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GHOST EFFECT FOR A VAPOR-VAPOR MIXTURE

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Abstract. This paper studies the non linear Boltzmann equation for a two component gas at the small Knudsen number regime. The solution is found from a truncated Hilbert expansion. The first order of the fluid equations shows the ghost effect. The fluid system is solved when the boundary conditions are close enough to each other. Next the boundary conditions for the kinetic system are satisfied by adding for the first and the second order terms of the expansion Knudsen terms. The construction of such boundary layers requires the study of a Milne problem for mixtures. In a last part the rest term of the expansion is rigorously controled by using a new decomposition into a low and a high velocity part.

1. **Introduction.** This paper is devoted to the rigorous asymptotic analysis of a kinetic system situated at a small Knudsen number regime with given indata boundary conditions. The physical model is described in ([29]). It is constituted by a mixture of vapor situated between two infinite parallel planes. Those two phases can condense or evaporate on the two infinite parallel planes of condensed phases kept at fixed temperatures. Moreover this model is supposed to be space homogeneous in the directions parallel to the planes. Next two remarkable situations are precisely investigated depending on the jump of the pressure of the total mixture between the two condensed phases. If this difference is of order $\mathcal{O}(1)$, the mixture is described at the continuum limit by the stationary Euler system corrected on each boundary by Knudsen layers. Hence the solution of this Euler system is constant exept at the boundary layers. In a second situation the jump is of the same order as the Knudsen number. In that case the macroscopic velocity of each specy is of order one w.r.t the Knudsen number and so disappears when Knudsen number tends to 0. But the continuum level is described by a convection diffusion system where zero order macroscopic quantites depend on the first order term of the macroscopic velocity. That means that a perturbation of order ε on the kinetic problem gives a finite effect on the fluid limit. This is an example of the ghost effect. It was pointed out in the situation of a one component gas in ([26]) and in the situation of a mixture of a condensable and a non condensable gas in ([1, 4, 3]). In the present paper, only the second situation is investigated and the solution of the kinetic system is constructed as an asymptotic expansion around a local Maxwellian. The Hilbert terms of the expansion are corrected from the first order by Knudsen layers

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in order to satisfy the boundary conditions (1.2, 1.3). The estimate on the rest term of the expansion remains the most delicate part of the work. The general technique is to linearize the problem satisfied by the rest term and to obtain the rest as the limit of a sequence of such linearized problem. But an important difficulty appears when the equilibrium state is a non local Maxwellian function due to the presence of third order terms in the velocity variable. If the equilibrium state is a global Maxwellian, the decomposition of the rest term performed in ([16, 20, 21, 14]) and in the present paper is not necessary because the third order term disappears. The first idea to treat this problem has been introduced by Caflish for a time dependant case and for a space periodic problem in [16]. The idea is to decompose the rest term into a low and a high velocity part. The method has been generalized in ([20, 21]) for the stationary Boltzmann equation for a single component gas in presence of a force term when macroscopic quantities satisfy Navier-Stokes system. But the technique is restricted to boundary conditions of Maxwell diffuse reflection type. In that case the type of boundary conditions is crucial because they lead to a normal flux of the distribution function equal to 0 and the approach breaks down for other types of boundary conditions. In the situation of a mixture this method has been generalized in ([14]) when one component satisfies boundary conditions of Maxwell-diffuse type and the other a given indata profile. Remark that when the equilibrium state of the system is a global Maxwellian function (see [8, 9, 6, 5]) the present decomposition is not useful. Moreover when the same system of kinetic equations is far from equilibrium the techniques of resolution are totally different. In that case compactness techniques are used (see [12, 13, 15]) and weak L^1 solutions are obtained when small velocities are truncated.

Next we mention some other related results to the present paper. In ([6, 5]), the authors consider the the Benard problem physically describded in [25]. They construct by means of perturbative arguments for small Knudsen number, a positive two dimensional solution to the stationary Boltzmann equations which is shown to satisfy a stability property for long times. Let us notice that in ([5]), the control of the rest term is performed thanks to a new spectral inequality. In ([7]) the ghost effect by curvature intoduced in ([27]) is rigorously analysed from perturbative arguments. The physical model corresponds to a Couette flow situated between two coaxial rotating cylinders at the small curvature and small Knudsen number regime. The comparison of the limiting model with the standard planar Couette flow shows that an infinitesimal variation on the curvature induces a finite effect on the solution.

Now let us describe the mathematical model studied in this paper. The molecules of both species are assumed to be mechanically identical that is the molecular mass and size are common to species. f^A, f^B are the distribution functions of the species A and B, solutions to the stationary Boltzmann equation for a two component gas ([17])

$$\begin{aligned}
\xi\frac{\partial}{\partial x}f^A(x,v) &= \frac{1}{\varepsilon}Q(f^A,f^A)(x,v)+\frac{1}{\varepsilon}Q(f^A,f^B)(x,v),\\
\xi\frac{\partial}{\partial x}f^B(x,v) &= \frac{1}{\varepsilon}Q(f^B,f^A)(x,v)+\frac{1}{\varepsilon}Q(f^B,f^B)(x,v),\\
&\qquad\qquad x\in[-1,1],\quad v\in\mathbb{R}^3, \qquad (1.1)
\end{aligned}$$

with

$$\varepsilon = \frac{\sqrt{\pi}}{2} K_n = \frac{\sqrt{\pi}}{2}\frac{l}{2} \text{ and } \quad l = \frac{1}{\sqrt{2}\pi d^2 n_I}.$$

l is the mean free path of the vapor molecules in the equilibrium state at rest with temperature T_I and density n_I, K_n is the Knudsen number and d corresponds to the diameter of the molecule. Q is called collision operator of the equation (1.1) and is defined by ([17], [18])

$$Q(f,g)(x,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathcal{B}(v-v_*,\omega)[f'g'_* - fg_*]d\omega dv_*,$$

with

$$f_* = f(x,v_*), \quad f' = f(x,v'), \quad f'_* = f(x,v'_*).$$

v, v_* and v', v'_* are the post-collisional and the pre-collisional velocities in an elastic collision:

$$v' = v - \langle v-v_*,\omega\rangle\omega, \quad v'_* = v_* + \langle v-v_*,\omega\rangle\omega.$$

The velocity $v \in \mathbb{R}^3$ has for coordinates (ξ,η,χ) and $\langle\ ,\ \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^3$. Let $\omega \in \mathbb{S}^2$ be represented by the polar angle (with axis along $v-v_*$) and the azimutal angle ϕ. The function $\mathcal{B}(v-v_*,\omega) = |\langle v-v_*,\omega\rangle|$ is the collision kernel of the collision operator Q considered in the situation of hard-sphere.

The boundary condition for the A and the B components satisfy the following given indata profile

$$f^A(-1,v) = \frac{p_I^A/T_I}{(\pi T_I)^{\frac{3}{2}}}\exp(-\frac{v^2}{T_I}),\ \xi>0,\quad f^A(1,v) = \frac{p_{II}^A/T_{II}}{(\pi T_{II})^{\frac{3}{2}}}\exp(\frac{-v^2}{T_{II}}),\ \xi<0, \tag{1.2}$$

$$f^B(-1,v) = \frac{p_I^B/T_I}{(\pi T_I)^{\frac{3}{2}}}\exp(-\frac{v^2}{T_I}),\ \xi>0,\quad f^B(1,v) = \frac{p_{II}^B/T_{II}}{(\pi T_{II})^{\frac{3}{2}}}\exp(\frac{-v^2}{T_{II}}),\ \xi<0. \tag{1.3}$$

T_I (resp. T_{II}) represents the temperature of the condensed phase situated at $x=-1$ (resp. $x=1$) and p_I^α is the saturation pressure of the species α at temperature T_I (resp. T_{II}). For the sake of simplicity, we take as in [29] $T_I = p_I^A = 1$. Moreover we assume that the pressures satisfy the relation $p_{II}^A = p_1^B + 1 - p_{II}^B + \frac{2}{\sqrt{\pi}}\Delta\,\varepsilon$, where Δ is a nonzero constant of order $\mathcal{O}(1)$ giving rise to the ghost-effect.

Next we define the macroscopic quantities n^α, u^α as the moments of the distribution function f^α, $\alpha \in \{A,B\}$ ([29]).

$$n^\alpha = \int_{\mathbb{R}^3_v} f^\alpha dv,\ \ nu_1^\alpha = \int_{\mathbb{R}^3_v}\xi f^\alpha dv,\ \ n^\alpha u^\alpha = \int_{\mathbb{R}^3_v} v f^\alpha dv,\ \ p^\alpha = T^\alpha n^\alpha = \frac{2}{3}\int_{\mathbb{R}^3_v}\left(v-u^\alpha\right)^2 f^\alpha dv. \tag{1.4}$$

Moreover the macroscopic quantites associated to the mixture can be defined by

$$n = n^A + n^B,\ \ nu = n^A u^A + n^B u^B,\ \ p = p^A + \frac{1}{3}n^A(u^A-u)^2 + p^B + \frac{1}{3}n^B(u^B-u)^2. \tag{1.5}$$

A usefull quantity is the concentration X^α of specy α defined by

$$X^\alpha = \frac{n^\alpha}{n}. \tag{1.6}$$

The main result of this paper is the following existence theorem for the system (1.1, 1.2, 1.3) proved by perturbative arguments.

Theorem 1.1. *For p_{II}^B (resp T_{II}) close enough to p_I^B (resp T_I), and ε small enough, there is a solution (f^A, f^B) to the system (1.1, 1.2, 1.3) of the form*

$$(f^A, f^B) = (f_{H0}^A + \varepsilon f_1^A + \varepsilon^2 f_2^A + \varepsilon^3 R^A,\, f_{H0}^B + \varepsilon f_1^B + \varepsilon^2 f_2^B + \varepsilon^3 R^B)$$

satisfying

$$\|R^A\|_\infty + \|R^B\|_\infty \leq \frac{c}{\varepsilon^{\frac{5}{2}}}.$$

Remark 1. In the situation investigated in ([14]) the A component satisfies a given indata profile whereas the B component satisfies Maxwell diffuse boundary conditions

$$f^B(-1, v) = \frac{1}{\pi T_I^2} \exp(-\frac{v^2}{T_I}) \int_{\xi'<0} |\xi'| f^B(-1, v') dv', \quad \xi > 0,$$
$$f^B(1, v) = \frac{1}{\pi T_{II}^2} \exp(-\frac{v^2}{T_{II}}) \int_{\xi'>0} |\xi'| f^B(1, v') dv', \quad \xi < 0.$$

And in this case the proof of Theorem 1.1 still holds. But the proof given in ([14]) cannot be generalized in the situation of the present paper.

This paper is organized as follows. In section 2 an asymptotic expansion in the parameter ε is performed. The lower term of the expansion is shown to be a local bi-Maxwellian. The next orders have to be corrected by adding Knudsen layers constructed by from Milne problems for mixtures ([2]). Moreover the construction of the boundary layers fixes the boundary conditions of some fluid quantities. Some estimates are also required on the boundary Knudsen terms and are obtained by arguing as in ([10]). At the end of the section a fluid system is derived and solved when boundary conditions for f^A and f^B are close enough to each other (Theorem 2.2). Finally section 3 deals with the control the rest term (1.1). The rest term is shown to satisfy a non linear Boltzmann problem. The estimates are firstly researched on a linearized problem and are obtained thanks to a decomposition into a low and a high velocity part ([20, 21, 14]). But in ([20, 21, 14]) the boundary conditions are of Maxwell-diffuse reflection type which plays a crucial role. Therefore the approach has to be modified here because the boundary conditions are different. Finally we find a decomposition which is working either in the present situation or in the situation developped in ([14]).

2. **Asymptotic expansion.**

In this section we perform an asymptotic expansions in the parameter ε of the solution of the system (1.1, 1.2, 1.3). The terms of the Hilbert expansion have to be modified in order to satisfy the boundary conditions (1.2, 1.3). That is why each term f_n^α of the expansion of the distribution function associated to specy α writes

$$f_n^\alpha = f_{Hn}^\alpha + f_{Kn}^{\alpha-} + f_{Kn}^{\alpha+}, \;\; \alpha \in \{A; B\}. \tag{2.7}$$

In (2.7), f_{Hn}^α is a smooth function depending on x whereas $f_{Kn}^{\alpha-}$ (resp. $f_{Kn}^{\alpha+}$) is a smooth exponentially fast decaying function depending on the rescaled variable $\frac{1+x}{\varepsilon}$ (resp. $\frac{1-x}{\varepsilon}$). At the end of the section, a fluid system is derived and solved when the boundary conditions are close to each other (Theorem 2.2).

2.1. **Hilbert expansion.** The distribution functions f^A and f^B are expanded in Hilbert series as follows

$$\begin{aligned} f_H^A(x,v) &= f_{H0}^A(x,v) + \varepsilon f_{H1}^A(x,v) + \cdots, \\ f_H^B(x,v) &= f_{H0}^B(x,v) + \varepsilon f_{H1}^B(x,v) + \cdots. \end{aligned} \tag{2.8}$$

Substitute f_H^A and f_H^B by the expressions given in (2.8) in the equation (1.1) leads to

$$\begin{aligned} \xi \frac{\partial}{\partial x}(f_{H0}^A + \varepsilon f_{H1}^A + \cdots) &= \frac{1}{\varepsilon} Q(f_{H0}^A + \varepsilon f_{H1}^A + \cdots, f_{H0}^A + \varepsilon f_{H1}^A + \cdots) \\ &+ \frac{1}{\varepsilon} Q(f_{H0}^A + \varepsilon f_{H1}^A + \cdots, f_{H0}^B + \varepsilon f_{H1}^B + \cdots), \end{aligned} \tag{2.9}$$

$$\begin{aligned} \xi \frac{\partial}{\partial x}\left(f_{H0}^B + \varepsilon f_{H1}^B + \cdots\right) &= \frac{1}{\varepsilon} Q(f_{H0}^B + \varepsilon f_{H1}^B, \cdots f_{H0}^A + \varepsilon f_{H1}^A + \cdots) \\ &+ \frac{1}{\varepsilon} Q(f_{H0}^B + \varepsilon f_{H1}^B + \cdots, f_{H0}^B + \varepsilon f_{H1}^B + \cdots). \end{aligned} \tag{2.10}$$

A important Hilbert term is

$$f_H = f_H^A + f_H^B. \tag{2.11}$$

It corresponds to the sum of the two components and satisfies the relation

$$\xi \frac{\partial}{\partial x}(f_{H0} + \varepsilon f_{H1} + \cdots) = \frac{1}{\varepsilon} Q(f_{H0} + \varepsilon f_{H1} + \cdots, f_{H0} + \varepsilon f_{H1} + \cdots). \tag{2.12}$$

By using the Hilbert expansions (2.8) for f_H^A and f_H^B and by identifying formally the different orders of ε in (1.4, 1.5, 1.6), the following relations are obtained on the macroscopic quantities for $\alpha \in \{A; B\}$

$$\int_{\mathbb{R}_v^3} f_{Hm}^\alpha dv = n_{Hm}^\alpha \ (m = 0, 1 \cdots), \quad \int_{\mathbb{R}_v^3} \xi f_{H0}^\alpha dv = n_{H0}^\alpha u_{1,H0}^\alpha, \tag{2.13}$$

$$\int_{\mathbb{R}_v^3} v f_{H0}^\alpha dv = n_{H0}^\alpha u_{H0}^\alpha, \quad \int_{\mathbb{R}_v^3} \xi^2 f_{H0}^\alpha dv = \frac{1}{2}\left(n_{H0}^\alpha T_{H0}^\alpha\right), \tag{2.14}$$

$$X_{H0}^\alpha = \frac{n_{H0}^\alpha}{n_{H0}}, \quad \int_{\mathbb{R}_v^3} v^2 f_{H0}^\alpha dv = n_{H0}^\alpha (u_{1,H0}^\alpha)^2 + \frac{3}{2} p_{H0}^\alpha, \tag{2.15}$$

$$\int_{\mathbb{R}_v^3} \xi f_{H1}^\alpha dv = n_{H0}^\alpha u_{1,H1}^\alpha + n_{H1}^\alpha u_{1,H0}^\alpha, \quad \int_{\mathbb{R}_v^3} v f_{H1}^\alpha dv = n_{H0}^\alpha u_{1,H1}^\alpha + n_{H1}^\alpha u_{1,H0}^\alpha, \tag{2.16}$$

$$\int_{\mathbb{R}_v^3} v^2 f_{H1}^\alpha dv = \frac{3}{2}(n_{H0}^\alpha T_{H1}^\alpha + n_{H1}^\alpha T_{H0}^\alpha) + 2 n_{H0}^\alpha u_{1,H0}^\alpha u_{H1}^\alpha + 2 n_{H0}^\alpha (u_{1,H0}^\alpha)^2. \tag{2.17}$$

2.2. **Determination of the Hilbert terms of the expansion.**

2.2.1. *Expression of* f_{H0}^A *and* f_{H0}^B. The identification of the terms of order -1 in the equations (2.9) and (2.10) leads to

$$Q(f_{H0}^A, f_{H0}^A) + Q(f_{H0}^B, f_{H0}^A) = 0, \tag{2.18}$$

$$Q(f_{H0}^A, f_{H0}^B) + Q(f_{H0}^B, f_{H0}^B) = 0. \tag{2.19}$$

The system (2.18, 2.19) is solved by using the following lemma.

