Galois module structure for wild extensions

Ph. Cassou-Noguès*, M.J. Taylor†

*Laboratoire A2X, Université Bordeaux1, 351 cours de la Libération, 33405 Talence-cedex, France

†University of Manchester, Institute of Science and Thechnology, Manchester, United Kingdom

Abstract

1. Introduction

The problem of studying rings of algebraic integers of number fields as a Galois module is an old and classical one. One of the high points of this study was Fröhlich's discovery of an unexpected connection between this problem and the behaviour of certain arithmetic invariants associated with representations of Galois groups.
For a number field or a local field F we denote by $\mathcal{O}_F$ its ring of integers.
Let N/K be a finite Galois extension of number fields and let G be its Galois group. One can associate to the symplectic representations of G their Artin root numbers and define, in the locally free class group $Cl(\mathbb{Z}[G])$ of $\mathbb{Z}[G]$ an element of order 1 or 2, the so called root number class, denoted by $W_{N/K}$. In [T$_1$], when N/K is at most tamely ramified (see also [F] for precise definitions), the second named author proved the following equality

$$U_{N/K} = W_{N/K} \tag{1.1}$$

where $U_{N/K}$ is defined by $[\mathcal{O}_N] - [\mathcal{O}_K[G]]$ in $Cl(\mathbb{Z}[G])$.
When N/K is a wild extension one can define, for any order Λ in $\mathbb{Q}[G]$, a root number class, an element of order 1 or 2 in $Cl(\Lambda)$, denoted again $W_{N/K}$. This raises the interesting question of defining, in this case, an analogue of $U_{N/K}$ and proving an analogue of (1.1). In [C] T.Chinburg introduced an invariant $\Omega(N/K, 2)$ which coincides with $U_{N/K}$ when N/K is tame and conjectured the following equality in $Cl(\mathbb{Z}[G])$

$$\Omega(N/K, 2) = W_{N/K}. \tag{1.2}$$

This conjecture has not been proved yet in full generality.
Let S be the set of those rational primes numbers below the wildly ramified primes of N/K. In a series of papers [H-W$_1$], [H-W$_2$], [H-W$_3$] D. Holland and S.M.J. Wilson developed their theory of factorizabilty defect in order to deal with this problem. In 1991 they finally proved, [H-W$_3$], Queyrut's conjecture, namely the equality

$$U^S_{N/K} = W_{N/K} \tag{1.3}$$

in Queyrut's class group. This group may be understood as the classical group $Cl(\mathcal{U}(N/K))$ where $\mathcal{U}(N/K)$ is the order of $\mathbb{Q}[G]$ such that $\mathcal{U}(N/K)_p = \mathbb{Z}_p[G]$ (resp. a maximal order in $\mathbb{Q}_p[G]$) if $p \notin S$ (resp. $p \in S$).
The aim of this paper is to develop a new approach to these questions by generalising to non abelian extensions $N/\mathbb{Q}$ the ideas first introduced in [T$_2$]. From now on we restrict ourselves to the case where $K = \mathbb{Q}$. Our strategy is the following: we first define a "good order" $\Lambda(N/\mathbb{Q})$, denoted by Λ, in $\mathbb{Q}[G]$, depending on the ramification of $N/\mathbb{Q}$, which coincides with $\mathbb{Z}[G]$ whenever $N/\mathbb{Q}$ is tame. It may happen that $\mathcal{O}_N$ is not be stable under Λ-action. Hence we have introduced our

“adjusted” ring of integers as a sub-lattice of $\mathcal{O}_N$ and a Λ-module, closely related to $\mathcal{O}_N$, that we denote by $\check{\mathcal{O}}_N$. After proving that $\check{\mathcal{O}}_N$ is a Λ-locally free module this article is devoted to the study of the class

$$\check{U}_N = [\check{\mathcal{O}}_N] - [\Lambda]$$

in $Cl(\Lambda)$.
We now can be more precise. For any $p \in S$ we fix a maximal ideal of $\mathcal{O}_N$ above p. We denote by N_p the completion of N at this prime ideal and we let $G(p)$ (resp. $G_0(p)$, resp. $G_1(p)$) be the Galois group of $N_p/\mathbb{Q}_p$ (resp. the inertia, resp. the first ramification) subgroup of $N_p/\mathbb{Q}_p$. We will say that hypothesis (H_p) is satisfied whenever $G_1(p)$ is a commutative, direct factor of $G(p)$ and a normal subgroup of G. The extension $N/\mathbb{Q}$ satisfies hypothesis (H) when it satisfies (H_p) for any $p \in S$. From now on we shall always assume (H) to be satisfied.

Definition 1.4 *Let Λ be the order of $\mathbb{Q}[G]$ such that for any prime divisor of $\mathbb{Q}$*

$$\Lambda_p = \mathfrak{M}(G_1(p)) \otimes_{\mathbb{Z}_p[G_1(p)]} \mathbb{Z}_p[G]$$

where $\mathfrak{M}(G_1(p))$ is the maximal order of $\mathbb{Q}_p[G_1(p)]$.

We now decompose $G_0(p)$ into a direct product

$$G_0(p) = G_1(p) \times G_0'(p) \tag{1.5}$$

where $G_0'(p)$ is cyclic of order prime to p.
We define

$$M_p = N_p^{G_0(p)}, N_p^{(t)} = N_p^{G_1(p)}, N_p^{(w)} = N_p^{G_0'(p)} \tag{1.6}$$

and we put

$$\check{\mathcal{O}}_{N_p} = \check{\mathcal{O}}_{N_w} \otimes_{\mathcal{O}_{M_p}} \mathcal{O}_{N_p^{(t)}} \tag{1.7}$$

where $\check{\mathcal{O}}_{N_w}$ is the largest $\mathcal{O}_{M_p}$ submodule of $\mathcal{O}_{N_w}$ stable under $\mathfrak{M}_{M_p}(G_1(p))$, the maximal order of $M_p[G_1(p)]$.

Definition 1.8 *Let $\check{\mathcal{O}}_N$ be the unique sublattice of $\mathcal{O}_N$ such that for any prime divisor of $\mathbb{Q}$*

$$\check{\mathcal{O}}_{N,p} = \check{\mathcal{O}}_{N_p} \otimes_{\mathbb{Z}_p[G(p)]} \mathbb{Z}_p[G].$$

Theorem *Under hypothesis (H) one has $\check{U}_N^2 = 1$. Moreover when 2 is at most tamely ramified in $N/\mathbb{Q}$, then $\check{U}_N = W_N$.*

Corollary *Under hypothesis (H) if $[N : \mathbb{Q}]$ is odd, then $\check{\mathcal{O}}_N$ is a Λ- free rank one module.*

Remarks

1. (H_p) is satisfied for any p in S when $N/\mathbb{Q}$ is either abelian or tame. In this last case our theorem is the equality (1.1).

2. We assume that $G_1(p)$ is a normal subgroup of G in order to obtain an order in $\mathbb{Q}[G]$. The fact that $G_1(p)$ is a direct factor of $G(p)$ will be used in the local case to assure the stability of $\mathrm{Det}({\Lambda_p}^\times)$ under Adams operations and a "fixed point theorem" [CN-T$_2$], Theorems 1 and 2. In the case where $G(p)$ is abelian, we can replace hypothesis (H_p) by hypothesis (H'_p) with

   (H'_p) $\qquad$ $G_1(p)$ is a normal subgroup of G.

   For example, suppose that $N/\mathbb{Q}$ is a dihedral extension of degree $2p^n$ with p an odd prime and assume that 2 is split in the quadratic subextension $k/\mathbb{Q}$ of $N/\mathbb{Q}$ then hypothesis (H) is satisfied.

Compared to [H-W] our result has the disadvantage that it is obtained under the restrictive hypothesis (H). Nevertheless it also has its advantages: the order $\Lambda(N/\mathbb{Q})$, that we are working with, is in general strictly contained in $\mathcal{U}(N/K)$ and in particular is not a maximal order at the primes in S. Therefore one of the ingredients of the proof of our main result provides a connection between local resolvents and local Gauss sums at the wild primes which is new and interesting.*
The organisation of this paper is as follows: In section 2 we deal with the local situation. In section 3 we use Adams operators to obtain a new representative for $\check{U}_N$ in Fröhlich's Hom-description of $Cl(\Lambda)$. It plays an important role since it allows us to work separately with tame and totally ramified wild extensions in the local wild situation. The proof of the theorem then splits into three parts; section 4 concerns the tame situation, section 5 deals with the wild situation while section 6 contains the final step of the proof.

2. The local case

In this section we consider a finite Galois extension $N/\mathbb{Q}_p$ whose Galois group we denote by G. We let G_0, G_1, G'_0 be the groups $G_0(p)$, $G_1(p)$, $G'_0(p)$ introduced in section 1. Under hypothesis (H_p) then G_1 is a commutative direct factor of G. For an extension $L/\mathbb{Q}_p$ and for a finite abelian group H, we let $\mathfrak{M}_L(H)$ denote the maximal order of $L[H]$.

Definition 2.1 *The order $\Lambda_L(N/\mathbb{Q}_p)$ in $L[G]$ is then defined by*

$$\Lambda_L(N/\mathbb{Q}_p) = \mathfrak{M}_L(G_1) \otimes_{\mathbb{Z}_p[G_1]} \mathbb{Z}_p[G].$$

*Moreover the proof of the equality between $\check{U}_N$ and the projection of $\Omega(N/\mathbb{Q}, 2)$ in $Cl(\Lambda)$ will provide further evidences towards conjecture (1.2)

In the sequel we frequently abbreviate $\Lambda_L(N/\mathbb{Q}_p)$ to Λ_L and $\Lambda_{\mathbb{Q}_p}(N/\mathbb{Q}_p)$ to Λ. Writing $\hat{H}$ for the group of abelian characters of H, we obtain $\hat{G}_0 = \hat{G}'_0 \times \hat{G}_1$. We then deduce at once the equalities:

$$\mathfrak{M}_L(G_0) = \mathfrak{M}_L(G_1) \otimes_{\mathcal{O}_L} \mathcal{O}_L[G'_0] \tag{2.2}$$

and therefore

$$\Lambda_L = \mathfrak{M}_L(G_0) \otimes_{\mathbb{Z}_p[G_0]} \mathbb{Z}_p[G]. \tag{2.3}$$

Let M be the maximal non-ramified sub-extension of $N/\mathbb{Q}_p$.

