Problem of evaporation-condensation for a two component gas in the slab.

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Abstract

This paper studies the non linear Boltzmann equation for a two component gas in the situation of hard spheres. A Hilbert expansion of the solution is performed. The first order of the fluid equations shows the ghost effect. The fluid system is solved when the boundary conditions are close to each other. The boundary conditions for the kinetic system are satisfied by adding for the first and the second order Knudsen layers. In a last part the rest term is rigorously controled by using a decomposition into a low part velocity and a high part velocity. This constitutes a generalization to the case of a two component gas of the results presented in [13, 14].

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1 Introduction.

Consider a mixture constituted by vapor and noncondensable gas whose the stationary behaviour is studied. The part of the space where the mixture is situated between two phases of a condensed gas represented by two vertical planes. Suppose that the model is homogeneous in space in the y and in the z direction. So we can consider that the space variable x belongs to $[-1, 1]$. The vertical planes are respectively kept at temperatures T_I and T_{II}. Denote n_I (resp. n_{II}) the density of saturation of the vapor at temperature T_I (resp. T_{II}). The first component of the gas denoted by A is constituted by vapor and can condense on each boundary. The other component denoted by B cannot condense. The molecules of the two gases are supposed mechanically identical i.e they have the same mass and the same diameter ([24]). The distribution functions f^A and f^B are solutions to the stationary Boltzmann equation for a two component gas ([10])

$$\begin{aligned}\xi\frac{\partial}{\partial x}f^A(x,v) &= \frac{1}{\varepsilon}Q(f^A,f^A)(x,v)+\frac{1}{\varepsilon}Q(f^A,f^B)(x,v),\\ \xi\frac{\partial}{\partial x}f^B(x,v) &= \frac{1}{\varepsilon}Q(f^B,f^A)(x,v)+\frac{1}{\varepsilon}Q(f^B,f^B)(x,v),\\ &\qquad x\in[-1,1],\quad v\in\mathbb{R}^3,\end{aligned} \tag{1.1}$$

with

$$\varepsilon = \frac{\sqrt{\pi}}{2}K_n = \frac{\sqrt{\pi}}{2}\frac{l}{2} \text{ and } \quad l = \frac{1}{\sqrt{2}\pi d^2 n_I}. \tag{1.2}$$

l is the mean free path of the vapor molecules in the equilibrium state at rest with temperature T_I and density n_I, K_n is the Knudsen number and d corresponds to the diameter of the molecule. Q is called collision operator and will be defined in the next section.
The boundary conditions for A have a given indatta profile and the boundary conditions for the B component are of diffuse reflection type.

In the present paper we are in the situation where ε is close to 0 and the distribution functions (f^A, f^B) of the two gases are researched as an asymptotic expansion plus a rest term. The same situation has been also considered away from equilibrium. In [B1, Bw], the author has obtained existence of weak and renormalized solutions in L^1 by using entropy flux compactness methods.

As a physical point of view this problem has been already studied in ([3, 1]) where two types of behaviour were pointed out. In a first situation the macroscopic velocity of the two gases is 0 ([3, 26]). That means physically that evaporation and condensation stop for the A component. But the Hilbert term of order 1 of the velocity of the A component keeps an influence at the hydrodynamical level. This is the ghost effect as defined for a one component gas in ([21]) and for a two component gas in ([3, 26, 25, 8]). In a second case the B component becomes negligeable and accumulates in a thin layer at the boundaries called Knudsen layer ([4]). In this paper only the first case will be treated (when the macroscopic velocity is 0). This paper is organized as follows.

Section 2 presents the model and the main result of this paper. Section 3 deals with the asymptotic expansion of the solutions. At the end of the section, a fluid system mixing 0 order terms and first order terms is derived and points out the ghost effect ([8, 24, 25]). The fluid system is solved when boundary conditions for f^A are close to each other Theorem (3.1). Section 4 is devoted to the boundary conditions of f^A and f^B. We show that Knudsen terms have to be added at first and second order terms of the Hilbert terms of f^A and f^B in order to satisfy the proper boundary conditions. Section 5 sudies the rest term which is decomposed as in [13, 14] into a high and a low velocity part. The main difficulty is to extend the approach of [13, 14] to the situation of a two component gas and to mix two different types of boundary conditions. Finally we control in section 6 the rest term of the expansion. The rest term of a linearized problem is first controled in a weighted L^2 norm and in a weighted L^∞ norm. In [13, 14], the authors consider only a one component gas satisfying boundary conditions of diffuse-reflection types and uses at a crucial point of the control that the total flux of the solution is zero. In this paper, we are not in this situation and this difficulty is solved thanks to the structure of the kernel of the linearized Boltzmann operator for a two component gas (see remarks 2, 4). At the end of the section the rest term of the full nonlinear problem is obtained as a limit of a sequence whose terms are solution to linearized problems (Proposition 6.1). Finally Theorem 2.1 can be deduced.

2 Presentation of the model.

The collision operator Q of the equation 1.1 is defined by ([10])

$$Q(f,g)(x,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\omega)[f'g'_* - fg_*]d\omega dv_*,$$

with

$$f_* = f(x,v_*), \quad f' = f(x,v'), \quad f'_* = f(x,v'_*),$$
$$v' = v - \langle v-v_*,\omega\rangle\omega, \quad v'_* = v_* + \langle v-v_*,\omega\rangle\omega.$$

The velocity $v \in \mathbb{R}^3$ has for coordinates (ξ,η,χ) and $\langle v-v_*,\omega\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^3$. Let $\omega \in \mathbb{S}^2$ be represented by the polar angle (with axis along $v-v_*$) and the azimutal angle ϕ. The function $B(v-v_*,\omega)$ is the collision kernel of the collision operator Q in the situation of hard-sphere. The boundary condition for the A component is the following given indatta profile

$$f^A(-1,v) = M_-(v), \quad \xi > 0, \quad f^A(1,v) = \frac{n_{II}}{n_I}M_+(v), \quad \xi < 0. \tag{2.3}$$

The boundary condition for the B component is of diffuse reflection type

$$f^B(-1,v) = M_-(v)\int_{\xi'<0}|\xi'|f^B(-1,v')dv', \quad \xi > 0, \tag{2.4}$$

$$f^B(1,v) = M_+(v)\int_{\xi'>0}|\xi'|f^B(1,v')dv', \quad \xi < 0, \tag{2.5}$$

where M_- and M_+ are the normalized Maxwellian distributions

$$M_-(v) = \frac{1}{\pi}\exp(-v^2) \quad \text{et} \quad M_+(v) = \frac{1}{\pi(\frac{T_{II}}{T_I})^2}\exp(-\frac{v^2}{\frac{T_{II}}{T_I}}).$$

Moreover the mass $m > 0$ for the B component is fixed as follows

$$\int_{-1}^{1} \int_{\mathbb{R}^3} f^B(x,v)dxdv = m. \tag{2.6}$$

The main result of this paper is

Theorem 2.1. *For n_{II} close enough to n_I, for some T_{II} close enough to T_I and ε small enough, there is a solution (f^A, f^B) to the system (1.1, 2.3, 2.4, 2.5, 2.6) of the form*

$$(f^A, f^B) = (f^A_{H0} + \varepsilon f^A_1 + \varepsilon^2 f^A_2 + \varepsilon^3 f^A_R,\ f^B_{H0} + \varepsilon f^B_1 + \varepsilon^2 f^B_2 + \varepsilon^3 f^B_R)$$

satisfying

$$\|f^A_R\|_\infty + \|f^B_R\|_\infty \leq \frac{c}{\varepsilon^{\frac{5}{2}}}.$$

3 Asymptotic expansion.

In this section after introducing the macroscopic quantities n, u_1, p, T, the distribution functions f^A and f^B are written as Hilbert expansions up to order 2. The Hilbert terms of this expansion are explicitely determined in section 3.6. At the end of the section, a fluid system mixing 0 order terms and first order terms is derived and closed for boundary conditions closed to each other (Theorem 3.1).

3.1 Macroscopic quantities.

For all distribution function f, the macroscopic quantities n, u, T et p are defined by ([23])

$$n = \int_{\mathbb{R}^3_v} f dv, \quad nu_1 = \int_{\mathbb{R}^3_v} \xi f dv, \quad nu = \int_{\mathbb{R}^3_v} v f dv,$$
$$p = Tn = \frac{2}{3} \int_{\mathbb{R}^3_v} \big((\xi - u_{1,H1})^2 + \eta^2 + \chi^2\big) f dv. \tag{3.1}$$

3.2 Hilbert expansion.

The distribution functions f^A and f^B are expanded in Hilbert series as follows

$$\begin{aligned} f^A_H(x,v) &= f^A_{H0}(x,v) + \varepsilon f^A_{H1}(x,v) + \cdots, \\ f^B_H(x,v) &= f^B_{H0}(x,v) + \varepsilon f^B_{H1}(x,v) + \cdots. \end{aligned} \tag{3.2}$$

Substitute f^A_H and f^B_H by the expressions given in (3.2) in the equation (1.1) leads to

$$\begin{aligned} \xi \frac{\partial}{\partial x}(f^A_{H0} + \varepsilon f^A_{H1} + \cdots) &= \frac{1}{\varepsilon} Q(f^A_{H0} + \varepsilon f^A_{H1} + \cdots, f^A_{H0} + \varepsilon f^A_{H1} + \cdots) \\ &+ \frac{1}{\varepsilon} Q(f^A_{H0} + \varepsilon f^A_{H1} + \cdots, f^B_{H0} + \varepsilon f^B_{H1} + \cdots), \end{aligned} \tag{3.3}$$

$$\begin{aligned} \xi \frac{\partial}{\partial x}\left(f^B_{H0} + \varepsilon f^B_{H1} + \cdots\right) &= \frac{1}{\varepsilon} Q(f^B_{H0} + \varepsilon f^B_{H1}, \cdots f^A_{H0} + \varepsilon f^A_{H1} + \cdots) \\ &+ \frac{1}{\varepsilon} Q(f^B_{H0} + \varepsilon f^B_{H1} + \cdots, f^B_{H0} + \varepsilon f^B_{H1} + \cdots). \end{aligned} \tag{3.4}$$

A important Hilbert term is

$$f_H = f^A_H + f^B_H. \tag{3.5}$$

It corresponds to the sum of the two components and satisfies the relation

$$\xi \frac{\partial}{\partial x}(f_{H0} + \varepsilon f_{H1} + \cdots) = \frac{1}{\varepsilon} Q(f_{H0} + \varepsilon f_{H1} + \cdots, f_{H0} + \varepsilon f_{H1} + \cdots). \tag{3.6}$$

By using the Hilbert expansions (3.2) for f_H^A and f_H^B and by identifying formally the different orders of ε in (3.1), the following relations are obtained for $\alpha \in \{A;B\}$

$$\int_{\mathbb{R}^3_v} f^\alpha_{Hm} dv = n^\alpha_{Hm}\ (m=0,1\cdots),\quad \int_{\mathbb{R}^3_v} \xi f^\alpha_{H0} dv = n^\alpha_{H0} u^\alpha_{1,H0},\quad \int_{\mathbb{R}^3_v} v f^\alpha_{H0} dv = n^\alpha_{H0} u^\alpha_{H0}, \tag{3.7}$$

$$\int_{\mathbb{R}^3_v} \xi^2 f^\alpha_{H0} dv = \frac{1}{2}\left(n^\alpha_{H0} T^\alpha_{H0}\right), \int_{\mathbb{R}^3_v} v^2 f^\alpha_{H0} dv = n^\alpha_{H0} (u^\alpha_{1,H0})^2 + \frac{3}{2} p^\alpha_{H0}, \tag{3.8}$$

$$\int_{\mathbb{R}^3_v} \xi f^\alpha_{H1} dv = n^\alpha_{H0} u^\alpha_{1,H1} + n^\alpha_{H1} u^\alpha_{1,H0},\quad \int_{\mathbb{R}^3_v} v f^\alpha_{H1} dv = n^\alpha_{H0} u^\alpha_{1,H1} + n^\alpha_{H1} u^\alpha_{1,H0}, \tag{3.9}$$

$$\int_{\mathbb{R}^3_v} v^2 f^\alpha_{H1} dv = \frac{3}{2}(n^\alpha_{H0} T^\alpha_{H1} + n^\alpha_{H1} T^\alpha_{H0}) + 2 n^\alpha_{H0} u^\alpha_{1,H0} u^\alpha_{H1} + 2 n^\alpha_{H0} (u^\alpha_{1,H0})^2. \tag{3.10}$$

3.3 Study of the terms of order -1.

The identification of the terms of order -1 in the equations (3.3) and (3.4) leads to

$$Q(f^A_{H0}, f^A_{H0}) + Q(f^B_{H0}, f^A_{H0}) = 0, \tag{3.11}$$

$$Q(f^A_{H0}, f^B_{H0}) + Q(f^B_{H0}, f^B_{H0}) = 0. \tag{3.12}$$

The system (3.11, 3.12) is solved by using the following lemma.

Lemma 3.1. *The solution to the system (3.11-3.12) is*

$$f^A_{H0}(x,v) = \frac{n^A_{H0}}{\pi^{\frac{3}{2}} (T_{H0})^{\frac{3}{2}}} \exp\left(-\frac{(\xi - u_{1,H0})^2 + \eta^2 + \chi^2}{T_{H0}}\right), \tag{3.13}$$

$$f^B_{H0}(x,v) = \frac{n^B_{H0}}{\pi^{\frac{3}{2}} (T_{H0})^{\frac{3}{2}}} \exp\left(-\frac{(\xi - u_{1,H0})^2 + \eta^2 + \chi^2}{T_{H0}}\right), \tag{3.14}$$

where $(n^A_{H0},\ n^B_{H0},\ T_{H0},\ u_{1,H0}) \in \mathbb{R}^{*3}_+ \times \mathbb{R}$.

The proof of Lemma 3.1 follows from ([2]).

3.4 Study of the 0 order terms.

The identification of the 0 order terms in the equation (3.4) yields

$$\xi \frac{\partial}{\partial x} f^B_{H0} = Q(f^B_{H1}, f^A_{H0}) + Q(f^B_{H0}, f^A_{H1}) + Q(f^B_{H1}, f^B_{H0}) + Q(f^B_{H0}, f^B_{H1}).$$

By integrating this equation on $\mathbb{R}^3_v$, it follows

$$\frac{\partial}{\partial x} \int_{\mathbb{R}^3_v} \xi f^B_{H0}(x,v) dv = 0.$$

But the boundary conditions for f^B being of diffuse reflection type, the total flux at each point of the boundary is zero. So

$$n^B_{H0}(x) u_{1,H0}(x) = 0, \quad x \in [-1,1]. \tag{3.15}$$

Among all the situations represented by (3.15) the following two cases are considered

$$u_{1,H0} \equiv 0 \text{ and } n^B_{H0} \neq 0 \quad \text{and} \quad n^B_{H0} \equiv 0 \text{ and } u^A_{1,H0} \neq 0. \tag{3.16}$$

These two situations are interesting because of the fluid equations that they give. In this paper only the first case ($u_{1,H1} \equiv 0$) is considered.

3.5 Fluid equations at zero order.

The identification of the 0 order terms in the equation (3.6) yields

$$\xi \frac{\partial}{\partial x} f_{H0} = Q(f_{H1}, f_{H0}) + Q(f_{H0}, f_{H1}), \tag{3.17}$$

Multiply (3.17) by ξ and integrate on $\mathbb{R}^3_v$ leads to

$$\frac{\partial}{\partial x}(n_{H0}T_{H0}) = \frac{\partial}{\partial x} p_{H0} = 0. \tag{3.18}$$

3.6 Decomposition of f_{H1}, f^A_{H1} and f^B_{H1}.

f_{H1} is split into its hydrodynamical and non hydrodynamical parts as follows

$$f_{H1} = f_{H0}\left(\frac{n_{H1}}{n_{H0}} + \frac{2u_{1,H1}}{T_{H0}}\xi + (\frac{v^2}{T_{H0}} - \frac{3}{2})\frac{T_{H1}}{T_{H0}} + \psi_{H1}\right)$$

with ψ_{H1} satisfying the orthogonality conditions

$$\int_{\mathbb{R}^3_v} f_{H0}\psi_{H1} dv = 0, \qquad \int_{\mathbb{R}^3_v} \xi f_{H0}\psi_{H1} dv = 0, \qquad \int_{\mathbb{R}^3_v} v^2 f_{H0}\psi_{H1} dv = 0.$$

According to ([19]) ψ_{H1} is solution to

$$\mathcal{L}_{T_{H0}}(\psi_{H1}(\tilde{v})) = \tilde{\xi}(\tilde{v}^2 - \frac{5}{2})\frac{1}{p_{H0}}\frac{\partial}{\partial x}T_{H0}. \tag{3.19}$$

where

$$\begin{aligned}\mathcal{L}_{T_{H0}}(\psi_{H1}(\tilde{v})) := \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2} E(\tilde{v_*})\Big(&\psi_{H1}(x,v') + \psi_{H1}(x,v'_*) - \psi_{H1}(x,v) \\ &-\psi_{H1}(x,v_*)\Big)\frac{B\left(|\tilde{v_*} - \tilde{v}|\sqrt{T_{H0}}, \langle \tilde{v_*} - \tilde{v}, \omega\rangle\sqrt{T_{H0}}\right)}{\sqrt{T_{H0}}} d\omega d\tilde{v_*}\end{aligned}$$

is called linearized Boltzmann operator.
Let $\xi A(|v|)$ be solution to ([12, 19])

$$\mathcal{L}_{T_{H0}}(\tilde{\xi}A(|\tilde{v}|)) = -\tilde{\xi}(\tilde{v}^2 - \frac{5}{2}), \qquad \int_0^{+\infty} r^4 A(r)E(r)dr = 0. \tag{3.20}$$

The non hydrodynamical part $f_{H0}\psi_{H1}$ of f_{H0} is then given by the expression

$$\psi_{H1}(\tilde{v}) = \frac{-\tilde{\xi}A(|\tilde{v}|)}{p_{H0}}\frac{\partial}{\partial x}T_{H0}.$$

Finally f_{H1} writes

$$f_{H1} = \left(\frac{n_{H1}}{n_{H0}} + \frac{2u_{1,H1}}{T_{H0}}\xi + (\frac{v^2}{T_{H0}} - \frac{3}{2})\frac{T_{H1}}{T_{H0}} - \frac{\tilde{\xi}A(|\tilde{v}|)}{p_{H0}}\frac{\partial}{\partial x}T_{H0}\right) f_{H0}. \tag{3.21}$$

Now let us determin (f^A_{H1}, f^B_{H1}). The identification of the 0 order terms in (3.3) and (3.4) gives the system

$$\xi\frac{\partial}{\partial x} f^A_{H0} = Q(f^A_{H0}, f_{H1}) + Q(f^A_{H1}, f_{H0}) \tag{3.22}$$

$$\xi\frac{\partial}{\partial x} f^B_{H0} = Q(f^B_{H0}, f_{H1}) + Q(f^B_{H1}, f_{H0}). \tag{3.23}$$

From ([2]) the kernel of the mapping

$$\lambda : (\phi_A, \phi_B) \mapsto \big(Q(\phi f_{H0}, f^A_{H0}) + Q(f_{H0}, \phi_A f^A_{H0}), Q(\phi f_{H0}, f^B_{H0}) + Q(f_{H0}, \phi_B f^B_{H0})\big) \tag{3.24}$$

is $\ker\lambda = \big\{\big(\alpha^A + \beta\xi + \gamma v^2, \alpha^B + \beta\xi + \gamma v^2\big), \quad (\alpha^A, \alpha^B, \beta, \gamma) \in \mathbb{R}^2_+ \times \mathbb{R}^2\big\}$.
(f^A_{H1}, f^B_{H1}) is split into its hydrodynamical part and its non hydrodynamical part as

$$f^A_{H1} = f^A_{H0}\left(\frac{p^A_{H1}}{p^A_{H0}} + 2\xi\frac{u_{1,H1}}{T_{H0}} + (\frac{v^2}{T_{H0}} - \frac{5}{2})\frac{T_{H1}}{T_{H0}} + \Psi_A\right) \tag{3.25}$$

$$f^B_{H1} = f^B_{H0}\left(\frac{p^B_{H1}}{p^B_{H0}} + 2\xi\frac{u_{1,H1}}{T_{H0}} + (\frac{v^2}{T_{H0}} - \frac{5}{2})\frac{T_{H1}}{T_{H0}} + \Psi_B\right), \tag{3.26}$$

where $(\Psi_A; \Psi_B) \in (Ker\lambda)^{\perp}$ has the expression

$$\Psi_A = -\frac{\tilde{\xi}A(|\tilde{v}|)}{p_{H0}}\frac{\partial}{\partial x}T_{H0} - \frac{\tilde{\xi}C(\tilde{v})}{n_{H0}p^A_{H0}}\frac{\partial}{\partial x}p^A_{H0}, \ \Psi_B = -\frac{\tilde{\xi}A(|\tilde{v}|)}{p_{H0}}\frac{\partial}{\partial x}T_{H0} - \frac{\tilde{\xi}C(\tilde{v})}{n_{H0}p^B_{H0}}\frac{\partial}{\partial x}p^B_{H0}$$

and C is a solution to the equation ([24, 27])

$$Q(E(\tilde{v}), E(\tilde{v})\tilde{\xi}C(\tilde{v})) = -\tilde{\xi}E(\tilde{v}).$$

3.7 First order fluid equations.

In this subsection we derive a fluid system mixing 0 order and first order terms and we solve it when the boundary conditions are close to each other.

