators $P_j(h)$, $P_{j,\theta}(h)$, $j = 1, 2$. From (5.5), (5.6) and (5.2) we deduce that

$$\|(P_{\cdot,\theta}(h) - P_{\cdot}(h))u\|^2 \leq C|\theta|^2 \|(P_{\cdot}(h) \pm i)u\|^2, \tag{5.7}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. Choose θ_0 small enough so that $C\theta_0^2 < 1$, where C is the constant at the r.h.s of (5.7). Therefore from (5.7) and the results on the perturbation of operators in [21] we deduce that $P_{\cdot,\theta}(h)$ is closed and $\mathcal{D}(P_{\cdot,\theta}(h)) = \mathcal{D}(P_1(h)) =: \mathcal{D}$ for all $\theta \in D(0, \theta_0)$. On the other hand, using the analytic assumption on V, we conclude that $\theta \to P_{\cdot,\theta}u$ is analytic for all $u \in \mathcal{D}$. Consequently, the self-adjoint operator $P_{j,\theta}$, defined for $\theta \in D(0, \theta_0) \cap \mathbb{R}$, extends to an analytic type-A family of operators on $D(0, \theta_0)$ with domain $\mathcal{D}$. Moreover, by an approximation it is easy to show that the estimate (5.7) remains true for $u \in \mathcal{D}$.

Now it is easy to see that for θ_0 small enough and $\theta \in D(0, \theta_0)$ we have $\pm i \notin \sigma(P_{\cdot,\theta}(h))$. Indeed,

$$P_{\cdot,\theta}(h) - i = \Big[I + (P_{\cdot,\theta}(h) - P_{\cdot}(h))(P_{\cdot}(h) - i)^{-1}\Big](P_{\cdot}(h) - i).$$

On the other hand,

$$\|(P_{\cdot,\theta}(h) - P_{\cdot}(h))(P_{\cdot}(h) - i)^{-1}u\| \leq C|\theta|\|u\|, \ \forall u \in L^2(\mathbb{R}^n)\,.$$

Thus $(P_{\cdot,\theta}(h) - i)$ is invertible and the same argument works for $(P_{\cdot,\theta}(h) + i)$.

Fix $\theta \in D(0, \theta_0)$ with $\theta_0 \leq \delta_0$, $\operatorname{Im}\theta < 0$ and fix the constants $R_0 > 0$ and $\epsilon > 0$ in the definition of $v(x_1)$. Consider an open simply connected relatively compact domain

$$\Omega_{\theta,\alpha} \subset \{z \in \mathbb{C} : \ \operatorname{Re} z \leq R_0 - 3\epsilon, \operatorname{Im} z \geq \alpha(1 - e^{-\epsilon}) \operatorname{Im}\theta\}, \ 0 < \alpha < 1.$$

We assume that $\Omega_{\theta,\alpha}$ is independent of h. The domain $\Omega_{\theta,\alpha}$ depends on θ and α but for simplicity of notation we will write below Ω instead of $\Omega_{\theta,\alpha}$.

Lemma 3. *There exist $\theta_0 > 0$, $h_0 > 0$ small enough such that for $\theta \in D(0, \theta_0)$ with $\operatorname{Im}\theta \leq 0$, $h \in]0, h_0]$ we have*

$$\|(z - P_{1,\theta})^{-1}\| \leq \frac{C_0}{\min\{\epsilon/2, \ (-c_1 \operatorname{Im}\theta + \operatorname{Im} z - C_1 h)\}}, \ c_1 = 1 - e^{-\epsilon} > 0, \tag{5.8}$$

uniformly with respect to $z \in \Omega$, provided $\operatorname{Im} z > c_1 \operatorname{Im}\theta + C_1 h$. The constants $C_0 \geq 1$, $C_1 > 0$ depend only on Ω.

Remark. The estimate (5.8) is similar to that in Lemma 3 in [2], where $\operatorname{Im} z$ is related to δ and Ch.

Proof. Let $\psi_0^2(x_1) + \psi_1^2(x_1) = 1$, where $\psi_0 \in C^\infty(\mathbb{R})$ is equal to 1 for $x_1 \geq R_0 - \epsilon$, $\psi_0 = 0$ for $x_1 \leq R_0 - 2\epsilon$. On the support of ψ_0, we have $x_1 - \operatorname{Re} z > \epsilon$ for all $z \in \Omega$. Combining this with the fact that $\operatorname{Re} a_\theta \geq 0$ for θ_0 small, we get

$$\begin{aligned}\operatorname{Re}\left((P_{1,\theta} - z)\psi_0 u, \psi_0 u\right) &\geq \operatorname{Re}\left((h\nabla \operatorname{Re} a_\theta h\nabla + x_1 - \operatorname{Re} z)\psi_0 u, \psi_0 u\right) - \mathcal{O}(h^2\theta)\|\psi_0 u\|^2 \\ &\geq \epsilon\|\psi_0 u\|^2 - \mathcal{O}(h^2\theta)\|\psi_0 u\|^2.\end{aligned}$$

Thus for $z \in \Omega$ and $u \in \mathcal{D}$ and for θ small we deduce

$$\|(P_{1,\theta} - z)\psi_0 u\| \geq \epsilon/2\|\psi_0 u\|, \tag{5.9}$$

uniformly on $z \in \Omega$.

On the support of ψ_1 we have

$$v(x_1) = 1 - e^{(x_1 - R_0)} \geq 1 - e^{-\epsilon} > 0$$

and

$$\operatorname{Im} a_{\theta,1,1} = \Big(2(\operatorname{Im}\theta) + \mathcal{O}(\operatorname{Re}\theta \operatorname{Im}\theta)\Big)e^{(x_1-R_0)},$$

$$\operatorname{Im} g_\theta(x) = \Big(-\frac{(\operatorname{Im}\theta)}{2} - (\operatorname{Re}\theta\operatorname{Im}\theta)\Big)e^{(x_1-R_0)}.$$

Choosing θ_0 small, we obtain

$$\operatorname{Im}\Big(-h^2\nabla a_\theta \nabla\psi_1 u, \psi_1 u\Big) \leq 0, \;\; |\operatorname{Im} h^2(g_\theta\psi_1 u, \psi_1 u)| \leq ch^2|\operatorname{Im}\theta|\|\psi_1 u\|^2$$

On the other hand, for $x_1 \in \operatorname{supp}\psi_1$ we get

$$\operatorname{Im}\left((x_1 + \theta v(x_1) - z)\psi_1 u, \psi_1 u\right) \leq \Big((c_1(\operatorname{Im}\theta) - \operatorname{Im} z)\psi_1 u, \psi_1 u\Big).$$

Consequently, for h_0, and θ_0 sufficiently small, we deduce the estimate

$$\|(P_{1,\theta} - z)\psi_1 u\| \geq (-c_1 \operatorname{Im}\theta + \operatorname{Im} z - c_2 h)\|\psi_1 u\|, \tag{5.10}$$

uniformly on $z \in \Omega$.

Let $\widetilde{\psi}_j \in C_0^\infty(\mathbb{R})$ be equal to 1 on the support of $\nabla\psi_j$. Since

$$[h^2\nabla\Big(a_\theta\nabla\Big), \psi_j] = \widetilde{\psi}_j[h^2\nabla a_\theta\nabla, \psi_j] = h\widetilde{\psi}_j\Big(h\nabla\Big(a_\theta(\nabla\psi_j)\Big) + 2(\nabla\psi_j)a_\theta(h\nabla)\Big),$$

as in the proof of (5.6), we obtain

$$\|[h^2\nabla\Big(a_\theta\nabla\Big),\, \psi_j]u\| \leq C_2 h\Big(\|(P_{1,\theta}(h) - z)u\| + \|u\|\Big) \tag{5.11}$$

for $z \in \Omega$ and $u \in \mathcal{D}$ with a constant $C_2 > 0$ depending on Ω.

Combining (5.9), (5.10), (5.11) with the estimate

$$\|(P_{1,\theta}(h) - z)u\|^2 = \sum_{j=0}^{1} \|\psi_j(P_{1,\theta}(h) - z)u\|^2$$

$$\geq \frac{1}{2}\sum_{j=0}^{1}\|(P_{1,\theta}(h) - z)\psi_j u\|^2 - \sum_{j=0}^{1}\|[h^2\nabla\Big(a_\theta\nabla\Big),\, \psi_j]u\|^2,$$

for h small enough and $z \in \Omega$ we deduce

$$C_0\|(P_{1,\theta}(h) - z)u\| \geq \min\{\epsilon/2,\; (-c_1\operatorname{Im}\theta + \operatorname{Im} z - C_1 h)\}\|u\|, \;\; u \in \mathcal{D}. \tag{5.12}$$

By the same argument, we prove an estimate similar to (5.12) for the adjoint operator $P_{1,\theta}^*(h) - \bar{z}$. Since $(P_{1,\theta}(h) - z)$ is closed, the operator $(P_{1,\theta}(h) - z)$ has a zero index, and we conclude that $(P_{1,\theta}(h) - z)$ is invertible for every $z \in \Omega$. Finally, (5.8) follows from (5.12) and this completes the proof. □

Now, it is easy to see that the operator $P_{2,\theta}(h) - z,\; z \in \Omega$, is a Fredholm operator and we have the following.

Lemma 4. *Let* $\operatorname{Im} z_0 > c_1 \operatorname{Im}\theta + C_1 h,\; z_0 \in \Omega$. *Then the operator* $P_{2,\theta}(h) - z_0$ *is a Fredholm one with index 0.*

Proof. For $z \in \Omega$ we have

$$P_{2,\theta}(h)-z = \Big(I+\Big(V\circ\Phi_\theta\Big)(1-\psi_z(x))\Big[(P_{1,\theta}(h)-z)+\Big(V\circ\Phi_\theta\Big)\psi_z(x)\Big]^{-1}\Big)\Big[(P_{1,\theta}(h)-z)+\Big(V\circ\Phi_\theta\Big)\psi_z(x)\Big],$$

where $\psi_z(x) \in C^\infty(\mathbb{R}^n)$ is a function such that $0 \leq \psi_z(x) \leq 1$, $\psi_z(x) = 0$ for $|x| \leq C_0$, $\psi_z(x) = 1$ for $|x| \geq C_0+1$. Choosing $C_0 > 0$ (depending on z) large enough, we may assume that $|\Big(V\circ\Phi_\theta\Big)|\psi_z(x)$ is small, so the operator

$$A_{1,\theta}(z) = P_{1,\theta}(h) - z + \Big(V\circ\Phi_\theta\Big)\psi_z(x)$$

is invertible for $z \in \Omega$. On the other hand,

$$K_\theta(z) = \Big(V\circ\Phi_\theta\Big)(1-\psi_z(x))A_{1,\theta}(z)^{-1}$$

is compact. Then

$$\dim\operatorname{Ker}\,(P_{2,\theta}(h)-z_0) = \dim\operatorname{Ker}\,(I + K_\theta(z_0)),$$

provided

$$\operatorname{Im} z_0 > c_1 \operatorname{Im}\theta + C_1 h.$$

A simple argument shows that $\operatorname{Im}(P_{2,\theta}(h)-z_0)$ is closed and

$$\operatorname{codim}\,(P_{2,\theta}(h)-z_0) = \dim\operatorname{Ker}\ (I + K^*_\theta(z_0)).$$