Lemma 2.4 *The order Λ_M is a free Λ-module of rank $[M : \mathbb{Q}_p]$.*

Proof. By definition $\mathfrak{M}_{\mathbb{Q}_p}(G_1) \subset \mathfrak{M}_M(G_1)$ and $\mathfrak{M}_{\mathbb{Q}_p}(G_1) \otimes_{\mathbb{Z}_p} M = \mathfrak{M}_M(G_1) \otimes_{\mathbb{Z}_p} M$. Therefore $\mathfrak{M}_M(G_1)$ is a free $\mathfrak{M}_{\mathbb{Q}_p}(G_1)$-module of rank $[M : \mathbb{Q}_p]$ and the result follows for Λ_M from the definitions. □

We now introduce our "adjusted ring of integers" in this local context. We set $N^{(t)} = N^{G_1}$, $N^{(w)} = N^{G'_0}$. We then consider $\mathcal{O}_{N^{(t)}}$ the ring of integers of $N^{(t)}$ and define $\check{\mathcal{O}}_{N^{(w)}}$ as the largest $\mathcal{O}_M$-submodule of $\mathcal{O}_{N^{(w)}}$ stable under $\mathfrak{M}_M(G_1)$.

Definition 2.5 *We define*

$$\check{\mathcal{O}}_N = \check{\mathcal{O}}_{N^{(w)}} \otimes_{\mathcal{O}_M} \mathcal{O}_{N^{(t)}}.$$

The rest of this section is devoted to the proof of the following result:

Theorem 2.6 *$\check{\mathcal{O}}_N$ is a free rank one Λ-module.*

Proof. The proof splits into three Lemmas.

Lemma 2.7 *$\check{\mathcal{O}}_N$ is a free $\mathfrak{M}_M(G_0)$-module of rank 1.*

Proof. Since $N^{(t)}/M$ is tame we know by Noether's Theorem that $\mathcal{O}_{N^{(t)}}$ is free over $\mathcal{O}_M[G'_0]$. It therefore suffices to show that $\check{\mathcal{O}}_{N^{(w)}}$ is free over $\mathfrak{M}_M(G_1)$. In fact since $\mathfrak{M}_M(G_1)$ is commutative and since $\check{\mathcal{O}}_{N^{(w)}} \otimes_{\mathcal{O}_M} M = N^{(w)}$ is free over $\mathfrak{M}_M(G_1) \otimes_{\mathcal{O}_M} M = M[G_1]$ by the normal basis theorem it suffices to prove that $\check{\mathcal{O}}_{N^{(w)}}$ is projective over $\mathfrak{M}_M(G_1)$, which is true since it is a finitely generated torsion free $\mathcal{O}_M$-module. □

Lemma 2.8 *$\check{\mathcal{O}}_N$ is a projective Λ-module of rank one.*

Proof. In order to show that $\check{\mathcal{O}}_N$ is a Λ-module, we start by proving that G acts on $\check{\mathcal{O}}_{N^{(w)}}$. It follows from the definition of $\check{\mathcal{O}}_{N^{(w)}}$ that

$$\check{\mathcal{O}}_{N^{(w)}} = \{x \in \mathcal{O}_{N^{(w)}} | x\lambda \in \mathcal{O}_{N^{(w)}}, \forall \lambda \in \mathfrak{M}_M(G_1)\}.$$

Let x (resp. g) be an element of $\check{\mathcal{O}}_{N^{(w)}}$ (resp. G). For any $\lambda \in \mathfrak{M}_M(G_1)$ one has the equality $(xg)\lambda = x(g\lambda g^{-1})g$.
Since G_1 is abelian $\mathfrak{M}_M(G_1)$ is the unique $\mathcal{O}_M$ maximal order of $M[G_1]$ and therefore is stable under conjugation. Hence $(g\lambda g^{-1}) \in \mathfrak{M}_M(G_1)$ and $(xg)\lambda \in \mathcal{O}_{N^{(w)}}$. We then conclude that $\check{\mathcal{O}}_{N^{(w)}}$ is stable under G-action. If we now let G act diagonally on the product $\check{\mathcal{O}}_{N^{(w)}} \times \mathcal{O}_{N^{(t)}}$ one can easily check that it induces an action of G on $\check{\mathcal{O}}_N$ and therefore induces a structure of Λ-module over $\check{\mathcal{O}}_N$.
Since $\check{\mathcal{O}}_N \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = N$ and $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}_p = \mathbb{Q}_p[G]$, it follows from the normal basis theorem that the rank of $\check{\mathcal{O}}_N$ over Λ is equal to 1. We deduce from Lemma 2.7 that there exists an isomorphism of $\mathfrak{M}(G_0)$-module

$$\pi : \mathfrak{M}_M(G_0) \to \check{\mathcal{O}}_N.$$

Following *mutatismutandis* the method of [T$_2$], Proposition 2.1, we obtain from π a split surjection

$$\overline{\pi} : \Lambda_M \to \check{\mathcal{O}}_N$$

of Λ-modules. Since, by Lemma 2.4, we know that Λ_M is Λ free we conclude that $\check{\mathcal{O}}_N$ is projective. □

We now wish to complete the proof of Theorem 2.6. Since $\check{\mathcal{O}}_N$ is $\mathbb{Z}_p$-torsion free, we first observe that it suffices to show that

$$\check{\mathcal{O}}_N \simeq \Lambda \tag{2.9}$$

as $\mathbb{Z}[G]$-modules. Moreover since ,$\mathcal{O}_M$ is $\mathbb{Z}_p$-free, (2.9) will follow from the isomorphism

$$\mathcal{O}_M \otimes_{\mathbb{Z}_p} \check{\mathcal{O}}_N \simeq \mathcal{O}_M \otimes_{\mathbb{Z}_p} \Lambda \tag{2.10}$$

as $\mathcal{O}_M[G]$-modules, [C-R]. For any $\mathcal{O}_M[G_0]$-module A we define the algebra $\mathrm{Map}_{G_0}(G, A)$ of set maps of G_0-sets $G \to A$, G_0 acting on G by right multiplication. This is an $\mathcal{O}_M$-module, with addition and $\mathcal{O}_M$-action being defined via that in A. Moreover this is a G-module where for $h \in G$ and $f \in \mathrm{Map}_{G_0}(G, A)$ we define

$$f^h(g) = f(hg), \forall g \in G.$$

Suppose that A and B are isomorphic as $\mathcal{O}_M[G_0]$-module then

$$\mathrm{Map}_{G_0}(G, A) \simeq \mathrm{Map}_{G_0}(G, B) \tag{2.11}$$

as $\mathcal{O}_M[G]$-modules. Hence, since we know from Lemma 2.7 that $\check{\mathcal{O}}_N$ and $\mathfrak{M}_M(G_0)$ are isomorphic $\mathcal{O}_M[G_0]$-modules, (2.10) will be a consequence of the lemma:

Lemma 2.12 *We have the following isomorphisms of $\mathcal{O}_M[G]$-modules*

i.) $\mathcal{O}_M \otimes_{\mathbb{Z}_p} \Lambda \simeq \mathrm{Map}_{G_0}(G, \mathfrak{M}_M(G_0))$;

ii.) $\mathcal{O}_M \otimes_{\mathbb{Z}_p} \check{\mathcal{O}}_N \simeq \mathrm{Map}_{G_0}(G, \check{\mathcal{O}}_N)$.

Proof. We first note that $\mathcal{O}_M \otimes_{\mathbb{Z}_p} \Lambda = \mathfrak{M}_M(G_0) \otimes_{\mathbb{Z}_p[G_0]} \mathbb{Z}_p[G]$. We then introduce the map

$$\theta : \begin{cases} \mathrm{Map}_{G_0}(G, \mathfrak{M}_M(G_0)) & \to & \mathcal{O}_M \otimes_{\mathbb{Z}_p} \Lambda \\ f & \mapsto & \sum_g f(g^{-1}) \otimes g \end{cases}$$

where g runs through a transversal of G/G_0. This is a homomorphism of $\mathcal{O}_M$-modules whose definition does not depend on the choice of this transversal. For $h \in G$ we obtain

$$\theta(f^h) = \sum_g f(hg^{-1}) \otimes g = \sum_u f(u^{-1}) \otimes uh = \theta(f)h.$$

Hence θ is a G-map.
It follows from $\theta(f) = 0$ that $f(g) = 0\ \forall g \in G/G_0$. As $f(gh) = f(g)h$ for $h \in G_0$ we deduce that $f = 0$. Thus we conclude that θ is injective.
Let $y = \sum_{g \in G/G_0} y_g \otimes g$ with $y_g \in \mathfrak{M}_M(G_0), \forall g \in G/G_0$. There exists a map $f \in \mathrm{Map}_{G_0}(G, \mathfrak{M}_M(G_0))$ which satisfies $f(g^{-1}) = y_g, \forall g \in G/G_0$ and consequently $\theta(f) = y$. We then have proved that θ is surjective. Therefore θ is an isomorphism of $\mathcal{O}_M[G]$-modules as we had to show. We now prove *ii.)* Following [F], III, Lemma 6.2, we introduce the algebra homomorphism

$$j : \begin{cases} M \otimes_{\mathbb{Q}_p} N & \to & \mathrm{Map}_{G_0}(G, N) \\ x \otimes y & \mapsto & j(x \otimes y) : g \mapsto xy^g \end{cases}.$$

This is an isomorphism of $M[G]$-modules. The restriction of j to $\mathcal{O}_M \otimes_{\mathbb{Z}_p} \check{\mathcal{O}}_N$ induces an injection

$$j : \ \mathcal{O}_M \otimes_{\mathbb{Z}_p} \check{\mathcal{O}}_N \ \hookrightarrow \ \mathrm{Map}_{G_0}(G, \check{\mathcal{O}}_N) \ . \tag{2.13}$$