Theorem 3.1. *The macroscopic quantities n_{H0}, $u^A_{1,H1}$, $u^B_{1,H1}$, p^A_{H0}, p^B_{H0}, T_{H0} satisfy the following fluid system*

$$\frac{\partial}{\partial x}p_{H0} = 0, \tag{3.27}$$

$$\frac{\partial}{\partial x}(n_{H0}u_{1,H1}) = 0, \tag{3.28}$$

$$\frac{\gamma_2}{2}\frac{\partial}{\partial x}\big(\frac{\partial}{\partial x}(T_{H0})T^{\frac{1}{2}}_{H0}\big) = -n_{H0}u_{1,H1}\frac{\partial}{\partial x}T_{H0}, \tag{3.29}$$

$$u_{1,H1} = -\gamma_c\frac{T^{\frac{1}{2}}_{H0}}{p^B_{H0}n_{H0}}\frac{\partial}{\partial x}p^A_{H0}, \tag{3.30}$$

$$u^B_{1,H1} = 0, \tag{3.31}$$

where $p_{H0} = n_{H0}T_{H0}$, $p^A_{H0} = n^A_{H0}T_{H0}$ and $p^B_{H0} = n^B_{H0}T_{H0}$.
Moreover, this system can be solved as follows
There are τ_0 and $\lambda > 0$ such that for all $\tau \in \mathbb{R}$ satisfying $|\tau| \leq \tau_0$ and all $m \geq 0$, the system (3.27, 3.28, 3.29, 3.30, 3.31) has a unique solution satisfying the the boundary conditions

$$n^A_{H0}(-1) = 1, \quad T_{H0}(-1) = 1, \quad n^A_{H0}(1) = 1 + \tau, \quad |T_{H0}(1) - 1| \leq \lambda\tau, \tag{3.32}$$

and the constraint 4.48. Moreover there is $\lambda > 0$ such that (for all $x \in [-1, 1]$)

$$\begin{aligned} &|T_{H0}(x) - 1| \leq \lambda\tau, \quad |n^A_{H0}(x) - 1| \leq \lambda\tau, \quad |u_{1,H1}(x)| \leq \lambda\tau, \\ &|(T_{H0})'(x)| \leq \lambda\tau, \quad |(n^A_{H0})'(x)| \leq \lambda\tau, \quad |(n^B_{H0})'(x)| \leq \lambda\tau. \end{aligned} \tag{3.33}$$

Remark 1. *When the Knudsen number tends to 0, the flow $u_{1,H}$ tends also to 0 ($u_{1,H0} \equiv 0$). At the level of the fluid mechanic if T_{H0} satisfies the Fourier law, the right-hand side of the equation (3.29) should be 0. But it is not the case because the right-hand side of (3.29) is*

$$n_{H0}u_{1,H1} = -\gamma_c\frac{T^{\frac{1}{2}}_{H0}}{p^B_{H0}}\frac{\partial}{\partial x}p^A_{H0} \neq 0.$$

That means that the flow $u_{1,H}$ keeps an influence on the 0 order term of the temperature at the limit. This points out the ghost effect as defined in ([21, 8]).

Proof. Derivation of the system (3.27, 3.28, 3.29, 3.30, 3.31).
By considering the terms of order 1 and by integrating (3.6) with respect to 1, ξ and v^2 on $\mathbb{R}^3_v$ we get the following equations

$$\frac{\partial}{\partial x}\left(\int_{\mathbb{R}^3_v} \xi f_{H1} dv\right) = 0, \quad \frac{\partial}{\partial x}\left(\int_{\mathbb{R}^3_v} \xi^2 f_{H1} dv\right) = 0, \quad \frac{\partial}{\partial x}\left(\int_{\mathbb{R}^3_v} \xi v^2 f_{H1} dv\right) = 0.$$

The first equation can be written by using the relation (3.8)

$$\frac{\partial}{\partial x}(n_{H0} u_{1,H1}) = 0.$$

According to ([19]) by setting

$$\gamma_2 = \frac{16}{15\pi^{\frac{1}{2}}} \int_{\mathbb{R}_+} r^6 A(r) \exp(-r^2) dr, \tag{3.34}$$

the third equation writes

$$n_{H0} u_{1,H1} \frac{\partial}{\partial x} T_{H0} = \frac{\gamma_2}{2} \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(T_{H0}) T_{H0}^{\frac{1}{2}}\right). \tag{3.35}$$

Moreover, multiply (3.25) by ξ, integrate on $\mathbb{R}^3_v$, use that $u^B_{1,H1} \equiv 0$ leads to

$$u_{1,H1} = -\gamma_c \frac{\sqrt{T_{H0}}}{p^B_{H0} n_{H0}} \frac{\partial}{\partial x} p^A_{H0}, \quad \text{with} \quad \gamma_c = \frac{4}{3}\int_{\mathbb{R}_+} C(r) r^4 exp(-r^2) dr. \tag{3.36}$$

Resolution of the system (3.27, 3.28, 3.29, 3.30, 3.31).
The system (3.27, 3.28, 3.29, 3.30, 3.31) is first solved for the boundary conditions $n^A_{H0}(-1) = 1$, $n^A_{H0}(1) = 1$, $T_{H0}(-1) = 1$, $T_{H0}(1) = 1$ and the constraint on n^B_{H0} (4.48). For this system, $\left(T_{H0}, n^A_{H0}, u_{H1}, n^B_{H0}\right) = \left(1, 1, 0, \frac{m}{2}\right)$ is a constant solution. Next a solution to the system is researched for the boundary conditions (3.32) and the constraint on the mass (4.48) as a perturbation of this constant solution.
First let us determine T_{H0}. From (3.28) there is a constant θ such that

$$n_{H0} u_{1,H1} = \theta. \tag{3.37}$$

So the equation (3.29) can be written by performing the change of unknown $T = T_{H0} - 1$,

$$\theta T' = \frac{\gamma_2}{2}\left(T''(1+T)^{\frac{1}{2}} + \frac{(T')^2}{(1+T)^{\frac{1}{2}}}\right).$$

Denote $c = T'(-1)$. T is the solution to the Cauchy problem

$$\begin{aligned} T'' &= \frac{2\theta}{\gamma_2}\frac{T'}{(1+T)^{\frac{1}{2}}} - \frac{(T')^2}{2(1+T)}. \\ T(-1) &= 0, \\ T'(-1) &= c. \end{aligned} \tag{3.38}$$

T satisfies the relation

$$T' = \frac{\frac{2\theta}{\gamma_2} T + c}{(1+T)^{\frac{1}{2}}}. \tag{3.39}$$

The Cauchy-Lipschitz Theorem garantees that the solution T to the Cauchy problem (3.38) is global on $[-1, 1]$.
A condition on c is now researched in order to get for all $x \in [-1, 1]$, $T(x) \leq \tau$. For $\frac{2\theta}{\gamma_2} > 0$,

$$\int_{-1}^{x} \frac{T'(s)}{\frac{2\theta}{\gamma_2} T(s) + c} ds \leq 2.$$

So for all $x \in [-1,1]$, $T(x) \leq c\dfrac{2\theta}{\gamma_2}(\exp(\dfrac{4\theta}{\gamma_2})-1) \leq \tau$ and by choosing c such that $0 < c \leq \dfrac{\tau\frac{2\theta}{\gamma_2}}{\exp(\frac{4\theta}{\gamma_2})-1}$, it holds that $T \leq \tau$. Another condition is next researched on c in order to get for all $x \in [-1,1]$, $T'(x) \leq \tau$. Divide (3.38) by T' and integrate on $[-1,x]$ leads to

$$T'(x) = c\exp\left(\frac{2\theta}{\gamma_2}\int_{-1}^{x}\frac{ds}{(1+T)} - \frac{1}{2}\int_{-1}^{x}\frac{T'}{(1+T)}ds\right). \tag{3.40}$$

As $\frac{2\theta}{\gamma_2} > 0$ (3.39) implies that for all $x \in [-1,1]$, $T'(x) > 0$. Moreover as

$$\int_{-1}^{x}\frac{ds}{(1+T)} \leq 2$$

and by choosing c such that

$$0 < c < \tau\exp(-\frac{4\theta}{\gamma_2}),$$

it holds that $T' \leq \tau$. The case $\dfrac{2\theta}{\gamma_2} < 0$ is similar.
n^A_{H0} is next determined. From the equation (3.27) there is a constant α such that

$$(n^A_{H0} + n^B_{H0})T_{H0} = \alpha. \tag{3.41}$$

So as for all $x \in [-1,1]$, $T_{H0}(x) \neq 0$,

$$n^B_{H0} = \left(\frac{\alpha}{T_{H0}} - n^A_{H0}\right). \tag{3.42}$$

The equation (3.30) implies that

$$\theta = -\frac{5}{2}\gamma_c\frac{(p^A_{H0})'}{\sqrt{T_{H0}}n^B_{H0}}. \tag{3.43}$$

Then by using (3.41),

$$(n^A_{H0})' + n^A_{H0}\frac{(T'_{H0} - \frac{2}{5\gamma_c}\theta\sqrt{T_{H0}})}{T_{H0}} + \frac{2\theta\alpha}{5\gamma_c(T_{H0})^{\frac{3}{2}}} = 0.$$

The solution to this equation with the boundary condition $n^A_{H0}(-1) = 1$ is

$$n^A_{H0}(x) = 1 - \frac{2\theta\alpha}{5}\int_{-1}^{x}\frac{1}{(T_{H0})^{\frac{3}{2}}}\exp\left(-\int_{y}^{x}(\frac{T'_{H0} - \frac{2}{5}T_{H0}\theta}{T_{H0}})ds)\right)dy.$$

The condition $n^A_{H0}(1) = 1+\tau$ gives the following relation between α and θ

$$\alpha = -\frac{\tau}{\frac{2\theta}{5}\int_{-1}^{1}\frac{1}{(T_{H0})^{\frac{3}{2}}}\exp(-\int_{y}^{1}(\frac{T'_{H0}-\frac{2}{5}T_{H0}\theta}{T_{H0}})(s)ds)dy}. \tag{3.44}$$

(4.48) and (3.42) provide another relation between α and θ,

$$\alpha = \frac{m+2}{\int_{-1}^{1}\frac{dx}{T_{H0}} + \frac{2\theta}{5}\int_{-1}^{1}\int_{-1}^{x}\frac{1}{(T_{H0})^{\frac{3}{2}}}\exp(\int_{y}^{x}\frac{T'_{H0}-\frac{2}{5}T_{H0}\theta}{T_{H0}}ds)dydx}. \tag{3.45}$$

So α and θ are determined.
An estimate on θ is next researched by supposing that $\theta > 0$, the case $\theta < 0$ being analogous. The relation (3.39) evaluated for $x = 1$ and $T' \leq \tau$ lead to $T'(1) = \frac{2\theta}{\gamma_2}\left(\frac{T(1)+c}{2+\tau}\right) \leq \tau$. So

$$0 \leq \frac{2\theta}{\gamma_2} \leq 2\tau. \tag{3.46}$$

From the estimate (3.46) applied to the equation (3.43), there is $\tilde{k}_1 \in \mathbb{R}_+$ such that $|(n^A_{H0})'| \leq \tilde{k}_1 \tau$. By differentiating (3.42) there is $\tilde{k}_2 \in \mathbb{R}_+$ such that $|(n^B_{H0})'| \leq \tilde{k}_2 \tau$. Finally from (3.30) there is $c_1 \in \mathbb{R}_+$ such that $|u_{1,H1}| \leq c_1 \tau$. □

4 Study of the boundary conditions.

In this section we show that f^A_{H0} and f^B_{H0} satisfy the boundary conditions (2.3, 2.4, 2.5). For the Hilbert terms f^A_{H1}, f^B_{H1}, f^A_{H2}, f^B_{H2}, Knudsen layers must be added at each boundary and these layers are solutions to Milne problems.

4.1 Closure of the system at the 0 order.

Recalling that the boundary conditions for f^A are

$$f^A(-1, v) = M_-(v),\ \xi > 0 \quad f^A(1, v) = M_+(v), \xi < 0$$

we restrict ourself to the situation where

$$\frac{n_{II}}{n_I} = 1 + \tau, \tag{4.47}$$

with τ small enough to be determined. From (2.6) the following constraint on the mass of the B component

$$\int_{-1}^{1} n^B_{H0} dx = m \tag{4.48}$$

is imposed, m being a fixed non negative constant. As $n^A_{H0}(-1) = 1$, $n^A_{H0}(1) = \frac{n_{II}}{n_I}$, $T^A_{H0}(-1) = 1$ and $T^A_{H0}(1) = \frac{T_{II}}{T_I}$, f^A_{H0} satisfies 2.3. For f^B_{H0}, since

$$\int_{\xi<0} \frac{1}{(\pi T_{H0}(-1))^{\frac{3}{2}}} \exp(-\frac{v^2}{T_{H0}(-1)})dv = 1,$$

it holds that for $\xi > 0$,

$$\left(\int_{\xi<0} |\xi| f^B_{H0}(-1, v) dv \right) \exp(-\frac{v^2}{T_{H0}(-1)}) = f^B_{H0}(-1, v).$$

The same result being also satisfied in 1, the boundary conditions for f^B_{H0} are of diffuse reflection type. Hence f^A_{H0} and f^B_{H0} satisfy the boundary conditions (2.3, 2.4, 2.5).

4.2 Knudsen layer at first and second orders.

f^A_{H1} and f^B_{H1} defined in (3.25) and (3.26) cannot satisfy the boundary conditions $f^A_{H1}(-1, v) = f^A_{H1}(1, v) = 0$ and $f^B_{H1}(-1, v) = f^B_{H1}(1, v) = 0$. Then Knudsen terms must be added at each boundary. By setting $x' = \frac{1+x}{\varepsilon}$, $x'' = \frac{1-x}{\varepsilon}$, f_1, f^A_1 and f^B_1 are written as follows

$$\begin{aligned} f_1(x, v) &= f_{H1}(x, v) + f^-_{K1}(x', v) + f^+_{K1}(x'', v), &(4.49)\\ f^A_1(x, v) &= f^A_{H1}(x, v) + f^{A-}_{K1}(x', v) + f^{A+}_{K1}(x'', v), &(4.50)\\ f^B_1(x, v) &= f^B_{H1}(x, v) + f^{B-}_{K1}(x', v) + f^{B+}_{K1}(x'', v). &(4.51)\end{aligned}$$

From here denote $\tilde{M} = \frac{1}{n^A_{H0}} f^A_{H0}$ i.e

$$\tilde{M} = \frac{1}{(\pi T_{H0})^{\frac{3}{2}}} \exp(-\frac{v^2}{T_{H0}}), \quad M^A = n^A_{H0} \tilde{M} \quad \text{and} \quad M^B = n^B_{H0} \tilde{M}.$$

Consider as in ([2]), the space $\mathcal{H}$ with the scalar product

$$\begin{aligned}\langle f, g \rangle &= \langle (f^A, f^B) ; (g^A, g^B) \rangle \\ &= n^A_{H0} \int_{\mathbb{R}^3} f^A(v) g^A(v) \tilde{M}(v) dv + n^B_{H0} \int_{\mathbb{R}^3} f^B(v) g^B(v) \tilde{M}(v) dv\end{aligned}$$

is introduced. Denote by $\| \, \|_{\mathcal{H}}$ the associated norm.

Proposition 1. *There are boundary conditions in $x=-1$ for the first order Hibert terms (f^A_{H1}, f^B_{H1}) defined by (3.25, 3.26) and Knudsen terms ($f^{A-}_{K1}(x',v)$, $f^{B-}_{K1}(x',v)$) solutions to*

$$\xi\frac{\partial}{\partial x'}f^{A-}_{K1}(x',v)=Q(M^A(-1,v),f^{-}_{K1}(x',v))+Q(f^{A-}_{K1}(x',v),M(-1,v)),$$
$$\xi\frac{\partial}{\partial x'}f^{B-}_{K1}(x',v)=Q(M^B(-1,v)f^{-}_{K1}(x',v))+Q(f^{B-}_{K1}(x',v),M(-1,v)), \tag{4.52}$$

where $M=M^A+M^B$ and $f^-_{K1}=f^{A-}_{K1}+f^{B-}_{K1}$.
Moreover the following asymptotic properties hold. f^{A-}_{K1} and f^{B-}_{K1} write as

$$f^{A-}_{K1}(x',v)=M^A(-1,v)b^{A-}_1(x',v),\quad f^{B-}_{K1}(x',v)=M^B(-1,v)b^{B-}_1(x',v),$$

where for x' tending to infinity b^{A-}_1 and b^{B-}_1 converge exponentially to 0 as

$$\|(1+|v|)^{\frac12}b^{A-}_1(x',v)\|_{\mathcal{H}}\le\exp(-\sigma x'),\quad \|(1+|v|)^{\frac12}b^{B-}_1(x',v)\|_{\mathcal{H}}\le\exp(-\sigma x'), \tag{4.53}$$

a.e $x'>0$ with $\sigma<2\nu_1$ where ν_1 is defined in (5.33).

