

Eigenfunctions of the laplacian on negatively curved manifolds : a semiclassical approach

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AN INTRODUCTION TO SEMICLASSICAL ANALYSIS.

1. Mechanics.

1.1. Three approaches to classical mechanics. The variational approach. The Maupertuis or Euler principle [M1744, E1744] is the mechanical analogue of the Fermat principle in optics: a solid of mass $m = 1$, moving under

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the effect of a force $F = -\text{grad}V$, with a total energy E , follows a trajectory γ which minimizes the action

$$(1.1) \quad S(\gamma) = \int \sqrt{2(E - V(\gamma))} \|d\gamma\|$$

among all curves with the same endpoints, and under the constraint that $\frac{\|\dot{\gamma}(t)\|^2}{2} + V(\gamma(t)) = E$ for all t . More precisely, we should look for *critical points* of S , among all paths with given endpoints, and constant total energy E . In these notes, we work on a riemannian manifold (X, g) , and $\|\cdot\|_x$ is the norm defined on $T_x X$ by the riemannian metric: $\|v\|_x^2 = g_x(v, v)$. In other words, the Maupertuis principle says that the trajectories of energy E are geodesics for a new, degenerate metric, $2(E - V(x))g_x$.

The dual formulation, due to Lagrange [L1788], is to find the extrema of the functional

$$(1.2) \quad A(\gamma) = \int_0^T \left(\frac{\|\dot{\gamma}(t)\|_{\gamma(t)}^2}{2} - V(\gamma(t)) \right) dt$$

among all curves going from x to y in a given time T . Let us introduce the lagrangian $L(x, v) = \frac{\|v\|_x^2}{2} - V(x)$; the movement is described by the Euler-Lagrange equation

$$(1.3) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial v}(\gamma, \dot{\gamma}) \right) = \frac{\partial L}{\partial x}(\gamma, \dot{\gamma}),$$

or more explicitly $D_{\dot{\gamma}}\dot{\gamma} = -\text{grad}V(\gamma)$. This second order equation defines a local flow (ϕ_{EL}^t) on the tangent bundle TX , called the Euler-Lagrange flow.

Hamiltonian point of view. The hamiltonian is the Fenchel–Legendre transform of L with respect to the variable v :

$$H(x, \xi) = \xi.v - L(x, v)$$

with $\xi = \frac{\partial L}{\partial v}$; we are in a nice situation where the Legendre transformation

$$\mathcal{L}eg : (x, v) \mapsto \left(x, \frac{\partial L}{\partial v} \right)$$

defines a diffeomorphism between the tangent bundle TX and the cotangent bundle T^*X . Its inverse is $\mathcal{L}eg^{-1} : (x, \xi) \mapsto \left(x, \frac{\partial H}{\partial \xi} \right)$. In fact, in our case, $\mathcal{L}eg$ is nothing else than the natural identification between TX and T^*X , provided by the riemannian metric. We can define a scalar product g^x on T_x^*X by $g^x(\xi, \xi) = g_x(v, v) = \|v\|_x^2$, with $\xi = \frac{\partial L}{\partial v}$. The vector ξ is called the momentum, and $H(x, \xi) = \frac{g^x(\xi, \xi)}{2} + V(x)$ is the total energy of the system. We shall also denote $g^x(\xi, \xi) = \|\xi\|_x^2$, but the reader should not confuse the norms $\|\cdot\|_x$ on T_x^*X and $T_x X$.

The Euler-Lagrange equation (1.3) is equivalent to Hamilton's system of equations,

$$(1.4) \quad \begin{cases} \dot{x} = \frac{\partial H}{\partial \xi} \\ \dot{\xi} = -\frac{\partial H}{\partial x}, \end{cases}$$

which define a local flow (ϕ_H^t) on T^*X , called the hamiltonian flow. This flow is conjugate to (ϕ_{EL}^t) via the diffeomorphism $\mathcal{L}eg$. It preserves the energy H , in the

sense that $H(x(t), \xi(t))$ is constant for any trajectory of the flow $(x(t), \xi(t))$. The hamiltonian flow also preserves the Liouville measure $dx d\xi$.

If a is a function on T^*X (an “observable quantity” in the language of Heisenberg), and if we denote $a_t = a \circ \phi_H^t$, we have

$$\frac{da}{dt} = \{H, a\},$$

where $\{., .\}$ denotes the Poisson bracket, $\{H, a\} = \sum \partial_{\xi_j} H \partial_{x_j} a - \partial_{x_j} H \partial_{\xi_j} a$.

A more intrinsic way of writing the Hamilton equations (1.4) would be to note that the vector field on the right hand side is the symplectic gradient of H , with respect to the *canonical symplectic form* on T^*X . Let us define the Liouville 1-form on the cotangent bundle, defined by

$$\alpha_{(x,\xi)}(P) = \xi \cdot d\pi(P) \text{ for all } P \in T_{(x,\xi)}(T^*X),$$

where $\pi : T^*X \rightarrow X$ is the usual projection, and $d\pi$ its tangent map. The cotangent bundle T^*X can be endowed with the symplectic form

$$(1.5) \quad \omega = -d\alpha.$$

In local coordinates, $\alpha = p \cdot dq$ and $\omega = dq \wedge dp$, if p and q denote respectively the “momentum” and “position” functions, $p(x, \xi) = \xi$, $q(x, \xi) = x$. The reader can check that the right hand side of (1.4) is the expression in local coordinates of the symplectic gradient X_H of H , defined by $dH = \omega(X_H, .)$. The Poisson bracket is given by $\{f, g\} = -\omega(X_f, X_g) = dg(X_f)$, for any two functions f, g on T^*X .

One can show that the flow ϕ_H preserves the symplectic form ω . In the language of symplectic geometry, a (local) diffeomorphism of T^*X which preserves ω is called a canonical transformation.

Hamilton–Jacobi equation, generating functions. This third point of view, called the Hamilton–Jacobi approach, meets many technical difficulties, but it is the key tool to understand the semiclassical analysis of the Schrödinger equation.

Around 1830, Hamilton introduced a new formalism, in which the action is seen as a function of the endpoints x and y [**H1830**, **H1834**]. Let $\gamma : [0, T] \rightarrow X$ be a solution of the Euler–Lagrange equation, joining x to y in time $T > 0$. To simplify the discussion, we consider here the nice, but usually unrealistic situation, where such a trajectory is unique. We can then consider the lagrangian action $A(x, y; T) = \int_0^T L(\gamma, \dot{\gamma}) dt$ as a function of x, y, T , and check that

$$(1.6) \quad \frac{\partial A}{\partial x} = -\dot{\gamma}(0); \quad \frac{\partial A}{\partial y} = \dot{\gamma}(T),$$

and

$$\frac{\partial A}{\partial T} = -E$$

where E is the energy $E = \frac{\|\dot{\gamma}\|^2}{2} + V(\gamma)$, constant along the trajectory γ . If we freeze the variable y (thus fixing an initial or rather “final” condition) and see A as a function of $x \in X$ and $T > 0$, we have

$$(1.7) \quad \frac{\partial A}{\partial T} + H(x, \partial_x A) = 0.$$

Hamilton then argues that being able to integrate the hamiltonian vector field (1.4) is equivalent to finding the generating function A , solution of the Hamilton–Jacobi equation (1.7) for any initial condition (or a large enough family of initial

conditions). By this procedure, the ordinary differential equations (1.3) or (1.4) have been replaced by a single PDE. Quoting Hamilton, “*even if it should be thought that no practical facility is gained, yet an intellectual pleasure may result from the reduction of [...] all researches respecting the forces and motions of body, to the study of one characteristic function*”.

Let us also consider the Legendre transform of $A(x, y; T)$ with respect to the variable T ,

$$(1.8) \quad S(x, y; E) = ET + A(x, y; T)$$

where T and E are related by

$$\frac{\partial A}{\partial T} = -E,$$

which implies that

$$\frac{\partial S}{\partial E} = T.$$

The function S is nothing else but the Maupertuis action (1.1) of the trajectory γ joining x to y with energy E :

$$S(x, y; E) = \int \sqrt{2(E - V(\gamma))} \|\dot{\gamma}\| dt = \int_0^T \|\dot{\gamma}\|^2 dt.$$

We still have

$$(1.9) \quad \frac{\partial S}{\partial x} = -\dot{\gamma}(0); \quad \frac{\partial S}{\partial y} = \dot{\gamma}(T).$$

If we freeze the final state y , the function S solves the stationary Hamilton–Jacobi equation,

$$(1.10) \quad H(x, \partial_x S) = E.$$

The solutions of the time–dependent Hamilton–Jacobi equation (1.7) and of the stationary equation (1.10) are related by the Legendre transform (1.8).

The Hamilton–Jacobi equation (1.7) has a simple geometrical interpretation. Consider a subset of the cotangent bundle T^*X , of the form $\mathcal{L}_0 = \{(x, d_x A_0), x \in \Omega_0\}$, with Ω_0 an open subset of X . This is a particular case of a *lagrangian submanifold* in T^*X (see Definition 12.1). Let \mathcal{L}_0 evolve under the hamiltonian flow, and consider $\mathcal{L}_t = \phi_H^t \mathcal{L}_0$: because ϕ_H^t preserves the symplectic form ω , \mathcal{L}_t is still a lagrangian manifold. Let us assume that, for $t \in [0, T]$, \mathcal{L}_t still projects diffeomorphically to an open subset of $\Omega_t \subset X$. This means exactly that \mathcal{L}_t is of the form $\mathcal{L}_t = \{(x, d_x A_t), x \in \Omega_t\}$ for some smooth function A_t . It can be shown that the relation $\mathcal{L}_t = \phi_H^t \mathcal{L}_0$ is equivalent to A_t solving the Hamilton–Jacobi equation (1.7), with the condition that Ω_t is the image of \mathcal{L}_0 under the “exponential” map associated with \mathcal{L}_0 :

$$(1.11) \quad \exp_{\mathcal{L}_0}^t : \mathcal{L}_0 \longrightarrow X,$$

$$(1.12) \quad \xi \mapsto \pi(\phi_H^t \xi)$$

(the notation π denotes the projection $T^*X \longrightarrow X$).

This approach suffers from the notorious problem of caustics (Figure 1). Usually, the exponential map will only be a diffeomorphism if the energy H is bounded on \mathcal{L}_0 , and if t is small enough. For large times two kinds of problems arise,

– exp is not injective (two trajectories starting in \mathcal{L}_0 land at the same point in X)

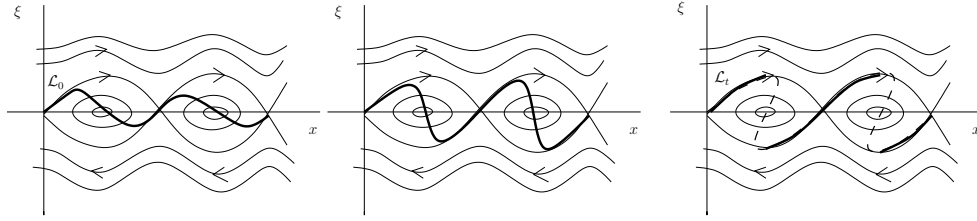


FIGURE 1. Appearance of caustics for large times.

– the tangent map $d\exp$ is not injective (focal points, conjugate points).

Geometrically, this means that after some time \mathcal{L}_t will cease to project diffeomorphically to X . From a PDE point of view, this means that the equation (1.7) does not, in general, have globally defined smooth solutions.

Although the problem of caustics makes the Hamilton–Jacobi equation rather difficult to work with, it is, nevertheless, the key tool to understand Schrödinger’s equation and its semiclassical analysis. Semiclassical methods often break down with the appearance of caustics, or a little after.

We now review Schrödinger’s view of mechanics, but also the work of Born, Heisenberg and Jordan, which lead to the idea of quantization.

1.2. Quantum/wave mechanics. At the beginning of the twentieth century, it became clear that classical mechanics was not applicable to certain problems, like the study of energy radiation in atoms. People started looking for new physical laws, but it was not until 1925 that theories judged as satisfactory were elaborated. These theories involve Planck’s constant $h = 2\pi\hbar = 6.626068 \times 10^{-34} m^2 \cdot kg/s$ (the “action quantum”), and one is supposed to recover classical mechanics when letting h tend to 0 in the equations.

Quantenmechanik. In 1925, Heisenberg, Born and Jordan gave some new laws of mechanics, supposed to replace the old Hamilton equations (1.4). Consider a hamiltonian system with d degrees of freedom, meaning that the manifold X has dimension d . In fact let us take $X = \mathbb{R}^d$ as in the paper [BHJ25-II]. In classical mechanics the time evolution is given by equation (1.4), defining a symplectic flow on the phase space T^*X . According to the quantum mechanics of [BHJ25-II], the time evolution of the system is ruled by the five following principles :

(0) The “phase space” is a Hilbert space \mathcal{H} .

(1) The “observable quantities” are described by linear operators (= infinite matrices). Heisenberg, Born and Jordan used a boldface letter \mathbf{a} to denote the quantum observable corresponding to the classical observable a ; if a is a real-valued function on T^*X then the corresponding operator \mathbf{a} is hermitian.

(2) **Main rules :** We consider, in particular, an algebra of operators generated by the momentum and position observables, $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_d)$ and $\mathbf{q} =$

$(\mathbf{q}_1, \dots, \mathbf{q}_d)$. These operators must obey the following commutation rules :

$$(1.13) \quad [\mathbf{p}_k, \mathbf{q}_l] = \frac{\hbar}{i} \delta_{kl} I,$$

$$(1.14) \quad [\mathbf{p}_k, \mathbf{p}_l] = 0,$$

$$(1.15) \quad [\mathbf{q}_k, \mathbf{q}_l] = 0.$$

Consider now a classical observable f defined by a power series

$$f(p, q) = \sum \alpha_{sr} p^s q^r.$$

Then the quantum observable \mathbf{f} should be defined by

$$\mathbf{f}(\mathbf{p}, \mathbf{q}) = \sum \alpha_{sr} \frac{1}{s+1} \sum_{l=0}^s \mathbf{p}^{s-l} \mathbf{q}^r \mathbf{p}^l.$$

This prehistoric “quantization rule” can be applied, in particular, to define the hamiltonian operator \mathbf{H} .

(3) A canonical transformation is a transformation that sends the observables (\mathbf{p}, \mathbf{q}) to new observables (\mathbf{P}, \mathbf{Q}) satisfying the same commutation relations. We ask that a canonical transformation preserve hermitian operators, and sends an observable of the form $\mathbf{f}(\mathbf{p}, \mathbf{q})$ to $\mathbf{f}(\mathbf{P}, \mathbf{Q})$. Such a transformation is of the form $\mathbf{P} = \mathbf{S}\mathbf{p}\mathbf{S}^{-1}$, $\mathbf{Q} = \mathbf{S}\mathbf{q}\mathbf{S}^{-1}$, where \mathbf{S} is a unitary operator.

(4) The equations of motion are

$$(1.16) \quad \begin{cases} \dot{\mathbf{p}} = -\frac{\partial \mathbf{H}}{\partial \mathbf{q}} \\ \dot{\mathbf{q}} = \frac{\partial \mathbf{H}}{\partial \mathbf{p}}, \end{cases}$$

where we define

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}_1} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\mathbf{f}(\mathbf{x}_1 + \varepsilon I, \mathbf{x}_2, \dots, \mathbf{x}_s) - \mathbf{f}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s))$$

for $\mathbf{f}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s)$ a power series in the s observables $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s$ (I is the identity operator).

It can be shown from formula (1.13) that we have the identity

$$[\mathbf{f}, \mathbf{g}] = \frac{\hbar}{i} \left(\frac{\partial \mathbf{f}}{\partial \mathbf{p}} \frac{\partial \mathbf{g}}{\partial \mathbf{q}} - \frac{\partial \mathbf{f}}{\partial \mathbf{q}} \frac{\partial \mathbf{g}}{\partial \mathbf{p}} \right)$$

holding for \mathbf{f}, \mathbf{g} power series in the operators \mathbf{p} and \mathbf{q} .

In particular, the equations (1.16) can be reexpressed as

$$\dot{\mathbf{f}} = \frac{i}{\hbar} [\mathbf{H}, \mathbf{f}]$$

for any observable \mathbf{f} .

(5) To integrate the equation of motion, we must find a unitary operator \mathbf{S} such that

$$(1.17) \quad \mathbf{S}\mathbf{H}\mathbf{S}^{-1} = \mathbf{W}$$

is diagonal. In other words, we look for a canonical transformation which allows to express the solutions of (1.16) as a superposition of periodic motions¹.

In a basis where \mathbf{H} is diagonal, we find, for any observable \mathbf{f} , that the matrix elements evolve according to

$$(1.18) \quad \mathbf{f}_{nm}(t) = \mathbf{f}_{nm}(0)e^{2i\pi\nu_{nm}t}$$

where the radiation spectrum ν_{nm} (“physical spectrum”) is related to the eigenvalues (E_n) of \mathbf{H} (“mathematical spectrum”) by

$$\nu_{nm} = \frac{E_n - E_m}{h}.$$

Wellenmechanik. In 1926, Erwin Schrödinger, independently of the work of Heisenberg, Born and Jordan, proposed a new equation, supposed to describe the state of our system submitted to a force field $-\text{grad } V$, when the value of the energy E is given : the “stationary” Schrödinger equation is a second order elliptic PDE,

$$(1.19) \quad -\frac{\hbar^2}{2}\Delta\psi + V\psi = E\psi,$$

where E is the energy. As we shall see, this equation is closely related to the stationary Hamilton–Jacobi equation (1.10). The corresponding evolution equation reads

$$(1.20) \quad i\hbar\frac{\partial\phi}{\partial t} = \left(-\frac{\hbar^2}{2}\Delta + V\right)\phi.$$

These two forms of the equation are related by a time/energy Fourier transform $\phi(t) = \int e^{-iEt/\hbar}\psi_E dE$, which recalls the relation (1.8).

According to Schrödinger’s theory, the energy spectrum can be computed by finding the values of E for which equation (1.19) admits solutions which are “single-valued, finite, and continuous throughout configuration space”.

Schrödinger, motivated by the works of De Broglie, gives an interpretation of ψ as a “wave function”. “*The true mechanical process is realised or represented in a fitting way by the wave processes in q -space, and not by the motion of image*”

¹The analogy with the theory of classical hamiltonian systems can be pushed further. In fact, equation (1.16) is a linear hamiltonian flow, in an infinite dimensional space. Such systems are completely integrable, due to the fact that a unitary transformation diagonalizing \mathbf{H} always exists. To be more explicit, let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space, seen as a real vector space endowed with the symplectic form $\omega(\phi, \psi) = \Im m\langle \phi, \psi \rangle$. If we use an orthonormal basis (e_n) to define coordinates, $\phi = \sum_n (x_n + i\xi_n)e_n$, then (x_n, ξ_n) are Darboux coordinates, meaning that $\omega = \sum_n dx_n \wedge d\xi_n$.

Let \mathbf{H} be a self-adjoint operator; it can be used to define a quadratic hamiltonian $H(\psi) = \frac{1}{2}\langle \psi, \mathbf{H}\psi \rangle$. If we consider quadratic observables, $f(\psi) = \frac{1}{2}\langle \psi, \mathbf{f}\psi \rangle$, then the Poisson bracket defined by ω correspond to the usual commutator bracket,

$$\{f, g\}(\psi) = \frac{1}{2}\langle \psi, i[\mathbf{f}, \mathbf{g}]\psi \rangle.$$

The Hamilton equations defined by H read $\frac{d\psi}{dt} = -i\mathbf{H}\psi$. Finally, linear transformations preserving ω are of the form $\psi \mapsto \mathbf{S}\psi$ where \mathbf{S} is unitary.

Thus, finding a unitary \mathbf{S} such that $\mathbf{S}^{-1}\mathbf{H}\mathbf{S}$ is diagonal amounts to finding a linear canonical transformation $\psi \mapsto \mathbf{S}\psi$ which transforms the hamiltonian H into $H(\mathbf{S}\psi) = \frac{1}{2}\sum(2\pi\nu_n)^2(x_n^2 + \xi_n^2)$. This means that we can integrate the equation of motion by decomposing it into a superposition of infinitely many independent harmonic oscillators.

points in this space. The study of the motion of image points, which is the object of classical mechanics, is only an approximate treatment, and has, as such, just as much justification as geometrical or “ray” optics has, compared with the true optical process”. This approximation is only justified when the dimensions of the system are very large compared to the wave length : “we inevitably became involved in irremovable contradictions if we tried, as was very natural, to maintain also the idea of paths of systems in these processes; just as we find the tracing of the course of a light ray to be meaningless, in the neighbourhood of a diffraction phenomenon”.

It is particularly interesting for us to note that Schrödinger derived the form of his equation by a heuristic argument, based on the desired asymptotic behaviour of the solutions when $\hbar \rightarrow 0$:

Assume that our mechanical phenomenon is described by a wave function ψ , and assume that this wave has the particular form : $\psi(x, 0) = \exp\left(i\frac{A(x,0)}{\hbar} + C\right)$ at $t = 0$. Assume also that for $t > 0$ the wave ψ looks like

$$(1.21) \quad \psi(x, t) \sim \exp\left(i\frac{A(x, t)}{\hbar} + C\right) + \text{small error.}$$

To find the form of the equation satisfied by ψ , Schrödinger postulates that the phase A must approximately satisfy the Hamilton-Jacobi equation (1.7), when the wave length is very small (semiclassical approximation). In other words, we must (almost) see the wave move according to the classical motion. Thus, the point is to find an equation, looking like a wave equation, and such that (1.21) is an approximate solution if $\frac{\partial A}{\partial t} + H(x, d_x A) = 0$ (equation (1.7)) and $\hbar \rightarrow 0$.

