On the wellposedness of gSQG equation in a half-plane

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Contents

gSQG equation in $\mathbb{R}^2$

We consider the gSQG (α-SQG) in $\mathbb{R}^2$:

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = -\nabla^{\perp}(-\Delta)^{-1+\frac{\alpha}{2}}\theta, \end{cases}$$

for $0 < \alpha \leq 1$. When $\alpha = 0$ and $\alpha = 1$, α-SQG reduces to the incompressible Euler and SQG equations, respectively. Note that

$$(-\Delta)^{-1+\frac{\alpha}{2}}\theta = C_\alpha \int_{\mathbb{R}^2} \frac{1}{|x-y|^{\alpha}}\theta(y)\,\mathrm{d}y \quad \text{for some } C_\alpha > 0.$$

Biot–Savart law:

$$u(x) = \int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^{2+\alpha}}\theta(y)\,\mathrm{d}y.$$

A priori estimate in $H^m(\mathbb{R}^2)$:

Divergence-free condition implies that

$$\frac{d}{dt}\|\theta(t)\|_{L^2} = -\int_{\mathbb{R}^2} \theta u \cdot \nabla\theta \, \mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}^2} u \cdot \nabla|\theta|^2 \, \mathrm{d}x = 0.$$

Let $m \in \mathbb{N}$. In a similar way, we have

$$\begin{aligned}\frac{d}{dt}\|\theta(t)\|_{H^m}^2 &= -\int_{\mathbb{R}^2} \nabla^m(u\cdot\nabla\theta)\nabla^m\theta \, \mathrm{d}x \\ &\lesssim \int_{\mathbb{R}^2} \left(|\nabla^m u||\nabla\theta| + \cdots + |\nabla u||\nabla^m\theta|\right)|\nabla^m\theta| \, \mathrm{d}x \\ &\lesssim \left(\|\nabla^m u\|_{L^p}\|\nabla\theta\|_{L^q} + \|\nabla u\|_{L^\infty}\|\nabla^m\theta\|_{L^2}\right)\|\nabla^m\theta\|_{L^2}\end{aligned}$$

for any non-negative p and q with $1/p + 1/q = 1/2$.

If we take $1/p = \alpha/2$ and $1/q = 1/2 - \alpha/2$, we have from $u \sim \nabla^{-1+\alpha}\theta$

$$\|\nabla^m u\|_{L^p(\mathbb{R}^2)} \lesssim \|\nabla^m u\|_{H^{1-\alpha}(\mathbb{R}^2)} \lesssim \|\theta\|_{H^m(\mathbb{R}^2)}$$

and

$$\|\nabla\theta\|_{L^q(\mathbb{R}^2)} \lesssim \|\theta\|_{H^{1+\alpha}(\mathbb{R}^2)}.$$

Thus,

$$\frac{d}{dt}\|\theta(t)\|_{H^m}^2 \lesssim \|\theta(t)\|_{H^m}^3 + \|\nabla u(t)\|_{L^\infty}\|\theta(t)\|_{H^m}^2, \qquad m \geq 1+\alpha.$$

By the inequality

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\theta\|_{H^m(\mathbb{R}^2)}, \qquad m > 1+\alpha,$$

we obtain

$$\frac{d}{dt}\|\theta(t)\|_{H^m}^2 \lesssim \|\theta(t)\|_{H^m}^3, \qquad m > 1+\alpha.$$

A priori estimate in $C_c^\beta(\mathbb{R}^2)$:

Let $\beta \in (\alpha, 1]$. We consider the flow map Φ defined by

$$\frac{d}{dt}\Phi(t,x) = u(t,\Phi(t,x)), \qquad \Phi(0,x) = x.$$

Using $\theta(t,\Phi(t,x)) = \theta_0(x)$, we have

$$\begin{aligned}\sup_{x\neq x'} \frac{|\theta(t,x)-\theta(t,x')|}{|x-x'|^\beta} &= \sup_{x\neq x'} \frac{|\theta(t,\Phi(t,x))-\theta(t,\Phi(t,x'))|}{|\Phi(t,x)-\Phi(t,x')|^\beta} \\ &= \sup_{x\neq x'} \frac{|\theta_0(x)-\theta_0(x')|}{|x-x'|^\beta}\left(\frac{|x-x'|}{|\Phi(t,x)-\Phi(t,x')|}\right)^\beta.\end{aligned}$$

Note that

$$\begin{aligned}\frac{d}{dt}|\Phi(t,x)-\Phi(t,x')|^2 &= 2(u(t,\Phi_x)-u(t,\Phi_{x'}))\cdot(\Phi(t,x)-\Phi(t,x')) \\ &\le 2\|\nabla u\|_{L^\infty}|\Phi(t,x)-\Phi(t,x')|^2.\end{aligned}$$

By Grönwall's inequality, it follows

$$e^{-\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,\mathrm{d}\tau} \le \frac{|\Phi(t,x)-\Phi(t,x')|}{|x-x'|} \le e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,\mathrm{d}\tau}.$$