Lemma 2.1. *The solution to the system (2.18-2.19) is*

$$f^A_{H0}(x,v) = \frac{n^A_{H0}}{\pi^{\frac{3}{2}}(T_{H0})^{\frac{3}{2}}}\exp\left(-\frac{(\xi-u_{1,H0})^2+\eta^2+\chi^2}{T_{H0}}\right), \tag{2.20}$$

$$f^B_{H0}(x,v) = \frac{n^B_{H0}}{\pi^{\frac{3}{2}}(T_{H0})^{\frac{3}{2}}}\exp\left(-\frac{(\xi-u_{1,H0})^2+\eta^2+\chi^2}{T_{H0}}\right), \tag{2.21}$$

where $(n^A_{H0},\, n^B_{H0},\, T_{H0},\, u_{1,H0}) \in \mathbb{R}^{*3}_+ \times \mathbb{R}$.

The proof of Lemma 2.1 follows from ([2]).

2.2.2. *Expression of* f^A_{H1} *and* f^B_{H1}. Firstly by inverting the relation

$$\xi\frac{\partial}{\partial x}f_{H0} = Q(f_{H0}, f_{H1}) + Q(f_{H1}, f_{H0}),$$

it holds that f_{H1} writes

$$f_{H1} = \left(\frac{n_{H1}}{n_{H0}} + \frac{2u_{1,H1}}{T_{H0}}\xi + (\frac{v^2}{T_{H0}} - \frac{3}{2})\frac{T_{H1}}{T_{H0}} - \frac{\tilde{\xi}A(|\tilde{v}|)}{p_{H0}}\frac{\partial}{\partial x}T_{H0}\right)f_{H0},$$

where

$$E(v) = \frac{1}{\pi^{\frac{3}{2}}}\exp(-v^2).$$

$\xi A(|v|)$ is the solution to ([14, 19, 25])

$$\mathcal{L}_{T_{H0}}(\tilde{\xi}A(|\tilde{v}|)) = -\tilde{\xi}(\tilde{v}^2 - \frac{5}{2}), \qquad \int_0^{+\infty} r^4A(r)E(r)dr = 0,$$

where $\mathcal{L}_{T_{H0}}$ is the linearized Boltzmann operator for a one component gas defined by

$$\mathcal{L}_{T_{H0}}(\psi_{H1}(\tilde{v})) := \int_{\mathbb{R}^3_{\tilde{v}_*}\times\mathbb{S}^2} E(\tilde{v}_*)\Big(\psi_{H1}(x,v') + \psi_{H1}(x,v'_*) - \psi_{H1}(x,v) - \psi_{H1}(x,v_*)\Big)\frac{\mathcal{B}\left(|\tilde{v_*}-\tilde{v}|\sqrt{T_{H0}}, \langle\tilde{v_*}-\tilde{v},\omega\rangle\sqrt{T_{H0}}\right)}{\sqrt{T_{H0}}}d\omega d\tilde{v}_*.$$

More precisely

$$\left(\frac{n_{H1}}{n_{H0}} + \frac{2u_{1,H1}}{T_{H0}}\xi + (\frac{v^2}{T_{H0}} - \frac{3}{2})\frac{T_{H1}}{T_{H0}}\right)f_{H0}$$

is the hydrodynamical part of f_{H1} and corresponds to the projection of f_{H1} on the kernel of $\mathcal{L}_{T_{H0}}$. The term

$$-\frac{\tilde{\xi}A(|\tilde{v}|)}{p_{H0}}\frac{\partial}{\partial x}T_{H0}\,f_{H0}$$

is the non hydrodynamical part of f_{H1} and corresponds to the projection of f_{H1} on the orthogonal of $\ker\mathcal{L}_{T_{H0}}$.

Next (f^A_{H1}, f^B_{H1}) is determined from the identification of the 0 order terms in (2.9) and (2.10). So

$$\xi\frac{\partial}{\partial x}f^A_{H0} = Q(f^A_{H0}, f_{H1}) + Q(f^A_{H1}, f_{H0}), \tag{2.22}$$

$$\xi\frac{\partial}{\partial x}f^B_{H0} = Q(f^B_{H0}, f_{H1}) + Q(f^B_{H1}, f_{H0}). \tag{2.23}$$

Therefore (f^A_{H1}, f^B_{H1}) can be computed after the inversion of the relations (2.22, 2.23). From ([2]) the kernel of the linear mapping

$$\lambda : (\phi_A, \phi_B) \mapsto \big(Q(\phi f_{H0}, f^A_{H0}) + Q(f_{H0}, \phi_A f^A_{H0}), Q(\phi f_{H0}, f^B_{H0}) + Q(f_{H0}, \phi_B f^B_{H0})\big)$$

is $\ker\lambda = \big\{\big(\alpha^A + \beta\xi + \gamma v^2, \alpha^B + \beta\xi + \gamma v^2\big)\,, \quad (\alpha^A, \alpha^B, \beta, \gamma) \in \mathbb{R}^2_+ \times \mathbb{R}^2\big\}$.
(f^A_{H1}, f^B_{H1}) is split into its hydrodynamical part and its non hydrodynamical part as

$$f^A_{H1} = f^A_{H0}\left(\frac{p^A_{H1}}{p^A_{H0}} + 2\xi\frac{u_{1,H1}}{T_{H0}} + (\frac{v^2}{T_{H0}} - \frac{5}{2})\frac{T_{H1}}{T_{H0}} - \frac{\tilde{\xi}A(|\tilde{v}|)}{p_{H0}}\frac{\partial}{\partial x}T_{H0} - \frac{\tilde{\xi}C(\tilde{v})}{n_{H0}p^A_{H0}}\frac{\partial}{\partial x}p^A_{H0}\right), \tag{2.24}$$

$$f^B_{H1} = f^B_{H0}\left(\frac{p^B_{H1}}{p^B_{H0}} + 2\xi\frac{u_{1,H1}}{T_{H0}} + (\frac{v^2}{T_{H0}} - \frac{5}{2})\frac{T_{H1}}{T_{H0}} - \frac{\tilde{\xi}A(|\tilde{v}|)}{p_{H0}}\frac{\partial}{\partial x}T_{H0} - \frac{\tilde{\xi}C(\tilde{v})}{n_{H0}p^B_{H0}}\frac{\partial}{\partial x}p^B_{H0}\right), \tag{2.25}$$

where C is a solution to the equation ([14, 30, 32])

$$Q(E(\tilde{v}), E(\tilde{v})\tilde{\xi}C(\tilde{v})) = -\tilde{\xi}E(\tilde{v}).$$

As previously f_{H2}, f^A_{H2} and f^B_{H2} can be computed in the following form

$$f_{H2} = f_{H0}(c_0 + c_1\xi + c_4 v^2 + \psi_{H2}),\; f^\alpha_{H2} = f^\alpha_{H0}(c^\alpha_0 + c_1\xi + c_4 v^2 + \psi_{H2} + \varphi^\alpha), \quad \alpha \in \{A, B\},$$

where

$$c_0 = \frac{p_{H2}}{p_{H0}} - \frac{5}{2}\left(\frac{T_{H2}}{T_{H0}} + \frac{n_{H1}T_{H1}}{n_{H0}T_{H0}}\right) - \frac{u^2_{1,H1}}{T_{H0}}, \quad c_1 = 2\left(\frac{u_{1,H2}}{T_{H0}} + \frac{n_{H1}}{n_{H0}}\frac{u_{1,H1}}{T_{H0}}\right)$$

$$c_4 = \frac{1}{T_{H0}}\left(\frac{T_{H2}}{T_{H0}} + \frac{n_{H1}}{n_{H0}}\frac{T_{H1}}{T_{H0}} + \frac{2}{3}\frac{u^2_{1,H1}}{T_{H0}}\right), c^\alpha_0 = \frac{p^\alpha_{H2}}{p^\alpha_{H0}} - \frac{5}{2}\left(\frac{T^\alpha_{H2}}{T_{H0}} + \frac{n^\alpha_{H1}T^\alpha_{H1}}{n^A_{H0}T_{H0}}\right) - \frac{u^2_{1,H1}}{T_{H0}}.$$

For the computation of the functions ψ_{H2} and φ^α, we refer to ([11]).

2.3. **Study of the boundary conditions for the Hilbert terms.** In this subsection we show that f^A_{H0} and f^B_{H0} satisfy the boundary conditions (1.2, 1.3). But for the other Hilbert terms f^A_{H1}, f^B_{H1}, f^A_{H2}, f^B_{H2}, Knudsen layers must be added at each boundary and these layers are solutions to Milne problems for mixtures ([2]).

2.3.1. *Closure of the system at the* 0 *order.* (f^A_{H0}, f^B_{H0}) satisfies (1.2, 1.3) when the macroscopic quantities p^A_{H0}, p^B_{H0}, T_{H0} satisfy the boundary conditions

$$\begin{aligned} p^A_{H0}(-1) = 1, \quad p^B_{H0}(-1) = p^B_I, \quad T_{H0}(-1) = 1,\\ p^A_{H0}(1) = p^B_{II} + p^B_I - 1, \quad p^B_{H0}(1) = p^B_{II}, \quad T_{H0}(1) = T_{II}, \end{aligned} \tag{2.26}$$

and $u_{1,H0} = 0$.

2.3.2. *Knudsen layer at first and second orders.* As f^A_{H1} and f^B_{H1} defined in (2.24) and (2.25) cannot satisfy the boundary conditions

$$f^A_{H1}(-1, v) = 0,\; f^A_{H1}(1, v) = \frac{2}{\sqrt{\pi}}\frac{\Delta}{p^A_{H0}(1)}f^A_{H0}(1, v),\; f^B_{H1}(-1, v) = f^B_{H1}(1, v) = 0,$$

Knudsen terms must be added at each boundary.

By setting $x' = \frac{1+x}{\varepsilon}$, $x'' = \frac{1-x}{\varepsilon}$, the modified Hilbert terms f_1, f_1^A and f_1^B are written as follows

$$\begin{aligned} f_1(x,v) &= f_{H1}(x,v) + f_{K1}^-(x',v) + f_{K1}^+(x'',v), &(2.27)\\ f_1^A(x,v) &= f_{H1}^A(x,v) + f_{K1}^{A-}(x',v) + f_{K1}^{A+}(x'',v), &(2.28)\\ f_1^B(x,v) &= f_{H1}^B(x,v) + f_{K1}^{B-}(x',v) + f_{K1}^{B+}(x'',v). &(2.29)\end{aligned}$$

Then we aim to construct the boundary layers f_{K1}^{A-}, f_{K1}^{B-}, f_{K1}^{A+} and f_{K1}^{B+} in order to impose the boundary conditions

$$f_{H1}^A(-1,v) + f_{K1}^{A-}(0,v) = 0, \quad f_{H1}^B(-1,v) + f_{K1}^{B-}(0,v) = 0 \quad \text{for} \quad \xi > 0 \tag{2.30}$$

and

$$f_{H1}^A(1,v) + f_{K1}^{A+}(0,v) = \frac{2}{\sqrt{\pi}} \frac{\Delta}{p_{H0}^A(1)} f_{H0}^A(1,v), \;\; f_{H1}^B(1,v) + f_{K1}^{B+}(0,v) = 0, \text{ for } \xi < 0. \tag{2.31}$$

From here denote $\widetilde{M} = \frac{1}{n_{H0}^A} f_{H0}^A$ i.e

$$\tilde{M} = \frac{1}{(\pi T_{H0})^{\frac{3}{2}}} \exp(-\frac{v^2}{T_{H0}}), \quad M^A = n_{H0}^A \tilde{M} \quad \text{and} \quad M^B = n_{H0}^B \tilde{M}.$$

Consider as in ([2]), the space $\mathcal{H}$ with the scalar product

$$\begin{aligned} \langle f, g\rangle &= \langle (f^A, f^B) ; (g^A, g^B)\rangle \\ &= n_{H0}^A \int_{\mathbb{R}^3} f^A(v) g^A(v) \widetilde{M}(v) dv + n_{H0}^B \int_{\mathbb{R}^3} f^B(v) g^B(v) \widetilde{M}(v) dv \end{aligned}$$

and $\| \, \|_{\mathcal{H}}$ the associated Hilbert norm.

