A generic exact solver for vehicle routing problems ans its applications

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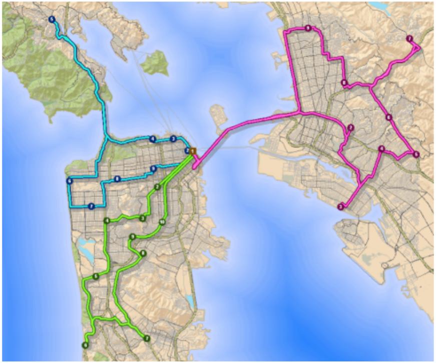
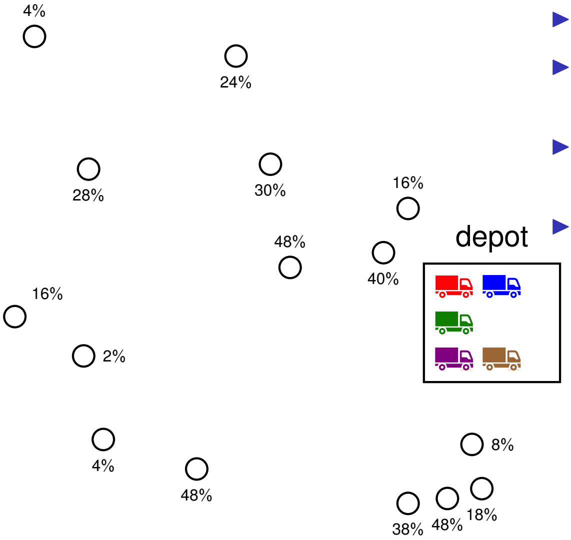
Contents

Vehicle Routing Problem (VRP)

- One of the most widely investigated optimization problems.
- Google Scholar finds +8,000 works published in 2019 ($>$1000 contain both “vehicle” and “routing” in the title)
- Direct application in the real-world systems that distribute goods and provide services



Capacitated Vehicle Routing Problem (CVRP)



- Depot
- Identical vehicles of capacity Q
- Clients $i \in V$ with demand d_i
- Matrix c of travelling costs

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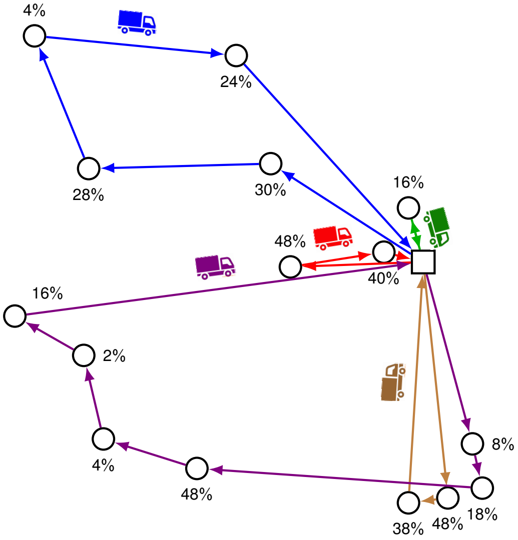


- Depot
- Identical vehicles of capacity Q
- Clients $i \in V$ with demand d_i
- Matrix c of travelling costs

Minimize the total travelling cost

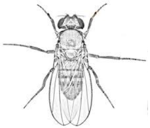
- such that every client is served
- total demand of clients served by the same vehicle does not exceed its capacity

Why do we care so much about CVRP?

First [Dantzig and Ramser, 1959] and the most basic VRP variant.

Common strategy in scientific research

- Study the simplest (bust still representative!) case of a phenomenon
- Generalize the discoveries for more complex cases



Drosophila Melanogaster

Hundreds of VRP variants

Vehicle capacities, time windows, heterogeneous fleet, multiple depots, split delivery, pickup and delivery, backhauling, optional customer service, arc routing, alternative delivery options, service levels, etc, etc

Some history

- [Balinski and Quandt, 1964] set-partitioning formulation for the CVRP
- [Laporte and Nobert, 1983] MIP formulation with edge variables, rounded capacity cuts, and branch-and-bound
- [Desrochers et al., 1992] first branch-and-price
- [Lysgaard et al., 2004] best branch-and-cut algorithm
- [Fukasawa et al., 2006] robust branch-cut-and-price
- [Baldacci et al., 2008] enumeration technique
- [Jepsen et al., 2008] (non-robust) subset-row cuts
- [Baldacci et al., 2011] *ng*-route relaxation
- [Pecin et al., 2017] limited-memory technique
- [Sadykov et al., 2021] bucket graph based labeling algorithm
- [Poggi and Uchoa, 2014] [Costa et al., 2019] recent surveys

Resource constrained paths to model feasible routes

- Complete directed graph $G = (V^0, A)$, $V^0 = \{0\} \cup V$.
- Capacity resource
- Resource consumption of arc $a = (i, j) \in A$ is d_j, $d_0 = 0$.
- Accumulated resource consumption interval for $v \in V^0$ is $[0, Q]$.