Since $\mathcal{O}_M$ is free over $\mathbb{Z}_p$, it follows from the inclusion $\check{\mathcal{O}}_N \subset \mathcal{O}_N$ that $\mathcal{O}_M \otimes_{\mathbb{Z}_p} \check{\mathcal{O}}_N \subset \mathcal{O}_M \otimes_{\mathbb{Z}_p} \mathcal{O}_N$.
Moreover, we know from [F], III, (6.5) that j induces, by restriction, an isomorphism

$$\mathcal{O}_M \otimes_{\mathbb{Z}_p} \mathcal{O}_N \simeq \mathrm{Map}_{G_0}(G, \mathcal{O}_N). \tag{2.14}$$

Thus, in order to prove that (2.13) is an isomorphism, it suffices to prove the equality of the indices

$$(2.15)\quad [\mathcal{O}_M \otimes_{\mathbb{Z}_p} \mathcal{O}_N : \mathcal{O}_M \otimes_{\mathbb{Z}_p} \check{\mathcal{O}}_N] = [\mathrm{Map}_{G_0}(G, \mathcal{O}_N) : \mathrm{Map}_{G_0}(G, \check{\mathcal{O}}_N)].$$

Let $\{e_k | 1 \leq k \leq m\}$ be a $\mathbb{Z}_p$-basis of $\mathcal{O}_M$. Since $\check{\mathcal{O}}_N$ is a $\mathbb{Z}_p$ sublattice of $\mathcal{O}_N$ of the same rank, there exists a $\mathbb{Z}_p$-basis $\{b_l | 1 \leq l \leq n\}$ of $\mathcal{O}_N$ and non-zero elements $\{a_l | 1 \leq l \leq n\}$ of $\mathbb{Z}_p$ such that $\{a_l b_l | 1 \leq l \leq n\}$ is a $\mathbb{Z}_p$-basis of $\check{\mathcal{O}}_N$. Using the basis $\{e_k \otimes b_l | 1 \leq k \leq m, 1 \leq l \leq n\}$ for $\mathcal{O}_M \otimes_{\mathbb{Z}_p} \mathcal{O}_N$ and $\{a_l(e_k \otimes b_l) 1 \leq k \leq m, 1 \leq l \leq n\}$ for $\mathcal{O}_M \otimes_{\mathbb{Z}_p} \check{\mathcal{O}}_N$ we immediately obtain that

$$(2.16)\qquad [\mathcal{O}_M \otimes_{\mathbb{Z}_p} \mathcal{O}_N : \mathcal{O}_M \otimes_{\mathbb{Z}_p} \check{\mathcal{O}}_N] = [\mathcal{O}_N : \check{\mathcal{O}}_N]^{[M:\mathbb{Q}_p]}.$$

We now fix a transversal $\{g_k | 1 \leq k \leq m\}$ of G/G_0. Let $\{u_{k,l} | 1 \leq k \leq m, 1 \leq l \leq n\}$ be the set of elements of $\mathrm{Map}_{G_0}(G, \mathcal{O}_N)$ defined by

$$u_{k,l}(g_t) = \begin{cases} b_l & \text{if} \quad k = t \\ 0 & \text{otherwise.} \end{cases}$$

One easily verifies that $\{u_{k,l} | 1 \leq k \leq m, 1 \leq l \leq n\}$ is a $\mathbb{Z}_p$-basis of $\mathrm{Map}_{G_0}(G, \mathcal{O}_N)$ and that $\{a_l u_{k,l} | 1 \leq k \leq m, 1 \leq l \leq n\}$ a $\mathbb{Z}_p$-basis of $\mathrm{Map}_{G_0}(G, \check{\mathcal{O}}_N)$. We then conclude that

$$(2.17)\qquad [\mathrm{Map}_{G_0}(G, \mathcal{O}_N) : \mathrm{Map}_{G_0}(G, \check{\mathcal{O}}_N)] = [\mathcal{O}_N : \check{\mathcal{O}}_N]^{[M:\mathbb{Q}_p]}.$$

Therefore (2.15) follows from (2.16) and (2.17). This completes the proof of the lemma and of Theorem 2.6. $\square$

Remarks

1.) One should notice that Theorem (2.6) remains true under the weaker hypothesis of G_1 being abelian. Nevertheless we will need (H_p) when dealing with the global situation.

2.) When $N/\mathbb{Q}_p$ is tame (H_p) is satisfied with $G_1 = \{e\}$. In this case $\Lambda = \mathbb{Z}_p[G]$, $\check{\mathcal{O}}_N = \mathcal{O}_N$ and Theorem (2.6) is the well known local normal integral basis theorem of E. Noether.

3.) Theorem (2.6) suggests the following natural question: when is $\check{\mathcal{O}}_N = \mathcal{O}_N$? It follows from [T$_2$], Proposition 1.12, that $\check{\mathcal{O}}_N = \mathcal{O}_N$ if and only if one of the two following conditions is satisfied:

   i.) $N/\mathbb{Q}_p$ is tame;

   ii.) $N/\mathbb{Q}_p$ has p-power ramification index and $N^{(w)} = N/M$ is maximally ramified (i.e. $\mathcal{O}_{N^{(w)}}$ admits $\mathfrak{M}_M(G_0)$).

3. A representative for $\check{U}_N$

An important tool in the proof of our main result is Fröhlich's "Hom-description" of class groups. After fixing the notation, the aim of this section is to define a suitable representative for our class $\check{U}_N$.
Let $\mathbb{Q}^c$ be an algebraic closure of $\mathbb{Q}$ and let $E/\mathbb{Q}$ be a "big enough" finite Galois extension contained in $\mathbb{Q}^c/\mathbb{Q}$. For any rational prime divisor p of $\mathbb{Q}$, we consider an algebraic closure $\mathbb{Q}_p^c$ of $\mathbb{Q}_p$ and once for all we fix an embedding $j_p : \mathbb{Q}^c \hookrightarrow \mathbb{Q}_p^c$. Let $L/\mathbb{Q}$ be a finite Galois subextension of $E/\mathbb{Q}$ with $H = \mathrm{Gal}(L/\mathbb{Q})$. We denote by L_p (resp. $H(p)$) the closure of $j_p(L)$ in $\mathbb{Q}_p^c$ (resp. $\mathrm{Gal}(L_p/\mathbb{Q}_p)$). In fact $L_p \simeq L_\mathfrak{p}$ where $L_\mathfrak{p}$ is the completion of L at the prime $\mathfrak{p}$ defined by the restriction of j_p to L. We let j_p^* be the group isomorphism from $H(p)$ into H induced by j_p and defined by the rule

$$x^{j_p^*(\omega)} = x^{j_p \omega j_p^{-1}}, \forall x \in L, \forall \omega \in H(p). \tag{3.1}$$

We write $\Omega_\mathbb{Q}$ (resp. $\Omega_{\mathbb{Q}_p}$) for $\mathrm{Gal}(E/\mathbb{Q})$ (resp. $\mathrm{Gal}(E_p/\mathbb{Q}_p)$.
We now recall that $N/\mathbb{Q}$ is a finite Galois extension with $\mathrm{Gal}(N/\mathbb{Q}) = G$. We previously introduced the order $\Lambda(N/\mathbb{Q})$ of $\mathbb{Q}[G]$ by prescribing its local components:

$$\Lambda(N/\mathbb{Q})_p = \begin{cases} \Lambda(N_p/\mathbb{Q}_p) \otimes_{\mathbb{Z}_p[G(p)]} \mathbb{Z}_p[G] & \text{if} \quad p \neq 0. \\ \mathbb{Q}[G] & \text{if} \quad p = 0. \end{cases} \tag{3.2}$$

and our "adjusted" ring of integers as the sublattice of $\mathcal{O}_N$ locally defined by

$$\check{\mathcal{O}}_{N,p} = \begin{cases} \check{\mathcal{O}}_{N_p} \otimes_{\mathbb{Z}_p[G(p)]} \mathbb{Z}_p[G] & \text{if} \quad p \neq 0. \\ N_p & \text{if} \quad p = 0. \end{cases} \tag{3.3}$$

when there in no risk of confusion we will write Λ for $\Lambda(N/\mathbb{Q})$ and Λ_p for $\Lambda(N_p/\mathbb{Q}_p)$.
Fröhlich's Hom-description of the class group of Λ is given by the following isomorphism

$$\frac{\mathrm{Hom}_{\Omega_\mathbb{Q}}(R_G, J(E))}{\mathrm{Hom}_{\Omega_\mathbb{Q}}(R_G, E^\times)\mathrm{Det}\ U(\Lambda)} \simeq Cl(\Lambda) \tag{3.4}$$

where we let R_G be the ring of virtual characters of G over $\mathbb{Q}^c$, we write J for idèles and U for unit idèles and we denote by Det Fröhlich's generalised determinant map, see [F], II. With this notation one has to distinguish between $\Lambda_p^\times$ and $U_p(\Lambda) = (\Lambda_p \otimes_{\mathbb{Z}_p[G(p)]} \mathbb{Z}_p[G])^\times$. Let $L/\mathbb{Q}$ be a subextension of $(E/\mathbb{Q})$ and we let $\mathfrak{p}$ be the prime divisor of L defined by j_p. We again obtain a homomorphism denoted by j_p

$$j_p : E \otimes_L L_\mathfrak{p} \to E_p.$$

On the other hand j_p induces an isomorphism

$$\chi \to \chi^{j_p}$$

between R_G and $R_{G,p}$, the ring of virtual $\mathbb{Q}_p^c$ characters of G. Using (3.1), we easily check that if we put $J_{\mathfrak{p}}(E) = (E \otimes_L L_{\mathfrak{p}})^\times$, then the map

$$(3.5) \qquad j^*_{L,p} : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\Omega_L}(R_G, J_{\mathfrak{p}}(E)) & \to & \mathrm{Hom}_{\Omega_{L_p}}(R_{G,p}, E(p)^\times) \\ f & \mapsto & j^*_{L,p}(f) : (\chi \mapsto f(\chi^{j^{-1}})^j) \end{array} \right.$$

is a group homomorphism .
From [F] II, Lemma 2.1 we know that $j^*_{L,p}$ is a group isomorphism. In the case where $L = \mathbb{Q}$, writing $j^*_{\mathbb{Q},p} = j^*_p$ we obtain a new group isomorphism also denoted by j^*_p

$$(3.6) \qquad j^*_p : \frac{\mathrm{Hom}_{\Omega_{\mathbb{Q}}}(R_G, J_p(E))}{\mathrm{Det}(U_p(\Lambda))} \simeq \frac{\mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(R_{G,p}, E_p^\times)}{\mathrm{Det}(U_p(\Lambda))}.$$

Remarks:

1.) Any element $a \in U_p(\Lambda)$ defines two determinant maps both denoted by $\mathrm{Det}(a)$, namely $R_G \to J(E)$ and $R_{G,p} \to E_p^\times$. Hence $\mathrm{Det}(U_p(\Lambda))$ has two different meanings in each side of (3.6). Nevertheless j^*_p maps isomorphically the lefthand group to the righthand group.