From the Biot–Savart law,

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\theta\|_{C^\beta\cap L^2(\mathbb{R}^2)}, \qquad \beta > \alpha.$$

This implies

$$e^{-\int_0^t \|\theta(\tau)\|_{C^\beta}\,\mathrm{d}\tau} \le \frac{|\Phi(t,x)-\Phi(t,x')|}{|x-x'|} \le e^{\int_0^t \|\theta(\tau)\|_{C^\beta}\,\mathrm{d}\tau}.$$

Combining the above, we have

$$\|\theta(t)\|_{C^\beta} \le \|\theta_0\|_{C^\beta} e^{\beta\int_0^t \|\theta(\tau)\|_{C^\beta}\,\mathrm{d}\tau}, \qquad \beta > \alpha.$$

Remarks

- α-SQG is locally well-posed in subcritical spaces. It was crucial that

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\theta\|_{H^m(\mathbb{R}^2)}, \qquad \|\nabla u\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\theta\|_{C_c^\beta(\mathbb{R}^2)}.$$

- By Cordoba and Martinez-Zoroa (2021) and Jeong and K. (2021), it was proved that α-SQG is ill-posed in the critical Sobolev space $H^{1+\alpha}$.
- C^1-illposedness of the critical SQG was proved by Elgindi and Masmoudi (2020).
- α-SQG with $\alpha \in (0,1)$ is ill-posed in the critical Hölder space C^α (in progress with Choi and Jung).

$$H^{1+\alpha}(\mathbb{R}^2) \hookrightarrow W^{1,p}(\mathbb{R}^2) \hookrightarrow C^\alpha(\mathbb{R}^2), \qquad \frac{1}{p} = \frac{1-\alpha}{2}.$$

gSQG equation in a half-plane

We consider α-SQG in the half-plane $\mathbb{R}^2_+ := (0, \infty) \times \mathbb{R}$:

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = -\nabla^\perp (-\Delta_D)^{-1+\frac{\alpha}{2}} \theta, \end{cases} \tag{α-SQG}$$

for $0 < \alpha \leq 1$. The velocity field u is given by

$$\begin{aligned} u(x) &= -\nabla^\perp (-\Delta_D)^{-1+\frac{\alpha}{2}} \theta \\ &= \int_{\mathbb{R}^2_+} \left[\frac{(x-y)^\perp}{|x-y|^{2+\alpha}} - \frac{(x-\tilde{y})^\perp}{|x-\tilde{y}|^{2+\alpha}} \right] \theta(y)\, \mathrm{d}y \\ &= \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+\alpha}} \overline{\theta(y)}\, \mathrm{d}y, \end{aligned} \tag{1}$$

where $\tilde{y} = (-y_1, y_2)$ for $y = (y_1, y_2)$ and $\overline{\theta(y)}$ is the odd extension of $\theta(y)$ in $\mathbb{R}^2$.

- $u_1 = 0$ on the boundary.
- In the half-plane case, it does not hold

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2_+)} \lesssim \|\theta\|_{C^\beta(\mathbb{R}^2_+)}.$$

  The velocity field u of the solution θ not vanishing at the boundary always does not have Lipschitz regularity (see velocity estimates in key lemmas for the details.)

Let us consider smooth initial data $\theta_0 \in C_c^\infty$.

- In $\mathbb{R}^2$ domain, it is well-known that
  1. Global regularity of solutions for $\alpha = 0$.
  2. Local regularity of solutions for $\alpha \in (0, 1]$.
- In $\mathbb{R}^2_+$, the global regularity was established for $\alpha = 0$ (for example, see Jiu, Li, and Zhang (2023), ...).

Solution spaces

For any $0 < \beta \leq 1$, let $X^\beta = X^\beta(\overline{\mathbb{R}^2_+})$ be a subspace of $C^\beta(\overline{\mathbb{R}^2_+})$ with anisotropic Lipschitz regularity in space: we say $f \in X^\beta$ if it belongs to C^β, differentiable almost everywhere, and satisfies

$$\|f\|_{X^\beta} := \|f\|_{L^\infty} + \|x_1^{1-\beta}\partial_1 f\|_{L^\infty} + \|\partial_2 f\|_{L^\infty} < \infty.$$

- X_c^β is a subset of X^β where $f \in X_c^\beta$ has a compact support.
- Let $\operatorname{supp} f \subset B(0;1)$. Then, $\|f\|_{X^{\beta_1}} \leq \|f\|_{X^{\beta_2}}$ for $\beta_1 \leq \beta_2$ and $\|f\|_{C^\beta} \lesssim \|f\|_{X^\beta}$ due to

$$\frac{|f(x_1,x_2) - f(x_1',x_2)|}{|x_1 - x_1'|^\alpha} \leq \frac{\left|\int_{x_1'}^{x_1} \tau^{-1+\alpha}\tau^{1-\alpha}\partial_1 f(\tau,x_2)\,\mathrm{d}\tau\right|}{|x_1 - x_1'|^\alpha}$$
$$\leq \|x_1^{1-\alpha}\partial_1 f\|_{L^\infty} \frac{\int_{x_1'}^{x_1} \tau^{-1+\alpha}\,\mathrm{d}\tau}{|x_1 - x_1'|^\alpha} \leq C\|x_1^{1-\alpha}\partial_1 f\|_{L^\infty}.$$

