Twisting in Hamiltonian Flows

In-Jee Jeong (Seoul National University)

September 18, 2023

Kinetic and fluid equations for collective behavior

Outline

- **Filamentation** ("creation of long and thin structures") for transported scalar: generic phenomenon – but how to prove?

Outline

- **Filamentation** ("creation of long and thin structures") for transported scalar: generic phenomenon – but how to prove?
- **Twisting** ("differential travel speed of nearby trajectories") for the flow map gives rise to filamentation.

Outline

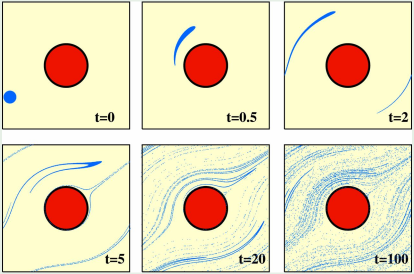
- **Filamentation** ("creation of long and thin structures") for transported scalar: generic phenomenon – but how to prove?
- **Twisting** ("differential travel speed of nearby trajectories") for the flow map gives rise to filamentation.
- Our contribution: **stability of twisting** for flows generated by *stable velocities*.

Outline

- **Filamentation** ("creation of long and thin structures") for transported scalar: generic phenomenon – but how to prove?
- **Twisting** ("differential travel speed of nearby trajectories") for the flow map gives rise to filamentation.
- Our contribution: **stability of twisting** for flows generated by *stable velocities*. No stability in terms of the flow map!

Outline

- **Filamentation** ("creation of long and thin structures") for transported scalar: generic phenomenon – but how to prove?
- **Twisting** ("differential travel speed of nearby trajectories") for the flow map gives rise to filamentation.
- Our contribution: **stability of twisting** for flows generated by *stable velocities*. No stability in terms of the flow map!
- Applications to PDE, including fluid and kinetic equations.