Proposition 1. *There are boundary conditions in $x = -1$ for the first order Hibert terms (f_{H1}^A, f_{H1}^B) defined by (2.24, 2.25) and Knudsen terms ($f_{K1}^{A-}(x',v)$, $f_{K1}^{B-}(x',v)$) solutions to*

$$\xi \frac{\partial}{\partial x'} f_{K1}^{A-}(x',v) = Q(M^A(-1,v), f_{K1}^-(x',v)) + Q(f_{K1}^{A-}(x',v), M(-1,v)), \tag{2.32}$$

$$\xi \frac{\partial}{\partial x'} f_{K1}^{B-}(x',v) = Q(M^B(-1,v) f_{K1}^-(x',v)) + Q(f_{K1}^{B-}(x',v), M(-1,v)), \tag{2.33}$$

where $M = M^A + M^B$ and $f_{K1}^- = f_{K1}^{A-} + f_{K1}^{B-}$.
Moreover the following asymptotic properties hold. f_{K1}^{A-} and f_{K1}^{B-} write as

$$f_{K1}^{A-}(x',v) = M^A(-1,v)\phi_1^{A-}(x',v), \quad f_{K1}^{B-}(x',v) = M^B(-1,v)\phi_1^{B-}(x',v),$$

where for x' tending to infinity ϕ_1^{A-} and ϕ_1^{B-} converge exponentially to 0 as

$$\|(1+|v|)^{\frac{1}{2}} \phi_1^{A-}(x',v)\|_{\mathcal{H}} \le \exp(-\sigma x'), \; \|(1+|v|)^{\frac{1}{2}} \phi_1^{B-}(x',v)\|_{\mathcal{H}} \le \exp(-\sigma x'), \tag{2.34}$$

a.e $x' > 0$ with $\sigma < 2\nu_1$ where ν_1 is defined in (3.82).

Moreover the construction of the Knudsen layers f_{K1}^{A-}, f_{K1}^{B-}, f_{K1}^{A+}, f_{K1}^{B+} define the boundary conditions for p_{H1}^A, p_{H1}^B and T_{H1}.

Proof. From [2] there are $\big(b_1^{A-},\, b_1^{B-}\big)$, $\big(g_1^{A-},\, g_1^{B-}\big)$ and $\big(d_1^{A-},\, d_1^{B-}\big)$ unique solutions to the Milne problems

$$\begin{aligned}
\xi\frac{\partial}{\partial x'}b_1^{A-}(x',v) &= \frac{1}{M^A(-1,v)}\big(Q(M^A(-1,v)M(-1,v)b_1^-(x',v))\\
&+ Q(M^A(-1,v)b_1^{A-}(x',v),M(-1,v))\big),\\
\xi\frac{\partial}{\partial x'}b_1^{B-}(x',v) &= \frac{1}{M^B(-1,v)}\big(Q(M^B(-1,v),M(-1,v)b_1^-(x',v))\\
&+ Q(M^B(-1,v)b_1^{B-}(x',v),M(-1,v))\big),
\end{aligned}$$

$$b_1^{A-}(0,v)=\frac{\xi}{\sqrt{T_{H0}(-1)}}A(|\tilde v|),\quad \xi>0,\quad b_1^{B}(0,v)=\frac{\xi}{\sqrt{T_{H0}(-1)}}A(|\tilde v|),\quad \xi>0,$$

$$\int_{\mathbb{R}^3}\xi M^A(-1,v)b_1^{A-}(x',v)dv=0,\quad \int_{\mathbb{R}^3}\xi M^B(-1,v)b_1^{B-}(x',v)dv=0,$$

$$\begin{aligned}
\xi\frac{\partial}{\partial x'}g_1^{A-}(x',v) &= \frac{1}{M^A(-1,v)}\big(Q(M^A(-1,v)M(-1,v)g_1^-(x',v))\\
&+ Q(M^A(-1,v)g_1^{A-}(x',v),M(-1,v))\big),\\
\xi\frac{\partial}{\partial x'}g_1^{B-}(x',v) &= \frac{1}{M^B(-1,v)}\big(Q(M^B(-1,v),M(-1,v)g_1^-(x',v))\\
&+ Q(M^B(-1,v)g_1^{B-}(x',v),M(-1,v))\big),
\end{aligned}$$

$$g_1^{A-}(0,v)=\frac{\frac{\xi}{\sqrt{T_{H0}(-1)}}C(|\tilde v|)}{X_{H0}^A(-1)},\quad \xi>0,\quad g_1^{B}(0,v)=-\frac{\frac{\xi}{\sqrt{T_{H0}(-1)}}C(|\tilde v|)}{X_{H0}^B(-1)},\quad \xi>0,$$

$$\int_{\mathbb{R}^3}\xi M^A(-1,v)g_1^{A-}(x',v)dv=0,\quad \int_{\mathbb{R}^3}\xi M^B(-1,v)g_1^{B-}(x',v)dv=0$$

and

$$\begin{aligned}
\xi\frac{\partial}{\partial x'}d_1^{A-}(x',v) &= \frac{1}{M^A(-1,v)}\big(Q(M^A(-1,v),M(-1,v)d_1^-(x',v))\\
&+ Q(M^A(-1,v)d_1^{A-}(x',v),M(-1,v))\big),\\
\xi\frac{\partial}{\partial x'}d_1^{B-}(x',v) &= \frac{1}{M^B(-1,v)}\big(Q(M^B(-1,v),M(-1,v)d_1^-(x',v))\\
&+ Q(M^B(-1,v)d_1^{B-}(x',v),M(-1,v))\big),
\end{aligned}$$

$$d_1^{A-}(0,v)=-2\frac{\xi}{\sqrt{T_{H0}(-1)}},\ \xi>0,\quad d_1^{B-}(0,v)=-2\frac{\xi}{\sqrt{T_{H0}(-1)}},\ \xi>0,$$

$$\int_{\mathbb{R}^3}\xi M^A(-1,v)d_1^{A-}(x',v)dv=0,\int_{\mathbb{R}^3}\xi M^B(-1,v)d_1^{B-}(x',v)dv=0,$$

with $b_1^-=b_1^{A-}+b_1^{B-}$, $g_1^-=g_1^{A-}+g_1^{B-}$ and $d_1^-=d_1^{A-}+d_1^{B-}$. Moreover

$$\begin{aligned}
&\lim_{x'\to+\infty}b_1^{A-}(x',v)=b_{1,\infty,0}^{A-}+b_{1,\infty,4}^{-}v^2, &&\lim_{x'\to+\infty}b_1^{B-}(x',v)=b_{1,\infty,0}^{B-}+b_{1,\infty,4}^{-}v^2,\\
&\lim_{x'\to+\infty}g_1^{A-}(x',v)=g_{1,\infty,0}^{A-}+g_{1,\infty,4}^{-}v^2, &&\lim_{x'\to+\infty}g_1^{B-}(x',v)=g_{1,\infty,0}^{B-}+g_{1,\infty,4}^{-}v^2,\\
&\lim_{x'\to+\infty}d_1^{A-}(x',v)=d_{1,\infty,0}^{A-}+d_{1,\infty,4}^{-}v^2, &&\lim_{x'\to+\infty}d_1^{B-}(x',v)=d_{1,\infty,0}^{B-}+d_{1,\infty,4}^{-}v^2,
\end{aligned}$$

where $b^{A-}_{1,\infty,0}$, $b^{B-}_{1,\infty,0}$, $b^{-}_{1,\infty,4}$, $g^{A-}_{1,\infty,0}$, $g^{B-}_{1,\infty,0}$, $g^{-}_{1,\infty,4}$, $d^{A-}_{1,\infty,0}$, $d^{B-}_{1,\infty,0}$ and $d^{-}_{1,\infty,4}$ are constants. Finally we define f^{A-}_{K1} as

$$\begin{aligned} f^{A-}_{K1}(x',v) &= \Big(\frac{u_{1,H1}(-1)}{\sqrt{T_{H0}(-1)}}(d^{A-}_{1}(x',v) - d^{A-}_{1,\infty,0} - d^{-}_{1,\infty,4}v^2) \\ &+ \frac{\partial_x T_{H0}(-1)}{p_{H0}(-1)}(b^{A-}_{1}(x',v) - b^{A-}_{1,\infty,0} - b^{-}_{1,\infty,4}v^2) \\ &+ \frac{\partial_x p^A_{H0}(-1)}{p_{H0}(-1)}(g^{A-}_{1}(x',v) - g^{A-}_{1,\infty,0} - g^{-}_{1,\infty,4}v^2)\Big) f^A_{H0}(-1,v). \end{aligned} \tag{2.35}$$

So from (2.24), (2.35) it comes that

$$\begin{aligned} f^{A-}_{K1}(0,v) + f^A_{H1}(-1,v) &= f^A_{H0}(-1,v)\Big(\frac{p^A_{H1}(-1)}{p^A_{H0}(-1)} + (\frac{v^2}{T_{H0}(-1)} - \frac{5}{2})\frac{T_{H1}(-1)}{T_{H0}(-1)} \\ &- \frac{u_{1,H1}(-1)}{\sqrt{T_{H0}(-1)}}(d^{A-}_{1,\infty,1} + d^{-}_{1,\infty,4}v^2) \\ &- \frac{\partial_x T_{H0}(-1)}{p_{H0}(-1)}(b^{A-}_{1,\infty,1} + b^{-}_{1,\infty,4}v^2) \\ &- \frac{\partial_x p^A_{H0}(-1)}{p_{H0}(-1)}(g^{A-}_{1,\infty,1} + g^{-}_{1,\infty,4}v^2)\Big). \end{aligned}$$

Therefore the boundary condition (2.30) is satisfied when $T_{H1}(-1)$ is defined by the relation

$$\frac{T_{H1}(-1)}{T^2_{H0}(-1)} = \frac{u_{1,H1}(-1)}{\sqrt{T_{H0}(-1)}}d^{-}_{1,\infty,4} + \frac{\partial_x T_{H0}(-1)}{p_{H0}(-1)}b^{-}_{1,\infty,4} + \frac{\partial_x p^A_{H0}(-1)}{p_{H0}(-1)}g^{-}_{1,\infty,4} \tag{2.36}$$

and the boundary condition $p^A_{H1}(-1)$ is defined as

$$\begin{aligned} \frac{p^A_{H1}(-1)}{p^A_{H0}(-1)} &= \frac{u_{1,H1}(-1)}{\sqrt{T_{H0}(-1)}}(d^{A-}_{1,\infty,0} + \frac{5}{2}T_{H0}(-1)d^{-}_{1,\infty,4}) \\ &+ \frac{\partial_x T_{H0}(-1)}{p_{H0}(-1)}(b^{A-}_{1,\infty,0} + \frac{5}{2}T_{H0}(-1)b^{-}_{1,\infty,4}) \\ &+ \frac{\partial_x p^A_{H0}(-1)}{p_{H0}(-1)}(g^{A-}_{1,\infty,0} + \frac{5}{2}T_{H0}(-1)g^{-}_{1,\infty,4}). \end{aligned}$$

So by using (2.36) $p^A_{H1}(-1)$ writes

$$\begin{aligned} p^A_{H1}(-1) &= p^A_{H0}(-1)\frac{u_{1,H1}(-1)}{\sqrt{T_{H0}(-1)}}(d^{A-}_{1,\infty,0} + \frac{5}{2}T_{H0}(-1)d^{-}_{1,\infty,4}) \\ &+ p^A_{H0}(-1)\frac{\partial_x T_{H0}(-1)}{p_{H0}(-1)}(b^{A-}_{1,\infty,0} + \frac{5}{2}T_{H0}(-1)b^{-}_{1,\infty,4}) \\ &- p^A_{H0}(-1)\frac{\partial_x p^A_{H0}(-1)}{p_{H0}(-1)}(g^{A-}_{1,\infty,0} + \frac{5}{2}T_{H0}(-1)g^{-}_{1,\infty,4}). \end{aligned}$$

Hence $p^A_{H1}(-1)$ can be rewritten in function of $X^A_{H0}(-1)$ as

$$\begin{aligned}p^A_{H1}(-1) &= X^A_{H0}(-1)p_{H0}(-1)\frac{u_{1,H1}(-1)}{\sqrt{T_{H0}(-1)}}(d^{A-}_{1,\infty,0} + \frac{5}{2}T_{H0}(-1)d^-_{1,\infty,4})\\ &+ X^A_{H0}(-1)\partial_x T_{H0}(-1)(b^{A-}_{1,\infty,0} + \frac{5}{2}T_{H0}(-1)b^-_{1,\infty,4})\\ &+ X^A_{H0}(-1)\partial_x p^A_{H0}(-1)(g^{A-}_{1,\infty,0} + \frac{5}{2}T_{H0}(-1)g^-_{1,\infty,4}).\end{aligned}$$

Next by setting for $\alpha \in \{A, B\}$,

$$a^{\alpha I}_V = d^{\alpha-}_{1,\infty,0} + \frac{5}{2}T_{H0}(-1)d^-_{1,\infty,4}, \quad a^{\alpha I}_T = b^{\alpha-}_{1,\infty,0} + \frac{5}{2}T_{H0}(-1)b^-_{1,\infty,4},$$
$$a^{\alpha I}_X = g^{\alpha-}_{1,\infty,0} + \frac{5}{2}T_{H0}(-1)g^-_{1,\infty,4},$$

we get the boundary condition for p^A_{H1}

$$\begin{aligned}p^A_{H1}(-1) &= X^A_{H0}(-1)p_{H0}(-1)\frac{u_{1,H1}(-1)}{\sqrt{T_{H0}(-1)}}a^{AI}_V\\ &+ X^A_{H0}(-1)\partial_x T_{H0}(-1)a^{AI}_T + X^A_{H0}(-1)\partial_x p^A_{H0}(-1)a^{AI}_X.\end{aligned}$$

In the same way we define f^{B-}_{K1} as (2.35) and we find the boundary condition for p^B_{H1},

$$\begin{aligned}p^B_{H1}(-1) &= X^B_{H0}(-1)p_{H0}(-1)\frac{u_{1,H1}(-1)}{\sqrt{T_{H0}(-1)}}a^{BI}_V\\ &+ X^B_{H0}(-1)\partial_x T_{H0}(-1)a^{BI}_T - X^B_{H0}(-1)\partial_x p^A_{H0}(-1)a^{BI}_X.\end{aligned}$$

Finally the boundary condition for p_{H1} at $x = -1$ writes

$$p_{H1}(-1) = p_{H0}(-1)\frac{u_{1,H1}(-1)}{\sqrt{T_{H0}(-1)}}a^I_V + \partial_x T_{H0}(-1)a^I_T + \partial_x p^A_{H0}(-1)a^I_X, \qquad (2.37)$$

with

$$a^I_V = X^A_{H0}(-1)a^{AI}_V + X^B_{H0}(-1)a^{BI}_V, \; a^I_T = X^A_{H0}(-1)a^{AI}_T + X^B_{H0}(-1)a^{BI}_T,$$
$$a^I_X = X^A_{H0}(-1)a^{AI}_X - X^B_{H0}(-1)a^{BI}_X.$$