2.) We shall in future use the symbol R_G rather than $R_{G,p}$ for $\mathbb{Q}_p^c$-characters unless there is a danger of confusion.

From now on we identify both sides of (3.4). For $x \in Cl(\Lambda)$, a representative of x will be any $f \in \mathrm{Hom}_{\Omega_{\mathbb{Q}}}(R_G, J(E))$ whose cosets equals x. We observe that f is specified by its values on the irreducible characters of G. We let $\check{U}_N$ be the class of $[\check{\mathcal{O}}_N] - [\Lambda]$ in $Cl(\Lambda)$. Let p be a prime divisor of $\mathbb{Q}$. For any character χ of G we denote by $T_p(\chi)$ the element of $E^\times$ defined by

$$(3.7) \qquad T_p(\chi) = \left\{ \begin{array}{lll} 1 & \text{if} & p = \infty \\ \tau_p(\chi) & \text{the local Galois Gauss sum at } p \text{ if} & p \neq \infty \end{array} \right.$$

and we let $R_p(\chi)$ be the idèle of E whose l-component in $J_l(E)$ for any prime divisor l of $\mathbb{Q}$ is given by :

$$(3.8) \qquad R_p(\chi)_l = \left\{ \begin{array}{ll} 1 & \text{if } \ l \neq p \\ (a_p|\chi) & \text{if } \ l = p \end{array} \right.$$

where a_p is a free basis of $\check{\mathcal{O}}_N \otimes_{\mathbb{Z}} \mathbb{Z}_p$ over $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We observe that T_p and R_p both define character maps, that is to say elements of $\mathrm{Hom}(R_G, J(E))$. By standard theory we know that

$$f = \prod_p f_p,$$

with $f_p = T_p R_p^{-1}$, is a representative for $\check{U}_N$.

For any finite prime divisor p of $\mathbb{Q}$ we denote by $\delta_p^{(t)}$ (resp. $\delta_p^{(w)}$) the Adams operations on characters of G defined by

$$\delta_p^{(t)}(\chi)(g) = \chi(g') \ (\text{resp.} \delta_p^{(w)}(\chi)(g) = \chi(g''))$$

where $g = g'g'' = g''g$ with g' (resp. g'') of order coprime to p (resp. a power of p). These operations induce natural endomorphisms on $\mathrm{Hom}(R_G, J(E))$ again denoted by $\delta_p^{(t)}$ and $\delta_p^{(w)}$. One can check that for any Adams operation Ψ on R_G and any character χ of G one has

$$\Psi(\det\chi) = \det(\Psi(\chi)) \tag{3.9}$$

Let p be a finite prime divisor of $\mathbb{Q}$. It follows the definition of the local Gauss sum that

$$\tau_p(\chi) = \tau_{G(p)}(\mathrm{Res}(\chi)) \quad ([\mathrm{T}_1], \text{section } 3)$$

Since $\delta_p^t(\mathrm{Res}(\chi))$ is a tame character of $G(p)$, we can define $\delta_p^{(t)}(T_p^*)$ by

$$\delta_p^{(t)}(T_p^*) = \tau^*_{G(p)}(\delta_p^{(t)}(\mathrm{Res}(\chi))$$

where for a tame character θ, $\tau^*_{G(p)}(\theta)$ denotes the modified Galois Gauss sum, see [T$_1$] (3.9). It then follows from (3.9) and the Galois action formula on local Gauss sums, [M], Theorem 7.2 that we have:

Lemma 3.10 *For any finite prime divisor p of $\mathbb{Q}$, the character map $T_p/\delta_p^{(t)}(T_p^*)\delta_p^{(w)}(T_p)$ is an element of* $\mathrm{Hom}_{\Omega_{\mathbb{Q}}}(R_G, E^\times)$.

Remark:
In fact the map belongs to $\mathrm{Ind}_{G(p)}^G(\mathrm{Hom}_{\Omega_{\mathbb{Q}}}(R_{G(p)}, E^\times))$.

Proposition 3.11 *For any prime divisor p of $\mathbb{Q}$, the character map $\dfrac{R_p}{\delta_p^{(t)}(R_p)\delta_p^{(w)}(R_p)}$ is an element of* $\mathrm{Hom}_{\Omega_{\mathbb{Q}}}(R_G, E^\times)\mathrm{Det}U(\Lambda)$.

Proof. We let x_p be the element of $\dfrac{\mathrm{Hom}_{\Omega_{\mathbb{Q}}}(R_G, J_p(E))}{\mathrm{Det}U_p(\Lambda)}$ represented by $\dfrac{R_p}{\delta_p^{(t)}(R_p)\delta_p^{(w)}(R_p)}$ considered with values in $J_p(E)$. Let Ind be the group homomorphism

$$\frac{\mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(R_{G(p)}, E_p^\times)}{\mathrm{Det}(\Lambda_p^\times)} \to \frac{\mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(R_G, E_p^\times)}{\mathrm{Det}U_p(\Lambda)} \tag{3.12}$$

induced by $f \to \mathrm{Ind}(f) : \chi \to f(\mathrm{Res}(\chi))$. Following Fröhlich's method , [F], III, Theorem 19, we find, that $j_p^*(x_p) = \mathrm{Ind}(y_p)$ where y_p is represented by

$$(3.13) \qquad \chi \to \frac{(b|\chi)}{(b|\delta_p^{(t)}(\chi))(b|\delta_p^{(w)}(\chi))}$$

where b is a basis element of $\check{\mathcal{O}}_{N_p}$ over Λ_p.
Using now a slight generalisation of Theorem 25 in [F] we may show that for any basis c of $\check{\mathcal{O}}_{N_p}$ over $\mathfrak{M}_{M_p}(G_0(p))$ there exists $\lambda \in \Lambda_{M_p}^{\times}$ such that

$$(3.14) \qquad (b|\chi)_{N_p/\mathbb{Q}_p} = (c|\mathrm{Res}(\chi))_{N_p/M_p}\mathrm{Det}_\chi(\lambda), \ \forall \chi \in R_{G(p)}.$$

Since $M_p/\mathbb{Q}_p$ is non ramified we know from [CN-T$_2$] Theorem 1 that $\mathrm{Det}(\Lambda_p^{\times})$ is stable under Adams operations. We thus deduce from (3.14) that there exists $\mu \in \Lambda_{M_p}^{\times}$ such that

$$(3.15) \qquad \frac{(b|\chi)}{(b|\delta_p^{(t)}(\chi))(b|\delta_p^{(w)}(\chi))} = \frac{(c|\mathrm{Res}(\chi))}{(c|\mathrm{Res}\delta_p^{(t)}(\chi))(c|\mathrm{Res}\delta_p^{(w)}(\chi))}\mathrm{Det}_\chi(\mu).$$

The first factor of the righthandside of (3.15) is clearly induced from $g \in \mathrm{Hom}_{\Omega_{M_p}}(R_{G_0(p)}, E_p^{\times})$ with

$$(3.16) \qquad g(\varphi) = \frac{(c|\varphi)}{(c|\delta_p^{(t)}(\varphi))(c|\delta_p^{(w)}(\varphi))}, \forall \varphi \in R_{G_0(p)}.$$

Since $G_0(p)$ is abelian, we first observe that every irreducible character φ of this group can be written

$$(3.17) \qquad \varphi = \delta_p^{(t)}(\varphi)\delta_p^{(w)}(\varphi).$$

Moreover we know that we obtain a basis of $\check{\mathcal{O}}_{N_p}$ over $\mathfrak{M}_{M_p}(G_0)$ by considering $c^{(t)} \otimes c^{(w)}$ where $c^{(t)}$ (resp. $c^{(w)}$) is a basis of $\mathcal{O}_{N_p^{(t)}}$ (resp. $\check{\mathcal{O}}_{N_p^{(w)}}$) over $\mathcal{O}_{M_p}[G_0'(p)]$ (resp. $\mathfrak{M}_{M_p}[G_1(p)]$). Hence, by an easy computation, we obtain for such a character the equality

$$(3.18) \qquad g(\varphi)^{-1} = \mathrm{Tr}_{N_p^{(t)}/M_p}(c^{(t)})\mathrm{Tr}_{N_p^{(w)}/M_p}(c^{(w)}).$$

Thus we can write the righthand side of (3.18) as a product

$$(3.19) \qquad g(\varphi) = p^m\varepsilon$$

where m is an integer and ε a unit of $\mathcal{O}_{M_p}$ both independent of φ.
It now follows from (3.15) that y_p is represented by

$$(3.20) \qquad \chi \to p^{m\chi(1)}\mathrm{Det}_\chi(\gamma), \ \forall \chi \in R_{G(p)}$$

with $\gamma \in \Lambda_{M_p}^{\times}$. This implies that $\mathrm{Det}(\gamma)$ is fixed under the action of $\Omega_{\mathbb{Q}_p}$. Using [CN-T$_2$] Theorem 2, we may conclude that $\mathrm{Det}(\gamma)$ belongs to $\mathrm{Det}(\Lambda_p^{\times})$. We finally obtain that $j_p^*(x_p)$ is represented by h with