Classical solutions to the gSQG in $\mathbb{R}^2_+$

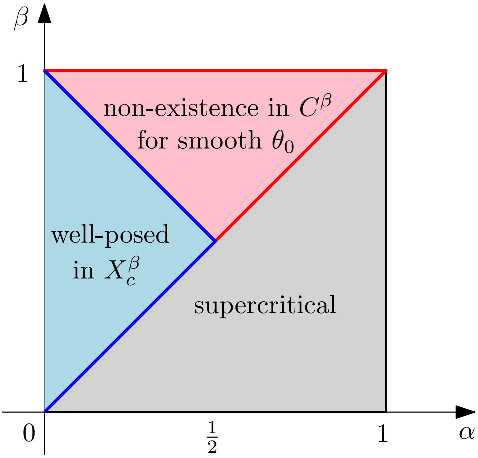

Figure 1: Well-posedness of α-SQG in X_c^β spaces

Previous results in $\mathbb{R}^2_+$

- Weak solutions: Resnick (1995) and Constantin and Nguyen (2018) proved the existence of weak solutions in $L^\infty_t L^2_x$ to the gSQG in $\mathbb{R}^2$ and open bounded set with smooth boundary, respectively. On the other hand, Buckmaster, Shkoller, and Vicol (2019), Isett and Ma (2021), and Cheng, Kwon, and Li (2021) proved the non-uniqueness of weak solutions in $\mathbb{R}^2$.
- Patch solutions in $\mathbb{R}^2_+$: Patch solutions have the form of

$$\theta(t,x) = \sum \theta_j \mathbf{1}_{\Omega_j(t)}(x),$$

where θ_j are some constants and $\Omega_j(t)$ are open sets with nonzero mutual distances and regular boundaries. Kiselev, Yao, and Zlatos (2017) proved the local wellposedness of H^3-patch solutions and their finite time blow-up. Gancedo and Patel (2021) proved similar results with H^2-patch solutions.

Velocity estimates

Lemma 1

Let $\alpha \in (0,1)$ and $\theta \in X_c^\alpha$. Then, the velocity $u = -\nabla^\perp(-\Delta_D)^{-1+\frac{\alpha}{2}}\theta$ satisfies

$$\|u_1\|_{C^{1,1-\alpha}} + \|\partial_2 u_2\|_{C^{1-\alpha}} + \|\partial_1(u_2 - U_2)\|_{L^\infty} \le C\|\theta\|_{X^\alpha}, \tag{2}$$

where

$$U_2(x) := -\frac{2}{\alpha}\int_{-\infty}^{\infty} \frac{\theta(0,y_2)}{|x-(0,y_2)|^\alpha}\,\mathrm{d}y_2$$

and

$$\left|\partial_1 U_2(x) - C_\alpha x_1^{-\alpha}\theta(0,x_2)\right| + |\partial_2 U_2(x)| \lesssim \|\partial_2\theta\|_{L^\infty}.$$

- $\nabla u_1,\ \partial_2 u_2 \in C^{1-\alpha}$, $\partial_1 u_2 \simeq x_1^{-\alpha}\theta(0,x_2)$
- In contrast, in the whole space $\mathbb{R}^2$ case, the velocity field produced by smooth solution $\theta \in C_c^\infty$ satisfies $u_1, u_2 \in C^\infty$

Lemma 2

Let $\alpha \in (0,1)$ and $\theta \in C_c^\alpha$. Let $\varphi \in C_c^\infty$ be a bump function such that $\varphi(x) = 1$ for $x \in \operatorname{supp}\theta$. Then, the velocity $u = -\nabla^\perp(-\Delta_D)^{-1+\frac{\alpha}{2}}\theta$ satisfies

$$
\begin{aligned}
&|u_1(x) - u_1(x')| + |u_2(x) - u_2(x') - \theta(x)(f(x) - f(x'))| \\
&\qquad \le C\|\theta\|_{C^\alpha}|x - x'|\log\left(10 + \frac{1}{|x - x'|}\right),
\end{aligned} \tag{3}
$$

where

$$
f(x) := -\frac{2}{\alpha}\int_{-\infty}^{\infty}\frac{\varphi(0,y_2)}{|x-(0,y_2)|^\alpha}\,\mathrm{d}y_2 \tag{4}
$$

and

$$
\partial_1 f(x) \simeq x_1^{-\alpha}, \qquad |\partial_2 f(x)| \lesssim 1.
$$

When $\alpha = 0$, one sholud replace (4) with

$$
f(x) := -2\int_{\mathbb{R}}\log(|x-(0,y_2)|)\varphi(0,y_2)\,\mathrm{d}y_2.
$$

Our results

We first consider $\alpha \in (0, \frac{1}{2}]$. In this case, we provide two main results.

