Hydrodynamic limit for granular gases: from Boltzmann equation to some modified Navier-Stokes-Fourier system

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Scope of the talk

Provide a (first) rigorous derivation of suitable Navier-Stokes hydrodynamic model from rapid granular flows described by the Boltzmann equation with *inelastic hard spheres*.

This is done by establishing

- A suitable Cauchy theory for close-to-equilibrium solutions for the Boltzmann equation with inelastic interactions.
- Identify the exact regime of weak inelasticity.
- Obtain estimates for the solutions which are uniform with respect to the Knudsen number (including *exponential stability*).
- Derive a new Navier-Stokes-Fourier system with self-consistent forcing terms and subject to Boussinesq relation. Model seems new in this context.

Granular gases

A granular material is a substance made of grains (!!), i.e. system of discrete particles characterized by the following features:

- Grains are ***macroscopic particles*** described by the rules of classical mechanics;
- the grains interaction (with each other or some background, boundaries...) are ***dissipative***: friction is always a relevant phenomena and collisions are ***inelastic***.
- ***Rapid granular flows*** described by suitable modifications of the ***Boltzmann equation*** which takes into account the *inelasticity* of collisions.

▶ Because of the frictional nature of granular gases, only hard-spheres interactions are physically relevant.

Ref.: I. Goldhirsch (1999), Pöschel & Brilliantov (2004), Garzò (2014)

No consensus on the limiting system that can derived from Boltzmann equation in the physics community.

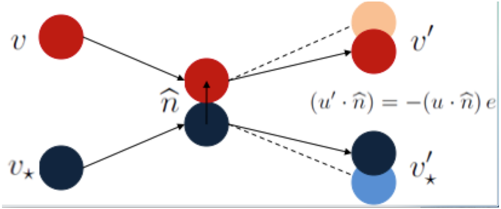
> *"the context of the hydrodynamic equations remains uncertain. What are the relevant space and time scales? How much inelasticity can be described in this way?"* *Brey & Dufty (2005).*

Microscopic description of inelastic collisions

$$\boxed{\text{Binary collision} \quad (v, v_*) \to (v', v_*')}$$

During the collision, a part of the *normal relative velocity* is dissipated while the tangential relative velocity is conserved, the loss of normal relative velocity is measured through the so-called ***restitution coefficient***

$$\boxed{u' \cdot n = -(u \cdot n)\, e, \qquad e \in (0, 1]}$$



($e = 1$ corresponding to elastic collision, in dotted line in the above figure).

Inelastic collisions

Then, the post-collisional velocities v', v'_* can be expressed as

$$v' = v - \frac{1+e}{2}(u \cdot n)\, n \qquad v'_* = v_* + \frac{1+e}{2}(u \cdot n)\, n$$

which satisfies the conservation of momentum

$$v + v_* = v' + v'_*.$$

However, microscopic kinetic energy is dissipated since

$$|v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 = -\frac{1-e^2}{2}|u \cdot n|^2 \leqslant 0.$$

Restitution coefficient

The restitution coefficient encodes all the microscopic properties of the inelastic collision mechanism. There are mainly two kinds of coefficients used in the mathematical and physics literature

- Constant restitution coefficient: e does not depend on the impact velocity

$$e = \alpha \in (0,1].$$

- Variable restitution coefficent: e depends on the (normal) relative velocity:

$$e = e(|u \cdot n|)$$

  for some suitable function

$$e \,:\, \mathbb{R}^+ \mapsto e(r) \in (0,1].$$

A particularly relevant example is the one of "visco-elastic hard-spheres" for which

$$e(r) + a\, r^{1/5} e(r)^{3/5} = 1 \qquad \forall r > 0.$$

Boltzmann collision operator for inelastic hard spheres

In weak-form

$$\int_{\mathbb{R}^d} \mathcal{Q}_\alpha(g,f)(v)\,\psi(v)\mathrm{d}v = \frac{1}{2}\int_{\mathbb{R}^{2d}} f(v)\,g(v_*)\,|v-v_*|\mathcal{A}_\alpha[\psi](v,v_*)\mathrm{d}v_*\mathrm{d}v,$$

where

$$\mathcal{A}_\alpha[\psi](v,v_*) = \int_{\mathbb{S}^{d-1}} (\psi(v') + \psi(v_*') - \psi(v) - \psi(v_*))b(\sigma\cdot\widehat{u})\mathrm{d}\sigma,$$

and the post-collisional velocities (v', v_*') are given by

$$v' = v + \frac{1+\alpha}{4}(|u|\sigma - u), \qquad v_*' = v_* - \frac{1+\alpha}{4}(|u|\sigma - u),$$

$$\text{where} \qquad u = v - v_*, \qquad \widehat{u} = \frac{u}{|u|}.$$

The Boltzmann equation

We consider here the (freely cooling) Boltzmann equation for *inelastic collisions*:

$$\partial_t F(t,x,v) + v \cdot \nabla_x F(t,x,v) = \mathcal{Q}_\alpha(F,F)$$

supplemented with initial condition $F(0,x,v) = F_{\rm in}(x,v)$.

- $F(t,x,v)$ density of granular particles having position $x \in \mathbb{T}^d_\ell$ and velocity $v \in \mathbb{R}^d$ at time $t \geqslant 0$ and $d \geqslant 2$.
- We consider the case of *flat torus*

  $$\mathbb{T}^d_\ell = \mathbb{R}^d/(2\pi\,\ell\,\mathbb{Z})^d$$

  for some typical length-scale $\ell > 0$.