To find such an equation, Schrödinger actually works with the stationary formulation : this means that $A(x, t)$ is of the form $A(x, t) = -Et + S(x)$ where S solves $H(x, d_x S) = E$ (equation (1.10)). If S satisfies (1.10), then the local speed of propagation of ψ is

$$u(x) = \frac{-\frac{\partial A}{\partial t}}{|\nabla A|} = \frac{E}{\sqrt{2(E - V(x))}}$$

and the wave length is $\lambda(x) = \frac{\hbar}{\sqrt{2(E - V(x))}}$. This encourages Schrödinger to propose the equation

$$\frac{\partial^2 \psi}{\partial t^2} = u^2 \Delta \psi.$$

From the expression of u , and since the formula (1.21) is supposed to give an approximate solution when $\lambda \rightarrow 0$, we find $-\frac{\hbar^2}{2}\Delta\psi + V\psi = E\psi$.

Let us now consider the propagation of an arbitrary wave ψ , and let us try to put Schrödinger’s discussion into mathematical words. At time $t = 0$, any initial state ψ can be written as

$$(1.22) \quad \psi(x) \sim \int a(x, \theta) \exp\left(\frac{i}{\hbar}A(x; \theta)\right) d\theta,$$

where θ varies in an open set of \mathbb{R}^d , $\exp\left(\frac{i}{\hbar}A(x; \theta)\right)$ is a generating family parametrized by θ , and a is a distribution. In \mathbb{R}^d , we can for instance take the plane waves $\exp\left(\frac{i}{\hbar}A(x; \theta)\right) = \exp\left(\frac{i}{\hbar}(x, \theta)\right)$, and the decomposition (1.22) is the Fourier decomposition. By linearity of the Schrödinger equation, and by the approximate form

of the solutions (1.21), after time t the wave looks like

$$(1.23) \quad \psi(t, x) \sim \int a(x, \theta) \exp\left(\frac{i}{\hbar} A(t, x; \theta)\right) d\theta,$$

where $A(t, x; \theta)$ is the solution of (1.7) with initial condition $A(x; \theta)$. If the oscillations are very rapid (λ small) we expect all these waves to interfere destructively, except at those points x where the phase has a stationary point,

$$\partial_\theta A(t, x, \theta_0) = 0$$

(for some θ_0). At such a point, we see essentially the wave $\exp\left(\frac{i}{\hbar} A(t, x; \theta_0)\right)$, with the frequency vector $\xi = \partial_x A(t, x, \theta_0)$. Thus, the wave front at time t can be represented by the subset of the cotangent space

$$(1.24) \quad \mathcal{L}(t) = \{(x, \xi), \text{ there exists } \theta_0, \partial_\theta A(t, x, \theta_0) = 0, \xi = \partial_x A(t, x, \theta_0)\}.$$

Assuming each $A(\cdot, \cdot, \theta)$ satisfies the Hamilton–Jacobi equation, one can check that $\mathcal{L}(t)$ is precisely the image of $\mathcal{L}(0)$ under the hamiltonian flow (1.4) at time t (see Exercise 12.9). In other words, the wave front is propagated according to the classical hamiltonian flow.

“The point of phase agreement for certain infinitesimal manifolds of wave systems, containing n parameters, moves according to the same laws as the image point of the mechanical system” [Schr26-II].

Recall that this is an approximation, valid when the wave length λ is very small; for mathematicians, this is the same as letting \hbar tend to 0, and this is called the semiclassical limit. Schrödinger’s heuristic discussion already contain the seeds of semiclassical analysis. Classical mechanics is obtained as a limiting case of wave mechanics by a phenomenon of constructive or destructive interferences.

Schrödinger writes : *“I consider it a very difficult task to give an exact proof that the superposition of these wave systems really produces a noticeable disturbance in only a relatively small region surrounding the point of phase agreement, and that everywhere else they practically destroy one another through interference”* [Schr26-II]. As we shall see in Section 4.2, this problem can in fact be handled by the stationary phase method, if we impose strong smoothness conditions on the distribution a .

2. Weyl quantization.

In [Schr26-III], Schrödinger realizes, in the case of $X = \mathbb{R}^d$, that his “wave mechanics” is equivalent to the “quantum mechanics” introduced by Born, Heisenberg and Jordan. The equivalence comes from the existence of an explicit *quantization procedure*, that is, a way to associate, to every function on the classical phase space $T^*X = \mathbb{R}^d \times \mathbb{R}^d$, an operator on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$, so that the commutation rules (1.13) hold. Schrödinger’s remark is that we can take the operators $\mathbf{q}_k =$ (multiplication by q_k) to the coordinate function q_k , and the operator $\mathbf{p}_k = \frac{\hbar}{i} \frac{\partial}{\partial q_k}$ to the function p_k .

One must then decide of a convention to define the operator $\mathbf{a}(\mathbf{q}, \mathbf{p})$ associated to an arbitrary function $a(q, p)$ of (q, p) . For instance, the function $p_k q_k$ could be represented by the operator $\mathbf{p}_k \mathbf{q}_k$ or by $\mathbf{q}_k \mathbf{p}_k$. Schrödinger leaves the issue open for general a , but recommends to quantize a hamiltonian of the form

$$H(q, p) = \frac{\|p\|^2}{2} + V(q),$$

where $\|\cdot\|$ is a riemannian metric, by the operator $\mathbf{H} = -\frac{\hbar^2}{2}\Delta + V$, where Δ is the laplacian associated to the metric. In this representation, Heisenberg's equation (1.17), requiring to diagonalize the operator \mathbf{H} , can be written $-\frac{\hbar^2}{2}\Delta\psi + V\psi = E\psi$, which is exactly Schrödinger's equation (1.19). Thus, the two theories will give the same values of the energy spectrum. Schrödinger suggests, however, that two theories can be mathematically equivalent without being physically equivalent.

Weyl quantization. Hermann Weyl [Weyl27] gave a quantization rule which defines $\mathbf{a}(\mathbf{q}, \mathbf{p})$ for any $a(q, p)$. He first proposed to quantize the observable $U_{p_0, q_0}(q, p) = e^{\frac{i}{\hbar}(p_0 \cdot q - q_0 \cdot p)}$ (with $q_0, p_0 \in \mathbb{R}^d$) by the operator $\mathbf{U}_{p_0, q_0}(\mathbf{q}, \mathbf{p}) = e^{\frac{i}{\hbar}(p_0 \cdot \mathbf{q} - q_0 \cdot \mathbf{p})}$ (where (\mathbf{q}, \mathbf{p}) are defined by Schrödinger's prescriptions). Then, the Fourier transform allows to quantize any observable : if a is decomposed into

$$a(q, p) = \int e^{\frac{i}{\hbar}(p_0 \cdot q - q_0 \cdot p)} \check{a}_{\hbar}(q_0, p_0) \frac{dq_0 dp_0}{(2\pi\hbar)^d},$$

then the Weyl quantization is

$$\mathbf{a}(\mathbf{q}, \mathbf{p}) = \int e^{\frac{i}{\hbar}(p_0 \cdot \mathbf{q} - q_0 \cdot \mathbf{p})} \check{a}_{\hbar}(q_0, p_0) \frac{dq_0 dp_0}{(2\pi\hbar)^d} =: \text{Op}_{\hbar}^W(a).$$

We used the ‘‘symplectic’’ Fourier transform,

$$\check{a}_{\hbar}(q, p) = \int e^{\frac{-i}{\hbar}(p_0 \cdot q - q_0 \cdot p)} a(q, p) \frac{dq dp}{(2\pi\hbar)^d}.$$

One can also check that the following expression holds [Foll],

$$\text{Op}_{\hbar}^W(a)f(x) = \frac{1}{(2\pi\hbar)^d} \int a\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{\hbar}\xi \cdot (x-y)} f(y) dy d\xi.$$

The Schrödinger representation. For $p = 0$, we have $U_{0, q}u(x) = u(x - q)$, so that $U_{0, q}$ corresponds to the translation of vector q in the position variable. Similarly, $U_{p, 0}$ translates the Fourier transform $\mathcal{F}_{\hbar}u$ (defined in Section 4.1) by the vector p , and $U_{p, 0}$ is interpreted as the translation of vector p in the momentum variable. To interpolate between these two cases, one usually says that $U_{p, q}$ is the operator corresponding to ‘‘translation of vector (q, p) in the phase space $\mathbb{R}^d \times \mathbb{R}^d$ ’’. The caveat is that $U_{0, q}$ and $U_{p, 0}$ do not commute, in fact the operators $\mathbf{U}_{p, q}$ obey the following composition rule,

$$(2.1) \quad \mathbf{U}_{p, q} \cdot \mathbf{U}_{p', q'} = \mathbf{U}_{p+p', q+q'} e^{\frac{i}{\hbar} \frac{1}{2}(pq' - q'p)}.$$

Consider the Heisenberg group \mathbf{H}_d with d degrees of freedom, defined as \mathbb{R}^{2d+1} endowed with the composition rule

$$(p, q, t) \cdot (p', q', t') = \left(p + p', q + q', t + t' + \frac{1}{2}(pq' - qp') \right), \quad (p, p', q, q' \in \mathbb{R}^d, t, t' \in \mathbb{R}).$$

Its Lie algebra is generated by $P_1, \dots, P_d, Q_1, \dots, Q_d, T$ with the relations

$$[P_j, P_k] = [Q_j, Q_k] = [P_j, T] = [Q_j, T] = 0; \quad [P_j, Q_k] = \delta_{jk}T.$$

The identity (2.1) can be reinterpreted by saying that

$$\rho_{\hbar}(p, q, t) = e^{\frac{it}{\hbar}} \mathbf{U}_{p, q}$$

defines a unitary representation from \mathbf{H}_d into $L^2(\mathbb{R}^d)$, called the Schrödinger representation of parameter \hbar . The associated infinitesimal representation is $P_k \mapsto \frac{\partial}{\partial q_k} = \frac{i}{\hbar} \mathbf{p}_k$, $Q_k \mapsto \frac{i}{\hbar} \mathbf{q}_k$, $T \mapsto \frac{i}{\hbar} I$.

THEOREM 2.1. (*Stone–von Neumann 1930 [St30, vN31], see [Foll]*) *Every irreducible unitary representation of \mathbf{H}_d is equivalent to exactly one of the following representations :*

- (a) ρ_\hbar ($\hbar \in \mathbb{R} \setminus \{0\}$) acting on $L^2(\mathbb{R}^d)$;
- (b) $\sigma_{ab}(p, q, t) = e^{2\pi i(ap+bq)}$, ($a, b \in \mathbb{R}^d$) acting on \mathbb{C} .

3. Born’s probabilistic interpretation of the Schrödinger equation.

Born discovered that the square modulus $|\psi|^2$ of the wave functions (satisfying the Schrödinger equation) could be used to predict the probability of where the “particle” would be found. More precisely, if ψ is normalized so that $\int |\psi(t, x)|^2 dx = 1$, then $|\psi(t, x)|^2$ gives the probability density of finding, in an experiment, the particle at x (at time t). This was the beginning of a tense philosophical (or physical) debate on the correct interpretation of the wave/particle duality.

“Let me say at the outset, that in this discourse, I am opposing not a few special statements of quantum physics held today (1950s), I am opposing as it were the whole of it, I am opposing its basic views that have been shaped 25 years ago, when Max Born put forward his probability interpretation, which was accepted by almost everybody. (E. Schrödinger, *The Interpretation of Quantum Physics*. Ox Bow Press, Woodbridge, CN, 1995).

“I don’t like it, and I’m sorry I ever had anything to do with it” (Erwin Schrödinger talking about quantum physics).

4. The semiclassical limit.

Let us now turn to much more recent mathematical preoccupations. The main subject of these notes is to try to describe the localization of the probability density $|\psi(x)|^2 dx$, for a Schrödinger eigenfunction, in the semiclassical limit $\hbar \rightarrow 0$. The quantum/classical correspondence tells us, intuitively, that the eigenfunctions, which are stationary solutions of the Schrödinger equation, should look like invariant probability measures of the classical hamiltonian flow. In this section we give a quick survey (without proofs) of the mathematical tools used to study this question.

It is not really satisfactory, and usually practically impossible, to study the density $|\psi(x)|^2 dx$ itself. This is because, when taking the modulus of ψ , we lose some precious information on the frequency vector of ψ (related to its phase, or complex argument). We need to study simultaneously the Fourier transform of ψ . Of course, rigorously speaking, one cannot study at the same time the *local* properties of a function and of its Fourier transform around some point $(x, \xi) \in T^*X$. This is expressed by Heisenberg’s uncertainty principle, saying that one cannot localize a function around the point x without perturbing a lot the momentum (and vice-versa). Microlocal analysis² is a collection of mathematical techniques allowing to study the joint localization of a function and its Fourier transform; because of the uncertainty principle, this can only be meaningful asymptotically, in the limit $\hbar \rightarrow 0$.

²More precisely, we will present here its \hbar -dependent version, also called semiclassical analysis, or “microlocal analysis with a small parameter”.

4.1. Fourier transform. The Fourier transform

$$\mathcal{F}_\hbar(u)(\xi) = \hat{u}_\hbar(\xi) = (2\pi\hbar)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{i}{\hbar}\xi \cdot x} u(x) dx$$

allows to analyze a signal u in terms of its frequencies, at the scale \hbar . For $u \in C_o^\infty$, we have the decomposition

$$u(x) = (2\pi\hbar)^{-d/2} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}\xi \cdot x} \hat{u}_\hbar(\xi) d\xi .$$

4.2. The stationary phase method. This is a result describing the asymptotic behaviour, as $\hbar \rightarrow 0$, of an integral of the form :

$$I(\hbar) = \int_{\mathbb{R}^D} e^{\frac{i}{\hbar}S(x)} a(x) dx$$

where $a \in C_o^\infty(\mathbb{R}^D)$ and $S \in C^\infty(\mathbb{R}^D, \mathbb{R})$.

The interferences between the different terms $e^{\frac{i}{\hbar}S(x)}$ are destructive, except at the stationary points of the phase S . The precise statement is :

- If S has no critical/stationary point in the support of a , then $I(\hbar) = O(\hbar^\infty)$ (this notation means that, for all $N > 0$, we have $I(\hbar) = O_N(\hbar^N)$).
- If S has a unique critical point x_0 , supposed to be non-degenerate, in the support of a , then there is an asymptotic development in powers of \hbar , up to any order,

$$(4.1) \quad I(\hbar) \sim (2\pi\hbar)^{D/2} \frac{e^{i\sigma\pi/4}}{|\det S''(x_0)|^{\frac{1}{2}}} e^{iS(x_0)/\hbar} \left(\sum_{j=0}^{\infty} \hbar^j a_j \right)$$

where $S''(x_0)$ is the hessian matrix of S at x_0 , $\sigma = n_+ - n_-$ is the index of $S''(x_0)$ (the difference between the number of positive and negative eigenvalues), and $a_0 = a(x_0)$. More generally, a_j can be expressed in terms of the derivatives of a up to order $2j$, at the point x_0 .

For technical developments, one usually needs to work with functions a which are not necessarily compactly supported, but have a well behaved behaviour at infinity, and can be allowed to depend on \hbar . The choice of a class of “symbols” is a technical issue, which depends on the aims, but also on the tastes of the authors. For the sake of completeness we give an example of a convenient class of symbols. However, it is not required to understand all the technical issues to read the next chapters.

Symbol spaces. Let $D, d > 0$ be two integers, and let U be an open subset of \mathbb{R}^D . Let us define *symbols of order m* (independent of \hbar) :

$$\Sigma^m(U \times \mathbb{R}^d) := \{a \in C^\infty(U \times \mathbb{R}^d; \mathbb{C}) /$$

for every compact $K \subset U$, there exists C such that

$$|D_z^\alpha D_\xi^\beta a(z, \xi)| \leq C(1 + |\xi|)^{m-|\beta|} \text{ for all } (z, \xi) \in K \times \mathbb{R}^d\}.$$

For instance, this class contains functions which are homogeneous in a neighbourhood of infinity. We denote $\Sigma^{-\infty} = \bigcap_{m \in \mathbb{Z}} \Sigma^m$ — this class contains the smooth compactly supported functions $C_o^\infty(U \times \mathbb{R}^d)$.

We also define *semiclassical symbols of order m and degree l* — thus called because they depend on a parameter \hbar :

$$(4.2) \quad \Sigma^{m,l} = \{a_{\hbar}(z, \xi) = \hbar^l \sum_{j=0}^{\infty} \hbar^j a_j(z, \xi), a_j \in \Sigma^{m-j}\}$$

This means that $a_{\hbar}(x, \xi)$ as an asymptotic development in powers of \hbar ; in the sense that

$$a - \hbar^l \sum_{j=0}^{N-1} \hbar^j a_j \in \hbar^{l+N} \Sigma^{m-N}$$

for all N , uniformly in \hbar . In this context, we denote $\Sigma^{-\infty, +\infty} = \bigcap_{m \geq 0} \Sigma^{-m, m}$.

In these definitions, $U \times \mathbb{R}^d$ can be replaced by a fiber bundle of rank d on a D -dimensional manifold.

Fresnel integrals, generalized stationary phase method. We can now describe the asymptotic behaviour, as $\hbar \rightarrow 0$, of the integral :

$$I_{\hbar}^S(a) = \int_{U \times \mathbb{R}^d} e^{\frac{i}{\hbar} S(z, \xi)} a(z, \xi) dz d\xi$$

where S is smooth, homogeneous of degree $n > 0$ near infinity with respect to ξ , and without critical points outside a compact subset of $U \times \mathbb{R}^d$. The integral $I_{\hbar}^S(a)$ is defined for $a \in \Sigma_o^{m,l}$, by continuous extension of the case $a \in C_o^{\infty}$. Here, the index $*_o$ in $\Sigma_o^{m,l}$ means that a is compactly supported with respect to z , with support independent of \hbar . Such non absolutely convergent oscillatory integrals are sometimes called Fresnel integrals, a well-known example is $\int_{\mathbb{R}^d} e^{\frac{i}{2\hbar} \|\xi\|^2} d\xi = (2\pi\hbar)^{d/2} e^{id\pi/4}$.

The previous asymptotic behaviour still holds in this setting.

4.3. Pseudodifferential operators. As we have seen, a quantization procedure is a way to associate an operator to a classical observable $a(p, q)$. Recall Schrödinger's prescriptions, $\mathbf{q}_k = (\text{multiplication by } q_k)$, and $\mathbf{p}_k = \frac{\hbar}{i} \frac{\partial}{\partial q_k}$, compatible with Heisenberg's commutation relations (1.13). To extend this definition to an arbitrary function of (p, q) , we meet an obvious problem : to quantize the function $p_k q_k^2$, for instance, we could propose any of the operators $\mathbf{p}_k \mathbf{q}_k^2$, $\mathbf{q}_k^2 \mathbf{p}_k$, or $\mathbf{q}_k \mathbf{p}_k \mathbf{q}_k$. There are many quantization procedures. We already met the Weyl quantization, which combines several remarkable features, like the fact that it associates a symmetric operator to a real symbol. Later on, we shall also define the anti-Wick positive quantization, which associates a nonnegative operator to a nonnegative symbol.

The theory of pseudodifferential operators with small parameter allows to describe the passage from the quantum theory to the classical theory when $\hbar \rightarrow 0$. This is also called \hbar -dependent microlocal analysis, microlocal analysis with small parameter, or semiclassical analysis. Pseudodifferential operators were first developed by Hörmander [**Ho**, **Ho79**] for the study of the regularizing properties of partial differential equations (without any small parameter). Pseudodifferential operators with small parameter, manipulated by Maslov [**Masl65**] in the framework of semiclassical analysis, developed by Voros in mathematical physics [**Vor**, **Vor78**], were perfected by Sjöstrand, Robert, Helffer, [**DimSjo**, **Rob**]... I advise to read

[**Helffer1**] for a history of the first years of this theory in the seventies and an exhaustive bibliography; see also [**Helffer2**] for a survey of applications.

Symbol spaces depend on authors, and can be extremely sophisticated. Hörmander's definition has no \hbar and involves symbols which are homogeneous near infinity, allowing to describe the regularizing properties of operators. The semiclassical symbol classes of [**DimSjo**] are rather aimed at describing the behaviour of operators when $\hbar \rightarrow 0$, say in L^2 norm. The symbols we use here combine both approaches (after an idea of Y. Colin de Verdière) : taking $\hbar = 1$ we would find (one of) Hörmander's symbol spaces.

Pseudodifferential operators. Let Ω be an open subset of \mathbb{R}^d , and let $a = a_{\hbar}(x, y, \xi) \in \Sigma_o^{m,l}(\Omega \times \Omega \times \mathbb{R}^d)$. Here the index $_o$ means that for every compact $K \subset \Omega$, there exists a compact K' such that $a(x, y, \xi) = 0$ for $x \in K, y \notin K', \xi \in \mathbb{R}^d$. Let u be a smooth function. We define :

$$\text{OP}_{\hbar}(a)u(x) = (2\pi\hbar)^{-d} \int e^{\frac{i}{\hbar}\xi \cdot (x-y)} a(x, y, \xi) u(y) \, dy d\xi,$$

the integral being well defined as a Fresnel integral. We denote $\Psi\text{DO}^{m,l}(\Omega)$ these operators, called (proper) pseudodifferential operators of *degree* l and *order* m , on Ω . The intersection $\Psi\text{DO}^{-\infty,\infty}$ of all the $\Psi\text{DO}^{m,l}(\Omega)$ are the negligible operators : they are the operators with a smooth kernel K_{\hbar} , and such that all derivatives of K_{\hbar} are $O(\hbar^{\infty})$ uniformly on compact sets.³

The class of pseudodifferential operators includes differential operators (corresponding to a symbol which is polynomial in ξ), but has the advantage of being stable under inversion, or more general smooth functional calculus.

Note that several symbols $a(x, y, \xi)$ can give the same operator $\text{OP}_{\hbar}(a)$. As a simple example, we note that $a(x, y, \xi) = V(x)$ and $a(x, y, \xi) = V(y)$ both give the operator of multiplication by V . It is often convenient to choose special representatives :

Weyl quantization. Left and right quantizations.