Thus $P_{2,\theta}(h) - z$ is a Fredholm operator with index 0 and the proof is complete. □

Let $\theta \in D(0,\theta_0)$, $\operatorname{Im}\theta \leq 0$. We say that $z \in \mathbb{C}$ is a *resonance* of $P_{2,\theta}(h)$ if

$$\dim\operatorname{Ker}\,(P_{2,\theta}(h)-z) > 0.$$

To examine the dependence on θ of the resonances, we will show that the operator $K_\theta(z_0)$ depends analytically on $\theta \in D(0,\theta_0)$. To do this, it is sufficient to show that the resolvent $(P_{1,\theta}(h)-z_0)^{-1}$ is analytic with respect to θ. Fix $\theta_1 \in D(0,\theta_0)$ and write

$$P_{1,\theta} - z_0 = P_{1,\theta_1} - z_0 - h^2\nabla(a_\theta - a_{\theta_1})\nabla + (\theta-\theta_1)v(x_1) + h^2(g_\theta - g_{\theta_1})$$

$$= P_{1,\theta_1} - z_0 + B_{\theta,\theta_1} = \Big(I + B_{\theta,\theta_1}(P_{1,\theta_1}-z_0)^{-1}\Big)(P_{1,\theta_1}-z_0).$$

Here the operator B_{θ,θ_1} depends analytically on θ. On the other hand, it is easy to see that for $|\theta-\theta_1|$ small enough we may arrange

$$\|B_{\theta,\theta_1}(P_{1,\theta_1}-z_0)^{-1}\| \leq C_1(\theta_1,z_0)|\theta-\theta_1| \leq 1/2.$$

For example, to estimate the terms involving a_θ, a_{θ_1}, we apply the equality

$$h^2\Big(\nabla(a_\theta - a_{\theta_1})\nabla\Big)(P_{1,\theta_1}-z_0)^{-1}$$

$$= h^2\Big(\nabla(a_\theta - a_{\theta_1})\nabla\Big)\Big[(P_1 - i)^{-1} + (P_1-i)^{-1}(i-z)(P_{1,\theta_1}-z_0)^{-1}\Big]$$

and we use the bound (5.6). Thus

$$(P_{1,\theta}-z_0)^{-1} = (P_{1,\theta_1}-z_0)^{-1}\Big(I + B_{\theta,\theta_1}(P_{1,\theta_1}-z_0)^{-1}\Big)^{-1}$$

and we obtain the analyticity for small $|\theta-\theta_1|$. Also let us point out that if the resolvent $(P_{2,\theta}-z)^{-1}$ exists, we have

$$(P_{2,\theta}-z)^{-1} = \Big(A_{1,\theta}(z)\Big)^{-1}(I + K_\theta(z))^{-1}.$$

For fixed z the invertibility of $(I + K_\theta(z))$ implies that the inverse operator $(I + K_\theta(z))^{-1}$ becomes an analytic function of θ, so the resolvent $(P_{2,\theta} - z)^{-1}$ will be also an analytic function of θ.

The resonances depend on h but they are independent on $\theta \in D(0, \theta)$, $\operatorname{Im} \theta < 0$. First, Lemma 3 implies easily that $P_{2,\theta}$ has no eigenvalues $z \in \Omega$, $\operatorname{Im} z \geq a_0 > 0$, where a_0 depends on V and Ω. The unitary operators $U(\theta)$, $\theta \in \mathbb{R}$, do not form a group. Nevertheless, the maps

$$L^2(\mathbb{R}^n) \ni f \longrightarrow f(x_1 + \theta, x'), \ \theta \in \mathbb{R}$$

form an unitary group. Then there exists a dense set $\mathcal{A} \subset L^2(\mathbb{R}^n)$ of analytic vectors so that

$$\sum_{n=0}^{\infty} \frac{\theta^n}{n!} \Big\| \frac{\partial^n f}{\partial x_1^n} \Big\|, \ f \in \mathcal{A}$$

is convergent for $\theta \in D(0, \theta_0)$. This implies that for θ_0 small and for $f \in \mathcal{A}$ the functions $U(\theta) f = J_\theta^{1/2}(x) f(\Phi_\theta(x))$ admit a holomorphic extension in $D(0, \theta_0)$. The same is true for

$$U_\theta^{-1}(\theta) f = J_\theta^{-1/2}(\Phi_\theta^{-1}(x)) f(\Phi_\theta^{-1}(x)),$$

since $\Phi_\theta^{-1}(x) = (x_1 + \theta w(\theta, x), x')$ with a function $w(\theta, x)$ holomorphic with respect to θ.

Now, we will follow an argument similar to that used by Wang [42]. Take $f, \, g \in \mathcal{A}$ and let $\theta \in \mathbb{R}$ be small. For z, $\operatorname{Im} z \geq a_0$, we have

$$(f, (P_2 - z)^{-1} g) = (U(\theta) f, (P_{2,\theta} - z)^{-1} U(\theta) g). \tag{5.13}$$

For $\operatorname{Im} z \geq a_0$ and $0 < h \leq h_0$ the right-hand side admits an analytic continuation for $\theta \in D(0, \theta_0)$. Consequently, for every fixed complex $\theta \in D(0, \theta_0)$, $\operatorname{Im} \theta < 0$, the left-hand side of (5.13) admits a meromorphic continuation with respect to z in $\Omega \cap \{z \in \mathbb{C} : \ \operatorname{Im} z > c_1 \operatorname{Im} \theta + C_1 h\}$, hence this is true for the right-hand side. Let us consider now the parameter $\theta \in \mathbb{C}$ satisfying $\operatorname{Im} \theta_1 \leq \operatorname{Im} \theta \leq \operatorname{Im} \theta_2 < 0$. Introduce the set

$$\mathcal{R}_{\theta_2}(P_2) = \bigcup_{f, g \in \mathcal{A}} \{z : \ z \ \text{is a pole of} \ (f, (P_2 - z)^{-1} g) \ \text{and} \ \operatorname{Im} z > c_1 \operatorname{Im} \theta_2 + C_1 h\}.$$

We claim that

$$z \in \mathcal{R}_{\theta_2}(P_2) \Longleftrightarrow z \in \bigcap_{\operatorname{Im} \theta_1 \leq \operatorname{Im} \theta \leq \operatorname{Im} \theta_2} \sigma_{\rm pp}\,(P_{2,\theta}).$$

If $z \in \mathcal{R}_{\theta_2}(P_2)$, then the left-hand side of (5.13) has a pole for some $f, g \in \mathcal{A}$ and we obtain the inclusion

$$\mathcal{R}_{\theta_2}(P_2) \subset \bigcap_{\operatorname{Im} \theta_1 \leq \operatorname{Im} \theta \leq \operatorname{Im} \theta_2} \sigma_{\rm pp}\,(P_{2,\theta}).$$

On the other hand, if z_0 is a pole of $(P_{2,\theta} - z)^{-1}$, then we can find $\varphi \neq 0$, $\psi \neq 0$, so that $(\psi, (P_{2,\theta} - z)^{-1} \varphi)$ has in a small complex neighborhood of z_0 an isolated singularity at $z = z_0$. The set $\mathcal{A}$ is dense in $L^2(\mathbb{R}^n)$ and by approximation, we construct functions $\psi_m \in \mathcal{A}$, $\varphi_m \in \mathcal{A}$ so that $\psi_m \longrightarrow \psi$, $\varphi_m \longrightarrow \varphi$. For m large enough $(\psi_m, (P_{2,\theta} - z)^{-1} \varphi_m)$ will have a pole at $z = z_0$. We fix a such m and setting $f = U^{-1}(\theta) \psi_m$, $g = U^{-1}(\theta) \varphi_m$, we deduce that

$$(U(\theta) f, (P_{2,\theta} - z)^{-1} U(\theta) g)$$

has also a singularity at $z = z_0$. This proves the inverse inclusion and the claim is established. This implies immediately that the resonances z with $\operatorname{Im} z > c_1 \operatorname{Im} \theta_2 + C_1 h$ are independent on the choice of $\operatorname{Im}\theta_1 \leq \operatorname{Im}\theta \leq \operatorname{Im}\theta_2 < 0$. Clearly, the above argument shows also that the operator $P_{2,\theta}(h)$ has no resonances z with $\operatorname{Im} z > 0$.

Finally, $P_2 = P_{2,\theta}(h)$ has no resonances $z \in \mathbb{R}\cap\Omega$. For this purpose, we may apply the argument of Theorem XIII.36 in [28]. Suppose that $\lambda \in \Omega\cap\mathbb{R}$ is an eigenvalue for $P_{2,\theta}$. Repeating the above argument based on the density of $\mathcal{A}$, we can find functions $F \neq 0$, $G \neq 0$ so that $(F, (P_2 - z)^{-1}G)$ has a pole at $z = \lambda$. Therefore,

$$\lim_{\epsilon\searrow 0} i\epsilon(F, (P_2 - \lambda - i\epsilon)^{-1}G) = (F, P_{\{\lambda\}}G) \neq 0,$$

$P_{\{\lambda\}}$ being the spectral projector of P_2 at λ. Consequently, $\lambda \in \sigma_{\rm pp}(P_2)$ and this leads to a contradiction with the absence of eigenvalues of P_2.

We define the multiplicity of a resonance z_0 by

$$m(z_0) = \operatorname{rank} \frac{1}{2\pi i}\int_{\gamma_\nu(z_0)} (z - P_{2,\theta})^{-1} dz,$$

where $\gamma_\epsilon(z_0) = \{z = z_0 + \nu e^{i\varphi},\ 0 \leq \varphi \leq 2\pi\}$ and $\nu > 0$ is small enough. The operator $P_{2,\theta}$ is of type (A), and we conclude that the multiplicity $m(z_0)$ is an analytic function of θ. Consequently, $m(z_0)$ is independent on $\theta \in D(0,\theta_0)$, $\operatorname{Im}\theta \leq \operatorname{Im}\theta_2 < 0$.

6. Representation of $\xi'(\lambda, h)$ for Stark hamiltonians

Let $\Omega = \Omega_{\theta,\alpha} \subset \Sigma_\theta$ be the domain introduced in the previous section and let W be an open relatively compact subset of Ω. We assume that W and Ω are symmetric with respect to $\mathbb{R}$ and independent of h and we suppose that $J = \Omega\cap\mathbb{R}$, $I = W\cap\mathbb{R}$ are intervals. The main result in this section is the following.

