Depth zero representations over $\bar{\mathbb{Z}}[\frac{1}{p}]$

(with Jean-François Dat)

I) Motivations

The local Langlands correspondence

$$\left\{ \begin{array}{l} \text{irreducible smooth representations} \\ \text{of p-adic group } G \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Langlands parameters} \\ W_F \to {}^L G \end{array} \right\}$$

. $\mathbb{C}$, $\overline{\mathbb{F}}_\ell$ $\ell \neq p$, $\bar{\mathbb{Z}}[\frac{1}{p}]$
$\bar{\mathbb{Z}}_\ell$

Dat, Helm, Kurinczuk, Moss (DHKM) have studied the moduli space $Z^1(W_F^\circ, \hat{G})$ of Langlands parameters over $\bar{\mathbb{Z}}[\frac{1}{p}]$.

Thm (DHKM): If G is tamely-ramified, $Z^1(W_F^\circ, \hat{G})_{\text{tame}}$ (parameters with trivial restriction to the wild inertia) is connected

On the group side, this suggest that the category of depth zero representations over $\bar{\mathbb{Z}}[\frac{1}{p}]$ is indecomposable.

↳ same for finite reductive groups.

II) Finite reductive groups

Let G be the $\mathbb{F}_q$-points ($q=p^n$) of a reductive group defined over $\mathbb{F}_q$.

R = commutative ring

$\mathrm{Rep}_R(G)$ = abelian category of RG-modules

A <u>block</u> is an indecomposable summand of $\mathrm{Rep}_R(G)$

Blocks $\longleftrightarrow$ indecomposable two-sided ideals of the ring RG

$\longleftrightarrow$ primitive idempotents in the centre $Z(RG)$

$RG = B_1 \oplus \cdots \oplus B_s \iff 1 = e_1 + \cdots + e_s \qquad B_i = e_i RG$

<u>$\mathbb{C}$:</u> $\pi \in \mathrm{Irr}(G) \qquad e_\pi = \dfrac{\dim(\pi)}{|G|} \sum_{g \in G} \mathrm{tr}(\pi(g))\, g \in \mathbb{C}G$

$$1 = \sum_{\pi \in \mathrm{Irr}(G)} e_\pi$$

Actually, $e_\pi \in \bar{\mathbb{Z}}[\frac{1}{|G|}]G$

<u>$\bar{\mathbb{Z}}_\ell$</u>: $\ell \neq p$ The ℓ-blocks (blocks over $\bar{\mathbb{Z}}_\ell$) correspond to minimal subsets $I \subseteq \mathrm{Irr}(G)$ such that $\sum_{\pi \in I} e_\pi \in \bar{\mathbb{Z}}[\frac{1}{|G|^{\ell'}}]G$ where $|G|^{\ell'}$ is the prime-to-ℓ factor of $|G|$.

<u>Thm</u> (Dat-L.): The category $\mathrm{Rep}_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G)$ is indecomposable. Equivalently, the central idempotent 1 in $\bar{\mathbb{Z}}[\frac{1}{p}]$ is primitive.

<u>Strategy of the proof:</u> $\pi, \pi' \in \mathrm{Irr}(G)$

For $\ell \neq p$, we say that $\pi \sim_\ell \pi'$ if π and π' are in the same ℓ-block.
We prove that for $\pi, \pi' \in \mathrm{Irr}(G)$ there exist prime numbers $\ell_1, \dots, \ell_r \neq p$ and $\pi_1, \dots, \pi_{r-1} \in \mathrm{Irr}(G)$ such that

$$\pi \sim_{\ell_1} \pi_1 \sim_{\ell_2} \pi_2 \sim_{\ell_3} \cdots \sim_{\ell_r} \pi'$$

III) p-adic groups

F: local non-archimedean field
$\mathbb{G}$: reductive group over F
$G = \mathbb{G}(F)$
R = commutative ring in which p is invertible

$\mathrm{Rep}_R(G)$ = category of smooth RG-modules

$\mathfrak{Z}_R(G)$ = Bernstein centre = centre of the category $\mathrm{Rep}_R(G)$
(endomorphisms of the identity functor)
$\mathfrak{z} \in \mathfrak{Z}_R(G)$ $\quad V \in \mathrm{Rep}_R(G)$ $\quad \mathfrak{z}_V : V \to V$

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle \mathfrak{z}_V}\downarrow & & \downarrow{\scriptstyle \mathfrak{z}_W} \\ V & \xrightarrow[f]{} & W \end{array}$$

The blocks of $\mathrm{Rep}_R(G)$ (indecomposable direct factor subcategories) correspond to primitive idempotents in $\mathfrak{Z}_R(G)$.

