Geometric effects on $W^{1,p}$ regularity of the stationary linearized Boltzmann equation

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This talk is based on a joint work with Daisuke Kawagoe, Chun-Hsiung Hsia, and Jhe-Kuan Su.

Stationary linearized Boltzmann equation in $\mathbb{R}^3$

Bounded domain:

$$\Omega \in \mathbb{R}^3. \tag{1}$$

We consider the linearized velocity distribution function:
$f : \Omega \times \mathbb{R}^3 \to \mathbb{R}$.
We define

$$\Gamma_- := \{(x, v) | \; x \in \partial\Omega, n(x) \cdot v < 0\}. \tag{2}$$

Incoming Boundary Value Problem for linearized stationary Boltzmann equation:

$$\begin{cases} v \cdot \nabla_x f = L(f), & x \in \Omega,\ v \in \mathbb{R}^3, \\ f(x,v) = g(x,v), & (x,v) \in \Gamma_-, \end{cases} \tag{3}$$

where L is the linearized collision operator.

Our gaol is to classify the range of $W^{1,p}$ solution space according to the geometry of the domain.
We focus on the stationary linearized Boltzmann equation in a convex domain. To our surprise, the flatness has a dramatic effect on the range of p.

Regularity of stationary Boltzmann equation in a bounded domain

Mixture lemma: Collision and free transport move regularity from velocity variable to space variable.

- (C. 2018 SIMA) Linearized equation, incoming boundary, locally Holder.
- (C., Kawagoe, Hsia 2019 Annales de l'Institut Henri Poincaré C) Linearized equation, diffuse reflection, pointwise estimate of derivatives.
- (Chen, Kim. 2022 ARMA) Nonlinear equation, diffuse boundary, locally $C^{1,\beta}$.
- (Chen 2022 SIMA) Cercignani–Lampis Boundary condition

C., Kawagoe, Hsia 2019 Annales de l'Institut Henri Poincaré C:

$$|\nabla_x f| + |\nabla_v f| \leq C|1 + d_x^{-1}|^{\frac{4}{3}+\epsilon},$$

where

$$d_x = \text{dist}(x, \partial\Omega).$$

Chen, Kim. 2022 ARMA:

$$\||v|^2 \nabla_v f\|_\infty \leq C\|T - T_w\|_\infty.$$

(Notice that the Lemma 2.13 in this paper is not correct)

Velocity averaging lemma

- (C.,Chung, Hsia, Su 2022 JSP) linearized equation, incoming boundary, $L^2_v(\mathbb{R}^3, H^{1-}_x(\Omega))$.

We can not recover $L^2_v(\mathbb{R}^3, H^1_x(\Omega))$ by this estimates by Bourgain-Brezis-Mironescu formula.

Time evolutional problem

Regularity for time evolutional problem:
(Guo, Kim, Tonon, Trescases 2017 Invent. Math.) Nonlinear equation, diffuse reflection, $W^{1,p}$ for $1 \leq p < 2$. Disprove H^1 result to free transport equation.

Motivation:
What is the situation for stationary solution?

Existence of H^1 solutions to Stationary Linearized Boltzmann Equation in a Small Domain.

Theorem (C.,Chung, Hsia, Kawagoe, Su April 19, 2023)

There exists a small $\epsilon > 0$ such that
Suppose domain Ω is of positive curvature with diam$(\Omega) < \epsilon$, the incoming boundary value problem for stationary linearized Boltzmann equation has a unique solution $f \in H^1(\Omega \times \mathbb{R}^3)$ if g is smooth enough.

Remark

Chen Kim investigate a related issue on asymptotic stability in $W_x^{1,p}$ for $1 \leq p < 3$ (ARMA 2024).

We classify the range of p for solution space $W^{1,p}$ according to the geomtry of the domain.