Filamentation in fluid flows

Evolution of elliptical vortex in incompressible flows

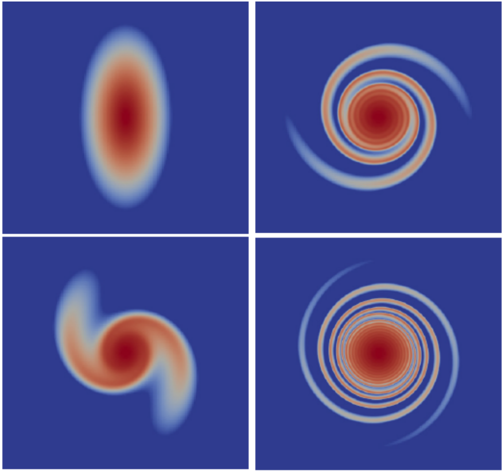


Figure: Krasny–Xu 2023

Filamentation in fluid flows

Evolution of elliptical vortex in incompressible flows

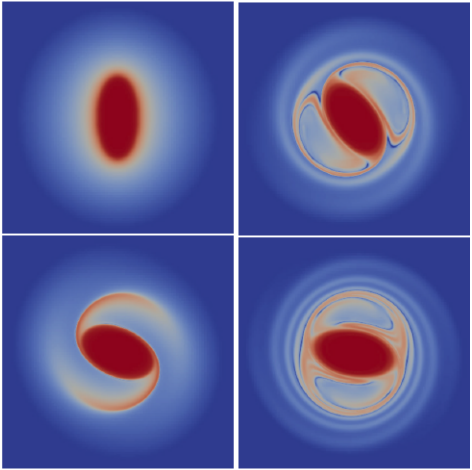


Figure: Krasny–Xu 2023

Motivation: Filamentation in fluid flows

Optimal mixing flows

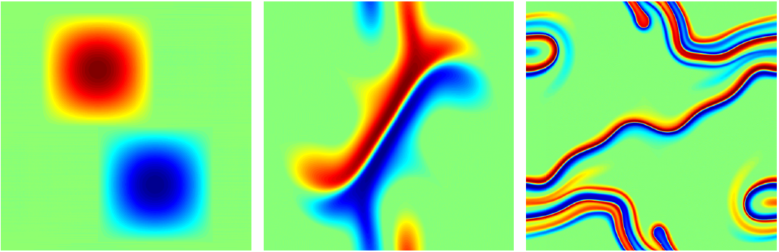


Figure: Iyer–Xu

Filamentation in plasma dynamics

Phase space evolution in Landau damping

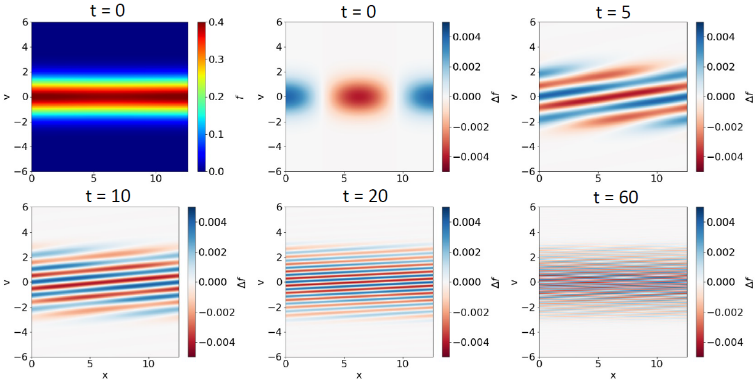


Figure: Krasny–Thomas–Sandberg 2023

Filamentation in plasma dynamics

Velocity distribution evolution in Landau damping

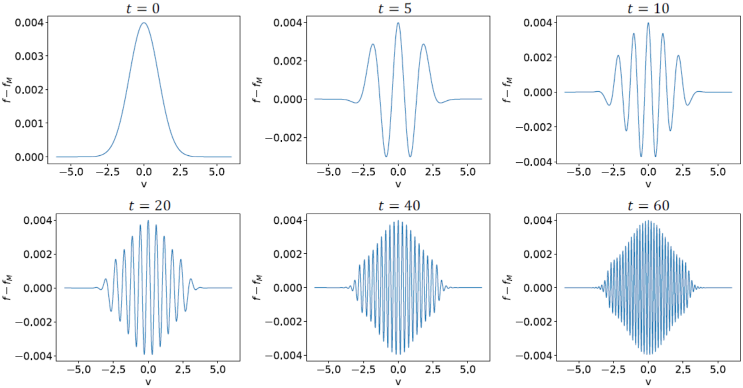


Figure: Krasny–Thomas–Sandberg 2023

Filamentation in plasma dynamics

Two-stream instability: phase space description

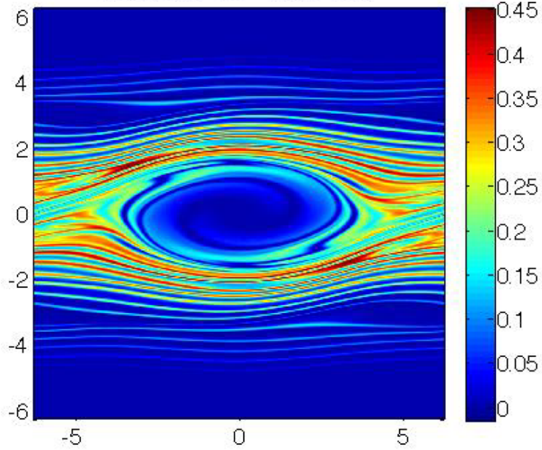
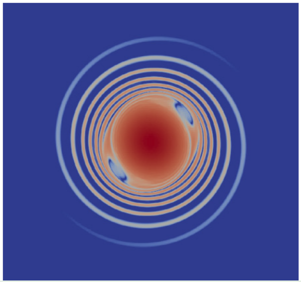


Figure: Liu–Chen–Quan–Zhou 2020

Q. How to verify generic filamentation?

- Intro. to Hamiltonian systems
- Steady Hamiltonian systems
- Main result: stability of twisting
- Applications of the main result
- Ideas of proof