Theorem 2. *Assume* (5.2) *with* $s_1 > \frac{n+1}{2}$ *and* $s_2 > n-1$*. Then* $\xi'(\lambda, h)$ *is real analytic in* I *and for* $\lambda \in I$ *we have the representation*

$$\xi'(\lambda, h) = \frac{1}{\pi}\operatorname{Im} r(\lambda, h) + \sum_{\substack{\omega\in \operatorname{Res}(P_2(h))\cap\Omega \\ \operatorname{Im}\omega\neq 0}} \frac{-\operatorname{Im}\omega}{\pi|\lambda-\omega|^2},$$

where $r(z,h)$ *is a function holomorphic in* Ω *and*

$$|r(z,h)| \leq C(W)h^{-n}, z\in W \tag{6.1}$$

with $C(W) > 0$ *independent on* $h \in]0, h_0[$.

Below we fix an integer $m \in \mathbb{N}$ large enough so that the statement i) of Lemma 2 holds and we define the functions

$$\sigma_\pm(z) = (z^2+1)^m \operatorname{tr}\Big[(P_.(h) - i)^{-m}(P_.(h)+i)^{-m}(z - P_.(h))^{-1}\Big]_1^2,\ \ \pm\operatorname{Im} z > 0.$$

For θ real the operator $(P_.(h) - i)^{-m}(P_.(h)+i)^{-m}(z - P_.(h))^{-1}$ is unitary equivalent to

$$(P_{.,\theta}(h) - i)^{-m}(P_{.,\theta}(h)+i)^{-m}(z - P_{.,\theta}(h))^{-1}.$$

Consequently, the cyclicity of the trace yields

$$\sigma_+(z) = (z^2+1)^m \mathrm{tr}\Big[(P_{\cdot,\theta}(h)-i)^{-m}(P_{\cdot,\theta}(h)+i)^{-m}(z-P_{\cdot,\theta}(h))^{-1}\Big]_1^2, \tag{6.2}$$

for all $z \in \Omega_+ = \Omega \cap \{\mathrm{Im}\, z > 0\}$, $\theta \in D(0,\theta_0) \cap \mathbb{R}$.

Now, fix $\delta > 0$ and let $z \in \Omega_\delta = \Omega \cap \{\mathrm{Im}\, z \geq \delta\}$. Since $P_{\cdot,\theta}(h)$ extends to an analytic type A family of operators on $D(0,\theta_0)$, for sufficiently small θ_0 and $z \in \Omega_\delta$, the r.h.s of (6.2) extends by analytic continuation in θ to the disk $D(0,\theta_0)$. For $\theta \in D(0,\theta_0)$ with $\mathrm{Im}\,\theta < 0$, both terms of (6.2) are analytic on Ω_+, and, consequently, (6.2) remains true for all z in Ω_+.

From now on, the number θ will be fixed in $D(0,\theta_0)$ with $\mathrm{Im}\,\theta < 0$. We drop the subscript θ most of the time and write $P_\cdot$, (resp. $\widehat{P}_\cdot$) instead of $P_{\cdot,\theta}(h)$, (resp. $\widehat{P}_{\cdot,\theta}(h)$).

In the Appendix, we will construct an operator $\widehat{P}_{2,\theta}(h) : \mathcal{D} \to L^2(\mathbb{R}^n)$ with the following properties:

$$K = \widehat{P}_{2,\theta}(h) - P_{2,\theta}(h) \text{ has rank } \mathcal{O}(h^{-n}), \tag{6.3}$$

$$(\widehat{P}_{2,\theta}(h) - z)^{-1} = \mathcal{O}(1) : L^2(\mathbb{R}^n) \to \mathcal{D}, \text{ uniformly on } z \in \overline{\Omega}. \tag{6.4}$$

Moreover, K is compactly supported in the sense that $K = \chi K \chi$ with $\chi \in C_0^\infty(\mathbb{R}^n)$.

Set $\tilde{K}(z) = K(z - \widehat{P}_2)^{-1}$. Then

$$(z - P_2) = (I + \tilde{K}(z))(z - \widehat{P}_2)$$

and the resonances $z \in \mathrm{Res}\, P_2$ coincide with their multiplicities with the zeros of the function

$$D(z,h) = \det(I + \tilde{K}(z)).$$

Repeating the argument of [37], we obtain easily an upper bound of the number of the resonances lying in Ω. For the sake of completeness we present the proof. First, we have the estimate

$$|D(z,h)| \leq e^{\|\tilde{K}(z)\|_{\mathrm{tr}}} \leq e^{C_0 h^{-n}},\ z \in \overline{\Omega}.$$

Next, for $\mathrm{Im}\, z \geq \delta > 0$, $z \in \overline{\Omega}$ we get

$$(I + \tilde{K}(z))^{-1} = (z - \widehat{P}_2)(z - P_2)^{-1},$$

hence $\|(I + \tilde{K}(z))^{-1}\| \leq C_1$. We write the operator $(I + \tilde{K}(z))^{-1}$ in the form

$$(I + \tilde{K}(z))^{-1} = \Big(I - \tilde{K}(z)(I + \tilde{K}(z))^{-1}\Big)$$

and we obtain

$$\Big|\det\Big((I + \tilde{K}(z))^{-1}\Big)\Big| \leq e^{C_2 h^{-n}},\ \mathrm{Im}\, z \geq \delta$$

which implies

$$|D(z,h)| \geq Ce^{-C_3 h^{-n}},\ z \in \overline{\Omega} \cap \{\mathrm{Im}\, z \geq \delta\}.$$

Now, applying the Jensen inequality in a slightly larger domain, we obtain the following.

Proposition 2. *Let $\Omega_{\theta,\alpha} \subset \mathbb{C}$ be a compact having the form given in Section 5. Then*

$$\#\{z \in \mathrm{Res}\, P_2(h),\ z \in \Omega_{\theta,\alpha}\} \leq C(\Omega_{\theta,\alpha}) h^{-n}. \tag{6.5}$$

Using the resolvent identity

$$(z-P_2)^{-1}-(z-\widehat{P}_2)^{-1}=-(z-P_2)^{-1}K(z-\widehat{P}_2)^{-1},$$

we decompose the r.h.s of (6.2) as a sum of two terms I_1+I_2, where

$$I_1= (z^2+1)^m\mathrm{tr}\Big((P_2-i)^{-m}(P_2+i)^{-m}(z-\widehat{P}_2)^{-1}-(P_1-i)^{-m}(P_1+i)^{-m}(z-P_1)^{-1}\Big),$$

$$I_2= (z^2+1)^m\mathrm{tr}\Big((P_2-i)^{-m}(P_2+i)^{-m}(P_2-z)^{-1}K(z-\widehat{P}_2)^{-1}\Big).$$

As in Section 3, by using the resolvent equation and the cyclicity of the trace, we show that I_2 is equal to $\mathrm{tr}\Big((P_2-z)^{-1}K(\widehat{P}_2-z)^{-1}\Big)$ modulo a function holomorphic in $\overline{\Omega}$ and bounded by $\mathcal{O}(h^{-n})$. From (5.3) and (6.3), we get

$$\widehat{P}_2-P_1=K+V\circ\Phi_\theta$$

which together with the first resolvent identity

$$(z-P_1)^{-1}-(z-\widehat{P}_2)^{-1}=(z-P_1)^{-1}(P_1-\widehat{P}_2)(z-\widehat{P}_2)^{-1},$$

yield

$$I_1=(z^2+1)^m\mathrm{tr}\Big(\big[(P_j-i)^{-m}(P_j+i)^{-m}\big]_{j=1}^{2}(z-\widehat{P}_2)^{-1}\Big)$$
$$+(z^2+1)^m\mathrm{tr}\Big((P_1-i)^{-m}(P_1+i)^{-m}(z-P_1)^{-1}K(z-\widehat{P}_2)^{-1}\Big)$$
$$+(z^2+1)^m\mathrm{tr}\Big((P_1-i)^{-m}(P_1+i)^{-m}(z-P_1)^{-1}\Big(V\circ\Phi_\theta\Big)(z-\widehat{P}_2)^{-1}\Big).$$

Exploiting (6.4) and Lemma 3, all terms on the right hand side of the above equality are holomorphic in Ω. Moreover, applying (6.3) and Lemma 2, we see that the first and the second terms are bounded by $\mathcal{O}(h^{-n})$. Since $V\circ\Phi$ satisfies (5.2), the last term can be estimated in the same way. For this purpose, it is sufficient to prove that the operator $(P_{1,\theta}-i)^{-m}\Big(V\circ\Phi_\theta\Big)$ is a trace class with trace norm bounded by $\mathcal{O}(h^{-n})$. The analysis, given in the proof of Lemma 3, implies that the operator $P_{1,\theta}-i$ is elliptic for $\mathrm{Im}\,\theta<0$. Then we decompose $(P_{1,\theta}-i)^{-m}\Big(V\circ\Phi_\theta\Big)$ as a sum of three terms, involving the functions $g(x_1)$ and $\gamma(x,\xi)$, introduced in the proof of Lemma 2, and repeat the argument of Lemma 2. Thus we have proved the following analogue to Proposition 1.

Proposition 3. *There exists a function $a_+(z,h)$ holomorphic in Ω, such that for $z\in\Omega_+$ we have*

$$\sigma_+(z)=\mathrm{tr}\Big((P_2-z)^{-1}K(\widehat{P}_2-z)^{-1}\Big)+a_+(z,h). \tag{6.6}$$

Moreover,

$$|a_+(z,h)|\leq C(\Omega)h^{-n},\;\; z\in\Omega \tag{6.7}$$

with a constant $C(\Omega)$ independent on $h\in]0,h_0]$.

Proof of Theorem 2. We repeat the argument of the proof of Lemma 1 and we get

$$\langle\xi',f\rangle=\lim_{\epsilon\searrow 0}\frac{i}{2\pi}\int f(\lambda)\Big[\sigma_+(\lambda+i\epsilon)-\sigma_-(\lambda-i\epsilon)\Big]d\lambda, \tag{6.8}$$

where the limit is taken in the sense of distributions. Next, we follow the proof of Theorem 1, applying (6.8) and Proposition 3. □

7. Local trace formula and spectral asymptotics

In this section we obtain a local trace formula in the spirit of [8] (see see [27] for compactly supported perturbations and [36], [37] for general long-range perturbations). Repeating the proof of Theorem 4 in [8], we get the following.

