Graph Limit for Interacting Particle Systems on Weighted Random Graphs

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Collective dynamics models

Interacting particle system (first-order)

$$\frac{d}{dt}u_i(t) = \frac{1}{N}\sum_{j=1}^{N} D(u_j(t) - u_i(t)), \tag{1}$$

- $u_i \in \mathbb{R}^d$ is the **state variable** (position, opinion, frequency, etc.)
- $D : \mathbb{R}^d \to \mathbb{R}^d$ is the **interaction function**.
  Often, $D(u_j(t) - u_i(t)) = a(\|u_j(t) - u_i(t)\|)(u_j(t) - u_i(t))$

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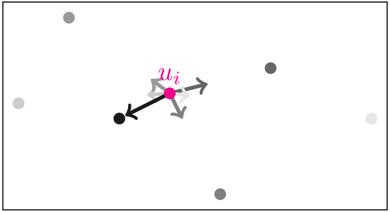

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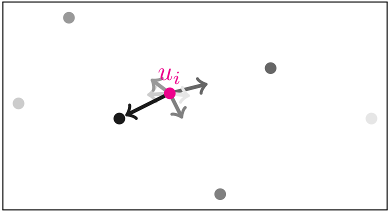

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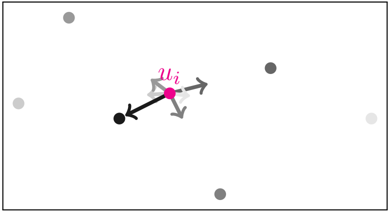

The interactions depend only on the particles' **positions** in the state space.
The particles are said to be **indistinguishable** (or **exchangeable**).

Indistinguishability for interacting particle systems

Definition: Indistinguishability

Indistinguishability is preserved if for all solutions $(u_1, \cdots, u_N)$ and $(\tilde{u}_1, \cdots, \tilde{u}_N)$,

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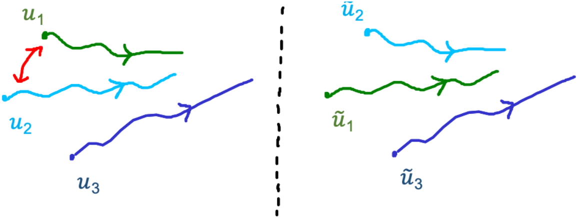

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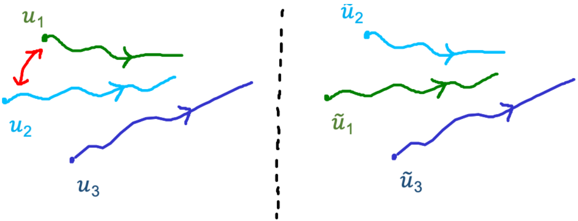
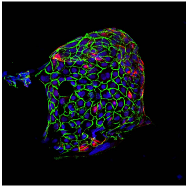


Trivially, System (1) preserves indistinguishability.

Need for of non-exchangeable particle systems



- Animal groups: leader-follower dynamics, animal personality
- Robotics: network communication
- Cell colonies: heterogeneous phenotypes

Non-exchangeable collective dynamics models

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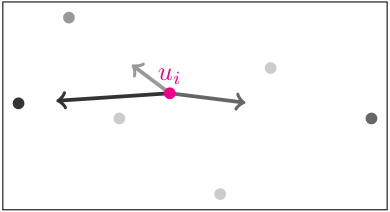

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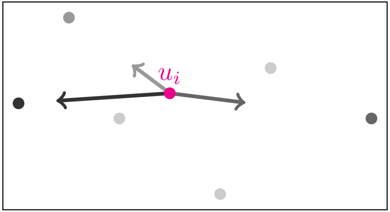

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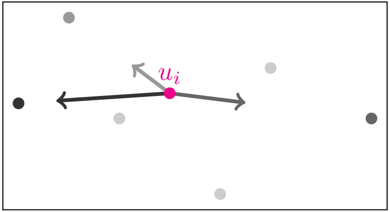

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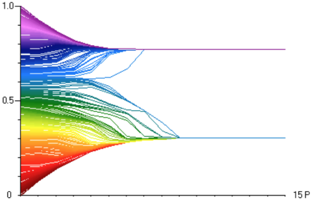
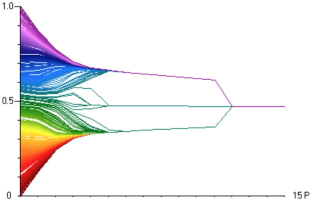


Clustering



Consensus

[Hegselmann and Krause, ’02]

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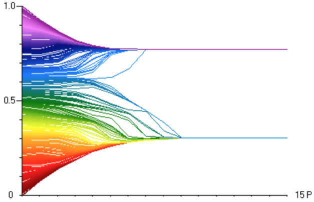

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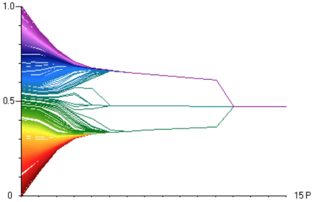

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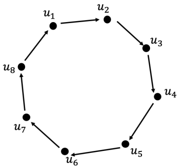
- **Large Population Limit**: N the number of agents goes to infinity.