Setup: Hamiltonian system

Definition (Hamiltonian flow)

Let Ω be a 2D domain and $\Psi(t,\cdot) : [0,\infty) \times \Omega \to \mathbb{R}$ be an at least C^2–smooth stream function. Consider the ODE

$$\dot{X} = -\partial_y \Psi(t, X, Y),$$
$$\dot{Y} = \partial_x \Psi(t, X, Y).$$

This defines an associated area-preserving **flow map**

$$\Phi(t,x,y) = (X(t,x,y), Y(t,x,y)) : [0,\infty) \times \Omega \to \Omega.$$

Setup: Hamiltonian system

Definition (Hamiltonian flow)

Let Ω be a 2D domain and $\Psi(t,\cdot) : [0,\infty) \times \Omega \to \mathbb{R}$ be an at least C^2–smooth stream function. Consider the ODE

$$\dot{X} = -\partial_y \Psi(t, X, Y),$$
$$\dot{Y} = \partial_x \Psi(t, X, Y).$$

This defines an associated area-preserving **flow map**

$$\Phi(t,x,y) = (X(t,x,y), Y(t,x,y)) : [0,\infty) \times \Omega \to \Omega.$$

Question (Twisting)

Under which conditions on $\Psi(t,\cdot)$ do we have **twisting**? E.g.

$$\|\nabla_{x,y}\Phi(t,\cdot)\|_{L^\infty(\Omega)} \to \infty \quad \text{as} \quad t \to \infty?$$

Setup: Hamiltonian system

Definition (Scalar advection)

Let $f_0 : \Omega \to \mathbb{R}$ be C^1–smooth, and define its Φ-pushforward

$$f(t, \Phi(t, x, y)) = f_0(x, y).$$

In other words $f(t, \cdot) = f_0 \circ \Phi^{-1}(t, \cdot)$.

Setup: Hamiltonian system

Definition (Scalar advection)

Let $f_0 : \Omega \to \mathbb{R}$ be C^1–smooth, and define its Φ-pushforward

$$f(t, \Phi(t, x, y)) = f_0(x, y).$$

In other words $f(t, \cdot) = f_0 \circ \Phi^{-1}(t, \cdot)$.

Question (Filamentation)

Under which conditions on Ψ, f_0 do we have **filamentation**? E.g.

$$\|\nabla_{x,y} f(t, \cdot)\|_{L^\infty(\Omega)} \to \infty \quad \text{as} \quad t \to \infty?$$

Key PDE Examples

Incompressible 2D Euler equations:

$$\begin{aligned} \dot{\Phi} &= \nabla^{\perp}\Psi, \\ \Psi &= -(-\Delta)^{-1}\omega, \\ \omega \circ \Phi &= \omega_0. \end{aligned}$$

Key PDE Examples

Incompressible 2D Euler equations:

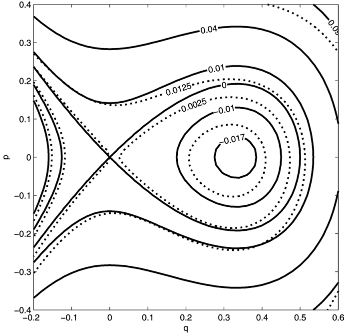
$$\begin{aligned} \dot{\Phi} &= \nabla^{\perp}\Psi, \\ \Psi &= -(-\Delta)^{-1}\omega, \\ \omega \circ \Phi &= \omega_0. \end{aligned}$$

Vlasov–Poisson equations:

$$\begin{aligned} \dot{\Phi} &= -\nabla^{\perp}_{x,v}\left(\frac{1}{2}|v|^2 + U(x) + U^{ext}(x)\right), \\ U &= \pm(-\Delta_x)^{-1}\int_{\mathbb{R}} f(t,x,v)dv, \\ f \circ \Phi &= f_0. \end{aligned}$$

Steady Hamiltonian

Generic C^2 steady Hamiltonian flow: periodic orbits separated by fix points and connecting orbits. Twisting can be defined in terms of difference in the period.