In order to satisfy the boundary conditions at $x = 1$ we proceed as for $x = -1$. In that case the Knudsen terms are defined as

$$\begin{aligned}f^{\alpha+}_{K1}(x'', v) &= \Big(\frac{u_{1,H1}(1)}{\sqrt{T_{H0}(1)}}(d^{\alpha+}_1(x'', v) - d^{\alpha+}_{1,\infty,0} - d^+_{1,\infty,4}v^2)\\ &+ \frac{\partial_x T_{H0}(1)}{p_{H0}(1)}(b^{\alpha+}_1(x'', v) - b^{\alpha+}_{1,\infty,0} - b^+_{1,\infty,4}v^2)\\ &+ \frac{\partial_x p^\alpha_{H0}(1)}{p_{H0}(1)}(g^{\alpha+}_1(x'', v) - g^{\alpha+}_{1,\infty,0} - g^+_{1,\infty,4}v^2)\Big) f^\alpha_{H0}(1, v), \quad \alpha \in \{A, B\},\end{aligned}$$

where $d^{\alpha+}_1$, $b^{\alpha+}_1$ and $g^{\alpha+}_1$ are solutions to Milne problems and the constants $d^{\alpha+}_{1,\infty,0}$, $d^+_{1,\infty,4}$, $b^{\alpha+}_{1,\infty,0}$, $b^+_{1,\infty,4}$, $g^{\alpha+}_{1,\infty,0}$ and $g^+_{1,\infty,4}$ are defined as previously. Therefore $p^A_{H1}(1)$

and $p^B_{H1}(1)$ are given by

$$\begin{aligned} p^A_{H1}(1) &= p^A_{H0}(1)\frac{u_{1,H1}(1)}{\sqrt{T_{H0}(1)}}a_V^{AII}X^A_{H0}(1) + \partial_x T_{H0}(1)a_T^{AII}X^A_{H0}(1) \\ &+ \partial_x p^A_{H0}(1)a_X^{AII}X^A_{H0}(1) + \frac{2}{\sqrt{\pi}}\Delta, \end{aligned}$$

and

$$p^B_{H1}(1) = p^B_{H0}(1)\frac{u_{1,H1}(1)}{\sqrt{T_{H0}(1)}}a_V^{BII}X^B_{H0}(1) + \partial_x T_{H0}(1)a_T^{BII}X^B_{H0}(1) - \partial_x p^A_{H0}(1)a_X^{BII}X^B_{H0}(1)$$

with

$$\begin{aligned} a_V^{\alpha II} = d^{\alpha+}_{1,\infty,0} + \frac{5}{2}T_{H0}(1)d^{+}_{1,\infty,4}, \quad a_T^{\alpha II} &= b^{\alpha+}_{1,\infty,0} + \frac{5}{2}T_{H0}(1)b^{+}_{1,\infty,4}, \\ a_T^{\alpha II} &= g^{\alpha+}_{1,\infty,0} + \frac{5}{2}T_{H0}(1)g^{+}_{1,\infty,4}. \end{aligned}$$

So adding the two previous equations gives $p_{H1}(1)$ as

$$p_{H1}(1) = p_{H0}(1)\frac{u_{1,H1}(1)}{\sqrt{T_{H0}(1)}}a_V^{II} + \partial_x T_{H0}(1)a_T^{II} + \partial_x p^A_{H0}(1)a_X^{II} + \frac{2}{\sqrt{\pi}}\Delta, \qquad (2.38)$$

with

$$\begin{aligned} a_V^{II} = X^A_{H0}(1)a_V^{AII} + X^B_{H0}(1)a_V^{BII},\ a_T^{II} &= X^A_{H0}(1)a_T^{AII} + X^B_{H0}(1)a_T^{BII}, \\ a_X^{II} &= X^A_{H0}(1)a_X^{AII} - X^B_{H0}(1)a_X^{BII}. \end{aligned}$$

□

Like previously f^A_{H2} and f^B_{H2} can be defined by identification of the first order terms in ε. f^A_{H2} and f^B_{H2} are computed in function of $(n^A_{H1}, n^B_{H1}, T^A_{H1}, T^B_{H1}, u^A_{1,H1}, u^B_{1,H1})$ which are solutions to a fluid system that can be solved by arguing as in Theorem 2.2. As for the first order, Knudsen terms f^{A-}_{K2}, f^{B-}_{K2}, f^{A+}_{K2}, f^{B+}_{K2} must be added to the Hilbert terms f^A_{H2} and f^B_{H2} in order to satisfy the boundary conditions $f^A_2(-1,v) = f^A_2(1,v) = f^B_2(-1,v) = f^B_2(1,v) = 0$. These Knudsen layers are also constructed by solving Milne problems for mixtures. In the following, we will use the notations

$$\begin{aligned} &\gamma^{A-}_{1,\varepsilon} = f^{A-}_{K1}(\frac{2}{\varepsilon},v),\ \gamma^{A+}_{1,\varepsilon} = f^{A+}_{K1}(\frac{2}{\varepsilon},v),\ \gamma^{B-}_{1,\varepsilon} = f^{B-}_{K1}(\frac{2}{\varepsilon},v), \\ &\gamma^{B+}_{1,\varepsilon} = f^{B+}_{K1}(\frac{2}{\varepsilon},v),\ \gamma^{-}_{1,\varepsilon} = \gamma^{A-}_{2,\varepsilon} + \gamma^{B-}_{2,\varepsilon},\ \gamma^{+}_{1,\varepsilon} = \gamma^{A+}_{1,\varepsilon} + \gamma^{B+}_{1,\varepsilon}, \\ &\gamma^{A-}_{2,\varepsilon} = f^{A-}_{K2}(\frac{2}{\varepsilon},v),\ \gamma^{A+}_{2,\varepsilon} = f^{A+}_{K2}(\frac{2}{\varepsilon},v),\ \gamma^{B-}_{2,\varepsilon} = f^{B-}_{K2}(\frac{2}{\varepsilon},v), \\ &\gamma^{B+}_{2,\varepsilon} = f^{B+}_{K2}(\frac{2}{\varepsilon},v),\ \gamma^{-}_{2,\varepsilon} = \gamma^{A-}_{2,\varepsilon} + \gamma^{B-}_{2,\varepsilon},\ \gamma^{+}_{2,\varepsilon} = \gamma^{A+}_{2,\varepsilon} + \gamma^{B+}_{2,\varepsilon}. \end{aligned} \qquad (2.39)$$

2.4. **First order fluid equations.** In this subsection we consider a fluid system mixing 0 order and first order terms which is derived from the kinetic system (1.1, 1.2, 1.3) ([29]). As in ([20], [21]), this system is solved for well prepared boundary conditions closed enough to each other (Theorem 2.2). This assumption is crucial for obtaining estimates on Knudsen terms given in Lemma 3.1.

Theorem 2.2. *The macroscopic quantities $u^A_{1,H1}$, $u^B_{1,H1}$, p^A_{H0}, p^B_{H0}, T_{H0} and p_{H1} satisfy the following fluid system*

$$\frac{\partial}{\partial x} p_{H0} = 0, \tag{2.40}$$

$$\frac{\partial}{\partial x}(n^A_{H0} u^A_{1,H1}) = 0, \tag{2.41}$$

$$\frac{\partial}{\partial x}(n^B_{H0} u^B_{1,H1}) = 0, \tag{2.42}$$

$$\frac{\gamma_2}{2}\frac{\partial}{\partial x}\Big(\frac{\partial}{\partial x}(T_{H0})T_{H0}^{\frac{1}{2}}\Big) = n_{H0} u_{1,H1}\frac{\partial}{\partial x}T_{H0}, \tag{2.43}$$

$$u^A_{1,H1} - u^B_{1,H1} = -\gamma_c \frac{T_{H0}^{\frac{1}{2}}}{p_{H0} n^A_{H0} n^B_{H0}}\frac{\partial}{\partial x} p^A_{H0}, \tag{2.44}$$

$$\frac{\partial}{\partial x} p_{H1} = 0, \tag{2.45}$$

where $p^A_{H0} = n^A_{H0} T_{H0}$ and $p^B_{H0} = n^B_{H0} T_{H0}$.
Moreover this system can be solved as follows. There are $\tau_0 > 0$ and $\lambda > 0$ such that for all $\tau \in \mathbb{R}$ satisfying $|\tau| \leq \tau_0$, there are T_{II}, p^B_{II} and Δ such that

$$|1 - T_{II}| \leq \tilde{\lambda}\tau, \quad |p^B_I - p^B_{II}| \leq \tilde{\lambda}\tau, \quad |\Delta| \leq \tilde{\lambda}\tau$$

and such that the system (2.40-2.45) has a unique solution T_{H0}, p^A_{H0}, p^B_{H0}, p_{H1}, $u^A_{1,H1}$, $u^B_{1,H1}$ satisfying the boundary conditions (2.26) and (2.37, 2.38).
Moreover there is $\lambda > 0$, such that (for all $x \in [-1, 1]$)

$$|p^A_{H0}(x) - 1| \leq \lambda\tau,\ |p^B_{H0}(x) - p^B_I| \leq \lambda\tau,\ |T_{H0}(x) - 1| \leq \lambda\tau,\ |u_{1,H1}| \leq \lambda\tau,$$
$$|(p^A_{H0})'(x)| \leq \lambda\tau, \quad |(p^B_{H0})'(x)| \leq \lambda\tau, \quad |(T_{H0})'(x)| \leq \lambda\tau. \tag{2.46}$$

Proof. (Theorem 2.2) The derivation of such a system is performed in ([29]). Next we focus on its closure. According to (2.41, 2.42) there are two constants θ^A and θ^B such that $\theta^A = n^A_{H0} u^A_{1,H1}$ and $\theta^B = n^B_{H0} u^B_{1,H1}$. Next we determine θ defined by $\theta = \theta^A + \theta^B = n_{H0} u_{1,H1}$. By using that $p_{H1}(-1) = p_{H1}(1)$ together with (2.37, 2.38), it holds that θ is given by

$$\theta = \frac{\partial_x T_{H0}(1) a^{II}_T - \partial_x T_{H0}(-1) a^I_T - \partial_x p^A_{H0}(1) a^{II}_X + \partial_x p^A_{H0}(-1) a^I_X + \frac{2}{\sqrt{\pi}}\Delta}{\sqrt{T_{H0}(-1)} a^I_V - \sqrt{T_{H0}(1)} a^{II}_V}.$$

According to the previous relation it is equivalent to find θ from Δ instead of the contrary. Therfore from a given θ, such that $|\theta| \leq \tau$, we define Δ by the prevous relation. Next in order to determine T_{H0}, we consider (2.43). By denoting $c = \partial_x T_{H0}(-1)$, T_{H0} is the solution of the Cauchy problem

$$\frac{\frac{\partial}{\partial x} T_{H0}}{\frac{2\theta}{\gamma_2}(T_{H0} - 1) + c} = \frac{1}{\sqrt{T_{H0}}}, \tag{2.47}$$

$$T_{H0}(-1) = 1, \tag{2.48}$$

$$\frac{\partial}{\partial x} T_{H0}(-1) = c. \tag{2.49}$$

In order to satisfy the inequalities (2.46) for T_{H0} an estimate is researched on c. By solving the Cauchy problem (2.47, 2.48, 2.49), it comes that

$$|T_{H0} - 1| \leq \frac{|c|\,\gamma_2}{2|\theta|}\left(\exp\Big(\frac{2\theta}{\gamma_2}\int_{-1}^{x} \frac{1}{\sqrt{T_{H0}}} ds\Big) + 1\right).$$

Next in order to get $|T_{H0}-1|\leq\tau$, it is enough to take $|c|\leq\frac{2|\theta|}{\gamma_2}\tau$ which implies

$$\frac{|c|\,\gamma_2}{2|\theta|}\left(\exp(\frac{2\theta}{\gamma_2}\int_{-1}^{x}\frac{1}{\sqrt{T_{H0}}}ds)+1\right)\leq\tau.$$

Moreover T_{II} defined by $T_{II}=T_{H0}(1)$ satisfies $|T_{II}-1|\leq\tilde{\lambda}\tau$, where $\tilde{\lambda}$ is a nonnegative constant. In order to estimate $\partial_x T_{H0}$, we use (2.47) and we chose again $|c|$ small enough. So $|\partial_x T_{H0}|\leq\tau$. Moreover from (2.40), n_{H0} writes

$$n_{H0}=\frac{\alpha}{T_{H0}},$$

where α is a free parameter. Hence the boundary condition $p_{H0}(-1)=1+p_I^B$ gives $\alpha=1+p_I^B$. In order to determine p_{H0}^B we look for an equation satisfied by the concentration X_{H0}^B. (2.44) can be rewritten

$$n_{H0}^B\theta_1^A-n_{H0}^A\theta_1^B=-\gamma_c T_{H0}^{\frac{1}{2}}\frac{\partial}{\partial x}X_{H0}^A.$$

Hence by multiplying the previous equation by T_{H0} and by deriving we get

$$\theta\frac{\partial}{\partial x}p_{H0}^B=-\gamma_c\frac{\partial}{\partial_x}\left(T_{H0}^{\frac{3}{2}}\frac{\partial}{\partial x}X_{H0}^B\right).$$

Then dividing by p_{H0}, it holds that X_{H0}^B satisfies

$$\theta\frac{\partial}{\partial x}X_{H0}^B=-\gamma_c\frac{\partial}{\partial x}(\frac{T_{H0}^{\frac{1}{2}}}{n_{H0}}\frac{\partial}{\partial x}X_{H0}^B). \tag{2.50}$$

To find X_{H0}^B, we proceed like for the resolution of (2.47, 2.48, 2.49). By setting $\phi=\frac{\partial}{\partial x}X_{H0}^B$ and $d=\frac{\partial}{\partial x}X_{H0}^B(-1)$, ϕ is solution to the Cauchy problem

$$\begin{aligned}(\theta+\gamma_c\frac{\partial}{\partial x}(\frac{T_{H0}^{\frac{1}{2}}}{n_{H0}}))\phi+\gamma_c\frac{T_{H0}^{\frac{1}{2}}}{n_{H0}}\frac{\partial}{\partial x}\phi &= 0,\\ \phi(-1) &= d.\end{aligned}$$

ϕ writes

$$\phi(x)=d\,\exp\Big(-\int_{-1}^{x}\frac{n_{H0}}{\gamma_c T_{H0}^{\frac{1}{2}}}(\theta+\gamma_c\frac{\partial}{\partial x}(\frac{T_{H0}^{\frac{1}{2}}}{n_{H0}}))\Big).$$

Hence by chosing d such that

$$|d|\leq\tau\exp\Big(\int_{-1}^{1}\frac{n_{H0}}{\gamma_c T_{H0}^{\frac{1}{2}}}(|\theta|+\gamma_c|\frac{\partial}{\partial x}(\frac{T_{H0}^{\frac{1}{2}}}{n_{H0}})|)\Big),$$

ϕ satisfies the estimate $|\phi|\leq\tau$. Finally X_{H0}^B is defined by

$$X_{H0}^B=\frac{p_I^B}{1+p_I^B}+\int_{-1}^{x}\phi(s)ds.$$

This determines p_{H0}^B and $p_{II}^B=p_{H0}^B(1)$ satisfying the estimates

$$|p_{H0}^B-p_I^B|\leq(1+p_I^B)\tau,\qquad|p_{II}^B-p_I^B|\leq\tilde{\lambda}\tau,\qquad|(p_{H0}^B)'|\leq\tilde{\lambda}\tau,$$

$\tilde{\lambda}$ being a nonnegative constant independant of τ. Finally
$p_{H0}^A=(1+p_I^B)-p_{H0}^B$ satisfies $p_{H0}^A(1)=(1+p_I^B)-p_{II}^B$ and the estimate

$$|\partial_x(p_{H0}^A)|\leq\tilde{\lambda}\tau.$$

□

3. **Study of the rest term.** This section devoted to the control of the rest term when p^A_{II} (resp.p^B_{II}, resp. T_{II}) is close to p^A_I (resp.p^B_I, resp.T_I) and ε is sufficiently small (Theorem 1.1). We first show that the rest term of the Hilbert expansion is the solution of a non linear Boltzmann system. Next the idea is to consider a linearization of such a problem and to estimate the solution of this linearized problem. Following the ideas of [16] this solution is decomposed into a low and a high velocity part, solutions to a system of equations. But the decomposition introduced in [20, 21] and generalized in ([14]) for mixtures has to be modified here. Indeed in [20, 21] one crucial point is that one of the distribution function satisfies Maxwell diffuse boundary conditions. So the flux of the solution is equal to zero. But this property is not true in the present situation of given indata profiles and the decomposition proposed in [14, 20, 21] has to be modified.