Assumption A. We say L satisfies the condition A if the operator $L(f)$ can be decomposed into the multiplicative term $-\nu(v)f$ and the integral term $K(f) = \int k(v, v_*)f(v_*)dv_*$ with the following estimates for some fixed $0 \leq \gamma \leq 1$.

$$\nu_0(1 + |v|)^\gamma \leq \nu(v) \leq \nu_1(1 + |v|)^\gamma \tag{4}$$

$$|k(v, v^*)| \leq C\frac{1}{|v - v^*|(1 + |v| + |v^*|)^{1-\gamma}}e^{\frac{1-\rho}{4}(|v-v^*|^2+(\frac{|v|^2-|v^*|^2}{|v-v^*|})^2)}, \tag{5}$$

$$|\nabla_v k(v, v^*)| \leq C\frac{1 + |v|}{|v - v^*|^2(1 + |v| + |v^*|)^{1-\gamma}}e^{\frac{1-\rho}{4}(|v-v^*|^2+(\frac{|v|^2-|v^*|^2}{|v-v^*|})^2)}, \tag{6}$$

$$|\nabla_v \nu(v)| \leq C(1 + |v|)^{\gamma-1}. \tag{7}$$

Remark

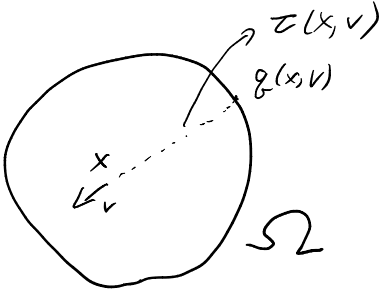
The crosssection $B = C|v|^\gamma \cos\theta$ *for* $0 \leq \gamma \leq 1$ *yields a linearized collision operator L satisfying Assumption A.*

Remark

For Grad's angular cutoff, $0 \leq B \leq C|v|^\gamma \cos\theta$ *for* $0 \leq \gamma \leq 1$*, (8) and the upper bound in (7) was proved by Caflisch (1980 CMP).*

$$\tau(x,v) := \inf_{t>0}\{t : \ x - vt \notin \Omega\},$$

$$q(x,v) := x - \tau(x,v)v.$$



We rewrite

$$\begin{cases} v \cdot \nabla_x f + \nu(v) f = K(f), & x \in \Omega,\ v \in \mathbb{R}^3, \\ f(x,v) = g(x,v), & (x,v) \in \Gamma_-. \end{cases} \tag{8}$$

Integral equation:

$$\begin{aligned} f(x,v) =& e^{-\nu(v)\tau(x,v)} g(q(x,v),v) \\ &+ \int_0^{\tau(x,v)} e^{-\nu(v)s} K(f)(x - sv, v)\, ds. \end{aligned} \tag{9}$$

Hereafter, we define

$$(Jg)(x,v) := e^{-\nu(v)\tau_-(x,v)} g(q(x,v),v), \tag{10}$$

$$(S_\Omega f)(x,v) := \int_0^{\tau(x,v)} e^{-\nu(v)s} f(x - sv, v) ds. \tag{11}$$

We can rewrite

$$f(x,v) = J(g) + S_\Omega K(f). \tag{12}$$

Definition

We say f is a solution to (8) *if f satisfies* (12)*.*

Let

$$L^p_\alpha(\Omega \times \mathbb{R}^3) := \{f \mid \|f\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} < \infty\},$$

where

$$\|f\|^p_{L^p_\alpha(\Omega\times\mathbb{R}^3)} := \int_\Omega \int_{\mathbb{R}^3} |f(x,v)|^p e^{p\alpha|v|^2}\, dxdv.$$

Also, for $1 \leq p < \infty$ and $\alpha \geq 0$, we define the function space $W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$ by

$$W^{1,p}_\alpha(\Omega \times \mathbb{R}^3) := \{f \mid \|f\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} < \infty\},$$

where

$$\|f\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} := \|f\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} + \|\nabla_x f\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} + \|\nabla_v f\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)}.$$

Notice that $W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$ with $\alpha = 0$ is the usual Sobolev space $W^{1,p}(\Omega \times \mathbb{R}^3)$.

Theorem (C., Hsia, Kawagoe, Su,11, 2023)

Suppose L satisfies **Assumption A**. *Let* $0 \leq \alpha < (1-\rho)/2$ *and* Ω *be a bounded convex domain with* C^2 *boundary. Then, the following statements hold.*

(i) *For* $1 \leq p < 2$, *there exists* $\epsilon > 0$ *depending on p and* α *such that: for any* Ω *with diam*$(\Omega) < \epsilon$, *the boundary value problem* (3) *has a unique solution* $f \in W^{1,p}_{\alpha}(\Omega \times \mathbb{R}^3)$ *if and only if* $Jg \in W^{1,p}_{\alpha}(\Omega \times \mathbb{R}^3)$.