Here $\Omega = \mathbb{R}^d$.

We already met the Weyl quantization⁴, $\text{Op}_{\hbar}^W(a) = \text{OP}_{\hbar}(a(\frac{x+y}{2}, \xi))$. If $a \in \Sigma_o^{m,l}(\mathbb{R}^d \times \mathbb{R}^d)$ is compactly supported with respect to the first variable, then $\text{Op}_{\hbar}^W(a) \in \Psi\text{DO}^{m,l}$.

The inverse of Weyl quantization is explicit, given by the Wigner transform : if $K(x, y)$ is the kernel of the operator A , we let :

$$W_A(x, \xi) = (2\pi\hbar)^{-d/2} \int e^{\frac{iv\xi}{\hbar}} K\left(x + \frac{v}{2}, x - \frac{v}{2}\right) \, dv.$$

Then $A = \text{Op}_{\hbar}^W(W_A)$. In particular, the Weyl symbol of an operator is unique.

Two other common quantizations are, the *left quantization*, defined by $\text{Op}_{\hbar}^L(a) = \text{OP}_{\hbar}(a(x, \xi))$ where $a \in \Sigma_o^{m,l}(\mathbb{R}^d \times \mathbb{R}^d)$ and the *right quantization*, $\text{Op}_{\hbar}^R(a) = \text{OP}_{\hbar}(a(y, \xi))$. The left and right symbols are both uniquely determined by the operator (there are explicit inversion formulas, too).

³Usually, in this theory, all the assertions about operators hold *modulo negligible operators*. Likewise, the assertions about functions hold *modulo negligible functions*. These are the smooth functions $u_{\hbar}(x)$ such that all derivatives are $O(\hbar^{\infty})$ uniformly on compact sets of X .

⁴I try to stick to the notation OP for symbols $a \in \Sigma(\Omega \times \Omega \times \mathbb{R}^d)$, and Op for symbols $a \in \Sigma(\Omega \times \mathbb{R}^d)$.

EXAMPLE 4.1. To quantize the observable $a(p, q) = pq^2$, the left quantization chooses $\mathbf{q}^2\mathbf{p}$, the right quantization chooses $\mathbf{p}\mathbf{q}^2$, and the Weyl quantization forms the combination $\frac{1}{4}(\mathbf{p}\mathbf{q}^2 + 2\mathbf{q}\mathbf{p}\mathbf{q} + \mathbf{q}^2\mathbf{p}) = \frac{1}{2}(\mathbf{p}\mathbf{q}^2 + \mathbf{q}^2\mathbf{p})$.

EXERCISE 4.2. On $\mathbb{R}^d \times \mathbb{R}^d$, consider a lagrangian $L(x, v)$ defined by a riemannian metric,

$$L(x, v) = \frac{1}{2}g_x(v, v) = \frac{1}{2} \sum_{i,j=1}^d g_{ij}(x)v_i v_j.$$

Check that the corresponding hamiltonian is

$$H(x, \xi) = \frac{1}{2}g^x(\xi, \xi) = \frac{1}{2} \sum_{i,j=1}^d g^{ij}(x)\xi_i \xi_j,$$

where $(g^{ij}(x))$ is the inverse of the matrix $(g_{ij}(x))$.

Write the explicit expression of the laplacian Δ associated to the metric g .

Choose a quantization procedure $\text{Op}_{\hbar} = \text{Op}_{\hbar}^W, \text{Op}_{\hbar}^L$ or Op_{\hbar}^R .

Show that

$$\text{Op}_{\hbar}(H) = -\frac{1}{2}\hbar^2 \Delta + \hbar \sum_{j=1}^d b_j(x) \frac{\hbar}{i} \frac{\partial}{\partial x_j} + \hbar^2 c(x)$$

for certain functions b_j, c , the expression of which depends on the choice of Op_{\hbar} .

Show that there are functions \tilde{b}_j, \tilde{c} such that

$$(4.3) \quad -\frac{1}{2}\hbar^2 \Delta = \text{Op}_{\hbar} \left(H(x, \xi) + \hbar \sum_j \tilde{b}_j(x) \xi_j + \hbar^2 \tilde{c}(x) \right).$$

Compare with (4.2) to find the order and the degree of $-\hbar^2 \Delta$ (of course, differential operators are pseudodifferential operators !).

The expression of b_j, c, d depends on the choice of Op_{\hbar} . The first term $H(x, \xi)$ does not, it is called the principal symbol of $-\frac{1}{2}\hbar^2 \Delta$.

Principal symbol. Let $a_{\hbar} \in \Sigma_o^{m,0}(\Omega \times \Omega \times \mathbb{R}^d)$. Applying the operator $A_{\hbar} = \text{Op}_{\hbar}(a_{\hbar}) \in \Psi\text{DO}^{m,0}$ to a function of the form $u(x)e^{iS(x)/\hbar}$, where u and S are smooth⁵, the method of stationary phase gives the following asymptotics :

$$A_{\hbar} \left(u(x)e^{iS(x)/\hbar} \right) = a_0(x, x, S'(x)) u(x)e^{iS(x)/\hbar} + O(\hbar).$$

This shows that the function $a_0(x, x, \xi)$ on $\mathbb{R}^d \times \mathbb{R}^d = T^*\mathbb{R}^d$ does not depend on the choice of the symbol $a_{\hbar}(x, y, \xi)$, but only on the operator A_{\hbar} . It is called the principal symbol of A_{\hbar} , denoted $\sigma^0(A_{\hbar})$. If $\sigma^0(A_{\hbar}) = 0$, then A_{\hbar} actually belongs to $\Psi\text{DO}^{m-1,1}$ (and conversely).

REMARK 4.3. For $a \in \Sigma_o^{m,0}$, we note that $\text{Op}_{\hbar}^W(a), \text{Op}_{\hbar}^L(a), \text{Op}_{\hbar}^R(a)$ all have the same principal symbol $a_0(x, \xi)$. In other words,

$$\text{Op}_{\hbar}^W(a) - \text{Op}_{\hbar}^{R/L}(a) \in \Psi\text{DO}^{m-1,+1}.$$

⁵Such a function is called a *WKB state*, see Section 14

Product. If $A_{\hbar} \in \Psi\text{DO}^{m_1,0}$ and $B_{\hbar} \in \Psi\text{DO}^{m_2,0}$, then the product $A_{\hbar}B_{\hbar}$ belongs to $\Psi\text{DO}_o^{m_1+m_2,0}$, and the principal symbols are multiplied : $\sigma^0(A_{\hbar}B_{\hbar}) = \sigma^0(A_{\hbar})\sigma^0(B_{\hbar})$. This is proved by the stationary phase method.

An equivalent statement : if $a \in \Sigma_o^{m_1,0}(\mathbb{R}^d \times \mathbb{R}^d)$ and $b \in \Sigma_o^{m_2,0}(\mathbb{R}^d \times \mathbb{R}^d)$, then $\text{Op}_{\hbar}(a)\text{Op}_{\hbar}(b) \in \Psi\text{DO}^{m_1+m_2,0}(\mathbb{R}^d)$, and

$$(4.4) \quad \text{Op}_{\hbar}(ab) - \text{Op}_{\hbar}(a)\text{Op}_{\hbar}(b) \in \Psi\text{DO}^{m_1+m_2-1,1}(\mathbb{R}^d).$$

Thanks to Remark 4.3, this statement does not depend on the choice of Op^W , Op^L or Op^R .

Brackets. If $A_{\hbar} \in \Psi\text{DO}^{m_1,0}$ and $B_{\hbar} \in \Psi\text{DO}^{m_2,0}$, then the bracket $[A_{\hbar}, B_{\hbar}]$ belongs to $\Psi\text{DO}^{m_1+m_2-1,1}$, and

$$\sigma^0(\hbar^{-1}[A_{\hbar}, B_{\hbar}]) = \frac{1}{i} \{ \sigma^0(A_{\hbar}), \sigma^0(B_{\hbar}) \};$$

where $\{.,.\}$ is the Poisson bracket.

Equivalently : if $a \in \Sigma_o^{m_1,0}(\mathbb{R}^d \times \mathbb{R}^d)$ and $b \in \Sigma_o^{m_2,0}(\mathbb{R}^d \times \mathbb{R}^d)$, we have

$$(4.5) \quad [\text{Op}_{\hbar}(a), \text{Op}_{\hbar}(b)] - \text{Op}_{\hbar}\left(\frac{\hbar}{i}\{a, b\}\right) \in \Psi\text{DO}^{m_1+m_2-2,2}$$

and again this statement does not depend on the choice of Op^W , Op^L or Op^R .

REMARK 4.4. There is also an integrated version of this result, called the Egorov Theorem. We will use it in the following form : assume the pseudodifferential operator A_{\hbar} is self-adjoint. Define the Schrödinger flow $(U_{\hbar}^t) = (\exp -\frac{it}{\hbar}A_{\hbar})$.

Let $a \in C_c^\infty(T^*\mathbb{R}^d)$. Then, for any given t in \mathbb{R} ,

$$(4.6) \quad U_{\hbar}^{-t}\text{Op}_{\hbar}(a)U_{\hbar}^t - \text{Op}_{\hbar}(a \circ \phi_{\sigma^0(A_{\hbar})}^t) \in \Psi\text{DO}^{-\infty,1}.$$

Here $\phi_{\sigma^0(A_{\hbar})}^t$ is the hamiltonian flow defined by the hamiltonian $\sigma^0(A_{\hbar})$. The estimate is usually not uniform in t , so that one cannot invert the limits $\hbar \rightarrow 0$ and $t \rightarrow \infty$. This is a notorious source of problems when one tries to use the semiclassical approximation to understand the large time behaviour of solutions of the Schrödinger equation.

Pseudodifferential operators on a compact manifold. Let X be a compact C^∞ manifold of dimension d . Let (Ω_i, φ_i) be a finite atlas of X ($X = \cup \Omega_i$, $\varphi_i : \Omega_i \rightarrow \mathbb{R}^d$). We use the φ_i to define local coordinates $\Phi_i : T^*\Omega_i \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ on T^*X as follows :

$$\Phi_i(x, p) = (\varphi_i(x), (d\varphi_i(x))^{-1}p).$$

These are symplectic (Darboux) coordinates on T^*X , i.e. the canonical symplectic form reads $\omega = \sum dx_j \wedge dp_j$ in these coordinates. Introduce a finite partition of unity $\chi_i \in C_o^\infty(\Omega_i)$ such that $\sum \chi_j^2 = 1$. For $a \in \Sigma_o^{m,l}(T^*X)$, we let :

$$(4.7) \quad \text{Op}_{\hbar}(a)u = \sum_i \chi_i [\text{Op}_{\hbar}(a \circ \Phi_i^{-1})(\chi_i u \circ \varphi_i^{-1})] \circ \varphi_i.$$

The map $a \mapsto \text{Op}_{\hbar}(a)$ thus defined depends on the partition of unity and on the local coordinates; but its range does not, modulo negligible operators. The algebra $\Psi\text{DO}^{m,l}(X)$ of pseudodifferential operators on X (modulo negligible operators) is thus well defined.

All the properties stated above can be extended to this case.

Continuity. Trace class and Hilbert-Schmidt operators. The domain of an operator in $\Psi\text{DO}^{m,0}(\Omega)$ depends a lot on m , that is, on the growth of the symbol when $\xi \rightarrow \infty$. When m decreases, the regularizing properties of the operator are improved. Without proof, let us mention that an operator in $\Psi\text{DO}^{0,0}(\Omega)$ is bounded from $L^2(\Omega)$ to $L^2_{loc}(\Omega)$, uniformly with respect to \hbar . On a compact manifold X , an operator in $\Psi\text{DO}^{m,0}$ is

- Hilbert-Schmidt if $m < -d/2$
- trace class if $m < -d$

In this latter case, the trace of $\text{Op}(a)$ is given by the convergent integral,

$$(4.8) \quad \text{Tr Op}(a) = (2\pi\hbar)^{-d} \int_{T^*X} a(x, \xi) dx d\xi .$$

5. Semiclassical measures, microlocal lifts.

A quantization procedure Op is said to be *nonnegative* if $\text{Op}(a)$ is a nonnegative operator as soon as a is a nonnegative function. The usual quantization procedures do not have this property.

Positive quantization on \mathbb{R}^d .

DEFINITION 5.1. (Coherent states) The coherent state (of size \hbar) centered at (x_0, ξ_0) is defined as the normalized gaussian state

$$e_{x_0, \xi_0}(x) = \frac{1}{(\pi\hbar)^{d/4}} e^{\frac{i}{\hbar} \xi_0 \cdot x} \exp\left(-\frac{\|x - x_0\|^2}{2\hbar}\right)$$

For $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$, we shall denote $\Pi_{(x, \xi)}$ the orthogonal projector onto $\mathbb{C}e_{(x, \xi)}$.

THEOREM 5.2. Let $a \in C^\infty(T^*\mathbb{R}^d)$. The operator defined by

$$\text{Op}^+(a) = (2\pi\hbar)^{-d} \int a(x, \xi) \Pi_{x, \xi} dx d\xi$$

belongs to the class $\Psi\text{DO}^{-\infty, 0}$, it is self-adjoint if a is real valued, and non-negative if a is non-negative. Its principal symbol is $a(x, \xi)$.

We have $\text{Op}^+(1) = I$, which allows to extend the definition of Op^+ to the case when a is constant in a neighbourhood of infinity in T^*X .

This quantization is called the anti-Wick quantization.

To define a positive quantization procedure on a compact manifold X , we choose an atlas of X and a non-negative subordinate partition of unity, $\sum \chi_j^2 = 1$. For $a \in C^\infty(T^*X)$, we let $\text{Op}_X^+(a) = \sum_j \chi_j \text{Op}_{\mathbb{R}^d}^+(a) \chi_j$ — where $\text{Op}_{\mathbb{R}^d}^+(a)$ is defined using local coordinates in the support of χ_j (see (4.7)). We can extend this definition to the case when a is constant in a neighbourhood of infinity in T^*X , by letting $\text{Op}_X^+(1) = I$.

Semiclassical measures. Let X be a compact riemannian manifold; we denote Vol the riemannian volume on X . To a family (u_\hbar) of normalized elements of $L^2(X, \text{Vol})$, we can associate a family of distributions μ_\hbar by the formula $\mu_\hbar(a) = \langle u_\hbar, \text{Op}_\hbar^+(a) u_\hbar \rangle_{L^2(X, \text{Vol})}$. They are in fact probability measures on T^*X . To be able to take weak limits when $\hbar \rightarrow 0$, we see them as probability measures

on the compactification $\overline{T^*X}$ of T^*X obtained by adding a sphere bundle at infinity. We will call the measures μ_{\hbar} the *Husimi measures* associated to the family (u_{\hbar}) . The term *Wigner transform* will be exclusively used in the case $X = \mathbb{R}^d$, for the distributions $a \mapsto \langle u_{\hbar}, \text{Op}_{\hbar}^W(a)u_{\hbar} \rangle$ defined thanks to the Weyl quantization. These distributions are also called *microlocal lifts* of the probability measures $|u_{\hbar}(x)|^2 d\text{Vol}(x)$. This means that their projection down to X is $|u_{\hbar}(x)|^2 d\text{Vol}(x) + O(\hbar)$.

Due to the uncertainty principle, these objects are not really meaningful for fixed $\hbar > 0$. In fact, their definition depends on a certain number of arbitrary choices, coming into play in the definition of Op : local coordinates, partition of unity, choice of the quantization procedure... However, the semiclassical limits of these distributions do not depend on all these arbitrary conventions : if $a \in \Sigma_o^{0,0}(T^*X)$, two definitions of $\text{Op}(a)$ only differ by $O(\hbar)$ in L^2 operator norm.

We shall call any limit point of the sequence (μ_{\hbar}) in the weak topology a *semiclassical measure* associated to the family (u_{\hbar}) .

EXAMPLE 5.3. (Coherent states)

$$u_{\hbar}(x) = e_{x_0, \xi_0}(x) = \frac{1}{(\pi\hbar)^{d/4}} e^{\frac{i}{\hbar}\xi_0 \cdot x} \exp\left(-\frac{\|x - x_0\|^2}{2\hbar}\right)$$

Then there is a unique semiclassical measure, the Dirac mass at (x_0, ξ_0) .

EXAMPLE 5.4. (Lagrangian states/WKB states) Let $u_{\hbar}(x) = b(x)e^{\frac{i}{\hbar}S(x)}$ where b and S are of class C^∞ . In Section 14, we will call such functions *lagrangian states* associated to the lagrangian manifold $\mathcal{L} = \{(x, dS(x))\}$.

There is a unique semiclassical measure associated to (u_{\hbar}) , it is carried by the lagrangian \mathcal{L} and projects to X as the measure $|b(x)|^2 d\text{Vol}(x)$.

EXERCISE 5.5. You have noted that we sometimes omit to indicate the dependence on \hbar in the definition of Op (which should be denoted Op_{\hbar}). The choice of scaling is, nevertheless, very important, and the properties observed vary a lot according to the scaling.

In the previous example, show that the measures defined by

$$\mu_{\hbar, \alpha}(a) = \langle u_{\hbar}, \text{Op}_{\hbar^\alpha}^+(a)u_{\hbar} \rangle$$

concentrate to the 0-section in T^*X if $\alpha > 1$, but concentrate to the sphere bundle at infinity $\overline{T^*X} \setminus T^*X$ if $\alpha < 1$.

When the u_{\hbar} are the eigenfunctions of a hamiltonian operator as in (1.19), one can apply the following theorem :

THEOREM 5.6. *Let P be a self-adjoint pseudodifferential operator, denote p_0 its principal symbol. Let (u_{\hbar}) be a family of tamed⁶ smooth functions, such that $Pu_{\hbar} = O(\hbar^\infty)$ and $\|u_{\hbar}\|_{L^2} = 1$. Let μ_{\hbar} be the Husimi measures associated to (u_{\hbar}) . Then, every weak limit μ_0 of the measures μ_{\hbar} on $\overline{T^*X}$*

- (1) *is a probability measure on $\overline{T^*X}$.*
- (2) *projects on X to a weak limit of the measures $|u_{\hbar}(x)|^2 d\text{Vol}(x)$.*
- (3) *is invariant under the hamiltonian flow of p_0 .*
- (4) *its restriction to T^*X is carried by the energy level $\{p_0 = 0\}$.*
- (5) *If p_0 is elliptic at infinity, then μ_0 is carried by T^*X .*

⁶meaning that, for all $N \in \mathbb{N}$, for any compact K , there exists $k \in \mathbb{N}$ such that the C^N norm of u_{\hbar} on K is $O(\hbar^{-k})$

The first two items have already been explained.

EXERCISE 5.7. Prove the third item by using the relation $\sigma^0(\hbar^{-1}[P, \text{Op}(a)]) = -i\{p_0, a\}$ to show that $\int \{p_0, a\} d\mu_0 = 0$ for any $a \in C_o^\infty(T^*X)$.

Prove the fourth item by using the relation $\sigma^0(\text{Op}(a)P) = a p_0$ to show that $\int a p_0 d\mu_0 = 0$ for any $a \in C_o^\infty(T^*X)$.

We do not give here the precise definition of “elliptic at infinity”. It implies that P is invertible in a neighbourhood of infinity in the class of pseudodifferential operators. More precisely, there exists a smooth a , taking the constant value 1 in a neighbourhood of infinity in T^*X , and a pseudodifferential operator $\text{Op}(b)$ such that

$$\text{Op}(a) = \text{Op}(b)P + R$$

where $R \in \Psi\text{DO}^{-\infty, \infty}$ is a negligible operator. From this fact, the last item follows easily. The ellipticity criterion is satisfied by the Schrödinger operator $\left(-\frac{\hbar^2 \Delta}{2} + V - E\right)$ on a compact manifold X .

Eigenfunctions of the laplacian. Let (X, g) be a compact riemannian manifold, and Δ the laplacian on X associated to the metric. If $(-\hbar^2 \Delta - 1)u_{\hbar} = 0$, and if we denote μ_{\hbar} the corresponding Husimi measures, then every limit point of the family $(\mu_{\hbar})_{\hbar \rightarrow 0}$ is a probability measure μ_0 carried by the unit cotangent bundle S^*X , invariant under the geodesic flow (apply Theorem 5.6 and remember Exercise 4.2). It is a widely open problem to find all the possible limits μ_0 among the invariant measures on S^*X .

In the case of the round sphere or a flat torus, it is easy to construct families of eigenfunctions (u_{\hbar}) for which μ_{\hbar} converges to the uniform measure on any given invariant lagrangian torus. On the flat torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ for instance, the family $(e^{\frac{i}{\hbar} \xi_0 \cdot x})$, where ξ_0 is a unit vector (and of course $\frac{\xi_0}{\hbar} \in 2\pi\mathbb{Z}^d$), has a unique semiclassical measure, the uniform measure on the lagrangian torus $\{(x, \xi_0), x \in \mathbb{T}^d\}$. More generally, for a completely integrable system, one can use WKB methods [Brill26, Kr26, Wtz26, Kell58, Masl65] to build quasimodes, in other words solutions of $\|(-\hbar^2 \Delta - 1)u_{\hbar}\| = O(\hbar^\infty)$, the Husimi measures of which concentrate to any given invariant torus⁷. Historically, the case of completely integrable systems, or perturbations thereof, was the most important, since it is related to the study of small atoms and ions. The “opposite” case of chaotic systems has been studied only more recently [Berr77, Vor77, Bo91], but the question of the localization of stationary motions in ergodic systems was already asked explicitly by Einstein [Ein17].