A set of feasible routes is modelled by set P of paths in G from node 0 to node 0 such that for each path $p \in P$

- each node $v \in V$ is visited at most once.
- accumulated resource consumption for every node v visited by p is within given intervals $[0, Q]$.

Path-based formulation

- Variable x_a — arc $a \in A$ is used in the solution or not
- Variable λ_p — path $p \in P$ is used in the solution or not
- $h_a^p = 1$ if and only if path p contains arc a, otherwise 0
- $\delta^-(v)$ — set of arcs in A incoming to $v \in V$

$$
\begin{aligned}
\text{Min} \quad & \sum_{a \in A} c_a x_a \\
\text{S.t.} \quad & \sum_{a \in \delta^-(v)} x_a = 1, && v \in V, \\
& Bx \leq b, \\
& x_a = \sum_{p \in P} h_a^p \lambda_p, && a \in A, \\
& \sum_{p \in P^k} \lambda_p \leq K, \\
& x_a \in \{0, 1\}, && a \in A, \\
& \lambda_p \in \{0, 1\}, && p \in P.
\end{aligned}
$$

Path-based formulation

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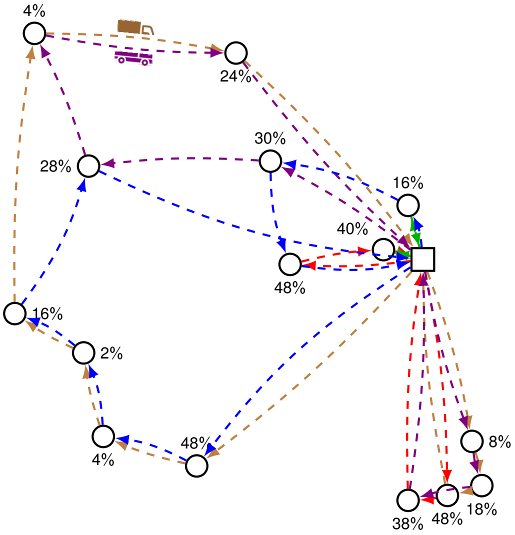
& Bx \leq b, & & \\
& x_a = \sum_{p \in P} h_a^p \lambda_p, & a \in A, & (\pi_a) \\
& \sum_{p \in P^k} \lambda_p \leq K, & & (\mu) \\
& 0 \leq x_a \leq 1, & a \in A, & \\
& 0 \leq \lambda_p \leq 1, & p \in P. &
\end{array}
$$

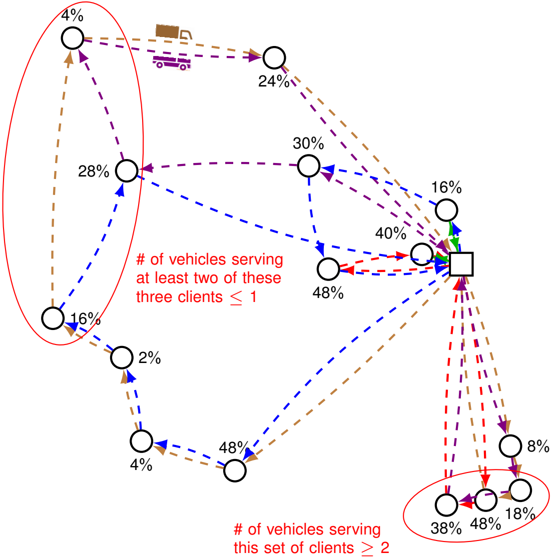
Column and cut generation: illustration



One continuous variable per feasible route.

Pricing problem is the Elementary Resource Constrained Shortest Path problem.

Column and cut generation: illustration



One continuous variable per feasible route.

Pricing problem is the Elementary Resource Constrained Shortest Path problem.

Additional constraints (cuts) are added to reduce the number of feasible non-integer solutions

Bad news

- Vehicle routing problems in practice are never “pure” CVRPs
- Designing and implementing a state-of-the-art BCP algorithm for a particular problem takes several months for an expert team
- One would like to have a generic algorithm that could be easily customised to many variants.
- Some attempts in the literature: [Desaulniers et al., 1998] [Baldacci and Mingozzi, 2009]

Generic model

Instead of implementing the algorithm to solve the ERCSPP, the user provides a graph-based model, i.e. for each graph it gives an implicit description of feasible paths:

- Nodes, arcs, the source and the sink
- Resources
- Resource consumption for arcs
- Accumulated resource consumption for vertices

In addition, a MIP model is given for “non-resource-related” constraints:

- Non-path variables, constraints, objective
- Mapping between variables and graph arcs (so that the coefficients of path variables in constraints can be determined)
- Optionally, separation algorithms for families of cutting planes (over non-path variables)

State-of-the-art Branch-Cut-and-Price for CVRP

- Stabilization techniques
- Primal heuristics
- Strong branching
- Bucket graph based labeling algorithm for the pricing
- Heuristic pricing
- Variable fixing by reduced costs
- (Dynamic) *ng*-route relaxation for the pricing
- Limited-memory rank-1 Chvátal-Gomory cuts
- Rounded capacity cuts
- Enumeration of elementary routes
- Ryan-and-Foster branching

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Generic model: collection of packing sets

Definition

A packing set is a subset of arcs (vertices) such that, in an optimal solution of the problem, at most one arc (vertex) in the subset appears at most once.

