Minimizing orbits in the discrete Aubry-Mather model

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Abstract

We consider a generalization of the Frenkel-Kontorova model in higher dimension. We give a wider applicability to Aubry's theory by studying models with vector-valued states over a one dimensional chain. This theory has a lot of similarities with Mather's twist approach over a multidimensional torus. Weakening the standard hypotheses used in one dimensional, we investigate properties (like boundness of jumps and definability of a rotation vector) of a special class of strong ground states: the calibrated configurations.

The main mathematical tool is to cast the study the minimizing configurations into the framework of discrete Lagrangian theory. We introduce forward and backward Lax-Oleinik problems and interpret their solutions as discrete viscosity solutions in the same spirit of Hamilton-Jacobi methods. With reduced hypotheses, we reproduce in this discrete setting some classical results of the Lagrangian Aubry-Mather theory. In particular, we obtain a graph property for the Aubry set, representation formulas for calibrated sub-actions and the existence of separating sub-actions.

Contents

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1 Introduction

One dimensional crystals. Via a *local interaction energy map* $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}$, the original Frenkel-Kontorova model describes a one dimensional chain of classical particles coupled to their neighbors and subjected to a periodic on-site potential. If $x_k \in \mathbb{R}$ denotes the position of the particle labeled by $k \in \mathbb{Z}$, the total energy of a chain $\{x_k\} \in \mathbb{R}^{\mathbb{Z}}$ is given by

$$\mathcal{L}_{\mathrm{tot}}(\{x_k\}) = \sum_{k\in\mathbb{Z}} \mathcal{L}(x_k, x_{k+1}),$$

which may *a priori* diverge. As an example of local interaction energy map, it is standard to study

$$\mathcal{L}(x_k, x_{k+1}) = \frac{\kappa}{2}\,(x_{k+1} - x_k - \upsilon)^2 + (1 - \cos 2\pi x_k),$$

when considering interactions of atoms via harmonic springs with elastic coupling constant κ and mean interatomic distance υ, in the presence of an external periodic potential $\mathcal{V}(x) = 1 - \cos 2\pi x$.

The main interest is to understand the set of *minimizing configurations*, or ground states in statistical physics, that is, the set of configurations $\{x_k\}_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ satisfying

$$\mathcal{L}(x_m, x_{m+1}, \ldots, x_n) := \sum_{k=m}^{n-1} \mathcal{L}(x_k, x_{k+1}) \le \mathcal{L}(y_m, y_{m+1}, \ldots, y_n)$$

for every $m < n$ and every configuration $\{y_k\}_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ with $y_m = x_m$ and $y_n = x_n$. Notice that, when $\mathcal{L}$ is supposed to be C^1, a minimizing configuration $\{x_k\}$ is *critical* in the sense that

$$\frac{\partial \mathcal{L}}{\partial y}(x_{k-1}, x_k) + \frac{\partial \mathcal{L}}{\partial x}(x_k, x_{k+1}) = 0, \quad \forall\, k \in \mathbb{Z}.$$

Central references, ramifications and further developments. S. Aubry and P. Y. Le Daeron [2] studied a large class of Frenkel-Kontorova models. Mainly assuming $\mathcal{L}$ to be C^2, periodic under the $\mathbb{Z}$ action and uniformly strictly convex, they proved that minimizing configurations do exist and have a well defined rotation number

$$\omega := \lim_{n\to+\infty} \frac{x_n - x_0}{n} = \lim_{n\to+\infty} \frac{x_0 - x_{-n}}{n}.$$

Moreover, they proved that any possible rotation number ω is achieved by a minimizing configuration $\{x_k\}$ which satisfies in addition $\sup_{k\in\mathbb{Z}} |x_k - x_0 - k\omega| < +\infty$.

