Some recent advances in the theory of moment models

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Context

1D kinetic PDE

$$\partial_t f + v\partial_x f = Q(f)$$

satisfying certain properties

- Well-posed with $f(t,x,v) \geq 0$ (together with IC and BC)
- Hyperbolic (at fixed v)
- Entropy decay

$$\partial_t \mathcal{H}(f) + \partial_x \mathcal{G}(f) = \mathcal{D}(f) \leq 0,$$

$$\mathcal{H}(f) = \int_v \eta(f), \quad \mathcal{G}(f) = \int_v v\eta(f), \quad \mathcal{D}(f) = \int_v \eta'(f)Q(f)$$

with $\mathcal{D}(f) = 0 \quad \Leftrightarrow \quad f = M$

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Objective: Discretize w.r.t. $v \in \mathbb{R}$ such that

- These properties are preserved
- Capture exactly physical regimes (equilibrium, purely anisotropic)

Context

Other (toy) models :

- Radiative transfer $\mu \in [-1, 1]$

$$\frac{1}{|v|}\partial_t f + \mu \partial_x f = L(f),$$

- Spray modelling $S \in \mathbb{R}^+$

$$\partial_t f + v\partial_x f + \partial_S (Kf) = 0,$$

Objective: Discretize w.r.t. $\mu \in [-1, +1]$ or $S \in \mathbb{R}^+$ such that

- These properties are preserved
- Capture exactly physical regimes (equilibrium, purely anisotropic)

State of the art and new alternatives

Alternatives (non-exhaustive):

- Brute force: numerical cost, no equilibrium
  - Monte-Carlo
  - Discrete velocities
- Moments methods:
  - Euler equations → restricted to low order
  - Grad's methods → non-hyperbolic, non-positive approximation
    ↪ regularizations (non-conservative)

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    $\hookrightarrow$ regularizations (non-conservative)

Novelties around

- Quadrature methods: HyQMOM
- Entropy method: φ-divergence
- Realizability method: Projection technique

1. Introduction
   - Kinetic models
   - Method of moments
2. Realizability domain
3. Closures
   - Quadrature approach
   - Entropy approach
   - Geometrical approach

Principle

Principle: $\partial_t f + v\partial_x f = Q(f)$

1. Choose basis of weight functions

$$\boldsymbol{w}(v) = \boldsymbol{w}_N(v) = (1, v, v^2, ..., v^N)^T$$

2. Integrate the equation against $\boldsymbol{w}(v)$ over v

$$\partial_t \boldsymbol{f} + \partial_x \boldsymbol{F} = \boldsymbol{Q},$$

$$\boldsymbol{f} = \int \boldsymbol{w}(v) f(v) dv, \quad \boldsymbol{F} = \int v \boldsymbol{w}(v) f(v) dv, \quad \boldsymbol{Q} = \int \boldsymbol{w}(v) Q(f)(v) dv$$

$\hookrightarrow$ Work with $\boldsymbol{f}$ instead of f

3. Express $\boldsymbol{F}(\boldsymbol{f})$ and $\boldsymbol{Q}(\boldsymbol{f})$ (**closure**) based on $\boldsymbol{f}$

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$\hookrightarrow$ Work with $\boldsymbol{f}$ instead of f

3. Express $\boldsymbol{F}(\boldsymbol{f})$ and $\boldsymbol{Q}(\boldsymbol{f})$ (**closure**) based on $\boldsymbol{f}$

Difficulty: Choose a **closure** that

- Preserves property
- Captures regimes

Construction and properties of the closure

Seek

$$\boldsymbol{F} = \int v\boldsymbol{w}(v)f(v)dv = (\boldsymbol{f}_1, \dots, \boldsymbol{f}_N, \boldsymbol{f}_{N+1}) \quad \text{knowing} \quad \boldsymbol{f} = \int \boldsymbol{w}(v)f(v)dv = (\boldsymbol{f}_0, \dots, \boldsymbol{f}_N)$$

Common idea:

- Solve the "problem of moments"

$$\text{from} \quad \boldsymbol{f} \in \mathbb{R}^{N+1}, \quad \text{find} \quad f_R \quad \text{s.t.} \quad \boldsymbol{f} = \int \boldsymbol{w} f_R \qquad (1)$$

- Closure: replace f by f_R in $\boldsymbol{F}$

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Problems:

- **Existence of a solution to** (1)?
  $\hookrightarrow$ Under condition $\Rightarrow$ When? $\Rightarrow$ Realizability
- **Uniqueness?**
  $\hookrightarrow$ Very rarely $\Rightarrow$ How to choose f_R? $\Rightarrow$ Closure

Definition and properties

Definition

Realizability domain

$$\mathcal{R}_{\boldsymbol{w}_N} := \left\{ \boldsymbol{f} \in \mathbb{R}^{N+1} \quad \text{s.t.} \quad \exists f_R \in L^1_{\boldsymbol{w}_{N+1}}(\mathbb{R})^+ \quad \boldsymbol{f} = \int \boldsymbol{w}_N f_R \right\}$$

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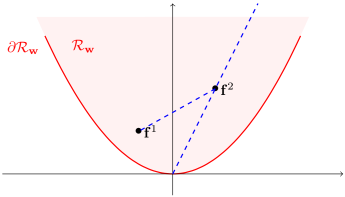
$$= \left\{ \int \boldsymbol{w}_N f, \quad f \in L^1_{\boldsymbol{w}_{N+1}}(\mathbb{R})^+ \right\}$$

Remark:

- $\mathcal{R}_{\boldsymbol{w}_N}$ is a convex cone,
  i.e. stable by positive combinations
- $\mathcal{R}_{\boldsymbol{w}_N}$ is open and $\partial\mathcal{R}_{\boldsymbol{w}_N} \not\subset \mathcal{R}_{\boldsymbol{w}_N}$



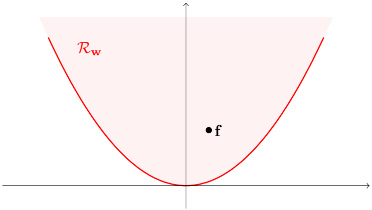
Hamburger moment problem

Proposition ([1])

Even case :	$\boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}_{2N}}$	$\Leftrightarrow$	$H_N(\boldsymbol{f}) = (f_{i+j})_{0\leq i,j\leq N}$	*is SPD*
Odd case :	$\boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}_{2N+1}}$	$\Leftrightarrow$	$H_N(\boldsymbol{f}) = (f_{i+j})_{0\leq i,j\leq N}$	*is SPD*

Example:

$$H_N(\boldsymbol{f}) = \begin{pmatrix} \boldsymbol{f}_0 & \boldsymbol{f}_1 & \dots & \boldsymbol{f}_N \\ \boldsymbol{f}_1 & \boldsymbol{f}_2 & \dots & \boldsymbol{f}_{N+1} \\ \vdots & & \ddots & \vdots \\ \boldsymbol{f}_N & \boldsymbol{f}_{N+1} & \dots & \boldsymbol{f}_{2N} \end{pmatrix}$$



$\hookrightarrow$ $\boldsymbol{f}_{2N+1}$ is free

[1] Hamburger (1920), Akhiezer, Krein

Hamburger moment problem

Remark: $\boldsymbol{f} \in \partial\mathcal{R}_{\boldsymbol{w}_{2N}} \quad \Leftrightarrow \quad H_N(\boldsymbol{f})$ symmetric positive singular

[2] Curto & Fialkow (1991-)

Hamburger moment problem

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Proposition ([2])

Suppose that $J := rank(H_N) < N$ and $H_J(\boldsymbol{f})$ is SPD
Then $\exists!$ representing measure

$$\gamma = \sum_{i=1}^{J} \alpha_i \delta_{v_i}, \qquad \alpha_i > 0,$$

Example:

$$H_N(\boldsymbol{f}) = \left(\begin{array}{ccc|ccc} \boldsymbol{f}_0 & \dots & \boldsymbol{f}_J & \boldsymbol{f}_{J+1} & \dots & \boldsymbol{f}_N \\ \vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\ \boldsymbol{f}_J & \dots & \boldsymbol{f}_{2J} & \boldsymbol{f}_{2J+1} & \dots & \boldsymbol{f}_{J+N} \\ \hline \boldsymbol{f}_J & \dots & \boldsymbol{f}_{2J+1} & \boldsymbol{f}_{2J+2} & \dots & \boldsymbol{f}_{J+N+1} \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{f}_N & \dots & \boldsymbol{f}_{J+N} & \boldsymbol{f}_{J+N+1} & \dots & \boldsymbol{f}_{2N} \end{array}\right)$$