- Definition of packing sets is a part of modeling
- Packing sets generalize customers in CVRP

Generic model: collection of packing sets

Definition

A packing set is a subset of arcs (vertices) such that, in an optimal solution of the problem, at most one arc (vertex) in the subset appears at most once.

- Definition of packing sets is a part of modeling
- Packing sets generalize customers in CVRP
- Generalization examples:
  - Heterogeneous Fleet: customer copies for each vehicle type
  - Multiple time windows: customer copies for each time window
  - Alternative delivery locations: all delivery locations for each client
  - Arc routing: two possible directions for a required edge

Generic BCP solver

Generic Branch-Cut-and-Price (BCP) state-of-the-art solver for Vehicle Routing Problems (VRPs) [Pessoa et al., 2020].

`vrpsolver.math.u-bordeaux.fr`

- Pre-compiled C++ code distributed in a docker image
- Open-source Julia-JuMP interface
- Demos for several VRPs and non-VRPs are available

Pessoa, A., Sadykov, R., Uchoa, E., and Vanderbeck, F. (2020).
A generic exact solver for vehicle routing and related problems.
Mathematical Programming, 183:483–523.

VRPSolver Julia-JuMP interface

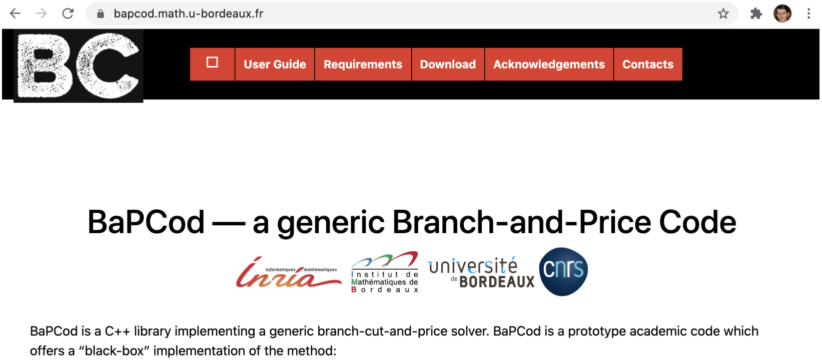
```
using VRPSolver, JuMP
function build_model(data::DataCVRP)
   A = arcs(data) # set of arcs of the input graph G'
   n = nb_customers(data)
   V = [i for i in 1:n] # set of customers of the input graph G'
   V0 = [i for i in 0:n] # set of vertices of the graphs G' and G
   Q = veh_capacity(data)

   cvrp = VrpModel()
   @variable(cvrp.formulation, x[a in A], Int)
   @objective(cvrp.formulation, Min, sum(c(data,a) * x[a] for a in A))
   @constraint(cvrp.formulation, setpart[i in V], sum(x[a] for a in inc(data, i)) == 1.0)

   function build_graph() # Build the model directed graph G=(V,A)
      v_source = v_sink = 0
      G = VrpGraph(cvrp, V0, v_source, v_sink, (0, n))
      cap_res_id = add_resource(G, main = true)
      for i in V
         set_resource_bounds(G, i, cap_res_id, 0, Q)
      end
      for (i,j) in A
         arc_id = add_arc(G, i, j, x[(i,j)])
         set_arc_consumption(G, arc_id, cap_res_id, d(data, j))
      end
      return G
   end

   G = build_graph()
   add_graph(cvrp, G)
   set_vertex_packing_sets(cvrp, [[(G,i)] for i in V])
   define_packing_sets_distance_matrix(cvrp, [[distance(data, (i, j)) for j in V] for i in V]
   add_capacity_cut_separator(cvrp, [ ( [(G,i)], d(data, i) ) for i in V], Q)
   set_branching_priority(cvrp, "x", 1)
   return (cvrp, x)
end
```

C++ interface through BaPCod



- User guide is available [Sadykov and Vanderbeck, 2021]
- BaPCod source code is available
- VRPSolver extension requires a precompiled RCSP library.