(ii) *We further assume that* $\partial\Omega$ *is of positive Gaussian curvature. Then, the range of p in* (i) *can be extended to* $1 \leq p < 3$.

(iii) *The conclusions in* (i) *and* (ii) *are optimal.*

arXiv:2311.12387

To be more precise, (iii) means:

Lemma (Counter example p=2)

For fixed $1 \leq p < 2$ *and* $0 \leq \alpha < (1-\rho)/2$*, there exist a bounded convex domain* Ω *and a boundary data g such that the boundary value problem* (3) *has a solution in* $L^2_\alpha(\Omega \times \mathbb{R}^3) \cap W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$ *but not in* $W^{1,2}_\alpha(\Omega \times \mathbb{R}^3)$*.*

Lemma (Counter example p=3)

For fixed $2 \leq p < 3$ *and* $0 \leq \alpha < (1-\rho)/2$*, there exist a bounded convex domain* Ω *with its boundary of positive Gaussian curvature and a boundary data g such that the boundary value problem* (3) *has a solution in* $L^3_\alpha(\Omega \times \mathbb{R}^3) \cap W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$ *but not in* $W^{1,3}_\alpha(\Omega \times \mathbb{R}^3)$*.*

Nonlinear case for small domain with positive Gaussian curvature is established in a norm that is a proper subspace of $W^{1,p}$ for $1 \leq p < 3$ by by C. Kawagoe, hsia,and Su in 3, 2024.

arXiv:2403.10016

Sketch of the proof

Recall

$$f(x,v) = J(g) + S_\Omega K(f). \quad (13)$$

Performing Picard iteration, formally we have

$$f = \sum_{i=0}^{\infty} (S_\Omega K)^i Jg. \quad (14)$$

Goal: To prove the series (14) converges in the desired norm.

For L^p_α space, we have

Lemma

Let $1 \le p < \infty$ and $0 \le \alpha < (1-\rho)/2$, where ρ is the constant in **Assumption A**. *Then, for any $h \in L^p_\alpha(\Omega \times \mathbb{R}^3)$, we have*

$$\|S_\Omega K h\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} \lesssim \mathrm{diam}(\Omega)^{\frac{1}{p}} \|h\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)}. \qquad (15)$$

If $\mathrm{diam}(\Omega)$ is small enough, by contraction mapping theorem, (3) has a solution in L^p_α.

Sobolev space case

We do not have a direct analogy for $W^{1,p}_\alpha$ case. Instead,

Lemma

Given $h \in W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$ with $1 \le p < 2$ and $0 \le \alpha < (1-\rho)/2$, where ρ is the constant in **Assumption A**, *we have*

$$\|S_\Omega K h\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} \lesssim \mathrm{diam}(\Omega)^{\frac{1}{p}} \|h\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} + \|h\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} + \mathrm{diam}(\Omega)^{\frac{1}{p}} \|h\|_{L^p_\alpha(\partial\Omega\times\mathbb{R}^3)},$$

where $\|h\|_{L^p_\alpha(\partial\Omega\times\mathbb{R}^3)}$ is defined by

$$\|h\|^p_{L^p_\alpha(\partial\Omega\times\mathbb{R}^3)} := \int_{\mathbb{R}^3}\int_{\partial\Omega} |h(z,v)|^p e^{p\alpha|v|^2}\, d\Sigma(z) dv,$$

and $d\Sigma$ denotes the surface measure on $\partial\Omega$.