In these notes, we shall focus on the case where the geodesic flow has a very chaotic behaviour. When the geodesic flow is ergodic, the semiclassical measures are essentially described by the Snirelman theorem [Sn74, Ze87, CdV85] (see Section 6). Let X be a compact riemannian manifold; call $0 < \lambda_1 \leq \lambda_2 \leq \dots$ the eigenvalues of the laplacian, and let (ψ_j) be an orthonormal basis of eigenfunctions : $-\Delta \psi_j = \lambda_j \psi_j$. Denote μ_j the corresponding Husimi measures (the semiclassical parameter is $\hbar = \lambda_j^{-1/2}$). We shall call $(L_E)_E$ the disintegration of the Liouville measure $dx d\xi$ with respect to the value E of the hamiltonian $\frac{\|\xi\|^2}{2}$. We normalize

⁷Note that $\|(-\hbar^2 \Delta - 1)u_{\hbar}\| \leq \varepsilon \|u_{\hbar}\|$ implies that 1 is an ε -neighbourhood of the spectrum of $-\hbar^2 \Delta$, but does not imply that u_{\hbar} is close to an eigenfunction of the laplacian.

L_E to be a probability measure on the energy layer $\{\frac{\|\xi\|^2}{2} = E\}$. If the geodesic flow on S^*X is ergodic with respect to $L_{\frac{1}{2}}$, then there is a “density 1” subsequence of the family (μ_j) converging to $L_{\frac{1}{2}}$:

THEOREM 5.8 (Snirelman theorem). [Sn74, Ze87, CdV85] *Assume that the action of S^*X is ergodic, with respect to the Liouville measure $L_{\frac{1}{2}}$. Then, there exists a subset $\mathcal{S} \subset \mathbb{N}$ of density 1, such that*

$$\mu_j \xrightarrow{j \rightarrow +\infty, j \in \mathcal{S}} L_{\frac{1}{2}}.$$

In specific examples, what we would like to know is whether the whole sequence μ_j converges to the Liouville measure, or if there can be exceptional subsequences converging to other invariant measures. In the case of nonpositively curved surfaces with flat cylinders, it is believed that certain sequences of eigenfunctions concentrate asymptotically on these cylinders. But in (strictly) negative curvature, it was conjectured by Rudnick and Sarnak [RudSa94] that the Liouville measure is the unique limit point of the μ_j s. It would imply, in particular, that the sequence of probability measures $|\psi_j(x)|^2 d\text{Vol}(x)$ on X converges weakly to the riemannian volume measure Vol.

ENTROPY AND LOCALIZATION OF EIGENFUNCTIONS.

6. Motivations.

The field of *quantum chaos* tries to understand how the chaotic behaviour of a classical hamiltonian system is translated in quantum mechanics. For instance, let X be a compact riemannian C^∞ manifold, with negative sectional curvature. The geodesic flow has the Anosov property, which is considered as the ideal chaotic behaviour in the theory of dynamical systems. The corresponding quantum dynamics is the unitary flow generated by the Laplace-Beltrami operator on $L^2(X)$. One expects that the chaotic features of the geodesic flow can be seen in the spectral properties of the laplacian. The Random Matrix conjecture [Bo91] asserts that the large eigenvalues should, after proper renormalization, statistically resemble those of a large random matrix, at least for a generic Anosov metric. The Quantum Unique Ergodicity conjecture [RudSa94] (see also [Berr77, Vor77]) deals with the corresponding eigenfunctions ψ : it claims that the probability density $|\psi(x)|^2 dx$ should approach (in a weak sense) the riemannian volume, when the eigenvalue tends to infinity. In fact, a corresponding (stronger) property should hold for the microlocal lift of this measure to the cotangent bundle T^*X , which describes the distribution of the wave function ψ on the classical phase space (position and momentum).

To describe the problem, we will adopt a semiclassical point of view, that is, consider the eigenstates of eigenvalue 1 of the semiclassical laplacian $-\hbar^2 \Delta$, in the semiclassical limit $\hbar \rightarrow 0$. We denote by $(\psi_k)_{k \in \mathbb{N}}$ an orthonormal basis of $L^2(X)$ made of eigenfunctions of the laplacian, and by $(-\frac{1}{\hbar_k^2})_{k \in \mathbb{N}}$ the corresponding eigenvalues:

$$(6.1) \quad -\hbar_k^2 \Delta \psi_k = \psi_k, \quad \text{with} \quad \hbar_{k+1} \leq \hbar_k.$$

We are interested in the high-energy eigenfunctions of $-\Delta$, in other words the semiclassical limit $\hbar_k \rightarrow 0$.

To an eigenfunction ψ_k corresponds a distribution on T^*X defined by

$$\mu_k(a) = \langle \psi_k, \text{Op}_{\hbar_k}(a)\psi_k \rangle_{L^2(X)}, \quad a \in C_o^\infty(T^*X).$$

Here Op_{\hbar_k} is a quantization procedure, set at the scale \hbar_k , which associates a bounded operator on $L^2(X)$ to any smooth phase space function a with nice behaviour at infinity. If a is a function on the manifold X , we have $\mu_k(a) = \int_X a(x)|\psi_k(x)|^2 dx + O(\hbar)$: the distribution μ_k is a *microlocal lift* of the probability measure $|\psi_k(x)|^2 dx$ into a phase space distribution. It contains the information about the frequency vector of ψ_k (in other words, the momentum), in addition to the position distribution $|\psi_k(x)|^2 dx$. The definition of μ_k is not canonical, it depends on a certain number of choices, like the choice of local coordinates, or of the quantization procedure (Weyl, anti-Wick, “right” or “left” quantization...); this somehow reflects the fact that, for $\hbar > 0$, it does not really make sense to study simultaneously the position and frequency of a wave. Mathematically speaking, one cannot study simultaneously the local properties of a function and of its Fourier transform around some point $(x, \xi) \in T^*X$. But the asymptotic behaviour of μ_k when $\hbar_k \rightarrow 0$ does not depend on the arbitrary conventions involved in its definition. We saw that it is possible to construct $\text{Op}_{\hbar_k}^+$ so that the μ_k are probability measures, in which case we call them *Husimi measures* associated to the eigenfunctions ψ_k . We call *semiclassical measures* the limit points of the sequence $(\mu_k)_{k \in \mathbb{N}}$, in the distribution topology.

The quantum hamiltonian $-\frac{\hbar^2 \Delta}{2}$ generates the Schrödinger flow

$$(U_{\hbar}^t) = (\exp(it\hbar \frac{\Delta}{2}))$$

acting unitarily on $L^2(X)$. A solution of (6.1) is an invariant state of the flow (U_{\hbar}^t) , corresponding to the energy $\frac{1}{2}$ of the hamiltonian. In the semiclassical limit $\hbar \rightarrow 0$, “quantum mechanics converges to classical mechanics”. We will denote $|\cdot|_x$ the norm on T_x^*M given by the metric. The geodesic flow $(g^t)_{t \in \mathbb{R}}$ is the hamiltonian flow on T^*X generated by the hamiltonian $H(x, \xi) = \frac{|\xi|_x^2}{2}$. In the previous chapter we saw the following :

PROPOSITION 6.1. *Any semiclassical measure is a probability measure carried on the energy layer $H^{-1}(\frac{1}{2})$, that is, the unit cotangent bundle S^*X . This measure is invariant under the geodesic flow.*

If the geodesic flow has the Anosov property — for instance if X has negative sectional curvature — then there exist many invariant probability measures on S^*X , in addition to the Liouville measure. The geodesic flow has countably many periodic orbits, each of them carrying an invariant probability measure. There are still many others, like the equilibrium states obtained by variational principles [KH].

For manifolds with an ergodic geodesic flow (with respect to the Liouville measure), it has been known for some time that *almost all* eigenfunctions become uniformly distributed over S^*X , in the semiclassical limit. This property is dubbed as Quantum Ergodicity :

THEOREM 6.2. [Sn74, Ze87, CdV85] *Let X be a compact riemannian manifold, and assume that the action of the geodesic flow on S^*X is ergodic with respect to the Liouville measure $L_{\frac{1}{2}}$. Let $(\psi_k)_{k \in \mathbb{N}}$ be an orthonormal basis of $L^2(X)$*

consisting of eigenfunctions of the laplacian (6.1), and let (μ_k) be the associated distributions on T^*X .

Then, there exists a subset $\mathcal{S} \subset \mathbb{N}$ of density 1, such that

$$\mu_k \xrightarrow[k \rightarrow \infty, k \in \mathcal{S}]{} L_{\frac{1}{2}}.$$

Proof: Let us give the main lines of the argument, and see where the ergodicity comes into play. For all $a \in C_o^\infty(T^*X)$, one first shows, without using any assumption on the dynamics, that

$$(6.2) \quad \sum_{j, \lambda_j \leq E} \int a d\mu_j \underset{E \rightarrow +\infty}{\sim} \frac{b_d}{(2\pi)^d} \text{Vol}(X) \int_{S^*X} a dL_{\frac{1}{2}} \times E^{d/2}.$$

The constant b_d is the volume of the euclidean d -dimensional ball. The idea is to express in two different ways the trace of $\text{Op}_{\sqrt{E}}(a)$: the trace can be expressed either as a spectral sum $\sum_k \langle \psi_k, \text{Op}(a)\psi_k \rangle$ or as the integral of the kernel on the diagonal (4.8). There are some technical details that we skip here.

From (6.2) one can deduce the Weyl asymptotics :

$$N(E) = \#\{j, \lambda_j \leq E\} \sim \frac{b_d}{(2\pi)^d} \text{Vol}(X) E^{d/2}$$

Thus, we have a Cesaro convergence :

$$(6.3) \quad \frac{1}{N(E)} \sum_{j, \lambda_j \leq E} \int a d\mu_j \xrightarrow{E \rightarrow +\infty} \int_{S^*X} a dL_{\frac{1}{2}}.$$

Using the ergodicity assumption, one can do better :

$$(6.4) \quad \frac{1}{N(E)} \sum_{j, \lambda_j \leq E} \left| \int a d\mu_j - \int_{S^*X} a dL_{\frac{1}{2}} \right|^2 \xrightarrow{E \rightarrow +\infty} 0.$$

Here is how. We know from Theorem 5.6 (3) that

$$\left| \int a d\mu_j - \int a \circ g^t d\mu_j \right| \rightarrow 0$$

as $j \rightarrow +\infty$, for any fixed t . Thus, we can write, for any given T ,

$$\begin{aligned} & \limsup_{E \rightarrow \infty} \frac{1}{N(E)} \sum_{j, \lambda_j \leq E} \left| \int a d\mu_j - \int_{S^*X} a dL_{\frac{1}{2}} \right|^2 \\ &= \limsup \frac{1}{N(E)} \sum_{j, \lambda_j \leq E} \left| \int M^T a d\mu_j - \int_{S^*X} a dL_{\frac{1}{2}} \right|^2 \\ &\leq \limsup \frac{1}{N(E)} \sum_{j, \lambda_j \leq E} \mu_j \left((M^T a - \int_{S^*X} a dL_{\frac{1}{2}})^2 \right) \\ &= L_{\frac{1}{2}} \left((M^T a - \int_{S^*X} a dL_{\frac{1}{2}})^2 \right). \end{aligned}$$

We denoted $M^T a = T^{-1} \int_0^T a \circ g^t dt$ the time average of a on the interval $[0, T]$. We used the Cauchy-Schwartz inequality, which requires to know that the μ_j can be assumed to be *probability* measures (see §5; this was the missing argument in Snirelman's original paper). In the last line, we used the Cesaro convergence (6.3)

of the sequence (μ_j) . Letting at the end T tend to $+\infty$, the ergodicity assumption means that

$$L_{\frac{1}{2}} \left((M^T a - \int_{S^*X} a dL_{\frac{1}{2}})^2 \right) \xrightarrow{T \rightarrow \infty} 0;$$

which proves (6.4).

Finally, the Snirelman theorem results from the classical lemma :

LEMMA 6.3. *Let (u_n) be a sequence of nonnegative numbers. If*

$$\frac{1}{n} \sum_{k=0}^n u_k \longrightarrow 0$$

then there exists $\mathcal{S} \subset \mathbb{N}$ of density 1 such that $u_n \xrightarrow[n \in \mathcal{S}]{} 0$.

For each $a \in C_o^\infty(T^*X)$, the lemma yields the Snirelman theorem for some density one set $\mathcal{S} \subset \mathbb{N}$, possibly depending on a . Using the fact that $C_o^\infty(T^*X)$ has a countable dense subset, one can find some $\mathcal{S} \subset \mathbb{N}$ that works for *all* $a \in C_o^\infty(T^*X)$. \square

REMARK 6.4. The result was subsequently extended to more general hamiltonians [HelMR87], to ergodic billiards [GL93, ZeZw96], and to certain discrete time symplectic dynamical systems.

The question of knowing, in particular cases, if there can exist “exceptional” subsequences with a different behaviour is widely open. On a negatively curved manifold, the geodesic flow satisfies the ergodicity assumption, and in fact much stronger properties : mixing, K -property,... In this case, the Quantum Unique Ergodicity conjecture [RudSa94] expresses the belief that there exists a unique semiclassical measure, namely the Liouville measure on S^*X : the whole sequence (μ_k) converges to $L_{\frac{1}{2}}$. In other words, in the semiclassical régime all eigenfunctions should become uniformly distributed over S^*X .

So far the most precise results on this question were obtained for manifolds X with constant negative curvature and *arithmetic* properties: see Rudnick–Sarnak [RudSa94], Wolpert [Wol01]. In that very particular situation, there exists a countable commutative family of self-adjoint operators commuting with the laplacian : the Hecke operators. One may thus decide to restrict the attention to bases of common eigenfunctions, often called “arithmetic” eigenstates, or Hecke eigenstates. A few years ago, Lindenstrauss [Li06] proved that the arithmetic eigenstates become asymptotically equidistributed (Arithmetic Quantum Unique Ergodicity). If there is some degeneracy in the spectrum of the laplacian, it could be possible that the Quantum Unique Ergodicity conjectured by Rudnick and Sarnak holds for one orthonormal basis but not for another. In the arithmetic case, it is believed that the spectrum of the laplacian has bounded multiplicity, in which case it would be a harmless assumption to consider only Hecke eigenstates.

Nevertheless, one may be less optimistic about the general conjecture. Faure–Nonnenmacher–De Bièvre exhibited in [FNDB03] a simple example of a symplectic Anosov dynamical system, namely the action of the linear hyperbolic automorphism $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on the 2-torus, the Weyl-quantization of which does not satisfy the Quantum Unique Ergodicity conjecture. In this model, it is known [KurRud00] that there is one orthonormal family of eigenfunctions satisfying Quantum Unique

Ergodicity, but, due to high degeneracies in the spectrum, one can also construct eigenfunctions with a different behaviour. Precisely, [FNDB03] construct a family of eigenstates for which the semiclassical measure consists in two ergodic components: half of it is the Liouville measure, while the other half is a Dirac peak on a single unstable periodic orbit. It was also shown that this half-localization on a periodic orbit is *maximal* for this model [FN04]: a semiclassical measure cannot have more than half the mass carried by a finite union of closed orbits. Another type of semiclassical measure was recently obtained by Kelmer for a quantized automorphism on a higher-dimensional torus [Kelm05]: it consists in the Lebesgue measure on some invariant co-isotropic subspace of the torus. For these torus automorphisms, the existence of exceptional eigenstates seems to be due to some nongeneric algebraic properties of the classical and quantized systems. It has been believed for a while that any perturbation of the system which lifts the degeneracies in the spectrum will also destroy the counterexamples to Quantum Unique Ergodicity. However, Kelmer has recently disproved this belief: he showed that a non-linear perturbation of his previous construction [Kelm05], if it is done so as to preserve the invariant co-isotropic subspaces, will still contradict Quantum Unique Ergodicity [Kelm06], in the same way as before the perturbation. Moreover, the spectrum of the perturbed quantum model is simple [Kelm08].

7. Main result.

We wish to consider the *Kolmogorov–Sinai* entropy of semiclassical measures. We work on a compact manifold X of arbitrary dimension, and assume that the geodesic flow has the Anosov property. In fact, our method is very general, and can without any doubt be adapted to more general Anosov hamiltonian systems.

The Kolmogorov–Sinai entropy, also called metric entropy, of a (g^t) -invariant probability measure μ is a nonnegative number $h_{KS}(\mu)$ that describes, in some sense, the complexity of a μ -typical orbit of the flow. The precise definition will be given later, but for the moment let us just give a few facts. A measure carried on a closed geodesic has zero entropy. In constant curvature, the entropy is known to be maximal for the Liouville measure. More generally, an upper bound on the entropy is given by the Ruelle inequality: since the geodesic flow has the Anosov property, the energy layer S^*X is foliated into unstable manifolds of the flow, and for any invariant probability measure μ one has

$$(7.1) \quad h_{KS}(\mu) \leq \left| \int_{S^*X} \log J^u(\rho) d\mu(\rho) \right|.$$

In this inequality, $J^u(\rho)$ is the *unstable Jacobian* of the flow at the point $\rho \in S^*X$, defined as the Jacobian of the map g^{-1} restricted to the unstable manifold at the point $g^1\rho$. The average of $\log J^u$ over any invariant measure is negative. In fact, if μ is an invariant probability measure,

$$\int_{S^*X} \log J^u(\rho) d\mu(\rho) = - \int_{S^*X} \sum \lambda_j^+(\rho) d\mu(\rho)$$

where $\lambda_j^+(\rho)$ are the positive Lyapunov exponents of ρ . If X has dimension d and has constant sectional curvature -1 , (7.1) just reads $h_{KS}(\mu) \leq d - 1$. Equality holds in (7.1) if and only if μ is the Liouville measure on S^*X [LY85].

Let μ be a (g^t) -invariant probability measure on S^*X . According to the Birkhoff ergodic theorem, for μ -almost every $\rho \in S^*X$, the weak limit

$$\mu^\rho = \lim_{|t| \rightarrow \infty} \frac{1}{t} \int_0^t \delta_{g^s \rho} ds$$

exists, and is an ergodic probability measure. We can then write

$$\mu = \int_{S^*X} \mu^\rho d\mu(\rho),$$

which is called the ergodic decomposition of μ . One can prove that the ergodic probability measures are the extremal points of the compact convex set of (g^t) -invariant probability measures.

To understand the connection of our results with the previous discussion, it is important to know that the entropy is an *affine* functional on the convex set of (g^t) -invariant probability measures :

$$h_{KS}(\mu) = \int_{S^*X} h_{KS}(\mu^\rho) d\mu(\rho).$$

In what follows, we consider a certain subsequence of eigenstates $(\psi_{k_j})_{j \in \mathbb{N}}$ of the laplacian, such that the corresponding sequence (μ_{k_j}) converges to a certain semiclassical measure μ (see the discussion preceding Proposition 6.1). The subsequence (ψ_{k_j}) will simply be denoted by $(\psi_{\hbar})_{\hbar \rightarrow 0}$, using the slightly abusive notation $\psi_{\hbar} = \psi_{\hbar_{k_j}}$ for the eigenstate ψ_{k_j} . Each state ψ_{\hbar} satisfies

$$(7.2) \quad (-\hbar^2 \Delta - 1)\psi_{\hbar} = 0.$$

It is proved in [A05] that the entropy of any semiclassical measure associated with eigenfunctions of the laplacian is strictly positive. In [AN07] more explicit lower bounds were obtained. We shall prove here the following lower bound :

THEOREM 7.1. *Let μ be a semiclassical measure associated to the eigenfunctions of the laplacian on X . Then its metric entropy satisfies*

$$(7.3) \quad h_{KS}(\mu) \geq \left| \int_{S^*X} \log J^u(\rho) d\mu(\rho) \right| - \frac{(d-1)}{2} \lambda_{\max},$$

where $d = \dim M$ and $\lambda_{\max} = \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \sup_{\rho \in S^*X} |dg_\rho^t|$ is the maximal expansion rate of the geodesic flow on S^*X .

In particular, if X has constant sectional curvature -1 , this means that

$$(7.4) \quad h_{KS}(\mu) \geq \frac{d-1}{2}.$$

The bound (7.4) in the above theorem is much sharper than the bound proved in [A05] in the case of constant curvature. On the other hand, if the curvature varies a lot (still being negative everywhere), the right hand side of (7.3) may be negative, in which case the above bound is trivial and the result of [A05] is better. We believe this to be but a technical shortcoming of our method, and would actually expect the following bound to hold:

$$(7.5) \quad h_{KS}(\mu) \geq \frac{1}{2} \left| \int_{S^*X} \log J^u(\rho) d\mu(\rho) \right|.$$

Our result is compatible with the kind of counter-examples obtained by Faure–Nonnenmacher–De Bièvre [FNDB03]. It allows certain ergodic components to

be carried by closed geodesics, but says that others must have positive entropy. Compare with the much stronger result obtained in the arithmetic case by Bourgain and Lindenstrauss :

THEOREM 7.2. [BLi03] *Let X be a congruence arithmetic surface, and (ψ_j) an orthonormal basis of eigenfunctions for the laplacian and the Hecke operators.*

*Let μ be a corresponding semiclassical measure, with ergodic decomposition $\mu = \int_{S^*X} \mu^\rho d\mu(\rho)$. Then for almost all ergodic components we have $h_{KS}(\mu^\rho) \geq \frac{1}{9}$.*

Quantum Unique Ergodicity would mean that $h_{KS}(\mu) = |\int_{S^*X} \log J^u(\rho) d\mu(\rho)|$ [LY85]. We believe however that (7.5) is the optimal result that can be obtained without using more precise information, like for instance upper bounds on the multiplicities of eigenvalues. Indeed, in the above mentioned examples of Anosov systems where the Quantum Unique Ergodicity conjecture is wrong, the bound (7.5) is actually *sharp* [FNDB03, Kelm05, AN06]. In those examples, the spectrum has very high degeneracies, which allows for much freedom to select the eigenstates, and could be responsible for the failure of Quantum Unique Ergodicity. Such high degeneracies are not expected to happen in the case of the laplacian on a negatively curved manifold. For the moment, however, there is no clear understanding of the precise relation between spectral degeneracies and failure of Quantum Unique Ergodicity. As explained above, at the time of revision of these notes, Kelmer had found a model (a non-linear symplectic diffeomorphism of \mathbb{T}^{2d} , $d \geq 2$ and its quantization), for which the spectrum is simple but Quantum Unique Ergodicity does not hold [Kelm08].