State-of-the-art performance for many problems

- Capacitated Vehicle Routing Problem (CVRP)
- CVRP with Time Windows
- Heterogeneous Fleet CVRP
- Multi-depot CVRP
- Pickup-and-Delivery Problem with Time Windows
- CVRP with Backhauls
- Multi-Trip Vehicle Routing Problem with Time Windows
- (Capacitated) Team Orienteering Problem
- Capacitated Profitable Tour Problem
- Vehicle Routing Problem With Service Levels
- Generalized Assignment Problem
- Vector Packing Problem
- (Variable Size) Bin Packing Problem
- Capacitated Arc Routing Problem
- Robust CVRP with Demand Uncertainty
- Location-Routing Problem
- Two-Echelon Vehicle Routing Problem
- Black-and-White Travelling Salesman Problem

World record for the CVRP exact solving

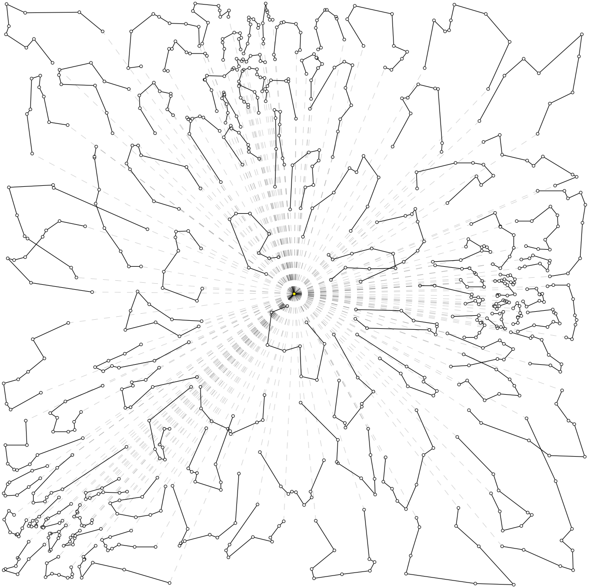


Figure: Optimal solution for X-n865-k95 (solved in 10 days)

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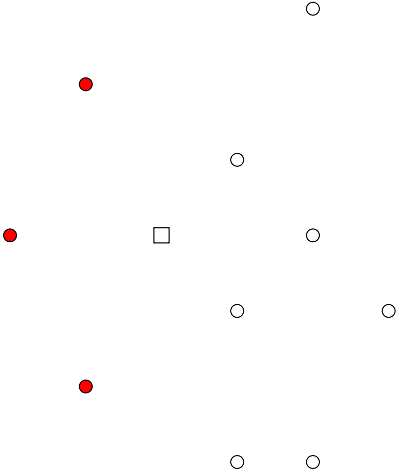
Robust counterpart of the CVRP



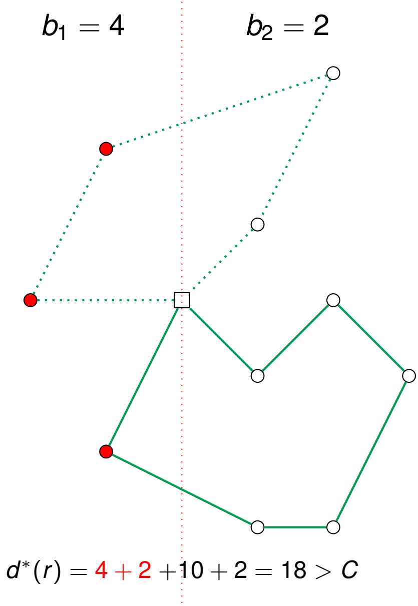
Demands are uncertain:

Each customer $i \in V \setminus \{0\}$ has a mean demand $\bar{d}_i$ and a demand deviation $\hat{d}_i$.

Example

- $K = 2$, $C = 16$
- $\bar{d}_i = 2$, $\hat{d}_i = 1$
- $\bar{d}_i = 4$, $\hat{d}_i = 2$

Partition-constrained uncertainty



Set $\mathcal{D}$ of demand vectors.

Each route must be robust to all demand scenarios of $\mathcal{D}$.

[Gounaris et al., 2013]:

$\mathcal{D} = \mathcal{D}^{part} = \{ d \in \mathbb{R}^n_+ \mid d_i = \bar{d}_i + \xi_i, i \in V^0, \sum_{i \in V_k} \xi_i \leq b_k, k = 1, \ldots, s, 0 \leq \xi \leq \hat{d} \}$.

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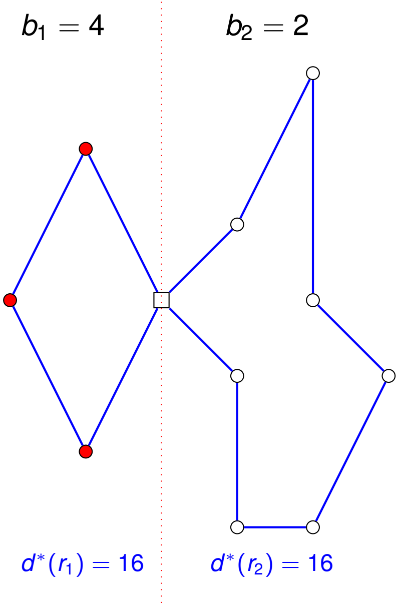