3.1. **The rest term.** In ([16]) (resp.[20, 21]), the authors solve the time dependant (resp. stationary) Boltzmann equation by splitting the distribution function into an asymptotic expansion and a rest term and by controlling the rest term. In [14], the proof developped in [20, 21] is adapted to the situation of a two component gas when one component satisfies Maxwell-diffuse boundary conditions. But here due to the two given indata profiles the decomposition has to be modified. As a result we obtain a decomposition which allows the control of the rest term in the present situation and in the situation of [14].

The rest term $\varepsilon^3 f^A_R$ (resp. $\varepsilon^3 f^B_R$) for f^A (resp. f^B) is defined as the difference of f^A (resp. f^B) and its asymptotic expansion as

$$\begin{aligned} f^A(x,v) &= M^A + \varepsilon\big(f^A_{H1}(x,v) + f^{A-}_{K1}(\frac{1+x}{\varepsilon},v) + f^{A+}_{K1}(\frac{1-x}{\varepsilon},v)\big) \\ &+ \varepsilon^2\big(f^A_{H2}(x,v) + f^{A-}_{K2}(\frac{1+x}{\varepsilon},v) + f^{A+}_{K2}(\frac{1-x}{\varepsilon},v)\big) + \varepsilon^3 R^A(x,v), \quad (3.51) \\ f^B(x,v) &= M^B + \varepsilon\left(f^B_{H1}(x,v) + f^{B-}_{K1}(\frac{1+x}{\varepsilon},v) + f^{B+}_{K1}(\frac{1-x}{\varepsilon},v)\right) \\ &+ \varepsilon^2\big(f^B_{H2}(x,v) + f^{B-}_{K2}(\frac{1+x}{\varepsilon},v) + f^{B+}_{K2}(\frac{1-x}{\varepsilon},v)\big) + \varepsilon^3 R^B(x,v). \quad (3.52) \end{aligned}$$

By plugging the expressions (3.51, 3.52) into (1.1) and by taking (2.22, 2.23) into account, (R^A, R^B) has to satisfy the system

$$\begin{aligned} \xi\frac{\partial}{\partial x}R^A &= \frac{1}{\varepsilon}\Big(Q(M^A,R) + Q(R^A,M)\Big) + Q(f^A_1 + \varepsilon f^A_2, R) + Q(R^A, f_1 + \varepsilon f_2) \\ &+ \varepsilon^2 Q(R^A,R) + \varepsilon^3 A, \quad (3.53) \\ \xi\frac{\partial}{\partial x}R^B &= \frac{1}{\varepsilon}\Big(Q(M^B,R) + Q(R^B,M)\Big) + Q(f^B_1 + \varepsilon f^B_2, R) + Q(R^B, f_1 + \varepsilon f_2) \\ &+ \varepsilon^2 Q(R^B,R) + \varepsilon^3 B, \quad (3.54) \end{aligned}$$

with $R = R^A + R^B$ and

$$\begin{aligned} A &= \frac{1}{\varepsilon}\bigg(-\xi\frac{\partial}{\partial x}f^A_{H2} + Q(f^A_1, f_2) + Q(f^A_2, f_1) + \varepsilon Q(f^A_2, f_2) \\ &+ Q(f^{A-}_{K2}(x',v), \Delta^+ M) + Q(\Delta^+ M^A, f^-_{K2}(x',v)) \\ &+ Q(\Delta^- M, f^{A+}_{K2}(x'',v)) + Q(\Delta^- M^A, f^+_{K2}(x'',v)) \\ &+ \frac{1}{\varepsilon}\Big(Q(f^{A+}_{K1}(x'',v), f^-_{K1}(x',v)) + Q(f^{A-}_{K1}(x',v), f^+_{K1}(x'',v))\Big)\bigg), \quad (3.55) \end{aligned}$$

$$\begin{aligned}
B &= \frac{1}{\varepsilon}\Big(-\xi\frac{\partial}{\partial x} f_{H2}^B + Q(f_1^B, f_2) + Q(f_2^B, f_1) + \varepsilon Q(f_2^B, f_2) \\
&+ \frac{1}{\varepsilon}\Big(Q(f_{K2}^{B-}(x',v), \Delta^+ M) + Q(\Delta^+ M^B, f_{K2}^-(x',v)) \\
&+ Q(f_{K2}^{B+}(x'',v), \Delta^- M) + Q(\Delta^- M^B, f_{K2}^+(x'',v))\Big)\Big) \\
&+ \frac{1}{\varepsilon}\Big(Q(f_{K1}^{B+}(x'',v), f_{K1}^-(x',v)) + Q(f_{K1}^{B-}(x',v), f_{K1}^+(x'',v))\Big)\Big), \quad (3.56)
\end{aligned}$$

with

$$\Delta^- M = \frac{M - M(-1,v)}{\varepsilon},\ \Delta^- M^A = \frac{M^A - M^A(-1,v)}{\varepsilon},$$
$$\Delta^- M^B = \frac{M^B - M^B(-1,v)}{\varepsilon},\ \Delta^+ M^B = \frac{M^B - M^B(1,v)}{\varepsilon},$$
$$\Delta^+ M = \frac{M - M(1,v)}{\varepsilon},\ \Delta^+ M^A = \frac{M^A - M^A(1,v)}{\varepsilon}.$$

Recall that the quantities f_1, f_1^A, f_1^B, f_2, f_2^A, f_2^B are defined by (2.27, 2.28, 2.29). On the other hand R^A and R^B satisfy the following boundary conditions

$$R^A(-1,v) = -\frac{\gamma_{1,\varepsilon}^{A,-} + \varepsilon\gamma_{2,\varepsilon}^{A,-}}{\varepsilon^2} = \zeta^{A-},\ \xi > 0, \quad R^A(1,v) = -\frac{\gamma_{1,\varepsilon}^{A,+} + \varepsilon\gamma_{2,\varepsilon}^{A,+}}{\varepsilon^2} = \zeta^{A+},\ \xi < 0, \tag{3.57}$$

$$R^B(-1,v) = -\frac{\gamma_{1,\varepsilon}^{B,-} + \varepsilon\gamma_{2,\varepsilon}^{B,-}}{\varepsilon^2} = \zeta^{B-},\ \xi > 0, \quad R^B(1,v) = -\frac{\gamma_{1,\varepsilon}^{B,+} + \varepsilon\gamma_{2,\varepsilon}^{B,+}}{\varepsilon^2} = \zeta^{B+},\ \xi < 0, \tag{3.58}$$

where the terms $\gamma_{1,\varepsilon}^-$, $\gamma_{1,\varepsilon}^+$, $\gamma_{1,\varepsilon}^{A,-}$, $\gamma_{1,\varepsilon}^{A,+}$, $\gamma_{1,\varepsilon}^{B,-}$, $\gamma_{1,\varepsilon}^{B,+}$, $\gamma_{2,\varepsilon}^-$, $\gamma_{2,\varepsilon}^+$, $\gamma_{2,\varepsilon}^{A,-}$, $\gamma_{2,\varepsilon}^{A,+}$, $\gamma_{2,\varepsilon}^{B,-}$, $\gamma_{2,\varepsilon}^{B,+}$ are defined by (2.39).
Moreover remark that according to (2.34) we have the estimate on the boundary terms ζ^{A-}, ζ^{A+}, ζ^{B-} and ζ^{B+}

$$\|\zeta^{A-}\| + \|\zeta^{A+}\| + \|\zeta^{B-}\| + \|\zeta^{B+}\| \le \tilde{c}\,\exp(\frac{c'}{\varepsilon}), \tag{3.59}$$

for $\tilde{c} > 0$.

3.2. **A linearized problem for the rest term.** The solutions (R^A, R^B) to the system (3.53, 3.54) are constructed as the respective limits to a sequence of iterations. The generic term of the iteration can be defined as a linear equation of the type

$$\begin{aligned}
\xi\frac{\partial}{\partial x}R^A &= \frac{1}{\varepsilon}\Big(Q(M^A,R) + Q(R^A,M)\Big) + \Big(Q(f_1^A + \varepsilon f_2^A, R) + Q(R^A, f_1 + \varepsilon f_2)\Big) \\
&+ \varepsilon^2 D^A, \qquad (3.60) \\
\xi\frac{\partial}{\partial x}R^B &= \frac{1}{\varepsilon}\left(Q(M^B,R) + Q(R^B,M)\right) + \Big(Q(f_1^A + \varepsilon f_2^A, R) + Q(R^A, f_1 + \varepsilon f_2)\Big) \\
&+ \varepsilon^2 D^B, \qquad (3.61)
\end{aligned}$$

satisfying the boundary conditions (3.57, 3.58). More precisely at the step k of the iteration, the term (D^A, D^B) is replaced by

$$(Q(R_{k-1}^A, R_{k-1}) + \varepsilon A, Q(R_{k-1}^B, R_{k-1}) + \varepsilon B).$$

In the following, the terms R, R^A and R^B will be estimated in terms of D, D^A, D^B and of the boundary conditions (3.57, 3.58).

3.3. **Decomposition of the rest term.** The natural way to deal with the linearized Boltzmann equation is to change the operator $f \mapsto Q(M, f)$ into the operator $f \mapsto -\frac{2}{M}Q(M, M^{-\frac{1}{2}}f)$. But when the Maxwellian is not homogeneous, this procedure produces the term $\xi M^{-\frac{1}{2}}\xi\frac{\partial}{\partial x}(M^{\frac{1}{2}}f)$ which behaves like $|v|^3 f$ and has no sign. So as in [14, 16, 20, 21], R, R^A and R^B are decomposed into a low and a high velocity part as follows

$$R = \sqrt{M}g + \sqrt{M_*}h, \quad R^A = \sqrt{M^A}g^A + \sqrt{M_*}h^A, \quad R^B = \sqrt{M^B}g^B + \sqrt{M_*}h^B, \tag{3.62}$$

where M_* is the global Maxwellian $M_*(v) = \frac{1}{(\pi T_*)^{\frac{3}{2}}}\exp(-\frac{v^2}{T_*})$, with $T_* > \sup_{x\in[-1,1]} T_{H0}(x)$. Hence there is $c > 0$ such that for all $(x, v) \in [-1, 1] \times \mathbb{R}^3$, $M_* \geq cM$, $M_* \geq cM^A$, $M_* \geq cM^B$. Since $R = R^A + R^B$,

$$g = \frac{\sqrt{n^A}}{\sqrt{n}}g^A + \frac{\sqrt{n^B}}{\sqrt{n}}g^B, \quad h = h^A + h^B. \tag{3.63}$$

Remark that in ([8, 9, 6, 5]) this decomposition is not useful because the equilibrium state is a global Maxwellian distribution.

In order to control g^A, g^B, h^A and h^B, the following L^2 norm is considered

$$\|f\| = \left(\int_{[-1,1]\times\mathbb{R}^3} (1 + |v|)f^2(x, v)dxdv\right)^{\frac{1}{2}} \tag{3.64}$$

and is extended to the boundary terms h^A_-, h^A_+, h^A_- and h^A_+ depending only on the v variable. As basis for the kernel of the linearized Boltzmann operator, we take $\psi_0 = \sqrt{M}$, $\psi_1 = \xi\sqrt{M}$ and $\psi_4 = (v^2 - \frac{3}{2}T)\sqrt{M}$. g is next decomposed into its hydrodynamical part Pg and non hydrodynamical part $\overline{g}$. Hence Pg writes

$$Pg = p_0(x)\psi_0 + p_1(x)\psi_1 + p_4(x)\psi_4. \tag{3.65}$$

For $\alpha \in \{A, B\}$ define

$$\psi_0^\alpha = \sqrt{M^\alpha}, \quad \psi_1^\alpha = \xi\sqrt{M^\alpha} \quad \text{and} \quad \psi_4^\alpha = (v^2 - \frac{3}{2}T)\sqrt{M^\alpha}.$$

Then (g^A, g^B) is split into its hydrodynamical part $(P^A g^A, P^B g^B)$ and its non hydrodynamical part $(\overline{g}^A, \overline{g}^B)$. $P^A g^A$ and $P^B g^B$ are decomposed into

$$P^A g^A = g_0^A + g_1^A + g_4^A, \quad P^B g^B = g_0^B + g_1^B + g_4^B, \tag{3.66}$$

with

$$g_i^\alpha(x, v) = p_i^\alpha(x)\psi_i^\alpha(v), \quad i \in \{0, 1, 4\}, \quad \alpha \in \{A, B\}.$$

Remark that according to the expression of the kernel of the linearized Boltzmann operator, we have $p_1^A = p_1^B$ and $p_4^A = p_4^B$. From now we set $p_1 = p_1^A = p_1^B$ and $p_4 = p_4^A = p_4^B$.