This work of Aubry and Le Daeron in solid-state physics and in an independent study of J. N. Mather [24] on twist homeomorphisms of the annulus gave rise the so called Aubry-Mather theory, which was later developed for Lagrangian systems (see [25]). An introduction to such theory is provided by the notes of G. Forni and J. N. Mather [13]. Furthermore, textbooks pertinent to the subject are [6, 11].

It is well known that area-preserving maps of the annulus occur as Poincaré section mappings of Hamiltonian systems with two degrees of freedom. Since such annulus maps play a role in the stability theory, it is quite natural to interpret certain aspects of the Aubry-Mather theory into the scene of the KAM theory, specially into the episode of the disintegration of invariant tori. Conversely, it is interesting to have in mind, as noticed by M. Herman in [20], the existence of connections between configurations with minimun energy and Lagrangian tori invariant under symplectic diffeormorphisms of the cotangent bundle of the d-dimensional torus.

The main objective of this paper is to extend the Frenkel-Kontorova model to the case where the state x_k of the atom at each site $k \in \mathbb{Z}$ of the lattice possesses d degrees of freedom, that is, to the case where $x_k \in \mathbb{R}^d$. First, we would like to clarify the natural mathematical setting where a such theory gives non trivial results, for instance by allowing the local interaction energy map $\mathcal{L}$ to have the lowest possible regularity and by avoinding the so called "twist condition". Our second purpose here is to study in detail a special class of minimizing configurations that we call calibrated and are strongly related to Fathi's theory of weak KAM or viscosity solutions. Another aim is an attempt to understand rotational theory in this general context.

Generalizations of Frenkel-Kontorova model have been pursued in several works. One can consider, for example, a multidimensional topology of interactions. The state of the system is still one-dimensional as in Aubry's theory, but the topology of the interactions is given by a lattice of higher dimension, by $\mathbb{Z}^d$ for instance. In this framework, $x_k \in \mathbb{R}$ is a real quantity representing the state of a particle at the site $k \in \mathbb{Z}^d$. By introducing a family of local interaction energies, the notions of minimizing configurations and rotation vectors can be defined similarly. In the context of these multidimensional models of Frenkel-Kontorova type, one still obtains a minimizing configuration having a given rotation vector when respecting an analogous bounded distance property. For precise definitions and statements, we refer the reader to the work of R. de la Llave and E. Valdinoci [22]. One should also consult the paper of H. Koch, R. de la Llave and C. Radin [21] for situations where the variables range over a more complicated lattice. In another direction, the potential could be assumed quasiperiodic instead of periodic as it is done in the work of J. M. Gambaudo, P. Guiraud and S. Petite [14].

Our hypotheses. From now on, we assume our local interaction energy map

$$\mathcal{L} = \mathcal{L}(x, y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

to be C^0 and invariant with respect to the diagonal action of $\mathbb{Z}^d$,

$$\mathcal{L}(x, y) = \mathcal{L}(x + s, y + s), \quad \forall\, s \in \mathbb{Z}^d.$$

The local interaction energy map is usually supposed to be C^2, superlinear and uniformly strictly convex. We will see that most of the theory can be done assuming only C^0 regularity and coerciveness. This later condition implies in particular compactness on any band $\|x - y\| \leq R$. We say that $\mathcal{L}(x, y)$ is coercive if

$$\lim_{R \to +\infty} \inf_{\|x-y\| \geq R} \mathcal{L}(x, y) = +\infty.$$

In the last section, will be interested in discussing rotational properties of minimizing configurations. We will then require $\mathcal{L}(x,y)$ to be superlinear, that is,

$$\lim_{R\to+\infty} \inf_{\|y-x\|\geq R} \frac{\mathcal{L}(x,y)}{\|x-y\|} = +\infty.$$

Of course superlinearity implies coerciveness.

