The local converse theorem for quasi-split $\mathrm{SO}(2n)$

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Three arithmetic invariants

There are three important arithmetic invariants in number theory.

- L-functions
- γ-functions
- ϵ-functions (or root numbers)

Typically, L-functions encodes the global arithmetic and γ-functions, ϵ-functions govern the local arithmetic.

The global converse theorem (GCT) and the local converse theorem (LCT) are prominent examples to show these phenomena.

Global Converse Theorem for $GL(n)$

For an irreducible admissible representation π and π' of $GL(n)(\mathbb{A})$ and $GL(m)(\mathbb{A})$, respectively, we can define its Rankin–Selberg L-function $L(s, \pi \times \pi')$.

We say that $L(s, \pi \times \pi')$ is nice if it satisfies

- (A.C) $L(s, \pi \times \pi')$ extends to an entire function on $\mathbb{C}$.
- (F.E) $L(s, \pi \times \pi') = \epsilon(s, \pi \times \pi')L(1 - s, \widetilde{\pi} \times \widetilde{\pi'})$
- (boundedness in vertical strips) $L(s, \pi \times \pi')$ is bounded in vertical strips.

If π and π' are cuspidal automorphic representations, then $L(s, \pi \times \pi')$ are nice!

Q) The converse holds?

$$T(m) = \bigcup_{1 \le d \le m} \{\pi' : \text{a cuspidal automorphic representation of } GL(d)\}$$

(Global Converse Theorem), Cogdell, P-S (1994)

Let π be an irreducible admissible representation of $GL_n(\mathbb{A})$.
Suppose $L(s, \pi \times \pi')$ are nice for all $\pi' \in T(n-1)$. Then π is a cuspidal automorphic representation.

The GCT tells that the family of GL-twisted L-functions determines the automorphy of global irreducible representations of $\mathrm{GL}_n(\mathbb{A})$.

The LCT shows that the local GL-twisted γ-functions uniquely determine the isomorphism classes of generic representations.

History of LCT

Let F be a p-adic local field of characteristic zero.

(weak LCT for GL_n) G. Henniart [Invent. M (1993)]

Let π_1, π_2 be irreducible generic admissible representations of GL_n with the same central characters. If the twisted γ-factors are same, that is,

$$\gamma(s, \pi_1 \times \rho, \psi) = \gamma(s, \pi_2 \times \rho, \psi)$$

for all irreducible supercuspidal representations ρ of $\mathrm{GL}_i(F)$ for $1 \leq i \leq n-1$, then $\pi_1 \simeq \pi_2$.

(weak LCT for SO_{2n+1}) D. Jiang and D. Soudry, [Ann. M (2003)]

Let π_1, π_2 be irreducible generic admissible representations of SO_{2n+1} with the same central characters. If the twisted γ-factors are same, that is,

$$\gamma(s, \pi_1 \times \rho, \psi) = \gamma(s, \pi_2 \times \rho, \psi)$$

for all irreducible supercuspidal representations ρ of $\mathrm{GL}_i(F)$ for $1 \leq i \leq 2n-1$, then $\pi_1 \simeq \pi_2$.

[Local Converse Conjecture] D. Jiang (2006)

Let G_n be one of the quasi-split group over F in the following $(\mathrm{GL}_{2n}, \mathsf{GL}_{2n+1},\ \mathrm{O}_{2n},\ \mathrm{SO}_{2n},\ \mathrm{Sp}_{2n},\ \mathrm{SO}_{2n+1},\ \mathrm{U}_{2n},\ \mathrm{U}_{2n+1}.)$

Let π_1, π_2 be irreducible generic admissible representations of G_n. If

$$\gamma(s, \pi_1 \times \rho, \psi) = \gamma(s, \pi_2 \times \rho, \psi)$$

for all irreducible supercuspidal representations ρ of $\mathrm{GL}_i(F)$ for all $1 \leq i \leq n$, then $\pi_1 \simeq \pi_2$.

Recent development of LCT for classical groups

- "LCT for GL_n" (J. Chai [2019, J.E.M.S], H. Jacquet and B. Liu [2018, A.J.M])
- "LCT for U_{2n}", (K. Morimoto, [2018, Trans. M])
  (He used the recent "LCT for GL_n" result via descent method.)