Proof. (Proposition 1.)
We adapt here the method developped for a one component gas in [6, 5] to the situation of a two component gas. From [2] there are $\left(b^{A-}_1,\, b^{B-}_1\right)$ and $\left(d^{A-}_1,\, d^{B-}_1\right)$ unique solutions to the Milne problems

$$\begin{aligned}\xi\frac{\partial}{\partial x'}b^{A-}_1(x',v) &= \frac{1}{M^A(-1,v)}\big(Q(M^A(-1,v)M(-1,v)b^-_1(x',v))\\ &+ Q(M^A(-1,v)b^{A-}_1(x',v),M(-1,v))\big),\\ \xi\frac{\partial}{\partial x'}b^{B-}_1(x',v) &= \frac{1}{M^B(-1,v)}\big(Q(M^B(-1,v),M(-1,v)b^-_1(x',v))\\ &+ Q(M^B(-1,v)b^{B-}_1(x',v),M(-1,v))\big),\end{aligned}$$

$$b^{A-}_1(0,v)=-\Psi^A_{H1}(-1,v),\quad \xi>0,\quad b^B_1(0,v)=-\Psi^B_{H1}(-1,v),\quad \xi>0,$$
$$\int_{\mathbb{R}^3}\xi M^A(-1,v)b^{A-}_1(x',v)dv=0,\quad \int_{\mathbb{R}^3}\xi M^B(-1,v)b^{B-}_1(x',v)dv=0$$

and

$$\begin{aligned}\xi\frac{\partial}{\partial x'}d^{A-}_1(x',v) &= \frac{1}{M^A(-1,v)}\big(Q(M^A(-1,v),M(-1,v)d^-_1(x',v))\\ &+ Q(M^A(-1,v)d^{A-}_1(x',v),M(-1,v))\big),\\ \xi\frac{\partial}{\partial x'}d^{B-}_1(x',v) &= \frac{1}{M^B(-1,v)}\big(Q(M^B(-1,v),M(-1,v)d^-_1(x',v))\\ &+ Q(M^B(-1,v)d^{B-}_1(x',v),M(-1,v))\big),\end{aligned}$$

$$d^{A-}_1(0,v)=0,\ \xi>0,\quad d^{B-}_1(0,v)=0,\ \xi>0,$$
$$\int_{\mathbb{R}^3}\xi M^A(-1,v)d^{A-}_1(x',v)dv=1,\int_{\mathbb{R}^3}\xi M^B(-1,v)d^{B-}_1(x',v)dv=1,$$

with $b^-_1=b^{A-}_1+b^{B-}_1$ and $d^-_1=d^{A-}_1+d^{B-}_1$. Moreover

$$\lim_{x'\to+\infty}b^{A-}_1(x',v)=b^{A-}_{1,\infty,0}+b^-_{1,\infty,4}v^2,\qquad \lim_{x'\to+\infty}b^{B-}_1(x',v)=b^{B-}_{1,\infty,0}+b^-_{1,\infty,4}v^2,$$
$$\lim_{x'\to+\infty}d^{A-}_1(x',v)=d^{A-}_{1,\infty,0}+\frac12\xi+d^-_{1,\infty,4}v^2,\qquad \lim_{x'\to+\infty}d^{B-}_1(x',v)=d^{B-}_{1,\infty,0}+\frac12\xi+d^-_{1,\infty,4}v^2,$$

where $b^{A-}_{1,\infty,0}$, $b^{B-}_{1,\infty,0}$, $b^-_{1,\infty,4}$, $d^{A-}_{1,\infty,0}$, $d^{B-}_{1,\infty,0}$ and $d^-_{1,\infty,4}$ are constants. The boundary conditions at -1 for p^A_{H1} and T_{H1} are chosen such that

$$T_{H1}(-1)=2u_{1,H1}(-1)d^-_{1,\infty,4}+b^-_{1,\infty,4},\qquad n^A_{H1}(-1)=\frac32T_{H1}(-1)+2u_{1,H1}(-1)d^{A-}_{1,\infty,0}+b^{A-}_{1,\infty,0}.$$

So $(f_{K1}^{A-}, f_{K1}^{B-})$ defined by

$$\begin{aligned} f_{K1}^{A-}(x,v) &= \big(2u_{1,H1}(-1)(d_1^{A-}(x',v) - d_{1,\infty,0}^{A+} - \xi - d_{1,\infty,0}^{+}v^2) \\ &+ (b_1^{A-}(x',v) - b_{1,\infty,0}^{A-} - b_{1,\infty,4}^{-}v^2)\big) f_{H0}^A, \\ f_{K1}^{B-}(x,v) &= \big(2u_{1,H1}(-1)(d_1^{B-}(x',v) - d_{1,\infty,0}^{B-} - \xi - d_{1,\infty,4}^{-}v^2) \\ &+ (b_1^{B-}(x',v) - b_{1,\infty,0}^{B-} - b_{1,\infty,4}^{-}v^2)\big) f_{H0}^B, \end{aligned}$$

satisfy (4.52) and (4.53) ([2]). □

In order to satisfy the boundary conditions in $x = 1$, we proceed as in $x = -1$. $p_{H1}^A(1)$ and $T_{H1}(1)$ are chosen as

$$T_{H1}(1) = \left(\frac{T_{II}}{T_I}\right)\left(2u_{1,H1}(1)d_{1,\infty,4}^{+} + \left(\frac{T_{II}}{T_I}\right) b_{1,\infty,4}^{+}\right), \tag{4.54}$$

$$n_{H1}^A(1) = \left(\frac{n_{II}}{n_I}\right)\left(\frac{3}{2}T_{H1}(1) + 2u_{1,H1}(1)d_{1,\infty,0}^{A+} + \left(\frac{T_{II}}{T_I}\right) b_{1,\infty,0}^{A+}\right) \tag{4.55}$$

and f_{K1}^{A+}, f_{K1}^{B+} are defined by

$$\begin{aligned} f_{K1}^{A+}(x,v) &= \big(2u_{1,H1}(1)(d_1^{A-}(x'',v) - d_{1,\infty,0}^{A+} - \xi - d_{1,\infty,0}^{+}v^2) \\ &+ (b_1^{A-}(x'',v) - b_{1,\infty,0}^{A-} - b_{1,\infty,4}^{-}v^2)\big) f_{H0}^A, \\ f_{K1}^{B+}(x,v) &= \big(2u_{1,H1}(-1)(d_1^{B-}(x',v) - d_{1,\infty,0}^{B-} - \xi - d_{1,\infty,4}^{-}v^2) \\ &+ (b_1^{B-}(x'',v) - b_{1,\infty,0}^{B-} - b_{1,\infty,4}^{-}v^2)\big) f_{H0}^B. \end{aligned}$$

From here we set

$$\begin{aligned} &\gamma_{1,\varepsilon}^{A-} = f_{K1}^{A-}(\frac{2}{\varepsilon},v),\ \gamma_{1,\varepsilon}^{A+} = f_{K1}^{A+}(\frac{2}{\varepsilon},v),\ \gamma_{1,\varepsilon}^{B-} = f_{K1}^{B-}(\frac{2}{\varepsilon},v), \\ &\gamma_{1,\varepsilon}^{B+} = f_{K1}^{B+}(\frac{2}{\varepsilon},v),\ \gamma_{1,\varepsilon}^{-} = \gamma_{1,\varepsilon}^{A-} + \gamma_{1,\varepsilon}^{B-},\ \gamma_{1,\varepsilon}^{+} = \gamma_{1,\varepsilon}^{A+} + \gamma_{1,\varepsilon}^{B+}. \end{aligned} \tag{4.56}$$

As for the first order, f_{H2}, f_{H2}^A and f_{H2}^B can be defined as

$$\begin{aligned} f_{H2} = f_{H0}(c_0 + c_1\xi + c_4 v^2 + \psi_{H2}), \qquad & f_{H2}^A = f_{H0}(c_0^A + c_1\xi + c_4 v^2 + \psi_{H2} + \varphi^A), \\ & f_{H2}^B = f_{H0}(c_0^B + c_1\xi + c_4 v^2 + \psi_{H2} + \varphi^B), \end{aligned} \tag{4.57}$$

with

$$\begin{aligned} c_0 = \frac{p_{H2}}{p_{H0}} - \frac{5}{2}(\frac{T_{H2}^A}{T_{H0}} + \frac{n_{H1}}{n_{H0}}\frac{T_{H1}}{T_{H0}}) - \frac{u_{1,H1}^2}{T_{H0}}, \quad & c_1 = 2(\frac{u_{1,H2}}{T_{H0}} + \frac{n_{H1}}{n_{H0}}\frac{u_{1,H1}}{T_{H0}}), \\ & c_4 = \frac{1}{T_{H0}}(\frac{T_{H2}}{T_{H0}} + \frac{n_{H1}T_{H1}}{n_{H0}T_{H0}} + \frac{2}{3}\frac{u_{H1}^2}{T_{H0}}). \end{aligned}$$

As for the first order, Knudsen terms f_{K2}^{A-}, f_{K2}^{B-}, f_{K2}^{A+}, f_{K2}^{B+} must be added to the Hilbert terms f_{H2}^A and f_{H2}^B in order to satisfy the boundary conditions $f_2^A(-1,v) = f_2^A(1,v) = f_2^B(-1,v) = f_2^B(1,v) = 0$. The macroscopic quantities n_{H1}^A, n_{H1}^B, T_{H1}^A, T_{H1}^B, $u_{1,H1}^A$, $u_{1,H1}^B$ are solutions to a fluid system which can be solved by reasonning as for the proof of Theorem 3.1. It can be also shown that $|T_{H1}| \le c\tau$. Analogously to (4.56), set

$$\begin{aligned} &\gamma_{2,\varepsilon}^{A-} = f_{K2}^{A+}(\frac{2}{\varepsilon},v),\ \gamma_{2,\varepsilon}^{A+} = f_{K2}^{A+}(\frac{2}{\varepsilon},v),\ \gamma_{2,\varepsilon}^{B-} = f_{K2}^{B-}(\frac{2}{\varepsilon},v), \\ &\gamma_{2,\varepsilon}^{B+} = f_{K2}^{B+}(\frac{2}{\varepsilon},v),\ \gamma_{2,\varepsilon}^{-} = \gamma_{2,\varepsilon}^{A-} + \gamma_{2,\varepsilon}^{B-},\ \gamma_{2,\varepsilon}^{+} = \gamma_{2,\varepsilon}^{A+} + \gamma_{2,\varepsilon}^{B+}. \end{aligned} \tag{4.58}$$

and

$$\begin{aligned} \Delta^- M = \frac{M - M(-1,v)}{\varepsilon},\ \Delta^- M^A = \frac{M^A - M^A(-1,v)}{\varepsilon},\ \Delta^- M^B = \frac{M^B - M^B(-1,v)}{\varepsilon} \\ \Delta^+ M = \frac{M - M(1,v)}{\varepsilon},\ \Delta^+ M^A = \frac{M^A - M^A(1,v)}{\varepsilon},\ \Delta^+ M^B = \frac{M^B - M^B(1,v)}{\varepsilon}. \end{aligned}$$

5 Study of the rest term.

In this section we first show that the rest term of the Hilbert expansion is the solution of a non linear system and we consider a linearized problem. Next we have to extend the method developped in [13, 14] for a one component gas the situation of a two component gas satisfying different boundary conditions. The rest term of the expansion is then decomposed into a low and a high velocity part solutions to a system of equations.

5.1 The rest term.

In ([9]) (resp.[13, 14]), the authors solve the time dependant (resp. stationary) Boltzmann equation by splitting the distribution function into an asymptotic expansion and a rest term and by controling the rest term. In the present case, the proof developped in [13, 14] is adapted to the situation of a two component gas. The rest term $\varepsilon^3 f^A_R$ (resp. $\varepsilon^3 f^B_R$) for f^A (resp. f^B) is defined as the difference of f^A (resp. f^B) and its asymptotic expansion as

$$\begin{aligned}
f^A(x,v) &= M^A + \varepsilon\big(f^A_{H1}(x,v) + f^{A-}_{K1}(\frac{1+x}{\varepsilon},v) + f^{A+}_{K1}(\frac{1-x}{\varepsilon},v)\big)\\
&+ \varepsilon^2\big(f^A_{H2}(x,v) + f^{A-}_{K2}(\frac{1+x}{\varepsilon},v) + f^{A+}_{K2}(\frac{1-x}{\varepsilon},v)\big) + \varepsilon^3 f^A_R(x,v), \qquad (5.1)\\
f^B(x,v) &= M^B + \varepsilon\left(f^B_{H1}(x,v) + f^{B-}_{K1}(\frac{1+x}{\varepsilon},v) + f^{B+}_{K1}(\frac{1-x}{\varepsilon},v)\right)\\
&+ \varepsilon^2\big(f^B_{H2}(x,v) + f^{B-}_{K2}(\frac{1+x}{\varepsilon},v) + f^{B+}_{K2}(\frac{1-x}{\varepsilon},v)\big) + \varepsilon^3 f^B_R(x,v). \qquad (5.2)
\end{aligned}$$

By plugging the expressions (5.1, 5.2) into (1.1) and by taking (4.52, 3.22, 3.23) into account, (f^A_R, f^B_R) has to satisfy the system

$$\begin{aligned}
\xi\frac{\partial}{\partial x} f^A_R &= \frac{1}{\varepsilon}\Big(Q(M^A, f_R) + Q(f^A_R, M)\Big) + Q(f^A_1 + \varepsilon f^A_2, f_R)\\
&+ Q(f^A_R, f_1 + \varepsilon f_2) + \varepsilon^2 Q(f^A_R, f_R) + \varepsilon^3 A,\\
\xi\frac{\partial}{\partial x} f^B_R &= \frac{1}{\varepsilon}\Big(Q(M^B, f_R) + Q(f^B_R, M)\Big) + Q(f^B_1 + \varepsilon f^B_2, f_R)\\
&+ Q(f^B_R, f_1 + \varepsilon f_2) + \varepsilon^2 Q(f^B_R, f_R) + \varepsilon^3 B,
\end{aligned}$$

with $f_R = f^A_R + f^B_R$ and

$$\begin{aligned}
A &= \frac{1}{\varepsilon}\Big(-\xi\frac{\partial}{\partial x} f^A_{H2} + Q(f^A_1, f_2) + Q(f^A_2, f_1) + \varepsilon Q(f^A_2, f_2)\\
&+ Q(f^{A-}_{K2}(x',v), \Delta^+ M) + Q(\Delta^+ M^A, f^-_{K2}(x',v))\\
&+ Q(\Delta^- M, f^{A+}_{K2}(x'',v)) + Q(\Delta^- M^A, f^+_{K2}(x'',v))\\
&+ \frac{1}{\varepsilon}\Big(Q(f^{A+}_{K1}(x'',v), f^-_{K1}(x',v)) + Q(f^{A-}_{K1}(x',v), f^+_{K1}(x'',v))\Big)\Big), \qquad (5.3)
\end{aligned}$$

$$\begin{aligned}
B &= \frac{1}{\varepsilon}\Big(-\xi\frac{\partial}{\partial x} f^B_{H2} + Q(f^B_1, f_2) + Q(f^B_2, f_1) + \varepsilon Q(f^B_2, f_2)\\
&+ \frac{1}{\varepsilon}\Big(Q(f^{B-}_{K2}(x',v), \Delta^+ M) + Q(\Delta^+ M^B, f^-_{K2}(x',v))\\
&+ Q(f^{B+}_{K2}(x'',v), \Delta^- M) + Q(\Delta^- M^B, f^+_{K2}(x'',v))\Big)\Big)\\
&+ \frac{1}{\varepsilon}\Big(Q(f^{B+}_{K1}(x'',v), f^-_{K1}(x',v)) + Q(f^{B-}_{K1}(x',v), f^+_{K1}(x'',v))\Big)\Big). \qquad (5.4)
\end{aligned}$$

Recall that the quantities f_1, f_1^A, f_1^B, f_2, f_2^A, f_2^B are defined by (5.32, 4.50, 4.51). On the other hand f_R^A and f_R^B satisfy the following boundary conditions

$$f_R^A(-1,v) = -\frac{\gamma_{1,\varepsilon}^{A,-} + \varepsilon\gamma_{2,\varepsilon}^{A,-}}{\varepsilon^2}, \quad \xi > 0, \quad f_R^A(1,v) = -\frac{\gamma_{1,\varepsilon}^{A,+} + \varepsilon\gamma_{2,\varepsilon}^{A,+}}{\varepsilon^2}, \quad \xi < 0,$$
$$f_R^B(-1,v) = \alpha_{R^B}^- M_-(v) - \frac{\gamma_{1,\varepsilon}^{B,-} + \varepsilon\gamma_{2,\varepsilon}^{B,-}}{\varepsilon^2}, \quad \xi > 0,$$
$$f_R^B(1,v) = \alpha_{R^B}^+ M_+(v) - \frac{\gamma_{1,\varepsilon}^{B,+} + \varepsilon\gamma_{2,\varepsilon}^{B,+}}{\varepsilon^2}, \quad \xi < 0,$$

where $\alpha_{R^B}^-$ and $\alpha_{R^B}^+$ are given by (5.11) and (5.13). Recall that the terms $\gamma_{1,\varepsilon}^-$, $\gamma_{1,\varepsilon}^+$, $\gamma_{1,\varepsilon}^{A,-}$, $\gamma_{1,\varepsilon}^{A,+}$, $\gamma_{1,\varepsilon}^{B,-}$, $\gamma_{1,\varepsilon}^{B,+}$, $\gamma_{2,\varepsilon}^-$, $\gamma_{2,\varepsilon}^+$, $\gamma_{2,\varepsilon}^{A,-}$, $\gamma_{2,\varepsilon}^{A,+}$, $\gamma_{2,\varepsilon}^{B,-}$, $\gamma_{2,\varepsilon}^{B,+}$ are defined by (4.56, 4.58).
In order to simplify the study of f_R^B, the unknown is changed as in [13] by using the decomposition: $L^2 = \mathbb{R}M^B \oplus (\mathbb{R}M^B)^\perp$. So for all $f_R^B \in L^2$, there is $\lambda \in \mathbb{R}$ such that $f_R^B = \lambda M^B + R^B$. As in [14], the condition

$$\int_{-1}^1 \int_{\mathbb{R}^3} f_R^B dvdx = 0$$

determines

$$\lambda = -\frac{1}{m}\int R^B dxdv. \tag{5.5}$$

For all function $R(x,v)$, $I(R)$ is defined by

$$I(R) = -\frac{1}{m}\int R\, dxdv.$$

By using the change of unknown $f_R^A = R^A$, $f_R^B = I(R^B)M^B + R^B$, (R^A, R^B) solves the system

$$\begin{aligned}
\xi\frac{\partial}{\partial x}R^A &= \frac{1}{\varepsilon}\big(Q(M^A,R) + Q(R^A,M)\big) + \mathcal{N}_A(R) + \widetilde{\mathcal{N}}_{A*}(R^A,R^B) \\
&+ \varepsilon^2\Big(Q(R^A,R) + I(R^B)Q(R^A,M^B) + \varepsilon A\Big), \qquad (5.6)\\
\xi\frac{\partial}{\partial x}R^B &= \frac{1}{\varepsilon}\big(Q(M^B,R) + Q(R^B,M)\big) + \mathcal{N}_B(R,R^B) \\
&+ \varepsilon^2\Big(I(R^B)\big(Q(M^B,R) + Q(R^B,M^B)\big) + Q(R^B,R) + \varepsilon B\Big) \qquad (5.7)
\end{aligned}$$

where $R = R^A + R^B$

$$\begin{aligned}
\mathcal{N}_A(R) &= Q(f_1^A + \varepsilon f_2^A, R), \qquad (5.8)\\
\widetilde{\mathcal{N}}_{A*}(R^A,R^B) &= Q(R^A, f_1 + \varepsilon f_2) + I(R^B)Q(f_1^A + \varepsilon f_2^A, M^B), \qquad (5.9)\\
\mathcal{N}_B(R^B,R) &= Q(f_1^B + \varepsilon f_2^B, R) + Q(R^B, f_1 + \varepsilon f_2)\\
&+ I(R^B)\left[Q(f_1^B + \varepsilon f_2^B, M^B) + Q(M^B, f_1 + \varepsilon f_2) - \xi\frac{\partial}{\partial x}M^B\right]. \qquad (5.10)
\end{aligned}$$

Hence we choose

$$\alpha_{R^B}^- = I(R^B)\sqrt{\pi} \tag{5.11}$$

and the boundary conditions for R^A and R^B write

$$\begin{aligned}
R^A(-1,v) = \zeta^{A-}, \quad \xi > 0, \quad R^A(1,v) = \zeta^{A+}, \quad \xi < 0,\\
R^B(-1,v) = \zeta^{B-}, \quad \xi > 0, \quad R^B(1,v) = \beta_{R^B}M_+ + \zeta^{B+}, \quad \xi < 0,
\end{aligned} \tag{5.12}$$

with

$$\beta_{R^B} = \alpha_{R^B}^+ - \alpha_{R^B}^-\left(\frac{T_{II}}{T_I}\right)^{\frac{1}{2}}\left(\frac{n_{II}}{n_I}\right), \tag{5.13}$$

$$\zeta^{A-} = -\frac{\gamma_{1,\varepsilon}^{A-} + \varepsilon\gamma_{2,\varepsilon}^{A-}}{\varepsilon^2}, \zeta^{A+} = -\frac{\gamma_{1,\varepsilon}^{A+} + \varepsilon\gamma_{2,\varepsilon}^{B+}}{\varepsilon^2},$$
$$\zeta^{B-} = -\frac{\gamma_{1,\varepsilon}^{B-} + \varepsilon\gamma_{2,\varepsilon}^{B-}}{\varepsilon^2}, \zeta^{B+} = -\frac{\gamma_{1,\varepsilon}^{B+} + \varepsilon\gamma_{2,\varepsilon}^{B+}}{\varepsilon^2}.$$

As in ([14]), the condition $\int_{\mathbb{R}^3} \xi R^B(1,v)dv = 0$ determines $\beta_{R^B} = \int_{\xi>0} \xi R^B(1,v)dv + \int_{\xi<0} \xi\zeta^+ dv$ and so $\alpha^+_{R^B}$.