Conservation of mass and momentum implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R}(t) := \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{T}^d_\ell} F(t,x,v)\mathrm{d}v\mathrm{d}x = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{U}(t) := \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{T}^d_\ell} vF(t,x,v)\mathrm{d}v\mathrm{d}x = 0\,.$$

No loss of generality in assuming that

$$\boldsymbol{R}(t) = \boldsymbol{R}(0) = 1, \qquad \boldsymbol{U}(t) = \boldsymbol{U}(0) = 0 \qquad \forall t \geqslant 0.$$

The *granular temperature*

$$\boldsymbol{T}(t) := \frac{1}{|\mathbb{T}^d_\ell|}\int_{\mathbb{R}^d\times\mathbb{T}^d_\ell} |v|^2 F(t,x,v)\mathrm{d}v\mathrm{d}x$$

is constantly decreasing

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{T}(t) = -(1-\alpha^2)\mathcal{D}_\alpha(F(t),F(t)) \leqslant 0\,, \qquad \forall t \geqslant 0.$$

Here $\mathcal{D}_\alpha(g,g)$ denotes the normalised energy dissipation associated to $\mathcal{Q}_\alpha$ given by

$$\mathcal{D}_\alpha(g,g) := \frac{\gamma_b}{4}\int_{\mathbb{T}^d_\ell}\frac{\mathrm{d}x}{|\mathbb{T}^d_\ell|}\int_{\mathbb{R}^d\times\mathbb{R}^d} g(x,v)g(x,v_*)|v-v_*|^3\mathrm{d}v\mathrm{d}v_*$$

Main consequences

- Cooling of the gas:

$$\boxed{\lim_{t\to\infty} T(t) = 0}$$

with some precise rate to be determined (Haff law) (S. Mischler, C. Mouhot, 2006–2009).

- No non trivial steady to the (spatially homogoneous) Boltzmann equation

$$\lim_{t\to\infty} F(t,v) = \delta_0(v).$$

- The temperature is the only known Lyapunov functional associated to the equation.

Navier-Stokes scaling

To capture some hydrodynamic behaviour of the gas, we need to write the above equation in *nondimensional form* introducing the dimensionless Knudsen number

$$\boxed{\varepsilon := \frac{\text{mean free path}}{\text{spatial length-scale}}}$$

which is assumed to be small. Re-scaled density

$$F_\varepsilon(t,x,v) = F\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, v\right), \qquad t \geqslant 0.$$

In this case, we choose for simplicity $\ell = \varepsilon$, i.e.

$$F_\varepsilon \ : \ \mathbb{R}^+ \times \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$$

with $\mathbb{T}^d = \mathbb{T}_1^d$.

$$\varepsilon^2\partial_t F_\varepsilon(t,x,v) + \varepsilon\, v\cdot\nabla_x F_\varepsilon(t,x,v) = \mathcal{Q}_\alpha(F_\varepsilon,F_\varepsilon), \qquad (x,v)\in\mathbb{T}^d\times\mathbb{R}^d\,,$$

supplemented with initial condition

$$F_\varepsilon(0,x,v) = F^\varepsilon_{\rm in}(x,v) = F_{\rm in}(\tfrac{x}{\varepsilon},v)\,.$$

Conservation of mass and density is preserved under this scaling whereas the cooling of the granular gas is now given by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{T}_\varepsilon(t) = -\frac{1-\alpha^2}{\varepsilon^2}\mathcal{D}_\alpha(F_\varepsilon(t),F_\varepsilon(t)),$$

where

$$\boldsymbol{T}_\varepsilon(t) = \int_{\mathbb{R}^d\times\mathbb{T}^d}|v|^2F_\varepsilon(t,x,v)\mathrm{d}v\mathrm{d}x.$$

Self-similar variables

$$\varepsilon^2\partial_t F_\varepsilon(t,x,v) + \varepsilon\, v\cdot\nabla_x F_\varepsilon(t,x,v) = \mathcal{Q}_\alpha(F_\varepsilon,F_\varepsilon), \qquad (x,v)\in\mathbb{T}^d\times\mathbb{R}^d,$$

supplemented with initial condition

$$F_\varepsilon(0,x,v) = F^\varepsilon_{\rm in}(x,v) = F_{\rm in}(\tfrac{x}{\varepsilon},v)\,.$$

Introduce the ansatz

$$F_\varepsilon(t,x,v) = V_\varepsilon(t)^d f_\varepsilon\Big(\tau_\varepsilon(t),x,V_\varepsilon(t)v\Big)\,,$$

with

$$\tau_\varepsilon(t) := \frac{1}{c_\varepsilon}\log(1+c_\varepsilon\, t)\,, \quad V_\varepsilon(t) = (1+c_\varepsilon\, t)\,, \quad t\geqslant 0,\ \ c_\varepsilon = \frac{1-\alpha}{\varepsilon^2} > 0\,.$$

we can prove that f_ε satisfies

$$\varepsilon^2\partial_t f_\varepsilon(t,x,v) + \varepsilon v\cdot\nabla_x f_\varepsilon(t,x,v) + \kappa_\alpha\,\nabla_v\cdot(v f_\varepsilon(t,x,v)) = \mathcal{Q}_\alpha(f_\varepsilon,f_\varepsilon)\,,$$

Here,

$$\kappa_\alpha = 1-\alpha\in(0,1)$$

- drift term $\kappa_\alpha\nabla_v\cdot(vf(t,x,v))$ acts as an energy supply which prevents the total cooling down of the gas.

Self-similar profile

Theorem (Mischler-Mouhot (2006–2009))

For $\alpha \in (\alpha_0, 1)$, there exists a unique solution G_α to the spatially homogeneous steady equation

$$\kappa_\alpha \nabla_v \cdot (v G_\alpha(v)) = \mathcal{Q}_\alpha(G_\alpha, G_\alpha)\,,$$

with unit mass and zero bulk velocity. Moreover,

$$\lim_{\alpha \to 1^-} \| G_\alpha - \mathcal{M} \|_{L^1(\langle v \rangle^2)} = 0\,,$$

where $\mathcal{M}$ is the Maxwellian distribution

$$\mathcal{M}(v) = G_1(v) = (2\pi\vartheta_1)^{-\frac{d}{2}} \exp\left(-\frac{|v|^2}{2\vartheta_1}\right), \qquad v \in \mathbb{R}^d\,,$$

for some explicit temperature $\vartheta_1 > 0$.