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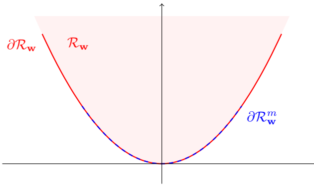
- Only on part of $\partial\mathcal{R}_{\boldsymbol{w}_N}$ (call it $\partial\mathcal{R}^m_{\boldsymbol{w}_N}$)
  Example: $\boldsymbol{f} = (0,0,1)^T \in \partial\mathcal{R}_{\boldsymbol{w}_2}$ since

$$H_2(\boldsymbol{f}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{is SPS,}$$



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Corollary

$\boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}_{2N+1}}$ *iff* $\exists \boldsymbol{g}_{2N+2}$ *s.t.* $(\boldsymbol{f},\ \boldsymbol{g}_{2N+2}) \in \partial\mathcal{R}^m_{\boldsymbol{w}_{2N+2}}$, *or equivalently*

$$\left(\begin{array}{ccc|c} \boldsymbol{f}_0 & \cdots & \boldsymbol{f}_N & \boldsymbol{f}_{N+1} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{f}_N & \cdots & \boldsymbol{f}_{2N} & \boldsymbol{f}_{2N+1} \\ \hline \boldsymbol{f}_{N+1} & \cdots & \boldsymbol{f}_{2N+1} & \boldsymbol{g}_{2N+2} \end{array}\right) \quad \text{symmetric positive singular}$$

$\hookrightarrow$ extends to multi-variate problems !

Corollary ([3])

$$\mathcal{R}_{\boldsymbol{w}_{2N+1}} = \left\{ \sum_{i=1}^{N} \alpha_i \boldsymbol{w}(v_i), \qquad \alpha_i \in \mathbb{R}^+, \quad v_i \in \mathbb{R} \right\}$$

[3] Curto-Fialkow (1998-)

Other sets of integration

Proposition (Stieltjes)

$$\boldsymbol{f} = \int_{\mathbb{R}^+} \boldsymbol{w}_N f \ \textit{iff}$$

- Even case $N = 2M$: $H_M(\boldsymbol{f})$ *and* $(\boldsymbol{f}_{i+j+1})_{i,j=0,\dots,M-1}$ *are SPD*
- Odd case $N = 2M + 1$: $H_{M-1}(\boldsymbol{f})$ *and* $(\boldsymbol{f}_{i+j+1})_{i,j=0,\dots,M}$ *are SPD*

Proposition (Hausdorff)

$$\boldsymbol{f} = \int_{-1}^{+1} \boldsymbol{w}_N f \ \textit{iff}$$

- Even case $N = 2M$: $H_M(\boldsymbol{f})$ *and* $(\boldsymbol{f}_{i+j+2} - \boldsymbol{f}_{i+j})_{i,j=0,\dots,M-1}$ *are SPD*
- Odd case $N = 2M + 1$: $(\boldsymbol{f}_{i+j} \pm \boldsymbol{f}_{i+j+1})_{i,j=0,\dots,M}$ *are SPD*

$\hookrightarrow$ **Property:** $\partial\mathcal{R}_{\boldsymbol{w}} = \partial\mathcal{R}_{\boldsymbol{w}}^m = \left\{ \sum_{i=1}^{M-1} \alpha_i \boldsymbol{w}(v_i) \right\}$

QMOM[4]: Principle

Considering $\boldsymbol{f} = (\boldsymbol{f}_0, \ldots, \boldsymbol{f}_{2N-1}) \in \mathcal{R}_{\boldsymbol{w}_{2N-1}} \subset \mathbb{R}^{2N}$, then

$$f_R \equiv \sum_{i=1}^{N} \alpha_i \delta_{v_i} \quad \Leftrightarrow \quad \boldsymbol{f} = \sum_{i=1}^{N} \alpha_i \boldsymbol{w}(v_i)$$

How to compute α_i and v_i?

[4] McGraw (1997), R. Fox, F. Laurent, D. Marchisio, M. Massot, C. Chalon

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How to compute α_i and v_i?