Introduce the quantities

$$\mu^A = \xi\frac{1}{2}\frac{\partial}{\partial x}(\ln(M^A)), \quad \mu^B = \xi\frac{1}{2}\frac{\partial}{\partial x}(\ln(M^B)).$$

The couples (g^A, h^A) and (g^B, h^B) are defined as the solutions to the systems

$$\xi\frac{\partial}{\partial x}g^A + \mu^A g^A = \frac{1}{\varepsilon}\mathcal{L}_A(g^A, g) + \mathcal{L}^1_A(g^A, g) + \frac{1}{\varepsilon}\chi_\gamma\sigma_A^{-1}\left(K^A_*(h) + K^1_*(h^A)\right), \quad (3.67)$$

$$\xi\frac{\partial}{\partial x}h^A = \frac{1}{\varepsilon}\overline{\chi}_\gamma K^A_*(h) + \frac{1}{\varepsilon}(-\nu + \overline{\chi}_\gamma K^1_*)h^A + N_{A*}(h) + \widetilde{N}_*(h^A) + \varepsilon^2 d^A \quad (3.68)$$

and

$$\xi\frac{\partial}{\partial x}g^B + \mu^B g^B = \frac{1}{\varepsilon}\mathcal{L}_B(g^B, g) + \mathcal{L}^1_B(g^B, g) + \frac{1}{\varepsilon}\chi_\gamma\sigma_B^{-1}\left(K^B_*(h) + K^1_*(h^B)\right), \quad (3.69)$$

$$\xi\frac{\partial}{\partial x}h^B = \frac{1}{\varepsilon}\overline{\chi}_\gamma K^B_*(h) + \frac{1}{\varepsilon}(-\nu + \overline{\chi}_\gamma K^1_*)h^B + N_{B*}(h) + \widetilde{N}_*(h^B) + \varepsilon^2 d^B, \quad (3.70)$$

where

$$d^A = M_*^{-\frac{1}{2}}D^A, \quad d^B = M_*^{-\frac{1}{2}}D^B,$$

$$\chi_\gamma(v) = 1, \quad \text{for} \quad |v| \leq \gamma, \qquad \chi_\gamma(v) = 0, \quad \text{for} \quad |v| \geq \gamma, \quad \text{and} \quad \overline{\chi}_\gamma = 1 - \chi_\gamma.$$

$\mathcal{L} = (\mathcal{L}_A, \mathcal{L}_B)$ is the linearized Boltzmann operator for a two component gas defined by

$$\mathcal{L}_A(g^A, g) = \frac{1}{\sqrt{M^A}}(Q(\sqrt{M^A}g^A, M) + Q(M^A, \sqrt{M}g)), \quad (3.71)$$

$$\mathcal{L}_B(g^B, g) = \frac{1}{\sqrt{M^B}}(Q(\sqrt{M^B}g^B, M) + Q(M^B, \sqrt{M}g)). \quad (3.72)$$

Moreover $\mathcal{L}^1_A$, $\mathcal{L}^1_B$, K^A_*, K^B_*, N^A_*, N^B_*, $\tilde{N}_*$ are defined by

$$\mathcal{L}^1_A(g^A, g) = \frac{1}{\sqrt{M^A}}(Q(\sqrt{M^A}g^A, f_1 + \varepsilon f_2) + Q(f^A_1 + \varepsilon f^A_2, \sqrt{M}g)), \quad (3.73)$$

$$\mathcal{L}^1_B(g^B, g) = \frac{1}{\sqrt{M^B}}(Q(\sqrt{M^B}g^B, f_1 + \varepsilon f_2) + Q(f^B_1 + \varepsilon f^B_2, \sqrt{M}g)), \quad (3.74)$$

$$K^A_*(f) = \frac{1}{\sqrt{M_*}}Q(M^A, \sqrt{M_*}f), \qquad K^B_*(f) = \frac{1}{\sqrt{M_*}}Q(M^B, \sqrt{M_*}f),$$

$$N_{A*}(g) = \frac{1}{\sqrt{M_*}}Q(f^A_1 + \varepsilon f^A_2, \sqrt{M_*}g), \; N_{B*}(g) = \frac{1}{\sqrt{M_*}}Q(f^B_1 + \varepsilon f^B_2, \sqrt{M_*}g), \quad (3.75)$$

$$\widetilde{N}_*(g) = \frac{1}{\sqrt{M_*}}Q(\sqrt{M_*}g, f_1 + \varepsilon f_2), \quad (3.76)$$

and $Q(M, \sqrt{M_*}h^\alpha)$ is decomposed into

$$\frac{1}{\sqrt{M_*}}Q(M, \sqrt{M_*}h^\alpha) = (-\nu + K^1_*)h^\alpha, \;\; \alpha \in \{A, B\}, \quad (3.77)$$

where ν, called collision frequency is defined by

$$\nu(x, v) = \int_{\mathbb{R}^3\times\mathbb{S}^2}\langle v_* - v, \omega\rangle M(x, v_*)dv_*d\omega.$$

Remark 2. In ([14]) the decomposition is different. In that case g^A, g^B, h^A and h^B have to solve

$$\begin{aligned}\xi\frac{\partial}{\partial x}g^A+\mu^A(g_0^A+g_4^A) &= \frac{1}{\varepsilon}\frac{1}{\sqrt{M^A}}(Q(\sqrt{M^A}g^A,M)+Q(M^A,\sqrt{M}g))\\ &+ \frac{1}{\varepsilon}\chi_\gamma\sigma_A^{-1}\left(K_*^A(h)+K_*^1(h^A)\right)+L_A^1(g_0^A+g_4^A,g_0+g_4)\\ &+ \tilde{L}_A^1(g_0^B+g_4^B), \qquad (3.78)\end{aligned}$$

$$\begin{aligned}\xi\frac{\partial}{\partial x}h^A+\mu^A\sigma^A(\overline{g}^A+g_1^A) &= \frac{1}{\varepsilon}\overline{\chi}_\gamma K_*^A(h)+\frac{1}{\varepsilon}(-\nu+\overline{\chi}_\gamma K_*^1)h^A+N_{A*}(\sigma(g_1+\overline{g})+h)\\ &+ \widetilde{N}_*^A(\sigma^A(\overline{g}^A+g_1^A)+h^A,(\sigma^B(\overline{g}^B+g_1^B)+h^B))\\ &+ \varepsilon^2 d^A \qquad (3.79)\end{aligned}$$

and

$$\begin{aligned}\xi\frac{\partial}{\partial x}g^B+\mu^B(g_0^B+g_4^B) &= \frac{1}{\varepsilon}\frac{1}{\sqrt{M^B}}(Q(\sqrt{M^B}g^B,M)+Q(M^B,\sqrt{M}g))\\ &+ \frac{1}{\varepsilon}\chi_\gamma\sigma_B^{-1}\left(K_*^B(h)+K_*^1(h^B)\right)+L_B^1(g_0^B+g_4^B,g_0+g_4), \quad (3.80)\end{aligned}$$

$$\begin{aligned}\xi\frac{\partial}{\partial x}h^B+\mu^B\sigma^B(\overline{g}^B+g_1^B) &= \frac{1}{\varepsilon}\overline{\chi}_\gamma K_*^B(h)+\frac{1}{\varepsilon}(-\nu+\overline{\chi}_\gamma K_*^1)h^B+N_{B*}(\sigma(\overline{g}+g_1)+h)\\ &+ \widetilde{N}_*^B(\sigma^B(\overline{g}^B+g_1^B)+h^B)+\varepsilon^2 d^B. \qquad (3.81)\end{aligned}$$

The operators

$$N_{B*}(f),\ \ \widetilde{N}_*^B(f),\ \ \widetilde{N}_*^A(f,g),\ \ L_A^1(f,g),\ \ \tilde{L}_A^1(f),$$

are analogous to the operators defined in (3.73, 3.74, 3.75, 3.76) and satisfy the bounds of Lemma 3.1. But this decomposition breaks down for the control of the rest term for the problem studied in the present paper. This fact is mainly due to the presence of the tems g_1 in the equations defining h^A and h^B. In ([14]) this problem is solved because of the boundary conditions which are of Maxwell-diffuse reflexion type which is not the case here. But the decomposition (3.67, 3.68, 3.69, 3.70) of the present paper can be applied to the case of [20, 21, 14].

Remark 3. In the hard-sphere case, there are two non negative constants ν_0 and ν_1 such that the collosion frequency ν satisfies

$$\nu_0(1+|v|)\le\nu(x,v)\le\nu_1(1+|v|). \qquad (3.82)$$

Moreover g^A, h^A, g^B, h^B satisfy the boundary conditions

$$\begin{aligned}g^A(-1,v)=0,\ \xi>0,\quad g^A(1,v)=0,\ \xi<0,\\ h^A(-1,v)=\zeta^{A-}M_*^{-\frac{1}{2}},\ \xi>0,\quad h^A(1,v)=\zeta^{A+}M_*^{-\frac{1}{2}},\ \xi<0, \qquad (3.83)\\ g^B(-1,v)=0,\ \xi>0,\quad g^B(1,v)=0,\ \xi<0,\\ h^B(-1,v)=M_*^{-\frac{1}{2}}\zeta^{B-},\ \xi>0,\quad h^B(1,v)=M_*^{-\frac{1}{2}}\zeta^{B+},\ \xi<0. \qquad (3.84)\end{aligned}$$

Define also the functions h_-^A, h_+^A, h_-^B and h_+^B as follows

$$h_-^A=M_*^{-\frac{1}{2}}\zeta^{A-},\ \ \xi>0,\quad h_-^A=0,\ \ \xi<0,\quad h_+^A=M_*^{-\frac{1}{2}}\zeta^{A+},\ \ \xi<0,\quad h_+^A=0,\ \ \xi>0,$$
$$h_-^B=M_*^{-\frac{1}{2}}\zeta^{B-},\ \ \xi>0,\quad h_-^B=0,\ \ \xi<0,\quad h_+^B=M_*^{-\frac{1}{2}}\zeta^{B+},\ \ \xi<0,\quad h_+^B=0,\ \ \xi>0.$$

We shall control the rest term (R^A, R^B) by using the norm

$$|f|_{r,\beta_0} = \sup_{x\in[-1,1]} \sup_{v\in\mathbb{R}^3} (1+|v|)^r |f(x,v)| \exp(\beta_0 v^2), \tag{3.85}$$

for a suitable β_0. The same notation will be used for the functions depending only on the v variable.

3.4. **L^2 estimates on the rest term.** Recall that the norm $\| \ \|$ had been defined in (3.64). First we have the following estimates

Lemma 3.1. *For τ defined in Theorem 2.2, the operators, $\mathcal{L}^1_A$, $\mathcal{L}^1_B$, N_{A*}, N_{B*}, N_*, defined by (3.73, 3.74, 3.75, 3.76) satisfy the inequalities*

$$\|(1+|v|)^{-1}\mathcal{L}^1_A(f^A, f)\| \leq \tau(\|f^A\| + \|f\|), \quad \|(1+|v|)^{-1}\mathcal{L}^1_B(f^B, f)\| \leq \tau(\|f^B\| + \|f\|),$$
$$\|(1+|v|)^{-1}N_*(f)\| \leq \tau\|f\|, \quad \|(1+|v|)^{-1}N_{A*}(f)\| \leq \tau\|f\|, \ \|(1+|v|)^{-1}N_{B*}(f)\| \leq \tau\|f\|.$$

For the proof of lemma 3.1, we refer to ([14]).

Next we will focus on the control of (R^A, R^B), solution to the linearized problem (3.60, 3.61) in the norm $\| \ \|$ which is resumed in the following proposition.

Proposition 2. *There are $\varepsilon_0 > 0$, τ_0 and $c > 0$ such that for all $\varepsilon < \varepsilon_0$ and $\tau < \tau_0$, the solutions to (3.67, 3.68, 3.69, 3.70, 3.83, 3.84) satisfy the estimates*

$$\begin{aligned}
\|h^A\| + \|h^B\| &\leq c\varepsilon^3 (\|\frac{d^A}{(1+|v|)}\| + \|\frac{d^B}{(1+|v|)}\|) \\
&+ c\sqrt{\varepsilon}\big(\|h^A_-\| + \|h^A_+\| + \|h^B_-\| + \|h^B_+\|\big), && (3.86)\\
\|\overline{g}^A\| + \|\overline{g}^B\| &\leq c\varepsilon^2 (\|\frac{d^A}{(1+|v|)}\| + \|\frac{d^B}{(1+|v|)}\|) \\
&+ \frac{c}{\varepsilon^{\frac{1}{2}}}\big(\|h^A_-\| + \|h^A_+\| + \|h^B_-\| + \|h^B_+\|\big), && (3.87)\\
\|P^A(g^A)\| + \|P^B(g^B)\| &\leq c\varepsilon (\|\frac{d^A}{(1+|v|)}\| + \|\frac{d^B}{(1+|v|)}\|) \\
&+ \frac{c}{\varepsilon^{\frac{3}{2}}}\big(\|h^A_-\| + \|h^A_+\| + \|h^B_-\| + \|h^B_+\|\big). && (3.88)
\end{aligned}$$

Remark 4. In the case of Maxwell-diffuse reflexion boundary conditions (see [14]) the estimate obtained for $\|h^A\| + \|h^B\|$ and $\|\overline{g}^A\| + \|\overline{g}^B\|$ are of the same order as in Proposition 2. But for the hydrodynamical part of g, $\|g^A_1\| + \|g^B_1\|$ are of the same magnitude as $\|\overline{g}^A\| + \|\overline{g}^B\|$ whereas $\|g^A_0\| + \|g^B_0\| + \|g^A_4\| + \|g^B_4\|$ is of the same order as $\|P^A(g^A)\| + \|P^B(g^B)\|$. In the situation of a one component gas, the estimate on g_1 is even of the same order as h. The reason is explained in Remark 5.

Proof. (Proposition 2). Multiply (3.67) by εg^A and (3.69) by εg^B, add the obtained equation and integrate on $[-1,1]\times\mathbb{R}^3$ leads to

$$\begin{aligned}\varepsilon(\mathcal{I}_{g^A}+\mathcal{I}_{g^B}) \quad &- \int_{\mathbb{R}^3}\int_{-1}^{1}\left(\mathcal{L}_A(g^A,g)g^A+\mathcal{L}_B(g^B,g)g^B\right)\,dxdv\\ &= \varepsilon\int_{\mathbb{R}^3}\int_{-1}^{1}\left(\mu^A\,(P^Ag^A)^2+\mu^B\,(P^Bg^B)^2\right)\,dxdv\\ &+ \varepsilon\int_{\mathbb{R}^3}\int_{-1}^{1}\left(\mu^A\,(P^Ag^A)\bar{g}^A+\mu^B\,(P^Bg^B)\bar{g}^B\right)\,dxdv\\ &+ \int_{\mathbb{R}^3}\int_{-1}^{1}\left(\mathcal{L}^1_A(g^A,g)g^A+\mathcal{L}^1_B(g^B,g)g^B\right)\,dxdv\\ &+ \varepsilon\int_{\mathbb{R}^3}\int_{-1}^{1}\left(\mathcal{D}^A\sqrt{M^A}g^A+\mathcal{D}^B\sqrt{M^B}g^B\right)\,dvdx,\end{aligned}$$

with for any $\alpha\in\{A,B\}$,

$$\mathcal{I}_{g^\alpha}=\int_{\mathbb{R}^3}\xi(g^\alpha(1,v))^2\,dv+\int_{\mathbb{R}^3}\xi(g^\alpha(-1,v))^2\,dv.$$

Recall the spectral inequality ([2]),

$$\langle\mathcal{L}(g^A,g^B),(g^A,g^B)\rangle\geq-\gamma_1(\|g^A\|^2+\|g^B\|^2),\qquad\text{with}\qquad\gamma_1>0. \tag{3.89}$$

We notice that a new spectral estimate involving the term $\mathcal{L}^1$ has been established in ([6]). By using the spectral inequality (3.89) we get

$$\begin{aligned}\varepsilon(\mathcal{I}_{g^A}+\mathcal{I}_{g^B})+\gamma_1(\|\bar{g}^A\|^2+\|g^B\|^2) \quad &\leq\quad \varepsilon\tau(\|P^Ag^A\|^2+\|P^Bg^B\|^2+\|\bar{g}^A\|^2+\|\bar{g}^B\|^2)\\ &+\quad \varepsilon\left(\|\mathcal{D}^A\|\|g^A\|+\|\mathcal{D}^B\|\|g^B\|\right),\end{aligned}$$

with

$$\mathcal{D}^A=\chi_\gamma\sigma_A^{-1}\left(K_*^A(h)+K_*^1(h^A)\right),\quad \mathcal{D}^B=\chi_\gamma\sigma_B^{-1}\left(K_*^B(h)+K_*^1(h^B)\right). \tag{3.90}$$