Theorem 1 (Local wellposedness)

Let $\alpha \in (0, \frac{1}{2}]$ and $\beta \in [\alpha, 1-\alpha]$. Then ($\alpha$-SQG) is locally well-posed on X_c^β: for any $\theta_0 \in X_c^\beta$, there exist $T = T(\|\theta_0\|_{X^\alpha}, |\operatorname{supp}\theta_0|) > 0$ and a unique solution θ to (α-SQG) in the class $L^\infty(0, T; X_c^\beta) \cap C([0, T]; C^{\beta'})$ for any $0 < \beta' < \beta$.

Remarks:

- Finite-time singularity formation within this class is possible at least for small $\alpha > 0$ as in the patch solution case (refer to Alexander Kiselev, Lenya Ryzhik, Yao Yao, and Andrej Zlatos (2016) and Francisco Gancedo and Neel Patel (2021)).
- Zlatos (2023) proved the local wellposedness and the finite-time blow-up for $\alpha \in (0, \frac{1}{2}]$.

- The $\beta = \alpha$ case with $\alpha > 0$ is interesting since it is known that (α-SQG) is ill-posed in the critical spaces $H^{1+\alpha}(\mathbb{R}^2)$ and $C^{\alpha}(\mathbb{R}^2)$ by Elgindi and Masmoudi (2020), Cordoba and Zoroa (2021), and Jeong and K. (2021). The differentiability of θ_0 (odd in x_1 variable) in the x_2 determines whether solutions instantaneously blow up or not.
- $\alpha = 0$ case: Well-posed in X^{β} with $\beta > 0$.
- Blow-up criterion: If the local solution blows up at the finite time $t^* > 0$, then

$$\sup_{t \in [0,t^*)} \|\partial_2 \theta(t)\|_{L^\infty} = \infty.$$

As a consequence of this theorem, if we consider C_c^∞-data, there is a unique local solution in $L^\infty([0,T);X_c^\beta)$ for $\beta \in [\alpha, 1-\alpha]$ when $\alpha \in (0, \frac{1}{2}]$.

Our next result shows that this regularity is sharp, even for C_c^∞-data. Note that this is in stark contrast to the global wellposedness result in $C_c^\infty(\overline{\mathbb{R}^2_+})$ for the 2D Euler equations ($\alpha = 0$ case). One can easily check that $\partial_1 u_2$ should not be singular in the Euler case since it holds when $\alpha = 0$ that

$$\begin{aligned} \partial_1 u_2 &= -\partial_1^2(-\Delta_D)^{-1}\theta \\ &= \theta + \partial_2^2(-\Delta_D)^{-1}\theta \\ &= \theta + \partial_2 u_1. \end{aligned}$$

Recalling that u is smooth in x_2 direction (even for all $\alpha \in [0,1]$),

$$\partial_2^k u(x) = \partial_2^k \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+\alpha}} \overline{\theta(y)}\,\mathrm{d}y = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+\alpha}} \partial_2^k \overline{\theta(y)}\,\mathrm{d}y,$$

u is smooth when $\theta \in C_c^\infty(\overline{\mathbb{R}^2_+})$.

Theorem 2 (instantaneous blow-up)

Let $\alpha \in (0, \frac{1}{2}]$ and assume $\theta_0 \in C_c^\infty$ does not vanish on the boundary. Then, the local-in-time solution θ to (α-SQG) given by Theorem 1 does not belong to $L^\infty(0, \delta; C^\beta)$ for any $\beta > 1 - \alpha$ and $\delta > 0$.

Theorem 3 (instantaneous blow-up)

Let $\alpha \in (\frac{1}{2}, 1]$ and assume $\theta_0 \in C_c^\infty$ does not vanish on the boundary. Then, there is no solution to (α-SQG) with initial data θ_0 belonging to $L^\infty(0, \delta; C^\alpha)$ for any $\delta > 0$.

Remarks:

- Since the solutions must satisfy $\|\partial_2 \theta(t)\|_{L^\infty} < \infty$ on some interval $[0, \delta]$ by Theorem 1, the smoothness of θ should break down in the x_1 variable.

- One can replace C^β and C^α with X^β and X^α, respectively, since $\|f\|_{C^\gamma} \lesssim \|f\|_{X^\gamma}$ when $\operatorname{supp} f \subset B(0;R)$ for some $R > 0$.
- The non-vanishing condition of initial data implies that there exists $x_0 \in \partial\mathbb{R}^2_+$ such that

$$\theta_0(x_0) \neq 0, \qquad \limsup_{x \to x_0, x \in \partial\mathbb{R}^2_+} \frac{|\theta_0(x_0) - \theta_0(x)|}{|x_0 - x|} > 0.$$

- Instantaneous blow-up comes from singular properties of fractional Laplacian operator on the half-plane. This kind of result can be extended to the logarithmically irregularized Euler equations, and the logarithmically regularized ones (ongoing with Jeong and Yao).