The problem at stake

Rescaled Boltzmann equation in self-similar variables

$$\partial_t f_\varepsilon(t,x,v) + \varepsilon^{-1} v \cdot \nabla_x f_\varepsilon(t,x,v) + \varepsilon^{-2}\kappa_\alpha \nabla_v \cdot (v f_\varepsilon(t,x,v)) = \varepsilon^{-2}\mathcal{Q}_\alpha(f_\varepsilon, f_\varepsilon),$$

Questions:

1. Well-posedness of the BE.
   a) In which sense ? (renormalized solutions ?, close-to-equilibrium solutions?)
   b) Estimates on the solution uniform with respect to ε.
2. Convergence of f_ε whenever $\varepsilon \to 0$.

   Do we have, at the limit

   $$f_\varepsilon(t,x,v) \simeq G_\alpha(v) + \varepsilon \Phi\left(v, \varrho(t,x), u(t,x), \theta(t,x)\right)$$

   for some universal profile Φ and where

   $$\varrho(t,x), \theta(t,x) \in \mathbb{R}, \qquad u(t,x) \in \mathbb{R}^d$$

   are macroscopic quantities satisfies some hydrodynamic equations.

In the elastic case $\alpha = 1$, well-known answers to this problem.

1. Well-posedness established in several frameworks. Ukai (1974), Di Perna & Lions (1989)
2. The hydrodynamic limit is well-understood and

$$\varepsilon^{-1}\left(f_{\varepsilon}(t,x,v) - \mathcal{M}(v)\right) \simeq \frac{\varrho(t,x)}{(2\pi\theta(t,x))^{\frac{d}{2}}} \exp\left(-\frac{|v-u(t,x)|^2}{2\theta(t,x)}\right)$$

   with ϱ, u, θ solutions to the Navier-Stokes-Fourier system with Boussinesq relation. De Masi, Esposito & Lebowitz (1989), Bardos & Ukai (1991), Bardos, Golse, Levermore (1991), Golse, Saint-Raymond (2004,2009), Levermore & Masmoudi (2010), Briant (2015).....

To answer these questions, we need to assume that α depends on ε to avoid the explosion of the term $\frac{1-\alpha}{\varepsilon^2}$.

Assumption (Nearly elastic assumption)

The restitution coefficient $\alpha(\cdot)$ is a continuously decreasing function of the Knudsen number ε satisfying the optimal scaling behaviour

$$\alpha = 1 - \lambda_0 \varepsilon^2 + o(\varepsilon^2)$$

with $\lambda_0 \geqslant 0$.

Case 1: If $\lambda_0 = 0$ The elastic regime occurs faster than the hydrodynamic convergence.

Case 2: If $0 < \lambda_0 < \infty$, This is the interesting case in which the elastic and hydrodynamic regimes work at the same pace and the limiting equation keeps track of the inelasticity through λ_0.

Cauchy Theory

Theorem (Existence and estimates – R. Alonso, I. Tristani, B.L. (2021))

One can construct two suitable Banach spaces $X_1 \subset X$ such that, for ε, λ_0 and η_0 sufficiently small with respect to the initial mass and energy, if

$$\|F^{\varepsilon}_{\rm in} - G_{\alpha(\varepsilon)}\|_X \leqslant \varepsilon\,\eta_0$$

then the inelastic Boltzmann equation has a unique solution

$$f_\varepsilon \in \mathcal{C}\big([0,\infty);X\big) \cap L^1\big([0,\infty);X_1\big)$$

satisfying

$$\left\|f_\varepsilon(t) - G_{\alpha(\varepsilon)}\right\|_X \leqslant C\varepsilon\eta_0 \exp\left(-\overline{\lambda}_\varepsilon\, t\right), \qquad \forall t > 0$$

for some positive constant $C > 0$ independent of ε and $-\lambda_\varepsilon < 0$ is the "energy" eigenvalue of the linearized operator, $\lambda_\varepsilon \simeq \frac{1-\alpha}{\varepsilon^2}$.

$X = \mathbb{W}^{k,1}_v \mathbb{W}^{m,2}_x(\langle v\rangle^q)$, $X_1 = \mathbb{W}^{k,1}_v \mathbb{W}^{m,2}_x(\langle v\rangle^{q+1})$ with $q > 3$, $m > d$, $m - 1 \geqslant k \geqslant 0$.

Hydrodynamic limit

Under the previous assumptions, set

$$f_\varepsilon(t,x,v) = G_\alpha + \varepsilon\, h_\varepsilon(t,x,v)\,,$$

with $h_\varepsilon(0,x,v) = h^\varepsilon_{\mathrm{in}}(x,v) = \varepsilon^{-1}\left(F^\varepsilon_{\mathrm{in}} - G_\alpha\right)$ such that

$$\lim_{\varepsilon\to 0} \left\|\boldsymbol{\pi}_0 h^\varepsilon_{\mathrm{in}} - h_0\right\|_{L^1_v \mathbb{W}^{m,2}_x} = 0\,,$$

where $\boldsymbol{\pi}_0$ stands for the projection over the kernel of the elastic linearized Boltzmann operator

$$\boldsymbol{\pi}_0 h = \sum_{i=1}^{d+2} \left(\int_{\mathbb{R}^d} h\,\Psi_i \mathrm{d}v\right) \Psi_i\,\mathcal{M} \qquad\qquad \Psi_{d+2}(v) = \frac{|v|^2 - d\vartheta_1}{\vartheta_1\sqrt{2d}}$$

$$h_0(x,v) = \left(\varrho_0(x) + u_0(x)\cdot v + \tfrac{1}{2}\theta_0(x)(|v|^2 - d\vartheta_1)\right)\mathcal{M}(v)\,,$$

with $\mathcal{M}$ being the Maxwellian distribution (with temperature ϑ_1) and

$$(\varrho_0, u_0, \theta_0) \in \left[\mathbb{W}^{m,2}_x(\mathbb{T}^d)\right]^{d+2}$$