- "LCT for Sp_{2n} for supercuspidal case", (Q. Zhang, [2018, Math. Ann])
- "LCT for U_{2n+1} for supercuspidal case", (Q. Zhang, [2019, Trans. M])
- "LCT for SO_{2n+1} and Sp_{2n} for both $\mathrm{char}(F) = 0, p \neq 2$ cases", (Y. Jo, [2022, preprint])
- "LCT for 'split' SO_{2n}", (B. Liu and A. Hazeltine, [2022, preprint])
  (The above four results uses the 'partial Bessel function theory' developed by Cogdell, Shahidi and Tsai [2017, Duke].)

- "LCT for Mp_{2n} for both $\mathrm{char}(F) = 0, p \neq 2$ cases", (H-, [2023, preprint])
  (use precise local theta correspondence between SO_{2n+1} and Mp_{2n})

Main result

We proved the "LCT for 'quasi-split' O_{2n}" for both $\mathrm{char}(F) = 0, p \neq 2$ cases.

(LCT for O_{2n}) H-K-K, (2023), preprint

Let π_1, π_2 be irreducible generic admissible representations of O_{2n} with the same central characters. If the twisted γ-factors are same, that is,

$$\gamma(s, \pi_1 \times \rho, \psi) = \gamma(s, \pi_2 \times \rho, \psi)$$

for all irreducible supercuspidal representations ρ of $\mathrm{GL}_i(F)$ for $1 \leq i \leq n$, then $\pi_1 \simeq \pi_2$.

As a corollary, we also proved the "LCT for 'quasi-split' SO_{2n}".

This is achieved by the precise study of the local theta correspondence between $\mathrm{O}(V)$ and $\mathrm{Sp}(W)$. To understand this theorem precisely, we first explain what it means 'generic' in the theorem.

From now on, F denotes a local field of characteristic zero or $p \neq 2$.

Comments on Arthur's results

The LCT is immediate from the local Langlands correspondence for quasi-split classical groups, which is done by Arthur and many others.

However, Arthur's results are conditional of weighted fundamental lemma and stabilization of the trace formula, which is not yet completed.

To make our results unconditional, we avoid to use any Arthur's results.

Generic characters

- G : a quasi-split reductive group over F
- $\mathrm{B} = \mathrm{TU}$: F-rational Borel subgroup of G
- T : maximal F-torus of B
- U : the unipotent radical of B
- Z : the center of G
- $\mathrm{Irr}(G)$: the set of equivalence classes of irreducible admissible representations of $G(F)$

Observation

$\mathrm{T}(F)$ acts on the set of characters on $\mathrm{U}(F)$ by conjugation.

For a character μ of $\mathrm{U}(F)$, we say μ is generic if its stabilizer in $\mathrm{T}(F)$ is $\mathrm{Z}(F)$. For $\pi \in \mathrm{Irr}(G)$, we say π is μ-generic if $\mathrm{Hom}_U(\pi, \mu) \neq 0$.

Fact

If π is μ-generic, then for any $t \in \mathrm{T}(F)$, π is μ^t-generic.

Orthogonal space

Let $(V, \langle \, , \, \rangle_V)$ be a $2n$-dimensional vector space over F with a non-degenerate symmetric form $\langle \, , \, \rangle_V$ on V.

- $\mathbb{H}$ be the hyperbolic plane over F
- For any $c, d \in F^\times$, let $(V_{c,d}, \langle \, , \, \rangle_2)$ be a 2-dimensional orthogonal space with a basis $\{e, e'\}$ satisfying

$$< e, e >= 2c, \; < e, e' >= 0, \; < e', e' >= -2cd$$

- $\epsilon \in \mathrm{O}(V_{c,d})$: the involution of $V_{c,d}$ satisfying

$$\epsilon(e) = e, \quad \epsilon(e') = -e'.$$

When $V \simeq \mathbb{H}^{n-1} \oplus V_{c,d}$, we says that V is associated to (c, d).

Orthogonal group

- $\mathrm{O}(V)$: the isometry group of V
- $\mathrm{SO}(V) = \{g \in \mathrm{O}(V) \mid \det(g) = 1\}$

Fact

- $\mathrm{O}(V)$ is quasi-split if and only if there are some $c, d \in F^\times$ such that $V \simeq \mathbb{H}^{n-1} \oplus V_{c,d}$.
- $\mathrm{O}(V)$ is split if and only if $d = 1$.

We may assume that V is associated to $(c, d) \in (F^\times)^2$.

Using the natural embedding $\mathrm{O}(V_{c,d}) \hookrightarrow \mathrm{O}(V)$, regard $\epsilon \in \mathrm{O}(V_{c,d})$ as an element of $\mathrm{O}(V)$ acting trivially on $\mathbb{H}^{n-1}$.