Trace inequalities

Lemma (Trace inequalities)

Let Ω *be a bounded domain with Lipschitz boundary. Also,* $\alpha \geq 0$*. Then,*
(i) For $1 < p < \infty$*, there exists a positive constant* $C_2(\Omega, p)$ *such that*

$$\|h\|_{L^p_\alpha(\partial\Omega\times\mathbb{R}^3)} \leq C_2(\Omega, p)\left(\delta^{\frac{p-1}{p}}\|\nabla_x h\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} + \delta^{-\frac{1}{p}}\|h\|_{L^p_a(\Omega\times\mathbb{R}^3)}\right)$$

for all $h \in W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$ *and* $0 < \delta < 1$*.*
(ii)

$$\|h\|_{L^1_\alpha(\partial\Omega\times\mathbb{R}^3)} \leq (1+\delta)\|\nabla_x h\|_{L^1_\alpha(\Omega\times\mathbb{R}^3)} + C_\delta(\Omega)\|h\|_{L^1_\alpha(\Omega\times\mathbb{R}^3)}$$

for all $h \in W^{1,1}_\alpha(\Omega \times \mathbb{R}^3)$*.*

For fixed $1 \leq p < 2$ and $0 \leq \alpha < (1-\rho)/2$, taking δ and $\mathrm{diam}(\Omega)$ sufficiently small and combining Lemmas above together, we have

$$\begin{aligned} \|(S_\Omega K)^i Jg\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} \leq & \frac{1}{2}\|(S_\Omega K)^{i-1} Jg\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} \\ & + C_3\|(S_\Omega K)^{i-1} Jg\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} \end{aligned} \tag{16}$$

For the case $2 \leq p < 3$, we need to use a good property of positive Gaussian curvature. We recall the following estimate.

Lemma (Proposition 5.9 in (C.,Chung, Hsia, Su 2022 JSP))

Let Ω be a C^2 bounded convex domain of positive Gaussian curvature. Then, there exists a positive constant $C_1(\Omega)$ depending only on Ω such that for any $z \in \partial\Omega$ and $v \in \mathbb{R}^3$ we have

$$|z - q(z, v)| \leq C_1(\Omega) N(z, v),$$

where

$$N(z, v) := |n(z) \cdot \hat{v}|, \quad \hat{v} := \frac{v}{|v|}.$$

From Lemma 7, we have the following estimate.

Lemma

Let Ω be a C^2 bounded convex domain of positive Gaussian curvature, and let $C_1(\Omega)$ be a constant defined in Lemma 7. Then, given $h \in W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$ with $2 \le p < 3$ and $0 \le \alpha < (1-\rho)/2$, where ρ is the constant in **Assumption A**, *we have*

$$\|S_\Omega K h\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} \lesssim \mathrm{diam}(\Omega)^{\frac{1}{p}} \|h\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} + \|h\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} + C_1(\Omega)^{\frac{1}{p}} \|h\|_{L^p_\alpha(\partial\Omega\times\mathbb{R}^3)}.$$

Counter-example for the case $p = 2$

We choose Ω as a small bounded convex domain such that

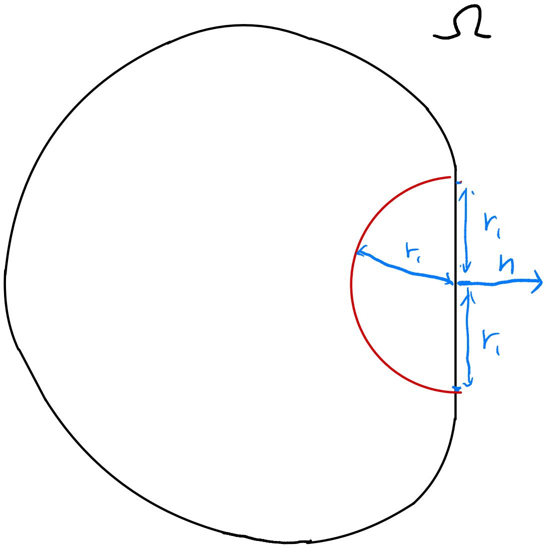
$$D_{r_1} := \{x = (0, x_2, x_3) \in \mathbb{R}^3 \mid |x| < r_1\} \subset \partial\Omega \tag{17}$$

with a small radius r_1 and

$$\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x| < r_1, x_1 < 0\} \subset \Omega. \tag{18}$$

We remark that $n(0) = (1, 0, 0)$.