Time-dependent case: no periodic orbits in general, and particles are free to travel essentially anywhere.

Example 1: Shear flows

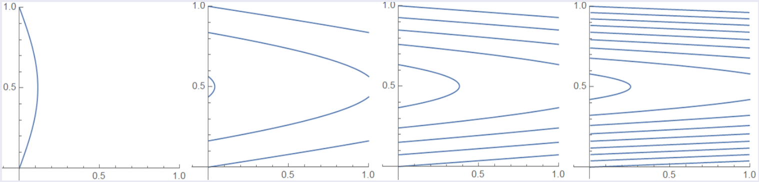
Domains $\Omega = \mathbb{T}^2, \mathbb{T} \times [0,1], \cdots$. Consider $\Psi(x,y) = G(y)$. Then

$$\dot{X} = -G'(Y), \qquad \dot{Y} = 0.$$

We have

$$X(t,x,y) = x - tG'(y) \ (\mathrm{mod} 2\pi), \quad \partial_y X(t,x,y) = -tG''(y);$$

we say twisting occurs if and only if $G'' \not\equiv 0$.

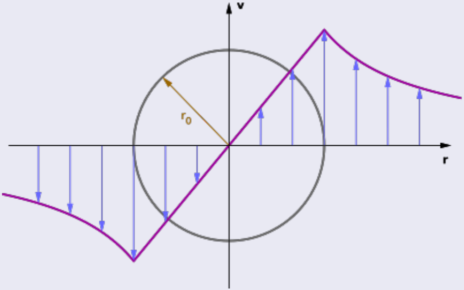


Example 2: Radial flows

Domains $\Omega = \mathbb{R}^2, B_0(1), \cdots$. Consider in polar coordinates

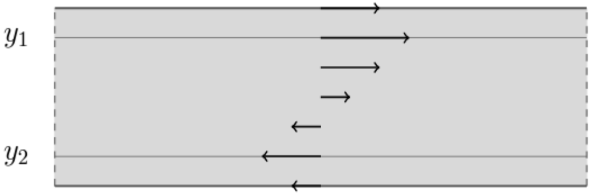
$$\dot{\Theta} = g(R), \qquad \dot{R} = 0.$$

We have $\Theta(t) = \theta + tg(r)$. Twisting occurs if and only if $g' \not\equiv 0$.



Twisting for steady Hamiltonian flows

Let $\bar{\Psi}$ be a steady Hamiltonian on Ω. We say it is **twisting** if there is an annular region $\mathbf{A} \subset \Omega$ foliated with streamlines such that the two connected components of $\partial\mathbf{A}$ have **different periods**.



Twisting for steady Hamiltonian flows

Let $\bar{\Psi}$ be a steady Hamiltonian on Ω. We say it is **twisting** if there is an annular region $\mathbf{A} \subset \Omega$ foliated with streamlines such that the two connected components of $\partial \mathbf{A}$ have **different periods**.

Twisting for steady Hamiltonian flows

Let $\bar{\Psi}$ be a steady Hamiltonian on Ω. We say it is **twisting** if there is an annular region $\mathbf{A} \subset \Omega$ foliated with streamlines such that the two connected components of $\partial \mathbf{A}$ have **different periods**.

Theorem (Drivas–Elgindi–J. 2023 preprint.)

There exists $\varepsilon_0 = \varepsilon_0(\bar{\Psi})$ such that if $\Psi(t, \cdot)$ be a time-dependent Hamiltonian on Ω satisfying

$$\|\bar{\Psi} - \Psi(t)\|_{L^\infty_t W^{1,1}(\Omega)} < \varepsilon_0,$$

then the flow Φ generated by $\Psi(t, \cdot)$ is **twisting**. *In particular,*

$$\|\nabla \Phi(t, \cdot)\|_{L^\infty(\Omega)} \geq c_0 t \quad \text{for all} \quad t \geq 0.$$

Twisting for steady Hamiltonian flows

Let $\bar{\Psi}$ be a steady Hamiltonian on Ω. We say it is **twisting** if there is an annular region $\mathbf{A} \subset \Omega$ foliated with streamlines such that the two connected components of $\partial \mathbf{A}$ have **different periods**.