Decompose $\mathbb{H}^{n-1} = X \oplus X^*$, where X is a $(n-1)$-dimensional maximal isotropic subspace of $\mathbb{H}^{n-1}$ and X^* is dual to X. Let $\{e_1, \cdots, e_{n-1}\}$ be a basis of X.

Borel subgroup of $\mathrm{SO}(V)$

- $\mathrm{B} = TU$: the F-rational Borel subgroup of $\mathrm{SO}(V)$ stabilizing the complete flag of X,

$$0 \subset \langle e_1 \rangle \subset \cdots \subset \langle e_1, \cdots, e_{n-1} \rangle = X.$$

- T : the F-rational torus stabilizing the lines $F \cdot e_i$ for $1 \leq i \leq n-1$
- U : the unipotent radical of B

Using the basis $\{e_1, \cdots, e_{n-1}, e, e', e^*_{n-1}, \cdots, e^*_1\}$ of V, we describe U as a $(n \times n)$-matrix group.

From now on, we fix a non-trivial additive character $\psi : F \to \mathbb{C}^\times$.

Definition

Let $E = F(\sqrt{d})$. Choose arbitrary $c' \in c\mathrm{N}_{E/F}(E^\times)/(F^\times)^2$.
Define a generic character $\mu_{c'} : \mathrm{U}(F) \to \mathbb{C}^\times$ of $\mathrm{SO}(V)(F)$ as

$$\mu_{c'}(u) = \psi(u_{1,2} + \cdots + u_{n-2,n-1} + u_{n-1,n}).$$

Note that $u_{n-1,n} = \langle ue, e^*_{n-1} \rangle_V$.

Fact

The map $c' \to \mu_{c'}$ gives a bijection between $c\mathrm{N}_{E/F}(E^\times)/(F^\times)^2$ and $\{\mathrm{T} - \text{orbits of generic characters of } \mathrm{SO}(V)(F)\}$.

When $\mathrm{O}(V)$ is split (i.e. $d = 1$), there is unique T-orbit of generic characters.

Generic representations of $\mathrm{SO}(V)$ and $\mathrm{O}(V)$

Def

For an irreducible admissible representation of $\mathrm{SO}(V)$, we say that π is $\mu_{c'}$-generic if $\mathrm{Hom}_{\mathrm{U}(F)}(\pi, \mu_{c'}) \neq 0$.

Denote by $\widetilde{\mathrm{U}} := \mathrm{U} \rtimes \langle \epsilon \rangle$. Define $\mu_{c'}^{\pm} : \widetilde{\mathrm{U}}(F) = \mathrm{U}(F) \rtimes \langle \epsilon \rangle \to \mathbb{C}^{\times}$ by

$$\mu_{c'}^{\pm}|_{\mathrm{U}(F)} = \mu_{c'} \text{ and } \mu_{c'}^{\pm}(\epsilon) = \pm 1.$$

Def

For an irreducible admissible representation of $\mathrm{O}(V)$, we say that π is $\mu_{c'}^{\pm}$-generic if $\mathrm{Hom}_{\mathrm{U}(F)}(\pi, \mu_{c'}^{\pm}) \neq 0$.

Note that if π is $\mu_{c'}^{\pm}$-generic, then $(\pi \otimes \det)$ is $\mu_{c'}^{\mp}$-generic.

(LCT for O_{2n}) H-K-K, (2023), preprint

Choose arbitrary $(c, d) \in (F^\times)^2$ and $c' \in c\mathrm{N}_{E/F}(E^\times)/(F^\times)^2$. Suppose that V is a quadratic space associated to (c, d).

Let π_1, π_2 be irreducible $\mu_{c'}^{\pm}$-generic admissible representations of $\mathrm{O}(V)$ with the same central characters and same signs.

If $\gamma(s, \pi_1 \times \rho, \psi) = \gamma(s, \pi_2 \times \rho, \psi)$ for all irreducible supercuspidal representations ρ of $\mathrm{GL}_i(F)$ for $1 \leq i \leq n$, then

$$\pi_1 \simeq \pi_2$$

As a corollary, we have

(LCT for SO_{2n}) H-K-K, (2023), preprint

Choose arbitrary $(c, d) \in (F^\times)^2$ and $c' \in c\mathrm{N}_{E/F}(E^\times)/(F^\times)^2$. Suppose that V is a quadratic space associated to (c, d).