$\mathrm{Rep}_R^0(G)$ = subcategory of depth zero representations
("representations comming from finite reductive groups")

↳ direct factor subcategory
↳ correspond to some idempotent $\varepsilon_0 \in \mathfrak{Z}_R(G)$

Thm (Dat + L.) G quasi-split tamely-ramified over F

$\mathrm{Rep}^0_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G)$ is indecomposable

$\Leftrightarrow$ ε_0 is primitive in $\mathfrak{Z}_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G)$

$\pi, \pi' \in \mathrm{Irr}_{\bar{\mathbb{Q}}}(G)$ $\quad \ell \neq p$

We say that π and π' are in the same ℓ-block if for any embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$ the base changed representation π, π' belongs to the same block over $\mathrm{Rep}_{\bar{\mathbb{Z}}_\ell}(G)$

Cor: π, π' two irreducible $\bar{\mathbb{Q}}G$-modules of depth zero. There exists a sequence of primes $\ell_1, \dots, \ell_r \neq p$ and of irreducible $\bar{\mathbb{Q}}G$-modules $\pi_0 = \pi$, $\pi_1, \dots, \pi_r = \pi'$ such that π_{i-1} and π_i belong to the same ℓ_i-block

G not tamely-ramified

T maximal torus defined over F
$\Phi^\vee \subseteq X_*(T)$ set of (absolute) coroots
Fr: geometric Frobenius in $\mathrm{Gal}(\bar{F}/F)/I_F$

$\pi_1(G) = \dfrac{X_*(T)}{\langle \Phi^\vee \rangle}$ $\pi_1(G) = \left(\pi_1(G)_{I_F}\right)^{Fr}$

Kottwitz $\kappa_G : G \twoheadrightarrow \pi_1(G)$

- surjectif
- kernel $G^\circ = \ker(\kappa_G)$ is the subgroup of G generated by parahoric subgroups
- inverse image $G^1 := \kappa_G^{-1}(\pi_1(G)_{tors})$ is the subgroup of G generated by compact subgroups

R : $\bar{\mathbb{Z}}[\frac{1}{p}]$-algebra $\Psi_G(R) = \mathrm{Hom}(\pi_1(G), R^\times)$

$\Psi_G(R)$ identifies via κ_G to a group of R-valued characters of G.
So it acts on $\mathrm{Rep}_R(G)$ by twisting representations, and on $\mathrm{Idem}(\mathfrak{Z}_R(G))$

G : quasi-split
T : centralizer of a maximal split torus

$$\Psi_T^{\delta}(\bar{\mathbb{Z}}[\tfrac{1}{p}])_p = \mathrm{Hom}\left(\pi_1(T)_{p\text{-tors}}, \mathbb{Z}[\tfrac{1}{p}]^\times\right)$$

Thm (Dat-L.) G quasi-split. The action of $\Psi_T^{\delta}(\bar{\mathbb{Z}}[\frac{1}{p}])_p$ on the set of primitive idempotents of $\mathfrak{Z}_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G)$ is simply-transitive

<u>Dual side</u>

We have a simply-transitive action of $\pi_0(\hat{T}^{I_F})_{F}$ on the set of connected components of $Z^1(W_F^0, \hat{G})_{\mathrm{Tame}}$.

$$\pi_0(\hat{T}^{I_F})_{Fr} \cong \Psi_T^{\delta}(\bar{\mathbb{Z}}[\tfrac{1}{p}])_p$$

Thm (Dat-L.) : G quasi-split over F. There is a natural bijection between connected components of $Z^1(W_F^0, \hat{G})_{\mathrm{Tame}}$ and blocks of $\mathrm{Rep}^0_{\bar{\mathbb{Z}}[\frac{1}{p}]}(G)$

IV) Applications

application to complex representations

idea: π, π' depth 0, we can link π and π' by a series of congruences for different primes $\pi \sim_{\ell_1} \pi_1 \sim \dots \sim_{\ell_r} \pi'$

Fargues and Scholze : $\pi \mapsto \varphi_\pi$

π: irreducible representation

φ_π: semi-simple Langlands parameter

↳ general construction

↳ difficult to prove properties (for cuspidal representation for instance)

expectation: if π has depth 0 then φ_π is tamely-ramified.

For $\ell \neq p$ $\quad FS_\ell$: $\mathcal{O}(Z^1(W_F^0, \hat{G})_{\bar{\mathbb{Z}}_\ell})^{\hat{G}} \to \mathfrak{Z}_{\bar{\mathbb{Z}}_\ell}(G)$

$\pi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G)$

φ_π is the $\overline{\mathbb{Q}}_\ell$-point of $Z^1(W_F^0, \hat{G})/\hat{G}$ obtained by

$$\mathcal{O}(Z^1(W_F^0, \hat{G})_{\overline{\mathbb{Z}}_\ell})^{\hat{G}} \to \mathfrak{Z}_{\overline{\mathbb{Z}}_\ell}(G) \longrightarrow \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(\pi) = \overline{\mathbb{Q}}_\ell$$

If π and π' are in the same ℓ-block then $\varphi_{\pi}|_{I_F^\ell} \sim \varphi_{\pi'}|_{I_F^\ell}$

⇝ π depth 0

control φ_π using a series of congruences with differents primes ℓ.

⚠ We need to have "independence of ℓ"

if $\pi \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(G)$, we want φ_π to be a semi-simple $\overline{\mathbb{Q}}$-parameter independant of the choice of ℓ and embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$