Theorem 3. *Assume that $P_j(h)$, $j = 1, 2$ satisfy the assumptions of Sections 4, 5. Let*

$$\Omega = \Omega_{\theta,\alpha} \subset \{z \in \mathbb{C} : \ \operatorname{Re} z \leq R_0 - 3\epsilon, \ \operatorname{Im} z \geq \alpha(1 - e^{-\epsilon}) \operatorname{Im} \theta\}, \ 0 < \alpha < 1$$

be an open, simply connected, relatively compact set defined in Section 5 such that $I = \Omega \cap \mathbb{R}$ is an interval. Suppose that f is holomorphic on a neighborhood of Ω and that $\psi \in C_0^\infty(\mathbb{R})$ satisfies

$$\psi(\lambda) = \begin{cases} 0, & d(I, \lambda) > 2\eta, \\ 1, & d(I, \lambda) < \eta, \end{cases}$$

where $\eta > 0$ is sufficiently small. Then

$$\operatorname{tr}\Big[(\psi f)(P_j(h))\Big]_{j=1}^{2} = \sum_{z \in \operatorname{Res} P_2(h) \cap \Omega} f(z) + E_{\Omega,f,\psi}(h) \tag{7.1}$$

with

$$|E_{\Omega,f,\psi}(h)| \leq M(\psi, \Omega) \sup \{|f(z)| \ : \ 0 < d(\Omega, z) \leq 2\eta \, , \ \operatorname{Im} z \leq 0\} h^{-n} \, .$$

For the applications we need an asymptotic development of the trace. For this purpose we will prove the following.

Theorem 4. *Assume (5.2) satisfied with $s_1 > \frac{n+1}{2}$ and $s_2 > n - 1$ and suppose that* $\operatorname{supp} V \subset \{x \in \mathbb{R}^n \, : \, x_1 > \delta_1\}$ *for some $\delta_1 \in \mathbb{R}$. Then for $f \in C_0^\infty(\mathbb{R})$, we have*

$$\operatorname{tr}(f(P_2(h)) - f(P_1(h))) \sim \sum_{j=0}^{\infty} a_j h^{j-n}, \ h \searrow 0, \tag{7.2}$$

with

$$a_0 = -(2\pi)^{-n} \int\int_{\mathbb{R}^{2n}} (\partial_{x_1} V(x)) f(|\xi|^2 + x_1 + V(x)) dx d\xi =: \langle \omega, f \rangle, \ a_1 = 0.$$

To obtain a Weyl-type asymptotics of $\xi(\lambda, h)$, we need a expansion for the trace involving the function

$$\breve{\theta}_h(\tau) = (2\pi h)^{-1} \int e^{i\tau t/h} \theta(t) dt$$

which is the semi-classical Fourier inverse transform of $\theta \in C_0^\infty(\mathbb{R})$.

Theorem 5. *In addition to the assumptions of Theorem 4 suppose that $p_2(x, \xi) = |\xi|^2 + V(x) + x_1$ is not critical for all $\tau \in [E_0, E_1]$. Then there exist $C_0 > 0$ and h_0 small enough such that for $\theta \in C_0^\infty(]-\frac{1}{C_0}, \frac{1}{C_0}[; \mathbb{R})$, $\theta = 1$ in a neighborhood of 0, $f \in C_0^\infty(]E_0, E_1[)$ and $h \in]0, h_0]$ we have*

$$\operatorname{tr}\Big(\Big[\breve{\theta}_h(\tau - P_j(h)) f(P_j(h))\Big]_{j=1}^{2}\Big) = (2\pi h)^{-n} \Big(f(\tau) \sum_{j=0}^{N-1} \gamma_j(\tau) h^j + \mathcal{O}(h^N \langle \tau \rangle^{-m})\Big), \ \forall m \in \mathbb{N}, \ \forall N \in \mathbb{N} \tag{7.3}$$

uniformly with respect to $\tau \in \mathbb{R}$, where

$$\gamma_0(\tau) = (2\pi i)^{-1} \int\int_{\mathbb{R}^{2n}} (\partial_{x_1} V(x)) \Big((\tau + i0 - p_2(x, \xi))^{-1} - (\tau - i0 - p_2(x, \xi))^{-1}\Big) dx d\xi.$$

The proof of Theorem 4 is a simple modification of that of Theorem 5, so we will establish below (7.3).

As we have noticed in Section 1, in the case where the operators $P_i(h)$ are elliptic, Theorem 4 and Theorem 5 are well known (see [10], [30], [20]). Our idea is to apply the results of [10], [30] and [20] after reducing the study of the left hand side of (7.2) and (7.3) to that of the trace of an elliptic operator (see (7.12)).

Proof. Let $f \in C_0^\infty(]E_0, E_1[)$ and let $\tilde{f} \in C_0^\infty(\mathbb{C})$ be an almost analytic extension of f with $\bar{\partial}_z \tilde{f}(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)$. We choose $\tilde{f}$ so that $\operatorname{supp} \tilde{f} \subset \{z \in \mathbb{C};\ \operatorname{Re} z \in [E_0, E_1]\}$. Let $R_1 > 0$ be a large constant such that

$$R_1 > \|V\|_\infty + E_1 + |\delta_1| + 3.$$

Introduce a partition of unity $\psi_0^2(x_1) + \psi_1^2(x_1) = 1$ on $[\delta_1 - 1, +\infty[$, where $\psi_1 \in C_0^\infty(\mathbb{R})$ and $\operatorname{supp} \psi_0 \subset [R_1, +\infty[$. Notice that $\psi_1(x_1) = 1$ for $\delta_1 - 1 \leq x_1 \leq \|V\|_\infty + E_1 + |\delta_1| + 3$.

As it was shown in Lemma 2.4, [32], for $f \in C_0^\infty(\mathbb{R})$ we have

$$\operatorname{tr}\Big(f(P_2(h)) - f(P_1(h))\Big) = -\operatorname{tr}\Big((\partial_{x_1} V) f(P_2(h))\Big).$$

Applying this to the l.h.s of (7.3) and using the cyclicity of the trace, we get

$$\operatorname{tr}\Big(\Big[\breve{\theta}_h(\tau - P_j(h)) f(P_j(h))\Big]_{j=1}^{2}\Big) = -\operatorname{tr}\Big[\psi_0 \breve{\theta}_h(\tau - P_2(h)) f(P_2(h)) (\partial_{x_1} V)\psi_0\Big] \tag{7.4}$$

$$-\operatorname{tr}\Big[\psi_1 \breve{\theta}_h(\tau - P_2(h)) f(P_2(h)) (\partial_{x_1} V)\psi_1\Big].$$

Let $\psi \in C_0^\infty(\mathbb{R}; [0,1])$ be equal to 1 for $|t| < 1$ and equal to 0 for $|t| > 2$ and introduce

$$\psi_Y(z) = \psi\Big(\frac{|\operatorname{Im} z|}{Y}\Big),\ \ Y = -Mh\log h,$$

where M is a large constant which we will choose below. Clearly, $(\psi_Y \tilde{f})(z) = f(z)$ for $z \in \mathbb{R}$ and the function $z \to \breve{\theta}_h(\tau - z)$ is analytic. Consequently, the Helffer-Sjöstrand formula yields

$$\operatorname{tr}\Big[\psi_1 \breve{\theta}_h(\tau - P_2(h)) f(P_2(h)) (\partial_{x_1} V)\psi_1\Big] \tag{7.5}$$

$$= -\frac{1}{\pi}\operatorname{tr}\Big(\int \bar{\partial}_z(\psi_Y \tilde{f})(z) \psi_1 \breve{\theta}_h(\tau - z)(z - P_2(h))^{-1} (\partial_{x_1} V)\psi_1 L(dz)\Big).$$

Let $G \in C_0^\infty(\mathbb{R})$ with $\psi_1 \prec G$. Introduce the operators

$$\tilde{P}_2(h) = -h^2\Delta + V(x) + G(x_1)x_1,\ \ \tilde{P}_1(h) = -h^2\Delta + G(x_1)x_1$$

and set

$$I = \operatorname{tr}\Big(\psi_1\Big[\breve{\theta}_h(\tau - P_2(h)) f(P_2(h)) - \breve{\theta}_h(\tau - \tilde{P}_2(h)) f(\tilde{P}_2(h))\Big](\partial_{x_1} V)\psi_1\Big).$$

It follows from (7.5) that

$$I = -\frac{1}{\pi}\int \bar{\partial}_z(\psi_Y \tilde{f})(z) \breve{\theta}_h(\tau - z) \operatorname{tr}\Big[\psi_1\Big((z - P_2(h))^{-1} - (z - \tilde{P}_2(h))^{-1}\Big)(\partial_{x_1} V)\psi_1\Big] L(dz).$$

We have $\tilde{G} := (G-1)x_1 = \tilde{P}_2(h) - P_2(h) = 0$ near $\operatorname{supp}\psi_1$ and this implies $\tilde{G}(z-\tilde{P}_2)^{-1}\psi_1 = [\tilde{G},(z-\tilde{P}_2)^{-1}]\psi_1$. Let $\tilde{\psi}_1 \in C^\infty(\mathbb{R})$ be a function with $\tilde{\psi}_1 = 1$ near $\operatorname{supp}\tilde{G}$ and $\tilde{\psi}_1 = 0$ near $\operatorname{supp}\psi_1$. Then

$$\psi_1\Big((z-P_2(h))^{-1} - (z-\tilde{P}_2(h))^{-1}\Big)(\partial_{x_1}V)\psi_1 = -\psi_1(z-P_2(h))^{-1}\tilde{\psi}_1\tilde{G}(z-\tilde{P}_2(h))^{-1}\psi_1(\partial_{x_1}V) \tag{7.6}$$

$$= -\psi_1(z-P_2(h))^{-1}\tilde{\psi}_1(z-\tilde{P}_2(h))^{-1}[\tilde{G},\tilde{P}_2(h)](z-\tilde{P}_2(h))^{-1}\psi_1(\partial_{x_1}V).$$

Let $\chi_1,...,\chi_N \in C_0^\infty(\mathbb{R}^n;[0,1])$ with $\psi_1 \prec \chi_1 \prec ... \prec \chi_N$ and $\chi_i[\tilde{G},\tilde{P}_2(h)] = 0$. By using the equalities $\chi_1...\chi_N\psi_1 = \psi_1$, $\chi_k\,[\tilde{G},\tilde{P}_2(h)] = 0$, $\chi_{k-1}[\chi_k,\ P_2(h)] = 0$ and the fact that

$$[\chi_k,(z-P_2(h))^{-1}] = (z-P_2(h))^{-1}[\chi_k,\ P_2(h)](z-P_2(h))^{-1},$$

we get

$$[\tilde{G},\tilde{P}_2(h)](z-\tilde{P}_2(h))^{-1}\psi_1$$
$$= [\tilde{G},\tilde{P}_2(h)](z-\tilde{P}_2(h))^{-1}[\chi_1,\tilde{P}_2(h)](z-\tilde{P}_2(h))^{-1}...[\chi_N,\tilde{P}_2(h)](z-\tilde{P}_2(h))^{-1}\psi_1 =: L_N(h).$$