(3.21) $$\chi \to h(\chi) = p^{m\chi(1)}$$

Since j_p^* is an isomorphism we find that there exists $\alpha_p \in U_p(\Lambda)$ such that

(3.22) $$\frac{R_p}{\delta_p^{(t)}(R_p)\delta_p^{(w)}(R_p)} = h_p \mathrm{Det}(\alpha_p)$$

with $h_p(\chi) = p^{m\chi(1)}$ in $J_p(E)$.
For any prime divisor l of $\mathbb{Q}$, $l \neq p$, (resp. $l = p$), we put $\alpha_l = p^{-m}$, (resp. $\alpha_l = 1$), as an element of $U_l(\Lambda)$. Writing $\alpha = (\alpha_l) \in U(\Lambda)$, we have thus proved that

(3.23) $$\frac{R_p}{\delta_p^{(t)}(R_p)\delta_p^{(w)}(R_p)} = k\mathrm{Det}(\alpha),$$

with $k \in \mathrm{Hom}_{\Omega_{\mathbb{Q}}}(R_G, E^\times)$ defined by $k(\chi) = p^{m\chi(1)}$. □
We now can define a new representative for $\check{U}_N$. Let S be the set of finite primes of $\mathbb{Q}$ which are wildly ramified in $N/\mathbb{Q}$. For any p, we know from the very definition of the local Gauss sum that τ_p, as an element of $\mathrm{Hom}(R_G, E^\times)$, is induced from $G_0(p)$ namely

$$\tau_p = \mathrm{Ind}_{G_0(p)}^G(\tau_{G_0(p)}).$$

If $p \notin S$ we can write $\tau_p^* = \mathrm{Ind}_{G_0(p)}^G(\tau^*{}_{G_0(p)})$ when $\tau_{G_0(p)}^*$ is the modified Galois Gauss sum. We define ${f_p}^* = T_p^*/R_p$ where T_p^* is associated to τ_p^* via (3.7).
For $p \in S$ we define

$$f_p^{(t)} = \delta_p^{(t)}(T_p^*)/\delta_p^{(t)}(R_p), \qquad f_p^{(w)} = \delta_p^{(w)}(T_p)/\delta_p^{(w)}(R_p).$$

It follows Lemma 3.10 and Proposition 3.11

Corollary 3.24
$$f = \prod_{p \notin S} f_p^* \prod_{p \in S} f_p^{(t)} f_p^{(w)}$$
is a representative of $\check{U}_N$.

For any character map g and any prime divisor q of $\mathbb{Q}$ we will denote by g_q the element of $\mathrm{Hom}(R_G, J_q(E))$ obtained by considering the q-component of g.
For such maps g and h we will write $g \sim h$ when $gh^{-1} \in \mathrm{Hom}_{\Omega_{\mathbb{Q}}}(R_G, E^\times)\mathrm{Det}U(\Lambda)$.

4. The character maps $f_p^{(t)}$

We start this section by introducing a little further notation. Let p be a prime of S and let $G^{(t)}(p)$ be the Galois group $\mathrm{Gal}(N_p^{(t)}/\mathbb{Q}_p)$. Since $G_1(p)$ is a direct factor of $G(p)$

$$G(p) = G_1(p) \times G^{(t)}(p) \tag{4.1}$$

Any irreducible character θ of $G(p)$ can therefore be written

$$\theta = \theta^{(w)}\theta^{(t)}$$

where $\theta^{(w)}$ (resp. $\theta^{(t)}$) is an irreducible character of $G_1(p)$ (resp. $G^{(t)}(p)$). It follows that

$$\delta_p^{(t)}(\theta) = \delta_p^{(t)}(\theta^{(t)}) \tag{4.2}$$

Theorem 4.3 *There exists a character map ${f'}_p^{(t)}$, ${f'}_p^{(t)} \sim f_p^{(t)}$ such that for any finite prime q*

i.) If $q \neq p$, then there exists a non ramified extension $F_q/\mathbb{Q}$ such that $j_q^*({f'}_{p,q}^{(t)}) \in \mathrm{Ind}_{G(p)}^G(\mathrm{Det}(\mathcal{O}_{F_q}[G(p)]^\times)$;

ii.) If $q = p$, then there exists a tamely ramified extension $L_p/\mathbb{Q}_p$ such that $j_p^*({f'}_{p,p}^{(t)}) \in \mathrm{Ind}_{G(p)}^G(\mathrm{Det}(\mathcal{O}_{L_p}[G(p)]^\times)$.

Proof. We start by considering $j_q^*(f_{p,q}^{(t)})$ for $p \neq q$. It follows from the definition of the local Gauss sum that

$$j_q^*(f_{p,q}^{(t)}) = \mathrm{Ind}_{G(p)}^G(g_q) \tag{4.4}$$

where $g_q \in \mathrm{Hom}(R_G, E_q^\times)$ is defined by

$$g_q(\theta) = \tau_{G(p)}^*(\delta_p^{(t)}(\theta))^{j_q}.$$

(From this point on we shall omit all mention of j_q in the remainder of the proof). We deduce from (4.2) that $\delta_p^{(t)}(\theta)$ is inflated from $\delta_p^{(t)}(\theta^{(t)})$, considered as a character of the tame extension $N_p^{(t)}/\mathbb{Q}_p$. From [T$_1$], Theorem 3, we deduce that there exists a non ramified finite extension $F_q/\mathbb{Q}_p$ and an element $\alpha_p \in \mathcal{O}_{F_q}[G(p)]^\times$ such that

$$\tau_{G^{(t)}(p)}^*(\varphi) = \mathrm{Det}(\alpha_p)(\varphi), \forall \varphi \in R_{G^{(t)}(p)}.$$

Since $F_q/\mathbb{Q}_p$ is unramified $\mathrm{Det}(\mathcal{O}_{F_p}[G^{(t)}(p)]^\times)$ is stable under Adams operations, [CN-T$_1$]. Therefore there exists $\beta_p \in \mathcal{O}_{F_p}[G^{(t)}(p)]^\times$ such that $\delta_p^{(t)}\mathrm{Det}(\alpha_p) = \mathrm{Det}(\beta_p)$. Considering β_p as an element of $\mathcal{O}_{F_p}[G(p)]^\times$ by (4-1) we obtain

(4.5) $\quad \mathrm{Det}(\beta_p)(\theta) = \mathrm{Det}(\beta_p)(\theta^{(t)}) = \mathrm{Det}(\alpha_p)(\delta_p^{(t)}(\theta)) = g_q(\theta), \forall\theta \in R_{G(p)}.$

Thus we have proved that

$$g_q \in \mathrm{Det}(\mathcal{O}_{F_p}[G(p)])^\times. \tag{4.6}$$

We now examine $j_p^*(f_{p,p}^{(t)})$. We obtain a result similar to (4.4), namely

$$j_p^*(f'^{(t)}_{p,p}) = \mathrm{Ind}_{G(p)}^G(g_p)$$

where g_p is defined by

$$g_p(\theta) = \tau^*_{G(p)}(\delta_p^{(t)}(\theta))(b|\delta_p^{(t)}(\theta))^{-1}_{N_p/\mathbb{Q}_p}, \forall\theta \in G(p) \tag{4.7}$$

with b a basis of $\check{\mathcal{O}}_{N_p}$ over Λ_p. Again using that $\delta^{(t)}(\theta) = \delta^{(t)}(\theta^{(t)})$ is inflated from $\mathrm{Gal}(N_p^{(t)}/\mathbb{Q}_p)$, we deduce from (4.7)

$$g_p(\theta) = \tau^*_{G^{(t)}(p)}(\delta_p^{(t)}(\theta^{(t)}))(\mathrm{Tr}_{N_p/N_p^{(t)}}(b)|\delta_p^{(t)}(\theta^{(t)}))^{-1}_{N_p^{(t)}/\mathbb{Q}_p}, \forall\theta \in R_{G(p)} \tag{4.8}$$

where for a field extension F/L, $\mathrm{Tr}_{F/L}$ denotes the trace.

Lemma 4.9 *There exists an integer $n \in \mathbb{Z}$, such that $p^{-n}\mathrm{Tr}_{N_p/N_p^{(t)}}(b)$ is a free basis of $\mathcal{O}_{N_p^{(t)}}$ as a $\mathbb{Z}_p[G(p)^{(t)}]$-module.*

Proof. It follows from $\check{\mathcal{O}}_{N_p} = b\Lambda_p$ that $\mathrm{Tr}_{N_p/N_p^{(t)}}(\check{\mathcal{O}}_{N_p}) = \mathrm{Tr}_{N_p/N_p^{(t)}}(b)\mathbb{Z}_p[G(p)^{(t)}]$. Since $\check{\mathcal{O}}_{N_p} = \mathcal{O}_{N_p^{(t)}} \otimes_{\mathcal{O}_{M_p}} \check{\mathcal{O}}_{N_p^{(w)}}$ we obtain that $\mathrm{Tr}_{N_p/N_p^{(t)}}(\check{\mathcal{O}}_{N_p}) = I\mathcal{O}_{N_p^{(t)}}$ where $I = \mathrm{Tr}_{N_p^{(w)}/M_p}(\check{\mathcal{O}}_{N_p})$.
Since $M_p/\mathbb{Q}_p$ is non-ramified, the ideal I of M_p is generated by p^n with $n \geq 0$. We thus deduce that

$$\mathcal{O}_{N_p^{(t)}} = p^{-n}\mathrm{Tr}_{N_p/N_p^{(t)}}(b)\mathbb{Z}_p[G^{(t)}_{(p)}] \tag{4.10}$$