5.2 A linearized problem.

The solutions $(R^A,\ R^B)$ to the system (5.6, 5.7) are constructed as the respective limits to a sequence of iterations. First, the following linearized problems are considered

$$\xi\frac{\partial}{\partial x}R^A = \frac{1}{\varepsilon}\Big(Q(M^A,R) + Q(R^A,M)\Big) + \mathcal{N}_A(R) + \widetilde{\mathcal{N}}_{A*}(R^A,R^B) + \varepsilon^2 D^A, \tag{5.14}$$
$$\xi\frac{\partial}{\partial x}R^B = \frac{1}{\varepsilon}\left(Q(M^B,R) + Q(R^B,M)\right) + \mathcal{N}_B(R^B,R) + \varepsilon^2 D^B, \tag{5.15}$$

satisfying the boundary conditions (5.12). Recall that the quantities $\mathcal{N}_A(R)$, $\widetilde{\mathcal{N}}_{A*}(R^A,R^B)$, $\mathcal{N}_B(R^B,R)$ are defined respectively by (5.8, 5.9, 5.10). The terms R, R^A and R^B will be estimated terms of D, D^A, D^B and of the boundary conditions (5.12). The nonlinear case is next considered.

5.3 Decomposition of the rest term.

The natural way to deal with the linearized Boltzmann equation is to change the operator $f \mapsto Q(M,f)$ into the operator $f \mapsto -\frac{2}{M}Q(M, M^{-\frac{1}{2}}f)$. But when the Maxwellian is not homogeneous, this procedure produces the term $\xi M^{-\frac{1}{2}}\xi\frac{\partial}{\partial x}(M^{\frac{1}{2}}f)$ which behaves like $|v|^3 f$ and has no sign. So as in [9, 13, 14, 11], R, R^A and R^B are decomposed into a low and a high velocity part as follows

$$R = \sqrt{M}g + \sqrt{M_*}h, \quad R^A = \sqrt{M^A}g^A + \sqrt{M_*}h^A, \quad R^B = \sqrt{M^B}g^B + \sqrt{M_*}h^B, \tag{5.16}$$

where M_* is the global Maxwellian $M_*(v) = \frac{1}{(\pi T_*)^{\frac{3}{2}}}\exp(-\frac{v^2}{T_*})$, with $T_* > \sup_{x\in[-1,1]} T_{H0}(x)$. Hence there is $c > 0$ such that for all $(x,v) \in [-1,1]\times\mathbb{R}^3$, $M_* \geq cM$, $M_* \geq cM^A$, $M_* \geq cM^B$. Since $R = R^A + R^B$,

$$g = \frac{\sqrt{n^A}}{\sqrt{n}}g^A + \frac{\sqrt{n^B}}{\sqrt{n}}g^B, \quad h = h^A + h^B. \tag{5.17}$$

The following norm is considered

$$\|f\| = \left(\int_{[-1,1]\times\mathbb{R}^3}(1+|v|)f^2(x,v)dxdv\right)^{\frac{1}{2}}. \tag{5.18}$$

This norm is extended to the boundary terms h^A_-, h^A_+, h^A_- and h^A_+ depending only on the v variable. As basis for the kernel of the linearized Boltzmann operator, we take $\psi_0 = \sqrt{M}$, $\psi_1 = \xi\sqrt{M}$ and $\psi_4 = (v^2 - \frac{3}{2}T)\sqrt{M}$. g is next decomposed into its hydrodynamical part $\hat{g}+g_1$ et non hydrodynamical part $\overline{g}$. $\hat{g}$ writes

$$\hat{g} = p_0(x)\psi_0 + p_4(x)\psi_4. \tag{5.19}$$

For $\alpha \in \{A,B\}$ define

$$\psi^\alpha_0 = \sqrt{M^\alpha}, \quad \psi^\alpha_1 = \xi\sqrt{M^\alpha} \quad \text{and} \quad \psi^\alpha_4 = (v^2 - \frac{3}{2}T)\sqrt{M^\alpha}.$$

(g^A, g^B) is split into its hydrodynamical part $(\hat{g}^A + g^A_1, \hat{g}^B + g^B_1)$ and its non hydrodynamical part $(\overline{g}^A, \overline{g}^B)$. $\hat{g}^A$ and $\hat{g}^B$ are decomposed into

$$\hat{g}^A = p^A_0\psi^A_0 + p^A_4\psi^A_4, \quad \hat{g}^B = p^B_0\psi^B_0 + p^B_4\psi^B_4 \tag{5.20}$$

and

$$g_1^A = p_1^A \psi_1^A, \quad g_1^B = p_1^B \psi_1^B. \tag{5.21}$$

Remark 2. *From the expression of the kernel of the linearized Boltzmann equation for a two component gas ([2]), $p_1^A = p_1^B$ and $p_4^A = p_4^B$. These two equalities are crucial for the proof of Proposition 1.*

By uniqueness of the decomposition of g,

$$\hat{g} = \frac{\sqrt{n^A}}{\sqrt{n}}\hat{g}^A + \frac{\sqrt{n^B}}{\sqrt{n}}\hat{g}^B, \quad g_1 = \frac{\sqrt{n^A}}{\sqrt{n}}g_1^A + \frac{\sqrt{n^B}}{\sqrt{n}}g_1^B, \quad \overline{g} = \frac{\sqrt{n^A}}{\sqrt{n}}\overline{g}^A + \frac{\sqrt{n^B}}{\sqrt{n}}\overline{g}^B.$$

The couples (g^A, h^A) and (g^B, h^B) are defined as the solutions to the systems

$$\begin{aligned} \xi\frac{\partial}{\partial x}g^A + \mu^A\hat{g}^A &= \frac{1}{\varepsilon}\frac{1}{\sqrt{M^A}}(Q(\sqrt{M^A}g^A, M) + Q(M^A, \sqrt{M}g)) \\ &+ \frac{1}{\varepsilon}\chi_\gamma\sigma_A^{-1}\left(K_*^A(h) + K_*^1(h^A)\right) + L_A^1(\hat{g}^A, \hat{g}) + \tilde{L}_A^1(\hat{g}^B), \end{aligned} \tag{5.22}$$

$$\begin{aligned} \xi\frac{\partial}{\partial x}h^A + \mu^A\sigma^A(\overline{g}^A + g_1^A) &= \frac{1}{\varepsilon}\overline{\chi}_\gamma K_*^A(h) + \frac{1}{\varepsilon}(-\nu + \overline{\chi}_\gamma K_*^1)h^A \\ &+ N_{A*}(\sigma(g_1 + \overline{g}) + h) \\ &+ \widetilde{N}_*^A(\sigma^A(\overline{g}^A + g_1^A) + h^A, (\sigma^B(\overline{g}^B + g_1^B) + h^B)) \\ &+ \varepsilon^2 d^A. \end{aligned} \tag{5.23}$$

and

$$\begin{aligned} \xi\frac{\partial}{\partial x}g^B + \mu^B\hat{g}^B &= \frac{1}{\varepsilon}\frac{1}{\sqrt{M^B}}(Q(\sqrt{M^B}g^B, M) + Q(M^B, \sqrt{M}g)) \\ &+ \frac{1}{\varepsilon}\chi_\gamma\sigma_B^{-1}\left(K_*^B(h) + K_*^1(h^B)\right) + L_B^1(\hat{g}^B, \hat{g}) \end{aligned} \tag{5.24}$$

$$\begin{aligned} \xi\frac{\partial}{\partial x}h^B + \mu^B\sigma^B(\overline{g}^B + g_1^B) &= \frac{1}{\varepsilon}\overline{\chi}_\gamma K_*^B(h) + \frac{1}{\varepsilon}(-\nu + \overline{\chi}_\gamma K_*^1)h^B + N_{B*}(\sigma(\overline{g} + g_1) + h) \\ &+ \widetilde{N}_*^B(\sigma^B(\overline{g}^B + g_1^B) + h^B) + \varepsilon^2 d^B, \end{aligned} \tag{5.25}$$

where $$d^A = M_*^{-\frac{1}{2}}D^A, \quad d^B = M_*^{-\frac{1}{2}}D^B,$$

$$\chi_\gamma(v) = 1, \quad \text{for} \quad |v| \leq \gamma, \qquad \chi_\gamma(v) = 0, \quad \text{for} \quad |v| \geq \gamma, \quad \text{and} \quad \overline{\chi}_\gamma = 1 - \chi_\gamma,$$

$$K_*^A(f) = \frac{1}{\sqrt{M_*}}Q(M^A, \sqrt{M_*}f), \qquad K_*^B(f) = \frac{1}{\sqrt{M_*}}Q(M^B, \sqrt{M_*}f),$$

$$\begin{aligned} L_B^1(\hat{g}, \hat{g}^B) &= \frac{1}{\sqrt{M^B}}(Q(f_1^B + \varepsilon f_2^B, \sqrt{M}\hat{g}) + Q(\sqrt{M^B}\hat{g}^B, f_1 + \varepsilon f_2)) \\ &- \frac{1}{m}\frac{1}{\sqrt{M^B}}(\int \sqrt{M^B}\hat{g}^B dvdx)(Q(f_1^B + \varepsilon f_2^B, M^B) \\ &+ Q(M^B, f_1 + \varepsilon f_2) - \xi\frac{\partial}{\partial x}M^B), \end{aligned} \tag{5.26}$$

$$L_A^1(\hat{g}, \hat{g}^B) = \frac{1}{\sqrt{M^A}}\Big(Q(\sqrt{M^A}\hat{g}^A, f_1 + \varepsilon f_2) + Q(f_1^A + \varepsilon f_2^A, \sqrt{M}\hat{g})\Big), \tag{5.27}$$

$$\tilde{L}_A^1(\hat{g}^B) = -\frac{1}{m}\frac{1}{\sqrt{M^A}}Q(f_1^A + \varepsilon f_2^A, M^B)(\int \sqrt{M^B}\hat{g}^B dvdx), \tag{5.28}$$

$$N_{A*}(f) = \frac{1}{\sqrt{M_*}}Q(f_1^A + \varepsilon f_2^A, \sqrt{M_*}f), \quad N_{B*}(f) = \frac{1}{\sqrt{M_*}}Q(f_1^B + \varepsilon f_2^B, \sqrt{M_*}f), \tag{5.29}$$

$$\begin{aligned}
\widetilde{N}_*^A(f^A, f^B) &= \frac{1}{\sqrt{M_*}}Q(\sqrt{M_*}f^A, f_1 + \varepsilon f_2) \\
&- \frac{1}{m}\frac{1}{\sqrt{M_*}}Q(f_1^A + \varepsilon f_2^A, M^B)\int_{\mathbb{R}^3}\int_{-1}^{1}\sqrt{M_*}f^B dvdx, \qquad (5.30) \\
\widetilde{N}_*^B(f^B) &= \frac{1}{\sqrt{M_*}}Q(\sqrt{M_*}f^B, f_1 + \varepsilon f_2) \\
&- \frac{1}{m}(\int_{\mathbb{R}^3}\int_{-1}^{1}\sqrt{M_*}f^B dvdx)\Big(Q(f_1^B + \varepsilon f_2^B, M^B) \\
&+ Q(M^B, f_1 + \varepsilon f_2) - \xi\frac{\partial}{\partial x}M^B\Big) \qquad (5.31)
\end{aligned}$$

and $Q(M, \sqrt{M_*}h^A)$ is decomposed into

$$\frac{1}{\sqrt{M_*}}Q(M, \sqrt{M_*}h^A) = (-\nu + K_*^1)h^A, \tag{5.32}$$

where ν, called collision frequency is defined by

$$\nu(x, v) = \int_{\mathbb{R}^3\times\mathbb{S}^2}\langle v_* - v, \omega\rangle M(x, v_*)dv_*dx.$$

Remark 3. *In the hard-sphere case, there are two non negative constants ν_0 and ν_1 such that*

$$\nu_0(1 + |v|) \le \nu(x, v) \le \nu_1(1 + |v|). \tag{5.33}$$

Moreover g^A, h^A, g^B, h^B satisfy the boundary conditions

$$\begin{aligned}
g^A(-1, v) = 0,\ \xi > 0, \quad g^A(1, v) = 0,\ \xi < 0, \\
h^A(-1, v) = \zeta^{A-}M_*^{-\frac{1}{2}},\ \xi > 0, \quad h^A(1, v) = \zeta^{A+}M_*^{-\frac{1}{2}},\ \xi < 0 \qquad (5.34) \\
g^B(-1, v) = 0,\ \xi > 0, \quad g^B(1, v) = \beta_{g^B}M_+(v)(M^B)^{-\frac{1}{2}}(1, v),\ \xi < 0, \\
h^B(-1, v) = M_*^{-\frac{1}{2}}\zeta^{B-},\ \xi > 0, \quad h^B(1, v) = M_*^{-\frac{1}{2}}(\beta_{h^B}M_+ + \zeta^{B+}),\ \xi < 0, \qquad (5.35)
\end{aligned}$$

together with the notations [13, 14]

$$\beta_{g^B} = \int_{\xi>0}\xi\sqrt{M^B}g^B(1, v)dv, \qquad \beta_{h^B} = \int_{\xi>0}\xi\sqrt{M_*}h^B(1, v)dv + \int_{\xi<0}\xi\zeta^+dv, \tag{5.36}$$

$$\mu^A = \xi\frac{1}{2}\frac{\partial}{\partial x}(\ln(M^A)),\ \ \sigma^A = \sqrt{\frac{M^A}{M_*}},\ \ \mu^B = \xi\frac{1}{2}\frac{\partial}{\partial x}(\ln(M^B)),\ \ \sigma^B = \sqrt{\frac{M^B}{M_*}}.$$

Define also the functions h_-^A, h_+^A, h_-^B and h_+^B as follows

$$\begin{aligned}
h_-^A = M_*^{-\frac{1}{2}}\zeta^{A-},\ \ \xi > 0, \quad h_-^A = 0,\ \ \xi < 0, \quad h_+^A = M_*^{-\frac{1}{2}}\zeta^{A+},\ \ \xi < 0, \quad h_+^A = 0,\ \ \xi > 0, \\
h_-^B = M_*^{-\frac{1}{2}}\zeta^{B-},\ \ \xi > 0, \quad h_-^B = 0,\ \ \xi < 0, \quad h_+^B = M_*^{-\frac{1}{2}}\zeta^{B+},\ \ \xi < 0, \quad h_+^B = 0,\ \ \xi > 0.
\end{aligned}$$

We shall control the rest term (R^A, R^B) by using the norm

$$|f|_{r,\beta_0} = \sup_{x\in[-1,1]}\sup_{v\in\mathbb{R}^3}(1 + |v|)^r|f(x, v)|\exp(\beta_0 v^2), \tag{5.37}$$

for a suitable β_0. The same notation will be used for the functions depending only on the v variable. First, the following estimate on the solution (R^A, R^B) to the linearized problem (5.14, 5.15), with (5.12) is established.

Proposition 1. *For all $r \geq 3$, there are c, ε_0, η_0 and β_0 such that for all $\varepsilon < \varepsilon_0$ and $\eta < \eta_0$, R^A and R^B satisfy the estimates*

$$\begin{aligned} |R^A|_{r,\beta_0} + |R^B|_{r,\beta_0} &\leq c\varepsilon^{\frac{1}{2}}(|D^A|_{r-1,\beta_0} + |D^B|_{r-1,\beta_0}) \\ &+ \frac{c}{\varepsilon^2}(|\zeta^{A-}|_{r,\beta_0} + |\zeta^{B-}|_{r,\beta_0} + |\zeta^{A+}|_{r,\beta_0} + |\zeta^{B+}|_{r,\beta_0}). \end{aligned}$$

And Theorem 2.1 can be deduced

5.4 Exponential form.

In order to estimate g^A, g^B, h^A and h^B, the exponential form of the equations (5.22, 5.23, 5.24, 5.25) is used. Consider f solution to

$$\xi \frac{\partial}{\partial x} f + \frac{1}{\varepsilon}\nu f = \frac{1}{\varepsilon} G, \tag{5.38}$$

satisfying the boundary conditions

$$f(-1, v) = f_-, \quad \xi > 0, \quad f(1, v) = f_+, \quad \xi < 0. \tag{5.39}$$

From here, we shall use the following notations ([13]),

$$\begin{aligned} \phi_{x,x'} &= \int_{x'}^{x} \nu(z, v) dz, \\ U_\varepsilon G(x, v) &= \frac{1}{\varepsilon\xi}\int_{-1}^{x} G(x', v)\exp(-\frac{\phi_{x,x'}}{\varepsilon\xi})dx', \quad \xi > 0, \\ U_\varepsilon G(x, v) &= -\frac{1}{\varepsilon\xi}\int_{x}^{1} G(x', v)\exp(-\frac{\phi_{x,x'}}{\varepsilon\xi})dx', \quad \xi < 0, \end{aligned}$$

$$V_\varepsilon^- f^- = \chi_{\{\xi>0\}} f^- \exp\left(-\frac{\phi_{x,-1}}{\varepsilon\xi}\right) \qquad \text{and} \qquad V_\varepsilon^+ f^+ = \chi_{\{\xi<0\}} f^+ \exp\left(\frac{\phi_{1,x}}{\varepsilon\xi}\right).$$

From the exponential form of the equation (5.38, 5.39), its solution can be written as $f = V_\varepsilon^+ f^+ + V_\varepsilon^- f^- + U_\varepsilon G$. The equations (5.22, 5.23, 5.24, 5.25) can be written in the form (5.38). Namely (5.22) writes

$$\xi \frac{\partial}{\partial x} g^A + \frac{\nu}{\varepsilon} g^A = \frac{1}{\varepsilon}(K g^A + S^A), \tag{5.40}$$

with

$$S^A = \frac{1}{\sqrt{M^A}} Q(M^A, \sqrt{M} g) + \chi_\gamma \sigma_A^{-1}(K_*^A h + K_*^1 h^A) - \varepsilon\mu^A \hat{g}^A + \varepsilon L_A^1(\hat{g}, \hat{g}^A) + \varepsilon L_A^1(\hat{g}^B). \tag{5.41}$$