Here

$$L_N(h) = \mathcal{O}_N(1)\Big(\frac{h^N}{|\operatorname{Im} z|^N}\Big) : H^{s-N}(\mathbb{R}^n) \to H^s(\mathbb{R}^n),$$

and we equip $H^N(\mathbb{R}^n)$ with the h-dependent norm $\|\langle hD\rangle^N\|_{L^2}$. Choosing $N > n$ our assumptions on V and Theorem 9.4 in [10] yield

$$\Big\|\Big(-h^2\Delta + 1\Big)^{-N/2}\partial_{x_1}V\Big\|_{\mathrm{tr}} = \mathcal{O}(h^{-n}).$$

Then

$$\|[\tilde{G},\tilde{P}_2](z-\tilde{P}_2)^{-1}\psi_1(\partial_{x_1}V)\|_{\mathrm{tr}} = \|L_N(h)\Big(-h^2\Delta+1\Big)^{N/2}\Big(-h^2\Delta+1\Big)^{-N/2}\partial_{x_1}V\|_{\mathrm{tr}} \tag{7.7}$$

$$\leq C\|\Big(-h^2\Delta+1\Big)^{-N/2}\partial_{x_1}V\|_{\mathrm{tr}}\Big(\frac{h^N}{|\operatorname{Im} z|^N}\Big) \leq C_1\Big(\frac{h^{N-n}}{|\operatorname{Im} z|^N}\Big).$$

Since $\bar{\partial}_z\tilde{f}(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)$, we have

$$\bar{\partial}_z(\psi_Y\tilde{f})(z) = \tilde{f}(z)\bar{\partial}_z(\psi_Y)(z) + \mathcal{O}(h^\infty)$$

and, consequently,

$$I = -\frac{1}{\pi}\int_{Y\leq|\operatorname{Im} z|\leq 2Y}(\bar{\partial}_z\psi_Y)(z)\tilde{f}(z)\breve{\theta}_h(\tau - z) \tag{7.8}$$

$$\times\operatorname{tr}\Big[\psi_1\Big((z-P_2(h))^{-1} - (z-\tilde{P}_2(h))^{-1}\Big)(\partial_{x_1}V)\psi_1\Big]L(dz) + \mathcal{O}(h^\infty)\,.$$

Next we take a real-valued function $K_0 \in C_0^\infty(\mathbb{R})$ such that

$$K_0 = 1 \text{ near } \operatorname{supp}\psi_1,\ \ K_0 = 0 \text{ near } \operatorname{supp}\tilde{\psi}_1.$$

Put $K = \alpha K_0$, $\alpha > 0$. By using that $K = \alpha$ near $\operatorname{supp}\psi_1$ and the fact that $K = 0$ near $\operatorname{supp}\tilde{\psi}_1$, we get in the operator norm

$$\psi_1(z-P_2(h))^{-1}\tilde{\psi}_1 = e^{-\alpha\log\frac{1}{h}}e^{K\log\frac{1}{h}}\psi_1(z-P_2(h))^{-1}(e^{-K\log\frac{1}{h}})\tilde{\psi}_1 \tag{7.9}$$

$$= e^{-\alpha\log\frac{1}{h}}\psi_1(z - e^{K\log\frac{1}{h}}P_2(h)e^{-K\log\frac{1}{h}})^{-1}\tilde{\psi}_1.$$

On the other hand, a simple calculus shows that

$$e^{K\log\frac{1}{h}}(z-P_2(h))e^{-K\log\frac{1}{h}}=$$
$$\Big(I+\big[2\alpha h\log\frac{1}{h}(\nabla K)\cdot\nabla+\mathcal{O}(\alpha h\log\frac{1}{h})\big](z-P_2(h))^{-1}\Big)(z-P_2(h))$$

in the operator norm for $h\leq h(\alpha)$, $h(\alpha)>0$ being a continuous function.

Let us choose α as

$$\alpha=\min\Big(\frac{|\operatorname{Im}z|}{\tilde{C}h\log\frac{1}{h}},\ \beta\Big),$$

where $\beta>0$ is some arbitrary large and fixed constant and $\tilde{C}$ is sufficiently large.

Since $\|(\nabla K)\nabla(z-P_2(h))^{-1}\|=\mathcal{O}(|\operatorname{Im}z|^{-1})$ and $\alpha h\log\frac{1}{h}|\operatorname{Im}z|^{-1}\leq\frac{1}{\tilde{C}}$, then for $\tilde{C}$ large enough the right hand side of (7.9) is $\mathcal{O}(e^{-\alpha\log\frac{1}{h}}|\operatorname{Im}z|^{-1})$ in $\mathcal{L}(L^2(\mathbb{R}^n))$. Combining this with (7.6), (7.7), we get for $|\operatorname{Im}z|\geq -Mh\log h$ with a new constant $C_2>0$

$$\Big\|\psi_1\Big((z-P_2(h))^{-1}-(z-\tilde{P}_2(h))^{-1}\Big)(\partial_{x_1}V)\psi_1\Big\|_{\mathrm{tr}}$$
$$=\mathcal{O}(h^{-n}|\operatorname{Im}z|^{-2}e^{-\alpha\log\frac{1}{h}})=\mathcal{O}\Big(h^{-n}|\operatorname{Im}z|^{-2}\max(h^{\beta},e^{-\frac{|\operatorname{Im}z|}{C_2h}})\Big).$$

On the other hand, the Paley-Wiener theorem yields

$$|\breve{\theta}_h(\tau-z)|=\mathcal{O}(\frac{1}{h}e^{\frac{1}{C_0h}|\operatorname{Im}z|}),$$

where we have used that $\operatorname{supp}\theta\subset]-\frac{1}{C_0},\frac{1}{C_0}[$. We choose $C_0>C_2$ and the r.h.s of (7.8) becomes

$$I=\mathcal{O}(1)\int_{\substack{Y\leq|\operatorname{Im}z|\leq 2Y\\ |\operatorname{Re}z|\leq\text{const.}}}Y^{-2}h^{-n-1}\max\Big(h^{\beta}\exp(\frac{|\operatorname{Im}z|}{C_0h}),\ \exp(-\frac{1}{h}|\operatorname{Im}z|(C_2^{-1}-C_0^{-1}))\Big)L(dz)+\mathcal{O}(h^{\infty})$$
$$=\mathcal{O}(1)Y^{-1}h^{-n-1}\max\Big(h^{\beta-\frac{2M}{C_0}},\ \exp\Big(-M(\log\frac{1}{h})(C_2^{-1}-C_0^{-1})\Big)\Big)+\mathcal{O}(h^{\infty}).$$

First choosing M sufficiently large and then the power in h^{β} large enough, we see that this expression is $\mathcal{O}_N(h^N)$ for any $N\in\mathbb{N}$ and $h\leq h(N)$.

Turning to the study of first term in the r.h.s. of (7.4), choose a real-valued smooth function $F(x_1)$ satisfying $F(x_1)=x_1$ for $x_1>R_1-1$ and $F(x_1)>R_1-2$ for all $x_1\in\mathbb{R}$. Using the inequality $R_1>\|V\|_{\infty}+E_1+|\delta_1|+3$, as well as the fact that $\operatorname{supp}\tilde{f}\subset\{z\in\mathbb{C};\ \operatorname{Re}z\in[E_0,E_1]\}$, we get

$$|\xi|^2+F(x_1)+V(x)-\operatorname{Re}z>1+|\delta_1|\geq 1,$$

uniformly on $(x,\,\xi)$ and $z\in\operatorname{supp}\tilde{f}$. Introduce the operator

$$\tilde{P}=-h^2\Delta+F(x_1)+V(x).$$

Clearly, $\tilde{P}$ is semi-bounded on $C_0^{\infty}(\mathbb{R}^n)$ and we denote also by $\tilde{P}$ the selfadjoint extension of $\tilde{P}$. By construction, $z\to(z-\tilde{P})^{-1}$ exists and is analytic in a neighborhood of $\operatorname{supp}\tilde{f}$. Combining this with Helffer-Sjöstrand formula, we get

$$\operatorname{tr}\Big[(\partial_{x_1}V)\psi_0\breve{\theta}_h(\tau-P_2(h))f(P_2(h))\psi_0\Big]\tag{7.10}$$
$$=-\frac{1}{\pi}\operatorname{tr}\Big(\int\bar{\partial}_z(\psi_Y\tilde{g})(z)\breve{\theta}_h(\tau-z)(\partial_{x_1}V)\psi_0(i-P_2(h))^{-m}\Big((z-P_2(h))^{-1}-(z-\tilde{P})^{-1}\Big)\psi_0L(dz)\Big),$$

where $\tilde{g}(z)=\tilde{f}(z)(i-z)^m$. Here m is fixed so that $(\partial_{x_1}V)(i-P_2(h))^{-m}$ is a trace class operator.

Since $P_2(h) - \tilde{P} = x_1 - F(x_1) =: \tilde{F} = 0$ near $\operatorname{supp}\psi_0$, we can repeat the argument of the proof of (7.6) and get

$$\Big((z - P_2(h))^{-1} - (z - \tilde{P})^{-1}\Big)\psi_0 = (z - P_2(h))^{-1}\tilde{\psi}_2\tilde{F}(z - \tilde{P}_2(h))^{-1}\psi_0$$

$$= (z - P_2(h))^{-1}\tilde{\psi}_2(z - \tilde{P})^{-1}[\tilde{F}, \tilde{P}](z - \tilde{P})^{-1}\psi_0,$$

where $\tilde{\psi}_2 \in C^\infty(\mathbb{R})$ with $\tilde{\psi}_2 = 1$ near supp $\tilde{F}$ and $\tilde{\psi}_2 = 0$ near $\operatorname{supp}\psi_0$.

To estimate the last term, notice that $(z - \tilde{P})^{-1}$ and $(h\nabla)(z - \tilde{P})^{-1} \in \mathrm{Op}_h^w(S^0(1))$ uniformly for $z \in \operatorname{supp}\tilde{f}$ (see the proof of Lemma 2 for the definition of $S^0(1)$). Consequently,

$$(z - \tilde{P})^{-1}[\tilde{F}, \tilde{P}](z - \tilde{P})^{-1} \in \mathrm{Op}_h^w(S^0(1)).$$

Since dist(supp$\tilde{\psi}_2$, supp ψ_0)> 0, it follows from Lemma 2.1 of [13] that

$$\|\tilde{\psi}_2(z - \tilde{P})^{-1}[\tilde{F}, \tilde{P}](z - \tilde{P})^{-1}\psi_0\| = \mathcal{O}(h^N), \ \ \forall N \in \mathbb{N}.$$

uniformly on $z \in \operatorname{supp}\tilde{f}$.