Let π_1, π_2 be irreducible generic admissible representations of $\mathrm{SO}(V)$ with the same central characters.

If $\gamma(s, \pi_1 \times \rho, \psi) = \gamma(s, \pi_2 \times \rho, \psi)$ for all irreducible supercuspidal representations ρ of $\mathrm{GL}_i(F)$ for $1 \leq i \leq n$, then

$$\pi_1 \simeq \pi_2 \text{ or } \pi_1 \simeq \pi_2^\epsilon, \quad \text{where } \pi^\epsilon(g) := \pi(\epsilon g \epsilon^{-1}).$$

When $\mathrm{char}(F) = 0$, a special case of this (i.e. $(c, d) = (-\frac{1}{8}, 1)$ and $c' = -\frac{1}{8}$) is proved by A. Hazeltine and B. Liu (preprint, (2022)).

We prove 'LCT for O_{2n}' by relating it with the 'LCT for Sp_{2n}' via the local theta correspondence.

Symplectic group

- $(W, \langle\ ,\ \rangle_W)$: a symplectic space over F
- $\mathrm{Sp}(W)$: the isometry group of W (it is always split)
- $\{f_i, f_j^*\}_{1\le i,j\le n}$ be the basis of W such that

$$\langle f_i, f_j\rangle_W = \langle f_i^*, f_j^*\rangle_W = 0, \quad \langle f_i, f_j^*\rangle_W = \delta_{ij}$$

- Y : the subspace of W generated by $\{f_1, \cdots, f_n\}$
- $\mathrm{B}' = \mathrm{T}'\mathrm{U}'$: the F-rational Borel subgroup of $\mathrm{Sp}(W)$ stabilizing the complete flag of Y,

$$0 \subset \langle f_1\rangle \subset \cdots \subset \langle f_1, \cdots, f_n\rangle = Y.$$

- T' : the F-rational torus stabilizing the lines $F\cdot f_i$ for $1\le i\le n$
- U' : the unipotent radical of B'

Using the basis $\{f_1, \cdots, f_{n-1}, f_n, f_n^*, f_{n-1}^*, \cdots, f_1^*\}$ of W, we describe U' as a $(n\times n)$-matrix group.

Generic representation of $\mathrm{Sp}(W)$

Definition

Choose an arbitrary $d \in F^\times/(F^\times)^2$. Define a generic character $\mu'_d : \mathrm{U}'(F) \to \mathbb{C}^\times$ of $\mathrm{Sp}(W)$ as

$$\mu'_d(u) = \psi(u_{1,2} + \cdots + u_{n-1,n} + d \cdot u_{n,n+1}).$$

Fact

The map $d \to \mu'_d$ gives a bijection between

$$F^\times/(F^\times)^2 \leftrightarrow \{\mathrm{T}'(F)\text{-orbits of generic characters of } \mathrm{U}'(F)\}$$

Definition

For $\pi' \in \mathrm{Irr}(\mathrm{Sp}(W))$, we says that π' is μ'_d-generic if $\mathrm{Hom}_{U'}(\pi', \mu'_d) \neq 0$.

(LCT for $\mathrm{Sp}(W)$) Jo, (2022), preprint

Let π_1, π_2 be irreducible μ_1'-generic admissible representations of $\mathrm{Sp}(W)$ with the same central characters.

If $\gamma(s, \pi_1 \times \rho, \psi) = \gamma(s, \pi_2 \times \rho, \psi)$ for all irreducible supercuspidal representations ρ of $\mathrm{GL}_i(F)$ for $1 \leq i \leq n$, then $\pi_1 \simeq \pi_2$.

To relate this with our theorem, our first task is to extend Jo's result.

(Extended LCT for $\mathrm{Sp}(W)$)

Choose arbitrary $d \in F^\times / F^{\times 2}$.

Let π_1, π_2 be irreducible μ_d'-generic admissible representations of $\mathrm{Sp}(W)$ with the same central characters.

If $\gamma(s, \pi_1 \times \rho, \psi) = \gamma(s, \pi_2 \times \rho, \psi)$ for all irreducible supercuspidal representations ρ of $\mathrm{GL}_i(F)$ for $1 \leq i \leq n$, then $\pi_1 \simeq \pi_2$.

Local theta correspondence between $\mathrm{O}(V)$ and $\mathrm{Sp}(W)$

There is a functor $\Theta_{V,W} : \mathrm{Irr}(\mathrm{O}(V)) \to \mathrm{Irr}(\mathrm{Sp}(W))$ induced by the local theta correspondence between $\mathrm{O}(V)$ and $\mathrm{Sp}(W)$.