We will also prove a graph theorem as in Mather's theory. Only in that part of the article, we will need $\mathcal{L}(x,y)$ to be ferromagnetic. A C^1 local interaction energy map $\mathcal{L}(x,y)$ is said to be *ferromagnetic* if, for every pair $(\bar{x},\bar{y}) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$x \in \mathbb{R}^d \mapsto \frac{\partial \mathcal{L}}{\partial y}(x,\bar{y}) \in \mathbb{R}^d \quad \text{and} \quad y \in \mathbb{R}^d \mapsto \frac{\partial \mathcal{L}}{\partial x}(\bar{x},y) \in \mathbb{R}^d$$

are homeomorphisms. This property is weaker than the twist property. It is implied for instance by C^2 regularity and uniform strict convexity in $y-x$. In section 2, we will show that the ferromagnetic condition allows us to introduce a discrete-time Lagrangian dynamics $\Phi_\tau : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$ (see definition 2.5).

Main results. Among the set of all minimizing configurations, calibrated configurations play a central role. We first introduce a notion of minimal mean energy per site, that we call *minimizing holonomic value*[1] $\bar{\mathcal{L}}$, in the following way

$$\bar{\mathcal{L}} = \inf \Big\{ \liminf_{n\to+\infty} \frac{1}{n} \mathcal{L}(x_0, x_1, \ldots, x_n) : \{x_k\}_{k\in\mathbb{Z}} \text{ any configuration} \Big\}.$$

We call thus *sub-action* any continuous $\mathbb{Z}^d$-periodic function $u : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$u(y) - u(x) \leq \mathcal{L}(x,y) - \bar{\mathcal{L}}, \quad \forall\, x, y \in \mathbb{R}^d.$$

We call *calibrated configuration* (or more precisely u-calibrated if needed) a configuration $\{x_k\}_{k\in\mathbb{Z}}$ such that, for some sub-action u,

$$u(x_{k+1}) - u(x_k) = \mathcal{L}(x_k, x_{k+1}) - \bar{\mathcal{L}}, \quad \forall\, k \in \mathbb{Z}.$$

It is then obvious that a calibrated configuration is a minimizing configuration.

In section 5, we show that calibrated configurations do exist, they have bounded jumps, $\sup_k \|x_{k+1} - x_k\| < +\infty$, and may be obtained using a notion of forward (or backward) calibrated sub-actions (see lemma 5.5). These specific sub-actions are similar to Fathi's weak KAM solutions or viscosity solutions of a discrete Hamilton-Jacobi equation. A sub-action is said to be *forward calibrated* if

$$\forall\, x \in \mathbb{R}^d, \quad \exists\, y \in \mathbb{R}^d \quad \text{s.t.} \quad u(y) - u(x) = \mathcal{L}(x,y) - \bar{\mathcal{L}}.$$

[1] Such nomenclature is due to the class of holonomic probabilities which we will study in section 3. As showed by D. A. Gomes in [17], even when no Lagrangian dynamics is clearly present and the notion of invariant measure has *a priori* no meaning, such set of probabilities is suitable for the minimization of the average action. As we will see, $\bar{\mathcal{L}}$ can be also characterized in terms of minimizing holonomic probabilities.

In particular, given x_0 and a forward calibrated sub-action u, there exists a forward configuration $\{x_k\}_{k\geq 0}$ of points of $\mathbb{R}^d$ such that $\mathcal{L}(x_k, x_{k+1}) = u(x_{k+1}) - u(x_k) + \bar{\mathcal{L}}$, $\forall\ k \geq 0$.

Calibrated sub-actions can be obtained as solutions of a suitable Lax-Oleinik problem. In section 4, the *forward Lax-Oleinik operator* is defined by

$$T_+u(x) := \sup_{y\in\mathbb{R}^d} [u(y) - \mathcal{L}(x,y)], \quad \forall\ x \in \mathbb{R}^d,\ \ \forall\ u \in C^0(\mathbb{R}^d)\ \ \mathbb{Z}^d\text{-periodic}.$$

So u is a forward calibrated sub-action if, and only if, $T_+u = u + c$ for some constant $c \in \mathbb{R}$ which necessarily equals $\bar{\mathcal{L}}$. We not only guarantee that calibrated sub-actions do exist, but we also discuss how the regularity of the local interaction energy map affects the regularity of a calibrated sub-action. We show (see proposition 4.7) that, if $\mathcal{L}$ is locally Lipschitz or C^2, then any forward (resp. backward) calibrated sub-action is Lipschitz or semiconvex (resp. semiconcave).