Brief proof of Theorem 1

We recall the definition of X^β:

Solution spaces

For any $0 < \beta \leq 1$, let $X^\beta = X^\beta(\overline{\mathbb{R}^2_+})$ be a subspace of $C^\beta(\overline{\mathbb{R}^2_+})$ with anisotropic Lipschitz regularity in space: we say $f \in X^\beta$ if it belongs to C^β, differentiable almost everywhere, and satisfies

$$\|f\|_{X^\beta} := \|f\|_{L^\infty} + \|x_1^{1-\beta}\partial_1 f\|_{L^\infty} + \|\partial_2 f\|_{L^\infty} < \infty,$$

and prove that gSQG is well-posed in X_c^β for $\alpha \in (0, \frac{1}{2}]$ and $\beta \in [\alpha, 1-\alpha]$. Let us fix α, β and $\theta_0 \in X_c^\beta$, thus, $\operatorname{supp}\theta_0 \subset B(0; R)$ for some $R > 0$. We give a priori estimate in X^β. From the equation (α-SQG),

$$\theta_t + (u \cdot \nabla)\theta = 0,$$

we have

$$\partial_t\partial_2\theta + (u \cdot \nabla)\partial_2\theta = -\partial_2 u_1 \partial_1\theta - \partial_2 u_2 \partial_2\theta.$$

Here, we consider the flow map Φ defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,x) = u(t,\Phi(t,x)), \qquad \Phi(0,x) = x.$$

Recall that ∇u_2, $\partial_1 u_1 \in C^{1-\alpha}$, but $u_2 \simeq x_1^{-\alpha}\theta(0,x_2)$. Note from the boundary condition $u_1(0,x_2) = 0$ that

$$-\|\partial_1 u_1\|_{L^\infty} \le \frac{\mathrm{d}}{\mathrm{d}t}\log\Phi_1(t,x) \le \|\partial_1 u_1\|_{L^\infty}.$$

Thus,

$$e^{-C\int_0^t \|\partial_2\theta(\tau)\|_{L^\infty}\,\mathrm{d}\tau} \le \frac{\Phi_1(t,x)}{x_1} \le e^{C\int_0^t \|\partial_2\theta(\tau)\|_{L^\infty}\,\mathrm{d}\tau},$$

and $\Phi(t,x)$ is well-defined for each $x \in \mathbb{R}^2_+$ for all $t \ge 0$. Using the flow map and $\partial_2 u_1(0,x_2) = 0$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_2\theta\|_{L^\infty} &\le \|\partial_2 u_1\partial_1\theta\|_{L^\infty} + \|\partial_2 u_2\partial_2\theta\|_{L^\infty} \\ &\le \|\partial_2 u_1\|_{C^{1-\alpha}}\|x_1^{1-\alpha}\partial_1\theta\|_{L^\infty} + \|\partial_2 u_2\|_{L^\infty}\|\partial_2\theta\|_{L^\infty}.\end{aligned}$$

Combining with $\|x_1^{1-\alpha}\partial_1\theta\|_{L^\infty} \lesssim \|x_1^{1-\beta}\partial_1\theta\|_{L^\infty}$ when $\beta \geq \alpha$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_2\theta\|_{L^\infty} \lesssim \|\partial_2\theta\|_{L^\infty}\|\theta\|_{X^\alpha} \lesssim \|\partial_2\theta\|_{L^\infty}\|\theta\|_{X^\beta}.$$

On the other hand,

$$\partial_t(x_1^{1-\beta}\partial_1\theta) + u\cdot\nabla(x_1^{1-\beta}\partial_1\theta) = -x_1^{1-\beta}\partial_1 u\cdot\nabla\theta + (1-\beta)u_1x_1^{-\beta}\partial_1\theta.$$

Recalling $\partial_1 u_2 \simeq x_1^{-\alpha}\theta(0,x_2)$ and $u_1(0,x_2)=0$, we have

$$\|x_1^{1-\beta}\partial_1 u_2\partial_2\theta\|_{L^\infty} \leq \|x_1^{1-\beta}\partial_1 u_2\|_{L^\infty}\|\partial_2\theta\|_{L^\infty} \lesssim \|x_1^{1-\alpha-\beta}\theta\|_{L^\infty}\|\partial_2\theta\|_{L^\infty}$$

and

$$\|(1-\beta)u_1x_1^{1-\beta}\partial_1\theta\|_{L^\infty} \leq (1-\beta)\|\partial_1 u_1\|_{L^\infty}\|x_1^{1-\beta}\partial_1\theta\|_{L^\infty},$$

respectively. Combining with $1-\alpha-\beta \geq 0$ and $\operatorname{supp}\theta(t) \subset B(0;2R)$ on some time interval $[0,T]$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x_1^{1-\beta}\partial_1\theta\|_{L^\infty} \lesssim \|\partial_2\theta\|_{L^\infty}\|\theta\|_{X^\beta}.$$