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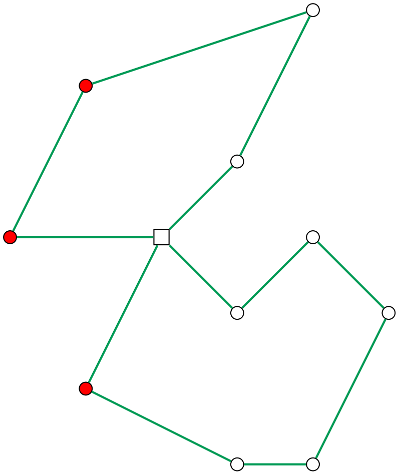
Cardinality-constrained uncertainty

$\Gamma = 2$



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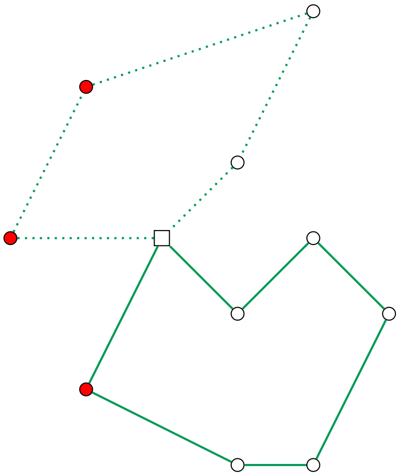

$d^*(r) = 4 + 2 + 10 + 1 = 17 > C$

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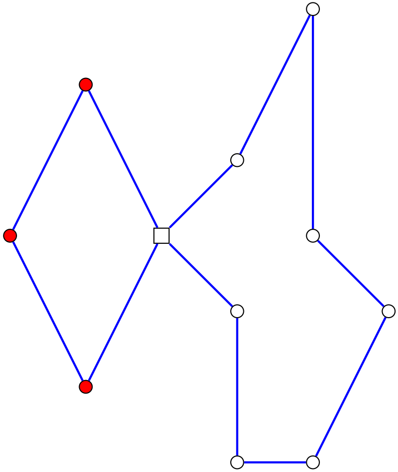

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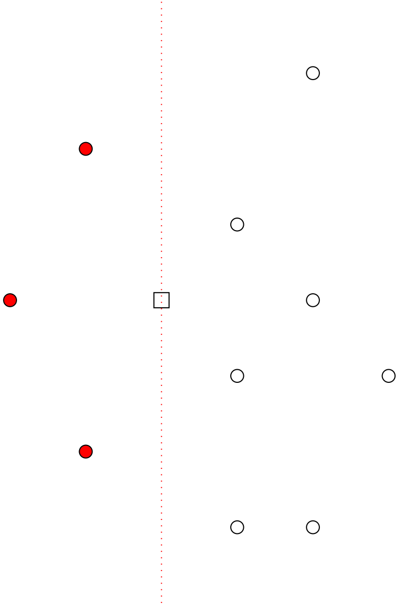
Example

- $K = 2$, $C = 16$
- $\bar{d}_i = 2$, $\hat{d}_i = 1$
- $\bar{d}_i = 4$, $\hat{d}_i = 2$

Knapsack-constrained uncertainty



Generalizes $\mathcal{D}^{part}$ and $\mathcal{D}^{card}$:

$$\mathcal{D} = \mathcal{D}^{knap} = \{d \in \mathbb{R}^n_+ \mid d_i = \overline{d}_i + \eta_i \hat{d}_i, i \in V^0, \sum_{i \in V_k} w_i \eta_i \leq b_k,\ k = 1, \ldots, s, 0 \leq \eta \leq 1\}.$$

For $\mathcal{D}^{part}$:

- $w_i = \hat{d}_i$, for all $i \in V^0$

For $\mathcal{D}^{card}$:

- $s = 1$, $b_1 = \Gamma$
- $w_i = 1$, for all $i \in V^0$

Main result

P^{knap} — set of robustly feasible elementary routes (paths).

$$P^{knap} = \left\{ p \in P_0 \;\middle|\; \sum_{i \in V_0} h_i^p d_i \le Q, \quad \forall d \in \mathcal{D}^{knap} \right\}.$$

New theorem (that extends known results for $\mathcal{D}^{card}$):

$$P^{knap} = \bigcup_{\theta \in \tilde{\Theta}} P^{\theta},$$

where

$$P^{\theta} = \left\{ p \in P_0 \;\middle|\; \sum_{i \in V_0} h_i^p d_i^{\theta} \le C - b^{\top}\theta \right\}.$$

and $\tilde{\Theta}$ is a discrete (small) vector set $\subset \mathbb{R}_+^s$.
$|\tilde{\Theta}| = 2^s$ for $\mathcal{D}^{part}$, and $|\tilde{\Theta}| = \lceil (n - \Gamma)/2 \rceil + 1$ for $\mathcal{D}^{card}$.

Heterogeneous Fleet Vehicle Routing Problem (HFVRP)

- Undirected graph $G' = (V, E)$, $V = \{0, \dots, n\}$, 0 is the depot, $V_0 = \{1, \dots, n\}$ are the customers; positive demands d_i^k, $i \in V_0$, $k \in K$; set of vehicle types K; edge costs c_e^k, $e \in E$, $k \in K$; vehicle type capacity Q^k, $k \in K$.
- Find a minimum cost set of routes, each route associated to a vehicle type, visiting all customers and such that the sum of the demands of the customers in a route does not exceed its vehicle type capacity.