If $\mathcal{L}(x,y)$ is in addition C^1, a similar graph property as in Aubry-Mather theory can be formulated. A *triple* (x_{-1}, x_0, x_1) is said critical if

$$\frac{\partial \mathcal{L}}{\partial y}(x_{-1}, x_0) + \frac{\partial \mathcal{L}}{\partial x}(x_0, x_1) = 0.$$

Notice ferromagnetism implies that, for x_0 fixed, the map $x_{-1} \mapsto x_1$ is a homeomorphism. Given a sub-action u, a triple (x_{-1}, x_0, x_1) is said *u-calibrated* if

$$\mathcal{L}(x_{-1}, x_0) - u(x_0) + u(x_{-1}) = \mathcal{L}(x_0, x_1) - u(x_1) + u(x_0) = \bar{\mathcal{L}}.$$

A calibrated triple is in particular critical. In section 6, we show that, for C^1 local interaction energy map $\mathcal{L}$, any sub-action u is differentiable at any mid point x_0 of some u-calibrated triple (see lemma 6.8) and that

$$Du(x_0) = \frac{\partial \mathcal{L}}{\partial y}(x_{-1}, x_0) = -\frac{\partial \mathcal{L}}{\partial x}(x_0, x_1).$$

One concludes that, in the ferromagnetic case, there exists at most one u-calibrated triple going throught any $x_0 \in \mathbb{R}^d$. This suggests to introduce the following set, called the *Aubry set*,

$$\mathcal{A}(\mathcal{L}) = \big\{\underline{x} = \{x_k\}_{k\in\mathbb{Z}} \in (\mathbb{R}^d)^{\mathbb{Z}} : \underline{x} \text{ is calibrated for any sub-action}\big\}$$

and its projection $pr^0(\mathcal{A}(\mathcal{L})) \subset \mathbb{R}^d$ called the *projected Aubry set*, where the map $pr^0 : (\mathbb{R}^d)^{\mathbb{Z}} \to \mathbb{R}^d$ denotes the zero coordinate projection. The Aubry set is not empty for a general C^0 coercive local interaction energy map $\mathcal{L}$. We give several properties in sections 6 and 7.

In the ferromagnetic case, the projected Aubry set satisfies in addition the graph property in the sense that, for any $x_0 \in pr^0(\mathcal{A}(\mathcal{L}))$, there exists an unique configuration $\underline{x} = \{x_k\}_{k\in\mathbb{Z}}$ going throught x_0, calibrated for any sub-action. In other words, $pr^0 : \mathcal{A}(\mathcal{L}) \to pr^0(\mathcal{A}(\mathcal{L}))$ is one-to-one. Besides, we are able to show that $Du : pr^0(\mathcal{A}(\mathcal{L})) \to \mathbb{R}^d$ is a continuous function for any sub-action u.

If one only assumes $\mathcal{L}$ to be C^0 and coercive, the projected Aubry set admits an equivalent characterization similar to the non-wandering set in dynamical systems,

$$pr^0(\mathcal{A}(\mathcal{L})) = \big\{x_0 \in \mathbb{R}^d : \forall\, \epsilon > 0,\ \exists\, n \geq 1,\ \exists\, x_0^\epsilon, \ldots, x_n^\epsilon \in \mathbb{R}^d,\ \exists\, s_n^\epsilon \in \mathbb{Z}^d \text{ s.t.} \\ |\mathcal{L}(x_0^\epsilon, x_1^\epsilon, \ldots, x_n^\epsilon) - n\bar{\mathcal{L}}| < \epsilon,\ \ x_0^\epsilon = x_0 \text{ and } x_n^\epsilon = x_0 + s_n^\epsilon\big\}.$$