1st idea:
Flat extension

$$\left(\begin{array}{ccc|c} \boldsymbol{f}_0 & \dots & \boldsymbol{f}_{N-1} & \boldsymbol{f}_N \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{f}_{N-1} & \dots & \boldsymbol{f}_{2N-2} & \boldsymbol{f}_{2N-1} \\ \hline \boldsymbol{f}_N & \dots & \boldsymbol{f}_{2N-1} & \boldsymbol{f}_{2N} \end{array}\right) \text{SPS} \quad \Leftrightarrow \quad \begin{array}{l} \boldsymbol{f}_{2N} = \boldsymbol{V}(\boldsymbol{f})^T H_{N-1}(\boldsymbol{f})^{-1} \boldsymbol{V}(\boldsymbol{f}) \\ \boldsymbol{V}(\boldsymbol{f}) = (\boldsymbol{f}_N, \dots, \boldsymbol{f}_{2N-1})^T \end{array}$$

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How to compute α_i and v_i?

2nd idea:
Change of basis

$$\boldsymbol{w}_N \to \tilde{\boldsymbol{w}}_N = P\boldsymbol{w}_N \quad \text{s.t.} \quad PH_N(\boldsymbol{f})P^T = Diag$$

[4] McGraw (1997), R. Fox, F. Laurent, D. Marchisio, M. Massot, C. Chalon

Properties

QMOM closure

$$f_R \equiv \sum_{i=1}^{N} \alpha_i \delta_{v_i}, \qquad \boldsymbol{f} = \sum_{i=1}^{N} \alpha_i \boldsymbol{w}(v_i), \qquad \boldsymbol{F} = \sum_{i=1}^{N} \alpha_i v_i \boldsymbol{w}(v_i)$$

Properties:

- Very efficient **algorithm**
- **Positivity**: $\boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}} \Rightarrow \alpha_i > 0$
- Capture Diracs, no equilibrium
- **Hyperbolicity**?
- **Entropy**?

Properties

Balance law

$$\partial_t \boldsymbol{f} + \partial_x \boldsymbol{F}(\boldsymbol{f}) = \boldsymbol{S}(\boldsymbol{f})$$

- (Strongly) hyperbolic

$$J_{\boldsymbol{f}} \boldsymbol{F} \text{ diagonalizable in } \mathbb{R}$$

- Symmetric hyperbolic[5]

$$\exists A \quad \text{SPD} \quad \text{s.t.} \quad A J_{\boldsymbol{f}} \boldsymbol{F}(\boldsymbol{f}) \text{ symmetric}$$

Symmetric dissipative[6] $A\boldsymbol{S}(\boldsymbol{f}) \leq 0 \Rightarrow \exists$ convex entropy

Natural framework for fluid dynamics

$\hookrightarrow$ "Some" well-posedness results
$\hookrightarrow$ "Correct" (physical) propagation of waves
$\hookrightarrow$ Numerical schemes

[5] Godunov-Mock
[6] Kawashima-Shizuta-Yong

Properties

Balance law

$$\partial_t \boldsymbol{f} + \partial_x \boldsymbol{F}(\boldsymbol{f}) = \boldsymbol{S}(\boldsymbol{f})$$

Non-hyperbolicity:

- Complex eigenvalues

$$\left\{ \begin{array}{l} \partial_t u + \partial_x v = 0, \\ \partial_t v - \partial_x u = 0, \end{array} \right. \qquad J_{\boldsymbol{f}} \boldsymbol{F} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

  - Elliptic problem $\partial_{tt} u + \partial_{xx} u = 0$
  - Unexpected from kinetic/physics (no wave propagation)

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  - Elliptic problem $\partial_{tt} u + \partial_{xx} u = 0$
  - Unexpected from kinetic/physics (no wave propagation)
- Non-diagonalizable → **Weakly hyperbolic**

$$\begin{cases} \partial_t u & +\partial_x u = 0 \\ \partial_t v & +\partial_x v + \partial_x u = 0 \end{cases} \quad J_f \boldsymbol{F} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \begin{cases} u_0(x) = a + b 1_{\mathbb{R}^+}(x) \\ v_0(x) = c + d 1_{\mathbb{R}^+}(x) \end{cases}$$

  - Solution

$$u(x,t) = a + b 1_{\mathbb{R}^+}(x - t), \qquad \partial_x u(x,t) \equiv (b-a)\delta(x-t)$$