Theorem (Drivas–Elgindi–J. 2023 preprint.)

There exists $\varepsilon_0 = \varepsilon_0(\bar{\Psi})$ such that if $\Psi(t, \cdot)$ be a time-dependent Hamiltonian on Ω satisfying

$$\|\bar{\Psi} - \Psi(t)\|_{L_t^\infty W^{1,1}(\Omega)} < \varepsilon_0,$$

then the flow Φ generated by $\Psi(t, \cdot)$ is **twisting**. *In particular,*

$$\|\nabla \Phi(t, \cdot)\|_{L^\infty(\Omega)} \geq c_0 t \quad \text{for all} \quad t \geq 0.$$

Difficulty: individual particles are free to move anywhere.

Counterexample?

On $\mathbb{T}^2$, we have that $\bar{\Psi}(x, y) = \cos(y)$ is twisting. However, consider its perturbation $\Psi(x, y) = \cos(y) + \varepsilon x$. Then

$$\dot{X} = \sin(Y), \qquad \dot{Y} = \varepsilon.$$

Counterexample?

On $\mathbb{T}^2$, we have that $\bar{\Psi}(x, y) = \cos(y)$ is twisting. However, consider its perturbation $\Psi(x, y) = \cos(y) + \varepsilon x$. Then

$$\dot{X} = \sin(Y), \qquad \dot{Y} = \varepsilon.$$

The solution is explicitly given by

$$X(t) = x + \frac{1}{\varepsilon}(\cos(y) - \cos(y + \varepsilon t)), \quad Y(t) = y + t\varepsilon.$$

Counterexample?

On $\mathbb{T}^2$, we have that $\bar{\Psi}(x, y) = \cos(y)$ is twisting. However, consider its perturbation $\Psi(x, y) = \cos(y) + \varepsilon x$. Then

$$\dot{X} = \sin(Y), \qquad \dot{Y} = \varepsilon.$$

The solution is explicitly given by

$$X(t) = x + \frac{1}{\varepsilon}(\cos(y) - \cos(y + \varepsilon t)), \quad Y(t) = y + t\varepsilon.$$

This flow is **time periodic**!! (NO twisting whatsoever.)

Counterexample?

On $\mathbb{T}^2$, we have that $\bar{\Psi}(x, y) = \cos(y)$ is twisting. However, consider its perturbation $\Psi(x, y) = \cos(y) + \varepsilon x$. Then

$$\dot{X} = \sin(Y), \qquad \dot{Y} = \varepsilon.$$

The solution is explicitly given by

$$X(t) = x + \frac{1}{\varepsilon}(\cos(y) - \cos(y + \varepsilon t)), \quad Y(t) = y + t\varepsilon.$$

This flow is **time periodic**!! (NO twisting whatsoever.)

Note that Ψ is actually not smooth on $\mathbb{T}^2$.

Counterexample?

On $\mathbb{T}^2$, we have that $\bar{\Psi}(x, y) = \cos(y)$ is twisting. However, consider its perturbation $\Psi(x, y) = \cos(y) + \varepsilon x$. Then

$$\dot{X} = \sin(Y), \qquad \dot{Y} = \varepsilon.$$

The solution is explicitly given by

$$X(t) = x + \frac{1}{\varepsilon}(\cos(y) - \cos(y + \varepsilon t)), \quad Y(t) = y + t\varepsilon.$$

This flow is **time periodic**!! (NO twisting whatsoever.)

Note that Ψ is actually not smooth on $\mathbb{T}^2$.

Indeed: $\bar{\Psi} - \Psi$ is not in $W^{1,1}(\mathbb{T}^2)$.