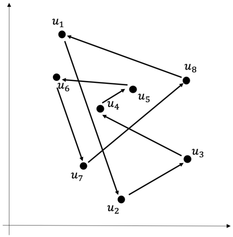
Particle systems on graphs: Example 1

Example: the monodirectional cycle

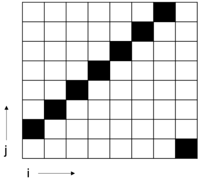
$$w_{ij} = \begin{cases} 1 & \text{if } j - i = 1 \\ 0 & \text{otherwise.} \end{cases}$$



Graph representation



State space



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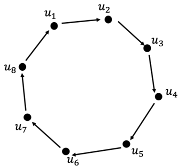

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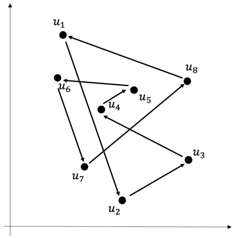

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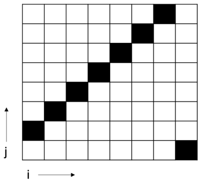

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Underlying weighted graph: $G_N = \langle V(G_N), E(G_N), W(G_N) \rangle$, where

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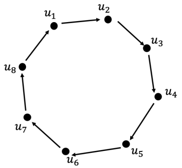

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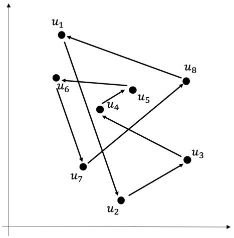

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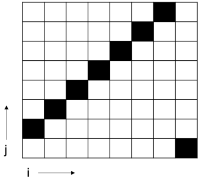

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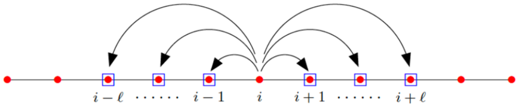

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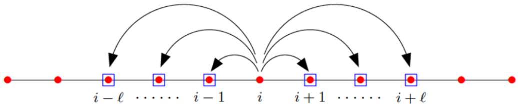

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How do we define the limit of (ℓ-nearest) as $N \to \infty$?

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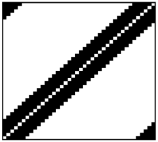

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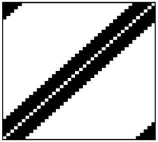

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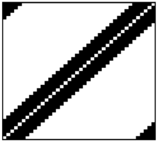

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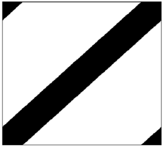

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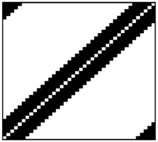

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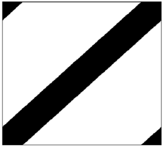

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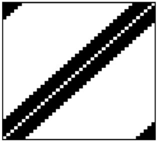

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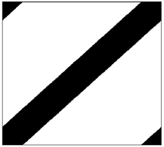

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The continuum limit

$$\partial_t u(t,x) = \int_I \chi_{[0,r]}(|x-y|)(u(t,x) - u(t,y))dy, \qquad x \in I := [0,1], t \geq 0.$$

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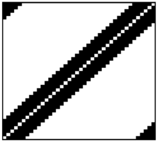

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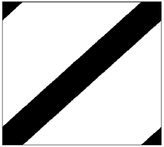

- $\{W_{G_N}\}$ converges to the $\{0,1\}$-valued function $W_G(x,y) = \chi_{[0,r]}(|x-y|)$.

The continuum limit

$$\partial_t u(t,x) = \int_I \chi_{[0,r]}(|x-y|)(u(t,x) - u(t,y))dy, \qquad x \in I := [0,1], t \geq 0.$$

Keep in mind: $u(t,x)$ is the **position** (or opinion) of agent with **label** x

From equations on graphs to a continuum equation on a graphon

Let $W : I^2 \to \mathbb{R}$ a *graphon* on I^2. Let $g : I \to \mathbb{R}$. Define a sequence of graphs G_N whose weights $\bar{W}^N$ are obtained by averaging W on the sets I_i^N:

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Theorem [Medvedev, '13]: Graph Limit

If $W \in L^\infty(I)$, it holds

$$\|u - u_N\|_{C([0,T];L^2(I))} \xrightarrow[N\to+\infty]{} 0$$

where u is the solution to the integro-differential equation

$$\partial_t u(t,x) = \int_I W(x,y)\, D(u(t,y) - u(t,x))dy, \quad u(0,x) = g(x).$$

What of Random Graphs?

Random graphs are neededed:

- when not all edges of a network are known
- when the number of vertices is large
  - Up to 700,000 starlings in a murmuration
  - About 55-70 billion neurons in the human cerebellum

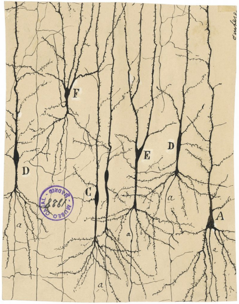

Figure: Left: Starling Murmuration (Bird Watch Ireland). Right: Pyramidal neurons of the cerebral cortex (illustration by Santiago Ramón y Cajal).

Random graphs

- **Random graph:** a graph which is generated by a random process.