The equation (5.23) can be written

$$\xi \frac{\partial}{\partial x} h^A + \frac{1}{\varepsilon}\nu h^A = \frac{1}{\varepsilon}(\overline{\chi}_\gamma K_*^1 h^A + Z^A), \tag{5.42}$$

with
$$\begin{aligned} Z^A &= -\varepsilon\mu^A\sigma^A(\overline{g}^A + g_1^A) + \overline{\chi}_\gamma K_*^A h + \varepsilon N_{A*}(\sigma(\overline{g} + g_1) + h) \\ &+ \varepsilon\widetilde{N}_*^A(\sigma^A(\overline{g}^A + g_1^A) + h^A, \sigma^B\overline{g}^B + h^B) + \varepsilon^3 d^A. \end{aligned} \tag{5.43}$$

The equation (5.24) writes

$$\xi \frac{\partial}{\partial x} g^B + \frac{\nu}{\varepsilon} g^B = \frac{1}{\varepsilon}(K g^B + S^B), \tag{5.44}$$

with
$$S^B = \frac{1}{\sqrt{M^B}} Q(\sqrt{M} g, M^B) + \chi_\gamma \sigma_B^{-1}(K_*^B h + K_*^1 h^B) - \varepsilon\mu^B \hat{g}^B + \varepsilon L_B^1(\hat{g}, \hat{g}^B). \tag{5.45}$$

The equation (5.25) writes

$$\xi \frac{\partial}{\partial x} h^B + \frac{1}{\varepsilon}\nu h^B = \frac{1}{\varepsilon}(\overline{\chi}_\gamma K^1_* h^B + Z^B), \tag{5.46}$$

with

$$\begin{aligned} Z^B &= -\varepsilon\mu^B\sigma^B(\overline{g}^B + g_1^B) + \overline{\chi}_\gamma K^B_* h + \varepsilon N_{B*}(\sigma(\overline{g} + g_1) + h) \\ &+ \varepsilon \widetilde{N}^B_*(\sigma^B(\overline{g}^B + g_1^B) + h^B) + \varepsilon^3 d^B. \end{aligned} \tag{5.47}$$

Multiply the equation (5.22) by $\sqrt{M^A}$ and (5.24) by $\sqrt{M^B}$ and add the two obtained equations. By using the relations (5.17), it holds that g and h are solutions to the equations

$$\xi \frac{\partial}{\partial x} g + \frac{1}{\varepsilon}\nu g = \frac{1}{\varepsilon}(Kg + S), \tag{5.48}$$

with

$$\begin{aligned} S = \chi_\gamma \sigma^{-1} K_* h - \varepsilon\mu\hat{g} + \varepsilon L^1(\hat{g}^B, \hat{g}), \\ \tilde{L} = K - \nu, \end{aligned} \tag{5.49}$$

$$\begin{aligned} L^1(\hat{g}^B, \hat{g}) &= \frac{1}{\sqrt{M}}(Q(f_1 + \varepsilon f_2, \sqrt{M}\hat{g}) + Q(\sqrt{M}\hat{g}, f_1 + \varepsilon f_2)) \\ &- \frac{1}{m}\frac{1}{\sqrt{M}}\Big(\int \sqrt{M^B}\hat{g}^B dvdx\Big)\Big(Q(f_1 + \varepsilon f_2, M^B) + Q(M^B, f_1 + \varepsilon f_2) \\ &- \xi \frac{\partial}{\partial x} M^B\Big). \end{aligned} \tag{5.50}$$

By adding (3.11) and (3.12) it holds that

$$\xi \frac{\partial}{\partial x} h + \frac{1}{\varepsilon}\nu h = \frac{1}{\varepsilon}(\overline{\chi}_\gamma K_* h + Z), \tag{5.51}$$

with

$$Z = -\varepsilon\mu\sigma(\overline{g} + g_1) + \varepsilon N_*(\sigma(\overline{g} + g_1) + h, \sigma^B(\overline{g}^B + g_1^B) + h^B) + \varepsilon^3 d, \tag{5.52}$$

$$\begin{aligned} N_*(f, f^B) &= \frac{1}{\sqrt{M_*}}\Big(Q(f_1 + \varepsilon f_2, \sqrt{M_*}f) + Q(\sqrt{M_*}f, f_1 + \varepsilon f_2)\Big) \\ &- \frac{1}{m}(\int \sqrt{M_*} f^B dvdx)\Big(Q(f_1 + \varepsilon f_2, M^B) + Q(M^B, f_1 + \varepsilon f_2) - \xi\frac{\partial}{\partial x} M^B\Big). \end{aligned} \tag{5.53}$$

6 Control of the rest term.

In this section, we first control the rest term of the linearized problem in L^2 and in L^∞ norms. In [13, 14], the authors consider only a one component gas satisfying boundary conditions of diffuse-reflection types and uses at a crucial point of the control that the total flux of the solution is zero. In this paper, we are not in this situation and this difficulty is solved thanks to the structure of the kernel of the linearized Boltzmann operator for a two component gas (see remarks 2 and 4). At the end of the section, the rest term of the full nonlinear problem is obtained as a limit of a sequence of rest terms of linearized problems (Proposition 6.1) and Theorem 2.1 can be deduced.

6.1 L^2 estimates on the rest term.

Recall that the norm $\| \ \|$ had been defined in (5.18).

Lemma 6.1. *For τ defined in Theorem 3.1, the operators L^1, N_*, L^1_B, L^1_A, $\tilde{L}^1_A$, N_{A*}, N_{B*}, $\widetilde{N}^A_*$, $\widetilde{N}^B_*$ defined by (5.50, 5.53, 5.26, 5.27), 5.28, 5.29, 5.30, 5.31) satisfy the inequalities*

$$\|(1+|v|)^{-1}L^1(f,f^B)\| \leq \tau(\|f\|+\|f^B\|), \quad \|(1+|v|)^{-1}L^1_B(f,f^B)\| \leq \tau(\|f\|+\|f^B\|),$$
$$\|(1+|v|)^{-1}L^1_A(f,f^A)\| \leq \tau(\|f\|+\|f^A\|), \quad \|(1+|v|)^{-1}\tilde{L}^1_A(f^B)\| \leq \tau\|f^B\|,$$
$$\|(1+|v|)^{-1}N_*(f,f^B)\| \leq \tau(\|f\|+\|f^B\|), \quad \|(1+|v|)^{-1}N_{A*}(f)\| \leq \tau\|f\|,$$
$$\|(1+|v|)^{-1}N_{B*}(f)\| \leq \tau\|f\|, \qquad \|(1+|v|)^{-1}\widetilde{N}^A_*(f^A,f^B)\| \leq \tau(\|f^A\|+\|f^B\|),$$
$$\|(1+|v|)^{-1}\tilde{N}^B_*(f^B)\| \leq \tau\|f^B\|.$$

Proof. (First inequality of Lemma 6.1).
As for all functions (φ,ψ) such that $(1+|v|)^{\frac{1}{2}}\varphi$ and $(1+|v|)^{\frac{1}{2}}\psi \in L^2$

$$\int_{\mathbb{R}^3} \frac{|Q(\sqrt{M}\varphi,\sqrt{M}\psi)|^2}{(1+|v|)M}dv \leq \int_{\mathbb{R}^3}(1+|v|)|\varphi|^2dv \int_{\mathbb{R}^3}(1+|v|)|\psi|^2dv,$$

it holds that

$$\|(1+|v|)^{-1}L^1(f,f^B)\| \leq (\|f_1\|+\epsilon\|f_2\|)(\|f\|+\|f^B\|)+c\|\xi\frac{\partial}{\partial x}M\|\,\|f^B\|.$$

Lemma **??** gives that there is $c>0$ such that $\|\xi\frac{\partial}{\partial x}M\| \leq c\tau$. In order to estimate $\|f_1\|$, f_1 is decomposed as in (4.49). First let us show that

$$\|f_{H1}\| \leq c\tau. \tag{6.1}$$

Let $\mathcal{L}$ be defined by

$$\mathcal{L}(\phi) = Q(f_{H0}\phi, f_{H0}) + Q(f_{H0}, f_{H0}\phi)$$

From (3.17), the function ϕ_{H1} defined by $f_{H1}=\phi_{H1}f_{H0}$ is solution to the equation

$$\mathcal{L}(\phi_{H1}) = \xi\frac{\partial}{\partial x}M.$$

and the restriction of $\mathcal{L}$ to the orthogonal of its kernel is invertible and such that $|||\mathcal{L}^{-1}|||=c$. So from ([13]), there is $c>0$ such that

$$\|\mathcal{L}^{-1}(\xi\frac{\partial}{\partial x}M)\| \leq c\|\xi\frac{\partial}{\partial x}M\|.$$

So by using Theorem 3.1, the non hydrodynamical part of ϕ_{H1} denoted by ψ_{H1} satisfies
$\|\psi_{H1}\| \leq \tau$. According to (3.21), the hydrodynamical part of ϕ_{H1} equal to

$$\left(\frac{n_{H1}}{n_{H0}} + 2\frac{u_{1,H1}}{T_{H0}}\xi + (\frac{v^2}{T_{H0}} - \frac{3}{2})\frac{T_{H1}}{T_{H0}}\right).$$

By using Lemma **??**, we get for all $x\in[-1,1]$, $|\frac{u_{1,H1}}{T_{H0}}(x)| \leq c\tau$, $|\frac{n_{H1}}{n_{H0}}(x)| \leq c\tau$ and $|\frac{T_{H1}}{T_{H0}}(x)| \leq c\tau$. So $\|\phi_{H1}\| \leq c\tau$ and f_{H1} satisfies (6.1). Recall that f^-_{K1} writes

$$f^-_{K1}(x',v) = \left(2u_{1,H1}(-1)(d^-_1(x',v) - d^-_{1,\infty,0} - \xi - d^-_{1,\infty,4}v^2) + b^-_1(x',v) - b^-_{1,\infty,0} - b^-_{1,\infty,4}v^2\right)f_{H0}.$$

Let us show that there is $c>0$, such that for all $x'\in[0,\frac{2}{\varepsilon}]$ and all $v\in\mathbb{R}^3$,

$$\|f^-_{K1}(x',v)\| \leq c\tau. \tag{6.2}$$

From ([7]), together with $d_1^-(0,v)=0$ for $\xi>0$ and $\int_{\mathbb{R}^3}\xi d_1^-(0,v)dv=1$, it holds that $|d_1^-(x',v)|\leq(\nu_0-\gamma)e^{-2\gamma x'}$ and $|d^-_{1,\infty,0}|+|d^-_{1,\infty,4}|\leq 1$ for all $\gamma\in]0,\nu_0[$. As $|u_{1,H1}(-1)|\leq\tau$,

$$\|2u_{1,H1}(-1)\left(d_1^-(x',v)-d^-_{1,\infty,0}-\xi-d^-_{1,\infty,4}\right)f_{H0}\|\leq c\tau.$$

Also from ([7]) together with $b_1^-(0,v)=\psi_{H1}(0,v)$ for $\xi>0$ and $\int_{\mathbb{R}^3}\xi d_1^-(0,v)dv=0$, it comes that $|b_1^-(x',v)|\leq\tau(\nu_0-\gamma)e^{-2\gamma x'}$ and $|b^-_{1,\infty,0}|+|b^-_{1,\infty,4}|\leq\tau$, for all $\gamma\in]0,\nu_0[$. So (6.2) follows. Analogously the same estimate is obtained on f^+_{K1}.
Reasonning in the same way, we show that $\|f_2\|$ is bounded. □

Next we will focus on the control of (R^A,R^B), solution to the linearized problem (5.14, 5.15) in the norm $\|\ \|$.

Proposition 1. *There are $\varepsilon_0>0$, τ_0 and $c>0$ such that for all $\varepsilon<\varepsilon_0$ and $\tau<\tau_0$, the solutions to (5.22, 5.23, 5.24, 5.25 5.34, 5.35) satisfy the estimates*

$$\begin{aligned}\|h^A\|+\|h^B\| &\leq c\varepsilon^3(\|\frac{d^A}{(1+|v|)}\|+\|\frac{d^B}{(1+|v|)}\|)\\ &\quad+c\sqrt{\varepsilon}\big(\|h^A_-\|+\|h^A_+\|+\|h^B_-\|+\|h^B_+\|\big), \qquad (6.3)\\ \|\hat{g}^A\|+\|\hat{g}^B\| &\leq c\varepsilon(\|\frac{d^A}{(1+|v|)}\|+\|\frac{d^B}{(1+|v|)}\|)\\ &\quad+\frac{c}{\varepsilon^{\frac{3}{2}}}\big(\|h^A_-\|+\|h^A_+\|+\|h^B_-\|+\|h^B_+\|\big), \qquad (6.4)\\ \|g_1^A\|+\|g_1^B\|+\|\overline{g}^A\|+\|\overline{g}^B\| &\leq c\varepsilon^2(\|\frac{d^A}{(1+|v|)}\|+\|\frac{d^B}{(1+|v|)}\|)\\ &\quad+\frac{c}{\sqrt{\varepsilon}}\big(\|h^A_-\|+\|h^A_+\|+\|h^B_-\|+\|h^B_+\|\big). \qquad (6.5)\end{aligned}$$

Remark 4. *In the situation of a one component gas ([13, 14]) with boundary conditions of diffuse-reflection type , the terms g_1 has the same order in ε as the high velocity part h. This fact will be explained during the proof of Proposition 1 for the control of g_1^A and g_1^B. This comes from the fact that for a one component gas, the flux $\int\xi\overline{g}dv$ is zero whereas for a two component gas, $\int_{\mathbb{R}^3}\xi\overline{g}^Adv$ and $\int_{\mathbb{R}^3}\xi\overline{g}^Bdv$ are not zero.*

Proof. (Proposition 1.)
First we will obtain a bound on $\|\overline{g}^A\|$, $\|\overline{g}^B\|$, $\|g_1^A\|$, $\|g_1^B\|$, $\|\hat{g}^A\|$ and $\|\hat{g}^B\|$ in terms of $\|h^A\|$ and $\|h^B\|$ and after we will control $\|h^A\|$ and $\|h^B\|$. Let us begin by $\|\overline{g}^A\|$ and $\|\overline{g}^B\|$. Let Λ be defined by

$$\Lambda:(g^A,g^B)\mapsto(\mathcal{L}_1(g^A,g^B),\mathcal{L}_2(g^A,g^B)),$$

with

$$\begin{aligned}\mathcal{L}_1(g^A,g^B)&=\frac{1}{\sqrt{M^A}}Q(\sqrt{M^A}g^A,M)+\frac{1}{\sqrt{M^A}}Q(M^A,\sqrt{M^A}g^A+\sqrt{M^B}g^B),\\ \mathcal{L}_2(g^A,g^B)&=\frac{1}{\sqrt{M^B}}Q(\sqrt{M^B}g^B,M)+\frac{1}{\sqrt{M^B}}Q(M^B,\sqrt{M^A}g^A+\sqrt{M^B}g^B).\end{aligned}$$

Let the scalar product $\langle.,.\rangle$ be defined by

$$\langle(f^A,f^B),(g^A,g^B)\rangle=\int_{\mathbb{R}^3}f^A(v)g^A(v)dv+\int_{\mathbb{R}^3}f^B(v)g^B(v)dv.$$

Multiply (5.22) by εg^A, (5.24) by εg^B, add the two obtained equation and integrate on $[-1,1]\times\mathbb{R}^3$,

$$
\begin{aligned}
&\varepsilon(\mathcal{I}_{g^A}+\mathcal{I}_{g^B})-\int_{\mathbb{R}^3}\int_{-1}^{1}\mathcal{L}_1(g^A,g^B)g^A dvdx-\int_{\mathbb{R}^3}\int_{-1}^{1}\mathcal{L}_2(g^A,g^B)g^B dvdx\\
&\quad=\varepsilon\int_{\mathbb{R}^3}\int_{-1}^{1}\left(\mu^A(\hat g^A)^2+\mu^B(\hat g^B)^2\right)dxdv+\varepsilon\int_{\mathbb{R}^3}\int_{-1}^{1}\left(\mu^A\hat g^A\overline{g}^A+\mu^B\hat g^B\overline{g}^B\right)dxdv\\
&\quad+\int_{\mathbb{R}^3}\int_{-1}^{1}\chi_\gamma\sigma_A^{-1}(K_*^1h^A+K_*^Ah)g^A dvdx+\int_{\mathbb{R}^3}\int_{-1}^{1}\chi_\gamma\sigma_B^{-1}(K_*^1h^B+K_*^Bh)g^B dvdx\\
&\quad+\int_{\mathbb{R}^3}\int_{-1}^{1}\varepsilon(\tilde L_A^1(\hat g^B)+L_A^1(\hat g,\hat g^A))g^A dvdx+\int_{\mathbb{R}^3}\int_{-1}^{1}\varepsilon L_B^1(\hat g,\hat g^B)g^B dvdx,
\end{aligned}
$$

with for $\alpha\in\{A,B\}$,

$$
\mathcal{I}_{g^\alpha}=\int_{\mathbb{R}^3}\xi(g^\alpha(1,v))^2dv-\int_{\mathbb{R}^3}\xi(g^\alpha(-1,v))^2dv.
$$

From (5.20), it follows that $\mu^A(\hat g^A)^2$ writes as the sum of the terms

$$
\frac{1}{2}\xi\frac{\partial}{\partial x}(\ln(M^A))p_i^A(x)p_j^A(x)\psi_i^A(v)\psi_j^A(v),\quad (i,j)\in\{0,4\}^2.
$$

These functions being odd in the ξ variable $\int_{\mathbb{R}^3}\mu^A(\hat g^A)^2dv=0$. Analogously $\int_{\mathbb{R}^3}\mu^B(\hat g^B)^2dv=0$. From the expression of μ^A and μ^B and Lemma 6.1, it holds that

$$
\left|\int_{\mathbb{R}^3}\int_{-1}^{1}\left(\mu^A\hat g^A\overline{g}^A+\mu^B\hat g^B\overline{g}^B\right)dxdv\right|\le c\tau(\|\hat g^A\|\|\overline{g}^A\|+\|\hat g^B\|\|\overline{g}^B\|).
$$

Recall the spectral inequality for a two component gas ([2])

$$
\langle\Lambda(g^A,g^B),(g^A,g^B)\rangle\le-(\gamma_1\|\overline{g}^A\|^2+\gamma_1\|\overline{g}^B\|^2).
$$

Thus (6.6) becomes

$$
\begin{aligned}
&\varepsilon(\mathcal{I}_{g^A}+\mathcal{I}_{g^B})+\frac{\gamma_1}{2}(\|\overline{g}^A\|^2+\|\overline{g}^B\|^2)\\
&\quad\le c\tau\varepsilon(\|\hat g^A\|\|\overline{g}^A\|+\|\hat g^B\|\|\overline{g}^B\|)+|\int_{\mathbb{R}^3}\int_{-1}^{1}\chi_\gamma\sigma_A^{-1}(K_*^1h^A+K_*^Ah)g^A dvdx|\\
&\quad+|\int_{\mathbb{R}^3}\int_{-1}^{1}\chi_\gamma\sigma_B^{-1}(K_*^1h^B+K_*^Bh)g^B dvdx|\\
&\quad+|\int_{\mathbb{R}^3}L_A^1(\hat g,\hat g^A)+\tilde L_A^1(\hat g^B))g^A dv|+|\int_{\mathbb{R}^3}L_B^1(\hat g,\hat g^B)g^B dv|.
\end{aligned}\tag{6.6}
$$

By using Remark 2, we obtain the relation

$$
\begin{aligned}
&\int_{\mathbb{R}^3}L_A^1(\hat g,\hat g^A)+\tilde L_A^1(\hat g^B))(g_1^A+\hat g^A)dv+\int_{\mathbb{R}^3}L_B^1(\hat g,\hat g^B)(g_1^B+\hat g^B)dv\\
&\qquad=\frac{1}{m}\left(\int\sqrt{M^B}\hat g^B dvdx\right)\left(\int_{\mathbb{R}^3}g_1^B(\xi\frac{\partial}{\partial x}M^B)dv\right),
\end{aligned}
$$