Hydrodynamic limit

Theorem (R. Alonso, I. Tristani, B. L. (2021))

Under these assumptions on the initial datum, for any $T > 0$, $\{h_\varepsilon\}_\varepsilon$ *converges in some weak sense to a limit* $\boldsymbol{h} = \boldsymbol{h}(t,x,v)$ *which is such that*

$$\boldsymbol{h}(t,x,v) = \Big(\varrho(t,x) + u(t,x)\cdot v + \frac{1}{2}\theta(t,x)(|v|^2 - d\vartheta_1)\Big)\,\mathcal{M}(v)\,,$$

where

$$(\varrho, u, \theta) \in \mathcal{C}\left([0,T];\ \left[\mathbb{W}_x^{m-2,2}(\mathbb{T}^d)\right]^{d+2}\right) \cap L^1\left((0,T);\ \left[\mathbb{W}_x^{m,2}(\mathbb{T}^d)\right]^{d+2}\right),$$

is solution to the following incompressible Navier-Stokes-Fourier system with forcing

$$\begin{cases} \partial_t u - \frac{\nu}{\vartheta_1}\,\Delta_x u + \vartheta_1\, u\cdot\nabla_x\, u + \nabla_x p = \lambda_0 u\,, \\ \partial_t\,\theta - \frac{\gamma}{\vartheta_1^2}\,\Delta_x\theta + \vartheta_1\, u\cdot\nabla_x\theta = \dfrac{\lambda_0\,\bar{c}}{2(d+2)}\sqrt{\vartheta_1}\,\theta\,, \\ \mathrm{div}_x u = 0, \qquad \varrho + \vartheta_1\,\theta = 0\,, \end{cases}$$

The above system is subject to initial conditions $(\varrho_{\rm in}, u_{\rm in}, \theta_{\rm in})$ related to $(\varrho_0, u_0, \theta_0)$. The viscosity $\boldsymbol{\nu} > 0$ and heat conductivity $\gamma > 0$ are explicit and $\lambda_0 > 0$ is the parameter appearing in our nearly elastic assumption. The parameter $\bar{c} > 0$ is depending on the collision kernel $b(\cdot)$.

Remark

- *If $\lambda_0 = 0$, then one recovers the classical Navier-Stokes-Fourier system. This confirms that, in this case, the elastic limit first occurs and then the hydrodynamic behaviour of the granular gas is that of a classical one.*
- *If $\lambda_0 > 0$, the systems maintains the memory of the inelasticity parameter α through $\lim_{\varepsilon\to 0} \frac{1-\alpha}{\varepsilon^2}$.*

Comments

1. Our result is, seemingly, the first result capturing the hydrodynamical limit for granular gases.
2. New Navier-Stokes-Fourier system derived in this context.
3. Approach is perturbative in many aspects.

Main features of the proof

- Doubly pertubative approach: close-to-equilibrium & close-to-elastic.
- Special role played by spectral theory of the linearized Boltzman model.
- Technical nonlinear estimates.
- Hydrodynamic limit in some perturbative regime.

Strategy of proof

Three steps

1. Spectral analysis of the full linearized Boltzmann operator.
2. Nonlinear estimates for small fluctuations.
3. Passage to the limit.

Study of fluctuations

Introducing

$$f_\varepsilon(t,x,v) = G_\alpha(v) + \varepsilon\, h_\varepsilon(t,x,v)\,,$$

the fluctuation h_ε satisfies

$$\begin{cases} \partial_t h_\varepsilon(t,x,v) + \dfrac{1}{\varepsilon} v\cdot\nabla_x h_\varepsilon(t,x,v) - \dfrac{1}{\varepsilon^2}\mathscr{L}_\alpha h_\varepsilon(t,x,v) = \dfrac{1}{\varepsilon}\mathcal{Q}_\alpha(h_\varepsilon,h_\varepsilon)(t,x,v)\,, \\ h_\varepsilon(0,x,v) = h_\varepsilon^{\rm in}(x,v)\,, \end{cases}$$

where $\mathscr{L}_\alpha$ is the linearized collision operator (local in the x-variable) defined as

$$\mathscr{L}_\alpha h = \mathcal{Q}_\alpha(h,G_\alpha) + \mathcal{Q}_\alpha(G_\alpha,h) - \kappa_\alpha \nabla_v\cdot(vh)\,,$$

Elastic case: $\mathscr{L}_1$ the usual linearized operator around $G_1 = \mathcal{M}$

$$\mathscr{L}_1(h) = \mathcal{Q}_1(\mathcal{M},h) + \mathcal{Q}_1(h,\mathcal{M}).$$

The (full) linearized operator is then

$$\mathcal{G}_{\alpha,\varepsilon} h = \varepsilon^{-2} \mathscr{L}_\alpha(h) - \varepsilon^{-1} v \cdot \nabla_x h$$

and the Botlzmann equation is re-written as a quasi-linear equation

$$\partial_t h_\varepsilon = \mathcal{G}_{\alpha,\varepsilon} h_\varepsilon + \frac{1}{\varepsilon} \mathcal{Q}_\alpha(h_\varepsilon, h_\varepsilon).$$

Question: Properties of $\mathcal{G}_{\alpha,\varepsilon}$? Spectrum, C_0-semigroup generation ?

The elastic case

For $\alpha = 1$, properties of $\mathcal{G}_{1,\varepsilon}$ are well-known. In a large class of Banach spaces, $\mathcal{G}_{1,\varepsilon}$ is the generator of a C_0-semigroup and admits a spectral gap.