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- **Example 1: Erdos-Rényi graph**: the edge between a pair of distinct nodes is inserted with probability p.

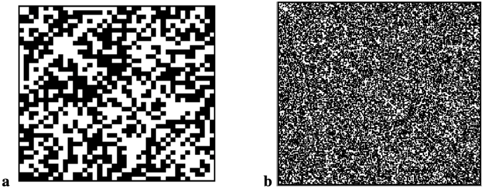

Figure: Pixel pictures of the Erdos-Rényi graph with $N = 40$ and $p = 0.5$ (left), $N = 600$ and $p = 0.5$ (right) [Medvedev, 2014]

Random graphs

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• **Example 2 : Small world graph**: replacing a random set of the local connections by randomly chosen long-range ones.

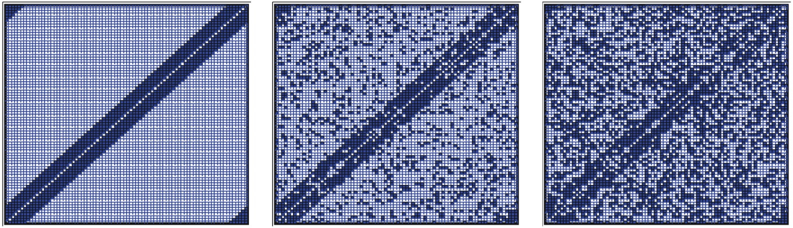

Figure: Pixel pictures of the Small world graph, p starts at 0 and increases from left to right [Medvedev, 2014]

Dynamical systems on W-random graph

- Let $X = (X_1, X_2, X_3, \dots)$ and $X^N = (X_1, X_2, \dots, X_N)$ where $X_i, i \in \mathbb{N}$ are i.i.d. random variables with $\mathcal{L}(X_1) = \mathcal{U}([0,1])$.
- Let $W : [0,1]^2 \to [0,1]$ be a graphon.

Definition [Medvedev, 2014]

A **W-random graph** on N nodes generated by the random sequence X, denoted $G_N = \mathbb{G}(X_N, W)$ is such that the edges of G_N are **selected at random** and

$$\mathbb{P}((i,j) \in E(G_N)) = W(X_i, X_j) \text{ for each } (i,j) \in \{1, \dots, N\}^2 \text{ for } i \neq j.$$

The decision wether to include a pair $(i,j) \in \{1, \dots, N\}^2$ is made **independently** from the decisions of other pairs.

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$$\frac{d}{dt} u_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} \xi_{ij} D(u_j^N(t) - u_i^N(t))$$

with $\mathcal{L}(\xi_{ij}|X) = \mathcal{B}(W(X_i, X_j))$.

Random graph limit

Dynamical systems on W-random graph

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(D_1) converges to (C_1) [Medvedev, '14]

Theorem [Medvedev, '14]: Random Graph Limit

Suppose $W \in \mathcal{W}_0$, a class of symmetric measurable function on I^2 with values on I. D is a **Lipschitz continuous function** on $\mathbb{R}$ and $g \in L^\infty(I)$. Let $T > 0$ and suppose that the solution of (C_1) $u(x,t)$ satisfies the following inequality

$$\min_{t\in[0,T]} \int_I \left\{ \int_I W(x,y) D(u(y,t) - u(x,t))^2 dy \right. \\ \left. - \left(\int_I W(x,y) D(u(y,t) - u(x,t) dy \right)^2 \right\} \geq c_1 > 0.$$

Then, the solution of (D_1) and (C_1) satisfy the following relation

$$\lim_{N\to+\infty} \mathbb{P}\{ \sup_{t\in[0,T]} \|u^N(t) - \mathbf{P}_{X^N} u(x,t)\|_{2,N} \leq \frac{C}{N^{1/2}} \} = 1$$

for some constant $C > 0$ with $\mathbf{P}_{X^N} u(x,t) = (u(X_1^N, t), u(X_2^N, t), \ldots, u(X_N^N, t))$ and

$$(u,v)_N := \frac{1}{N} \sum_{i=1}^{N} u_i v_i, \text{ and the corresponding norm } \|u\|_{2,N} := \sqrt{(u,u)_N}.$$

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What about weighted graphs ?

Weighted random graph

Example [Garlaschelli, '18]
A **weighted random graph** model in which the **probability of drawing an edge** of discrete weight $w \in \mathbb{N}$ between vertices i and j is given by

$$\mathbb{P}(\xi_{ij} = w) = q_{ij}(w) = p^w(1-p).$$

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Lack of a general framework !

1. Weighted random graphs generated by random sequences
2. Weighted random graphs generated by deterministic sequences
3. Blinking systems

Definition [Ayi, P.D., 2023]

A **q-weighted random graph** on N nodes generated by the **random** sequence X, denoted G_N, is such that the weight of an edge of G_N is randomly attributed. More precisely, the **law for the weight of the edge** (i,j) is $q(X_i, X_j, .)$ where

$$\begin{aligned} q : \quad I \times I &\rightarrow \mathcal{P}(\mathbb{R}_+) \\ (x,y) &\mapsto q(x,y;.). \end{aligned}$$

The decision of the attribution of the weight of a pair $(i,j) \in \{1,\ldots,N\}^2$, $i \neq j$, is made **independently** from the decision for other pairs.