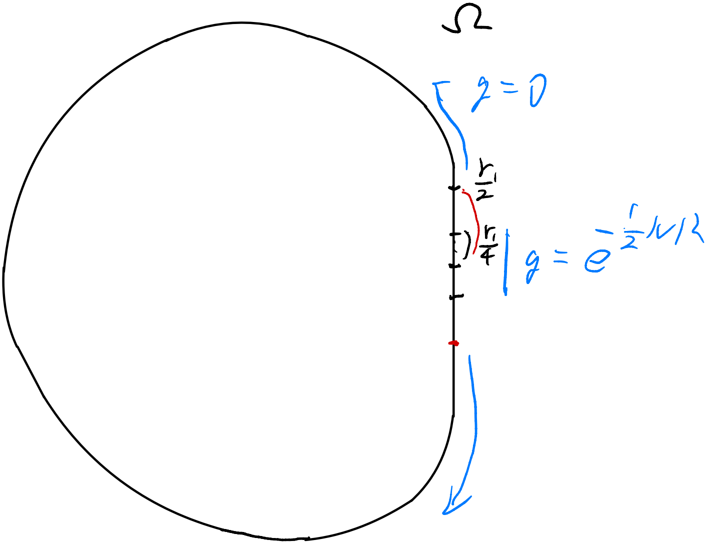
Counter-example for $p = 2$



Let φ_1 be a smooth cut-off function on $\partial\Omega$ such that $0 \le \varphi_1 \le 1$, $\varphi_1(x) = 1$ for $x \in D_{r_1/4}$, and $\varphi_1(x) = 0$ for $x \in \partial\Omega \setminus D_{r_1/2}$. We pose the boundary data g of the form:

$$g(x, v) = \varphi_1(x) e^{-\frac{1}{2}|v|^2}, \quad (x, v) \in \Gamma^-. \tag{19}$$

Counter-example for $p = 2$



We assume $f \in W^{1,2}_\alpha$, then derive a contradiction. Recall

$$(Jg)(x,v) := e^{-\nu(v)\tau_-(x,v)} g(q(x,v),v), \tag{20}$$

$$(S_\Omega f)(x,v) := \int_0^{\tau(x,v)} e^{-\nu(v)s} f(x-sv,v)ds. \tag{21}$$

Thus,

$$\begin{aligned}\nabla_x f(x,v) = &- \nu(|v|)(\nabla_x \tau(x,v)) Jg(x,v) + (\nabla_x q(x,v)) J(\nabla_X g)(x,v)\\ &+ S_{\Omega,x} Kf(x,v) + S_\Omega K(\nabla_x f)(x,v).\end{aligned}$$

By assumption, we see that $S_\Omega K(\nabla_x f) \in L^2_\alpha(\Omega \times \mathbb{R}^3)$, and therefore the integral

$$\int_\Omega \int_{\mathbb{R}^3} \left|\nabla_x f - S_\Omega K \nabla_x f\right|^2 e^{2\alpha|v|^2}\, dxdv \tag{22}$$

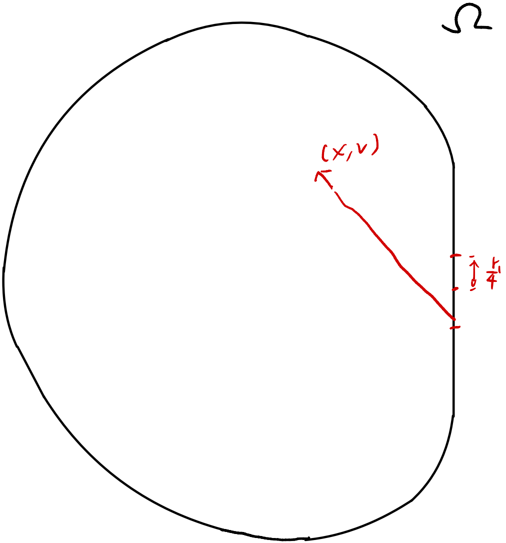
is bounded.