  - δ-schocks (measure solution)

Properties

QMOM closure

$$f_R \equiv \sum_{i=1}^{N} \alpha_i \delta_{v_i}, \qquad \boldsymbol{f} = \sum_{i=1}^{N} \alpha_i \boldsymbol{w}(v_i), \qquad \boldsymbol{F} = \sum_{i=1}^{N} \alpha_i v_i \boldsymbol{w}(v_i)$$

Properties:

- Very efficient **algorithm**
- **Positivity**: $\boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}} \Rightarrow \alpha_i > 0$
- Capture Diracs, no equilibrium
- **Weak hyperbolicity**:

$$J_{\boldsymbol{f}} \boldsymbol{F} = P Diag(J_1, \ldots, J_N) P^{-1}, \quad J_i = \begin{pmatrix} v_i & \alpha_i \\ 0 & v_i \end{pmatrix}$$

  δ-shocks of mass α_i and velocity v_i

- **Entropy** $\rightarrow \sum_i \alpha_i E(v_i)$ Not related to kinetic

Extensions

- Algorithm: DQMOM
- Multi-variate $v \in \mathbb{R}^3$: CQMOM

$$f_R = \sum_i \alpha_i \delta(v_1 - v_{i,1}) \delta(v_2 - v_{i,2} | v_1) \delta(v_3 - v_{i,3} | v_1, v_2)$$

$\hookrightarrow$ principal directions

- Multi-Gaussian a.k.a. EQMOM

$$f_R = \sum_i \alpha_i \exp\left(-\sigma(v - v_i)^2\right)$$

- Strongly hyperbolic: HyQMOM

$$\boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}_{2N}} \quad \text{and} \quad f_R = \sum_{i=1}^{N+1} \alpha_i \delta_{v_i}$$

$\hookrightarrow$ $2N + 2$ parameters for $2N + 1$ moments

Minimum entropy

Consider $\partial_t f + v\partial_x f = Q(f)$ such that

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$$\mathcal{H}(f) = \int_v \eta(f), \quad \mathcal{G}(f) = \int_v v\eta(f), \quad \mathcal{D}(f) = \int_v \eta'(f)Q(f)$$

Proposition ([7])

Choice: $f_R = \underset{\int \boldsymbol{w} f = \boldsymbol{f}}{argmin}\ \mathcal{H}(f)$,

Boltzmann/Shanon entropy: $\eta(f) = f \log f - f$, then $(\eta^*)' = \exp$

[7] Levermore (1996), Junk, Schneider, Borwein & Lewis, Mead & Papanicolaou
[8] Lax & Friedrichs (1971), S. Kawashima, Y. Shizuta, W.-A. Yong

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- $f_R \equiv (\eta^*)'(\boldsymbol{\lambda}^T \boldsymbol{w}(v))$
- $\partial_t \left(\int \boldsymbol{w} f_R\right) + \partial_x \left(\int v\boldsymbol{w} f_R\right)$ *is symmetric hyperbolic*[8]
  $A\partial_t \boldsymbol{\lambda} + B\partial_x \boldsymbol{\lambda}$

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Choice: $f_R = \underset{\int \boldsymbol{w} f = \boldsymbol{f}}{argmin}\ \mathcal{H}(f)$, *then*

- $f_R \equiv (\eta^*)'(\boldsymbol{\lambda}^T \boldsymbol{w}(v))$
- $\partial_t \left(\int \boldsymbol{w} f_R\right) + \partial_x \left(\int v\boldsymbol{w} f_R\right)$ *is symmetric hyperbolic*[8]
- $\int \boldsymbol{\lambda}^T \boldsymbol{w} Q(f_R) = \int \eta'(f_R)Q(f_R) = \mathcal{D}(f_R) \leq 0$

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Issues

Two difficulties:

- Computational costs
- Undefined for some $\boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}}$:

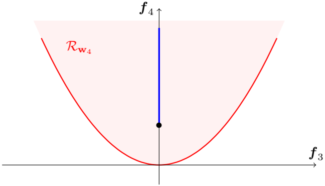
Proposition ([5])

$$\exists \boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}} \quad s.t \quad \nexists \boldsymbol{\lambda} \text{ satisfying } \int_{\mathbb{R}} \boldsymbol{w}(v)(\eta^*)'(\boldsymbol{\lambda}^T \boldsymbol{w}(v))dv = \boldsymbol{f}.$$