- Well-known facts in spaces with Maxwellian weights based upon $L^2_v(\mathcal{M}^{-\frac{1}{2}})$ (on which $\mathscr{L}_1$ is self-adjoint).
- Careful study of the spectrum due to Ellis & Pinsky (1975). Crucial brick in the study of the Navier-Stokes limit by Bardos & Ukai (1991). Extended recently by Gallagher, Tristani (2020).
- Results extended to smaller spaces Gualdani, Mischler, Mouhot (2017), Guo (2006), Gervais (2021).
- Estimates uniform with respect to ε obtained very recently by Briant, Merino-Aceituno, Mouhot (2019).

$$\left\|\mathcal{V}_{1,\varepsilon}(t)\left[h - \mathbf{P}_0 h\right]\right\|_{\mathbb{W}^{s,1}_v \mathbb{W}^{\ell,2}_x(\langle v\rangle^q)} \leqslant C_0 \exp(-\mu_\star t)\left\|h - \mathbf{P}_0 h\right\|_{\mathbb{W}^{s,1}_v \mathbb{W}^{\ell,2}_x(\langle v\rangle^q)}, \qquad \forall\, t \geqslant 0,$$

with $\mathcal{V}_{1,\varepsilon}(t) = \exp\left(t\mathcal{G}_{1,\varepsilon}\right)$, $\ell \geqslant s \geqslant 0$, $q > q^\star$.

$\mathbf{P}_0$ is the spectral projection onto $\mathrm{Ker}(\mathcal{G}_{1,\varepsilon}) = \mathrm{Ker}(\mathscr{L}_1)$ which is *independent of* ε

$$\mathbf{P}_0 h = \boldsymbol{\pi}_0 \left(\int_{\mathbb{T}^d} h \mathrm{d}x \right).$$

In the elastic case, the nonlinear dynamics occurs on $\mathrm{Range}(\mathbf{I} - \mathbf{P}_0)$.

$$\mathbf{P}_0 \mathcal{Q}_1(h, h) = 0$$

So, Duhamel formula says that

$$h_\varepsilon(t) = \mathcal{V}_{1,\varepsilon}(t) h_{\mathrm{in}} + \frac{1}{\varepsilon} \int_0^t \mathcal{V}_{1,\varepsilon}(t-s)(\mathbf{I} - \mathbf{P}_0)\mathcal{Q}_1(h_\varepsilon(s), h_\varepsilon(s)) \mathrm{d}s$$

In the inelastic case, we do not know what could be the equivalent of the spectral projection $\mathbf{P}_0$ and, more importantly, we do not expect $\mathcal{Q}_{\alpha,\varepsilon}(h, h)$ to stay on the kernel of this spectral projection.

Perturbing the elastic case

Goal: exploit this to deduce properties of $\mathcal{G}_{\alpha,\varepsilon}$.
Crucial point

$$\|\mathcal{G}_{\alpha,\varepsilon} - \mathcal{G}_{1,\varepsilon}\| = \mathbf{O}\left(\frac{1-\alpha}{\varepsilon^2}\right).$$

- This is not a standard pertubation argument (in the sense of Kato, say) because the domain of $\mathscr{L}_\alpha$ is much smaller than that of $\mathscr{L}_1$.

Not enough to deduce directly any kind of spectral structure.

Theorem (In $\mathbb{W}_v^{s,1}\mathbb{W}_x^{\ell,2}(\langle v\rangle^q)$ $\ell \in \mathbb{N}$, $s \geqslant 0$, $\ell \geqslant s$, $q > q^\star$)

For $\mu_\star - \mu > 0$ sufficiently small and ε small enough

$$\mathfrak{S}(\mathcal{G}_{\alpha,\varepsilon}) \cap \{z \in \mathbb{C} \,;\, \mathrm{Re} z \geqslant -\mu\} = \{\lambda_1(\varepsilon), \dots, \lambda_{d+2}(\varepsilon)\},$$

where $\lambda_1(\varepsilon) = 0$, $\lambda_j(\varepsilon) = \varepsilon^{-2}\kappa_\alpha > 0$, $j = 2, \dots, d+1$, and

$$\lambda_{d+2}(\varepsilon) = -\lambda_\varepsilon = -\frac{1-\alpha}{\varepsilon^2} + \mathrm{O}(\varepsilon^2), \qquad \textit{for } \varepsilon \simeq 0$$

are eigenvalues of $\mathcal{G}_{\alpha,\varepsilon}$ with $|\lambda_j(\varepsilon)| < \mu_\star - \mu$.

- These eigenvalues are actually associated to the (spatially homogenous) collision operator.
- The negative eigenvalue $-\lambda_\varepsilon$ will be enough to get the asymptotic stability.
- The difficulty is to prove that, for ε small enough, there is nothing more than these eigenvalues.

Nonlinear equation

We solve the equation

$$\partial_t h_\varepsilon = \mathcal{G}_{\alpha,\varepsilon} h_\varepsilon + \frac{1}{\varepsilon}\mathcal{Q}_\alpha(h_\varepsilon, h_\varepsilon), \qquad h = h_\varepsilon \in \mathcal{E} = \mathbb{W}^{k,1}_v \mathbb{W}^{m,2}_x(\langle v\rangle^q).$$

Main ideas

- The projection $\mathbf{P}_\varepsilon$ associated to the above set of eigenvalues does not kill the collision operator. No nice energy estimates (no symmetry space).
- Splitting of the linearized operator (Gualdani, Mischler, Mouhot (2017) and introduced by Tristani (2016) for granular gases).

  $$\mathcal{G}_{\alpha,\varepsilon} = \frac{1}{\varepsilon^2}\mathcal{A} + \mathcal{B}_{\alpha,\varepsilon}$$

  with $\mathcal{A}$ regularizing in velocity and

  $$\mathcal{B}_{\alpha,\varepsilon} + \varepsilon^{-2}\nu_0 \text{ (hypo)-dissipative}$$

- Important: typically, $\mathcal{A}_\alpha$ maps continuously L^1_v into any Sobolev space $\mathbb{W}^{k,p}_v(\varpi_q)$ for any k, p, q. **No regularizing effect in spatial variable.**