On the other hand, the argument of the proof of Lemma 2 yields

$$\|(\partial_{x_1}V)(i - P_2(h))^{-m}\|_{\mathrm{tr}} = \mathcal{O}(h^{-n}).$$

Combining the above three equalities with the estimate $|(z - P_2(h))^{-1}\| = \mathcal{O}(|\operatorname{Im} z|^{-1})$, we obtain

$$\|\Big((\partial_{x_1}V)(i - P_2(h))^{-m}\Big((z - P_2(h))^{-1} - (z - \tilde{P})^{-1}\Big)\psi_0\|_{\mathrm{tr}} = \mathcal{O}(h^{N-n}/|\operatorname{Im} z|).$$

Going back to the integral in (7.10), and using that $\bar{\partial}_z(\psi_Y\tilde{f})(z) = \tilde{f}(z)\bar{\partial}_z(\psi_Y)(z) + \mathcal{O}(h^\infty)$, we deduce

$$\mathrm{tr}\Big[(\partial_{x_1}V)\psi_0\breve{\theta}_h(\tau - P_2(h))f(P_2(h))\psi_0\Big] = \mathcal{O}(h^\infty).$$

Summing up, we have proved that

$$\mathrm{tr}\Big(\Big[\breve{\theta}_h(\tau - P_j(h))f(P_j(h))\Big]_{j=1}^2\Big) = -\mathrm{tr}\Big[(\partial_{x_1}V)\breve{\theta}_h(\tau - \tilde{P}_2(h))f(\tilde{P}_2(h))\psi_1\Big] + \mathcal{O}(h^\infty). \tag{7.11}$$

In the same way, we obtain

$$\mathrm{tr}\Big(\Big[f(P_j(h))\Big]_{j=1}^2\Big) = -\mathrm{tr}\Big[(\partial_{x_1}V)f(\tilde{P}_2(h))\psi_1\Big] + \mathcal{O}(h^\infty). \tag{7.12}$$

The operator $\tilde{P}_2(h)$ is a short-range perturbation with decreasing potential V of $\tilde{P}_1(h)$, so Theorem 4 and Theorem 5 follow from the h-pseudodifferential calculus and the analysis of elliptic operators in Chapters 8, 9, 12, [10] (see also [29]). The leading term a_0 has the form

$$a_0 = (2\pi)^{-n}\int\int_{\mathbb{R}^{2n}}\Big(f(|\xi|^2 + G(x_1)x_1 + V(x)) - f(|\xi|^2 + G(x_1)x_1)\Big)\psi_1(x_1)dxd\xi$$

$$= (2\pi)^{-n}\int\int_{\mathbb{R}^{2n}}\Big(f(|\xi|^2 + x_1 + V(x)) - f(|\xi|^2 + x_1)\Big)dxd\xi,$$

since the integration with respect to x_1 in the second integral is over the set $\delta_1 \le x_1 \le E_1 + |\delta_1| + \|V\|_\infty$ and $G(x_1) = \psi_1(x_1) = 1$ on this set. This proves that the leading terms is independent on the choice of G, ψ_1. A similar argument implies the independence of the asymptotic expansion (7.2) on the choice of G, ψ_1, provided that $\psi_1(x_1) = 1$ for $\delta_1 - 1 \le x_1 \le E_1 + |\delta_1| + \|V\|_\infty + 3$. Next, we have

$$a_0 = (2\pi)^{-n} \int\int_{\mathbb{R}^{2n}} \Big(f(|\xi|^2 + x_1 + V(x)) - f(|\xi|^2 + x_1) \Big) dx d\xi$$
$$= -(2\pi)^{-n} \int\int_{\mathbb{R}^{2n}} (\partial_{x_1} V) f(|\xi|^2 + x_1 + V) dx d\xi$$
$$= (2\pi)^{-n} \int\int_{\mathbb{R}^{2n}} \int_0^V f'(|\xi|^2 + x_1 + t) dt dx d\xi.$$

□

Now we will apply Theorems 2, 4, 5 to obtain a lower bounds for the number of the resonances and a Weyl-type asymptotics for $\lambda \in [E_0, E_1]$.

Theorem 6. *Assume the assumptions of Theorem 5 fulfilled and suppose that* $E_1 < \delta_1 = \inf\{x_1 \in \mathbb{R} : x_1 \in \operatorname{supp}_{x_1} V\}$. *Then there exists* $h_0 > 0$ *small enough such that for* $h \in]0, h_0]$ *we have*

$$\xi(\lambda, h) = (2\pi h)^{-n} c_0(\lambda) + \mathcal{O}(h^{-n+1}), \tag{7.13}$$

uniformly on $\lambda \in [E_0, E_1]$, *where*

$$c_0(\lambda) = -\frac{1}{n}\omega_n \int_{\mathbb{R}^n} \partial_{x_1} V(x)(\lambda - V(x) - x_1)_+^{\frac{n}{2}} dx \tag{7.14}$$

with $\omega_n = \operatorname{vol} S^{n-1}$. *Moreover, if for all* $\mathbb{N} \in \mathbb{N}$,

$$\operatorname{Res}(P_2(h)) \cap \Big([E_0, E_1] - i[0, Nh \ln(1/h)] \Big) = \emptyset, \quad 0 < h < h(N)),$$

then

$$\xi'(\lambda, h) \sim \sum_{j=0}^{\infty} \gamma_j(\lambda) h^{j-n}, \ h \searrow 0$$

with $\gamma_0(\lambda) = c_0'(\lambda)$.

Proof. Following [8], [12], the proof is rather similar to that in these papers and for this reason we will present only the main steps. The reader may consult [8], [12] for more details. Let $\Omega \subset \mathbb{C}$ be a compact domain having the properties described in Section 5 and let $[E_0, E_1] \subset \Omega \cap \mathbb{R}$. We assume that $z \in \Omega \Rightarrow \operatorname{Re} z < \delta_1$. Let $\varphi \in C_0^\infty(\mathbb{R})$, $\operatorname{supp} \varphi \subset [E_0 - \epsilon, E_1 + \epsilon], \subset \Omega \cap \mathbb{R}$, $\epsilon > 0$. Introduce the functions

$$M_\varphi(\lambda) = -\frac{1}{\pi} \sum_{w \in (\operatorname{Res} P_2(h)) \cap \Omega} \int_{-\infty}^{\lambda} \frac{\operatorname{Im} w}{|w - \mu|^2} \varphi(\mu) d\mu,$$
$$G_\varphi(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\lambda} \varphi(\mu) \operatorname{Im} r(\mu, h) d\mu,$$

$r(\lambda, h)$ being the holomorphic function in Theorem 2 related to the domain Ω. Applying Theorem 2, we get

$$\int_{-\infty}^{\lambda} \varphi \xi'(\mu) d\mu = M_\varphi(\lambda) + G_\varphi(\lambda), \ \lambda \in [E_0, \ E_1].$$

The function $M_\varphi(\lambda)$ is increasing and, as in [8], [12], we may apply a Tauberian theorem based on the estimates

$$M_\varphi(\lambda) = \mathcal{O}(h^{-n}), \ \frac{d}{d\lambda}(\hat{\theta}_h * M_\varphi)(\lambda) = \mathcal{O}(h^{-n}).$$

The first estimate follows from Proposition 2, while for the second one we exploit Theorem 5 and the argument of [12]. Consequently, we have

$$M_\varphi(\lambda) = (\hat{\theta}_h * M_\varphi)(\lambda) + \mathcal{O}(h^{1-n}),$$
$$G_\varphi(\lambda) = (\hat{\theta}_h * G_\varphi)(\lambda) + \mathcal{O}(h^{1-n}).$$

Without loss of the generality, we may assume that every $\lambda \in \operatorname{supp} \varphi$ is a non-critical value of the symbol $|\xi|^2 + x_1 + V(x)$. Applying Theorem 5, we deduce

$$\int_{-\infty}^{\lambda} \varphi\xi'(\mu)d\mu = \hat{\theta}_h * \int_{-\infty}^{\lambda} \varphi\xi'(\mu)d\mu + \mathcal{O}(h^{1-n}) = \Big(\int_{-\infty}^{\lambda} \gamma_0(\mu)d\mu\Big)h^{-n} + \mathcal{O}(h^{1-n}),$$

where the function $\gamma_0(\mu)$ is given by Theorem 5.

Now we choose $C_1 = \delta_1 - \|V\|_\infty$ and we apply the above argument for a cut-off function $\varphi(\lambda)$ for which we have

$$\operatorname{supp} \varphi \subset \{\lambda \in \mathbb{R} : \lambda < C_1\}.$$

For this purpose we choose a domain Ω as above and observe that $|\xi|^2 + x_1 + V(x)$ has no critical values $\lambda \in \operatorname{supp} \varphi$. It is easy to see that $\gamma_j(\lambda) = 0,\ j = 0, ..., N-1$ for $\lambda < C_1$, so for such cut-off function we have

$$\int_{-\infty}^{\lambda} \varphi\xi'(\mu)d\mu = \mathcal{O}(h^{N-n}),\ \forall N \in \mathbb{N}.$$

Consider a partition of unity $\varphi_1(\lambda) + \varphi_2(\lambda) + \varphi_3(\lambda) = 1$, $\varphi_j(\lambda) \in C_0^\infty(\mathbb{R})$, $j = 1, 2, 3$, on the interval $[C_2, E_1 + \epsilon]$, $E_1 + \epsilon < \delta_1$, $C_2 < C_1$. We assume that

$$\varphi_3 = 1 \text{ on } [E_0,\ E_1],\ \operatorname{supp} \varphi_3 \subset\ [E_0 - \epsilon,\ E_1 + \epsilon],$$
$$\operatorname{supp} \varphi_1\ \subset]-\infty,\ C_1 - \epsilon[,\ \epsilon > 0.$$

For $\lambda \in [E_0,\ E_1]$, we get

$$\begin{aligned}\xi(\lambda, h) - \xi(C_2, h) = &-\frac{1}{\pi} \sum_{w \in (\operatorname{Res} P_2(h)) \cap \Omega} \int_{C_2}^{\infty} \frac{\operatorname{Im} w}{|w-\mu|^2} \varphi_1(\mu)d\mu \\ &+ \frac{1}{\pi} \int_{C_2}^{\infty} \varphi_1(\mu) \operatorname{Im} r(\mu, h)d\mu \\ &+ \operatorname{tr}\Big[\varphi_2(P_j)(h)\Big]_{j=1}^{2} + M_{\varphi_3}(\lambda) + G_{\varphi_3}(\lambda).\end{aligned}$$

For the terms involving φ_3 and φ_1 we apply the above argument and we observe that

$$\int_{C_2}^{C_1-\epsilon} \varphi_1\xi' d\mu = \int_{-\infty}^{C_1-\epsilon} \varphi_1\xi' d\mu - \int_{-\infty}^{C_2} \varphi_1\xi' d\mu = \mathcal{O}(h^{N-n}),\ \forall N \in \mathbb{N}.$$