So (6.6) can be simplified into

$$
\begin{aligned}
&\varepsilon(\mathcal{I}_{g^A}+\mathcal{I}_{g^B})+\frac{\gamma_1}{2}(\|\overline{g}^A\|^2+\|\overline{g}^B\|^2)\\
&\quad\le c\tau\varepsilon(\|\hat g^A\|\|\overline{g}^A\|+\|\hat g^B\|\|\overline{g}^B\|)+|\int_{\mathbb{R}^3}\int_{-1}^{1}\chi_\gamma\sigma_A^{-1}(K_*^1h^A+K_*^Ah)g^A dvdx|\\
&\quad+|\int_{\mathbb{R}^3}\int_{-1}^{1}\chi_\gamma\sigma_B^{-1}(K_*^1h^B+K_*^Bh)g^B dvdx|\\
&\quad+c\tau\varepsilon\left(\|\hat g^B\|\|g_1^B\|+(\|\hat g^A\|+\|\hat g^B\|)(\|\overline{g}^A\|+\|\overline{g}^B\|)\right).
\end{aligned}\tag{6.7}
$$

In order to deal with the terms $\tau\varepsilon \left(\|\hat{g}^B\|\|g_1^B\| + (\|\hat{g}^A\| + \|\hat{g}^B\|)(\|\overline{g}^A\| + \|\overline{g}^B\|)\right)$ and $c\tau\varepsilon(\|\hat{g}^A\|\|\overline{g}^A\| + \|\hat{g}^B\|\|\overline{g}^B\|)$, the following property is used (for all $\sigma > 0$),

$$|ab| \leq \sigma a^2 + \frac{b^2}{4\sigma}. \tag{6.8}$$

So for all $\sigma > 0$,

$$\begin{aligned}&\tau\varepsilon \left(\|\hat{g}^B\|\|g_1^B\| + (\|\hat{g}^A\| + \|\hat{g}^B\|)(\|\overline{g}^A\| + \|\overline{g}^B\|)\right)\\ &\quad\leq \sigma(\|g_1^B\|^2 + \|\overline{g}^A\|^2 + \|\overline{g}^B\|^2) + \frac{\tau^2\varepsilon^2}{4\sigma}(\|\hat{g}^A\|^2 + \|\hat{g}^B\|^2)\end{aligned}$$

and the inequality (6.7) becomes for σ small enough

$$\begin{aligned}\varepsilon(\mathcal{I}_{g^A} + \mathcal{I}_{g^B}) + \frac{\gamma_1}{2}(\|\overline{g}^A\|^2 + \|\overline{g}^B\|^2) &\leq |\int_{\mathbb{R}^3}\int_{-1}^{1} \chi_\gamma \sigma_A^{-1}(K_*^1 h^A + K_*^A h) g^A dv dx|\\ &+ |\int_{\mathbb{R}^3}\int_{-1}^{1} \chi_\gamma \sigma_B^{-1}(K_*^1 h^B + K_*^B h) g^B dv dx|\\ &+ \sigma\|g_1^B\|^2 + \frac{\tau^2\varepsilon^2}{4\sigma}(\|\hat{g}^A\|^2 + \|\hat{g}^B\|^2).\end{aligned}$$

By continuity of the operators, K_*^1, K_*^A and K_*^B, it holds that

$$\begin{aligned}\varepsilon(\mathcal{I}_{g^A} + \mathcal{I}_{g^B}) + \frac{\gamma_1}{4}(\|\overline{g}^A\|^2 + \|\overline{g}^B\|^2) &\leq c(\|h^A\| + \|h^B\|)(\|g^A\| + \|g^B\|)\\ &+ \sigma\|g_1^B\|^2 + c\tau\varepsilon^2(\|\hat{g}^A\|^2 + \|\hat{g}^B\|^2).\end{aligned} \tag{6.9}$$

From the boundary conditions (5.34) satisfied by g^A, $\mathcal{I}_{g^A} \geq 0$ and by reasonning as in [14], we get $\mathcal{I}_{g^B} \geq 0$. In order to achieve the control of $\|\overline{g}^A\|$ and $\|\overline{g}^B\|$, we need to estimate $\|g_1^A\|$, $\|g_1^B\|$, $\|\hat{g}^A\|$ and $\|\hat{g}^B\|$. Let us begin by $\|g_1^A\|$ and $\|g_1^B\|$. Recall that from subsection 5.1, we have $\int_{\mathbb{R}^3} \xi R^B(x,v)dv = 0$. By splitting R^B as in (5.16) and by using that $\int_{\mathbb{R}^3} \xi \hat{g}^B dv = 0$, it holds that

$$\int_{\mathbb{R}^3} \left(\xi^2\sqrt{M^B}p_1^B + \xi\sqrt{M^B}\overline{g}^B(x,v) + \xi\sqrt{M_*}h^B\right) dv = 0.$$

So

$$cp_1^B = -\int_{\mathbb{R}^3} \xi\sqrt{M^B}\overline{g}^B dv - \int_{\mathbb{R}^3} \xi\sqrt{M_*}h^B dv$$

and it comes that

$$\|g_1^B\| \leq c(\|\overline{g}^B\| + \|h^B\|). \tag{6.10}$$

Moreover from the expression of the kernel of the linearized Boltzmann operator for a two component gas ([2]), $p_1^A = p_1^B$. Hence

$$\|g_1^A\| \leq c(\|\overline{g}^B\| + \|h^B\|). \tag{6.11}$$

Next in order to estimate $\|\hat{g}^A\|$ and $\|\hat{g}^B\|$, multiply (5.22) by $\xi\psi_i^A$ and integrate on $[-1,1]\times\mathbb{R}^3$ yields

$$\begin{aligned}\phi_i^A(x) &= \phi_i^A(-1) - \int_{-1}^{x}\int_{\mathbb{R}^3} g^A(y,v)\left(\xi^2\frac{\partial}{\partial y}\psi_i^A(y,v)\right) dv dy\\ &+ \frac{1}{\varepsilon}\int_{-1}^{x}\int_{\mathbb{R}^3} \frac{1}{\sqrt{M_A}}\left(Q(\sqrt{M^A}\overline{g}^A, M) + Q(M^A, \sqrt{M}\overline{g})\right)\xi\psi_i^A dv dy\\ &+ \frac{1}{\varepsilon}\int_{-1}^{x}\int_{\mathbb{R}^3} \chi_\gamma\sigma_A^{-1}\left[K_*^A h + K_*^1 h^A\right]\xi\psi_i^A dv dy\\ &+ \int_{-1}^{x}\int_{\mathbb{R}^3} (L_A^1(\hat{g}, \hat{g}^A) + \tilde{L}_A^1(\hat{g}^B))\xi\psi_i^A dv dy,\end{aligned}$$

with

$$\phi_i^A(x) = \int_{-1}^{x}\int_{\mathbb{R}^3} \xi^2 g^A \psi_i^A dv dy.$$

In order to control the term $\phi_i^A(-1)$ Cauchy-Schwartz inequality is used. So

$$\begin{aligned}
|\int_{\mathbb{R}^3} \xi^2 g^A(-1,v)\psi_i^A(-1,v)dv| &\leq (\int_{\mathbb{R}^3} \xi (g^A(-1,v))^2 dv)^{\frac{1}{2}} (\int_{\mathbb{R}^3} |\xi|^3 (\psi_i^A)^2(-1,v)dv)^{\frac{1}{2}},\\
&\leq c(\int_{\mathbb{R}^3} \xi (g^A(-1,v))^2 dv)^{\frac{1}{2}}.
\end{aligned}$$

Then we get for $i \in \{0,4\}$, $\left|\phi_i^A(-1)\right| \leq \left(\mathcal{I}_{g^A}\right)^{1/2}$ and the same result holds for $\phi_i^B(-1)$. Hence for $i \in \{0,4\}$ it holds that

$$\begin{aligned}
\left|\phi_i^A(x)\right| &\leq \left(\mathcal{I}_{g^A}\right)^{1/2} + \tau\|g^A\| + \frac{c}{\varepsilon}\|\overline{g}^A\| + \frac{c}{\varepsilon}\|\overline{g}^B\| + \frac{c}{\varepsilon}\|h^A\| + \frac{c}{\varepsilon}\|h^B\|\\
&+ c\tau(\|\hat{g}^A\| + \|\hat{g}^B\|), \qquad (6.12)\\
\left|\phi_i^B(x)\right| &\leq \left(\mathcal{I}_{g^B}\right)^{1/2} + \tau\|g^B\| + \frac{c}{\varepsilon}\|\overline{g}^B\| + \frac{c}{\varepsilon}\|\overline{g}^A\| + \frac{c}{\varepsilon}\|h^A\| + \frac{c}{\varepsilon}\|h^B\|\\
&+ c\tau(\|\hat{g}^A\| + \|\hat{g}^B\|). \qquad (6.13)
\end{aligned}$$

The inequalities (6.12, 6.13) give the control of the terms $\phi_i^A(x)$ and $\phi_i^B(x)$ for $i \in \{0,4\}$. By reasonning as in [13, 14] it comes that

$$\|\hat{g}^A\|^2 \leq c\int_{-1}^{1} \left(|\phi_0^A|^2 + |\phi_4^A|^2\right) dx + c\|\overline{g}^A\|^2, \quad \|\hat{g}^B\|^2 \leq c\int_{-1}^{1} \left(|\phi_0^B|^2 + |\phi_4^B|^2\right) dx + c\|\overline{g}^B\|^2$$

From (6.12) and (6.13),

$$\begin{aligned}
\|\hat{g}^A\|^2 \leq \mathcal{I}_{g^A} + c\tau\|\hat{g}^A\|^2 + \frac{c}{\varepsilon^2}\left(\|\overline{g}^A\|^2 + \|\overline{g}^B\|^2 + \|h^A\|^2 + \|h^B\|^2\right) + c\tau(\|\hat{g}^A\|^2 + \|\hat{g}^B\|^2).\\
\|\hat{g}^B\|^2 \leq \mathcal{I}_{g^B} + c\tau\|\hat{g}^B\|^2 + \frac{c}{\varepsilon^2}\left(\|\overline{g}^A\|^2 + \|\overline{g}^B\|^2 + \|h^A\|^2 + \|h^B\|^2\right) + c\tau(\|\hat{g}^A\|^2 + \|\hat{g}^B\|^2).
\end{aligned}$$

By adding the two last inequalities and by choosing ε and τ small enough,

$$\|\hat{g}^A\|^2 + \|\hat{g}^B\|^2 \leq \mathcal{I}_{g^A} + \mathcal{I}_{g^B} + \frac{c}{\varepsilon^2}\left(\|\overline{g}^A\|^2 + \|\overline{g}^B\|^2 + \|h^A\|^2 + \|h^B\|^2\right).$$

By bounding $\mathcal{I}_{g^A} + \mathcal{I}_{g^B}$ from the inequality (6.9) and by choosing ε small enough we get

$$\begin{aligned}
\|\hat{g}^A\|^2 + \|\hat{g}^B\|^2 &\leq \frac{\sigma}{\varepsilon}\|g_1^B\|^2 + \frac{c}{\varepsilon}(\|h^A\| + \|h^B\|)(\|g^A\| + \|g^B\|)\\
&+ \frac{c}{\varepsilon^2}\left(\|\overline{g}^A\|^2 + \|\overline{g}^B\|^2 + \|h^A\|^2 + \|h^B\|^2\right).
\end{aligned}$$

According to the inequality (6.10) and by splitting g^A and g^B into $g^A = g_1^A + \hat{g}^A + \overline{g}^A$ and $g^B = g_1^B + \hat{g}^B + \overline{g}^B$ it holds that

$$\begin{aligned}
\|\hat{g}^A\|^2 + \|\hat{g}^B\|^2 &\leq \frac{c}{\varepsilon}(\|h^A\| + \|h^B\|)(\|\hat{g}^A\| + \|\hat{g}^B\|)\\
&+ \frac{c}{\varepsilon}(\|h^A\| + \|h^B\|)(\|g_1^A\| + \|g_1^B\| + \|\overline{g}^A\| + \|\overline{g}^B\|)\\
&+ \frac{c}{\varepsilon^2}\left(\|\overline{g}^A\|^2 + \|\overline{g}^B\|^2 + \|h^A\|^2 + \|h^B\|^2\right).
\end{aligned}$$

Use again the inequalities (6.9, 6.10, 6.11) and choose τ small enough leads to

$$\begin{aligned}
\|\hat{g}^A\|^2 + \|\hat{g}^B\|^2 &\leq \frac{c}{\varepsilon}(\|h^A\| + \|h^B\|)(\|\hat{g}^A\| + \|\hat{g}^B\|) + c\tau(\|\overline{g}^A\|^2 + \|\overline{g}^B\|^2)\\
&+ \frac{c}{\varepsilon^2}\left(\|\overline{g}^A\|^2 + \|\overline{g}^B\|^2 + \|h^A\|^2 + \|h^B\|^2\right). \qquad (6.14)
\end{aligned}$$

Property 6.8 is again applied to the term $\frac{c}{\varepsilon}(\|h^A\| + \|h^B\|)(\|\hat{g}^A\| + \|\hat{g}^B\|)$. So

$$\|\hat{g}^A\| + \|\hat{g}^B\| \leq \frac{c}{\varepsilon}(\|\overline{g}^A\| + \|\overline{g}^B\| + \|h^A\| + \|h^B\|). \tag{6.15}$$

Now let us show that $\|\overline{g}^A\| + \|\overline{g}^B\|$ is bounded in terms of $\|h^A\|$ and $\|h^B\|$ by controling the right-hand side of (6.9). By using the inequality (6.15) it follows that

$$\begin{aligned}(\|h^A\| + \|h^B\|)(\|\hat{g}^A\| + \|\hat{g}^B\|) &\leq \frac{c}{\varepsilon}(\|h^A\| + \|h^B\|)(\|\overline{g}^A\| + \|\overline{g}^B\| + \|g_1^A\| + \|g_1^B\|)\\ &+ \frac{c}{\varepsilon}(\|h^A\| + \|h^B\|)^2.\end{aligned}$$

And from (6.10, 6.11), we get that

$$\begin{aligned}&\frac{c}{\varepsilon}(\|h^A\| + \|h^B\|)(\|\overline{g}^A\| + \|\overline{g}^B\| + \|g_1^A\| + \|g_1^B\|)\\ &\qquad \leq \frac{c}{\varepsilon}(\|h^A\| + \|h^B\|)(\|\overline{g}^A\| + \|\overline{g}^B\| + \|h^A\| + \|h^B\|).\end{aligned}$$

So by using inequality (6.8) to (6.9) it holds that

$$\|\overline{g}^A\| + \|\overline{g}^B\| \leq \frac{c}{\varepsilon}(\|h^A\| + \|h^B\|). \tag{6.16}$$

and (6.15) leads to

$$\|\hat{g}^A\| + \|\hat{g}^B\| \leq \frac{c}{\varepsilon^2}(\|h^A\| + \|h^B\|). \tag{6.17}$$

Now let us control $\|h^A\|$ and $\|h^B\|$. Multiply (5.23) by εh^A, (5.25) by ϵh^B and integrate on $\mathbb{R}^3\times[-1,1]$. By setting for $\alpha \in \{A, B\}$,

$$\mathcal{I}_{h^\alpha} = \int_{\mathbb{R}^3} \xi(h^\alpha(1,v))^2 dv - \int_{\mathbb{R}^3} \xi(h^\alpha(-1,v))^2 dv,$$

it holds that

$$\begin{aligned}\varepsilon(\mathcal{I}_{h^A} + \mathcal{I}_{h^B}) + \int_{\mathbb{R}^3}\int_{-1}^1 \nu(h^A)^2 + (h^B)^2)dxdv &= -\varepsilon \int_{\mathbb{R}^3}\int_{-1}^1 \mu^A\sigma^A(\overline{g}^A + g_1^A)h^A dvdx\\ &+ \int_{\mathbb{R}^3}\int_{-1}^1 ((\overline{\chi}_\gamma K_*^A)h)h^A dvdx + \int_{\mathbb{R}^3}\int_{-1}^1 ((\overline{\chi}_\gamma K_*^1)h^A)h^A dvdx\\ &+\varepsilon \int_{\mathbb{R}^3}\int_{-1}^1 N_{A*}(\sigma(\overline{g} + g_1) + h)h^A dvdx\\ &+\varepsilon \int_{\mathbb{R}^3}\int_{-1}^1 \widetilde{N}_*^A(\sigma^A(\overline{g}^A + g_1^A) + h^A, \sigma^B(\overline{g}^B + g_1^B) + h^B)h^A dvdx\\ &+\varepsilon \int_{\mathbb{R}^3}\int_{-1}^1 N_{B*}(\sigma(\overline{g} + g_1) + h)h^B + \widetilde{N}_*^B(\sigma^B(\overline{g}^B + g_1^B) + h^B)h^B dvdx\\ &+\varepsilon^3 \int_{\mathbb{R}^3}\int_{-1}^1 (d^A h^A + d^B h^B)dvdx.\end{aligned}$$

From (5.33) and Lemma 6.1, we get

$$\begin{aligned}\varepsilon(\mathcal{I}_{h^A} + \mathcal{I}_{h^B}) + \nu_0(\|h^A\|^2 + \|h^B\|^2) &\leq \left|\int_{\mathbb{R}^3}\int_{-1}^1 (\overline{\chi}_\gamma K_*^1 h^A)h^A dvdx\right|\\ + \left|\int_{\mathbb{R}^3}\int_{-1}^1 (\overline{\chi}_\gamma K_*^A h)h^A dvdx\right| &+ \left|\int_{\mathbb{R}^3}\int_{-1}^1 (\overline{\chi}_\gamma K_*^1 h^B)h^B dvdx\right| + \left|\int_{\mathbb{R}^3}\int_{-1}^1 (\overline{\chi}_\gamma K_*^B h)h^B dvdx\right|\\ &+c\tau\varepsilon(\|\overline{g}^A\| + \|g_1^A\| + \|h^A\| + \|\overline{g}^B\| + \|g_1^B\| + \|h^B\|)(\|h^A\| + \|h^B\|)\\ &+\varepsilon^3(\|d^A\|\,\|h^A\| + \|d^B\|\,\|h^B\|).\end{aligned}$$

By continuity of K_*^1, K_*^A and K_*^B, it holds that

$$\int_{-1}^{1}\int_{\mathbb{R}^3}(\overline{\chi}_\gamma K_*^1 h^A)hdvdx \leq \frac{\|h\|\|h^A\|}{(1+\gamma)^{\frac{1}{2}}},\ |\int_{\mathbb{R}^3}\int_{-1}^{1}(\overline{\chi}_\gamma K_*^A h)h^A dvdx| \leq \frac{\|h^A\|\|h\|}{(1+\gamma)^{\frac{1}{2}}},$$
$$|\int_{\mathbb{R}^3}\int_{-1}^{1}(\overline{\chi}_\gamma K_*^B h)h^B dvdx| \leq \frac{\|h\|\|h^A\|}{(1+\gamma)^{\frac{1}{2}}},$$

Moreover, according to the boundary conditions (5.34, 5.35) satisfied by h^A and h^B,

$$\mathcal{I}_{h^A} \geq -c(\|h_-^A\|^2 + \|h_+^A\|^2),\quad \mathcal{I}_{h^B} \geq -c\left(|\beta_{h^B}|^2 + \|h_-^B\|^2 + \|h_+^B\|^2\right).$$

Hence

$$\begin{aligned}
\|h^A\|^2 + \|h^B\|^2 \quad &\leq \quad c(\|h_-^A\|^2 + \|h_+^A\|^2 + \|h_-^B\|^2 + \|h_+^B\|^2 + (\beta_{h^B})^2)\\
&+ \quad \frac{c}{(1+\gamma)^{\frac{1}{2}}}(\|h^A\|^2 + \|h^B\|^2) + c\varepsilon(\|h_-^A\|^2 + \|h_+^A\|^2) + (\beta_{h^B})^2\\
&+ \quad c\tau\varepsilon(\|\overline{g}^A\| + \|g_1^A\| + \|\overline{g}^B\| + \|g_1^B\| + \|h^A\| + \|h^B\|)\|h^A\|\\
&+ \quad \varepsilon^3(\|h^A\|\,\|\frac{d^A}{1+|v|}\| + \|h^B\|\,\|\frac{d^B}{1+|v|}\|).
\end{aligned}$$

It remains to control $|\beta_{h^B}|$. By using the exponential form of (5.46) and by reasonning as in [14], β_{h^B} satisfies

$$|\beta_{h^B}| \leq \frac{c}{\sqrt{\varepsilon}}\left(\|\overline{\chi}_\gamma K_*^1 h^B\| + \|\nu^{-1}Z^B\|\right) + (\|h_-^B\| + \|h_+^B\|). \tag{6.18}$$

Moreover by definition of Z^B (5.47) and by using Lemma 6.1, it comes

$$\begin{aligned}
\|\nu^{-1}Z^B\| \quad &\leq \quad c\tau\varepsilon(\|\overline{g}^B\| + \|g_1^B\|) + \|\overline{\chi}_\gamma K_*^B h\|\\
&+ \quad c\tau\varepsilon(\|\overline{g}^A\| + \|g_1^A\| + \|h^A\| + \|\overline{g}^B\| + \|g_1^B\| + \|h^B\|) + \varepsilon^6\|\frac{d^B}{(1+|v|)}\|^2.
\end{aligned}$$

So (6.3) holds. From (6.15, 6.16, 6.11, 6.10) and by taking ε and τ small enough and γ big enough, the inegalities (6.4) and (6.5) follow easyly. $\square$

6.2 L^∞ estimates on the rest term.

In order to control in L^∞ of (R^A, R^B), we shall use the norms

$$|f|_r = \sup_{x\in[-1,1]}\sup_{v\in\mathbb{R}^3}(1+|v|)^r|f(x,v)|,\ r \geq 0, \qquad N(f) = \sup_{x\in[-1,1]}\left(\int_{\mathbb{R}^3}|f(x,v)|^2 dv\right)^{\frac{1}{2}}.$$

The aim of this section is to control g^A, g^B, h^A, h^B with the norm $|\quad|_r$. First, let us give the two following propositions whose the proof is in ([13]).