Since Noether's theorem implies that $\mathcal{O}_{N_p^{(t)}}$ is of free rank one on $\mathbb{Z}_p[G^{(t)}(p)]$, we deduce from (4.10) that $p^{-n}\mathrm{Tr}_{N_p/N_p^{(t)}}(b)$ is a free basis of $\mathcal{O}_{N_p^{(t)}}$. Setting $c = p^{-n}\mathrm{Tr}_{N_p/N_p^{(t)}}(b)$, (4.8) can now be written as

$$g_p(\theta) = \tau^*_{G^{(t)}_{(p)}}(\delta_p^{(t)}(\theta^{(t)}))(c|\delta_p^{(t)}(\theta^{(t)}))^{-1}p^{n\theta(1)}. \tag{4.11}$$

Using [F$_1$], IV, Theorem 31 and the stability of $\mathrm{Det}(\mathcal{O}_F[G(p)^{(t)}]^\times)$ under Adams operations, we obtain a tamely ramified extension $L_p/\mathbb{Q}_p$ and $\gamma_p \in \mathcal{O}_{L_p}[G(p)]^\times$ such that

$$g_p(\theta) = \mathrm{Det}(\gamma_p)(\theta)p^{n\theta(1)}.$$

We define

$$u_p = \mathrm{Hom}_{\Omega_{\mathbb{Q}}}(R_G, E^\times) \tag{4.12}$$

by $u_p(\theta) = p^{n\theta(1)}$. It follows (4.6) and (4.12) that ${f'}_p^{(t)} = f_p^{(t)} u_p^{-1}$ has the required properties. □

Remark:
One has to observe that we have used the stability of $\mathrm{Det}(\mathcal{O}_F[G]^\times)$ under Adams operations with $F/\mathbb{Q}_p$ tamely ramified. This can be deduced from [CN-T$_1$] after observing that [CN-T$_1$], Theorem2, remains true for any extension $F/\mathbb{Q}_p$ whenever the p-Sylow subgroup of G is abelian.

5. The character maps $f_p^{(w)}$

We consider a prime p of S and we keep the notation of section 4. We start this section by describing the image of $R_{G(p)}$ under the Adams operation $\delta_p^{(w)}$.

Proposition 5.1 *Let θ be an irreducible character of $G(p)$, $\theta = \theta^{(w)}\theta^{(t)}$ with $\theta^{(w)}$ and $\theta^{(t)}$ defined as in section 4. Then $\delta_p^{(w)}(\theta) = \theta^{(w)}\alpha(\theta)$ where $\alpha(\theta)$ is a sum of non ramified abelian characters of $G_{(p)}^{(t)}$ of order a power of p.*

Proof. For simplicity henceforth in the proof we shall omit the dependence on p; we will write $G^{(t)}$, G'_0, $\delta^{(w)}, \ldots$ etc. for $G_{(p)}^{(t)}$, $G'_0(p)$, $\delta_p^{(w)}$. It follows from the decomposition of θ that the proof is reduced to showing, that for an irreducible character χ of $G^{(t)}$, $\delta^{(w)}(\chi)$ is the sum of abelian non-ramified characters of $G^{(t)}$. Let χ be such a character. By a slight generalisation of [S$_1$], Proposition 25, we obtain a subgroup Σ, $G'_0 \subset \Sigma \subset G^{(t)}$ and an abelian character φ of Σ such that $\chi = \mathrm{Ind}_\Sigma^{G^{(t)}}\varphi$. Let $p^n q$,with $(p,q) = 1$, be the order of $G^{(t)}$. We choose, once for all, an integer A, $A \equiv 1 \bmod p^n$, $A \equiv 0 \bmod q^r$ with r big enough. Thus $\delta^{(w)}$ is the Adams operator Ψ_A. Since $G^{(t)}/G'_0$ is cyclic, we denote by T the unique subgroup of $G^{(t)}$ with $\Sigma \subset T \subset G^{(t)}$, $[G^{(t)} : T] = p^m$, $[T : \Sigma] = B$ and $(p, B) = 1$. As an abelian character of Σ, we may decompose φ into a product $\varphi'\varphi''$ with φ' (resp. φ'') of order coprime to p (resp. power of p). We denote $\varphi_p = (\varphi'\varphi'')^{A/B}$. We observe that, since r has been chosen big enough, $\varphi_p = {\varphi''}^C$ with $\frac{1}{B} \equiv \frac{A}{B} \equiv C \bmod p^n$.

Lemma 5.2 $\delta^{(w)}(\chi) = [T : \Sigma]\mathrm{Ind}_T^{G^{(t)}}(\varphi \circ v)$ *where v is the transfer map $T^{(ab)} \to \Sigma^{(ab)}$*

Proof. Since T is an invariant subgroup such that $(A, [G^{(t)} : T]) = 1$ we know that $\delta^{(w)}(\chi) = \mathrm{Ind}_T^{G^{(t)}}(\Psi_A(\mathrm{Ind}_\Sigma^T(\varphi)))$. For $x \in T$ it follows from the definition that

$$\Psi_A(\mathrm{Ind}_\Sigma^T(\varphi))(x) = \sum_{y \in T/\Sigma} \varphi(yx^A y^{-1}) = \sum_{y \in T/\Sigma} \varphi_p(yx^B y^{-1})$$

Since T/G_0' is abelian, $yx^B y^{-1} \equiv x^B \bmod G_0'$. Using the fact that φ_p is trivial on G_0', we obtain that $\Psi_A(\mathrm{Ind}_\Sigma^T(\varphi))(x) = [T : \Sigma]\varphi_p(x^B), \forall x \in T$. We complete the proof of the lemma by showing that $\varphi_p(x^B) = \varphi_p \circ v(x), \forall x \in T$. This follows easily from the definition of the transfer (see [S$_2$], VII, section 8) and using that $G^{(t)}/G_0'$ is cyclic. □

For $s \in G^{(t)}$, since $sx^B s^{-1} \equiv x^B \bmod G_0'$, we get $\varphi_p \circ v(sxs^{-1}) = \varphi_p \circ v(x)$. We then conclude that $G^{(t)}/T$ acts trivially on $\varphi_p \circ v$ and thus that this character can be extended into an abelian character of $G^{(t)}$, which we denote $\tilde{\varphi}_p$. We have shown that $\delta^{(w)}(\chi) = [T : \Sigma] \sum_\beta \tilde{\varphi}_p \beta$ where β runs through the abelian characters of $G^{(t)}/T$ inflated to $G^{(t)}$, which is the required result. □

Theorem 5.3 *There exists a character map ${f'}_p^{(w)}$, ${f'}_p^{(w)} \sim f_p^{(w)}$ such that for a finite prime q:*

1.) *If $q \neq p$, then $j_q^*({f'}_{p,q}^{(w)}) \in \mathrm{Det}(\mathbb{Z}_q(p^n)[G]^\times)$;*

2.) *If $q = p$, then $j_q^*({f'}_{p,q}^{(w)}) \in \mathrm{Det}(U_p(\Lambda))\mathrm{Det}(\mathcal{O}_{M_p'}[G]^\times)$, where $M_q'/\mathbb{Q}_p$ is a non-ramified extension.*

Proof. Let q be a prime number $q \neq p$. As we saw previously $j_q^*(f_{p,q}^{(w)}) = \mathrm{Ind}_{G(p)}^G(h_q)$ with $h_q \in \mathrm{Hom}(R_{G(p)}, E_q^\times)$ is defined by $h_q(\theta) = \tau_{G(p)}(\delta_p^{(w)}(\theta))$.
By decomposing θ into $\theta^{(w)}\theta^{(t)}$, we deduce that $\delta_p^{(w)}(\theta) = \theta^{(w)}\delta_p^{(w)}(\theta^{(t)})$. It now follows from Proposition 5.1 that there exist non-ramified abelian characters α_i of $G(p)$ such that $\delta_p^{(w)}(\theta^{(t)}) = \alpha_1 + ... + \alpha_n$ with $n = \theta(1)$. Let $L_p^{(w)} = N_p^{G^{(t)}(p)}$. Thus $\theta^{(w)}$ may be viewed as a character of $\mathrm{Gal}(L_p^{(w)}/\mathbb{Q}_p)$. A generator of its Artin conductor defines, by local class field theory, an element of $G(p)^{(ab)}$; we let $u_p(\theta^{(w)})$ be its component on the p-Sylow subgroup of $G(p)^{(ab)}$. We now can express

$$h_q(\theta) = \tau_{G(p)}(\theta^{(w)})^{\theta(1)} \prod_{1 \le i \le n} \alpha_i(u_p(\theta^{(w)})) \tag{5.4}$$

which we write as

$$h_q(\theta) = k_q(\theta) l_q(\theta). \tag{5.5}$$

We now fix a p-Sylow subgroup of $G(p)$, $H(p) = G_1(p) \times H'(p)$. This is an abelian subgroup of $G(p)$. For any irreducible character φ of $H(p)$, we decompose it as a product $\varphi''\varphi'$ with φ'' (resp. φ') a character of $G_1(p)$ (resp. $H'(p)$). For such φ we fix a lifting of $u_p(\varphi'')$ into $H(p)$ and denote by $v_p(\varphi)$ its projection on $H'(p)$. Let m_q

$$m_q : R_{H(p)} \to \mathcal{O}_{E_q}^{\times}$$

be defined on irreducible characters by

$$m_q(\varphi) = \mathrm{Det}(v_p(\varphi))(\varphi). \tag{5.6}$$

For an irreducible character θ of $G(p)$ we have

$$\theta|_{H(p)} = \theta^{(w)}\theta^{(t)}|_{H'(p)} = \theta^{(w)}\delta_p^{(w)}(\theta^{(t)})|_{H'(p)}$$