8. Definition of entropy, and main idea of the proof.

Let μ be a probability measure on T^*X . Let (P_1, \dots, P_K) be a finite measurable partition of the unit tangent bundle : $T^*X = P_1 \sqcup \dots \sqcup P_K$. The Shannon entropy of μ with respect to the partition P is

$$(8.1) \quad h_P(\mu) = - \sum_{k=1}^K \mu(P_k) \log \mu(P_k).$$

Assume now that μ is (g^t) -invariant. For any integer n , denote $P^{\vee n}$ the partition formed by the sets $P_{\alpha_0} \cap g^{-1}P_{\alpha_1} \dots \cap g^{-n+1}P_{\alpha_{n-1}}$. Denote

$$(8.2) \quad \begin{aligned} h_n(\mu, P) &= h_{P^{\vee n}}(\mu) \\ &= - \sum_{(\alpha_j) \in \{1, \dots, K\}^n} \mu(P_{\alpha_0} \cap g^{-1}P_{\alpha_1} \dots \cap g^{-n+1}P_{\alpha_{n-1}}) \log \mu(P_{\alpha_0} \cap g^{-1}P_{\alpha_1} \dots \cap g^{-n+1}P_{\alpha_{n-1}}). \end{aligned}$$

If μ is (g^t) -invariant, it follows from the concavity of $x \mapsto -x \log x$ that

$$(8.3) \quad h_{n+m}(\mu, P) \leq h_n(\mu, P) + h_m(\mu, P),$$

in other words the sequence $(h_n(\mu, P))_{n \in \mathbb{N}}$ is subadditive. The entropy of μ with respect to the action of geodesic flow and to the partition P is defined by

$$(8.4) \quad h_{KS}(\mu, P) = \lim_{n \rightarrow +\infty} \frac{h_n(\mu, P)}{n} = \inf_{n \in \mathbb{N}} \frac{h_n(\mu, P)}{n}.$$

The existence of the limit, and the fact that it coincides with the infimum, follow from a standard subadditivity argument. Note that $\mu(P_{\alpha_0} \cap g^{-1}P_{\alpha_1} \dots \cap g^{-n+1}P_{\alpha_{n-1}})$ measures the μ -probability to visit successively $P_{\alpha_0}, P_{\alpha_1}, \dots, P_{\alpha_{n-1}}$ at times 1, 2, ...,

$n - 1$ of the geodesic flow. The entropy measures the average exponential decay of these probabilities when n gets large. In particular, if there is a uniform exponential decay, that is, if there exists $C, \beta \geq 0$ such that $\mu(P_{\alpha_0} \cap g^{-1}P_{\alpha_1} \dots \cap g^{-n}P_{\alpha_n}) \leq Ce^{-\beta n}$, for all n and all $\alpha_0, \dots, \alpha_n$, then it is easy to see that $h_{KS}(\mu, P) \geq \beta$.

The entropy of μ with respect to the action of the geodesic flow is defined as

$$(8.5) \quad h_{KS}(\mu) = \sup_P h_{KS}(\mu, P),$$

the supremum running over all finite measurable partitions P . Assume μ is carried on the energy layer S^*X . Due to the Anosov property of the geodesic flow on S^*X , it is known that the supremum (8.5) is reached as soon as the maximum diameter of the sets $P_k \cap S^*X$ is small enough.

We will restrict our attention to partitions P which are actually partitions of the base X (lifted to T^*X): $X = \sqcup_{k=1}^K P_k$. This choice is not crucial, but it simplifies certain aspects of the analysis.

The fact that the limit in (8.4) coincides with the infimum has a crucial consequence, the upper semicontinuity property of $h_{KS}(\cdot, P)$: if (μ_k) is a sequence of (g^t) -invariant probability measures converging weakly to μ , then

$$(8.6) \quad h_{KS}(\mu, P) \geq \limsup_k h_{KS}(\mu_k, P)$$

(provided μ does not charge the boundary of P).

Since our semiclassical measure μ is defined as a limit of Husimi measures associated to ψ_{\hbar} , a naive idea would be to estimate from below the entropy of ψ_{\hbar} and then take the limit.

A first issue is to decide how to define the ψ_{\hbar} -probability to visit successively $P_{\alpha_0}, P_{\alpha_1}, \dots, P_{\alpha_{n-1}}$ at times 1, 2, ..., $n - 1$.

From the definition of the Husimi measures, a first idea could be to consider

$$(8.7) \quad \left\langle \psi_{\hbar}, \text{Op}_{\hbar} \left((\mathbb{1}_{P_{\alpha_0}}) (\mathbb{1}_{P_{\alpha_1}} \circ g^1) \dots (\mathbb{1}_{P_{\alpha_{n-1}}} \circ g^{n-1}) \right) \psi_{\hbar} \right\rangle.$$

To avoid dealing with characteristic functions (which are not quantized to pseudodifferential operators), we can smooth them by convolution and try replacing $\mathbb{1}_{P_k}$ by a smooth $\mathbb{1}_{P_k}^{sm}$. Even so, studying the large- n behaviour of (8.7) is very problematic. In fact, the derivatives of $(\mathbb{1}_{P_{\alpha_0}}^{sm}) (\mathbb{1}_{P_{\alpha_1}}^{sm} \circ g^1) \dots (\mathbb{1}_{P_{\alpha_{n-1}}}^{sm} \circ g^{n-1})$ grow like e^n , so that when n reaches the size $|\log \hbar|$ this function no longer belongs to any reasonable symbol space (the operator is not a pseudodifferential operator).

We also note that an overlap of the form (8.7) is a *hybrid* expression: this is a *quantum* matrix element, but the operator is defined in terms of the *classical* flow! From the point of view of quantum mechanics, it is more natural to consider, instead, the operator obtained as the product of Heisenberg-evolved quantized functions, namely

$$(8.8) \quad \hat{P}_{\alpha_{n-1}}(n-1) \hat{P}_{\alpha_{n-2}}(n-2) \dots \hat{P}_{\alpha_1}(1) \hat{P}_{\alpha_0}.$$

Here we used the shorthand notation $\hat{P}_k \stackrel{\text{def}}{=} \text{Op}(\mathbb{1}_{P_k}^{sm})$, $k \in [1, K]$, and $\hat{P}_k(t) = U_{\hbar}^{-t} \hat{P}_k U_{\hbar}^t$. Instead of (8.7), a second idea is to consider

$$(8.9) \quad \left\langle \psi_{\hbar}, \hat{P}_{\alpha_{n-1}}(n-1) \dots \hat{P}_{\alpha_1}(1) \hat{P}_{\alpha_0} \psi_{\hbar} \right\rangle.$$

as the ψ_{\hbar} -probability to visit successively $P_{\alpha_0}, P_{\alpha_1}, \dots, P_{\alpha_{n-1}}$ at times 1, 2, ..., $n-1$. However, the scalar product is a complex number, and can not be directly manipulated as a probability.

Our third and final try is to consider

$$(8.10) \quad \|\hat{P}_{\alpha_{n-1}}(n-1) \dots \hat{P}_{\alpha_1}(1) \hat{P}_{\alpha_0} \psi_{\hbar}\|^2.$$

In fact, if we do the smoothing of $\mathbb{1}_{P_k}$ so that

$$\sum_k (\mathbb{1}_{P_k}^{sm})^2 \equiv 1$$

then the norms (8.10) can actually be manipulated like probability measures :

$$(8.11) \quad \sum_{\alpha_0, \dots, \alpha_{n-1}} \|\hat{P}_{\alpha_{n-1}}(n-1) \hat{P}_{\alpha_{n-2}}(n-2) \dots \hat{P}_{\alpha_1}(1) \hat{P}_{\alpha_0} \psi_{\hbar}\|^2 = 1,$$

and

$$\begin{aligned} \sum_{\alpha_{n-1}} \|\hat{P}_{\alpha_{n-1}}(n-1) \hat{P}_{\alpha_{n-2}}(n-2) \dots \hat{P}_{\alpha_1}(1) \hat{P}_{\alpha_0} \psi_{\hbar}\|^2 \\ = \|\hat{P}_{\alpha_{n-2}}(n-2) \dots \hat{P}_{\alpha_1}(1) \hat{P}_{\alpha_0} \psi_{\hbar}\|^2. \end{aligned}$$

Finally, using the Egorov theorem (4.6), we see that, for fixed n ,

$$\|\hat{P}_{\alpha_{n-1}}(n-1) \dots \hat{P}_{\alpha_1}(1) \hat{P}_{\alpha_0} \psi_{\hbar}\|^2 \xrightarrow[\hbar \rightarrow 0]{} \mu \left((\mathbb{1}_{P_{\alpha_0}}^{sm})^2 (\mathbb{1}_{P_{\alpha_1}}^{sm} \circ g^1)^2 \dots (\mathbb{1}_{P_{\alpha_{n-1}}}^{sm} \circ g^{n-1})^2 \right)$$

if the Husimi measures of ψ_{\hbar} converge to μ . Apart from the smoothing, this is the quantity we are interested in when computing entropy of μ (8.2).

It is proved in [A05] that

$$\|\hat{P}_{\alpha_{n-1}}(n-1) \dots \hat{P}_{\alpha_1}(1) \hat{P}_{\alpha_0} \psi_{\hbar}\|^2 \leq \frac{C}{\hbar^d} e^{-(d-1)n},$$

say, in dimension d and constant curvature -1 , and assuming the diameter of the P_k is small enough⁸. From this, it would be tempting to deduce that the entropy of the ψ_{\hbar} -Husimi measures is bounded below by $d-1$, then use the semicontinuity property (8.6) to deduce that $h_{KS}(\mu) \geq d-1$ (thus proving quantum unique ergodicity).

Of course, we can not apply (8.6), since we are not in the situation of a sequence (μ_k) of g^t -invariant probability measures converging to μ . To use (8.6) we need to know if a similar property holds in our quantum framework, using expressions such as (8.10) to evaluate entropies. This is, in fact, NOT the case : a factor of 2 is lost somewhere in the proof, and we will end up proving

$$h_{KS}(\mu) \geq \frac{d-1}{2}.$$

As we shall see, this is due to the fact that the operators $\hat{P}_{\alpha}(t)$ appearing in (8.10) do not commute : for non commuting operators, the interpretation of $\|\hat{P}_{\alpha_{n-1}}(n-1) \dots \hat{P}_{\alpha_1}(1) \hat{P}_{\alpha_0} \psi_{\hbar}\|^2$ as “the probability to visit successively $P_{\alpha_0}, P_{\alpha_1}, \dots, P_{\alpha_{n-1}}$ at times 1, 2, ..., $n-1$ ” is not allowed. It is only acceptable for a time n small enough so that the operators almost commute (up to some small error). This restriction of the time range is responsible for the loss of a factor 2.

⁸To prove this estimate, we assume, without any loss of generality, that the injectivity radius of X is larger than 1.

THE ENTROPIC UNCERTAINTY PRINCIPLE.

In this chapter, we give the steps of the proof of our main result, inequality (7.3). To simplify the presentation we restrict ourselves to the case of constant curvature $\equiv -1$.

We start with a functional inequality called the “entropic uncertainty principle”.

9. The abstract result...

We consider a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, and denote $\|\psi\| = \sqrt{\langle \psi, \psi \rangle}$ the associated norm. The same notation $\|\cdot\|$ will also be used for the operator norm on $\mathcal{L}(\mathcal{H})$.

We define the following family of l_p norms on \mathcal{H}^N : for $\Psi = (\Psi_1, \dots, \Psi_N) \in \mathcal{H}^N$, we let

$$(9.1) \quad \|\Psi\|_p \stackrel{\text{def}}{=} \left(\sum_{k=1}^N \|\Psi_k\|^p \right)^{1/p}, \quad 1 \leq p < \infty, \quad \text{and} \quad \|\Psi\|_\infty \stackrel{\text{def}}{=} \max_k \|\Psi_k\|.$$

For $p = 2$, this norm coincides with the Hilbert norm deriving from the scalar product

$$\langle \Psi, \Phi \rangle_{\mathcal{H}^N} = \sum_{k=1}^N \langle \Psi_k, \Phi_k \rangle_{\mathcal{H}}.$$

We can define similarly a family of l_p norms on $\mathcal{H}^M \ni \Phi = (\Phi_1, \dots, \Phi_M)$:

$$(9.2) \quad \|\Phi\|_p \stackrel{\text{def}}{=} \left(\sum_{j=1}^M \|\Phi_j\|^p \right)^{1/p}, \quad 1 \leq p < \infty, \quad \text{and} \quad \|\Phi\|_\infty \stackrel{\text{def}}{=} \max_j \|\Phi_j\|.$$

For $\Psi \in \mathcal{H}^N$ with $\|\Psi\|_2 = 1$, we define its entropy,

$$h(\Psi) = - \sum_{k=1}^N \|\Psi_k\|^2 \log \|\Psi_k\|^2;$$

(and we define similarly the entropy of a normalized vector $\Phi \in \mathcal{H}^M$). The entropy is related to the l_p norms by the fact that $-\frac{1}{4}h(\Psi)$ is the derivative of $\|\Psi\|_p$ at $p = 2$.

Consider the action of a bounded operator $T : \mathcal{H}^N \rightarrow \mathcal{H}^M$, which we present as a $M \times N$ matrix (T_{jk}) of bounded operators on \mathcal{H} . We denote $\|T\|_{p,q}$ the norm of T from $l_p(\mathcal{H}^N)$ to $l_q(\mathcal{H}^M)$, for $1 \leq p, q \leq \infty$.

THEOREM 9.1 (Riesz interpolation theorem). [**DunSchw**, Section VI.10] *The function $\log \|T\|_{1/a, 1/b}$ is a convex function of (a, b) in the square $0 \leq a, b \leq 1$.*

From this theorem, Maassen and Uffink derived a new form of uncertainty relations [**MaaUff88**].

THEOREM 9.2. *Assume that $\|T\|_{2,2} = 1$, which implies in particular that $\|T_{jk}\| \leq 1$ for all j, k . Introduce the real number $c(T) = \max_{j,k} \|T_{jk}\|$, where the norm is the operator norm in $\mathcal{L}(\mathcal{H})$.*

For all $\Psi \in \mathcal{H}$ such that $\|\Psi\|_2 = 1$ and $\|T\Psi\|_2 = 1$, we have

$$h(T\Psi) + h(\Psi) \geq -2 \log c(T).$$

REMARK 9.3. The theorem proved in [MaaUff88] deals with $\mathcal{H} = \mathbb{C}$. The operator T is then simply represented by an $M \times N$ complex matrix, and $c(T)$ is the supremum of its entries. According to the theorem, if a matrix has small entries, then either $h(T\Psi)$ or $h(\Psi)$ must be large. Here $h(\Psi) = -\sum |\Psi_k|^2 \log |\Psi_k|^2$ measures the entropy of the probability vector ($|\Psi_k|^2$), given by the coordinates of Ψ in the canonical basis of \mathbb{C}^N .

Proof: In the case $a = 1, b = 0$, we have for any Ψ ,

$$\|T\Psi\|_\infty = \sup_j \|(T\Psi)_j\| \leq \sup_{j,k} \|T_{j,k}\| \sum_{k'} \|\Psi_{k'}\| = \sup_{j,k} \|T_{j,k}\| \|\Psi\|_1,$$

which can be written as $\|T\|_{1,\infty} \leq \sup_{j,k} \|T_{j,k}\| \stackrel{\text{def}}{=} c(T)$.

Let us assume that T is contracting on l_2 : $\|T\|_{2,2} \leq 1$. We take $t \in [0, 1]$ and $a_t = \frac{1+t}{2}$, $b_t = \frac{1-t}{2}$ to interpolate between $(1/2, 1/2)$ and $(1, 0)$; Theorem 9.1 implies that

$$\|T\|_{1/a_t, 1/b_t} \leq c(T)^t.$$

COROLLARY 9.4. *Let the operator $T : \mathcal{H}^N \rightarrow \mathcal{H}^M$ satisfy $\|T\|_{2,2} \leq 1$ and call $c(T) \stackrel{\text{def}}{=} \sup_{j,k} \|T_{j,k}\|$. Then, for all $t \in [0, 1]$, for all $\Psi \in \mathcal{H}^N$,*

$$\|T\Psi\|_{\frac{2}{1-t}} \leq c(T)^t \|\Psi\|_{\frac{2}{1+t}}.$$

We define for any $r > 0$ or $-1 < r < 0$ the ‘‘moments’’

$$M_r(\Psi) \stackrel{\text{def}}{=} \left(\sum_j \|\Psi_j\|^{2+2r} \right)^{1/r}.$$

Corollary 9.4 leads to the following family of ‘‘uncertainty relations’’:

$$(9.3) \quad \forall t \in (0, 1), \forall \Psi \in \mathbb{C}^N, \quad M_{\frac{-t}{1-t}}(T\Psi) M_{\frac{-t}{1+t}}(\Psi) \leq c(T)^2.$$

In the case $\|\Psi\|_2 = 1$, we notice that the moments converge to the same value when $r \rightarrow 0$ from above or below:

$$\lim_{r \rightarrow 0} M_r(\Psi) = e^{-h(\Psi)}, \quad \text{where } h(\Psi) = -\sum_j \|\Psi_j\|^2 \log \|\Psi_j\|^2.$$

If, furthermore, $\|T\Psi\|_2 = 1$, then the limit $t \rightarrow 0$ of the inequalities (9.3) yield the Entropic Uncertainty Principle stated in Theorem 9.2. \square

We shall use Theorem 9.2 in the following particular case :

EXAMPLE 9.5. Suppose we have two partitions of unity $(\pi_k)_{k=1}^N$ and $(\tau_j)_{j=1}^M$, that is, two families of operators on \mathcal{H} such that

$$(9.4) \quad \sum_{k=1}^N \pi_k \pi_k^* = Id, \quad \sum_{j=1}^M \tau_j \tau_j^* = Id.$$

Let U be a unitary operator on \mathcal{H} . We can take $T_{jk} = \tau_j^* U \pi_k$.

LEMMA 9.6. *Let $T_{jk} = \tau_j^* U \pi_k$, for some bounded operator $U : \mathcal{H} \rightarrow \mathcal{H}$. Then we have the identity*

$$\|T\|_{2,2} = \|U\|_{\mathcal{L}(\mathcal{H})}.$$

PROOF. The operator T may be described as follows. Consider the following row and column vectors of operators on \mathcal{H} :

$$L \stackrel{\text{def}}{=} (\pi_1, \dots, \pi_N), \quad \text{as well as } C = \begin{pmatrix} \tau_1^* \\ \dots \\ \tau_M^* \end{pmatrix}.$$

We can write $T = CUL$. We insert this formula in the identity

$$\|T\|_{2,2}^2 = \|T^*T\|_{\mathcal{L}(\mathcal{H}^N)} = \|L^\dagger U^* C^\dagger CUL\|_{\mathcal{L}(\mathcal{H}^N)}$$

Using (9.4) for the τ_j , we notice that $C^\dagger C = Id_{\mathcal{H}}$, so that the norm above reads

$$\|L^\dagger U^* UL\|_{\mathcal{L}(\mathcal{H}^N)}.$$

Then, we use the identities

$$\|(UL)^\dagger(UL)\|_{\mathcal{L}(\mathcal{H}^N)} = \|(UL)(UL)^\dagger\|_{\mathcal{L}(\mathcal{H})} = \|(UL)L^\dagger U^*\|_{\mathcal{L}(\mathcal{H})} = \|UU^*\|_{\mathcal{L}(\mathcal{H})},$$

where we used (9.4) for the π_k . \square

Therefore, if U is contracting (resp. $\|U\|_{\mathcal{L}(\mathcal{H})} = 1$) one has $\|T\|_{2,2} \leq 1$ (resp. $\|T\|_{2,2} = 1$).

We also specify the vector Ψ by taking $\Psi_k = \pi_k^* \psi$ for some normalized $\psi \in \mathcal{H}$. From (9.4), we check that $\|\Psi\|_2 = \|\psi\|$, and also that $(T\Psi)_j = \tau_j^* U\psi$. Thus, if $\|U\psi\| = 1$, the relation (9.4) also implies $\|T\Psi\|_2 = \|U\psi\| = 1$. With this choice for T and Ψ , Theorem 9.2 reads as follows:

THEOREM 9.7. *Let U be an isometry on \mathcal{H} , and let π, τ be two quantum partitions of unity as in (9.4).*

Define $c_{\tau,\pi}(U) \stackrel{\text{def}}{=} \sup_{j,k} \|\tau_j^ U \pi_k\|_{\mathcal{L}(\mathcal{H})}$.*

Then, for any normalized $\psi \in \mathcal{H}$, we have

$$h_\tau(U\psi) + h_\pi(\psi) \geq -2 \log c_{\tau,\pi}(U)$$

where $h_\pi(\psi) = -\sum_{k=1}^N \|\pi_k^* \psi\|^2 \log \|\pi_k^* \psi\|^2$ and $h_\tau(\psi) = -\sum_{j=1}^M \|\tau_j^* \psi\|^2 \log \|\tau_j^* \psi\|^2$.

Note that the definition $h_\pi(\psi) = -\sum_{k=1}^N \|\pi_k^* \psi\|^2 \log \|\pi_k^* \psi\|^2$ is somewhat analogous to (8.1), π playing the role of the partition P and ψ the role of the measure μ . The quantity $h_\pi(\psi)$ may be called the ‘‘Shannon entropy’’ of the state ψ with respect to the partition π .

10. ... applied to eigenfunctions of the laplacian...