Proposition 2. *For all $r \geq 0$, there is a constant c such that for all function G such that $(1+|v|)^r G \in L^\infty$, U_ε satisfies the inequality*

$$|U_\varepsilon G|_r \leq c|\frac{G}{\nu}|_r.$$

Proposition 3. *For all function G such that $(1+|v|)^r G \in L^\infty$ and $\delta > 0$ and for all $r \geq 2$, there is C_δ such that*

$$N(U_\varepsilon G) \leq \frac{C_\delta}{\sqrt{\varepsilon}}\|\nu^{-\frac{1}{2}}G\| + \delta|G|_r.$$

In order to control $|g^A|_r$ and $|g^B|_r$, we need a bound on $|g|_r$.

Proposition 4. *For all $r \geq 1$, there are nonnegative constants c and H_γ such that*

$$\begin{aligned} |g|_r &\leq c(N(g^A)+N(g^B)) + H_\gamma(N(h^A)+N(h^B)) \\ &+ c\sqrt{\varepsilon}(\|\frac{d^A}{(1+|v|)}\| + \|\frac{d^B}{(1+|v|)}\|) + \frac{c}{\varepsilon^2}(\|h^A_-\| + \|h^A_+\| + \|h^B_-\| + \|h^B_+\|). \end{aligned}$$

Proof. (Proposition 4.)
From the equation (5.48) written in the exponential form,

$$g = V^+_\varepsilon(g^{B+}) + U_\varepsilon(Kg+S), \quad \text{with} \quad g^{B+} = \beta_{g^B} M_+ (M^B(1,v))^{-\frac{1}{2}}. \tag{6.19}$$

Proposition 2 applied to the equation (6.19) leads to

$$|g|_r \leq c\left|\nu^{-1}Kg\right|_r + c\left|\nu^{-1}S\right|_r + c|\beta_{g^B}|. \tag{6.20}$$

The continuity of K gives for all $r \geq 1$ ([13]),

$$\left|\nu^{-1}Kg\right|_r \leq c \sup_{x\in[-1,1]} \sup_{v\in\mathbb{R}^3} (1+|v|)^{r-1}|g(x,v)| = c|g|_{r-1}, \tag{6.21}$$

$$\left|\nu^{-1}Kg\right|_0^2 \leq c \sup_{x\in[-1,1]} \int_{\mathbb{R}^3} g^2(x,v)dv = c(N(g))^2. \tag{6.22}$$

Then by using (6.20), it holds that

$$|g|_r \leq c\,|g|_{r-1} + c\,|S|_r + c|\beta_{g^B}|. \tag{6.23}$$

So, from (6.22) and by induction, it holds that

$$|g|_r \leq cN(g) + c\sum_{k=0}^{r} |S|_k + c|\beta_{g^B}| \leq cN(g) + c|S|_r + c|\beta_{g^B}|. \tag{6.24}$$

Let us find a majoration on $|S|_r$. By definition of S (5.49),

$$|S|_r \leq \left|\chi_\gamma \sigma^{-1} K_* h\right|_r + \varepsilon(|\mu\hat{g}|_r + |L^1(\hat{g}^B,\hat{g})|_r). \tag{6.25}$$

But, by continuity of K_*,

$$\left|\chi_\gamma\sigma^{-1}K_*h\right|_r \leq \sup_{x\in[-1,1]} \sup_{v\in\mathbb{R}^3} |(1+|v|)^r \chi_\gamma \sigma^{-1}| \sup_{x\in[-1,1]} \sup_{v\in\mathbb{R}^3} |K_*h| \;\leq\; H_\gamma N(h).$$

On the other hand, according to ([9]), we have

$$\left|L^1(\hat{g}^B,\hat{g})\right|_r \leq c(|\hat{g}^B|_r + |\hat{g}|_r) \leq c\left(N(\hat{g}^A) + N(\hat{g}^B)\right).$$

Moreover the functions $(1+|v|)^r\psi_i(v)$ being bounded on $\mathbb{R}^3$ for all $i \in \{0,4\}$, it holds that

$$|\hat{g}|_r \leq c \sup_{x\in[-1,1]} (|p_0(x)| + |p_4(x)|) \leq cN(\hat{g}).$$

So by using the inequality (6.25)

$$|S|_r \leq c\varepsilon\left(N(\hat{g}^A) + N(\hat{g}^B)\right) + H_\gamma\left(N(h^A) + N(h^B)\right). \tag{6.26}$$

By using the inequality (6.26) in the right-hand side of (6.24),

$$|g|_r \leq cN(g) + c\varepsilon\left(N(\hat{g}^A) + N(\hat{g}^B)\right) + H_\gamma\left(N(h^A) + N(h^B)\right) + c|\beta_{g^B}|. \tag{6.27}$$

A bound on $N(g)$ is now researched. From Proposition 3 applied to the equation (5.48) , it holds that for all $\delta > 0$,

$$N(g) \leq \frac{C_\delta}{\sqrt{\varepsilon}}\|\nu^{-1}Kg\| + \delta|Kg|_r + \frac{C_\delta}{\sqrt{\varepsilon}}\|\nu^{-1}S\| + \delta|S|_r + c|\beta_{g^B}|. \tag{6.28}$$

But from (6.21) and (6.27), we get

$$|Kg|_r \leq cN(g) + c\varepsilon(N(\hat{g}^A) + N(\hat{g}^B)) + H_\gamma(N(h^A) + N(h^B)) + c|\beta_{g^B}|.$$

Hence by using the previous inequality in (6.28) and by choosing δ small enough, it comes that

$$N(g) \leq \frac{C_\delta}{\sqrt{\varepsilon}}\|\nu^{-1}Kg\| + c\varepsilon(N(\hat{g}^A) + N(\hat{g}^B)) + H_\gamma(N(h^A) + N(h^B)) + \frac{C_\delta}{\sqrt{\varepsilon}}\|\nu^{-1}S\| + c|\beta_{g^B}|.$$

But by continuity of K we have $\|\nu^{-1}Kg\| \leq c\|g\|$ and the definition of S (5.49) gives

$$\|\nu^{-1}S\| \leq C_\gamma\|h\| + c\tau\varepsilon(\|\hat{g}^A\| + \|\hat{g}^B\|).$$

Hence

$$\begin{aligned} N(g) &\leq \frac{C_\delta}{\sqrt{\varepsilon}}\|g\| + c\varepsilon(N(\hat{g}^A) + N(\hat{g}^B)) + H_\gamma(N(h^A) + N(h^B)) \\ &+ \frac{C_\delta}{\sqrt{\varepsilon}}\|h\| + C_\delta\tau\sqrt{\varepsilon}(\|\hat{g}^A\| + \|\hat{g}^B\|) + c|\beta_{g^B}|. \end{aligned}$$

Moreover by reasonning as in [14] and by using Proposition 1 $|\beta_{g^B}|$ is controled as follows

$$|\beta_{g^B}| \leq c\sqrt{\varepsilon}\left(\|\frac{d^A}{(1+|v|)}\| + \|\frac{d^B}{(1+|v|)}\|\right) + \frac{c}{\varepsilon^2}\left(\|h_-^A\| + \|h_+^A\| + \|h_-^B\| + \|h_+^B\|\right). \tag{6.29}$$

□

Proposition 5. *For all $r \geq 3$ there are nonegative constants c and H_γ such that*

$$\begin{aligned} \left(|g^A|_r + |g^B|_r\right) &\leq c\sqrt{\varepsilon}\left(\|\frac{d^A}{(1+|v|)}\| + \|\frac{d^B}{(1+|v|)}\|\right) + cH_\gamma(|h^A|_r + |h^B|_r) \\ &+ \frac{c}{\varepsilon^2}(|h_-^A|_r + |h_+^A|_r + |h_-^B|_r + |h_+^B|_r). \end{aligned}$$

Proof. (Proposition 5.)
We proceed as for the proof of Proposition 4. The solutions to the equations (5.40) and (5.44) are written in the exponential form as follows

$$g^A = U_\varepsilon(Kg^A + S^A), \quad g^B = V_\varepsilon^+(g^{B+}) + U_\varepsilon(Kg^B + S^B), \tag{6.30}$$

with g^{B+} defined in (6.19). Reasonning as in the proof of the inequality (6.23), we get

$$|g^A|_r \leq cN(g^A) + c|S^A|_r, \qquad |g^B|_r \leq cN(g^B) + c|S^B|_r + c|\beta_{g^B}|. \tag{6.31}$$

The definitions of S^A and S^B (5.41, 5.45) together with the inequality

$$|\frac{1}{\sqrt{M^A}}Q\left(\sqrt{M}g, M^A\right)|_r \leq c\,|g|_r \quad ([17]),$$

lead to

$$\begin{aligned} |S^A|_r &\leq c|g|_r + (|\chi_\gamma\sigma_A^{-1}K_*^Ah|_r + |\chi_\gamma\sigma_A^{-1}K_*^1h^A|_r) + \tau\varepsilon|\hat{g}^A|_r + \varepsilon\left(|L_1^A(\hat{g},\hat{g}^A)|_r + |L_A^1(\hat{g}^B)|_r\right), \\ |S^B|_r &\leq c|g|_r + (|\chi_\gamma\sigma_B^{-1}K_*^Bh|_r + |\chi_\gamma\sigma_B^{-1}K_*^1h^B|_r) + \tau\varepsilon|\hat{g}^B|_r + \varepsilon|L_B^1(\hat{g},\hat{g}^B)|_r. \end{aligned}$$

Reasonning as for the proof of the inequality (6.26), it holds that

$$\left|S^A\right|_r + \left|S^B\right|_r \leq c|g|_r + C_\gamma\left(N(h^A) + N(h^B)\right) + c\varepsilon\left(N(\hat{g}^A) + N(\hat{g}^B)\right),$$

So by bounding $|g|_r$ thanks to Proposition 4, we get

$$\begin{aligned} |S^A|_r + |S^B|_r &\leq c(N(g^A) + N(g^B)) + H_\gamma(N(h^A) + N(h^B)) \\ &+ c\sqrt{\varepsilon}(\|\frac{d^A}{(1+|v|)}\| + \|\frac{d^B}{(1+|v|)}\|) + \frac{c}{\varepsilon^2}(\|h_-^A\| + \|h_+^A\| + \|h_-^B\| + \|h_+^B\|). \end{aligned} \tag{6.32}$$

From (6.31) together with Proposition 4, it follows that

$$\begin{aligned}|g^A|_r+|g^B|_r &\leq c(N(g^A)+N(g^B))+cH_\gamma(N(h^A)+N(h^B))\\ &+ c\sqrt{\varepsilon}(\|\frac{d^A}{(1+|v|)}\|+\|\frac{d^B}{(1+|v|)}\|)+\frac{c}{\varepsilon^2}(\|h_-^A\|+\|h_+^A\|+\|h_-^B\|+\|h_+^B\|).\end{aligned} \tag{6.33}$$

In order to achieve the control of $|g^A|_r+|g^B|_r$, we need to estimate $N(g^A)+N(g^B)$.
By using (6.30) and from Proposition 3, it follows that for all $\delta>0$,

$$\begin{aligned}N(g^A)+N(g^B) &\leq \frac{C_\delta}{\sqrt{\varepsilon}}(\|\nu^{-1}Kg^A\|+\|\nu^{-1}Kg^B\|)+\delta(|Kg^A|_r+|Kg^B|_r)\\ &+ \frac{C_\delta}{\sqrt{\varepsilon}}(\|\nu^{-1}S^A\|+\|\nu^{-1}S^B\|)+\delta(|S^A|_r+|S^B|_r).\\ &+ |\beta_{g^B}|N(M_+(M^B(1,v))^{-\frac{1}{2}}),\end{aligned} \tag{6.34}$$

Moreover $|Kg^A|_r\leq c|g^A|_{r-1}\leq cN(g^A)+c|S^A|_r$, and $|Kg^B|_r\leq c|g^B|_{r-1}\leq cN(g^B)+c|S^B|_r$. From (6.29) and by choosing $\delta>0$ small enough,

$$\begin{aligned}N(g^A)+N(g^B) &\leq \frac{C_\delta}{\sqrt{\varepsilon}}(\|\nu^{-1}Kg^A\|+\|\nu^{-1}Kg^B\|+\|\nu^{-1}S^A\|+\|\nu^{-1}S^B\|)\\ &+ \delta(|S^A|_r+|S^B|_r)+H_\gamma(N(h^A)+N(h^B))\\ &+ c\sqrt{\varepsilon}(\|\frac{d^A}{(1+|v|)}\|+\|\frac{d^B}{(1+|v|)}\|)+\frac{c}{\varepsilon^2}(\|h_-^A\|+\|h_+^A\|+\|h_-^B\|+\|h_+^B\|).\end{aligned}$$

So by choosing δ small enough and by using (6.32)

$$\begin{aligned}N(g^A)+N(g^B) &\leq \frac{C_\delta}{\sqrt{\varepsilon}}(\|\nu^{-1}Kg^A\|+\|\nu^{-1}Kg^B\|+\|\nu^{-1}S^A\|+\|\nu^{-1}S^B\|)\\ &+ H_\gamma(N(h^A)+N(h^B))+c\sqrt{\varepsilon}(\|\frac{d^A}{(1+|v|)}\|+\|\frac{d^B}{(1+|v|)}\|)\\ &+ \frac{c}{\varepsilon^2}(\|h_-^A\|+\|h_+^A\|+\|h_-^B\|+\|h_+^B\|).\end{aligned} \tag{6.35}$$

By continuity of K and from Proposition 1,

$$\|Kg^A\|+\|Kg^B\|\leq c\varepsilon\big(\|\frac{d^A}{(1+|v|)}\|+\|\frac{d^B}{(1+|v|)}\|\big)+\frac{c}{\varepsilon^{\frac{3}{2}}}(\|h_-^A\|+\|h_+^A\|+\|h_-^B\|+\|h_+^B\|)$$

and by definitions of S^A and S^B (5.41, 5.45) and from Proposition 1, $\|\nu^{-1}S^A\|+\|\nu^{-1}S^B\|$ satisfies the same previous estimate as $\|Kg^A\|+\|Kg^B\|$. So (6.35) reads

$$\begin{aligned}N(g^A)+N(g^B) &\leq \delta(|g^A|_r+|g^B|_r)+\sqrt{\varepsilon}\left(\|\frac{d^A}{(1+|v|)}\|+\|\frac{d^B}{(1+|v|)}\|\right)\\ &+ \frac{c}{\varepsilon^2}(\|h_-^A\|+\|h_+^A\|+\|h_-^B\|+\|h_+^B\|)+H_\gamma(N(h^A)+N(h^B)).\end{aligned} \tag{6.36}$$

From the inequalities (6.33, 6.36),

$$\begin{aligned}|g^A|_r+|g^B|_r &\leq c\sqrt{\varepsilon}(\|\frac{d^A}{(1+|v|)}\|+\|\frac{d^B}{(1+|v|)}\|)\\ &+ \frac{c}{\varepsilon^2}(\|h_-^A\|+\|h_+^A\|+\|h_-^B\|+\|h_+^B\|)+H_\gamma(N(h^A)+N(h^B)).\end{aligned}$$

But for all f such that $(1+|v|)^r f\in L^\infty$ it holds that for $r\geq 1$,

$$[N(f)]^2\leq \sup_{x\in[-1,1]}\sup_{v\in\mathbb{R}^3}(f^2(x,v)(1+|v|)^{2r})\int_{\mathbb{R}^3}\frac{dv}{(1+|v|)^{2r}}\leq |f|_r^2.$$

Hence for all $r \geq 1$,

$$N(h^A) + N(h^B) \leq c(|h^A|_r + |h^B|_r). \tag{6.37}$$

Moreover for all function f such that $(1+|v|)f \in L^2$ and $(1+|v|)^3 f \in L^\infty$, it holds that $\|f\| \leq |f|_3$. $\square$

In order to achieve the control of $|g^A|_r + |g^B|_r$ it remains to estimate $|h^A|_r + |h^B|_r$.