Reduction of the robust CVRP to a HFVRP

$K = \tilde{\Theta}$, $Q^k = C - b^\top \theta(k)$, $d_i^k = d_i^\theta, \forall k \in K, i \in V_0$, $c_e^k = c_e$, $\forall k \in K, e \in E$.

VRPSolver Model for Heterogeneous Fleet Vehicle Routing Problem (HFVRP)

Graphs G^k

$G^k = (V^k, A^k)$, $V^k = \{v_0^k, \ldots, v_n^k\}$, $v^k_{\text{source}} = v^k_{\text{sink}} = v_0^k$, $k \in K$
$A^k = \{(v_i^k, v_j^k), (v_j^k, v_i^k) : \{i,j\} \in E\}$
$q^k_{a,1} = d_j^k$, $a = (v_i^k, v_j^k) \in A^k$, $k \in K$ (define $d_0^k = 0$);
$l_{v_i^k,1} = 0, u_{v_i^k,1} = Q^k, v_i^k \in V^k$, $k \in K$.

Formulation

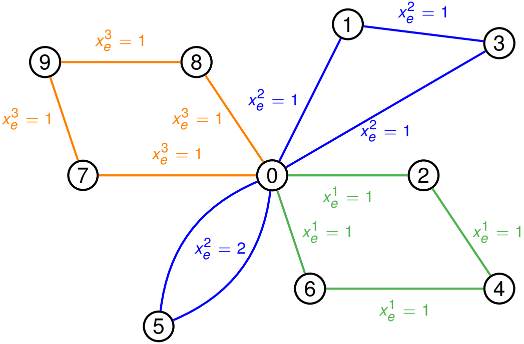
Integer variables x_e^k, $e \in E$, $k \in K$.

$$
\begin{array}{rll}
\text{Min} & \sum_{k\in K}\sum_{e\in E} c_e^k x_e^k & \\
\text{S.t.} & \sum_{k\in K}\sum_{e\in\delta(i)} x_e^k = 2, & i \in V_0;
\end{array}
$$

$M(x_e^k) = \{(v_i^k, v_j^k), (v_j^k, v_i^k)\}$, $e = \{i,j\} \in E$, $k \in K$.
Packing sets defined on vertices: $\mathcal{B}^{\mathcal{V}} = \cup_{i\in V_+}\{\{v_i^k : k \in K\}\}$

VRPSolver Model for the HFVRP : illustration

Integer variables x_e^k, $e \in E$, $k \in K$ - how many times edge e is used in a route of a type k vehicle.



Additional results for the robust CVRP

Our paper [Pessoa et al., 2021]

- Reduction from the robust CVRP to a deterministic HFVRP (already presented)
- Pre-processing to reduce $|\tilde{\Theta}|$ (sufficient conditions for $P^\theta = \emptyset$ which can be verified in polynomial time)
- A new families of cutting planes, expressed over arc variables x, which is provably stronger than those proposed by [Gounaris et al., 2013].
- An iterated local search algorithm to find initial feasible solutions

Pessoa, A., Poss, M., Sadykov, R., and Vanderbeck, F. (2021).
Branch-and-cut-and-price for the robust capacitated vehicle routing problem with knapsack uncertainty.
Operations Research, 69(3):739–754.

Computational results for $\mathcal{D}^{part}$ (30-150 clients)

Comparison with the state-of-the-art algorithm

Inst. class	# in.	VRPSolver			[Gounaris et al., 2016]		
		#n.	t.	#opt.	gap	t.	#opt.
A	26	1.00	2.91	26	1.97%	3440.31	12
B	23	1.05	5.98	23	1.39%	250.96	13
E	11	1.00	11.40	11	2.19%	573.01	5
F	3	5.37	833.42	2	1.10%	55.76	2
M	3	3.33	153.51	3	2.70%	86700.00	1
P	24	1.00	1.48	24	2.09%	976.36	10
all	90	1.11	4.75	89	1.87%	981.90	43

Effect of pre-processing (reduction of $|\tilde{\Theta}|$)

Inst. class	#in.	Initial $\vert\tilde{\Theta}\vert$	Reduced $\vert\tilde{\Theta}\vert$	%red.	# of $\vert\tilde{\Theta}\vert = 1$
A	26	7.1	2.7	83.4%	7
B	23	7.2	3.9	75.8%	0
E	11	7.3	3.6	77.3%	4
F	3	5.0	5.0	68.8%	0
M	3	9.7	1.3	91.7%	2
P	24	7.3	4.3	73.2%	9
all	90	7.2	3.6	77.8%	22

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A property of modern BCP algorithms for the CVRP

If a good upper bound is known on the optimal solution value, instances with small and moderate size can now be rapidly solved to optimality.