In this section we define the data to input into Theorem 9.7, in order to obtain information on the eigenstates ψ_{\hbar} (7.2) and the semiclassical measures μ considered in the previous chapters. Only the Hilbert space is fixed, $\mathcal{H} \stackrel{\text{def}}{=} L^2(X)$. All other data depend on the semiclassical parameter \hbar : the quantum partitions π, τ , the unitary operator U . Besides, we will need yet another technical variant of Theorem 9.7.

10.1. Smooth partition of unity. To evaluate the Kolmogorov–Sinai entropy of a semiclassical measure, we must start by decomposing T^*X into a finite partition. We actually specify the form of the partition we want to use. We work with a measurable partition $(P_k)_{k=1,\dots,K}$ of the base $X : X = \sqcup P_k$, that we lift to a partition of the phase space T^*X .

For semiclassical methods we actually need to work with smooth functions, and so we introduce a smooth partition of unity $(\mathbb{1}_{P_k}^{sm})$, obtained by smoothing the characteristic functions $(\mathbb{1}_{P_k})$ with a convolution kernel. We require that the smoothing be done so that $\sum_{k=1}^K (\mathbb{1}_{P_k}^{sm})^2 \equiv 1$.

We finally denote $\hat{P}_k = \text{Op}(\mathbb{1}_{P_k}^{sm})$: it is simply the operator of multiplication by $\mathbb{1}_{P_k}^{sm}$. We have

$$(10.1) \quad \sum_{k=1}^K \hat{P}_k^2 = I,$$

which means that these operators form a quantum partition of unity as in (9.4), which we will call $\mathcal{P}^{(0)}$.

10.2. Refinement of the partition under the Schrödinger flow. We denote by $U^t = \exp(it\hbar \Delta / 2)$ the quantum propagator. With no loss of generality, we will assume that the injectivity radius of X is much greater than 1, and work with the propagator at time one, $U = U^1$. This propagator quantizes the geodesic flow at time one, g^1 . The \hbar -dependence of U will be implicit in our notations.

As one does to compute the Kolmogorov–Sinai entropy of an invariant measure, we define a new quantum partition of unity by evolving and refining the initial partition $\mathcal{P}^{(0)}$ under the quantum evolution. For each time $n \in \mathbb{N}$ and any sequence of symbols $\alpha = (\alpha_0 \cdots \alpha_{n-1})$, $\alpha_i \in [1, K]$ (we say that the sequence α is of *length* $|\alpha| = n$), we define the operators

$$(10.2) \quad \hat{P}_\alpha = \hat{P}_{\alpha_{n-1}}(n-1) \hat{P}_{\alpha_{n-2}}(n-2) \cdots \hat{P}_{\alpha_0}.$$

We keep using the notation $A(t) = U^{-t} A U^t$ for the quantum evolution of an operator A . From (10.1) and the unitarity of U , the family of operators $\{\hat{P}_\alpha\}_{|\alpha|=n}$ obviously satisfies the relation $\sum_{|\alpha|=n} \hat{P}_\alpha \hat{P}_\alpha^* = Id_{L^2}$, and therefore forms a quantum partition which we call $\mathcal{P}^{(n)}$. We also have $\sum_{|\alpha|=n} \hat{P}_\alpha^* \hat{P}_\alpha = Id_{L^2}$, and we denote $\mathcal{T}^{(n)}$ the partition of unity given by the family of operators $\{\hat{P}_\alpha^*\}_{|\alpha|=n}$.

10.3. In the entropic uncertainty principle, Theorem 9.7, we shall input the following data :

- the quantum partition $\pi = \mathcal{P}^{(n)}$ is given by the family of operators $\{\hat{P}_\alpha, |\alpha| = n\}$. The quantum partition $\tau = \mathcal{T}^{(n)}$ is given by the family of operators $\{\hat{P}_\alpha^*, |\alpha| = n\}$. The integer n will always be of order $\mathcal{K} |\log \hbar|$, where \mathcal{K} will be determined later.
- the isometry will be $\mathcal{U} = U^n$.

To apply Theorem 9.7 we will need an upper bound on

$$c_{\mathcal{T}^{(n)}, \mathcal{P}^{(n)}}(\mathcal{U}) = \max_{|\alpha|=|\alpha'|=n} \|\hat{P}_{\alpha'} U^n \hat{P}_\alpha\|.$$

We remark that $\hat{P}_{\alpha'} U^n \hat{P}_{\alpha}$ can be developed as

$$U^{-n+1} \hat{P}_{\alpha'_{n-1}} U \cdots U \hat{P}_{\alpha'_1} U \hat{P}_{\alpha'_0} U \hat{P}_{\alpha_{n-1}} \cdots U \hat{P}_{\alpha_1} U \hat{P}_{\alpha_0}$$

or equivalently

$$U^n \hat{P}_{\alpha'_{n-1}}(2n-1) \cdots \hat{P}_{\alpha'_1}(n+1) \hat{P}_{\alpha'_0}(n) \hat{P}_{\alpha_{n-1}}(n-1) \cdots \hat{P}_{\alpha_1}(1) \hat{P}_{\alpha_0}.$$

10.4. The main estimate. Let us assume (without loss of generality) that the injectivity radius of X is greater than 1; and that the diameter of each P_k is small enough so that, for every j, k , for every $x, y \in P_j, P_k$, there is at most one unit speed geodesic joining x and y in time 1.

The estimate essentially proven in [A05] is :

THEOREM 10.1. *Let χ be an energy cut-off, that is, a smooth compactly supported function vanishing outside $H^{-1}([1/2 - \varepsilon, 1/2 + \varepsilon])$.*

Given $\mathcal{K} > 0$ and a partition $\mathcal{P}^{(0)}$, there exists $\hbar_{\mathcal{K}, \mathcal{P}^{(0)}, \chi}$ such that, for any $\hbar \leq \hbar_{\mathcal{K}, \mathcal{P}^{(0)}, \chi}$, for any positive integer $n \leq \mathcal{K} |\log \hbar|$, and any pair of sequences α, α' of length n ,

$$(10.3) \quad \|\hat{P}_{\alpha'} U^n \hat{P}_{\alpha} \text{Op}(\chi)\| \leq C \hbar^{-\frac{d}{2}} e^{-(d-1)n} (1 + O(\varepsilon))^n.$$

The constant C only depends on the riemannian manifold.

REMARK 10.2. The best bound we can hope for on the norm of the operators $\hat{P}_{\alpha'} U^n \hat{P}_{\alpha}$ is certainly the trivial one : $c_{\mathcal{T}^{(n)}, \mathcal{P}^{(n)}}(\mathcal{U}) \leq 1$. We can only improve this bound if we insert the energy cut-off $\text{Op}_{\hbar}(\chi)$, which has the effect of restricting our operators to functions oscillating at a certain (high) frequency. In fact, if we take χ of the form $f((2H - 1))$, where f is a smooth function on \mathbb{R} , supported in $[1 - 2\varepsilon, 1 + 2\varepsilon]$, we can take $\text{Op}(\chi)$ to be the corresponding function of the laplacian, obtained by functional calculus, $f((-\hbar^2 \Delta - 1))$. Thus, $\text{Op}(\chi)$ is a smoothed version of the projection to the spectral window $[\frac{1-2\varepsilon}{\hbar^2}, \frac{1+2\varepsilon}{\hbar^2}]$ of the laplacian.

Since we have no good estimate on $\|\hat{P}_{\alpha'} U^n \hat{P}_{\alpha}\|$, but only on $\|\hat{P}_{\alpha'} U^n \hat{P}_{\alpha} \text{Op}(\chi)\|$, we will need to modify accordingly the statement of the entropic uncertainty principle : see later Theorem 10.5

REMARK 10.3. If we were in variable curvature, instead of the exponent $d - 1$ we would have a variable exponent depending on the local Lyapunov exponents.

The proof of Theorem 10.1 will be given in Section 15. The idea is rather simple, although the technicalities may seem difficult. We first show that any state in the image of $\text{Op}(\chi)$ may be decomposed as a superposition of essentially $\hbar^{-\frac{d}{2}}$ normalized lagrangian states, supported on lagrangian manifolds transverse to the stable foliation. The lagrangian states we work with are truncated δ -functions, supported on spheres $S_z^* X$. The action of the operator $U^{n-1} \hat{P}_{\alpha'} U^n \hat{P}_{\alpha} = \hat{P}_{\alpha'_{n-1}} U \cdots U \hat{P}_{\alpha'_0} U \hat{P}_{\alpha_{n-1}} \cdots U \hat{P}_{\alpha_0}$ on such lagrangian states is translated, by the theory of Fourier integral operators (WKB methods), into a classical picture : an application of U corresponds, classically, to applying g^1 , so that it stretches the lagrangian in the unstable direction. Each multiplication by \hat{P}_{α} corresponds to cutting out a small piece of lagrangian. This iteration of stretching and cutting is responsible for the exponential decay.

In [AN07] the estimate of Theorem 10.1 is modified by optimizing the shape of the cutoff χ . We want to take into account the fact that we are dealing with

eigenfunctions of the laplacian, and not merely spectral packets in the window $[\frac{1-2\epsilon}{\hbar^2}, \frac{1+2\epsilon}{\hbar^2}]$. But we also want $\text{Op}(\chi)$ to stay in a reasonable class of pseudodifferential operators. We consider a smooth function $\chi \in C^\infty(\mathbb{R}; [0, 1])$, with $\chi(t) = 1$ for $|t| \leq 1$ and $\chi(t) = 0$ for $|t| \geq 2$. Then, for some fixed $\delta \in (0, 1)$, we rescale that function to obtain an \hbar -dependent cutoff near S^*X :

$$(10.4) \quad \forall \hbar \in (0, 1), \forall n \in \mathbb{N}, \forall \rho \in T^*X, \quad \chi_\delta(\rho; \hbar) \stackrel{\text{def}}{=} \chi(\hbar^{-1+\delta}(H(\rho) - 1/2)).$$

The cutoff χ_δ is localized in a tubular neighbourhood of S^*X of width $2\hbar^{1-\delta}$

THEOREM 10.4. [AN07] *Given $\mathcal{K} > 0$ a partition $\mathcal{P}^{(0)}$ and $\delta > 0$ small enough, there exists $\hbar_{\mathcal{K}, \mathcal{P}^{(0)}, \delta}$ such that, for any $\hbar \leq \hbar_{\mathcal{K}, \mathcal{P}^{(0)}, \delta}$, for any positive integer $n \leq \mathcal{K} \lfloor \log \hbar \rfloor$, and any pair of sequences α, α' of length n ,*

$$(10.5) \quad \|\hat{P}_{\alpha'} U^n \hat{P}_\alpha \text{Op}(\chi_\delta)\| \leq C \hbar^{-\frac{d-1}{2}-\delta} e^{-(d-1)n} (1 + O(\hbar^\delta))^n.$$

The constant C only depends on the riemannian manifold (M, g) .

Theorem 10.4 essentially improves the prefactor $\hbar^{-\frac{d}{2}}$ of Theorem 10.1. Its proof is similar, the main difficulty being to define $\text{Op}(\chi_\delta)$ — the function χ_δ does not fall into one of the usual “nice” classes of symbols, since its derivatives explode quite fast when $\hbar \rightarrow 0$. To define $\text{Op}(\chi_\delta)$ would be much beyond the scope of these notes (see [SZ99, AN07]). We shall admit Theorem 10.4, and only prove the simpler version of Theorem 10.1.

10.5. Technical variant of the entropic uncertainty principle. As explained in Remark 10.2, we cannot apply Theorem 9.7 directly, because we need to insert our energy cut-off $\text{Op}(\chi)$. On the other hand, this frequency cut-off does not really bother us, since it hardly modifies our eigenfunctions.

We generalize the statement of Theorem 9.7 by introducing an auxiliary operator \mathcal{O} .

THEOREM 10.5. [AN07] *Let \mathcal{O} be a bounded operator on \mathcal{H} . Let \mathcal{U} be an isometry on \mathcal{H} .*

Define $c_{\mathcal{O}}^{\tau, \pi}(\mathcal{U}) \stackrel{\text{def}}{=} \sup_{j, k} \|\tau_j^ \mathcal{U} \pi_k \mathcal{O}\|_{\mathcal{L}(\mathcal{H})}$.*

Then, for any $\theta \geq 0$, for any normalized $\psi \in \mathcal{H}$ satisfying

$$\forall k = 1, \dots, \mathcal{N}, \quad \|(Id - \mathcal{O})\pi_k^* \psi\| \leq \theta,$$

the entropies $h_\tau(\mathcal{U}\psi)$, $h_\pi(\psi)$ satisfy

$$h_\tau(\mathcal{U}\psi) + h_\pi(\psi) \geq -2 \log(c_{\mathcal{O}}^{\pi, \tau}(\mathcal{U}) + \mathcal{N} \theta).$$

10.6. Applying the entropic uncertainty principle. We now precise all the data we will use in the entropic uncertainty principle, Theorem 10.5:

- the quantum partition $\pi = \mathcal{P}^{(n)}$, $\tau = \mathcal{T}^{(n)}$ have already been defined in 10.3. The integer n will be of order $\mathcal{K} \lfloor \log \hbar \rfloor$, where the choice of \mathcal{K} will be determined later. In the semiclassical limit, these partitions have cardinality $\mathcal{N} = K^n \asymp \hbar^{-K_0}$ for some fixed $K_0 > 0$.
- the isometry will be $\mathcal{U} = U^n$.
- the operator \mathcal{O} is $\mathcal{O} = \text{Op}(\chi_\delta)$. Since we are, at the end, interested in eigenfunctions of the laplacian, we need to know that this operator hardly

modifies them. In fact, for any $L > 0$, there exists \hbar_L such that, for any $\hbar \leq \hbar_L$, any solution of $(-\hbar^2 \Delta - I)\psi_\hbar = 0$ satisfies

$$(10.6) \quad \forall \alpha, |\alpha| = n \leq \mathcal{K} |\log \hbar|, \quad \|(\text{Op}(\chi_\delta) - Id)\hat{P}_\alpha^* \psi_\hbar\| \leq \hbar^L \|\psi_\hbar\|.$$

This means that, for an eigenfunction ψ_\hbar , all the states $\hat{P}_\alpha^* \psi_\hbar$ are very sharply microlocalized near the energy layer S^*X .

- $\theta = \hbar^L$, and L will be chosen very large.

All these quantities are defined for $n = \mathcal{K} |\log \hbar|$, \mathcal{K} will be determined later, but fixed.

The entropies associated with a state $\psi \in \mathcal{H}$ are given by

$$h_{\mathcal{P}^{(n)}}(\psi) = - \sum_{|\alpha|=n} \|\hat{P}_\alpha^* \psi\|^2 \log (\|\hat{P}_\alpha^* \psi\|^2)$$

and

$$h_{\mathcal{T}^{(n)}}(\psi) = - \sum_{|\alpha|=n} \|\hat{P}_\alpha \psi\|^2 \log (\|\hat{P}_\alpha \psi\|^2).$$

We may apply Theorem 10.5 to any sequence of states satisfying (10.6).

COROLLARY 10.6. *Define*

$$(10.7) \quad c_{\text{Op} \chi_\delta}(U^n) \stackrel{\text{def}}{=} \max_{|\alpha|=|\alpha'|=n} \|\hat{P}_{\alpha'} U^n \hat{P}_\alpha \text{Op}(\chi_\delta)\|.$$

Then for any normalized state ϕ satisfying (10.6),

$$h_{\mathcal{T}^{(n)}}(U^n \phi) + h_{\mathcal{P}^{(n)}}(\phi) \geq -2 \log (c_{\text{Op} \chi_\delta}(U^n) + h^{L-K_0}).$$

We now apply Corollary 10.6 to the particular case of the eigenstates ψ_\hbar . The estimate (10.5) can be rewritten as

$$c_{\text{Op} \chi_\delta}(U^n) \leq C \hbar^{-\frac{d-1}{2}-\delta} e^{-(d-1)n} (1 + O(\hbar^\delta))^n.$$

We choose L large enough such that \hbar^{L-K_0} is negligible in comparison with $\hbar^{-\frac{d-1}{2}-\delta} e^{-(d-1)n}$.

PROPOSITION 10.7. *Let $(\psi_\hbar)_{\hbar \rightarrow 0}$ be any sequence of eigenstates (7.2). Then, in the semiclassical limit, we have*

$$(10.8) \quad h_{\mathcal{T}^{(n)}}(\psi_\hbar) + h_{\mathcal{P}^{(n)}}(\psi_\hbar) \geq 2(d-1)n + (d-1+2\delta) \log \hbar + \mathcal{O}(1).$$

This holds for $n \leq \mathcal{K} |\log \hbar|$ (\mathcal{K} arbitrary) and $\hbar \leq \hbar_{\mathcal{K}, \mathcal{P}^{(0)}, \delta}$.

11. ...and the conclusion.

Before taking the limit $\hbar \rightarrow 0$, we prove that a similar lower bound holds if we replace $n \asymp |\log \hbar|$ by some fixed n_o , and $\mathcal{P}^{(n)}$ by the corresponding partition $\mathcal{P}^{(n_o)}$. Proposition 11.1 below is the semiclassical analogue of the classical subadditivity (8.3) of entropy for invariant measures.

We introduce the Ehrenfest time $n_E(\hbar) = \frac{(1-\delta') |\log \hbar|}{\lambda_{\max}}$ (δ' fixed, arbitrarily small). In constant curvature -1 , the maximal expansion rate of the geodesic flow on S^*X is $\lambda_{\max} = 1$. The Ehrenfest time is the main limitation to use semiclassical methods to understand the large time behaviour of the Schrödinger flow : roughly speaking, we have $U^{-t} \text{Op}_\hbar(a) U^t \sim \text{Op}(a \circ g^t)$ for $|t| \leq \frac{n_E(\hbar)}{2}$, but for larger t we can no longer refer to the classical dynamics to understand $U^{-t} \text{Op}_\hbar(a) U^t$.

PROPOSITION 11.1 (Subadditivity). *Let $\delta' > 0$. There is a function $R(n_o, \hbar)$ such that, for all integer n_o ,*

$$\lim_{\hbar \rightarrow 0} |R(n_o, \hbar)| = 0$$

and such that, for all $n_o, n \in \mathbb{N}$ with $n_o + n \leq n_E(\hbar)$, for any (ψ_{\hbar}) normalized eigenstates satisfying (7.2), the following inequality holds:

$$h_{\mathcal{P}(n_o+n)}(\psi_{\hbar}) \leq h_{\mathcal{P}(n_o)}(\psi_{\hbar}) + h_{\mathcal{P}(n)}(\psi_{\hbar}) + R(n_o, \hbar).$$

We do not prove this proposition in these notes, but just make a few more comments. The non-commutative dynamical system formed by (U^t) acting on pseudodifferential operators is (approximately) commutative on time intervals of length $n_E(\hbar)$:

$$\|[\text{Op}_{\hbar}(a)(t), \text{Op}_{\hbar}(b)(-t)]\|_{L^2(X)} = \mathcal{O}(\hbar^{c\delta'}),$$

for any time $|t| \leq \frac{n_E(\hbar)}{2}$, or equivalently (using the unitarity of U^t)

$$\|[\text{Op}_{\hbar}(a)(t), \text{Op}_{\hbar}(b)]\|_{L^2(X)} = \mathcal{O}(\hbar^{c\delta'}),$$

for any time $|t| \leq n_E(\hbar)$. On such a time interval, we almost have a commutative dynamical system, up to small errors tending to 0 with \hbar . This roughly explains why the quantum entropy $h_{\mathcal{P}(n_o+n)}(\psi_{\hbar})$ has the same subadditivity property as the classical entropy (8.3), up to small errors, as long as $n_o + n$ remains bounded by the Ehrenfest time.

Thanks to this subadditivity, we may finish the proof of Theorem 7.1. Although Proposition 10.7 holds for $n \leq \mathcal{K}|\log \hbar|$ and \mathcal{K} arbitrary, we are now limited by Proposition 11.1 to $\mathcal{K} = \frac{1-\delta'}{\lambda_{\max}}$. For $n = n_E(\hbar)$, Proposition 10.7 can be written

$$(11.1) \quad h_{\mathcal{P}(n)}(\psi_{\hbar}) + h_{\mathcal{T}(n)}(\psi_{\hbar}) \geq 2(d-1)n - \frac{(d-1+2\delta)\lambda_{\max}}{(1-\delta')}n + \mathcal{O}(1).$$

Let $n_o \in \mathbb{N}$ be fixed and $n = n_E(\hbar)$. Using the Euclidean division $n = qn_o + r$ (with $r \leq n_o$), Proposition 11.1 implies that for \hbar small enough,

$$\frac{h_{\mathcal{P}(n)}(\psi_{\hbar})}{n} \leq \frac{h_{\mathcal{P}(n_o)}(\psi_{\hbar})}{n_o} + \frac{h_{\mathcal{P}(r)}(\psi_{\hbar})}{n} + \frac{R(n_o, \hbar)}{n_o}.$$

A similar inequality holds with \mathcal{P} replaced by \mathcal{T} .

Using (10.8) and the fact that $h_{\mathcal{P}(r)}(\psi_{\hbar})$ stays uniformly bounded (by a quantity depending on n_o) when $\hbar \rightarrow 0$, we find

$$(11.2) \quad \frac{1}{2} \left[\frac{h_{\mathcal{P}(n_o)}(\psi_{\hbar})}{n_o} + \frac{h_{\mathcal{T}(n_o)}(\psi_{\hbar})}{n_o} \right] \geq (d-1) - \frac{(d-1+2\delta)\lambda_{\max}}{2(1-\delta')}n + \mathcal{O}(1) - \frac{R(n_o, \hbar)}{n_o} + \mathcal{O}_{n_o}(1/n).$$

We are now dealing with the partition $\mathcal{P}^{(n_o)}$, n_o being fixed.