It is relatively easy to prove that a point x_0, satisfying the second definition of the projected Aubry set, is the projection of a configuration $\{x_k\}_{k\in\mathbb{Z}}$, calibrated for any sub-action, which is made of limit points when ϵ tends to 0 of the sequences $(x_1^\epsilon, x_2^\epsilon, \ldots)$ and $(x_n^\epsilon - s_n^\epsilon, x_{n-1}^\epsilon - s_n^\epsilon, \ldots)$.

The converse is more difficult to prove and uses the notion of *separating sub-action*. A sub-action u is called separating when, for any pair of points (x, y) verifying $u(y) - u(x) = \mathcal{L}(x, y) - \bar{\mathcal{L}}$, there necessarily exists $\{x_k\}_{k\in\mathbb{Z}} \in \mathcal{A}(\mathcal{L})$ with $x_0 = x$ and $x_1 = y$. We establish in section 10 the existence of these sub-actions and their generic condition. The existence result is a discrete and topological version of the critical subsolutions of the Hamilton-Jacobi equation determined by A. Fathi and A. Siconolfi in [12]. We actually obtain a stronger statement: for every $\{x_k\}_{k\in\mathbb{Z}} \in \mathcal{A}(\mathcal{L})$, for any integer $m \geq 1$, for all $\epsilon > 0$, there are $n \geq m$, $x_m^\epsilon, \ldots, x_n^\epsilon \in \mathbb{R}^d$ and $s_n^\epsilon \in \mathbb{Z}^d$, with $x_n^\epsilon = x_0 + s_n^\epsilon$, such that

$$|\mathcal{L}(x_0, \ldots, x_{m-1}, x_m^\epsilon, \ldots, x_n^\epsilon) - n\bar{\mathcal{L}}| < \epsilon.$$

The second definition of the projected Aubry set suggests to introduce the *Peierls barrier*,

$$\mathbf{h}(x, y) = \liminf_{n\to+\infty}\ \inf \big\{\mathcal{L}(x_0, \ldots, x_n) - n\bar{\mathcal{L}} : x_k \in \mathbb{R}^d,\ x_0 = x,\ x_n - y \in \mathbb{Z}^d\big\}.$$

In section 8, assuming C^0 regularity and coerciveness, we show that $\mathbf{h}(x, y)$ is well defined on $\mathbb{R}^d \times \mathbb{R}^d$, continuous, $\mathbb{Z}^d \times \mathbb{Z}^d$-periodic and satisfies $u(y) - u(x) \leq \mathbf{h}(x, y)$, for all $x, y \in \mathbb{R}^d$ and any sub-action u. We also prove that, for any $x \in \mathbb{R}^d$, $\mathbf{h}(x, \cdot)$ is backward calibrated and that, for any $y \in \mathbb{R}^d$, $-\mathbf{h}(\cdot, y)$ is forward calibrated (see theorem 8.10). The projected Aubry set admits then a third characterization,

$$pr^0(\mathcal{A}(\mathcal{L})) = \{x_0 \in \mathbb{R}^d : \mathbf{h}(x_0, x_0) = 0\}.$$

The set of forward (resp. backward) calibrated sub-actions is completely determined by the projected Aubry set. In section 9, we show that any calibrated sub-action is characterized by its values on the projection of the Aubry set and the values of the Peierls barrier. We show (see theorem 9.3) that u is forward calibrated if, and only if,

$$u(x) = \sup_{y \in pr^0(\mathcal{A}(\mathcal{L}))} [\psi(y) - \mathbf{h}(x, y)], \quad \forall\, x \in \mathbb{R}^d,$$

for some C^0 and $\mathbb{Z}^d$-periodic function $\psi : \mathbb{R}^d \to \mathbb{R}$ satisfying $\psi(y) - \psi(x) \leq \mathbf{h}(x, y)$ for all $x, y \in pr^0(\mathcal{A}(\mathcal{L}))$. Moreover $u(x) = \psi(x)$ for all $x \in pr^0(\mathcal{A}(\mathcal{L}))$. Such result shall be understood as the analogous of the one obtained by G. Contreras for weak KAM solutions (see [5]).