Therefore, we can deduce for $\beta \in [\alpha, 1-\alpha]$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{X^\beta} \lesssim \|\partial_2\theta\|_{L^\infty}\|\theta\|_{X^\beta} \lesssim \|\theta\|_{X^\alpha}\|\theta\|_{X^\beta}.$$

As a corollary, we obtain that: For any $T \in (0,\infty)$,

$$\int_0^T \|\partial_2\theta(t)\|_{L^\infty}\,\mathrm{d}t < \infty \quad \Longleftrightarrow \quad \sup_{t\in[0,T]} \|\theta(t)\|_{X^\beta} < \infty. \quad \square$$

Brief proof of Theorem 2

Let $\alpha \in (0, \frac{1}{2}]$ and $\theta_0 \in C_c^\infty(\mathbb{R}^2_+)$. Then from Theorem 1, we obtain $T > 0$ and the unique solution $\theta \in L^\infty(0, T; X_c^{1-\alpha})$. We prove that

$$\sup_{t\in[0,\delta]} \|\theta(t)\|_{C^\beta} = \infty \quad \text{for all } \beta > 1-\alpha \text{ and } \delta > 0.$$

From the assumption of initial data, we have $x_0 = (0, a) \in \partial\mathbb{R}^2_+$ such that $\theta_0(x_0) \neq 0$ and $\partial_2\theta_0(x_0) > 0$. For simplicity, let $\theta_0(x_0) = 1$. Take $x = x_0 + (\ell^{-1}, -\ell^{-(1-\gamma)})$ for large $\ell > 0$, where $\gamma > 0$ will be specified later. We claim that there exists $t^* = t^*(\ell) \searrow 0$ and an arbitrary constant $\varepsilon > 0$ such that

$$\frac{\theta(t^*, \Phi(t^*, x_0)) - \theta(t^*, \Phi(t^*, x))}{|\Phi(t^*, x_0) - \Phi(t^*, x)|^\beta} = \frac{\theta_0(x_0) - \theta_0(x)}{|x_0 - x|} \frac{|x_0 - x|}{|\Phi(t^*, x_0) - \Phi(t^*, x)|^\beta}$$
$$\gtrsim \ell^\varepsilon,$$

where Φ is the flow map defined in the proof of Theorem 1.

- $\dfrac{\theta_0(x_0) - \theta_0(x)}{|x_0 - x|} \gtrsim \partial_2\theta_0(x_0)$

From $|x_0 - x| = \sqrt{\ell^{-2} + \ell^{-2(1-\gamma)}} = \left(\frac{1+\ell^{2\gamma}}{\ell^{2\gamma}}\right)^{\frac{1}{2}} |a - x_2| = x_1(1 + \ell^{2\gamma})^{\frac{1}{2}}$,

$$\frac{\theta_0(x_0) - \theta_0(x)}{|x_0 - x|} = \frac{\theta_0(a, 0) - \theta_0(0, x_2)}{|x_0 - x|} + \frac{\theta_0(0, x_2) - \theta_0(x)}{|x_0 - x|}$$

$$= \frac{\theta_0(0, a) - \theta_0(0, x_2)}{|a - x_2|}\left(\frac{\ell^{2\gamma}}{1 + \ell^{2\gamma}}\right)^{\frac{1}{2}} + \frac{\theta_0(0, x_2) - \theta_0(x)}{|x_1|}\left(\frac{1}{1 + \ell^{2\gamma}}\right)^{\frac{1}{2}}$$

$$\gtrsim \frac{\theta_0(0, a) - \theta_0(0, x_2)}{|a - x_2|} \gtrsim \partial_2\theta_0(x_0)$$

for sufficiently large ℓ.

We recall the velocity estimate:

$$\|\nabla u_1\|_{C^{1-\alpha}} + \|\partial_2 u_2\|_{C^{1-\alpha}} \le C\|\theta\|_{X^\alpha}, \qquad \partial_1 u_2 \simeq x_1^{-\alpha}\theta(0,x_2).$$

- $\dfrac{|x_0 - x|}{|\Phi(t^*,x_0) - \Phi(t^*,x)|^\beta} \gtrsim \ell^\varepsilon$

Let us consider Φ_1 first. Using $u_1(0,x_2) = 0$, we have

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1(t,x)\right| &= \left|u_1(t,\Phi(t,x))\right| \\
&= \left|u_1(t,\Phi(t,x)) - u_1(t,0,\Phi_2(t,x))\right| \\
&\le \|\partial_1 u_1\|_{L^\infty}\Phi_1(t,x) \\
&\lesssim \|\theta\|_{X^\alpha}\Phi_1(t,x),
\end{aligned}$$

which implies that $\Phi_1(t,x) \sim x_1$ on the sufficiently short time interval $[0, T]$, not depending on the choice of $x_1 > 0$.