11.1. End of the proof. Let us take a subsequence of (ψ_{\hbar_k}) such that the Husimi measures $\mu_k = \mu_{\psi_{\hbar_k}}$ converge to a semiclassical measure μ on S^*X , invariant under the geodesic flow (see Prop. 6.1). We may take the limit $\hbar_k \rightarrow 0$ (so

that $n \rightarrow \infty$) in the expression above. The norms appearing in the definition of $h_{\mathcal{P}(n_o)}(\psi_{\hbar_k})$ and $h_{\mathcal{T}(n_o)}(\psi_{\hbar_k})$ can be written as

$$(11.3) \quad \|\hat{P}_{\alpha} \psi_{\hbar_k}\| = \|\hat{P}_{\alpha_{n_o}}(n_o) \cdots \hat{P}_{\alpha_1}(1) \hat{P}_{\alpha_0} \psi_{\hbar_k}\|$$

$$(11.4) \quad \|\hat{P}_{\alpha}^* \psi_{\hbar_k}\| = \|\hat{P}_{\alpha_0} \hat{P}_{\alpha_1}(1) \cdots \hat{P}_{\alpha_{n_o}}(n_o) \psi_{\hbar_k}\|.$$

For any sequence α of length n_o , the laws of pseudodifferential calculus imply the convergence of $\|\hat{P}_{\alpha}^* \psi_{\hbar_k}\|^2$ and $\|\hat{P}_{\alpha} \psi_{\hbar_k}\|^2$ to the same quantity $\mu(\{\alpha\})$, where $\{\alpha\}$ is the function $(\mathbb{1}_{P_{\alpha_0}}^{sm})^2 (\mathbb{1}_{P_{\alpha_1}}^{sm})^2 \circ g^1 \cdots (\mathbb{1}_{P_{\alpha_{n_o-1}}}^{sm})^2 \circ g^{n_o-1}$ on T^*X . Thus $h_{\mathcal{P}(n_o)}(\psi_{\hbar_k})$ and $h_{\mathcal{T}(n_o)}(\psi_{\hbar_k})$ both semiclassically converge to the classical entropy

$$h_{n_o}(\mu) = h_{n_o}(\mu, (\mathbb{1}_{P_k}^{sm^2})) = - \sum_{|\alpha|=n_o} \mu(\{\alpha\}) \log \mu(\{\alpha\}).$$

We have thus obtained the lower bound

$$(11.5) \quad \frac{h_{n_o}(\mu)}{n_o} \geq (d-1) - \frac{(d-1+2\delta)\lambda_{\max}}{2(1-\delta')}.$$

δ and δ' could be taken arbitrarily small, and at this stage they can be let vanish. Remember also that $\lambda_{\max} = 1$ for constant sectional curvature -1 .

The Kolmogorov–Sinai entropy of μ (with respect to the partition $X = \sqcup P_k$) is by definition the limit of the first term $\frac{h_{n_o}(\mu)}{n_o}$ when n_o goes to infinity (8.2) (8.4), with the notable difference that the smooth functions $(\mathbb{1}_{P_k}^{sm})^2$ should be replaced by the characteristic functions $(\mathbb{1}_{P_k})$. We note, however, that the lower bound (11.5) does not depend on the derivatives of $(\mathbb{1}_{P_k}^{sm})^2$: as a result, the same bound carries over to the characteristic functions $(\mathbb{1}_{P_k})$.

We can finally let n_o tend to $+\infty$, to obtain (7.4). □

The proof is finished, save for Theorem 10.1.

WKB METHODS.

To prove our main estimate (Theorem 10.1), we need to describe the action of the operator $U^t = \exp(it\hbar\frac{\Delta}{2})$ on “rapidly oscillating” functions, in the limit $\hbar \rightarrow 0$. The idea, already used by Schrödinger, is to describe the action of $\exp(it\hbar\frac{\Delta}{2})$ on functions of the form $e^{\frac{i}{\hbar}S(x)}$, called WKB functions or lagrangian functions; and to use the fact that all the functions we consider can be represented as integral combinations of lagrangian functions.

12. Lagrangian submanifolds of T^*X and generating functions.

We have seen that T^*X is endowed with a “canonical” symplectic form ω , defined as follows. Let $\Omega \subset X$ be an open subset of X , endowed with a coordinate chart $\phi : X \rightarrow \mathbb{R}^d$. Then $T^*\Omega \subset T^*X$ can be endowed with the coordinate chart

$$(12.1) \quad \Phi : T^*\Omega \rightarrow \mathbb{R}^d \times (\mathbb{R}^d)^*$$

$$(12.2) \quad (x, p) \mapsto (\phi(x), (d\phi_x^*)^{-1}p).$$

On $T^*\Omega$, ω is defined as the pullback by Φ of the symplectic form $\sum_{i=1}^d dq_i \wedge dp_i$ of $\mathbb{R}^d \times (\mathbb{R}^d)^*$. We leave it to the reader to check that this definition does not depend on the choice of local coordinates. Thus, by choosing an atlas of X , one can define

ω on T^*X , and the definition does not depend on the atlas. In fact, ω can also be defined in an intrinsic way, as was done in Section 1 by formula (1.5).

DEFINITION 12.1. In the $2d$ -dimensional symplectic manifold (T^*X, ω) , a lagrangian submanifold is a d -dimensional submanifold on which the restriction of ω vanishes.

Equivalently, a submanifold $\mathcal{L} \subset T^*X$ is lagrangian if and only if, for all $\rho \in \mathcal{L}$, $T_\rho \mathcal{L}$ is its own ω -orthogonal in $T_\rho(T^*X)$.

EXAMPLE 12.2. On $T^*\mathbb{R}^d = \mathbb{R}^d \times (\mathbb{R}^d)^*$ endowed with the symplectic form $\sum_{i=1}^d dq_i \wedge dp_i$, affine subspaces of the form $\mathbb{R}^d \times \{\xi_0\}$ or $\{x_0\} \times \mathbb{R}^d$ are examples of lagrangian submanifolds. More generally, for any manifold X , the zero section $\{(x, 0), x \in X\} \subset T^*X$ is a lagrangian submanifold of T^*X . For any $x \in X$, the fiber T_x^*X is also lagrangian.

Generating functions.

EXERCISE 12.3. In $\mathbb{R}^d \times (\mathbb{R}^d)^*$ endowed with the symplectic form $\sum_{i=1}^d dq_i \wedge dp_i$, consider an linear subspace of the form $\text{Graph} A = \{(x, Ax)\}$ where A is a linear operator from \mathbb{R}^d to itself. Show that $\text{Graph} A$ is lagrangian if and only if A is symmetric for the canonical euclidean structure on \mathbb{R}^d : $\langle Ax, y \rangle = \langle x, Ay \rangle$.

The following gives us many examples of lagrangian submanifolds.

EXERCISE 12.4. Let X be a smooth manifold and consider T^*X endowed with its usual symplectic structure. Let $\Omega \subset X$ be an open subset of X , and let a be a smooth 1-form on Ω . Consider the graph $\text{Graph} a = \{(x, a_x)\} \subset T^*\Omega$. Show that $\text{Graph} a$ is lagrangian if and only if the 1-form a is closed: $da = 0$.

In particular, if Ω is simply connected, this implies the existence of a smooth function $S : \Omega \rightarrow \mathbb{R}$ such that $a = dS$. The function S is called a generating function of the lagrangian manifold $\text{Graph} a$.

We denote $\pi : T^*X \rightarrow X$ the canonical projection.

DEFINITION 12.5. Let $\mathcal{L} \subset T^*X$ be a lagrangian submanifold. The *caustic* of \mathcal{L} is the set of points $\rho \in \mathcal{L}$ such that the restriction of π to \mathcal{L} is not a local diffeomorphism at ρ .

In Example 12.2, the zero section in T^*X has empty caustic, whereas the case $\mathcal{L} = T_x^*X$ gives an example of a lagrangian submanifold for which the caustic is all of \mathcal{L} .

If ρ does not belong to the caustic, Exercise 12.4 shows there is a neighbourhood of ρ in which \mathcal{L} is the graph of the differential dS , for some function S defined locally up to an additive constant. We say S is a generating function of \mathcal{L} near ρ .

What happens on the caustic?

Let $S(x, \theta)$ be a real-valued function on $\Omega_X \times \Omega_{\mathbb{R}^N}$ where Ω_X is an open subset of X and $\Omega_{\mathbb{R}^N}$ an open subset of \mathbb{R}^N . Let $C_S = \{(x, \theta), \frac{\partial S}{\partial \theta} = 0\}$. On C_S we assume that all the differentials $d_{(x, \theta)} \frac{\partial S}{\partial \theta_i}$ ($i = 1, \dots, N$) are linearly independent: then C_S is a smooth d -dimensional submanifold of $\Omega_X \times \Omega_{\mathbb{R}^N}$ (recall $d = \dim X$). Define $j_S : C_S \rightarrow T^*X$ by $j_S(x, \theta) = (x, \partial_x S(x, \theta))$.

PROPOSITION 12.6. *The map j_S is an immersion. Its image,*

$$\mathcal{L}_S = \{(x, \xi) \in T^*X, \text{ there exists } \theta \text{ with } \partial_\theta S(x, \theta) = 0 \text{ and } \xi = \partial_x S(x, \theta)\}$$

is a lagrangian submanifold of T^*X .

One calls $S(., \theta)$ a generating family (or generating function) of \mathcal{L}_S . Compare with (1.24).

THEOREM 12.7. *Every lagrangian submanifold \mathcal{L} of T^*X admits, locally, a generating family. More precisely : for $\rho_0 \in \mathcal{L}$, let $N = \dim \text{Ker } d\pi_{\rho_0}$. Then there is a neighbourhood Ω of ρ_0 in T^*X , there are open subsets $\Omega_X \subset X$ and $\Omega_{\mathbb{R}^N} \subset \mathbb{R}^N$, and a function $S : \Omega_X \times \Omega_{\mathbb{R}^N} \rightarrow \mathbb{R}$ satisfying all the required conditions, such that*

$$\mathcal{L} \cap \Omega = \mathcal{L}_S.$$

Proof: Using (12.1) we see it is enough to consider the case $X = \mathbb{R}^d$. Let $\rho_0 = (x_0, \xi_0) \in T^*\mathbb{R}^d = \mathbb{R}^d \times (\mathbb{R}^d)^*$ and let $L = T_{\rho_0}\mathcal{L}$. It is a lagrangian linear subspace of $\mathbb{R}^d \times (\mathbb{R}^d)^*$. Let $\pi : \mathbb{R}^d \times (\mathbb{R}^d)^* \rightarrow \mathbb{R}^d$ be the projection on the first coordinate. By assumption, $F = \pi(L)$ is a linear subspace of \mathbb{R}^d of dimension $d - N$. Let G be a supplementary subspace of F in $\mathbb{R}^d : \mathbb{R}^d = F \oplus G$. We have a corresponding decomposition of the dual space, $\mathbb{R}_d^* = G^\circ \oplus F^\circ$, where F° is the space of linear forms vanishing on F , and similarly for G° . We leave it to the reader to check that the projection $P : L \rightarrow F \times F^\circ$ is an isomorphism.

Since L is tangent to \mathcal{L} at ρ_0 , there is a neighbourhood Ω of ρ_0 such that $P : \mathcal{L} \rightarrow F \times F^\circ$ is a diffeomorphism. In other words, there is a smooth map $\varphi : (F \times F^\circ) \cap \Omega \rightarrow G \times G^\circ$ such that $\mathcal{L} \cap \Omega$ is the graph of φ . Writing $\varphi = (f, g)$ we have

$$\mathcal{L} \cap \Omega = \{(x_F, f(x_F, \xi_{F^\circ}), g(x_F, \xi_{F^\circ}), \xi_{F^\circ}), x_F \in F, \xi_{F^\circ} \in F^\circ\}.$$

For \mathcal{L} to be lagrangian we must have $df \wedge d\xi_{F^\circ} + dx_F \wedge dg = 0$, in other words $d(fd\xi_{F^\circ} - gdx_F) = 0$. This means there exists, in a neighbourhood of ρ_0 , a function $S(x_F, \xi_{F^\circ})$ such that $dS = fd\xi_{F^\circ} - gdx_F$ (equivalently, $f = \partial_{\xi_{F^\circ}} S, g = -\partial_{x_F} S$). Consider the function

$$\mathcal{S}(x_F, x_G, \xi_{F^\circ}) = \xi_{F^\circ} \cdot x_G - S(x_F, \xi_{F^\circ})$$

defined on an open subset of $F \times G \times F^\circ = \mathbb{R}^d \times F^\circ$. It is now straightforward to check that this is a generating function of $\mathcal{L} \cap \Omega$, and $\dim F^\circ = N$ as announced. \square

EXAMPLE 12.8. Here is a fundamental example : in $T^*\mathbb{R}^d = \mathbb{R}^d \times (\mathbb{R}^d)^*$ endowed with the canonical symplectic form $\sum_{i=1}^d dq_i \wedge dp_i$, a generating function for $T_x^*\mathbb{R}^d = \{x\} \times (\mathbb{R}^d)^*$ is $S(y, \theta) = \sum_{i=1}^d \theta_i(y_i - x_i) = \langle \theta, y - x \rangle$ (here $N = d$ and θ varies in \mathbb{R}^d).

EXERCISE 12.9. A crucial thing : (i) Show that a (connected) lagrangian submanifold \mathcal{L} is invariant under the hamiltonian flow of H if and only if it is contained in some fixed energy layer $\{H = E\}$.

(ii) As a particular case, deduce that if $H(x, d_x S(x, \theta)) = E$ for any (x, θ) , then the lagrangian manifold \mathcal{L}_S generated by S is invariant under the hamiltonian flow (ϕ_H^t) .

(iii) Assume now that S is a smooth function of (t, x, θ) , and assume that

$$\frac{\partial S}{\partial t} + H(x, d_x S) = 0$$

for all (t, x, θ) . Denote $S_t(x, \theta) = S(t, x, \theta)$. Show that the lagrangian manifold \mathcal{L}_{S_t} is the image of \mathcal{L}_{S_0} under $\phi_{\overline{H}}^t$. *Hint : reduce the problem to the previous one by considering the hamiltonian $\overline{H}(x, t, \xi, E) = H(x, \xi) + E$ on $T^*(X \times \mathbb{R})$.*

13. Lagrangian distributions.

Let \mathcal{L} be a lagrangian submanifold of T^*X . For our applications we shall only be interested in the case where \mathcal{L} is relatively compact and where it has a global generating function $S : \mathcal{L} = \mathcal{L}_S$, S being defined on $\Omega_X \times \Omega_{\mathbb{R}^N}$. In this case we define the (semiclassical) notion of a lagrangian function associated to \mathcal{L} as follows :

DEFINITION 13.1. We denote $O^m(X, \mathcal{L}_S)$ the space of functions of the form

$$u_{\hbar}(x) = \frac{e^{i\alpha(\hbar)}}{(2\pi\hbar)^{N/2}} \int_{\Omega_{\mathbb{R}^N}} e^{i\frac{S(x,\theta)}{\hbar}} a_{\hbar}(x, \theta) d\theta$$

where

- $\alpha(\hbar)$ is a real number that may depend on \hbar ,
- the function a defined on $\Omega_X \times \Omega_{\mathbb{R}^N}$ is smooth and has an asymptotic development when $\hbar \rightarrow 0$,

$$a \sim \sum_{j=0}^{\infty} \hbar^{j+m} a_{j+m},$$

the asymptotic development holds in all C^k -norms on compact subsets,

- we assume that a is compactly supported with respect to the variable θ .

As usual, the class $O^m(X, \mathcal{L}_S)$ should actually be defined *modulo negligible functions*, which, we recall, are smooth functions u_{\hbar} for which all the C^k -norms on compact sets are $\mathcal{O}(\hbar^\infty)$. Then, one can prove [GS94] that the definition of $O^m(X, \mathcal{L}_S)$ does not depend on the choice of the generating function S :

THEOREM 13.2. *If $\mathcal{L}_S = \mathcal{L}_{S'}$ then $O^m(X, \mathcal{L}_S) = O^m(X, \mathcal{L}_{S'})$.*

EXAMPLE 13.3. On $X = \mathbb{R}^d$, the Dirac mass at x

$$\delta_x(y) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{\frac{\langle \xi, y-x \rangle}{\hbar}} d\xi$$

can be seen as a lagrangian distribution associated with the lagrangian submanifold $T_x^*\mathbb{R}^d$ ($S(y, \xi) = \langle \xi, y-x \rangle$), save for the fact that the symbol $a(y, \xi) \equiv 1$ is not compactly supported. Let $\chi(y, \xi)$ be a smooth, positive, compactly supported function, call Ω a bounded open set containing the support of χ . Then the function

$$\delta_x^\chi(y) = \frac{1}{(2\pi\hbar)^d} \int e^{\frac{\langle \xi, y-x \rangle}{\hbar}} \chi(y, \xi) d\xi$$

falls into the class $O^{-d/2}(\mathbb{R}^d, T_x^*\mathbb{R}^d \cap \Omega)$. Assume χ takes the constant value 1 in a neighbourhood of a certain compact subset $\mathcal{E} \subset T_x^*\mathbb{R}^d$. Then δ_x^χ is often called a “delta-function truncated away from \mathcal{E} ” : it is a Dirac mass whose frequencies near \mathcal{E} have not been touched, while the frequencies out of Ω have been suppressed.

14. WKB description of the operator $U^t = \exp(i\hbar\frac{\Delta}{2})$.

REMARK 14.1. The initials WKB stand for Wentzel, Kramers and Brillouin, who independently proposed this method to find approximate solutions of a 1-d *stationary* Schrödinger equation — in other words, to find approximate eigenfunctions [Wtz26, Kr26, Brill26]. The method was later generalized by Keller and Maslov to find approximate eigenfunctions (quasimodes) of higher dimensional, completely integrable systems [Kell58, Masl65].

Here we present the WKB method applied to the *evolutionary* Schrödinger equation. It was first used by Van Vleck [VV28].

Recall that we are interested in the hamiltonian flow generated by $H(x, \xi) = \frac{\|\xi\|_x^2}{2}$, namely the geodesic flow, denoted g^t in Section 6. We wish to study the Schrödinger flow $U^t = e^{i\hbar\frac{\Delta}{2}}$ and to relate it to the geodesic flow as $\hbar \rightarrow 0$.

Consider an initial state $u(0)$ of the form $u(0, x) = a_{\hbar}(0, x) e^{\frac{i}{\hbar}S(0, x)}$, where $S(0, \bullet)$, $a_{\hbar}(0, \bullet)$ are smooth functions defined on a subset of $\Omega \subset X$, a_{\hbar} has a fixed compact support in Ω and has an asymptotic development $a_{\hbar} \sim \sum_k \hbar^k a_k$, valid in all C^n -norms. This represents a WKB (or lagrangian) state, supported on the lagrangian manifold $\mathcal{L}(0) = \{(x, d_x S(0, x)), x \in \Omega\}$.

The WKB method consists in looking for an approximate expression⁹ for the state $\tilde{u}(t) \stackrel{\text{def}}{=} U^t u(0)$, in the form

$$(14.1) \quad u(t, x) = e^{\frac{iS(t, x)}{\hbar}} a_{\hbar}(t, x) = e^{\frac{iS(t, x)}{\hbar}} \sum_{k=0}^{N-1} \hbar^k a_k(t, x)$$

where N is a fixed, arbitrarily large integer. We want $u(t)$ to solve $\frac{\partial u}{\partial t} = i\hbar\frac{\Delta_x u}{2}$ up to a remainder of order \hbar^N . Computing explicitly both sides of the equation, and identifying the successive powers of \hbar , we see that the functions S and a_k must satisfy the following partial differential equations:

$$(14.2) \quad \begin{cases} \frac{\partial S}{\partial t} + H(x, d_x S) = 0 & \text{(Hamilton-Jacobi equation)} \\ \frac{\partial a_0}{\partial t} = -\langle d_x a_0, d_x S(t, x) \rangle - a_0 \frac{\Delta_x S(t, x)}{2} & \text{(0-th transport equation)}, \\ \frac{\partial a_k}{\partial t} = \frac{i\Delta a_{k-1}}{2} - \langle da_k, dS \rangle - a_k \frac{\Delta S}{2} & \text{(k-th transport equation)}. \end{cases}$$

Assume that, on a certain time interval — say $s \in [0, 1]$ — the above equations have a well defined smooth solution $S(s, x)$, meaning that the transported lagrangian manifold $\mathcal{L}(s) = \phi_H^s \mathcal{L}(0)$ is of the form $\mathcal{L}(s) = \{(x, d_x S(s, x))\}$, where $S(s)$ is a smooth function defined on the open set $\pi\mathcal{L}(s)$. Under these conditions, we denote as follows the induced flow on X :

$$(14.3) \quad G_s^t : x \in \pi\mathcal{L}(s) \mapsto \pi g^{t-s}(x, d_x S(s, x)) \in \pi\mathcal{L}(t),$$

In the first chapter we introduced the *exponential map* associated to \mathcal{L}_s : we have $G_s^t = \exp_{\mathcal{L}_s}^{t-s} \circ \pi^{-1}$. Note that $G_s^s = I$ and that the following composition rule holds : $G_{t_1}^{t_2} \circ G_{t_0}^{t_1} = G_{t_0}^{t_2}$.