Let $r_2 > 0$ and

$$D_{r_1,r_2} := \{(x,v) \in \Omega \times \mathbb{R}^3 \mid q(x,v) \in D_{r_1/4}, \tau(x,v) \leq 1, |v| < r_2\}. \tag{23}$$

In this region, we have $J(\nabla_X g) = 0$

Counter-example for $p = 2$



$|v| < r_2$

$q(x,v) \in D_{\frac{r_1}{4}}$

$\tau(x,v) \leq 1$

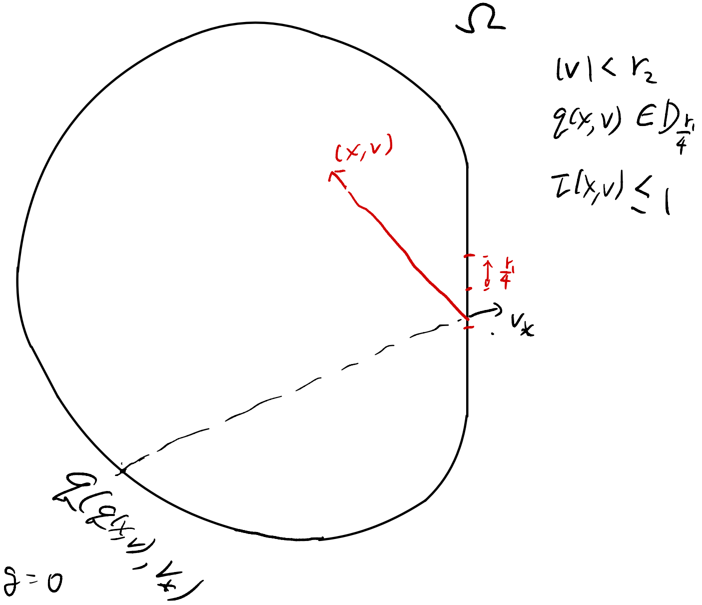
$$
\begin{aligned}
S_{\Omega,x}Kf(x,v) =& (\nabla_x \tau(x,v)) e^{-\nu(v)\tau(x,v)} \int_{\Gamma^+_{q(x,v)}} k(v,v^*) f(q(x,v),v^*)\, dv^* \\
&+ (\nabla_x \tau(x,v)) e^{-\nu(v)\tau(x,v)} \int_{\Gamma^-_{q(x,v)}} k(v,v^*) e^{-\frac{1}{2}|v^*|^2}\, dv^*.
\end{aligned}
$$

We substitute the function f in the first term of the right hand side by the integral equation again to obtain

$$
\begin{aligned}
&\int_{\Gamma^+_{q(x,v)}} k(v,v^*) f(q(x,v),v^*)\, dv^* \\
=& \int_{\Gamma^+_{q(x,v)}} k(v,v^*) Jg(q(q(x,v),v^*),v^*)\, dv^* \\
&+ \int_{\Gamma^+_{q(x,v)}} k(v,v^*) S_\Omega Kf(q(q(x,v),v^*),v^*)\, dv^*.
\end{aligned}
$$

Here, since $q(q(x, v), v^*) \notin D_{r_1}$ for $(x, v) \in D_{r_1,r_2}$ and $v^* \in \Gamma^+_{q(x,v)}$, the first term in the right hand side is zero.

Counter-example for $p = 2$



On the other hand, we have

$$\left| \int_{\Gamma^+_{q(x,v)}} k(v, v^*) S_\Omega K f(q(q(x, v), v^*), v^*) \, dv^* \right|$$
$$\lesssim \|f\|_{C_\alpha((\Omega \times \mathbb{R}^3) \cup \Gamma^\pm)} \operatorname{diam}(\Omega)$$
$$\lesssim \operatorname{diam}(\Omega).$$

Therefore, we can make contribution from the integral

$$\int_{\Gamma^+_{q(x,v)}} k(v, v^*) f(q(x, v), v^*) \, dv^*$$

arbitrary small by taking $\operatorname{diam}(\Omega)$ sufficiently small.