Then by choosing τ small enough, it comes that

$$\begin{aligned}\varepsilon(\mathcal{I}_{g^A}+\mathcal{I}_{g^B})+\gamma_1(\|\bar{g}^A\|^2+\|g^B\|^2) \quad &\leq\quad \varepsilon\tau(\|P^Ag^A\|^2+\|P^Bg^B\|^2)\\ &+\quad \varepsilon\left(\|\mathcal{D}^A\|\|g^A\|+\|\mathcal{D}^B\|\|g^B\|\right).\end{aligned} \tag{3.91}$$

In order to control the terms g_1^A and g_1^B we use the relation

$$\xi\partial_x\left(\sqrt{M^\alpha}g^\alpha\right)=\mu^\alpha g^\alpha+\sqrt{M^\alpha}\xi\partial_xg^\alpha\quad \alpha\in\{A,B\}.$$

Multiply (3.67) by $\sqrt{M^A}$, (3.68) by $\sqrt{M^B}$, integrate in v and use the previous relation leads to

$$\begin{aligned}\frac{\partial}{\partial x}\left(\int_{\mathbb{R}^3}\xi g^A\sqrt{M^A}\,dv\right)&=\frac{1}{\varepsilon}\left(\int_{\mathbb{R}^3}\sqrt{M^A}\mathcal{D}^A\,dv\right),\\ \frac{\partial}{\partial x}\left(\int_{\mathbb{R}^3}\xi g^B\sqrt{M^B}\,dv\right)&=\frac{1}{\varepsilon}\left(\int_{\mathbb{R}^3}\sqrt{M^B}\mathcal{D}^B\,dv\right).\end{aligned}$$

Hence after integration between -1 and x of the two previous equations, we get

$$\begin{aligned}
|\int_{\mathbb{R}^3} \xi\, g_1^A \sqrt{M^A}\, dv| &\leq |\int_{\mathbb{R}^3} g_1^A(-1,v) \sqrt{M^A(-1,v)}\, dv| \\
&+ |\int_{\mathbb{R}^3} \xi\, \bar{g}^B \sqrt{M^A}\, dv| + \frac{1}{\varepsilon}|\int_{\mathbb{R}^3} \sqrt{M^A}\mathcal{D}^A\, dv|, \\
|\int_{\mathbb{R}^3} \xi\, g_1^B \sqrt{M^B}\, dv| &\leq |\int_{\mathbb{R}^3} g_1^B(-1,v) \sqrt{M^B(-1,v)}\, dv| \\
&+ |\int_{\mathbb{R}^3} \xi\, \bar{g}^B \sqrt{M^B}\, dv| + \frac{1}{\varepsilon}|\int_{\mathbb{R}^3} \sqrt{M^B}\mathcal{D}^B\, dv|.
\end{aligned}$$

Then

$$|\int_{\mathbb{R}^3} \xi\, g_1^A \sqrt{M^A}\, dv| \leq c\left(\mathcal{I}_{g^A} + \|\bar{g}^A\| + \frac{1}{\varepsilon}\|\mathcal{D}^A\|\right),$$
$$|\int_{\mathbb{R}^3} \xi\, g_1^B \sqrt{M^B}\, dv| \leq c\left(\mathcal{I}_{g^B} + \|\bar{g}^B\| + \frac{1}{\varepsilon}\|\mathcal{D}^B\|\right)$$

and finally we obtain the following estimates on $\|g_1^A\|$ and $\|g_1^B\|$,

$$\|g_1^A\| \leq \left(\mathcal{I}_{g^A} + \|\bar{g}^A\| + \frac{1}{\varepsilon}\|\mathcal{D}^A\|\right), \quad \|g_1^B\| \leq \left(\mathcal{I}_{g^B} + \|\bar{g}^B\| + \frac{1}{\varepsilon}\|\mathcal{D}^B\|\right). \tag{3.92}$$

Remark 5. In ([14]) and in ([20, 21]) the terms g_1 are controled by using the Maxwell diffuse boundary conditions. More precisely in ([14]), the B component satisfying diffuse reflection boundary conditions, its flux satisfies

$$\int_{\mathbb{R}^3} \xi(g^B + h^B)dv = 0. \tag{3.93}$$

Hence we get the inequality

$$\|g_1^B\| \leq \|\bar{g}^B\| + \|h^B\|. \tag{3.94}$$

Moreover due to the expression of the kernel of the linearized Boltzmann operator, the estimate (3.94) is also satisfied by g_1^A. In the situation of a one component gas ([20, 21]), the inequality $\|g_1\| \leq \|h\|$ is obtained from the same arguments. But in the present case, the relation (3.93) is not true.

Multiply (3.67) by $\xi\sqrt{M^A}$, (3.69) by $\xi\sqrt{M^B}$ and add the two obtained equations

$$\frac{\partial}{\partial x}\left(\int_{\mathbb{R}^3} \xi^2 \sqrt{M^A} g^A\, dv + \int_{\mathbb{R}^3} \xi^2 \sqrt{M^B} g^B\, dv\right) = \frac{1}{\varepsilon}\int_{\mathbb{R}^3} \xi\left(\mathcal{D}^A + \mathcal{D}^B\right)\, dv.$$

Next by setting

$$g^A_{x^2} = \int_{\mathbb{R}^3} \xi^2 \sqrt{M^A}\, \bar{g}^A\, dv, \quad g^B_{x^2} = \int_{\mathbb{R}^3} \xi^2 \sqrt{M^B}\, \bar{g}^B\, dv,$$

and after integration in the x variable between -1 and x, it holds that

$$\begin{aligned}
&|\int_{\mathbb{R}^3} \xi^2 \sqrt{M^A} g^A\, dv + \int_{\mathbb{R}^3} \xi^2 \sqrt{M^B} g^B\, dv| \\
&\qquad \leq |\int_{\mathbb{R}^3} \xi^2 \sqrt{M^A} g^A(-1,v)\, dv + \int_{\mathbb{R}^3} \xi^2 \sqrt{M^B} g^B(-1,v)\, dv| \\
&\qquad + |\int_{\mathbb{R}^3} \xi(\sqrt{M^A}\mathcal{D}^A + \sqrt{M^B}\mathcal{D}^B)\, dv| + \frac{1}{\varepsilon}|\int_{-1}^{1} \xi^2\, (g^A_{x^2} + g^B_{x^2})\, dx|.
\end{aligned}$$

Therefore

$$\|g_0^A+g_0^B+3g_4^A+3g_4^B\|_2\leq c\left(\mathcal{I}_{g^A}+\mathcal{I}_{g^B}+\|\bar{g}^A\|+\|\bar{g}^B\|+\frac{1}{\varepsilon}(\|\mathcal{D}^A|+\|\mathcal{D}^B\|)\right).$$

In order to obtain an extimate on $\|g_4^A\|+\|g_4^B\|$, consider $(\xi\mathcal{B},\xi\mathcal{B})\in Ker(\mathcal{L})^{\perp}$ solution to

$$(\mathcal{L}_A(\xi\mathcal{B}),\mathcal{L}_B(\xi\mathcal{B}))=(\xi(v^2-\frac{5}{2})\sqrt{M^A},\xi(v^2-\frac{5}{2})\sqrt{M^B}).$$

Hence multiplying (3.67) by $\frac{\xi}{\sqrt{T}}\mathcal{B}(\frac{|v|}{\sqrt{T}})$ and (3.69) by $\frac{\xi}{\sqrt{T}}\mathcal{B}(\frac{|v|}{\sqrt{T}})$ gives

$$\begin{aligned}\int_{\mathbb{R}^3}\mathcal{B}(\frac{|v|}{\sqrt{T}})\,\frac{\xi^2}{\sqrt{T}}\frac{\partial}{\partial x}(\sqrt{M^A}g^A)dv+\int_{\mathbb{R}^3}\mathcal{B}(\frac{|v|}{\sqrt{T}})\,\frac{\xi^2}{\sqrt{T}}\frac{\partial}{\partial x}(\sqrt{M^B}g^B)dv\\
=\int_{\mathbb{R}^3}\frac{\xi}{\sqrt{T}}\,\mathcal{B}(\frac{|v|}{\sqrt{T}})\sqrt{M^A}\mathcal{L}_A(g^A,g)dv+\int_{\mathbb{R}^3}\frac{\xi}{\sqrt{T}}\,\sqrt{M^B}\mathcal{B}(\frac{|v|}{\sqrt{T}})\mathcal{L}_B(g^B,g)dv\\
+\int_{\mathbb{R}^3}\frac{\xi}{\sqrt{T}}\,\mathcal{B}(\frac{|v|}{\sqrt{T}})\sqrt{M^A}\mathcal{L}_A^1(g^A,g)dv+\int_{\mathbb{R}^3}\frac{\xi}{\sqrt{T}}\,\mathcal{B}(\frac{|v|}{\sqrt{T}})\sqrt{M^B}\mathcal{L}_B^1(g^B,g)dv\\
+\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\frac{\xi}{\sqrt{T}}\,\mathcal{B}(\frac{|v|}{\sqrt{T}})\Big(\sqrt{M^A}\mathcal{D}^A+\sqrt{M^B}\mathcal{D}^B\Big)\,dv.\end{aligned}\tag{3.95}$$

Moreover $\mathcal{L}$ being self adjoint, we have

$$\begin{aligned}\int_{\mathbb{R}^3}\frac{\xi}{\sqrt{T}}\,\mathcal{B}(\frac{|v|}{\sqrt{T}})\sqrt{M^A}\mathcal{L}_A(g^A,g)dv+\int_{\mathbb{R}^3}\frac{\xi}{\sqrt{T}}\,\sqrt{M^B}\mathcal{B}(\frac{|v|}{\sqrt{T}})\mathcal{L}_B(g^B,g)dv\\
=\int_{\mathbb{R}^3}(\sqrt{M^A}\bar{g}^A+\sqrt{M^B}\bar{g}^B)\frac{\xi}{\sqrt{T}}\frac{|v|^2}{T}\,dv.\end{aligned}$$

Therefore by using the previous relation, (3.95) writes

$$\begin{aligned}\frac{\partial}{\partial x}\left(\int_{\mathbb{R}^3}\mathcal{B}(\frac{|v|}{\sqrt{T}})\,\frac{\xi^2}{\sqrt{T}}\,\sqrt{M^A}g^A\,dv\right)+\frac{\partial}{\partial x}\left(\int_{\mathbb{R}^3}\mathcal{B}(\frac{|v|}{\sqrt{T}})\,\frac{\xi^2}{\sqrt{T}}\,\sqrt{M^B}g^B\,dv\right)\\
=\int_{\mathbb{R}^3}\frac{\partial}{\partial x}\left(\frac{\xi^2}{\sqrt{T}}\,\mathcal{B}(\frac{|v|}{\sqrt{T}})\right)\left(\sqrt{M^A}g^A+\sqrt{M^B}g^B\right)\,dv\\
+\frac{1}{\varepsilon}\int_{\mathbb{R}^3}(\sqrt{M^A}\bar{g}^A+\sqrt{M^B}\bar{g}^B)\frac{\xi}{\sqrt{T}}\frac{|v|^2}{T}\,dv\\
+\int_{\mathbb{R}^3}\left(\frac{\xi}{\sqrt{T}}\mathcal{B}(\frac{|v|}{\sqrt{T}})\right)\left(\sqrt{M^A}\mathcal{L}_A^1(g^A,g)+\sqrt{M^B}\mathcal{L}_B^1(g^B,g)\right)\,dv\\
+\int_{\mathbb{R}^3}\frac{\xi}{\sqrt{T}}\mathcal{B}(\frac{|v|}{\sqrt{T}})\Big(\sqrt{M^A}\mathcal{D}^A+\sqrt{M^B}\mathcal{D}^B\Big)\,dv.\end{aligned}\tag{3.96}$$

But

$$\int_{\mathbb{R}^3}\xi^2\mathcal{B}(|v|)\sqrt{M^A}g^A\,dv+\int_{\mathbb{R}^3}\xi^2\mathcal{B}(|v|)\sqrt{M^B}g^B\,dv=k_2\,p_4+g^A_{x^2B}+g^B_{x^2B},$$

with

$$k_2=\int_{\mathbb{R}^3}\xi^2\psi_4\,\mathcal{B}(|v|)\sqrt{M^A}\,dv+\int_{\mathbb{R}^3}\xi^2\psi_4\,\mathcal{B}(|v|)\sqrt{M^B}\,dv.$$

Moreover by using the spectral inequality (3.89), it comes that $k_2 < 0$. Then the equation (3.96) reads

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{T}}(k_2 p_4(x) + g^A_{x^2B} + g^B_{x^2B})\right) &= \frac{1}{\varepsilon}(g^A_{x2} + g^B_{x2}) + \int_{\mathbb{R}^3} \frac{\partial}{\partial x}(\tilde{\xi}\tilde{\mathcal{B}})\,(\sqrt{M^A}D^A + \sqrt{M^B}D^B)\,dv \\ &+ \int_{\mathbb{R}^3}\left(\frac{\xi}{\sqrt{T}}\mathcal{B}(\frac{|v|}{\sqrt{T}})\right)\left(\sqrt{M^A}\mathcal{L}^1_A(g^A,g) + \sqrt{M^B}\mathcal{L}^1_B(g^B,g)\right)dv \\ &+ \frac{1}{\varepsilon}\int_{\mathbb{R}^3}\xi\mathcal{B}(|v|)\,(\sqrt{M^A}D^A + \sqrt{M^B}D^B)\,dv, \end{aligned} \tag{3.97}$$

with

$$g^A_{x2} = \int_{\mathbb{R}^3}\xi v^2\sqrt{M^A}\bar{g}^A\,dv, \quad g^B_{x2} = \int_{\mathbb{R}^3}\xi v^2\sqrt{M^B}\bar{g}^B\,dv.$$

Next we aim to determine g^A_{x2} and g^B_{x2}. Multiply (3.67) by $|v|^2\sqrt{M^A}$, (3.69) by $|v|^2\sqrt{M^B}$, integrate with respect to the v variable and add the two equations gives

$$\int_{\mathbb{R}^3}\xi|v|^2\frac{\partial}{\partial x}\left(\sqrt{M^A}g^A + \sqrt{M^B}g^B\right)dv = \frac{1}{\varepsilon}\int_{\mathbb{R}^3}|v|^2\left(\sqrt{M^A}\mathcal{D}^A + \sqrt{M^B}\mathcal{D}^B\right)dv.$$

Hence by integrating between -1 and x, there is a nonnegative constant c_1 such that

$$(g^A_{x2} + g^B_{x2}) = c_1 + \frac{1}{\varepsilon}\int_1^x\int_{\mathbb{R}^3}|v|^2\left(\sqrt{M^A}\mathcal{D}^A + \sqrt{M^B}\mathcal{D}^B\right)dv.$$