Examples

- *Erdös-Rényi* **weighted** *random graph* (Garlaschelli, 09): Generate between any two nodes an edge with weight $w \in \mathbb{N}$, with probability $p^w(1-p)$.

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Weighted random graph limit

- Let $X = (X_1, X_2, X_3, \dots)$ and $X^N = (X_1, X_2, \dots, X_N)$ where $X_i, i \in \mathbb{N}$ are i.i.d. random variables with $\mathcal{L}(X_1) = \mathcal{U}(I)$.

Dynamical systems on q-weighted random graph

$$\begin{cases} \frac{d}{dt} u_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} \xi_{ij} D(u_j^N(t) - u_i^N(t)), \\ u_i^N(0) = g(X_i^N), \qquad i \in \{1, \dots, N\} \end{cases} \qquad (\mathcal{S}_N^{\mathrm{r-r}})$$

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where $\bar{w}(x,y) := \int_{\mathbb{R}_+} wq(x,y;dw)$ is the **expected value** of the edge (x,y).

Our framework

Hypothesis 1

Let $D \in L^\infty(\mathbb{R})$ be bounded and Lipschitz continuous, with $\|D\|_{L^\infty(\mathbb{R})} := K$ and $\|D'\|_{L^\infty(\mathbb{R})} := L$.

Hypothesis 2

There exists $M > 0$ such that for all $(x, y) \in I^2$, for all $k \in \{1, \cdots, 4\}$,

$$\left(\int_{\mathbb{R}_+} w^k q(x, y; dw)\right)^{1/k} \leq M,$$

i.e. the first four moments of the probability measure $q(x, y; \cdot)$ are bounded uniformly in x and y.

Our result

Theorem [Ayi, P.D., 2023]: Weighted Random Graph Limit

Let D satisfy Hyp. 1, let $g \in L^\infty(I)$ and let q be a weighted random graph law satisfying Hyp. 2. Then, **solution u^N to the discrete system** $(S_N^{\mathrm{r-r}})$ **converges** to the **solution u of the continuous model** (C). More precisely,

$$\mathbb{P}\left[\sup_{t\in[0,T]} \|u^N(t) - \mathbf{P}_{\tilde{X}^N} u(\cdot,t)\|_{2,N} \geq \frac{C_1(T)}{\sqrt{N}}\right] \leq \frac{\tilde{C}_1}{N}$$

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- The convergence is quantitative.

Sketch of the proof

For all $i \in \{1, \ldots, N\}$ and $t \in [0, T]$, we denote

$$\zeta_i^N(t) := u(X_i, t) - u_i^N(t)$$

and $\zeta^N(t) = (\zeta_1^N(t), \ldots, \zeta_N^N(t))$.

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We substract $(S_N^{\mathrm{r-r}})$ from (C) evaluated at $x = X_i$ and obtain

$$\frac{d}{dt}\zeta_i^N(t) = \int_I \bar{w}(X_i, y) D(u(y,t) - u(X_i,t))dy - \frac{1}{N}\sum_{j=1}^N \xi_{ij} D(u_j^N(t) - u_i^N(t))$$

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$$+ \frac{1}{N}\sum_{j=1}^N \xi_{ij} \left[D(u(X_j,t) - u(X_i,t)) - D(u_j^N(t) - u_i^N(t)) \right].$$

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We denote, for all $t \geq 0$,

$$Z_i^N(t) := \frac{1}{N}\sum_{j=1}^N \xi_{ij} D(u(X_j,t) - u(X_i,t)) - \int_I \bar{w}(X_i, y) D(u(y,t) - u(X_i,t))dy.$$

Multiplying by $\frac{1}{N}\zeta_i^N$ and summing over i yields

$$\frac{1}{2}\frac{d}{dt}\|\zeta^N\|_{2,N}^2 = -(Z^N, \zeta^N)_N + \frac{1}{N^2}\sum_{i=1}^N\sum_{j=1}^N \xi_{ij}[D(u(X_j)-u(X_i)) - D(u_j^N - u_i^N)]\zeta_i^N.$$

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with $\alpha_N := \left(\frac{1}{N^2}\sum_{i=1}^N\sum_{j=1}^N \xi_{ij}^2\right)^{\frac{1}{2}}$ and $\gamma_N = \max_{i\in\{1,\cdots,N\}} \frac{1}{N}\sum_{j=1}^N \xi_{ij}$.

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From **Gronwall's lemma** and using the fact that $\|\zeta^N(0)\|_{2,N}^2 = 0$, we obtain:

$$\|\zeta^N(t)\|_{2,N}^2 \leq T\sup_{s\in[0,T]}\|Z^N(s)\|_{2,N}^2\, e^{(1+2L(\alpha_N+\gamma_N))T}.$$

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Bienaymé-Chebyshev's inequality (general form)

$$\mathbb{P}[|X-\mathbb{E}[X]| \geq C] \leq \frac{1}{C^k}\mathbb{E}[|X-\mathbb{E}[X]|^k]$$

Using **Bienaymé-Chebyshev's inequality** and the bounds on the first four moments of q, we can prove that:

- $\mathbb{P}[\alpha_N \geq 2M] \leq \dfrac{1}{N^2}$
- $\mathbb{P}[\gamma_N \geq 2M] \leq \dfrac{5}{N}$
- $\mathbb{P}\left[\|Z^N(t)\|_{2,N} \geq \sqrt{\dfrac{1+K^2M^2}{N}}\right] \leq \dfrac{3M^4K^4}{N}$

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And we obtain the desired result:

$$\mathbb{P}\left[\|\zeta^N(t)\|_{2,N} \geq \frac{C_1}{\sqrt{N}}\right] \leq \frac{\tilde{C}_1}{N}.$$

Link with previous results

Dynamical systems on q-weighted random graph

$$\frac{d}{dt}u_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} \xi_{ij} D(u_j^N(t) - u_i^N(t)) \qquad (S_N^{\mathrm{r-r}})$$

with $\mathcal{L}(\xi_{ij}|X) = q(X_i, X_j, \cdot)$

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$$\partial_t u(x,t) = \int_I W(x,y)D(u(y,t) - u(x,t))dy. \qquad (C_1)$$

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- The two match since

$$\int_{\mathbb{R}} wq(x,y;dw) = W(x,y).$$

Weighted Random Graphs generated by Deterministic Sequences

We consider the deterministic sequence

$$x^N = \{x_1^N, \dots, x_N^N\},$$

$x_i^N \in [\frac{i-1}{N}, \frac{i}{N}) =: I_i^N$, $i \in \{1, \dots, N\}$.

Definition [Ayi, P.D., 2023]

A **q-weighted random graph** on N nodes generated by the **deterministic** sequence x^N, denoted $\overline{G}_N$, is such that the weight of an edge of $\overline{G}_N$ is randomly attributed. More precisely, the **law for the weight of the edge** (i,j) is $q(x_i^N, x_j^N, .)$ where

$$\begin{aligned} q: \quad I \times I &\to \mathcal{P}(\mathbb{R}_+) \\ (x,y) &\mapsto q(x,y;.). \end{aligned}$$

The decision of the attribution of the weight of a pair $(i,j) \in \{1, \dots, N\}^2$, $i \neq j$, is made **independently** from the decision for other pairs.

Weighted random graph limit

- Let $x^N = \{x_1^N, \dots, x_N^N\}$, $x_i^N \in [\frac{i-1}{N}, \frac{i}{N}) =: I_i^N$, $i \in \{1, \dots, N\}$.

Dynamical systems on q-weighted random graph generated by deterministic sequence

$$\begin{cases} \frac{d}{dt} u_i^N(t) = \frac{1}{N} \sum_{j=1}^N \xi_{ij} D(u_j^N(t) - u_i^N(t)), \\ u_i^N(0) = g(x_i^N), \qquad i \in \{1, \dots, N\} \end{cases} \qquad (S_N^{\mathrm{r-d}})$$

with $\mathcal{L}(\xi_{ij} | x^N) = q(x_i^N, x_j^N; \cdot)$.

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We prove the convergence towards the continuum limit

The weighted random graph limit equation

$$\begin{cases} \partial_t u(x,t) = \int_I \left(\int_{\mathbb{R}_+} w q(x,y;dw) \right) D(u(y,t) - u(x,t)) dy \\ u(x,0) = g(x), \quad x \in I, \end{cases} \tag{C}$$

Our result

Let $I_i^N := I_i^N$ and $u_N \in \mathcal{C}(0, T; L^2(I))$ be defied from u^N by:

$$\forall t \in \mathbb{R},\ \forall x \in I, \qquad u_N(x,t) = \sum_{i=1}^{N} u_i^N(t) \mathbf{1}_{I_i^N}(x)$$

Theorem [Ayi, P.D., 2023]

Let D satisfy Hyp. 1, and let $g \in C^{0,\frac{1}{2}}(I)$. Suppose that the weighted random graph law satisfies Hyp. 2 and that $(x,y) \mapsto \int_{\mathbb{R}_+} wq(x,y;dw)$ is $\frac{1}{2}-$Hölder on I^2. Then, u_N converges to the solution u of the continuous model (C). More precisely,

$$\mathbb{P}\left[\|u_N - u\|_{\mathcal{C}(0,T;L^2(I))} \geq \frac{C_2}{\sqrt{N}} \right] \leq \frac{\tilde{C}_2}{N}$$

for some explicit constants $C_2, \tilde{C}_2 > 0$.

Numerical Illustration 1: the weighted Erdös-Rényi random graph

- *Erdös-Rényi random graph*: **unweighted** graph constructed by randomly linking any two nodes with a given probability $p \in (0, 1)$.