- Important: typically, $\mathcal{A}_\alpha$ maps continuously L^1_v into any Sobolev space $\mathbb{W}^{k,p}_v(\varpi_q)$ for any k, p, q. **No regularizing effect in spatial variable.**

- One writes

$$\partial_t h_\varepsilon = \mathcal{B}_{\alpha,\varepsilon} h_\varepsilon + \frac{1}{\varepsilon}\left(\mathcal{Q}_\alpha(h_\varepsilon, h_\varepsilon) - \mathcal{Q}_1(h_\varepsilon, h_\varepsilon)\right) + \frac{1}{\varepsilon}\mathcal{Q}_1(h_\varepsilon, h_\varepsilon) + \frac{1}{\varepsilon^2}\mathcal{A} h_\varepsilon$$

The first term is nicely dissipative, the second is not as stiff as expected in the regime we investigate

$$\|\mathcal{Q}_\alpha(h_\varepsilon, h_\varepsilon) - \mathcal{Q}_1(h_\varepsilon, h_\varepsilon)\| = \mathrm{O}(1-\alpha).$$

The last two terms are purely elastic ! Even if they are stiff.

We use an approach borrowed from Briant, Merino-Aceituno, Mouhot (2019) and splits our solution

$$h_\varepsilon = h^0_\varepsilon + h^1_\varepsilon$$

$$\begin{cases} \partial_t h^0 & = \mathcal{B}_{\alpha,\varepsilon} h^0 + \varepsilon^{-1}\mathcal{Q}_\alpha(h^0,h^0) + \varepsilon^{-1}\Big[\mathcal{Q}_\alpha(h^0,h^1) + \mathcal{Q}_\alpha(h^1,h^0)\Big] \\ & \quad + \Big[\mathcal{G}_{\alpha,\varepsilon} h^1 - \mathcal{G}_{1,\varepsilon} h^1\Big] + \varepsilon^{-1}\Big[\mathcal{Q}_\alpha(h^1,h^1) - \mathcal{Q}_1(h^1,h^1)\Big], \\ h^0(0) & = h^\varepsilon_{\mathrm{in}} \in \mathcal{E}. \end{cases}$$

and

$$\begin{cases} \partial_t h^1 & = \mathcal{G}_{1,\varepsilon} h^1 + \varepsilon^{-1}\mathcal{Q}_1(h^1,h^1) + \varepsilon^{-2}\mathcal{A}h^0, \\ h^1(0) & = 0. \end{cases}$$

We look for $h^1 \in \mathcal{H} = \mathbb{W}^{m,2}_{v,x}\left(\mathcal{M}^{-1/2}\right)$. Recall the regularizing effect to $\mathcal{A}$!

$$\begin{cases} \partial_t h^0 & = \mathcal{B}_{\alpha,\varepsilon} h^0 + \varepsilon^{-1}\mathcal{Q}_\alpha(h^0,h^0) + \varepsilon^{-1}\Big[\mathcal{Q}_\alpha(h^0,h^1) + \mathcal{Q}_\alpha(h^1,h^0)\Big] \\ & \qquad + \Big[\mathcal{G}_{\alpha,\varepsilon} h^1 - \mathcal{G}_{1,\varepsilon} h^1\Big] + \varepsilon^{-1}\Big[\mathcal{Q}_\alpha(h^1,h^1) - \mathcal{Q}_1(h^1,h^1)\Big], \\ h^0(0) & = h^\varepsilon_{\rm in} \in \mathcal{E}. \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h^0(t)\|_{\mathcal{E}} \leqslant -\varepsilon^{-2}\nu_0\|h^0(t)\|_{\mathcal{E}_1} + C\varepsilon^{-1}\Big(\|h^0(t)\|_{\mathcal{E}} + \|h^1(t)\|_{\mathcal{E}_1}\Big)\|h^0(t)\|_{\mathcal{E}_1}$$
$$+ C(1-\alpha)\varepsilon^{-2}\|h^1(t)\|_{\mathcal{E}_2} + C(1-\alpha)\varepsilon^{-1}\|h^1(t)\|^2_{\mathcal{E}_2}$$

with $\mathcal{E}_2 \subset \mathcal{E}_1 \subset \mathcal{E}$. Choosing, for ε small enough,

$$\nu_0 - \varepsilon\, C\Big(\|h^0(t)\|_{\mathcal{E}} + \|h^1(t)\|_{\mathcal{E}_1}\Big) \geqslant \mu_0 > 0$$

we obtain that

$$\|h^0(t)\|_{\mathcal{E}} \lesssim \|h^0(0)\|_{\mathcal{E}}\, e^{-\frac{\mu_0}{\varepsilon^2}t} + \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)}\|h^1(s)\|_{\mathcal{E}_2}\,\mathrm{d}s$$
$$+ \varepsilon\lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)}\|h^1(s)\|^2_{\mathcal{E}_2}\,\mathrm{d}s\,.$$

with $\lambda_\varepsilon \simeq \frac{1-\alpha(\varepsilon)}{\varepsilon^2}$.

Estimating h^1

Difficulty: how to apply the spectral projection $\mathbf{P}_\varepsilon$ associated to the energy eigenvalue $-\lambda_\varepsilon$?