Thus, in D_{r_1,r_2}

$$
\begin{aligned}
&-(\nabla_x f(x,v) - S_\Omega K(\nabla_x f)(x,v)) \geq \\
&\nu(|v|)(\nabla_x \tau(x,v)) e^{-\nu(v)\tau(x,v)} \left(e^{-\frac{1}{2}|v|^2} - \int_{\Gamma^-_{q(x,v)}} k(v,v^*) e^{-\frac{1}{2}|v^*|^2} - \epsilon \right) \\
&\geq \nu(|v|)(\nabla_x \tau(x,v)) e^{-\nu(v)} \left(e^{-\frac{1}{2}|v|^2} - \int_{\Gamma^-_{q(x,v)}} k(v,v^*) e^{-\frac{1}{2}|v^*|^2} - \epsilon \right)
\end{aligned} \tag{24}
$$

Lemma

There exist $\eta_0 > 0$ *and* $r_2 > 0$ *such that*

$$\nu(|v|)e^{-\frac{1}{2}|v|^2} - \int_{\Gamma^-_{q(x,v)}} k(v, v^*)e^{-\frac{1}{2}|v^*|^2}\,dv^* > \eta_0$$

for all $(x, v) \in D_{r_1,r_2}$.

$$\int_\Omega \int_{\mathbb{R}^3} |\nabla_x f - S_\Omega K \nabla_x f|^2 e^{2\alpha|v|^2} dxdv \gtrsim \int_{D_{r_1,r_2}} |\nabla_x \tau(x,v)|^2 dxdv.$$

Here, we perform the same change of variable.

$$\int_{D_{r_1,r_2}} |\nabla_x \tau(x,v)|^2 dxdv$$
$$= \int_{D_{r_1/4}} \int_{\{v_1<0\}\cap\{|v|<r_2\}} \int_0^{\min\{\tau(z,-v),1\}} |\nabla_x \tau(z+tv,v)|^2 dtN(z,v)|v|dv$$

It is known that

$$\nabla_x \tau(x, v) = \frac{-n(q(x, v))}{N(x, v)|v|}. \tag{25}$$

$$\int_0^{\min\{\tau(z,-v),1\}} |\nabla_x \tau(z + tv, v)|^2 \, dt N(z, v)|v| = \frac{\min\{\tau(z, -v), 1\}}{N(z, v)|v|}$$

We restrict ourselves to the case $|v| < r_1/2$. In this case, we have $\tau(z, -v) > 1$. Let $r_3 := \min\{r_1/2, r_2\}$. Then, we have

$$\int_{D_{r_1/4}} \int_{\{v_1<0\}\cap\{|v|<r_2\}} \int_0^{\min\{\tau(z,-v),1\}} |\nabla_x \tau(z + tv, v)|^2 \, dt N(z, v)|v| dv$$

$$\geq \int_{D_{r_1/4}} \int_{\{v_1<0\}\cap\{|v|<r_3\}} \frac{1}{N(z, v)|v|} \, dv \, d\Sigma(z).$$

Introducing the spherical coordinates to v so that $\theta = 0$ corresponds to $(-1, 0, 0)$, we have

$$\int_{\{v_1<0\}\cap\{|v|<r_3\}} \frac{1}{N(z, v)|v|}\, dv = \pi r_3^2 \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}\, d\theta,$$

which is divergent for all $z \in D_{r_1/4}$. Therefore the integral (22) is not bounded, and it is a contradiction.

Thank you!

Counter example for $p = 3$

We consider $B(0, r)$. We parametrize the boundary by $x = (r\cos\theta, r\sin\theta\cos\phi, r\sin\theta\sin\phi)$ for $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. With these coordinates, for $\theta_0 \in (0, \pi)$, let $\partial\Omega_{\theta_0} := \{x \in \partial\Omega \mid 0 \le \theta < \theta_0\}$. Take $0 < \theta_1 < \theta_2 < \pi$ and a smooth cut-off function φ_2 on $\partial\Omega$ such that $\varphi_2(x) = 1$ for $x \in \partial\Omega_{\theta_1}$, $\varphi_2(x) = 0$ for $x \in \partial\Omega \setminus \partial\Omega_{\theta_2}$, and $0 \le \varphi_2(x) \le 1$ for $x \in \partial\Omega_{\theta_2} \setminus \partial\Omega_{\theta_1}$. We pose the boundary data g of the form:

$$g(x, v) = \varphi_2(x) e^{-\frac{1}{2}|v|^2}, \quad (x, v) \in \Gamma^-. \tag{26}$$

Thank you!