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$$q(x,y;\cdot) = (1-p)\sum_{i=0}^{+\infty} p^i \delta_i, \qquad \text{for all } x,y \in \mathbb{R}. \tag{2}$$

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$$\begin{cases} \partial_t u(x,t) = \frac{p}{1-p} \int_I D(u(y,t) - u(x,t)) dy \\ u(x,0) = g(x), \quad x \in I. \end{cases}$$

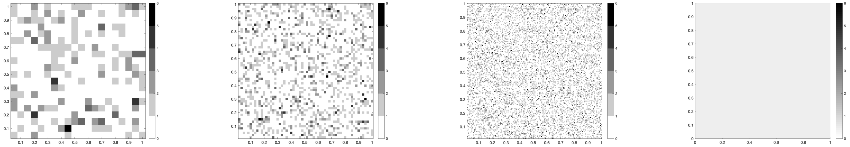
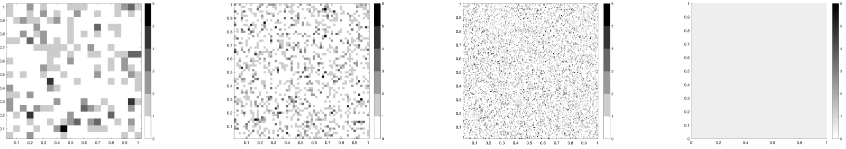


Figure: Left and Center: Random interaction matrices generated by deterministic sequences for $N = 20$, $N = 60$ and $N = 150$, and corresponding continuous graphon $\bar{w}$ for (2).



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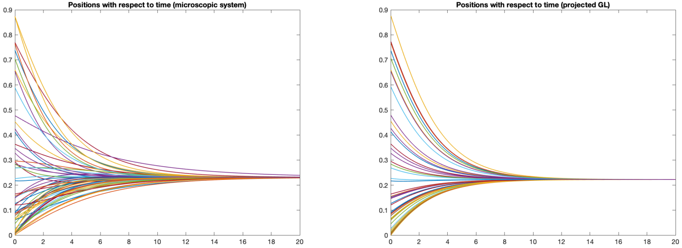

Figure: Time evolution of microscopic system $(\mathcal{S}_N^{\mathrm{r-r}})$ for $N = 60$ (left), and of the corresponding projection of the graph limit (right) with (2).

$$
\begin{cases} \dfrac{d}{dt} u_i^N(t) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \xi_{ij} D(u_j^N - u_i^N), \quad i \in \{1, \dots, N\} \\ u_i^N(0) = g(x_i^N), \quad i \in \{1, \dots, N\} \end{cases} \qquad (\mathcal{S}_N^{\mathrm{r-d}})
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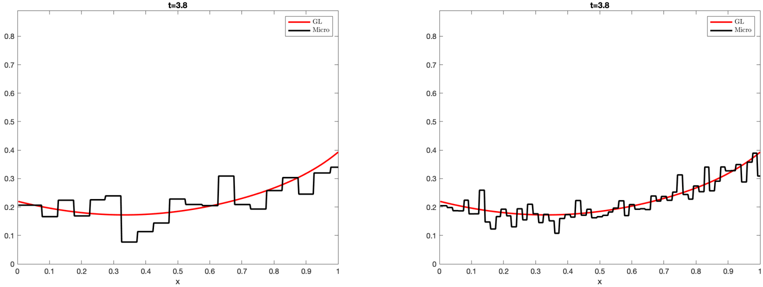

Figure: Evolution of the graph limit $u(\cdot, t)$ (red) and of $u_N(\cdot, t)$ solution to $(\mathcal{S}_N^{\mathrm{r-d}})$ (black) for $N = 20$ and $N = 60$, with the random weighted graph law (2).

Convergence of $(\mathcal{S}_N^{\mathrm{r-r}})$ and $(\mathcal{S}_N^{\mathrm{r-d}})$

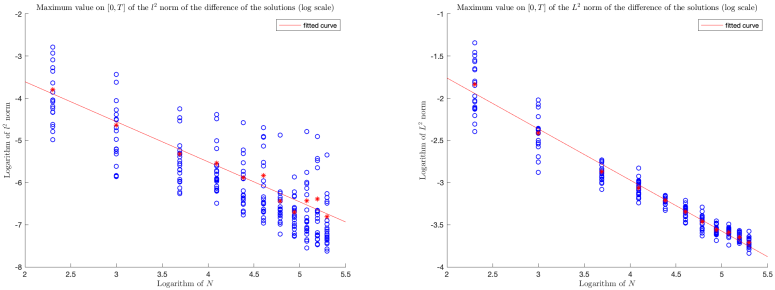

Figure: Left: Convergence of $(\mathcal{S}_N^{\mathrm{r-r}})$ quantified by $\sup_{t\in[0,T]} \|u^N(t) - \mathbf{P}_{\tilde{X}_N} u(\cdot,t)\|_{2,N}$. Right: Convergence of $(\mathcal{S}_N^{\mathrm{r-d}})$ quantified by $\sup_{t\in[0,T]} \|u_N(\cdot,t) - u(\cdot,t)\|_{L^2}$ for different values of N, with 20 runs for each value of N. Case of the random weighted graph law (2).

Numerical Illustration 2: Weighted "Small World" network

- *Model for a "small-world" network* (Watts and Strogatz, '98): Connect each node with its k closest neighbors to form a ring lattice. Then, rewire each edge at random with probability p.

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$$q(x, y; dw) = \begin{cases} \frac{\rho(x,y)}{r} d\lambda_{[0,1]} + (1 - \frac{\rho(x,y)}{r})\delta_1 & \text{if } \rho(x - y) \leq r \\ d\lambda_{[0,1]} & \text{otherwise} \end{cases} \quad (3)$$

where $\rho(x, y) = \min\{|x - y|, |x - y - 1|, |y - x - 1|\}$.