We cheat a bit and apply the one associated to the elastic operator:

$$\partial_t \mathbf{P}_0 h(t) = \mathbf{P}_0 \mathcal{G}_\varepsilon h(t) + \frac{1}{\varepsilon}\mathbf{P}_0 \mathcal{Q}_\alpha(h,h)$$

with

$$\mathbf{P}_0\left[\mathcal{G}_\varepsilon h(t)\right] \simeq -\lambda_\varepsilon \mathbf{P}_0 h(t) + \mathrm{O}\left(\frac{1-\alpha(\varepsilon)}{\varepsilon^2}\right)\|(\mathbf{I}-\mathbf{P}_0)\,h(t)\|_{\mathcal{E}} \qquad t \geqslant 0$$

and

$$\|\mathbf{P}_0 \mathcal{Q}_\alpha(h,h)\|_{\mathcal{E}} \simeq (1-\alpha)\,|\mathcal{D}_\alpha(h,h)|\,\|\phi_0\|_{\mathcal{E}} \lesssim (1-\alpha)\|h\|_{\mathcal{E}}^2$$

Thus

$$\|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}} \lesssim \|\mathbf{P}_0 h(0)\|_{\mathcal{E}} e^{-\lambda_\varepsilon t} + \|h^0(t)\|_{\mathcal{E}} + \varepsilon\lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)}\left(\|h^1(s)\|_{\mathcal{E}}^2 + \|h^0(s)\|_{\mathcal{E}}^2\right)\mathrm{d}s + \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)}\left(\left\|h^0(s)\right\|_{\mathcal{E}} + \left\|(\mathbf{I}-\mathbf{P}_0)\,h^1(s)\right\|_{\mathcal{E}}\right)\mathrm{d}s.$$

Estimating $\Psi(t) = (\mathbf{I} - \mathbf{P}_0)h^1(t)$

$$\partial_t \Psi = \mathcal{G}_{1,\varepsilon}\Psi + \varepsilon^{-1}\mathcal{Q}_1(h^1, h^1) + \varepsilon^{-2}(\mathbf{I} - \mathbf{P}_0)\mathcal{A}h^0 ,$$

Known estimates for the elastic semigroup on the "symmetric space" $\mathcal{H}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Psi(t)\|_{\mathcal{H}}^2 \leqslant -c_0\|\Psi(t)\|_{\mathcal{H}_1}^2 + C\|h^1(t)\|_{\mathcal{H}}^2\,\|h^1(t)\|_{\mathcal{H}_1}^2 + \varepsilon^{-2}\|\Psi(t)\|_{\mathcal{H}}\,\|\mathbf{P}_0^{\perp}h^0(t)\|_{\mathcal{H}}.$$

with $\mathcal{H}_1 \subset \mathcal{H}$. Then, we conclude with a Gronwall argument, that for any $r \in (0,1)$,

$$\|h^1(t)\|_{\mathcal{H}}^2 \leqslant C\,\eta_0 \exp\left(-2(1-r)\lambda_\varepsilon\, t\right) , \qquad \forall\, t \geqslant 0 .$$

Hydrodynamic limit

Theorem (Peculiar weak convergence)

Fix $T > 0$, with the splitting $h_\varepsilon = h_\varepsilon^0 + h_\varepsilon^1 \subset L^1\big((0,T); L^1_v \mathbb{W}^{m,2}_x(\langle v\rangle^q)\big)$, up to extraction of a subsequence, one has

$$\begin{cases} \left\{h_\varepsilon^0\right\}_\varepsilon \text{ converges to } 0 \text{ strongly in } L^1((0,T)\,;\,\mathcal{E}) \\ \\ \left\{h_\varepsilon^1\right\}_\varepsilon \text{ converges to } \boldsymbol{h} \text{ weakly in } L^2\left((0,T)\,;\,L^2_v \mathbb{W}^{m,2}_x\left(\mathcal{M}^{-\frac{1}{2}}\right)\right) \end{cases}$$

where $\boldsymbol{h} = \boldsymbol{\pi}_0(\boldsymbol{h})$. In particular, there exist

$$\varrho \in L^2\left((0,T);\ \mathbb{W}^{m,2}_x(\mathbb{T}^d)\right), \qquad \theta \in L^2\left((0,T);\ \mathbb{W}^{m,2}_x(\mathbb{T}^d)\right),$$

$$u \in L^2\left((0,T);\ \left(\mathbb{W}^{m,2}_x(\mathbb{T}^d)\right)^d\right)$$

such that

$$\boldsymbol{h}(t,x,v) = \left(\varrho(t,x) + u(t,x)\cdot v + \frac{1}{2}\theta(t,x)(|v|^2 - d\vartheta_1)\right)\mathcal{M}(v).$$

Classical estimates

Average

$$\left\langle f \right\rangle = \int_{\mathbb{R}^d} f(t,x,v)\mathrm{d}v.$$

For any function $\psi = \psi(v)$ such that $|\psi(v)| \lesssim \langle v \rangle^q(v)$ one has

$$\left\langle \psi\, h_\varepsilon \right\rangle \longrightarrow \left\langle \psi\, \boldsymbol{h} \right\rangle.$$

First consequences:

- *incompressibility condition*

$$\mathrm{div}_x u(t,x) = 0\,, \qquad t \in (0,T)\,,$$

- *Boussinesq relation*

$$\nabla_x \left(\varrho + \vartheta_1 \theta\right) = 0\,.$$

Introducing

$$E(t) = \int_{\mathbb{T}^d} \theta(t,x)\mathrm{d}x, \qquad t \in (0,T)\,,$$

one has *strengthened Boussinesq relation*

$$\varrho(t,x) + \vartheta_1 \left(\theta(t,x) - E(t)\right) = 0\,, \qquad \text{for a.e. } (t,x) \in (0,T) \times \mathbb{T}^d.$$

As in the classical elastic case, we write

$$\left\langle v \otimes v\, h_\varepsilon \right\rangle = \left\langle \boldsymbol{A}\, h_\varepsilon \right\rangle + p_\varepsilon \mathbf{Id}, \qquad p_\varepsilon = \left\langle \frac{1}{d}|v|^2\, h_\varepsilon \right\rangle,$$

with the traceless tensor $\boldsymbol{A} = \boldsymbol{A}(v) = v \otimes v - \frac{1}{d}|v|^2\mathbf{Id}$. One has

$$\partial_t \left\langle h_\varepsilon \right\rangle + \frac{1}{\varepsilon}\mathrm{div}_x \left\langle v\, h_\varepsilon \right\rangle = 0\,,$$