$\hookrightarrow$ lack of integrability

$$\exp\left(\sum_{i=0}^{3} \boldsymbol{\lambda}_i v^i + \lambda_4 v^4\right)$$

when $\lambda_4 \to 0^-$



[5] Junk (1998), Schneider, McDonald, Groth

Alternative: φ-divergence[8]

Idea (via minimization techniques): Case $\eta(f) = f \log f - f$

- Use relative entropy[9]

$$D(f,M) = \int M\eta\left(\frac{f}{M}\right) = \int f \log\left(\frac{f}{M}\right)$$

- Replace η by

$$\eta_N(f) = f\left(\frac{N^2}{N+1} f^{1/N} - N\right) + \frac{N}{N+1} \approx f \log f - f$$

s.t. $D_N(f,M) = \int M\eta_N(\frac{f}{M})$ and obtain

$$f_R^N = \underset{\int \boldsymbol{w}f=\boldsymbol{f}}{\operatorname{argmin}}\ D_N(f,M) = M\left(1 + \frac{\boldsymbol{\lambda}^T\boldsymbol{w}}{N}\right)_+^N \approx \exp(\boldsymbol{\lambda}^T\boldsymbol{w})$$

[8] M. Abdelmalik (2016-), H. Van Brummelen
[9] Kullback & Leibler (1951)

Alternative: φ-divergence[9]

Entropy inequality

$$\partial_t \int \eta(f) + \partial_x \int v\eta(f) = \int \eta'(f) Q(f) \leq 0$$

Reconstruction

$$f_R^N = M \left(1 + \frac{\boldsymbol{\lambda}^T \boldsymbol{w}}{N}\right)_+^{N+1}$$

Properties:

- $\partial_t \left(\int \boldsymbol{w} f_R\right) + \partial_x \left(\int v \boldsymbol{w} f_R\right)$ is symmetric hyperbolic
- Construction of Q_N s.t.

$$\int \boldsymbol{\lambda}^T \boldsymbol{w} Q_N(f_R^N) = \int \eta_N'(f_R^N) Q_N(f_R^N) \leq 0$$

- Still need to compute $\boldsymbol{\lambda}$

[9] M. Abdelmalik (2016-), H. Van Brummelen

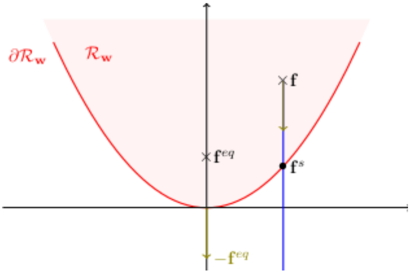
General idea[11]

Receipe:

1. Choose an equilibrium $f^{eq} \in L^1_{\boldsymbol{w}_{N+1}}(\mathbb{R})^+$
2. Project $\boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}_N}$ toward $\partial\mathcal{R}_{\boldsymbol{w}_N}$ along $-\boldsymbol{f}^{eq}$

$$\boldsymbol{f} = \alpha_0 \boldsymbol{f}^{eq} + \boldsymbol{f}^s, \qquad \boldsymbol{f}^s \in \partial\mathcal{R}_{\boldsymbol{w}_N}$$

3. Reconstruct f_R from f^{eq} and $\boldsymbol{f}^s$



[11] T.P. (2020-)

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Problem: Choose $f^{eq}(\boldsymbol{f})$ such that

- Maxwellians $f^{eq} \equiv M(\rho, u, T) \equiv (\eta^*)'(\boldsymbol{\lambda}^T \boldsymbol{w})$
- $\boldsymbol{f}^s \in \partial\mathcal{R}^m_{\boldsymbol{w}_N} \subset \partial\mathcal{R}_{\boldsymbol{w}_N}$ to reconstruct $\boldsymbol{f}^s = \sum_i \alpha_i \boldsymbol{w}(v_i)$

[11] T.P. (2020-)

A parametrization of $\mathcal{R}_{\boldsymbol{w}}$

Suppose that $f^{eq}(\boldsymbol{\lambda}) \equiv (\eta^*)'(\boldsymbol{\lambda}^T \boldsymbol{w}_J(v))$ and $2M = N - (J+1)$, seek

$$\boldsymbol{f} = \boldsymbol{f}^{eq}(\boldsymbol{\lambda}) + \sum_{i=1}^{M} \alpha_i \boldsymbol{w}(v_i)$$

Is $\left\{ \begin{array}{rll} & \Lambda \times (\mathbb{R}^{*+} \times \mathbb{R})^M & \to \mathcal{R}_{\boldsymbol{w}} \\ \boldsymbol{v} = & (\boldsymbol{\lambda}, \alpha_1, v_1, \dots, \alpha_K, v_K) & \mapsto \boldsymbol{f} \end{array} \right.$ a bijection ?