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- First moment:

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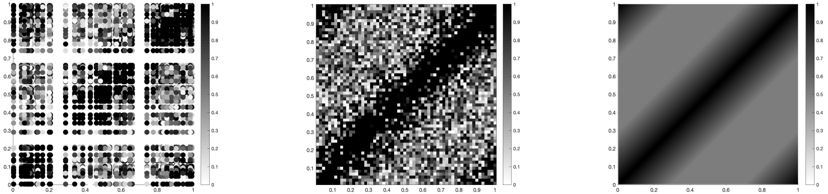
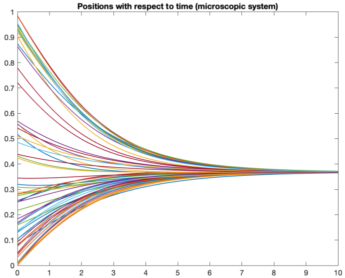


Figure: Values of the random interaction matrices generated from a random sequence (left) and a deterministic sequence (right) according to the random weighted graph law (3) for $N = 60$. Right: Corresponding continuous graphon $(x, y) \mapsto \bar{w}(x, y)$.

$$\begin{cases} \frac{d}{dt} u_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} \xi_{ij} D(u_j^N(t) - u_i^N(t)), \\ u_i^N(0) = g(X_i^N), \quad i \in \{1, \dots, N\} \end{cases} \qquad (\mathcal{S}_N^{\mathrm{r-r}})$$



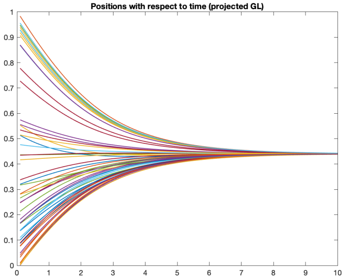


Figure: Time evolution of the microscopic system $(\mathcal{S}_N^{\mathrm{r-r}})$ for $N = 60$ (left), and of the corresponding projection of the graph limit (right), for the random weighed graph law (3).

$$D(z) = \frac{z}{1 + \|z\|^2} \text{ and } g(x) = \sin(4x)^2.$$

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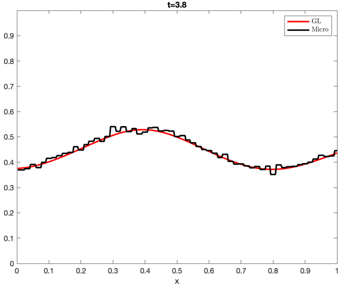

Figure: Evolution of the graph limit $u(\cdot,t)$ (red) and of $u_N(\cdot,t)$ solution to ($\mathcal{S}_N^{\mathrm{r-d}}$) (black) for $N = 60$, with the random weighted graph law (3).

Blinking systems

$\forall k \in \mathbb{N}$, $\forall t \in [k, k+1)$, $\xi_{ij}(t) = \xi_{ij}^k$ with $\mathcal{L}(\xi_{ij}^k | \tilde{X}) = q(X_i, X_j, \cdot)$.
Let $T > 0$, $n \in \mathbb{N}^*$, $\varepsilon = \frac{T}{n}$.

Definition: blinking system

$$\begin{cases} \dfrac{d}{dt} u_i^{N,\varepsilon}(t) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \xi_{ij}\left(\frac{t}{\varepsilon}\right) D(u_j^{N,\varepsilon}(t) - u_i^{N,\varepsilon}(t)), \\ u_i^{N,\varepsilon}(0) = g(X_i), \qquad i \in \{1, \dots, N\} \end{cases} \tag{$S_{N,\varepsilon}$}$$

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$$\begin{cases} \frac{d}{dt} u_i^{N,\varepsilon}(t) = \frac{1}{N} \sum_{j=1}^{N} \xi_{ij}\left(\frac{t}{\varepsilon}\right) D(u_j^{N,\varepsilon}(t) - u_i^{N,\varepsilon}(t)), \\ u_i^{N,\varepsilon}(0) = g(X_i), \qquad i \in \{1, \ldots, N\} \end{cases} \tag{$S_{N,\varepsilon}$}$$

Applying results from Averaging Theory (Skorokhod, '02), it holds

$$\mathbb{P}\left\{ \lim_{\varepsilon \to 0} \sup_{s \leq T} \left| u^{N,\varepsilon}(s) - u^{N,\text{Av}}(s) \right| = 0 \right\} = 1,$$

Blinking systems

$\forall k \in \mathbb{N}$, $\forall t \in [k, k+1)$, $\xi_{ij}(t) = \xi_{ij}^k$ with $\mathcal{L}(\xi_{ij}^k | \tilde{X}) = q(X_i, X_j, \cdot)$.
Let $T > 0$, $n \in \mathbb{N}^*$, $\varepsilon = \frac{T}{n}$.

Definition: blinking system

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where $u^{N,\text{Av}}$ is the solution to

Definition: Averaged System

$$\begin{cases} \dfrac{d}{dt} u_i^{N,\text{Av}}(t) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \left(\int_{\mathbb{R}_+} w q(X_i, X_j; dw) \right) D(u_j^{N,\text{Av}}(t) - u_i^{N,\text{Av}}(t)), \\ u_i^{N,\text{Av}}(0) = g(X_i), \qquad i \in \{1, \dots, N\} \end{cases} \qquad (S_{N,\text{Av}})$$

How are all the systems related?