Next, we may choice the value of $\xi(\lambda, h)$ at one point and we suppose that $\xi(C_2, h) = 0$. Thus we obtain the asymptotics (7.13). To find the coefficient $c_0(\lambda)$, notice that from Theorem 4 and the definition of $\xi(\lambda, h)$ we have

$$\begin{aligned}\langle c_0, f'\rangle &= \int\int_{\mathbb{R}^{2n}} (\partial_{x_1} V(x)) f(|\xi|^2 + x_1 + V(x))dxd\xi \\ &= \frac{1}{2}\omega_n \int_0^\infty \int_{\mathbb{R}^n} \partial_{x_1} V(x) f(t + V(x) + x_1) t^{\frac{n}{2}-1} dtdx \\ &= -\frac{1}{2}\omega_n \int_0^\infty \int_{\mathbb{R}^n} \partial_{x_1} V(x) \int_{t+V(x)+x_1}^{\infty} f'(\lambda) t^{\frac{n}{2}-1} d\lambda dtdx\end{aligned}$$

$$= -\frac{1}{n}\omega_n \int_{\mathbb{R}_\lambda}\int_{\mathbb{R}^n} \partial_{x_1} V(x)(\lambda - V(x) - x_1)_+^{\frac{n}{2}} f'(\lambda) d\lambda dx,$$

which gives (7.14). The second assertion of Theorem 6 can be established combining Theorems 2, 5 with the argument of the proof of Theorem 3 in [12]. As in Theorem 3 in [12], we show that for $\lambda \in [E_0,\, E_1]$ we have

$$\Big(\hat{\theta}_h * \frac{-\operatorname{Im} w}{|.-w|^2}\Big)(\lambda) = \frac{-\operatorname{Im} w}{|\lambda - w|^2} + \mathcal{O}\Big(\frac{1}{|\operatorname{Im} w|} e^{-\frac{C}{2h}|\operatorname{Im} w|}\Big),$$

$$\Big(\hat{\theta}_h * (1-\varphi_3(.))\frac{\operatorname{Im} w}{|.-w|^2}\Big)(\lambda) = (1-\varphi_3(\lambda))\frac{\operatorname{Im} w}{|\lambda - w|^2} + \mathcal{O}(h^\infty),$$

$$\hat{\theta}_h * G_{\varphi_3} = G_{\varphi_3} + \mathcal{O}(h^\infty),$$

where we assume that $\theta = 1$ on $[-\frac{1}{2C},\, \frac{1}{2C}]$. Since for all resonances $w \in \Omega$ we have the lower bound $|\operatorname{Im} w| \geq Nh\log(1/h)$, we may estimate the exponent involving $|\operatorname{Im} w|$. Thus we conclude that

$$\hat{\theta}_h * (\varphi_3 \xi')(\lambda, h) = \varphi_3(\lambda)\xi'(\lambda, h) + \mathcal{O}(h^{NC/2-n-1}),\ \forall N \in \mathbb{N}$$

and this implies easily the second assertion of Theorem 6. □

Let V, $[E_0, E_1]$ and δ_1 be as in Theorem 6. For $f \in C_0^\infty(\mathbb{R})$, introduce the measure

$$\langle \mu, f\rangle = \int \Big(f(x_1 + V(x)) - f(x_1)\Big) dx. \tag{7.15}$$

Since

$$|\langle \mu, f\rangle| \leq \sup|f'| \int |V(x)| dx,$$

μ is a distribution of first order. We denote by $\operatorname{singsupp}_a \mu$ the analytic singular support of μ.

Theorem 7. *Suppose the assumptions of Theorem 4 fulfilled. Let $\delta_1 > E_1 > \delta_1 - \|V\|_\infty$ and let $\lambda \in]E_0, E_1[\cap \operatorname{singsupp}_a \mu$. Then for every h-independent complex neighborhood Ω of λ there exist $h_0 = h(\Omega) > 0$ sufficiently small and $C = C(\Omega) > 0$ so that for $h \in]0, h_0[$ we have*

$$\#\{z \in \Omega :\ z \in \operatorname{Res}(P_2(h))\} \geq C(\Omega) h^{-n}.$$

Proof. Let $\psi \in C_0^\infty(\mathbb{R})$ be a cut-off function so that $\psi = 1$ near $[\delta_1 - \|V\|_\infty - 1,\ E_1 + |\delta_1| + \|V\|_\infty + 3]$. For $f \in C_0^\infty(]E_0, E_1[)$ introduce the distributions

$$\langle \tilde{\mu}, f\rangle = \int \Big(f(\psi(x_1)x_1 + V(x)) - f(\psi(x_1)x_1)\Big) dx,$$

$$\langle \tilde{\omega}, f\rangle = (2\pi)^{-n} \int \Big(f(|\xi|^2 + \psi(x_1)x_1 + V(x)) - f(|\xi|^2 + \psi(x_1)x_1)\Big) dx d\xi.$$

Clearly, $\tilde{\mu} \in \mathcal{E}'(\mathbb{R})$ and $\tilde{\mu} = \mu$, $\tilde{\omega} = \omega$ on $]E_0, E_1[$, where the distribution ω has been introduced in Theorem 4. Next the proof follows with minor modifications that in [35] and we will present only the main steps.

Denote by $WF_a(.)$ the analytic wave front. As it was shown in [35], we have

$$\mathrm{WF}_a(\tilde{\mu}) = \mathrm{WF}_a(\tilde{\omega}).$$

Since $\tilde{\omega}$ is real, it is clear that $(\lambda, 1)$ and $(\lambda, -1)$ are in $\mathrm{WF}_a(\tilde{\omega})$. From the definition of the analytic wave front set by the F.B.I transformation, it follows that there exist sequences $(\alpha_j, \beta_j) \to (\lambda, 1)$, $\gamma_j \nearrow +\infty$ and $\epsilon_j \searrow 0$ such that

$$\Big| \int (f_j \widetilde{\chi})(t) \tilde{\omega}(t)\, dt \Big| \geq e^{-\epsilon_j \gamma_j}, \tag{7.16}$$

where $f_j(t) = e^{i\gamma_j(\alpha_j - t)\beta_j} e^{-\frac{\gamma_j}{2}(\alpha_j - t)^2}$ and $\widetilde{\chi}$ is a cut-off function supported in a small real neighborhood of λ, and equal to 1 near λ.

Let a be a small positive constant. Set $\Omega_0 :=]\lambda - 2a, \lambda + 2a[+ i] - 2a^2, a^2]$ and $\Omega :=]\lambda - a, \lambda + a[+ i] - a^2, a^2]$. By construction, there exist $C_0 > 0$ such that for j large enough we have

$$\big| f_j(t) \big| \leq e^{-\frac{1}{C_0}\gamma_j} \quad \text{uniformly for} \quad t \in \Omega_0 \setminus \Omega,\, \mathrm{Im}\, t \leq 0. \tag{7.17}$$

Let $\chi(t) \in C_0^\infty(]\lambda - 2a, \lambda + 2a[; [0, 1])$ with $\chi(t) = 1$ on $]\lambda - a, \lambda + a[$. For $a > 0$ sufficiently small the inequality (7.16) remains true for $\widetilde{\chi}$ replaced by χ.

Applying Theorem 4, we get

$$\mathrm{tr}\big[(\chi f_j)(P_2(h)) - (\chi f_j)(P_1(h))\big] = (2\pi h)^{-n} \int (\chi f_j)(t) \tilde{\omega}(t)\, dt + \mathcal{O}_j(h^{1-n}). \tag{7.18}$$

Here we have used that we may write the leading term with the cut-off function $\psi(x_1)$ and for this reason we use the distribution $\tilde{\omega}$. On the other hand, an application of Theorem 3 with $f = f_j, \psi = \chi$ and $\Omega_{\theta,\alpha}$ replaced by Ω, yields

$$\mathrm{tr}\big[(\chi f_j)(P_2(h)) - (\chi f_j)(P_1(h))\big] = \sum_{z \in \mathrm{Res}(P_2(h)) \cap \Omega_-} f_j(z) + \mathcal{O}(1) h^{-n} e^{-\frac{1}{C_0}\gamma_j}, \tag{7.19}$$

where $\Omega_- = \{w \in \Omega;\ \mathrm{Im}(w) < 0\}$. Combining (7.18) and (7.19), we get

$$\sum_{z \in \mathrm{Res}(P_1(h)) \cap \Omega_-} f_j(z) = (2\pi h)^{-n} \int (\chi f_j)(t) \tilde{\omega}(t)\, dt + \mathcal{O}(h^{-n}) e^{-\frac{1}{C_0}\gamma_j} + \mathcal{O}_j(h^{1-n}),$$

which together with (7.16) imply

$$\Big| \sum_{z \in \mathrm{Res}(P_2(h)) \cap \Omega_-} f_j(z) \Big| \geq (2\pi h)^{-n} \Big[e^{-\epsilon_j \gamma_j} - \mathcal{O}(1) e^{-\frac{1}{C_0}\gamma_j} \Big] + \mathcal{O}_j(h^{1-n}).$$

Fixing j large enough and then taking h sufficiently small, we conclude that

$$\Big| \sum_{z \in \mathrm{Res}(P_2(h)) \cap \Omega_-} f(z) \Big| \geq \frac{1}{C} h^{-n}, \tag{7.20}$$

where $f = f_j$ (with j fixed) is independent of h. Thus we obtain a lower bound on the number of resonances in Ω_- and since we can choose Ω as small as we wish, the proof is complete. □

8. Appendix

Our purpose is to construct an operator $\widehat{P}_{2,\theta}$ which satisfies (6.3) and (6.4). Introduce

$$0 < c_0 = \frac{1}{C_0} \min\{\epsilon/2,\ (\alpha - 1) c_1 \,\mathrm{Im}\, \theta\} \leq \min\{\epsilon/2,\ (\alpha - 1) c_1 \,\mathrm{Im}\, \theta\},$$

where C_0, c_1 are the constants of Lemma 3, ϵ is given by the analytic distortion and $0 < \alpha < 1$ is the constant in the definition of Ω. We assume below all these constants fixed.

Since $\Big(V \circ \Phi_\theta\Big)$ satisfies (5.2), we may choose $R_1 > 0$ large enough so that

$$\sup_{|x| \geq R_1 - 2} \Big| \Big(V \circ \Phi_\theta\Big)\Big| \leq \frac{c_0}{3}. \tag{8.1}$$

We fix below R_1 with the above property. Let $\chi_0 + \chi_1 = 1$, where $\chi_0 \in C_0^\infty(\mathbb{R}^n;\, [0,1])$ is equal to 1 near $B(0, R_1)$ and $\chi_0 = 0$ for $|x| \geq R_1 + 1$. Let $\chi_j \prec \widetilde{\chi}_j$, where the function $\widetilde{\chi}_j$ has a support close to that of χ_j.

Choose a real-valued smooth function $f(t)$ satisfying $f(t) - t \geq M$ for $0 \leq t \leq M$, $f(t) = t$ for $t > 2M$ and $f(t) - t \geq 0$, $f(t) \geq M > 0$ for all $t \in \mathbb{R}^+$. Here M is a large constant which will be fixed in the formula (8.5). Recall that, taking θ_0 small enough for $|\theta| \leq \theta_0$ we have

$$\langle \operatorname{Re} a_\theta(x)\xi, \xi\rangle \geq \frac{1}{2}|\xi|^2 \tag{8.2}$$

uniformly on $x \in \mathbb{R}^n$.