- Instances with 50 clients $\rightarrow$ $\approx$ 1 second
- Instances with 100 clients $\rightarrow$ $\approx$ 1 minute
- Instances with 150 clients $\rightarrow$ several minutes

Idea to use this property

If an instance is large, we can decompose it into sub-instances of smaller size, and solve them (optimally or sub-optimally) to try to improve the current solution.

Partial OPtimization Metaheuristic Under Special Intensification Conditions [Taillard and Voss, 2002]

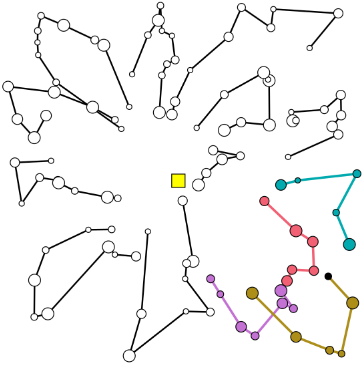
An overview of our POPMUSIC matheuristic

[Queiroga et al., 2021]

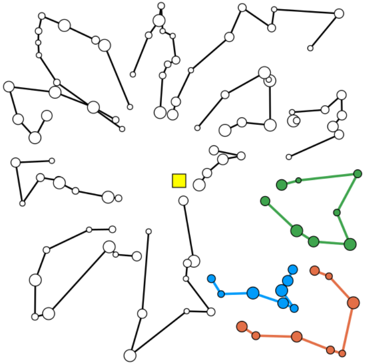
1. Obtain a (good) initial solution using a known heuristic
2. Fix initial target dimension
3. For every “seed” client, construct and solve a restricted instance using a metric:
   - add to the restricted instance closest routes in the current solution while the target dimension is not exceeded
   - if the restricted instance has not yet been solved, solve it
   - if an improved solution for the restricted instance is obtained, update the current “global” solution
4. Increase the target dimension and go to Step 3

Queiroga, E., Sadykov, R., and Uchoa, E. (2021).
A POPMUSIC matheuristic for the capacitated vehicle routing problem.
Computers & Operations Research, 136:105475.

Our POPMUSIC matheuristic : illustration



(a) Initial solution and a constructed subproblem. Seed client is marked in black.



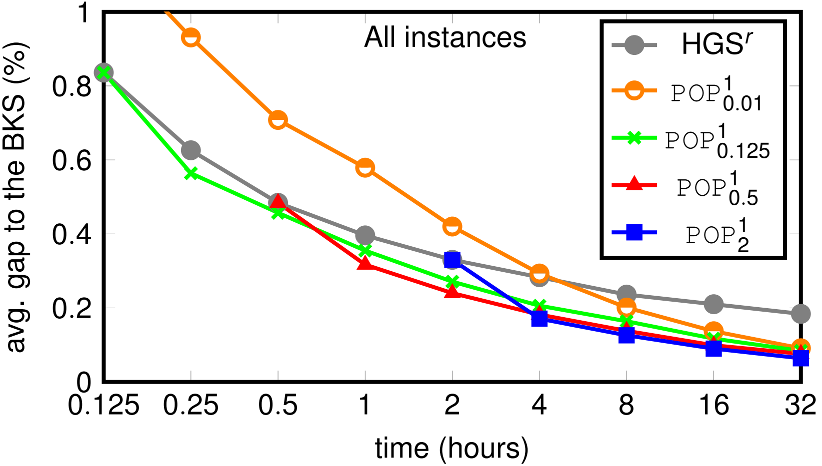
(b) Improved solution after finding a better subsolution

Figure: Constructing and solving a subproblem. Depot is the yellow square, and customers are circles with diameter proportional to its demand.

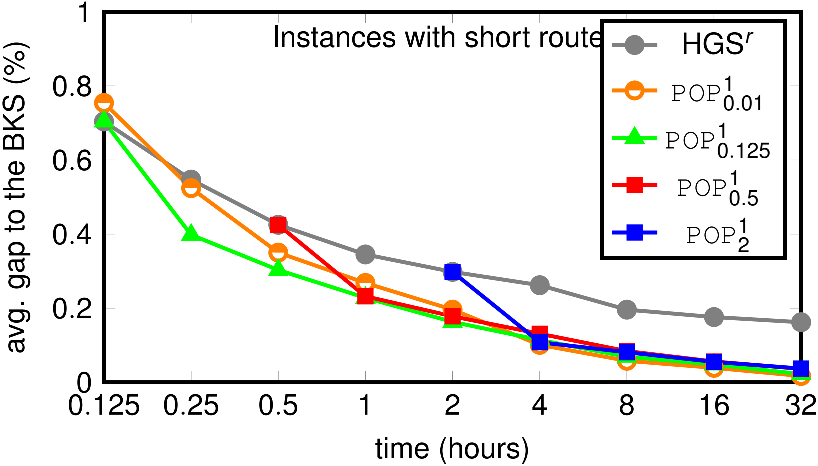
Computational comparison with [Vidal et al., 2012]

[Vidal et al., 2012] is probably the most known heuristic for classic VRP problems (>500 citations on Google Scholar)

Instances with 300–1000 clients.