For $\pi \in \mathrm{Irr}(\mathrm{O}(V))$, $\Theta_{V,W}(\pi)$ may be zero.

Put $\mathrm{Irr}^{supp}(\mathrm{O}(V)) := \{\pi \in \mathrm{Irr}(\mathrm{O}(V)) \text{ such that } \Theta_{V,W}(\pi) \neq 0\}$

The Howe duality theorem says that $\Theta_{V,W}|_{\mathrm{Irr}^{supp}(\mathrm{O}(V))}$ is injective.

The functor $\Theta_{V,W}$ preserves genericity and γ-factors (up to twist).

Main Theorem, (H-K-K)

1. If $\pi \in \mathrm{Irr}(\mathrm{O}(V))$ is tempered and $\mu^{+}_{c'}$-generic, then $\pi \in \mathrm{Irr}^{supp}(\mathrm{O}(V))$ and $\Theta_{V,W}(\pi)$ is $(\mu'_{-c'})^{-1}$-generic.
2. Let χ_V be the quadratic character associated to $(F(\sqrt{\mathrm{disc}(V)})/F$.

   Suppose $\pi \in \mathrm{Irr}^{supp}(\mathrm{O}(V))$ is $\mu^{+}_{c'}$-generic. Then for any irreducible supercuspidal representation ρ of $\mathrm{GL}_r(F)$,

   $$\gamma(s, \Theta_{V,W}(\pi) \times \rho, \psi) = \gamma(s, \pi \times \rho\chi_V, \psi) \cdot \gamma(s, \rho\chi_V, \psi).$$

Sketch of the proof

(Proof of LCT for $\mathrm{O}(V)$)

(Step 1) Given $\mu^+_{c'}$-generic tempered $\pi_1, \pi_2 \in \mathrm{Irr}(\mathrm{O}(V))$, suppose that

$$\gamma(s, \pi_1 \times \rho, \psi) = \gamma(s, \pi_2 \times \rho, \psi)$$

for all supercuspidal $\rho \in \mathrm{Irr}(\mathrm{GL}_i)$ for $1 \leq i \leq n$.

By the second property of $\Theta_{V,W}$,

$$\gamma(s, \Theta_{V,W}(\pi_1) \times \rho, \psi) = \gamma(s, \Theta_{V,W}(\pi_2) \times \rho, \psi)$$

for all supercuspidal $\rho \in \mathrm{Irr}(\mathrm{GL}_i)$ for $1 \leq i \leq n$.

(Step 2) By the first property of $\Theta_{V,W}$, $\Theta_{V,W}(\pi_1), \Theta_{V,W}(\pi_1)$ are nonzero and $(\mu'_{-c'})^{-1}$-generic. Then by the extended LCT for $\mathrm{Sp}(W)$, we have $\Theta_{V,W}(\pi_1) \simeq \Theta_{V,W}(\pi_2)$ and so $\pi_1 \simeq \pi_2$.

(Step 3) Reduce the general cases to tempered cases.

Main Theorem, (H-K-K)

1. If $\pi \in \mathrm{Irr}(\mathrm{O}(V))$ is tempered and $\mu^+_{c'}$-generic, then $\pi \in \mathrm{Irr}^{supp}(\mathrm{O}(V))$ and $\Theta_{V,W}(\pi)$ is $(\mu'_{-c'})^{-1}$-generic.
2. Let χ_V be the quadratic character associated to $(F(\sqrt{\mathrm{disc}(V)})/F$.

   Suppose $\pi \in \mathrm{Irr}^{supp}(\mathrm{O}(V))$ is $\mu^+_{c'}$-generic. Then for any irreducible supercuspidal representation ρ of $\mathrm{GL}_r(F)$,

$$\gamma(s, \Theta_{V,W}(\pi) \times \rho, \psi) = \gamma(s, \pi \times \rho\chi_V, \psi) \cdot \gamma(s, \rho\chi_V, \psi).$$

Remark : The proof of (1) requires the computation of twisted Jacquet module of the Weil representation. It is independent of Arthur's results on the local Langlands correspondence for $\mathrm{O}(V)$.

The proof of (2) consists of showing the non-vanishing and cuspidality of global theta lift from $\mathrm{O}(V)$ to $\mathrm{Sp}(W)$.

Thank you!