Introduce the operator

$$\widetilde{P}_\theta := \widetilde{\chi}_0(x)\Big(f(h^2 D_x^2) - (hD_x)^2\Big)\chi_0(x) + P_{2,\theta}.$$

Let $\psi_0^2 + \psi_1^2 = 1$, where ψ_j has the same support properties as χ_j. We assume that $\psi_1 \prec \chi_1$. Combining (8.1) with the estimate (5.9), we obtain

$$\|(P_{2,\theta} - z)\psi_1 u\| > C_1 \|\psi_1 u\|,\ \ \forall u \in \mathcal{D} \tag{8.3}$$

for all $z \in \overline{\Omega}$. On the other hand, since $\chi_0\psi_1 = 0$ and $\chi_1\psi_1 = \psi_1$, we have

$$(\widetilde{P}_\theta - z)\psi_1 u = (\widetilde{P}_\theta - z)\chi_1\psi_1 u = (P_{2,\theta} - z)\chi_1\psi_1 u.$$

Then, for $z \in \overline{\Omega}$ and $u \in \mathcal{D}$, the estimate (8.3) yields

$$\|(\widetilde{P}_\theta - z)\psi_1 u\| \geq C_1\|\psi_1 u\|. \tag{8.4}$$

Let $\widetilde{\psi}_0 \in C_0^\infty(\mathbb{R}^n;\, [0,1])$ have support close to that of ψ_0 and assume $\psi_0 \prec \widetilde{\psi}_0$. Let $\mathcal{G} \in C_b^\infty(\mathbb{R};\mathbb{R})$ be bounded with all its derivatives. We choose $\mathcal{G}$ so that $\mathcal{G}(x_1) \geq 2\epsilon$ for $x_1 \geq 2\epsilon$ and $\mathcal{G}(x_1 - \operatorname{Re} z) = x_1 - \operatorname{Re} z$ for $x_1 \in \operatorname{supp}_{x_1}\psi_0$ and $z \in \overline{\Omega}_1$.

Next, we choose M so that

$$f(|\xi|^2) - |\xi|^2 + \langle \operatorname{Re} a_\theta(x)\xi, \xi\rangle - \sup_{|x|\leq R_1}|V \circ \Phi_\theta(x)| - \|\mathcal{G}\|_\infty - |\theta_0|\|v\|_\infty > c(1 + |\xi|^2), \tag{8.5}$$

for some positive constant c independent on $x \in \mathbb{R}^n$ and $z \in \overline{\Omega}$.

We define

$$\widetilde{P}_{1,\theta,z} := \widetilde{\chi}_0(x)\Big(f(h^2D_x^2) - (hD_x)^2\Big)\chi_0(x)$$

$$-h^2\nabla\Big(a_\theta(x)\nabla\Big) + \mathcal{G}(x_1 - \operatorname{Re} z) + \Big(V \circ \Phi_\theta\Big)(x)\widetilde{\psi}_0(x) + \theta v(x_1) - i \operatorname{Im} z + h^2 g(x_1).$$

Clearly,

$$(\widetilde{P}_\theta - z)\psi_0 u = \widetilde{P}_{1,\theta,z}\psi_0 u,\ \forall u \in \mathcal{D},\ z \in \overline{\Omega}. \tag{8.6}$$

Exploiting (8.5), we will show that $\widetilde{P}_{1,\theta,z}$ is globally elliptic for $z \in \overline{\Omega}$. By using the h-pseudodifferential calculus, we will construct a parametrix for $\widetilde{P}_{1,\theta,z}$ and estimate its norm.

The principal symbol of $\widetilde{P}_{1,\theta,z}$ has the form

$$\tilde{p}_{1,\theta,z}(x,\xi) := \Big(f(|\xi|^2) - |\xi|^2\Big)\chi_0(x) + \langle a_\theta(x)\xi, \xi\rangle + \mathcal{G}(x_1 - \operatorname{Re} z) + \Big(V \circ \Phi_\theta\Big)(x)\tilde{\psi}_0(x) + \theta v(x_1) - i \operatorname{Im} z.$$

Obviously, $\widetilde{p}_{1,\theta,z}(x,\xi) \in S((1+|\xi|^2))$ and

$$|\tilde{p}_{1,\theta,z}(x,\xi)| \geq \tilde{c}|\xi|^2, \;\; \tilde{c} > 0, \; |\xi| >> 1. \tag{8.7}$$

Here we use fairly the notations and the terminology for symbol spaces (see for instance, [10]). We claim that

$$|\widetilde{p}_{1,\theta}(x,\xi,z)| \geq c_2(1+|\xi|^2), \; c_2 > 0 \tag{8.8}$$

uniformly on $z \in \overline{\Omega}$ and $(x,\, \xi) \in \mathbb{R}^{2n}$. For $|x| \leq R_1$, we have $\chi_0(x) = 1$, and from (8.5) we deduce

$$|\operatorname{Re} \widetilde{p}_{1,\theta}(x,\xi,z)| \geq c(1+|\xi|^2), \tag{8.9}$$

uniformly on $z \in \overline{\Omega}$ and $|x| \leq R_1$.

For $|x| \geq R_1$ and $x_1 - R_0 > -\epsilon$, we have $x_1 - \operatorname{Re} z > 2\epsilon$, since $\operatorname{Re} z \leq R_0 - 3\epsilon$ for $z \in \Omega$ (here R_0 is given by the definition of the set $\Omega = \Omega_{\theta,\alpha}$). Combining this with (8.1), (8.2), and using $\big(f(|\xi|^2) - |\xi|^2\big)\chi_0(x) \geq 0$ as well as the inequality $\mathcal{G}(t) \geq 2\epsilon$ for $t > 2\epsilon$, we obtain

$$\operatorname{Re} \widetilde{p}_{1,\theta}(x,\xi,z) = \Big(f(|\xi|^2 - |\xi|^2\Big)\chi_0(x) \tag{8.10}$$

$$+\langle \operatorname{Re} a_\theta(x)\xi, \xi\rangle + \mathcal{G}(x_1 - \operatorname{Re} z) + \operatorname{Re}\Big(V \circ \Phi_\theta\Big)(x)\tilde{\psi}_0(x) \geq c_3(\epsilon + |\xi|^2), \; c_3 > 0,$$

uniformly on $z \in \overline{\Omega}$, $|x| \geq R_1$ and $x_1 - R_0 > -\epsilon$.

For $|x| \geq R_1$ and $x_1 - R_0 \leq -\epsilon$, we repeat the arguments of the proof of Lemma 3. More precisely, applying the inequalities $\operatorname{Im} a_\theta \leq 0$, $\operatorname{Im} \theta < 0$ and $\operatorname{Im} z \geq \alpha(1 - e^{-\epsilon}) \operatorname{Im} \theta$, we get

$$|\operatorname{Im} \widetilde{p}_{1,\theta}(x,\xi,z)| = |\langle \operatorname{Im} a_\theta(x)\xi, \xi\rangle \tag{8.11}$$

$$+ \operatorname{Im}\Big(V \circ \Phi_\theta\Big)(x)\tilde{\psi}_0(x) + \operatorname{Im} \theta v(x_1) - \operatorname{Im} z| \geq 2c_0/3,$$

uniformly on $z \in \overline{\Omega}$, $|x| \geq R_1$ and $x_1 - R_0 \leq -\epsilon$. Summing up the estimates (8.7), (8.9), (8.10) and (8.11), we obtain (8.8).

Applying a classical result for elliptic operators, we deduce from (8.8) that for h_0 small enough the operator $\widetilde{P}_{1,\theta,z}$ is invertible for $h \in]0, h_0]$ and $\|\widetilde{P}_{1,\theta,z}^{-1}\| = \mathcal{O}(1)$, uniformly on $z \in \overline{\Omega}$. Combining this with (8.6), we get

$$\|(\widetilde{P}_\theta - z)\psi_0 u\| \geq C_2\|\psi_0 u\|, \; \forall u \in \mathcal{D}, \tag{8.12}$$

uniformly on $z \in \overline{\Omega}$.

Taking together (8.4), (8.12), and using an estimate for the commutator $[\psi_j, \widetilde{P}_\theta]$, similar to (5.11), for $z \in \overline{\Omega}$ and h small we deduce

$$\|(\widetilde{P}_\theta - z)u\|^2 = \sum_{j=0}^{1} \|\psi_j(\widetilde{P}_\theta - z)u\|^2 \tag{8.13}$$

$$\geq \frac{1}{2}\sum_{j=0}^{1}\|\widetilde{P}_\theta - z)\psi_j u\|^2 - \sum_{j=0}^{1}\|[\psi_j, \widetilde{P}_\theta]u\|^2 \geq c_4\|u\|^2,\ c_4 > 0,\ u \in \mathcal{D}.$$

Exploiting once more the fact that the Weyl symbol of $\tilde{P}_\theta - P_{2,\theta}$ has compact support in x and ξ, we conclude that $\tilde{P}_\theta - z$ is a Fredholm operator with index 0. Consequently, we have proved the following.

Lemma 5. *For h small enough and $z \in \overline{\Omega}$ the operator $(\widetilde{P}_\theta - z)$ is invertible and*

$$(z - \widetilde{P}_\theta)^{-1} = \mathcal{O}(1) : L^2(\mathbb{R}^n) \to \mathcal{D}.$$

By construction, we have $\widetilde{P}_\theta - P_{2,\theta} = \tilde{\chi}_0\Big(f(-h^2\Delta) + h^2\Delta\Big)\chi_0 = \tilde{\chi}_0 Q(-h^2\Delta)\chi_0$ with $Q \in C_0^\infty(\mathbb{R})$. Let $W \in C^\infty(\mathbb{R}^n; \mathbb{R}^+)$ with $W = 0$ near $\operatorname{supp}\tilde{\chi}_0$ and $W = |x|^2$ near infinity. Obviously, the operator $(-h^2\Delta + W)$ has discrete spectrum. This implies that

$$K = \tilde{\chi}_0 Q(-h^2\Delta + W)\chi_0 \text{ has rank } \mathcal{O}(h^{-n}).$$

On the other hand, the h-pseudodifferential calculus (see [10]) shows that

$$\tilde{\chi}_0 Q(-h^2\Delta)\chi_0 - K = \mathcal{O}(h^\infty)$$

in $\mathcal{L}(L^2(\mathbb{R}^n))$. Thus we conclude that

$$\tilde{\chi}_0 Q(-h^2\Delta)\chi_0 = K + T, \tag{8.14}$$

where K has rank $\mathcal{O}(h^{-n})$ and $\|T\| = \mathcal{O}(h^\infty)$.

Now it is clear that (6.3) and (6.4) are fulfilled with $\widehat{P}_{2,\theta} = \widetilde{P}_\theta - T$.

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