Proposition 6. *For all $r \geq 3$ there is $c > 0$ such that*

$$\begin{aligned} |h^A|_r + |h^B|_r &\leq c\varepsilon^{\frac{3}{2}}(\|\frac{d^A}{(1+|v|)}\| + \|\frac{d^B}{(1+|v|)}\|) + \varepsilon^{\frac{5}{2}}(|\nu^{-1}d^A|_r + |\nu^{-1}d^B|_r) \\ &+ \frac{c}{\varepsilon^2}(|h^A_-|_r + |h^A_+|_r + |h^B_-|_r + |h^B_+|_r). \end{aligned}$$

Proof. (Proposition 6.)
h^A et h^B can be written as

$$\begin{aligned} h^A &= V_\varepsilon^-(h^A_-) + V_\varepsilon^+(h^A_+) + U_\varepsilon(\overline{\chi}_\gamma K^1_* h^A + Z^A), \\ h^B &= V_\varepsilon^-(h^B_-) + V_\varepsilon^+(h^B_+ + \beta_{h^B} M_+ (M_*)^{-\frac{1}{2}}) + U_\varepsilon(\overline{\chi}_\gamma K^1_* h^B + Z^B). \end{aligned}$$

From Proposition 2, by continuity of K^1_*, K^A_*, K^B_* and by taking $|V_\varepsilon^- h^A_-|_r \leq |h^A_-|_r$, $|V_\varepsilon^+ h^A_+|_r \leq |h^A_+|_r$, $|V_\varepsilon^- h^B_-|_r \leq |h^B_-|_r$, $|V_\varepsilon^+(h^B_+ + \beta_{h^B} M_+(M_*)^{-\frac{1}{2}})|_r \leq |h^B_+|_r + c|\beta_{h^B}|$ into account, it holds that

$$\begin{aligned} |h^A|_r + |h^B|_r &\leq \frac{c}{1+\gamma}|h^A|_r + \frac{c}{1+\gamma}|h^B|_r + \frac{c}{1+\gamma}|h|_r \\ &+ c\tau\varepsilon\big(|\overline{g}^B|_r + |g^B_1|_r + |h^B|_r + |\overline{g}^A|_r + |g^A_1|_r + |h^A|_r\big) \\ &+ \varepsilon^3(|\nu^{-1}d^A|_r + |\nu^{-1}d^B|_r) + |h^A_-|_r + |h^A_+|_r + |h^B_-|_r + |h^B_+|_r + c|\beta_{h^B}|. \end{aligned}$$

From the inequalities (6.18, 6.19) and by using Proposition 1, $|\beta_{h^B}|$ is controled as follows

$$|\beta_{h^B}| \leq c(\|h^A_-\| + \|h^A_+\| + \|h^B_-\| + \|h^B_+\|) + \varepsilon^{\frac{5}{2}} \left(\|\frac{d^A}{(1+|v|)}\| + \|\frac{d^B}{(1+|v|)}\| \right).$$

Moreover for all function f such that $\|f\|$ and $|f|_r$ are defined, $\|f\| \leq |f|_3$. So, by choosing τ and ε small enough and γ big enough in (6.38) it holds that

$$\begin{aligned} |h^A|_r + |h^B|_r &\leq \tau\varepsilon(|\overline{g}^B|_r + |g^B_1|_r + |\overline{g}^A|_r + |g^A_1|_r) \\ &+ \varepsilon^{\frac{5}{2}}(|\nu^{-1}d^A|_r + |\nu^{-1}d^B|_r) + |h^A_-|_r + |h^A_+|_r + |h^B_-|_r + |h^B_+|_r. \end{aligned}$$

In order to control the term $\tau\varepsilon(|\overline{g}^A|_r + |g^A_1|_r + |\overline{g}^B|_r + |g^B_1|_r)$, we use that $|\overline{g}^A|_r \leq |g^A|_r + |g^A_1|_r + |\hat{g}^A|_r$ and $|\overline{g}^B|_r \leq |g^B|_r + |g^B_1|_r + |\hat{g}^B|_r$, with for all $i \in \{0, 1, 4\}$, $|g^A_i|_r \leq N(g^A_i)$ and $|g^B_i|_r \leq N(g^B_i)$. So,

$$\varepsilon\tau(|\overline{g}^A|_r + |g^A_1|_r + |\overline{g}^B|_r + |g^B_1|_r) \leq \varepsilon\tau(|g^A|_r + N(g^A) + |g^B|_r + N(g^B)).$$

Proposition 5 applied to the inequality (6.36) gives

$$\begin{aligned} N(g^A) + N(g^B) &\leq c\sqrt{\varepsilon}\left(\|\frac{d^A}{(1+|v|)}\| + \|\frac{d^B}{(1+|v|)}\| \right) + H_\gamma(|h^A|_r + |h^B|_r) \\ &+ \frac{c}{\varepsilon^2}(|h^A_-|_r + |h^A_+|_r + |h^B_-|_r + |h^B_+|_r). \end{aligned}$$

Then by choosing ε and τ small enough in the inequality (6.38), Proposition 6 follows. $\square$

Proof. (Proposition 1.)
σ_A and σ_B being bounded, R^A and R^B satisfy

$$M_*^{-\frac{1}{2}}(|R^A| + |R^B|) \leq (|h^A| + c|g^A| + |h^B| + c|g^B|).$$

Recall that $M_* = \frac{1}{(\pi T_*)^{\frac{3}{2}}} \exp(-\frac{v^2}{T_*})$ with $T_* > T_{H0}$. Set $\beta_0 = \frac{1}{2T_*}$.

$$|M_*^{-\frac{1}{2}} R^A|_r + |M_*^{-\frac{1}{2}} R^B|_r \leq (|h^A|_r + c|g^A|_r + |h^B|_r + c|g^B|_r).$$

Then Propositions 5 and 6 imply that, for all $r \geq 3$,

$$\begin{aligned} |R^A|_{r,\beta_0} + |R^B|_{r,\beta_0} &\leq c\sqrt{\varepsilon}(\|d^A\| + \|d^B\|) + \varepsilon^3 \left(|\nu^{-1} d^A|_r + |\nu^{-1} d^B|_r\right) \\ &+ \frac{c}{\varepsilon^2}(|h_-^A|_r + |h_+^A|_r + |h_-^B|_r + |h_+^B|_r). \end{aligned}$$

Finally the definition of h_-^A, h_+^A, h_-^B, h_+^B, d^A, d^B and the estimates $\|d\| \leq |\frac{d}{\nu}|_3$ and (5.33) lead to the conclusion. □

6.3 Convergence of the iterative process.

This subsection deals with the rest terms (R^A, R^B) of the non linear problems, solutions to the system (5.6, 5.7). They are constructed as the limit of a sequence of iterations of linearized problems.

Theorem 6.1. *For all $r \geq 3$, there is c, c', ε_0, τ_0 and β_0 such that for all $\varepsilon < \varepsilon_0$, and $\tau < \tau_0$, the problem (5.6, 5.7) has a unique solution (R^A, R^B) satisfying*

$$|R^A|_{r,\beta_0} + |R^B|_{r,\beta_0} \leq c\left(\varepsilon^{\frac{3}{2}}(|A|_{r,\beta_0} + |B|_{r,\beta_0}) + \exp(-\frac{c'}{\varepsilon})\right).$$

Recall that the norm $| \ |_{r,\beta_0}$ is defined by the formula (5.37).

Proof. (Theorem 6.1.)
The solution (R^A, R^B) to the problem (5.6, 5.7) shall be obtained as the limit to the sequences (R_k^A, R_k^B) defined by $R_0^A = R_0^B = 0$ and for all $k \geq 1$,

$$\begin{aligned} \xi \frac{\partial}{\partial x} R_k^A &= \frac{1}{\varepsilon}(Q(R_k^A, M) + Q(M^A, R_k)) + \mathcal{N}_A(R_k) + \tilde{\mathcal{N}}_{A*}(R_k^A, R_k^B) \\ &+ \varepsilon^2 \Big(Q(R_k^B, R_{k-1}) + I(R_{k-1}^B)\, Q(R_{k-1}^A, M^B)\Big) + \varepsilon^3 A, \qquad (6.38) \\ \xi \frac{\partial}{\partial x} R_k^B &= \frac{1}{\varepsilon}(Q(R_k^B, M) + Q(M^B, R_k)) + \mathcal{N}_B(R_k^B, R_k) \\ &+ \varepsilon^2 \Big(I(R_{k-1}^B)(Q(M^B, R_{k-1}) + Q(R_{k-1}^B, M^B)) + Q(R_{k-1}^B, R_{k-1})\Big) \\ &+ \varepsilon^3 B, \qquad (6.39) \end{aligned}$$

satisfying the boundary conditions

$$\begin{aligned} R_k^A(-1, v) = \zeta^{A-}, \quad \xi > 0, \qquad R_k^A(1, v) = \zeta^{A+}, \quad \xi < 0, \\ R_k^B(-1, v) = \zeta^{B-}, \quad \xi > 0, \qquad R_k^B(1, v) = \beta_{R_k^B} M_+ + \zeta^{B+}, \quad \xi < 0. \qquad (6.40) \end{aligned}$$

From Proposition 1 applied to the equations (6.38, 6.39, 6.40),

$$\begin{aligned} |R_k^A|_{r,\beta_0} + |R_k^B|_{r,\beta_0} &\leq c\varepsilon^{\frac{1}{2}}(|D^A|_{r-1,\beta_0} + |D^B|_{r-1,\beta_0}) \\ &+ \frac{c}{\varepsilon^2}(|\zeta^{A-}|_{r,\beta_0} + |\zeta^{A+}|_{r,\beta_0} + |\zeta^{B-}|_{r,\beta_0} + |\zeta^{B+}|_{r,\beta_0}), \end{aligned}$$

with

$$\begin{aligned} D^A &= \varepsilon A + Q(R_{k-1}, R_{k-1}^A) + I(R_{k-1}^B) Q(M^B, R_{k-1}^A), \\ D^B &= \varepsilon B + I(R_{k-1}^B)\left(Q(M^B, R_{k-1}) + Q(R_{k-1}^B, M^B)\right) + Q(R_{k-1}^B, R_{k-1}). \end{aligned}$$

The inequality ([13]),

$$|M^{-\frac{1}{2}} Q(R, S)|_{r-1} \leq |M^{-\frac{1}{2}} R|_r |M^{-\frac{1}{2}} S|_r. \qquad (6.41)$$

leads to

$$|Q(R^A_{k-1},R_{k-1})+I(R^B_{k-1})Q(R^A_{k-1},M^B)|_{r-1,\beta_0}\le(|R_{k-1}|_{r,\beta_0}+|R^B_{k-1}|_{r,\beta_0})|R^A_{k-1}|_{r,\beta_0},$$
$$|I(R^B_{k-1})(Q(M^B,R_{k-1})+Q(R^B_{k-1},M^B))+Q(R_{k-1},R^B_{k-1})|_{r-1,\beta_0}$$
$$\le|R^B_{k-1}|_{r,\beta_0}|R_{k-1}|_{r,\beta_0}+|R^B_{k-1}|^2_{r,\beta_0}.$$

So

$$\begin{aligned}|D^A|_{r-1,\beta_0}&\le\varepsilon|A|_{r-1,\beta_0}+\big(|R_{k-1}|_{r,\beta_0}+|R^B_{k-1}|_{r,\beta_0}\big)\,|R^A_{k-1}|_{r,\beta_0},\\ |D^B|_{r-1,\beta_0}&\le\varepsilon|B|_{r-1,\beta_0}+\big(|R_{k-1}|_{r,\beta_0}+|R^B_{k-1}|_{r,\beta_0}\big)\,|R^B_{k-1}|_{r,\beta_0}.\end{aligned}$$

Hence for all $k\ge 0$, R^A_k and R^B_k satisfy

$$\begin{aligned}|R^A_k|_{r,\beta_0}+|R^B_k|_{r,\beta_0}&\le\varepsilon^{\frac12}|R_{k-1}|_{r,\beta_0}\big(|R^A_{k-1}|_{r,\beta_0}+|R^B_{k-1}|_{r,\beta_0}\big)\\&+\ c\varepsilon^{\frac32}\big((|A|_{r,\beta_0}+|B|_{r,\beta_0})+\frac{c}{\varepsilon^2}\exp(-\frac{c'}{\varepsilon}\big).\end{aligned}\tag{6.42}$$

Therefore we get for ε small enough, uniformly in k and for all $c''<c'$,

$$|R^A_k|_{r,\beta_0}+|R^B_k|_{r,\beta_0}\le c_1\varepsilon^{\frac32}(|A|_{r,\beta_0}+|B|_{r,\beta_0})+c\exp(-\frac{c''}{\varepsilon})\tag{6.43}$$

Moreover by using Lemma 6.1 we get the estimate

$$|A|_{r,\beta_0}+|B|_{r,\beta_0}=\mathcal{O}(\frac{1}{\varepsilon^4})\tag{6.44}$$

whose proof is left in appendix. Set $W^A_k=R^A_k-R^A_{k-1}$ and $W^B_k=R^B_k-R^B_{k-1}$. From (6.38, 6.39), (W^A_k,W^B_k) satisfies the system

$$\begin{aligned}\xi\frac{\partial}{\partial x}W^A_k&=\frac{1}{\varepsilon}(Q(M^A,W_k)+Q(W^A_k,M))+\mathcal{N}_A(W_k)+\widetilde{\mathcal{N}}_{A*}(W^A_k,W^B_k)\\&+\ \varepsilon^2\Big(Q(R^A_{k-1},W_k)+Q(W^A_k,R_{k-2})\\&+\ I(W^B_k)Q(R^A_{k-1},M^B)+I(R^B_{k-2})Q(W^A_{k-1},M^B)\Big)\\ \xi\frac{\partial}{\partial x}W^B_k&=\frac{1}{\varepsilon}(Q(M^B,W_k)+Q(W^B_k,M))+\mathcal{N}_B(W_k)\\&+\ \varepsilon^2\Big(Q(R^B_{k-1},W_{k-1})+Q(W^B_{k-1},R_{k-2})+I(W^B_k)Q(R^B_{k-1},M^B)\\&+\ I(R^B_{k-2})Q(W^B_{k-1},M^B)\Big)\end{aligned}$$

with the boundary conditions

$$W^A_k(-1,v)=0,\ \xi>0,\quad W^A_k(1,v)=0,\ \xi<0,$$
$$W^B_k(-1,v)=0,\ \xi>0,\quad W^B_k(1,v)=\beta_{W^B_k}M_+,\ \xi<0.$$

From proposition 1, (W^A_k,W^B_k) satisfies the majoration

$$|W^A_k|_{r,\beta_0}+|W^B_k|_{r,\beta_0}\le c\sqrt{\varepsilon}(|\widetilde{D}^A|_{r,\beta_0}+|\widetilde{D}^B|_{r,\beta_0})$$

with

$$\widetilde{D}^A=Q(R^A_{k-1},W_k)+Q(W^A_k,R_{k-2})+I(W^B_k)Q(R^A_{k-1},M^B)+I(R^B_{k-1})Q(W^A_{k-1},M^B),$$
$$\widetilde{D}^B=Q(R^B_{k-1},W_{k-1})+Q(W^B_{k-1},R_{k-2})+I(W^B_k)Q(R^B_{k-1},M^B)+I(W^B_{k-1})Q(W^B_{k-1},M^B).$$

Hence by using the inequality (6.41) and the estimate (6.43), it holds that

$$|W^A_k|_{r,\beta_0}+|W^B_k|_{r,\beta_0}\le c\sqrt{\varepsilon}\Big(\varepsilon^{\frac32}(|A|_{r,\beta_0}+|B|_{r,\beta_0})+\exp(-\frac{c'}{\varepsilon})\Big)(|W^A_{k-1}|_{r,\beta_0}+|W^B_{k-1}|_{r,\beta_0}).$$

So from (6.44) and by choosing ε small enough,

$$|W_k^A|_{r,\beta_0} + |W_k^B|_{r,\beta_0} \leq c\varepsilon(|W_{k-1}^A|_{r,\beta_0} + |W_{k-1}^B|_{r,\beta_0}).$$

So by choosing again ε small enough, we show that the sequence $\Big((R_k^A, R_k^B)\Big)_{k\in\mathbb{N}}$ is a Cauchy sequence in a weighted $L^\infty \times L^\infty$ space and so converges.
Now let us show the uniqueness of the solution to the problem (5.6, 5.7). Let (R_1^A, R_1^B) and (R_2^A, R_2^B) be two solutions to the problem (5.6, 5.7). By considering the quantities $R_2^A - R_1^A$ and $R_2^B - R_1^B$ and by proceeding like for the existence step, it comes

$$|R_2^A - R_1^A|_{r,\beta_0} + |R_2^B - R_1^B|_{r,\beta_0} \leq c\varepsilon(|R_2^A - R_1^A|_{r,\beta_0} + |R_2^B - R_1^B|_{r,\beta_0}).$$

So by choosing ε small enough, the uniqueness of the solution follows. □

Proof. (Theorem 2.1.)
For n_{II} close enough to n_I and for some T_{II} close enough to T_I, the asymptotic expansion

$$(f_{H0}^A + \varepsilon f_1^A + \varepsilon^2 f_2^A + \varepsilon^3 f_R^A,\ f_{H0}^B + \varepsilon f_1^B + \varepsilon^2 f_2^B + \varepsilon^3 f_R^B)$$

has been determined. For ε small enough Proposition 6.1 controls the rest term (f_R^A, f_R^B). This shows Theorem 2.1. □

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A Proof of 6.44.

For the proof of 6.44, we will give only the estimate of $\frac{1}{\varepsilon}Q(f_{K1}^{A-}(x',v), f_{K1}^{A+}(x'',v))$. The other terms of A and B can be treated analogously. $[-1,1]$ is split as $[-1,1] = \Omega_- \cup \Omega \cup \Omega_+$, with η small enough where

$$\Omega_- = [-1,\,-1+\eta] \times \mathbb{R}^3, \quad \Omega = [-1-\eta,\,1-\eta] \times \mathbb{R}^3, \quad \Omega_+ = [1-\eta,\,1] \times \mathbb{R}^3.$$

$(1+|v|)^{r-1}M_0^{-\frac{1}{2}}Q(f_{K1}^{A-}(x',v), f_{K1}^{A+}(x'',v))$ will be estimated successively on Ω_-, Ω et Ω_+. The inequality (6.41) applied on the domain Ω_+ writes

$$\begin{aligned}\sup_{(x,v)\in\Omega_+} &|(1+|v|)^{r-1}M_*^{-\frac{1}{2}}Q(f_{K1}^{A-}(x',v), f_{K1}^{A+}(x'',v))| \\ &\le \sup_{(x,v)\in\Omega_+} |(1+|v|)^{r}M_*^{-\frac{1}{2}}f_{K1}^{A-}(x',v)| \\ &\times \sup_{(x,v)\in\Omega_+} |(1+|v|)^{r}M_*^{-\frac{1}{2}}M^A(1,v)b_1^{A+}(x'',v))|.\end{aligned}$$

By definition of M_* there is $c>0$ such that $M_*^{-\frac{1}{2}}\,M^A(-1,v) \le c$ and $M_*^{-\frac{1}{2}}\,M^A(1,v) \le c$. Moreover

$$\begin{aligned}&\sup_{(x,v)\in\Omega_+} |\frac{1}{\varepsilon}(1+|v|)^{r}M_*^{-\frac{1}{2}}f_{K1}^{A-}(x',v)| \\ &\le c \sup_{(x,v)\in[-1,1]\times\mathbb{R}^3} |(1+|v|)^{r}\,e^{\gamma\frac{1+x}{\varepsilon}}b_1^{A-}(\frac{1+x}{\varepsilon},v)|\;|\frac{1}{\varepsilon}e^{-\gamma\frac{2-\eta}{\varepsilon}}|, \\ &\le c \sup_{(x,v)\in[-1,1]\times\mathbb{R}^3} |(1+|v|)^{r}\,e^{\gamma\frac{1+x}{\varepsilon}}b_1^{A-}(\frac{1+x}{\varepsilon},v)|.\end{aligned}$$

But from ([7, 2]), there is $c>0$ such that for all $\gamma \in]0,\nu_0[$,

$$\sup_{(x,v)\in[-1,1]\times\mathbb{R}^3} |(1+|v|)^{r}\,e^{\gamma\frac{1+x}{\varepsilon}}b_1^{A-}(\frac{1+x}{\varepsilon},v)| \le c.$$

So there is $\tilde{c}>0$ such that

$$\sup_{(x,v)\in\Omega_+} |\frac{1}{\varepsilon}(1+|v|)^{r-1}M_*^{-\frac{1}{2}}Q(f_{K1}^{A-}(x',v), f_{K1}^{A+}(x'',v))| \le \tilde{c}.$$

Analogously we show that $Q(f_{K1}^{A-}(x',v), f_{K1}^{A+}(x'',v))$ satisfies the same estimate on Ω_- and Ω. □