To obtain the claim, we show that there exists $t^* \lesssim \ell^{\gamma-\alpha}$ such that

$$\Phi_2(t^*, x) = \Phi_2(t^*, x_0),$$

where $\Phi_2(0, x) = x_2 = a - \ell^{-(1-\gamma)} < a = \Phi_2(0, x_0)$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_2(t, x_0) - \Phi_2(t, x)) = u_2(t, \Phi(t, x_0)) - u_2(t, \Phi(t, x))$$
$$= u_2(t, \Phi(t, x_0)) - u_2(t, 0, \Phi_2(t, x)) + u_2(t, 0, \Phi_2(t, x)) - u_2(t, \Phi(t, x)).$$

For the first and second terms, we have from $\Phi(t, x_0) = (0, \Phi_2(t, x_0))$ that

$$\begin{aligned} u_2(t, 0, \Phi_2(t, x_0)) - u_2(t, 0, \Phi_2(t, x)) &\leq \|\partial_2 u_2\|_{L^\infty}(\Phi_2(t, x_0) - \Phi_2(t, x)) \\ &\lesssim \|\theta\|_{X^\alpha}(\Phi_2(t, x_0) - \Phi_2(t, x)) \end{aligned}$$

until $\Phi_2(t, x) \leq \Phi_2(t, x_0)$.

On the other hand, using

$$\theta(t,0,\Phi_2(t,x)) \simeq \theta(t,\Phi(t,x)) = \theta_0(x) \simeq \theta_0(x_0) = 1$$

for sufficiently large ℓ, we have

$$\begin{aligned} u_2(t,0,\Phi_2(t,x)) - u_2(t,\Phi(t,x)) &= -\int_0^{\Phi_1(t,x)} \partial_1 u_2(t,\tau,\Phi_2(t,x))\,\mathrm{d}\tau \\ &\simeq -\int_0^{\Phi_1(t,x)} \tau^{-\alpha}\theta(t,0,\Phi_2(t,x))\,\mathrm{d}\tau \\ &\simeq -\Phi_1(t,x)^{1-\alpha} \simeq -x_1^{1-\alpha}. \end{aligned}$$

Combining the above, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_2(t,x_0) - \Phi_2(t,x)) \lesssim -x_1^{1-\alpha} + \|\theta\|_{X^\alpha}(\Phi_2(t,x_0) - \Phi_2(t,x)).$$

Grönwall's inequality gives

$$\Phi_2(t,x_0) - \Phi_2(t,x) \le \left(a - x_2 - \frac{1}{C} x_1^{1-\alpha} t\right) e^{C\|\theta\|_{X^\alpha} t}$$
$$= \left(\ell^{-(1-\gamma)} - \frac{1}{C}\ell^{-(1-\alpha)} t\right) e^{C\|\theta\|_{X^\alpha} t}.$$

Thus, there exists $t^* \le C\ell^{\gamma-\alpha}$ such that $\Phi_2(t^*,x) = \Phi_2(t^*,x_0)$.

Now, we have $\Phi_1(t^*,x) \sim x_1$ and $\Phi_2(t^*,x) = \Phi_2(t^*,x_0)$ with $t^* \lesssim \ell^{\gamma-\alpha}$, for sufficiently large ℓ. Thus,

$$\frac{|x_0 - x|}{|\Phi(t^*,x_0) - \Phi(t^*,x)|^\beta} \simeq \frac{a - (a - \ell^{-(1-\gamma)})}{\Phi_1(t^*,x)^\beta}$$
$$\simeq \frac{\ell^{-(1-\gamma)}}{\ell^{-\beta}}$$
$$= \ell^{\gamma+\beta-1}.$$

Taking $\gamma \in (1-\beta, \alpha)$ and $\varepsilon = \gamma + \beta - 1 > 0$, we finish the proof.

Proof of Lemma 1

Let $\alpha \in (0,1)$. We recall $\tilde{y} = (-y_1, y_2)$ and

$$u(x) = \int_{\mathbb{R}^2_+} \left[\frac{(x-y)^\perp}{|x-y|^{2+\alpha}} - \frac{(x-\tilde{y})^\perp}{|x-\tilde{y}|^{2+\alpha}} \right] \theta(y)\,\mathrm{d}y.$$

- $\|\nabla u_1\|_{C^{1-\alpha}} + \|\partial_2 u_2\|_{C^{1-\alpha}} \lesssim \|\partial_2 \theta\|_{L^\infty}$

Given $f \in L^\infty$ with $\operatorname{supp} f \subset B(0;1)$, we can see that

$$\left\| \int_{\mathbb{R}^2_+} \frac{x-y}{|x-y|^{2+\alpha}} f(y)\,\mathrm{d}y \right\|_{C^{1-\alpha}} + \left\| \int_{\mathbb{R}^2_+} \frac{x-\tilde{y}}{|x-\tilde{y}|^{2+\alpha}} f(y)\,\mathrm{d}y \right\|_{C^{1-\alpha}} \lesssim \|f\|_{L^\infty}.$$

This gives $\|\partial_2 u\|_{C^{1-\alpha}} \lesssim \|\partial_2 \theta\|_{L^\infty}$ and $\|\partial_1 u_1\|_{C^{1-\alpha}} = \|\partial_2 u_2\|_{C^{1-\alpha}} \lesssim \|\partial_2 \theta\|_{L^\infty}$.