Theorem (From twisting to filamentation)

In the same setting, there is **generic** *filamentation for advected scalars; that is, for generic C^1 initial data f_0,*

$$\|\nabla f(t,\cdot)\|_{L^\infty} \gtrsim t^{1-}, \quad \text{as} \quad t \to \infty.$$

Applications to PDE

Consider the PDEs of the form

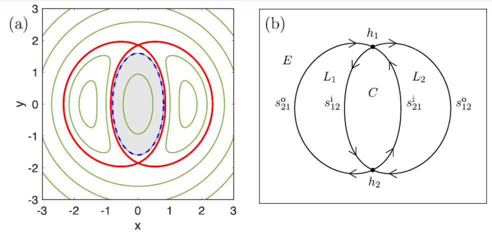
$$\dot{\Phi} = \nabla^\perp \Psi, \quad f(t) = f_0 \circ \Phi^{-1}(t)$$

and $f(t) \mapsto \Psi(t)$ by a functional relation. We need a steady solution $(\bar{f}, \bar{\Psi})$ which is **stable** just in the $W^{1,1}$ norm of $\bar{\Psi}$.

Application to incompressible 2D Euler

Collection of stable steady Euler flows

- Monotone radial vortex in $\mathbb{R}^2$, $B_0(R)$.
- Kirchhoff Ellipses with aspect ratio < 3.
- First eigenfunctions on $\mathbb{T}^2$ under a symmetry.
- Second eigenfunctions on $\mathbb{T}^2$ under two symmetries.
- Constant vorticity flow on bounded domains.
- ...



Application to Vlasov–Poisson equation

Example of a VP stable steady state

Existence and Stability: Marchioro–Pulvirenti ('86), Wan ('90), Rein ('92, '94), Batt–Morrison–Rein ('95), Guo–Rein ('99), ...

$$\bar{f}(x,v) = \varphi(\frac{1}{2}|v|^2 + \bar{U}(x) + U^{ext}(x)), \qquad \varphi' \le 0.$$

Application to Vlasov–Poisson equation

Example of a VP stable steady state

Existence and Stability: Marchioro–Pulvirenti ('86), Wan ('90), Rein ('92, '94), Batt–Morrison–Rein ('95), Guo–Rein ('99), ...

$$\bar{f}(x,v) = \varphi(\frac{1}{2}|v|^2 + \bar{U}(x) + U^{ext}(x)), \qquad \varphi' \leq 0.$$

Consequence of stability of twisting

Infinite gradient growth for generic perturbations of $\bar{f}$.

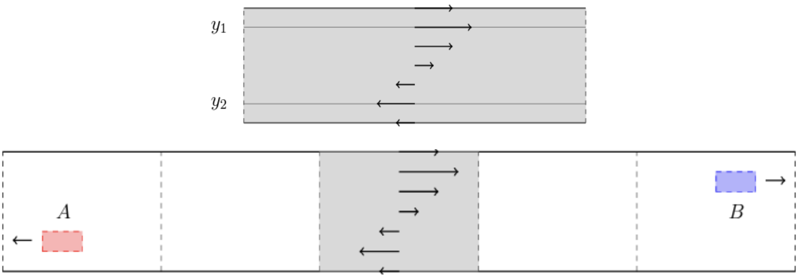
Proof: twisting quantity in the case $\mathbb{T} \times [0,1]$

Computation of **localized averaged winding number**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_i(t) := \frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{T}\times[0,1]} \tilde{X}(t,x,y) F_i(Y(t,x,y))\,\mathrm{d}x\mathrm{d}y$$

Steady case: $\tilde{X} = x - t(\partial_y \bar{\Psi})(y)$, $\bar{\mathcal{I}}_i(t) \simeq \mathcal{I}_i(0) - t(\partial_y \bar{\Psi})(y_i)$.

Key inequality: $\left|\bar{\mathcal{I}}_i(t) - \mathcal{I}_i(t)\right| \lesssim \|\bar{\Psi} - \Psi\|_{L^\infty_t W^{1,1}}$.



Summary

- Filamentation common in Hamiltonian systems
- Result of twisting for the flow map
- Stability of twisting in the time-dependent case
- Weak requirement $W^{1,1}$ facilitates PDE applications

Thank you for your attention!