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Current work:

- Description of $\mathcal{R}_{\boldsymbol{w}} \backslash \boldsymbol{f}\left(\Lambda \times (\mathbb{R}^{+} \times \mathbb{R})^M\right)$
  $\hookrightarrow$ density in $\mathcal{R}_{\mathrm{w}}$ $\to$ similar to Junk line
- Lose uniqueness when $\alpha_K = 0$ (any v_K)

Hyperbolicity

Suppose $2M = N - (J+1)$

$$\boldsymbol{f} \quad = \quad \int \boldsymbol{w}(\eta^*)'(\boldsymbol{\lambda}^T \boldsymbol{w}_J(v)) \quad + \quad \sum_{i=1}^{M} \alpha_i \boldsymbol{w}(v_i)$$

Proposition

- *Weak hyperbolicity*

$$J_{\boldsymbol{f}} \boldsymbol{F} = P \left(\begin{array}{c|c} A^{-1}B & 0 \\ \hline * & Diag(J_{\alpha_1,v_1}, \ldots, J_{\alpha_K,v_K}) \end{array} \right) P^{-1}$$

- *Symmetric hyperbolicity for* $\boldsymbol{\lambda}$

$$A(\boldsymbol{\lambda}, \alpha_i, v_i)\partial_t \boldsymbol{\lambda} + B(\boldsymbol{\lambda}, \alpha_i, v_i)\partial_x \boldsymbol{\lambda}$$

with

$$A(\boldsymbol{\lambda}, \alpha_i, v_i) = \int \prod_i (v - v_i)^2 \boldsymbol{w}_J \boldsymbol{w}_J^T (\eta^*)'' \left(\boldsymbol{\lambda}^T \boldsymbol{w}_J\right)$$

$$B(\boldsymbol{\lambda}, \alpha_i, v_i) = \int v \prod_i (v - v_i)^2 \boldsymbol{w}_J \boldsymbol{w}_J^T (\eta^*)'' \left(\boldsymbol{\lambda}^T \boldsymbol{w}_J\right)$$

On going work: Part on (α_i, s_i) ?

Properties and on-going work

Properties:

- Positivity

$$\boldsymbol{f} = \int \boldsymbol{w}(v)\left(\alpha_0 f^{eq}(v)dv + \sum_i \alpha_i \delta_{v_i}\right)$$

- Captures f^{eq} and δ_v
- Weak hyperbolicity
- Entropy decay: on-going work

Current work:

- Symmetrization: Find a basis $\tilde{\boldsymbol{w}}$ such that

$$A\partial_t \begin{pmatrix} \boldsymbol{\lambda} \\ \alpha_i \\ v_i \end{pmatrix} + B\partial_x \begin{pmatrix} \boldsymbol{\lambda} \\ \alpha_i \\ v_i \end{pmatrix}$$

- Choice of closure $\boldsymbol{Q}(\boldsymbol{f}) \rightarrow$ Entropy decay and symmetric dissipation (relation with $\tilde{\boldsymbol{w}}$)
- Computation (numerical) of $\boldsymbol{\lambda}, (\alpha_i, v_i)_{i=1,\ldots,J}$

In a nutshell

- **Quadrature**:
  - Weakly hyperbolic $\rightarrow$ strongly
  - Positive reconstruction
  - Capture Diracs, not equilibria
  - Entropy $\rightarrow$ not kinetic
  - **Algorithm !**
- **Entropy**:
  - **Symmetric hyperbolic and dissipative**
  - Choice for a positive reconstruction
  - Capture equilibria and Diracs (on the boundary)
- **Projection**:
  - Weakly hyperbolic
  - Positive reconstruction
  - **Capture equilibria and Diracs**