- $\|\partial_1(u_2 - U_2)\|_{L^\infty} \lesssim \|\theta\|_{X^\alpha}$

We recall

$$u_2 = \int_{\mathbb{R}^2_+} \left[\frac{x_1 - y_1}{|x-y|^{2+\alpha}} - \frac{x_1 - \tilde{y}_1}{|x-\tilde{y}|^{2+\alpha}} \right] \theta(y)\,\mathrm{d}y.$$

Since

$$\frac{x_1 - y_1}{|x-y|^{2+\alpha}} = \partial_{y_1} \frac{1}{|x-y|^\alpha}, \qquad -\frac{x_1 - \tilde{y}_1}{|x-\tilde{y}|^{2+\alpha}} = \partial_{y_1} \frac{1}{|x-\tilde{y}|^\alpha},$$

we have

$$u_2(x) = -\int_{\mathbb{R}^2_+} \left[\frac{1}{|x-y|^\alpha} + \frac{1}{|x-\tilde{y}|^\alpha} \right] \partial_1\theta(y)\,\mathrm{d}y + U_2(x),$$

where

$$U_2(x) := -\frac{2}{\alpha} \int_{-\infty}^{\infty} \frac{\theta(0, y_2)}{|x-(0,y_2)|^\alpha}\,\mathrm{d}y_2.$$

Note that

$$\partial_1(u_2(x) - U_2(x)) = \int_{\mathbb{R}^2_+} \left[\frac{x_1 - y_1}{|x-y|^{2+\alpha}} + \frac{x_1 + y_1}{|x - \tilde{y}|^{2+\alpha}} \right] \partial_1 \theta(y) \, \mathrm{d}y.$$

We estimate

$$\begin{aligned} \left| \int \frac{x_1 - y_1}{|x-y|^{2+\alpha}} \partial_1 \theta(y) \, \mathrm{d}y \right| &= \left| \int \frac{x_1 - y_1}{y_1^{1-\alpha}|x-y|^{2+\alpha}} y_1^{1-\alpha} \partial_1 \theta(y) \, \mathrm{d}y \right| \\ &\lesssim \| x_1^{1-\alpha} \partial_1 \theta \|_{L^\infty} \end{aligned}$$

on the two regions $\{0 \le y_1 \le \frac{1}{2} x_1\} \cup \{|x_1 - y_1| \le \frac{1}{2} x_1\}$, and

$$\int_{\{y_1 \ge \frac{3}{2} x_1\}} \left[\frac{x_1 - y_1}{|x-y|^{2+\alpha}} + \frac{x_1 + y_1}{|x - \tilde{y}|^{2+\alpha}} \right] \partial_1 \theta(y) \, \mathrm{d}y \lesssim \| x_1^{1-\alpha} \partial_1 \theta \|_{L^\infty},$$

using the cancellation property. Thus, we have $\|\partial_1(u_2(x) - U_2(x))\|_{L^\infty} \lesssim \|x_1^{1-\alpha} \partial_1 \theta\|_{L^\infty}$ and $\|\partial_1(u_2 - U_2)\|_{L^\infty} \lesssim \|\theta\|_{X^\alpha}$.

- $\left|\partial_1 U_2(x) - C_\alpha x_1^{-\alpha}\theta(0,x_2)\right| + |\partial_2 U_2(x)| \lesssim \|\partial_2\theta\|_{L^\infty}$

We recall

$$U_2(x) := -\frac{2}{\alpha}\int_{-\infty}^{\infty} \frac{\theta(0,y_2)}{|x-(0,y_2)|^\alpha}\,\mathrm{d}y_2.$$

Then, we have

$$\partial_1 U_2 = 2\int_{-\infty}^{\infty} \frac{x_1\theta(0,y_2)}{|x-(0,y_2)|^{2+\alpha}}\,\mathrm{d}y_2$$

$$= 2x_1\theta(0,x_2)\int_{-\infty}^{\infty} \frac{1}{|x-(0,y_2)|^{2+\alpha}}\,\mathrm{d}y_2 + 2\int_{-\infty}^{\infty} \frac{x_1(\theta(0,y_2)-\theta(0,x_2))}{|x-(0,y_2)|^{2+\alpha}}\,\mathrm{d}y_2$$

$$= C_\alpha x_1^{-\alpha}\theta(0,x_2) + 2\int_{-\infty}^{\infty} \frac{x_1(\theta(0,y_2)-\theta(0,x_2))}{|x-(0,y_2)|^{2+\alpha}}\,\mathrm{d}y_2$$

Using the change of variable gives

$$\left|2\int_{-\infty}^{\infty} \frac{x_1(\theta(0,y_2)-\theta(0,x_2))}{|x-(0,y_2)|^{2+\alpha}}\,\mathrm{d}y_2\right| \lesssim \min\{x_1^{-\alpha}\|\theta\|_{L^\infty},\ x_1^{1-\alpha}\|\partial_2\theta\|_{L^\infty}\}. \qquad \square$$

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Thank you very much