⁹often called an Ansatz

We then introduce the following unitary operator T_s^t , which transports functions on $\pi\mathcal{L}(s)$ into functions on $\pi\mathcal{L}(t)$:

$$(14.4) \quad T_s^t(a)(x) = a \circ G_t^s(x) J_t^s(x)^{1/2}.$$

Here $J_t^s(x)$ is the Jacobian of the map G_t^s at the point x , measured with respect to the riemannian volume on X . It is given by

$$(14.5) \quad J_s^t(x) = \exp \left\{ \int_s^t \Delta S(\tau, G_s^\tau(x)) d\tau \right\}.$$

We leave it as an exercise to check this formula, and to deduce that the 0-th transport equation in (14.2) is explicitly solved by

$$(14.6) \quad a_0(t) = T_0^t a_0, \quad t \in [0, 1].$$

The higher-order terms $k \geq 1$ are given by

$$(14.7) \quad a_k(t) = T_0^t a_k + \int_0^t T_s^t \left(\frac{i \Delta a_{k-1}(s)}{2} \right) ds.$$

The function $u(t, x)$ defined by (14.1) satisfies the approximate equation

$$\frac{\partial u}{\partial t} = i\hbar \frac{\Delta u}{2} - i\hbar^N e^{\frac{i}{\hbar} S(t, x)} \frac{\Delta a_{N-1}}{2}(t, x).$$

By Duhamel's principle, the difference between $u(t)$ and the exact solution $\tilde{u}(t)$ is

$$u(t) - \tilde{u}(t) = -i\hbar^N \int_0^t U^{t-s} \left(e^{\frac{i}{\hbar} S(s, x)} \frac{\Delta a_{N-1}}{2}(s, x) \right) ds,$$

and from the unitarity of U^t , this is bounded, for $t \in [0, 1]$, by

$$(14.8) \quad \|u(t) - \tilde{u}(t)\|_{L^2} \leq \frac{\hbar^N}{2} \int_0^t \|\Delta a_{N-1}(s)\|_{L^2} ds \leq C t \hbar^N \left(\sum_{k=0}^{N-1} \|a_k(0)\|_{C^{2(N-k)}} \right).$$

The constant C is controlled by the volumes of the sets $\pi\mathcal{L}(s)$ ($0 \leq s \leq t \leq 1$), and by a certain number of derivatives of the flow G_t^s ($0 \leq s \leq t \leq 1$).

REMARK 14.2. Elaborating on these methods, one proves the following : if u is a lagrangian state in $O^m(X, \mathcal{L})$, then $U^t u$ is a lagrangian state in $O^m(X, g^t \mathcal{L})$. We have proved it in the particular case when $g^t \mathcal{L}$ is a graph over X for all t . The operator U^t is called a *Fourier Integral Operator* associated with the transformation g^t .

This is the property Schrödinger had looked for when introducing his equation. We have, in addition, found the explicit formula for all the $a_k(t)$. For $k = 0$, equation (14.6) is called the Van Vleck formula.

15. Proof of the main estimate.

15.1. Decomposition of $\text{Op}(\chi)u$ into truncated delta-functions. We can now prove Theorem 10.1, which estimates the norm of the operator

$$\hat{P}_{\alpha_n}(n) \hat{P}_{\alpha_{n-1}}(n-1) \dots \hat{P}_{\alpha_0} \text{Op}(\chi) = U^{-n} \hat{P}_{\alpha_n} U \hat{P}_{\alpha_{n-1}} \dots U \hat{P}_{\alpha_0} \text{Op}(\chi)$$

(where we denote $U^t = \exp(i\hbar t \frac{\Delta}{2})$ and $U = U^1$). Since U^t is unitary, the norm of this operator is also the same as the norm of $\hat{P}_{\alpha_n} U \hat{P}_{\alpha_{n-1}} \dots U \hat{P}_{\alpha_0} \text{Op}(\chi)$.

The pseudo-differential operator $\text{Op}(\chi)$ is defined in §4.3 :

$$\text{Op}(\chi) = \sum_l \varphi_l \text{OP}(\chi) \varphi_l$$

where (φ_l) is an auxiliary partition of unity on X (i.e. $\sum_l \varphi_l(x)^2 \equiv 1$) such that the support of each φ_l is endowed with local coordinates in \mathbb{R}^d . In local coordinates in the support of φ_l , $\text{OP}(\chi)$ is then defined by the usual formula,

$$(15.1) \quad \text{OP}(\chi)u(x) = (2\pi\hbar)^{-d} \int u(z) e^{i\frac{\langle \xi, x-z \rangle}{\hbar}} \chi(z, \xi) dz d\xi.$$

The function χ will be chosen of the form $\chi(z, \xi) = \chi_1(|\xi|_z)$ where χ_1 is a smooth function on \mathbb{R}_+ supported in $[1 - \varepsilon/2, 1 + \varepsilon/2]$ with $\chi_1 \equiv 1$ in a neighbourhood of 1. For $x \in \Omega_{\alpha_0}$, we can write

$$(15.2) \quad \text{Op}(\chi)u(x) = \sum_l \int u(z) \delta_z^l(x) dz,$$

where we denote δ_z^l the truncated δ -function

$$(15.3) \quad \delta_z^l(x) = \varphi_l(x) \varphi_l(z) \int e^{i\frac{\langle \xi, x-z \rangle}{\hbar}} \chi(z, \xi) \frac{d\xi}{(2\pi\hbar)^d}.$$

Each δ_z^l is a lagrangian state associated with the lagrangian manifold $T_z^*X \cap H^{-1}((\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon))$. Equation (15.2) means that every state in the image of $\text{Op}(\chi)$ can be decomposed as an integral combination of the lagrangian states δ_z^l . We shall first estimate the norm of $\hat{P}_{\alpha_n} U \hat{P}_{\alpha_{n-1}} \dots U \hat{P}_{\alpha_0} \delta_z^l$ for any z , and then use (15.2) to write, for an arbitrary function u ,

$$\begin{aligned} \|\hat{P}_{\alpha_n} U \hat{P}_{\alpha_{n-1}} \dots U \hat{P}_{\alpha_0} \text{Op}(\chi)u\| &\leq \sum_l \sup_z \|\hat{P}_{\alpha_n} U \hat{P}_{\alpha_{n-1}} \dots U \hat{P}_{\alpha_0} \delta_z^l\| \int_X |u(y)| dy \\ &\leq \sum_l \sup_z \|\hat{P}_{\alpha_n} U \hat{P}_{\alpha_{n-1}} \dots U \hat{P}_{\alpha_0} \delta_z^l\| \sqrt{\text{Vol } X} \|u\|_{L^2(X)} \end{aligned}$$

The estimates will be done by induction on n : we will propose an Ansatz – that is, an approximate expression – for $\hat{P}_{\alpha_n} U \hat{P}_{\alpha_{n-1}} \dots U \hat{P}_{\alpha_0} \delta_z^l$, valid for “large” n . In what follows we omit the l superscript and just write δ_z .

15.2. The Ansatz for $n = 1$. At $n = 0$ we know that $\hat{P}_{\alpha_0} \delta_z(x)$ is a lagrangian state associated with the lagrangian manifold $\mathcal{L}^0 = T_z^*X \cap H^{-1}((\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon))$, a union of spheres $H^{-1}(\frac{1}{2} + \eta) \cap T_z^*X$.

From Remark 14.2, we know that $U^t \hat{P}_{\alpha_0} \delta_z$ is a lagrangian state associated to

$$\mathcal{L}^0(t) = g^t \left(T_z^*X \cap H^{-1}((\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)) \right).$$

If we assume that the injectivity radius of X is greater than $1 + 100\varepsilon$, then this is a graph over X for $0 < t < 1 + \varepsilon$. This is just saying that the exponential map \exp_z^t is a diffeomorphism from $T_z^*X \cap H^{-1}((\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon))$ onto its image, for $0 < t < 1 + \varepsilon$.

This means we have an Ansatz

$$(15.4) \quad U^t \hat{P}_{\alpha_0} \delta_z \sim (2\pi\hbar)^{-d/2} e^{\frac{iS^0(t, x|z)}{\hbar}} \left(\sum_{k=0}^{\infty} \hbar^k b_k^0(t, x|z) \right),$$

where the function $S^0(t, x|z)$ is a generating function of the lagrangian manifold $\mathcal{L}^0(t)$.

We denote

$$(15.5) \quad v^0(t; x|z) = e^{\frac{iS^0(t, x|z)}{\hbar}} b_{\hbar}^0(t, x|z),$$

$$(15.6) \quad b_{\hbar}^0(t, x|z) \stackrel{\text{def}}{=} \left(\sum_{k=0}^{N-1} \hbar^k b_k^0(t, x|z) \right)$$

For $t = 1$, the function $v^0(1; x|z)$ gives us an approximation to $U\hat{P}_{\alpha_0}\delta_z$, the difference being bounded in L^2 -norm by $\mathcal{O}(\hbar^{N-\frac{d}{2}})$.

15.3. Iteration of the WKB Ansatz. In this section we will obtain an approximate Ansatz for $\hat{P}_{\alpha_n} \dots U\hat{P}_{\alpha_1} U\hat{P}_{\alpha_0} \delta_z$. Above we have already performed the first step, obtaining an approximation $v^0(1, \cdot|z)$ of $U\hat{P}_{\alpha_0}\delta_z$. Until Lemma 15.1 we will fix the base point z , and omit it in our notations when no confusion may arise; at the end we will obtain an estimate which is uniform in z .

Applying the multiplication operator \hat{P}_{α_1} to the state $v^0(1, x) := v^0(1, x|z)$, we obtain another WKB state which we denote as follows:

$$v^1(0, x) = b_{\hbar}^1(0, x) e^{\frac{i}{\hbar} S^1(0, x)}, \quad \text{with} \quad \begin{cases} S^1(0, x) := S^0(1, x|z), \\ b_{\hbar}^1(0, x) := \hat{P}_{\alpha_1}(x) b_{\hbar}^0(1, x|z). \end{cases}$$

This state is associated with the lagrangian manifold

$$\mathcal{L}^1(0) = \mathcal{L}^0(1) \cap T^*\Omega_{\alpha_1}.$$

If this intersection is empty, then $v^1(0) = 0$, which means that $\hat{P}_{\alpha_1} v^0(1) = \mathcal{O}(\hbar^N)$ in L^2 norm. In the opposite case, we can evolve $v^1(0)$ following the procedure described in §14. For $t \in [0, 1]$, and up to an error $\mathcal{O}_{L^2}(\hbar^N)$, the evolved state $U^t v^1(0)$ is given by the WKB Ansatz

$$v^1(t, x) = b_{\hbar}^1(t, x) e^{\frac{i}{\hbar} S^1(t, x)}, \quad b_{\hbar}^1(t) = \sum_{k=0}^{N-1} b_k^1(t).$$

The state $v^1(t)$ is associated with the lagrangian $\mathcal{L}^1(t) = g^t \mathcal{L}^1(0)$, and the function $b_{\hbar}^1(t)$ is supported inside $\pi \mathcal{L}^1(t)$.

15.3.1. *Evolved lagrangians.* We can iterate this procedure, obtaining a sequence of approximations

$$(15.7) \quad v^j(t) = U^t \hat{P}_{\alpha_j} v^{j-1}(1) + \mathcal{O}(\hbar^N), \quad \text{where } v^j(t, x) = v^j(t, x|z) = b_{\hbar}^j(t, x|z) e^{\frac{i}{\hbar} S^j(t, x|z)}.$$

(Again, the initial position z is fixed for the moment, and we do not always indicate in the notations the z -dependence). To show that this procedure is consistent, we must check that the lagrangian manifold $\mathcal{L}^j(t)$ supporting $v^j(t)$ does not develop caustics through the evolution ($t \in [0, 1]$), and that the projection $\pi : \mathcal{L}^j(t) \rightarrow X$ remains injective. These were the two conditions required to apply the method of §14. We now show that these properties hold, due to our assumption that the curvature is negative (in fact, it is enough to assume that the geodesic flow has the Anosov property).

The manifolds $\mathcal{L}^j(t)$ are obtained by the following procedure. Knowing $\mathcal{L}^{j-1}(1)$, which is generated by the phase function $S^{j-1}(1)$, we take for $\mathcal{L}^j(0)$ the intersection

$$\mathcal{L}^j(0) = \mathcal{L}^{j-1}(1) \cap T^*\Omega_{\alpha_j}.$$

If this set is empty, then we stop the construction. Otherwise, this lagrangian is evolved into $\mathcal{L}^j(t) = g^t \mathcal{L}^j(0)$ for $t \in [0, 1]$. Notice that the lagrangian $\mathcal{L}^j(t)$ corresponds to evolution at time $j+t$ of a piece of $\mathcal{L}^0(0)$: it is made up of the image under the geodesic flow of a compact piece of the fiber T_z^*X . If the geodesic flow is Anosov, the geodesic flow has no conjugate points — by a result of Klingenberg [Kl74]. This means precisely that $g^t \mathcal{L}^0(0)$ will not develop caustics.

On a negatively curved manifold, there cannot be two homotopic geodesics joining two points x and z . As a consequence, there cannot be two geodesics joining x and z in time $j+t$ and which fall in the same Ω_{α_k} for all integer times k (this holds if the injectivity radius is larger than 1 and if the diameter of the Ω_{α_k} is small enough). This means that, for any $j \geq 1$, $0 \leq t \leq 1$, the manifold $\mathcal{L}^j(t)$ projects injectively to $\pi \mathcal{L}^j$.

Finally, we recall that $\mathcal{L}^0(0)$ was obtained by propagating a piece of $T_z^*X \cap H^{-1}(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$. Since the geodesic flow on each energy layer $H^{-1}(1/2 + \eta)$ is Anosov, the sphere bundle $H^{-1}(1/2 + \eta) \cap T_z^*X$ is uniformly transverse to the stable foliation in $H^{-1}(1/2 + \eta)$ — also a result of [Kl74]. As a consequence, the action of the geodesic flow carries a piece of sphere $H^{-1}(1/2 + \eta) \cap T_z^*X$ exponentially close to a piece of unstable leaf of $H^{-1}(1/2 + \eta)$ when $t \rightarrow +\infty$. This transversality of spheres with the stable foliation is crucial in our choice of the “basis” δ_z .

15.3.2. *Exponential decay.* We now analyze the behaviour of the symbols $b_h^j(t, x)$ appearing in (15.7), when $j \rightarrow \infty$. These symbols are constructed iteratively: starting from the function $b_h^{j-1}(1) = \sum_{k=0}^{N-1} b_k^{j-1}(1)$ supported inside $\pi \mathcal{L}^{j-1}(0)$, we define

$$(15.8) \quad b_h^j(0, x) = \hat{P}_{\alpha_j}(x) b_h^{j-1}(1, x), \quad x \in \pi \mathcal{L}^j(0).$$

The WKB procedure of §14 shows that for any $t \in [0, 1]$,

$$(15.9) \quad U^t v^j(0) = v^j(t) + R_N^j(t),$$

where the transported symbol $b_h^{j-1}(t) = \sum_{k=0}^{N-1} \hbar^k b_k^{j-1}(t)$ is supported inside $\pi \mathcal{L}^j(t)$. The remainder satisfies

$$(15.10) \quad \|R_N^j(t)\| \leq C t \hbar^N \left(\sum_{k=0}^{N-1} \|b_k^j(0)\|_{C^{2(N-k)}} \right).$$

To control this remainder when $j \rightarrow \infty$, we need to bound from above the derivatives of b_h^j . Lemma 15.1 below shows that all terms $b_k^j(t)$ and their derivatives decay exponentially when $j \rightarrow \infty$, due to the Jacobian appearing in (14.4).

To understand the reasons of the decay, we first look at the principal symbol $b_0^j(1, x)$. It satisfies the following recurrence:

$$(15.11) \quad b_0^j(1, x) = T_j^{j+1}(\hat{P}_{\alpha_j} \times b_0^{j-1}(1))(x) = (\hat{P}_{\alpha_j} \times b_0^{j-1}(1)) \circ G_{j+1}^j(x) \sqrt{J_{j+1}^j(x)};$$

using similar notations as above, the transport map $G_{s,t}$ is defined, for $j \leq s$, $t \leq j+1$, by $G_{s,t} := \exp_{\mathcal{L}^j(s-j)}^{t-s} \circ \pi^{-1}$, and maps $\pi \mathcal{L}^j(s-j)$ to $\pi \mathcal{L}^j(t-j)$. We denote J_s^t the jacobian of G_s^t . We recall that $G_{n-1}^n G_{n-2}^{n-1} \dots G_1^2 = G_1^n$, where both sides are defined.

Iterating this expression, and using the fact that $0 \leq \hat{P}_{\alpha_j} \leq 1$, we get at time n and for any $x \in \pi \mathcal{L}^n(0)$:

$$(15.12) \quad |b_0^n(0, x)| \leq |b_0^0(1, G_n^1(x))| \times \left(J_n^{n-1}(x) J_{n-1}^{n-2}(G_n^{n-1}x) \dots J_2^1(G_3^2(x)) \right)^{1/2}.$$

By the chain rule, this product of jacobians is simply $J_n^1(x)^{1/2} = J_1^n(G_n^1(x))^{-1/2}$.

Recall that $\mathcal{L}^0(0)$ intersected with each energy layer $S^{1+\eta}X := \{\xi \in T^*X, \|\xi\| = 1 + \eta\}$ is just a piece of the sphere $S_z^{1+\eta}X$. Thus, if $d(x, z) = 1 + \eta$, the jacobian $J_1^n(G_n^1(x))$ measures the expansion rate of the sphere $g^n(S_z^{1+\eta}X)$: in dimension d and curvature $\equiv -1$, it grows asymptotically like $e^{(d-1)(1+\eta)n}$ when $n \rightarrow \infty$. If $x \in \pi\mathcal{L}^1(0)$ (and if this last set is non-empty) we have $d(x, z) \geq 1 - \varepsilon$ (because $\mathcal{L}^0(0)$ is contained in $H^{-1}(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$). We obtain the following estimate on the principal symbol $b_0^n(0)$:

$$(15.13) \quad \forall n \geq 1 \quad \|b_0^n(0)\|_\infty \leq \|b_0^0(1)\|_\infty [\exp(-n(d-1)(1-\varepsilon))]^{1/2}$$

The following lemma, which we shall not prove here, shows that the upper bound extends to the full symbol $b_\hbar^n(0, x)$ and its derivatives.

LEMMA 15.1. *Take any index $0 \leq k \leq N$ and $m \leq 2(N-k)$. Then there exists a constant $C(k, m)$ such that*

$$(15.14) \quad \forall n \geq 1, \quad \forall x \in \pi\mathcal{L}^n(0), \\ |d^m b_k^n(0, x|z)| \leq C(k, m) n^{m+3k} [\exp(-n(d-1)(1-\varepsilon))]^{1/2}.$$

This bound is uniform with respect to the initial point z . For $(k, m) \neq (0, 0)$, the constant $C(k, m)$ depends on the partition $\mathcal{P}^{(0)}$, while $C(0, 0)$ does not.

Taking into account the fact that the remainders $R_N^j(1)$ are dominated by the derivatives of the b_k^j (see (15.10)), the above statement translates into

$$\forall j \geq 1, \quad \|R_N^j(1)\|_{L^2} \leq C(N) j^{3N} [\exp(-n(d-1)(1-\varepsilon))]^{1/2} \hbar^N.$$

A crucial fact for us is that the above bound also holds for the propagated remainder $\hat{P}_{\alpha_n} U \cdots U \hat{P}_{\alpha_{j+1}} R_N^j(1)$, due to the fact that the operators $\hat{P}_{\alpha_j} U$ have norms less than 1. As a result, the total error at time n is bounded from above by the sum of the errors $\|R_N^j(1)\|$. We obtain the following estimate for any $n > 0$:

$$(15.15) \quad \|\hat{P}_{\alpha_n} U \hat{P}_{\alpha_{n-1}} \cdots \hat{P}_{\alpha_1} v^0(1|z) - v^n(0|z)\| \leq C(N) \hbar^N \sum_{j=0}^n j^{3N} [\exp(-n(d-1)(1-\varepsilon))]^{1/2}.$$

The last term is bounded by $C(N)\hbar^N$. This bound is uniform with respect to the initial point z .

15.4. Conclusion. From (15.15), we see that we can use our Ansatz $v^n(0|z)$ to estimate the norm of $\hat{P}_{\alpha_n} U \hat{P}_{\alpha_{n-1}} \cdots \hat{P}_{\alpha_1} v^0(1|z)$, up to an error $\mathcal{O}(\hbar^N)$. From (15.14) and the definition (15.7) of $v^n(0|z)$, we have

$$(15.16) \quad \|v^n(0|z)\|_{L^2(X)} \leq [\exp(-n(d-1)(1-\varepsilon))]^{1/2} \sum_{k=0}^{N-1} C(k, 0) \hbar^k n^{3k}.$$

As required in Theorem 10.1, let us now take an arbitrary large \mathcal{K} , and $n = \mathcal{K} \lfloor \log \hbar \rfloor$. In the inequalities (15.13) and (15.16), the right hand term is bounded below by a fixed power of \hbar (more precisely, $\hbar^{-\frac{1}{2}\mathcal{K}(d-1)}$). Thus, we will choose N , the order of our WKB expansion, large enough so that the remainder (15.15) is negligible compared to $\hbar^{-\frac{1}{2}\mathcal{K}(d-1)}$.

Now, remember the relation (15.4) between $v^0(1|z)$ and $U\hat{P}_{\alpha_0}\delta_z$: note in particular the normalization factor $(2\pi\hbar)^{-d/2}$. The combination of (15.15) and (15.16) gives us

$$(15.17) \quad \|\hat{P}_{\alpha_n}U\hat{P}_{\alpha_{n-1}}\cdots\hat{P}_{\alpha_1}U\hat{P}_{\alpha_0}\delta_z\|_{L^2(X)} \leq \frac{2}{(2\pi\hbar)^{d/2}}[\exp(-n(d-1)(1-\varepsilon))]^{1/2}$$

for $n = \mathcal{K}|\log \hbar|$ and $\hbar \leq \hbar_{\mathcal{K}}$.

Combined with (15.2) and the subsequent discussion, we find

$$\|\hat{P}_{\alpha_n}U\hat{P}_{\alpha_{n-1}}\cdots U\hat{P}_{\alpha_0}\text{Op}(\chi)u\| \leq \frac{2l\sqrt{\text{Vol } X}}{(2\pi\hbar)^{d/2}}\|u\|_{L^2(X)}[\exp(-n(d-1)(1-\varepsilon))]^{1/2}$$

which is the announced result.

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