. Therefore, with the notation of (5.5), we obtain that

$$l_q = \mathrm{Ind}_{H(p)}^{G(p)}(m_q). \tag{5.7}$$

Since the Artin conductors of θ and θ^{ω} coincide for $\omega \in \Omega_{\mathbb{Q}}$, $v_p(\varphi)$ can be chosen so that $v_p(\varphi^{\omega}) = v_p(\varphi), \forall \omega \in \Omega_{\mathbb{Q}_q}$. It follows that $m_q \in \mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(R_{H(p)}, \mathcal{O}_{E_q}^{\times})$. Since $p \neq q$, $\mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(R_{H(p)}, \mathcal{O}_{E_q}^{\times}) = \mathrm{Det}(\mathbb{Z}_q[H(p)]^{\times})$, which implies that

$$l_q \in \mathrm{Det}(\mathbb{Z}_q[G(p)]^{\times}). \tag{5.8}$$

It now follows from the definition of Galois Gauss sums associated to abelian characters that

$$n_q : \left\{ \begin{array}{ccc} R_{G_1(p)} & \to & \mathcal{O}_{E_q}^{\times} \\ \varphi & \mapsto & \tau_{G(p)}(\varphi) \end{array} \right.$$

defines an element of $\mathrm{Hom}_{\Omega_{\mathbb{Q}_p(p^n)}}(R_{G_1(p)}, \mathcal{O}_{E_q}^{\times}) = \mathrm{Det}(\mathbb{Z}_q(p^n)[G_1(p)]^{\times})$. Since $k_q = \mathrm{Ind}_{G_1(p)}^{G(p)}(n_q)$, we conclude that $k_q \in \mathrm{Det}(\mathbb{Z}_q(p^n)[G(p)]^{\times})$ and hence, via (5.5) and (5.8) that

$$h_q \in \mathrm{Det}(\mathbb{Z}_q(p^n)[G(p)]^{\times}) \tag{5.9}$$

We now consider $j_p^*(f_{p,p}^{(w)})$ which can be written

$$\mathrm{Ind}_{G_1(p)}^{G(p)}(h_p)$$

with

$$h_p(\theta) = \tau_{G(p)}(\delta_p^{(w)}(\theta))(b|\delta_p^{(w)}(\theta))^{-1}, \forall \theta \in R_{G(p)} \tag{5.10}$$

where b is a basis of $\check{\mathcal{O}}_{N_p}$ over Λ_p. We write $h_p'(\theta) = \tau_{G(p)}(\delta_p^{(w)}(\theta))$ and $h_p''(\theta) = (b|\delta_p^{(w)}(\theta))$. By the same reasoning as for (5.5) we obtain a decomposition $h_p'(\theta) =$

$k_p(\theta)l_p(\theta)$ with the map $l_p \in \mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(R_{G(p)}, \mathcal{O}_{E_p}^{\times})$. Moreover $l_p = \delta_p^{(w)}(l'_p)$ with l'_p defined by $l'_p(\theta) = \mathrm{Det}(u_p(\theta^{(w)}))(\theta)$ for θ irreducible. It follows from the definition of l'_p that, if θ and θ' are irreducible characters of $G(p)$ conjugate in the sense of [CN-T$_2$], section 2, then $u_p(\theta^{(w)}) = u_p(\theta'^{(w)})$. This implies that $l'_p \in \mathrm{Det}(\Lambda_p^{\times})$ ([CN-T$_2$], Theorem 2.2) and hence by [CN-T$_2$], Theorem 1, that $l_p \in \mathrm{Det}(\Lambda_p^{\times})$. We thus have proved that there exists $\delta \in \Lambda_p^{\times}$ such that

$$h'_p = k_p \mathrm{Det}(\delta). \tag{5.11}$$

We now consider $h''_p(\theta)$. Proposition 5.1 implies that $\delta_p^{(w)}(\theta)$ is inflated from $\mathrm{Gal}(N_p^{(w)}/\mathbb{Q}_p)$. Therefore

$$(b|\delta_p^{(w)}(\theta)) = (\mathrm{Tr}_{N_p/N_p^{(w)}}(b)|\delta_p^{(w)}(\theta))_{N_p^{(w)}/\mathbb{Q}_p}.$$

Since $\check{\mathcal{O}}_{N_p} = \check{\mathcal{O}}_{N_p^{(w)}} \otimes_{\mathcal{O}_{M_p}} \mathcal{O}_{N_p^{(t)}}$ we immediately get that

$$\mathrm{Tr}_{N_p/N_p^{(w)}}(\check{\mathcal{O}}_{N_p}) = \check{\mathcal{O}}_{N_p^{(w)}} \otimes_{\mathcal{O}_{M_p}} \mathrm{Tr}_{N_p^{(t)}/M_p}(\mathcal{O}_{N_p^{(t)}}).$$

Since $N_p^{(t)}/M_p$ is tame we obtain that $\mathrm{Tr}_{N_p/N_p^{(w)}}(\check{\mathcal{O}}_{N_p}) = \check{\mathcal{O}}_{N_p^{(w)}}$ and that $c = \mathrm{Tr}_{N_p/N_p^{(w)}}(b)$ is a basis of $\check{\mathcal{O}}_{N_p^{(w)}}$ on $\Lambda_{\mathbb{Q}_p}(N_p^{(w)}/\mathbb{Q}_p)$. We now observe that, since $N_p^{(w)}$ is the compositum of arithmetically disjoint abelian extensions namely $L_p^{(w)}/\mathbb{Q}_p$ and $M_p/\mathbb{Q}_p$, c can be chosen as a product $c = d \otimes e$ where d is a basis of $\check{\mathcal{O}}_{L_p^{(w)}}$ over $\mathfrak{M}_{\mathbb{Q}_p}(G_1(p))$ and e is a normal integral basis of $M_p/\mathbb{Q}_p$. We finally obtain

$$(b|\delta_p^{(w)}(\theta)) = (d|\theta^{(w)})^{\theta(1)}_{L_p^{(w)}/\mathbb{Q}_p} \prod_{1\le i\le n} (e|\alpha_i)_{M_p/\mathbb{Q}_p} \tag{5.12}$$

if $\delta_p^{(w)}(\theta^t) = \sum_{1\le i\le n} \alpha_i$. Let $M'_p/\mathbb{Q}_p$ be the maximal p-subextension of $M_p/\mathbb{Q}_p$ and $U = \mathrm{Gal}(M'_p/\mathbb{Q}_p)$. It follows from the very definition of the α_i that

$$\prod_{1\le i\le n} (e|\alpha_i)_{M_p/\mathbb{Q}_p} = \prod_{1\le i\le n} (f|\alpha_i)_{M'_p/\mathbb{Q}_p}$$

with $f = \mathrm{Tr}_{M_p/M'_p}(e)$. Let $\lambda = \sum_{g\in U} f^g g^{-1}$. This is an invertible element of $\mathcal{O}_{M'_p}[U]^{\times}$. By standard arguments we conclude that λ has a lifting μ in $\mathcal{O}_{M'_p}[G(p)]^{\times}$ such that

$$\mathrm{Det}(\mu)(\delta_p^{(w)}(\theta)) = \mathrm{Det}(\lambda)(\delta_p^{(w)}(\theta^t)), \forall \theta \in R_{G(p)}.$$

Using once again the stability of $\mathrm{Det}(\mathcal{O}_{M'_p}[G(p)]^{\times})$ under Adams operations, we have proved that

$$h_p = v_p \mathrm{Det}(\nu) \mathrm{Det}(\delta) \tag{5.13}$$

with $v_p = \mathrm{Ind}_{G_1(p)}^{G(p)} s_p$, $s_p \in \mathrm{Hom}(G_1(p), E_p^\times)$ defined by $\theta \to s_p(\theta) = \tau_{G(p)}(\theta)(d|\theta)^{-1}$ and $\nu \in \mathcal{O}_{M'_p}[G(p)]^\times, \delta \in \Lambda_p^\times$.
We now want to show that s_p commutes with the action of $\Omega_{\mathbb{Q}_p}$. We let ω be an element of $\Omega_{\mathbb{Q}_p}$. From the definition of abelian resolvents we obtain that

$$(5.14) \qquad (d|\theta^{\omega^{-1}})^\omega = \sum_{g \in G_1(p)} d^{g\omega} \theta(g^{-1}) = (d|\theta)\theta(\omega')$$

with $\omega' = \omega|_{L_p^{(w)}}$.
We now consider $\tau_{G(p)}(\theta)$. Since θ is inflated from $\mathrm{Gal}(L_p^{(w)}/\mathbb{Q}_p)$, $\tau_{G(p)}(\theta)$ is the local Galois Gauss sum associated to the character θ of $\mathrm{Gal}(L_p^{(w)}/\mathbb{Q}_p)$. Let $\mathbb{Q}_p^{(ab)}$ denote the maximal abelian extension of $\mathbb{Q}_p$ in a chosen algebraic closure of $\mathbb{Q}_p$. Since $(L_p^{(w)}/\mathbb{Q}_p)$ is totally ramified, there exists $\tilde{\omega}$ in the inertia group of $\mathbb{Q}_p^{(ab)}/\mathbb{Q}_p$ such that $\tilde{\omega}|_{L_p^{(w)}} = \omega$. It thus follows [M] Corollary 5.2, Remark 1 that

$$(5.15) \qquad \tau(\theta^{\omega^{-1}})^\omega = \tau(\theta)\theta(\tilde{\omega}) = \tau(\theta)\theta(\omega').$$

We have thefore proved, via (5.14) and (5.15), that $s_p \in \mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(G_1(p), E_p^\times)$.
We must observe that, since p is wildly ramified in $L_p^{(w)}/\mathbb{Q}_p$, $s_p(\theta)$ is not in general a unit; nevertheless it can be written as the product of a global element by a unit. More precisely, let θ be an irreducible character of $G_1(p)$. Since the prime ideal of $\mathbb{Q}(\theta^{j_p^{-1}})$ above p is principal there exists $\alpha_p(\theta^{j_p^{-1}}) \in \mathbb{Q}(\theta^{j_p^{-1}}) \subset E$ such that

$$(5.16) \qquad s_p(\theta) = \alpha_p(\theta^{j_p^{-1}})\varepsilon_p(\theta) \text{ with } \varepsilon_p(\theta) \in \mathcal{O}_{E_p}^\times.$$