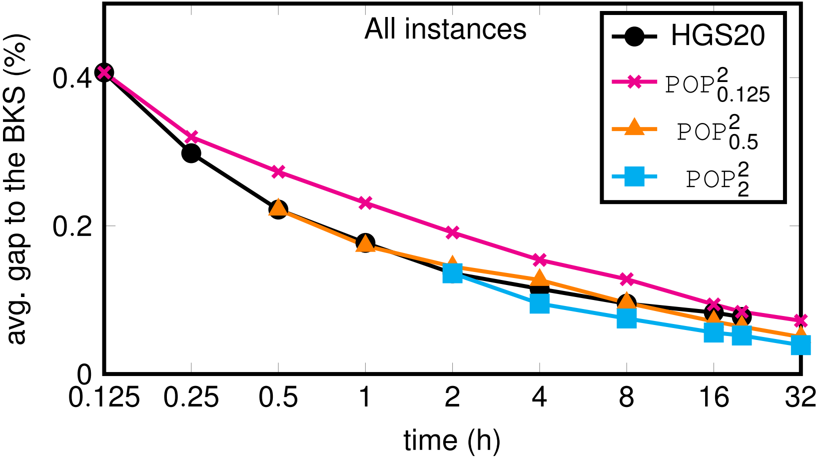


Computational comparison with [Vidal et al., 2012]



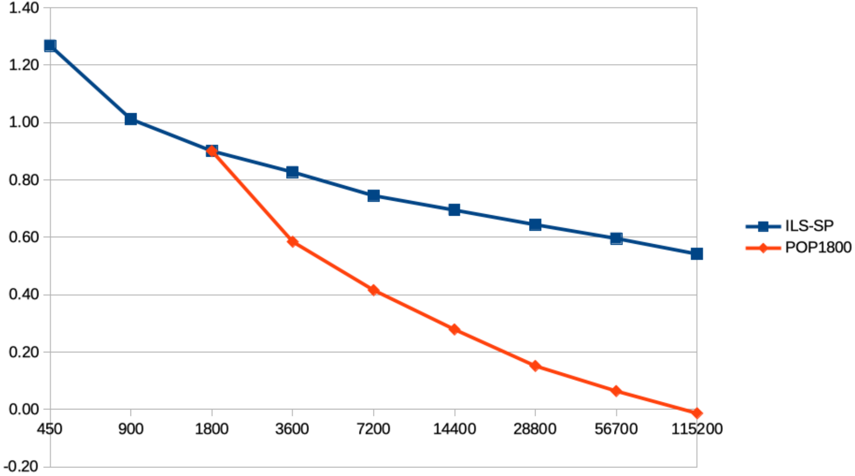
Computational comparison with [Vidal, 2020]

[Vidal, 2020] is an improved version of [Vidal et al., 2012] heuristic, with specialised implementation for the CVRP (`https://github.com/vidalt/HGS-CVRP`)



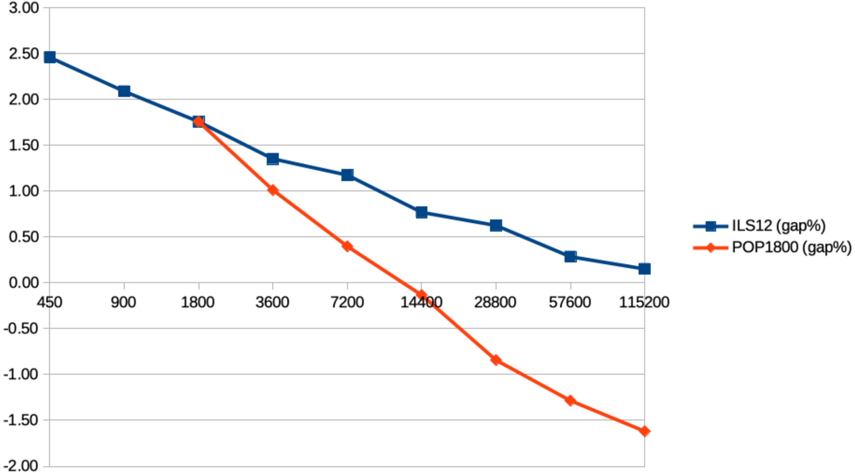
Computational results for the CVRP with backhauls

Comparison with [Subramanian and Queiroga, 2020].



Computational results for the HFVRP

Comparison with [Subramanian et al., 2012].



Contents

Conclusions and perspectives

- Generic Branch-Cut-and-Price solver combines an outstanding performance for exact solution of many VRPs with a (relative) ease of use
- Exact deterministic solver may be useful for problems with uncertainty and as a base for (mat)heuristics
- The solver is an excellent tool (much better than MIP solvers) for estimating the “real” quality of VRP heuristics
- It can be used for testing new families of robust cutting planes within a state-of-the-art BCP algorithm
- We are working now on extending modelling capabilities of VRPSolver

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