So far we have considered four models :

- the system on a fixed weighted random graph ($\mathcal{S}_N^{\mathrm{r-r}}$)
- its limit as N goes to infinity, i.e. the graph limit equation (C)
- the blinking system ($S_{N,\varepsilon}$)
- its limit as ε goes to zero, i.e. the associated averaged system ($S_{N,\mathrm{Av}}$).

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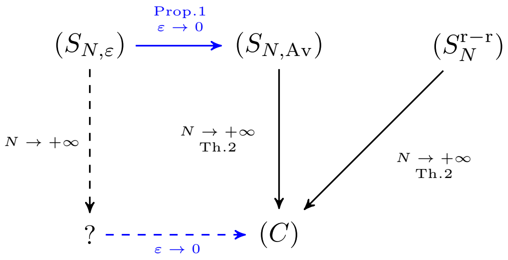

Figure: Existing and missing links between systems ($S_{N,\varepsilon}$), ($S_{N,\mathrm{Av}}$), ($\mathcal{S}_N^{\mathrm{r-r}}$) and ($C$)

Convergence of $(S_{N,\varepsilon})$ to (C)

Theorem [Ayi, P.D., '23]

Let $T>0$, $\varepsilon>0$ be given. Let $X=(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables and for all $N\in\mathbb{N}$, let $X^N=(X_i)_{1\leq i\leq N}$. Let $\xi_{ij}(t)=\xi_{ij}^k$ for all $t\in[k\varepsilon,(k+1)\varepsilon)$, $k\in\{0,\dots,n-1\}$ where $\mathcal{L}(\xi_{ij}^k|X^N)=q(X_i,X_j,\cdot)$. Let $u^{N,\varepsilon}$ be the solution to $(S_{N,\varepsilon})$ and let u be the solution to (C). Then,

$$\mathbb{P}\left[\sup_{t\in[0,T]}\|u^{N,\varepsilon}(t)-P_{X^N}u(t,\cdot)\|_{2,N}\geq\frac{C_3(T)}{\sqrt{N\varepsilon}}\right]\leq\frac{\tilde{C}_3(T)}{N\varepsilon}.$$

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$$\mathbb{P}\left[\sup_{t\in[0,T]} \|u^{N,\varepsilon}(t) - P_{X^N}u(t,\cdot)\|_{2,N} \ge \frac{C_3(T)}{\sqrt{N\varepsilon}}\right] \le \frac{\tilde{C}_3(T)}{N\varepsilon}.$$

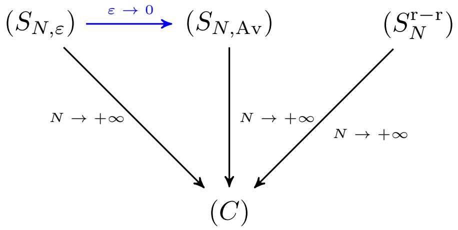
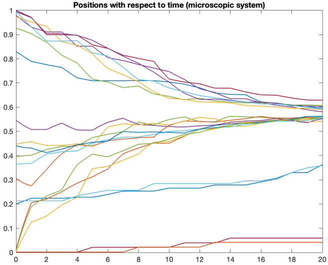
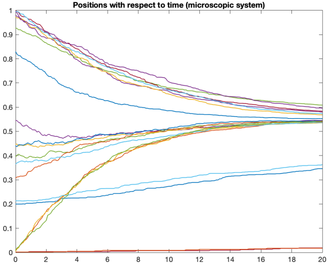
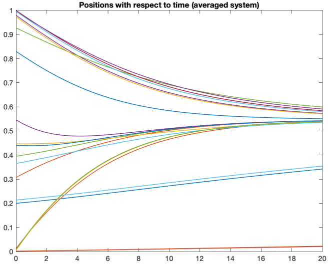


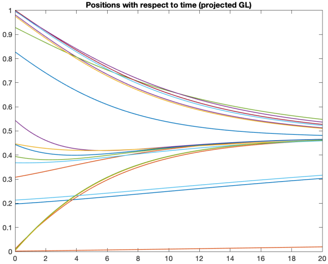
Figure: Links between systems $(S_{N,\varepsilon})$, $(S_{N,\mathrm{Av}})$, $(\mathcal{S}_N^{\mathrm{r-r}})$ and (C)

Numerical Simulations









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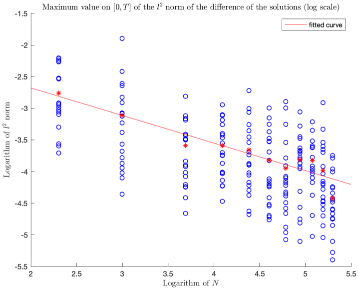

Figure: Quantification of the convergence of the microscopic systems $(S_{N,\varepsilon})$ given by $\sup_{t\in[0,T]} \|u^{N,\varepsilon}(t) - \mathbf{P}_{\tilde{X}_N} u(\cdot,t)\|_{2,N}$ for a fixed $\varepsilon = 0.1$, with 20 runs for each value of N, for the weighted random graph law (**??**) (logarithmic scale).

Thank you for your attention !