We once again identify $R_{G_1(p)}$ and $R_{G_1(p),p}$ via j_p. We let $\{\theta_1, \theta_2, ..., \theta_m\}$ be representatives of the distinct $\Omega_{\mathbb{Q}}$- (i.e. $\Omega_{\mathbb{Q}_p}$-) orbits of $\hat{G}_1(p)$. We define $r_p \in \mathrm{Hom}_{\Omega_{\mathbb{Q}}}(R_{G_1(p)}, E^\times)$ by setting $r_p(\theta_i) = \alpha_p(\theta_i), 1 \leq i \leq m$. We denote $\hat{r}_p = \mathrm{Ind}_{G_1(p)}^G(r_p)$. As usual $\hat{r}_p$ can be decomposed as a product

$$(5.17) \qquad \hat{r}_p = \prod_q \hat{r}_{p,q}.$$

We let $f_p'^{(w)}$ be defined by

$$(5.18) \qquad f_p^{(w)} = f_p'^{(w)} \hat{r}_p.$$

It follows from (5.13), (5.16) and (5.18) that

$$(5.19) \qquad j_p^*(f_{p,p}'^{(w)}) = \mathrm{Ind}_{G_1(p)}^G(s_p r_{p,p}^{-1}) \mathrm{Ind}_{G(p)}^G(\mathrm{Det}(\nu)\mathrm{Det}(\delta)).$$

Since $s_p r_{p,p}^{-1} \in \mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(R_{G_1(p)}, \mathcal{O}_{E_p}^\times)$ it belongs to $\mathrm{Det}(\mathfrak{M}_{\mathbb{Q}_p}(G_1(p)^\times))$. We thus conclude that $\mathrm{Ind}_{G_1(p)}^G(s_p r_{p,p}^{-1})$ and $\mathrm{Ind}_{G(p)}^G(\mathrm{Det}(\delta)) \in \mathrm{Det}U_p(\Lambda)$. Morover

$$(5.20) \qquad \mathrm{Ind}_{G(p)}^G \mathrm{Det}(\nu) \in \mathrm{Det}(\mathcal{O}_{M'_p}[G]^\times)$$

On the other hand for any finite prime q

$$j_q^*(\hat{r}_{p,q}) \in \mathrm{Ind}_{G_1(p)}^G(\mathrm{Hom}_{\Omega_{\mathbb{Q}_q}}(R_{G_1(p)}, \mathcal{O}_{E_q}^\times))$$

Since $q \neq p$, $\mathrm{Hom}_{\Omega_{\mathbb{Q}_q}}(R_{G_1(p)}, \mathcal{O}_{E_q}^\times) = \mathrm{Det}(\mathbb{Z}_q[G_1(p)]^\times)$. Hence we conclude that

$$j_q^*(\hat{r}_{p,q}) \in \mathrm{Det}(\mathbb{Z}_q[G]^\times). \tag{5.21}$$

From (5.9), (5.13), (5.20) and (5.21) we deduce that $f'^{(w)}_p$, defined in (5.18), has indeed the required properties. □

6. The final step

From Corollary 3.24, Theorem 4.3 and 5.3 we obtain the representative

$$\check{f}' = \prod_{p \notin S} f_p^* \prod_{p \in S} f'^{(t)}_p f'^{(w)}_p \tag{6.1}$$

for $\check{U}_N$.

For q a prime divisor of $\mathbb{Q}$ we denote by $\check{f}'_q$ the element of $\mathrm{Hom}_{\Omega_{\mathbb{Q}}}(R_G, J_q(E))$ obtained by composing $\check{f}'$ with the projection $J(E) \to J_q(E)$.

Our first aim is to show for any finite prime divisor q

$$\check{f}'_q \in \mathrm{Det}(U_q(\Lambda)). \tag{6.2}$$

We first deduce from [T$_1$] that if p is a finite prime which is tamely ramified in $N/\mathbb{Q}$ then there exists a tamely ramified extension $T_q/\mathbb{Q}_q$ and $a_{p,q} \in \mathcal{O}_{T_q}[G]^\times$ such that

$$j_q^*(f_{p,q}^*) = \mathrm{Det}(a_{p,q}) \tag{6.3}$$

with $a_{p,q} = 1$ for almost every p.

We consider two cases:

Case 1 : $q \notin S$

It follows from (6.3), Theorem 4.3 i) and 5.3 i) that there exists a tame extension $\tilde{T}_q/\mathbb{Q}_q$ such that

$$j_q^*(\check{f}'_q) \in \mathrm{Det}(\mathcal{O}_{\tilde{T}_q}[G]^\times).$$

Since $j_q^*(\check{f}'_q)$ is $\Omega_{\mathbb{Q}_q}$-fixed we use the fixed point theorem for group rings to deduce that

$$j_q^*(\check{f}'_q) \in \mathrm{Det}(\mathbb{Z}_q[G]^\times). \tag{6.4}$$

Case 2 : $q \in S$

From (6.3), Theorems 4.3 and 5.3 we know there exists a tame extension, again denoted by $\tilde{T}_q$, $\lambda \in U_q(\Lambda)$ and $c \in \mathcal{O}_{\tilde{T}_q}[G]^\times$ such that

$$j_q^*(\check{f}'_q) = \mathrm{Det}(\lambda)\mathrm{Det}(c).$$

Since $j_q^*(\check{f}'_q)$ and $\mathrm{Det}(\lambda)$ are both $\Omega_{\mathbb{Q}_q}$-fixed, we deduce that the same is true for $\mathrm{Det}(c)$ and, via the "fixed point theorem" for group algebras, [F], Theorem 10A, that $\mathrm{Det}(c) \in \mathrm{Det}(\mathbb{Z}_q[G]^\times)$. We thus conclude that

$$j_q^*(\check{f}'_q) \in \mathrm{Det}(U_q(\Lambda)). \tag{6.5}$$

It follows from (6.4) and (6.5) that $\check{U}_N$ is represented by $(\check{f}'_\infty)$ which belongs to $\mathrm{Hom}_{\Omega_\mathbb{Q}}(R_G, J_\infty(E))$. Since $(\check{f}'_\infty)$ takes real values on real valued characters, it follows that $({f'}^2_\infty) \in \mathrm{Det}\ U_\infty(\Lambda)$. Therefore we have proved that $\check{U}_N^2 = 1$.
We now want to examine more precisely $(\check{f}'_\infty)$. From section 2, and from (4.12) and (5.18) we deduce the decomposition of $(\check{f'}_\infty)$ into a product

$$(\check{f}'_\infty) = uv \tag{6.6}$$

where u and v both belong to $\mathrm{Hom}_{\Omega_\mathbb{Q}}(R_G, J_\infty(E))$ with

$$u(\chi) = \tau(\chi)(a_\infty|\chi)^{-1}, \ \ v(\chi) = \prod_{p\in S} \frac{\tau_p(\delta_p^{(t)}(\chi))\tau_p(\delta_p^{(w)}(\chi))}{\tau_p(\chi)} \times \hat{r}_p u_p \tag{6.7}$$

We now assume that $2 \notin S$.

Lemma 6.8 $v \in \mathrm{Det}\ U_\infty(\Lambda)$

Proof. We decompose v as a product $\prod_{p\in S} v_p$. Let $R_G^{(s)}$ be the subgroup of R_G generated by symplectic characters. From [F], I, Proposition 2 we know that it suffices to show that for $p \in S$, $v_p(\chi)_\mathfrak{p}$ is real and positive for all $\chi \in R_G^{(s)}$ and all infinite prime divisors $\mathfrak{p}$ of E. Since p, is odd it is easy to check from their definitions that the infinite components of both $\hat{r}_p$ and u_p have the required property. The map

$$\chi \to \frac{\tau_p(\delta_p^{(t)}(\chi))\tau_p(\delta_p^{(w)}(\chi))}{\tau_p(\chi)}$$

is induced from $G_0(p)$. For simplicity we denote by τ the map $\tau_{G_0(p)}$. Since p is odd, any irreducible real valued character is inflated from a character φ of $G(p)^{(t)} = \mathrm{Gal}(N_p^{(t)}/\mathbb{Q}_p)$. Let φ be such an irreducible symplectic character. It suffices now to prove that

$$\tau^*(\delta_p^{(t)}(\varphi))/\tau^*(\varphi) \tag{6.9}$$

is real and positive when replacing τ by the modified Gauss sum τ^*. It is an easy exercise to show that there exists a subgroup H with $G'_0(p) \subset H \subset G(p)^{(t)}$ and $u \in \hat{H}$ such that

$$\varphi = \mathrm{Ind}_H^{G(p)^{(t)}}(u). \tag{6.10}$$

Moreover, even if the Adams operation $\delta_p^{(t)}$ does not commute with induction, one can prove that

$$\tau^*(\delta_p^{(t)}(\varphi)) = \tau^*(\mathrm{Ind}_H^{G(p)^{(t)}}(\delta_p^{(t)}(u))).$$

Using now the inductive properties of τ^* in degree 0, we obtain that (6.9) can be written

$$\tau^*(\delta_p^{(t)}(u))/\tau^*(u). \tag{6.11}$$

Since p is tame in $N_p^{(t)}/\mathbb{Q}_p$, $u = (\delta_p^{(t)}u)\alpha$ where α is a non ramified abelian character. We deduce from [F], IV, Proposition 1.1 that

$$\tau^*(\delta_p^{(t)}(u)) = \tau^*(u) \tag{6.12}$$

which implies the required result. □

Using [H-W$_3$], proof of Theorem 2.2, we know that W_N can be represented by the function W defined by

$$W(\chi)_q = \begin{cases} 1 & \text{if } q \neq \infty \text{ and } \chi \text{ non symplectic} \\ \tau(\chi)W_\infty(\chi) & \text{otherwise.} \end{cases} \tag{6.13}$$

Since the signs of $W_\infty(\chi)$ and $(a_\infty|\chi)$ for symplectic characters are the same, we conclude from (6.6), (6.7), (6.8) that

$$\check{f}'_\infty W^{-1} \in \mathrm{Det}\ U_\infty(\Lambda) \tag{6.14}$$

We then have proved, as required, that when 2 is at most tamely ramified

$$\check{f}' \sim \check{f}'_\infty \sim W \tag{6.15}$$

□

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