$$\partial_t \left\langle v\, h_\varepsilon \right\rangle + \frac{1}{\varepsilon}\mathrm{Div}_x \left\langle \boldsymbol{A}\, h_\varepsilon \right\rangle + \frac{1}{\varepsilon}\nabla_x p_\varepsilon = \frac{\kappa_\alpha}{\varepsilon^2}\left\langle v\, h_\varepsilon \right\rangle,$$

$$\partial_t \left\langle \tfrac{1}{2}|v|^2 h_\varepsilon \right\rangle + \frac{1}{\varepsilon}\mathrm{div}_x \left\langle \tfrac{1}{2}|v|^2 v\, h_\varepsilon \right\rangle = \frac{1}{\varepsilon^3}\mathscr{J}_\alpha(f_\varepsilon, f_\varepsilon) + \frac{2\kappa_\alpha}{\varepsilon^2}\left\langle \tfrac{1}{2}|v|^2 h_\varepsilon \right\rangle,$$

where

$$\mathscr{J}_\alpha(f,f) = \int_{\mathbb{R}^d} \left[\mathcal{Q}_\alpha(f,f) - \mathcal{Q}_\alpha(G_\alpha, G_\alpha)\right]\, |v|^2 \mathrm{d}v.$$

The LHS converges (in the distributional sense) as in the elastic case.

The RHS is treated as a source term which takes into account the *drift term* and the *dissipation of kinetic energy* at the microscopic level.
It holds

$$\frac{1}{\varepsilon^3}\mathscr{I}_\alpha(f_\varepsilon, f_\varepsilon) \longrightarrow \mathcal{J}_0 \qquad \text{in } \mathscr{D}'_{t,x}\,,$$

where

$$\mathcal{J}_0(t,x) = -\lambda_0\,\bar{c}\,\vartheta_1^{\frac{3}{2}}\left(\varrho(t,x) + \frac{3}{4}\vartheta_1\,\theta(t,x)\right) = -\lambda_0\,\bar{c}\,\vartheta_1^{\frac{5}{2}}\left(E(t) - \frac{1}{4}\theta(t,x)\right)$$

The limit velocity $u(t,x)$ satisfies

$$\partial_t u - \frac{\nu}{\vartheta_1}\,\Delta_x u + \vartheta_1 \mathrm{Div}_x\,(u \otimes u) + \nabla_x p = \lambda_0 u$$

while the limit temperature $\theta(t,x)$ satisfies

$$\partial_t \theta - \frac{\gamma}{\vartheta_1^2}\,\Delta_x \theta + \vartheta_1\, u\cdot\nabla_x\theta = \frac{2}{(d+2)\vartheta_1^2}\mathscr{J}_0 + \frac{2d\lambda_0}{d+2}E(t) + \frac{2}{d+2}\frac{\mathrm{d}}{\mathrm{d}t}E(t)\,,$$

where

$$E(t) = \int_{\mathbb{T}^d}\theta(t,x)\mathrm{d}x, \qquad t \geqslant 0\,.$$

The limit velocity $u(t,x)$ satisfies

$$\partial_t u - \frac{\nu}{\vartheta_1}\Delta_x u + \vartheta_1 \mathrm{Div}_x(u \otimes u) + \nabla_x p = \lambda_0 u$$

while the limit temperature $\theta(t,x)$ satisfies

$$\partial_t \theta - \frac{\gamma}{\vartheta_1^2}\Delta_x \theta + \vartheta_1 \, u \cdot \nabla_x \theta = \frac{2}{(d+2)\vartheta_1^2}\mathcal{J}_0 + \frac{2d\lambda_0}{d+2}E(t) + \frac{2}{d+2}\frac{\mathrm{d}}{\mathrm{d}t}E(t),$$

where

$$E(t) = \int_{\mathbb{T}^d} \theta(t,x)\mathrm{d}x, \qquad t \geqslant 0.$$

One has

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = \frac{2}{d\vartheta_1^2}\int_{\mathbb{T}^d} \mathcal{J}_0(t,x)\mathrm{d}x + 2\lambda_0 E(t) = \bar{c}_0 E(t)$$

and

$$E(0) = \lim_{\varepsilon \to 0}\int_{\mathbb{T}^d}\left\langle \tfrac{1}{2}\left(|v|^2 - (d+2)\vartheta_1\right)h_\varepsilon \right\rangle \mathrm{d}x = 0!$$

Thus, $E(t) = 0$ for any $t \geqslant 0$.

This gives the final system

$$\begin{cases} \partial_t u - \frac{\nu}{\vartheta_1}\,\Delta_x u + \vartheta_1 \mathrm{Div}_x\,(u \otimes u) + \nabla_x p = \lambda_0 u \\ \\ \partial_t\,\theta - \frac{\gamma}{\vartheta_1^2}\,\Delta_x \theta + \vartheta_1\,u \cdot \nabla_x \theta = \frac{\lambda_0\,\bar{c}}{2(d+2)}\sqrt{\vartheta_1}\,\theta \\ \\ \mathrm{div}_x u(t,x) = 0, \qquad \varrho(t,x) + \vartheta_1 \theta(t,x) = 0\,, \qquad x \in \mathbb{T}^d\,. \end{cases}$$

Summary

- We proved the existence and uniqueness of close-to-equilibrum solution to the BE for granular gases.
- Exponential stability of these solutions with decay rate prescribed by the energy eigenvalue $-\lambda_\varepsilon$.
- Prove the convergence of fluctuations to some limiting function $\boldsymbol{h}$ depending on t, x only through macroscopic quantities solving a modified *incompressible Navier-Stokes-Fourier system*.
- Results obtained in a nearly elastic regime and the limiting hydrodynamic system keeps track of this regime.

Open problems/Research projects

- Study the case of viscoelastic hard spheres. In this case, the nearly elastic regime should emerge naturally with the scaling.
- Can one be more precise in the spectral description and derive the equivalent of Ellis & Pinski (1975) result for granular gases ? This would allow for instance to adapt the work of Bardos & Ukai (1991), Gallagher & Tristani (2020).
- Other kinds of scalings (Euler).
- Understand the role of entropy for granular gases.