So by plugging the previous expression of $g^A_{x2} + g^B_{x2}$ into (3.97), it holds that

$$\begin{aligned}\frac{\partial}{\partial x}(k_4 p_4(x) + g^A_{x^2B} + g^B_{x^2B}) &= \frac{c_1}{\varepsilon} + \int_{\mathbb{R}^3}\xi\mathcal{B}(|v|)\left(\mathcal{L}^1_A(g^A,g)\sqrt{M^A} + \mathcal{L}^1_B(g^B,g)\sqrt{M^B}\right)dv \\ &- \int_{\mathbb{R}^3}\frac{\partial}{\partial x}(\mathcal{B}(\frac{|v|}{\sqrt{T}}))\frac{\xi}{\sqrt{T}})\left(\sqrt{M^A}g^A + \sqrt{M^B}g^B\right)dv \\ &+ \frac{1}{\varepsilon}\int_1^x\int_{\mathbb{R}^3}|v|^2\left(\sqrt{M^A}\mathcal{D}^A + \sqrt{M^B}\mathcal{D}^B\right)dv.\end{aligned}$$

Next by setting

$$\tilde{p}_4 = k_2 p_4 + g^A_{x^2B} + g^B_{x^2B}$$

and

$$\begin{aligned}\mathcal{D}_2 &= \int_{\mathbb{R}^3}\xi\mathcal{B}\left(\mathcal{L}^1_A(g^A,g)\sqrt{M^A} + \mathcal{L}^1_B(g^B,g)\sqrt{M^B}\right)dv \\ &- \int_{\mathbb{R}^3}\frac{\partial}{\partial x}(\tilde{\mathcal{B}}\tilde{\xi})\left(\sqrt{M^A}g^A + \sqrt{M^B}g^B\right)dv \\ &+ \int_{\mathbb{R}^3}\tilde{\xi}\mathcal{B}(|\tilde{v}|)\left(\sqrt{M^A}\mathcal{D}^A + \sqrt{M^B}\mathcal{D}^B\right)dv,\end{aligned} \tag{3.98}$$

we get the relation

$$\tilde{p}_4'(x) = \frac{c_1}{\varepsilon} + \mathcal{D}_2. \tag{3.99}$$

By integrating (3.99) between 1 and -1 we get

$$\tilde{p}_4(-1) - \tilde{p}_4(1) = -\frac{2c_1}{\varepsilon} + \int_{-1}^1\mathcal{D}_2(s)\,ds$$

and by integrating (3.99) between 1 and x we get

$$\tilde{p}_4(x) - \tilde{p}_4(1) = -\frac{x-1}{\varepsilon} c_1 + \int_1^x \mathcal{D}_2(s)\, ds.$$

Then by eliminating c_1 in the previous equation we get for any $x \in [-1,1]$

$$\tilde{p}_4(x) = \tilde{p}_4(1) + \frac{x-1}{2}\left(\tilde{p}_4(1) - \tilde{p}_4(-1) + \int_{-1}^1 \mathcal{D}_2(s)\, ds\right) + \int_1^x \mathcal{D}_2(s)\, ds. \quad (3.100)$$

Next we aim to control $\|\mathcal{D}_2\|$. Firstly

$$\partial_x(\frac{\xi}{\sqrt{T}}\mathcal{B}(\frac{|v|}{\sqrt{T}})) = -\frac{\xi}{T^{\frac{3}{2}}}\partial_x T\, \mathcal{B}(\frac{|v|}{\sqrt{T}}) + -\frac{\xi|v|}{T^2}\mathcal{B}'(\frac{|v|}{\sqrt{T}})\partial_x T.$$

Hence according to the estimate (2.46) on $\partial_x T$, it holds that

$$\|\partial_x(\frac{\xi}{\sqrt{T}}\mathcal{B}(\frac{|v|}{\sqrt{T}}))\| \leq c\,\tau.$$

Moreover according to Lemma 3.1, we have

$$|\int_{\mathbb{R}^3} \xi\mathcal{B}(|v|)\left(\mathcal{L}^1_A(g^A,g)\sqrt{M^A} + \mathcal{L}^1_B(g^B,g)\sqrt{M^B}\right) dv| \leq c\,\tau\,(\|g^A\| + \|g^B\|).$$

So $\|\mathcal{D}_2\|$ satisfies the estimate

$$\begin{aligned}\|\mathcal{D}_2\| \;&\leq\; c\,\tau(\|g_0^A\| + \|g_0^B\| + \|g_1^A\| + \|g_1^B\| + \|g_4^A\| + \|g_4^B\| + \|\bar{g}^A\| + \|\bar{g}^B\|)\\ &+\; c(\|\mathcal{D}^A\| + \|\mathcal{D}^B\|).\end{aligned}$$

Therefore from relation (3.100) we obtain

$$\begin{aligned}\|g_4^A\| + \|g_4^B\| \;&\leq\; c\tau\left(\|g_0^A\| + \|g_0^B\| + \|g_1^A\| + \|g_1^B\| + \|g_4^A\| + \|g_4^B\|\right) + c(\|\bar{g}^A\| + \|\bar{g}^B\|)\\ &+\; c\,(\|\mathcal{D}^A\| + \|\mathcal{D}^B\|).\end{aligned}$$

So by using (3.91) and by taking τ small enough we get

$$\|g_0^A\| + \|g_0^B\| + \|g_1^A\| + \|g_1^B\| + \|g_4^A\| + \|g_4^B\| \leq \frac{c}{\varepsilon}(\|\mathcal{D}^A\| + \|\mathcal{D}^B\|).$$

Moreover by using again (3.91), it holds that

$$\|\bar{g}^A\| + \|\bar{g}^B\| \leq c\,(\|\mathcal{D}^A\| + \|\mathcal{D}^B\|).$$

Then g^A and g^B have been estimated in terms of $\|\mathcal{D}^A\|$ and $\|\mathcal{D}^B\|$. Hence it remains to control h^A and h^B.

Control of h^A and h^B.

Multiply (3.68) by εh^A, (3.70) by εh^B and integrate on $\mathbb{R}^3 \times [-1,1]$. By setting for $\alpha \in \{A,B\}$,

$$\mathcal{I}_{h^\alpha} = \int_{\mathbb{R}^3} \xi(h^\alpha(1,v))^2 dv - \int_{\mathbb{R}^3} \xi(h^\alpha(-1,v))^2 dv,$$

it holds that

$$\varepsilon(\mathcal{I}_{h^A}+\mathcal{I}_{h^B})+\int_{\mathbb{R}^3}\int_{-1}^{1}\nu(h^A)^2+(h^B)^2)dxdv=\int_{\mathbb{R}^3}\int_{-1}^{1}((\overline{\chi}_\gamma K_*^A)h)h^A dvdx$$
$$+\int_{\mathbb{R}^3}\int_{-1}^{1}((\overline{\chi}_\gamma K_*^1)h^A)h^A dvdx+\int_{\mathbb{R}^3}\int_{-1}^{1}((\overline{\chi}_\gamma K_*^B)h)h^B dvdx+\int_{\mathbb{R}^3}\int_{-1}^{1}((\overline{\chi}_\gamma K_*^1)h^B)h^B dvdx$$
$$+\varepsilon\int_{\mathbb{R}^3}\int_{-1}^{1}(N_{A*}(h)+\widetilde{N}_*(h^A))h^A dvdx+\varepsilon\int_{\mathbb{R}^3}\int_{-1}^{1}(N_{B*}(h)+\widetilde{N}_*(h^B))h^B dvdx$$
$$+\varepsilon^3\int_{\mathbb{R}^3}\int_{-1}^{1}(d^Ah^A+d^Bh^B)dvdx.$$

From (3.82) and Lemma 3.1, we get

$$\varepsilon(\mathcal{I}_{h^A}+\mathcal{I}_{h^B})+\nu_0(\|h^A\|^2+\|h^B\|^2)\leq\left|\int_{\mathbb{R}^3}\int_{-1}^{1}(\overline{\chi}_\gamma K_*^1h^A)h^A dvdx\right|$$
$$+\left|\int_{\mathbb{R}^3}\int_{-1}^{1}(\overline{\chi}_\gamma K_*^Ah)h^A dvdx\right|+\left|\int_{\mathbb{R}^3}\int_{-1}^{1}(\overline{\chi}_\gamma K_*^1h^B)h^B dvdx\right|+\left|\int_{\mathbb{R}^3}\int_{-1}^{1}(\overline{\chi}_\gamma K_*^Bh)h^B dvdx\right|$$
$$+c\tau\varepsilon(\|h^A\|+\|h^B\|)(\|h^A\|+\|h^B\|)+\varepsilon^3(\|d^A\|\,\|h^A\|+\|d^B\|\,\|h^B\|).$$

By continuity of K_*^1, K_*^A and K_*^B, it holds that

$$\int_{-1}^{1}\int_{\mathbb{R}^3}(\overline{\chi}_\gamma K_*^1h^A)h\,dvdx\leq\frac{\|h\|\|h^A\|}{(1+\gamma)^{\frac12}},\ |\int_{\mathbb{R}^3}\int_{-1}^{1}(\overline{\chi}_\gamma K_*^Ah)h^A dvdx|\leq\frac{\|h^A\|\|h\|}{(1+\gamma)^{\frac12}},$$
$$|\int_{\mathbb{R}^3}\int_{-1}^{1}(\overline{\chi}_\gamma K_*^Bh)h^B dvdx|\leq\frac{\|h\|\|h^A\|}{(1+\gamma)^{\frac12}}.$$

Moreover, according to the boundary conditions (3.83, 3.84) satisfied by h^A and h^B,

$$\mathcal{I}_{h^A}\geq-c(\|h_-^A\|^2+\|h_+^A\|^2),\quad\mathcal{I}_{h^B}\geq-c\left(\|h_-^B\|^2+\|h_+^B\|^2\right).$$

Hence

$$\begin{aligned}\|h^A\|^2+\|h^B\|^2\quad&\leq\quad c\varepsilon(\|h_-^A\|^2+\|h_+^A\|^2+\|h_-^B\|^2+\|h_+^B\|^2+\frac{c}{(1+\gamma)^{\frac12}}(\|h^A\|^2+\|h^B\|^2)\\&+\quad c\tau\varepsilon(\|h^A\|+\|h^B\|)\|h^A\|+\varepsilon^3(\|h^A\|\,\|\frac{d^A}{1+|v|}\|+\|h^B\|\,\|\frac{d^B}{1+|v|}\|)\end{aligned}$$

and (3.86) follows. After recalling that $\mathcal{D}^A$ and $\mathcal{D}^B$ have been defined in (3.90) we finally get an estimate on $\|\mathcal{D}^A\|+\|\mathcal{D}^B\|$ which leads to the control of P^Ag^A, P^Bg^B, $\bar{g}^A$ and $\bar{g}^B$. □

3.5. L^∞ **estimates on the rest term.** This subsection is devoted to the L^∞ estimate of the linearized rest term (R^A,R^B) solution to (3.60, 3.61). This control is performed by using first a L^∞ bound on g^A, g^B, h^A and h^B with the norm

$$|f|_r=\sup_{x\in[-1,1]}\sup_{v\in\mathbb{R}^3}(1+|v|)^r|f(x,v)|.$$

The arguments are the same as the ones developped in [14]. But for the sake of clarity we will recall some elements. The control is performed by introducing the following intermediate norm between $|\ |_r$ and $\|\ \|$

$$N(f)=\sup_{x\in[-1,1]}\left(\int_{\mathbb{R}^3}|f(x,v)|^2dv\right)^{\frac12}.$$

By considering the exponential formulation of (3.67, 3.69) together with the estimates (3.86, 3.87, 3.88) we obtain the L^∞ estimate

$$\begin{aligned}(|g^A|_r + |g^B|_r) &\leq c\sqrt{\varepsilon}\left(\|\frac{d^A}{(1+|v|)}\| + \|\frac{d^B}{(1+|v|)}\|\right) + cH_\gamma(|h^A|_r + |h^B|_r)\\ &+ \frac{c}{\varepsilon^2}(|h^A_-|_r + |h^A_+|_r + |h^B_-|_r + |h^B_+|_r),\end{aligned} \tag{3.101}$$

$$\begin{aligned}|h^A|_r + |h^B|_r &\leq c\varepsilon^{\frac{3}{2}}(\|\frac{d^A}{(1+|v|)}\| + \|\frac{d^B}{(1+|v|)}\|) + \varepsilon^{\frac{5}{2}}(|\nu^{-1}d^A|_r + |\nu^{-1}d^B|_r)\\ &+ \frac{c}{\varepsilon^2}(|h^A_-|_r + |h^A_+|_r + |h^B_-|_r + |h^B_+|_r).\end{aligned} \tag{3.102}$$

As a consequence we get the following bounds on the solution $(R^A,\, R^B)$ to the linearized problem (3.60, 3.61).

Proposition 3. *For all $r \geq 3$, there are c, ε_0, η_0 and β_0 such that for all $\varepsilon < \varepsilon_0$ and $\eta < \eta_0$, $(R^A,\, R^B)$ solutions to (3.60, 3.61) satisfy the estimates*

$$\begin{aligned}|R^A|_{r,\beta_0} + |R^B|_{r,\beta_0} &\leq c\varepsilon^{\frac{1}{2}}(|D^A|_{r-1,\beta_0} + |D^B|_{r-1,\beta_0})\\ &+ \frac{c}{\varepsilon^2}(|\zeta^{A-}|_{r,\beta_0} + |\zeta^{B-}|_{r,\beta_0} + |\zeta^{A+}|_{r,\beta_0} + |\zeta^{B+}|_{r,\beta_0}).\end{aligned}$$

The proof is analogous to the one given in [14]. It uses the L^∞ bounds on g^A, h^A, g^B, h^B (3.101, 3.102) and the properties on the Boltzmann operator given in ([22], [23]). For more precisions we refer to this paper.

3.6. **Convergence of the iterative process.** This subsection deals with the rest terms (R^A, R^B) of the expansion given in Theorem 1.1. We recall that (R^A, R^B) is solutions to the non linear system (3.53, 3.54) and is constructed as the limit of a sequence of iterations of linearized problems of the type (3.60, 3.61). By using Proposition 3, this sequence is proved to be a converging sequence and satisfies the following estimates

Proposition 4. *For all $r \geq 3$, there is c, c', ε_0, τ_0 and β_0 such that for all $\varepsilon < \varepsilon_0$, and $\tau < \tau_0$, the problem (3.53, 3.54) has a unique solution (R^A, R^B) satisfying*

$$|R^A|_{r,\beta_0} + |R^B|_{r,\beta_0} \leq c\left(\varepsilon^{\frac{3}{2}}(|A|_{r,\beta_0} + |B|_{r,\beta_0}) + \exp(-\frac{c'}{\varepsilon})\right).$$

For the proof of Proposition 4 we refer to ([14]). Therefore we deduce Theorem 1.1.

Proof. (Theorem 1.1). By arguing as in ([14]), it can be shown that $(|A|_{r,\beta_0} + |B|_{r,\beta_0}) = \mathcal{O}(\frac{1}{\varepsilon^4})$. For p^B_{II} close enough to p^B_I and T_{II} close enough to 1, the asymptotic expansion

$$(f^A_{H0} + \varepsilon f^A_1 + \varepsilon^2 f^A_2 + \varepsilon^3 R^A,\ f^B_{H0} + \varepsilon f^B_1 + \varepsilon^2 f^B_2 + \varepsilon^3 R^B)$$

has been determined to define (f^A, f^B). For ε small enough Proposition 4 controls the rest term (R^A, R^B) of the expansion. □

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