As in Fathi's weak KAM theory, we say that a forward u_+ and a backward u_- calibrated sub-actions are *conjugated*, and we write $u_+ \sim u_-$, if they coincide on the projected Aubry set. We then show in proposition 9.6 that the Peierls barrier can be defined by

$$\mathbf{h}(x,y) = \sup_{u_+ \sim u_-} [u_-(y) - u_+(x)], \quad \forall\, x, y \in \mathbb{R}^d.$$

In particular, we obtain a forth characterization of the projected Aubry set: a point x_0 is outside $pr^0(\mathcal{A}(\mathfrak{L}))$ if, and only if, there exist conjugated calibrated sub-actions, u_+ and u_-, such that $u_+(x_0) \neq u_-(x_0)$.

In section 11, we introduce the notion of rotation vector ω of a configuration $\{x_k\}_{k\in\mathbb{Z}}$

$$\omega = \lim_{n-m\to+\infty} \frac{x_n - x_m}{n-m},$$

when the limit exists. We show in particular that there exists minimizing configuration with rotation vector of arbitrarily large norm.

We have chosen to translate the Frenkel-Kontorova model into the framework of Aubry-Mather theory mainly to be able in a subsequent article to reach Fathi's approach of weak KAM solutions (or the problem viscosity solutions of Hamilton-Jacobi equations) using a more dynamical discretization scheme. Let $\mathbb{T}^d$ denote the d dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$. The main object we are interested in is thus a Lagrangian $L(x,v)$ defined on $\mathbb{T}^d \times \mathbb{R}^d$ and a family of local interaction energies parametrized by $\tau > 0$,

$$\mathfrak{L}_\tau(x,y) = \tau L\Big(x \ (\mathrm{mod}\ \mathbb{Z}^d), \frac{y-x}{\tau}\Big), \quad \forall\, x,y \in \mathbb{R}^d.$$

Notice that $\mathfrak{L}_\tau$ is invariant under the diagonal action of $\mathbb{Z}^d$,

$$\mathfrak{L}_\tau(x+s, y+s) = \mathfrak{L}_\tau(x,y), \quad \forall\, s \in \mathbb{Z}^d.$$

The two approches are complementary. While the Lagrangian formulation will be more adapted in the description of the support of minimizing measures, the Frenkel-Kontorova setting will be used in the construction of sub-actions (or discrete weak KAM solutions or discrete viscosity solutions), as well as in the definition of two major notions of action potential between two points: the Mañé potential and the Peierls barrier. We intend later to beter understand the limit when the step τ tends to zero and the thermodynamic formalism approach when the temperature goes to zero.

2 A discrete-time Lagrangian dynamics

We fix from now on a C^0 coercive Lagrangian $L(x,v) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $\mathbb{Z}^d$-periodic in x, and its associated local interaction energy map

$$\mathfrak{L}_\tau(x,y) = \tau L\Big(x, \frac{y-x}{\tau}\Big)$$

defined on $\mathbb{R}^d \times \mathbb{R}^d$ and invariant by the diagonal action of $\mathbb{Z}^d$. We begin by recalling some well known notions of divergence type at infinity. Coerciveness is our basic assumption, superlinerarity will be used when homology will play a role.

Definition 2.1. *Let $L(x,v) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^0-Lagrangian.*

i. $L(x,v)$ is said to be coercive if $\lim_{R\to+\infty} \inf_{\|v\|\geq R} \inf_{x\in\mathbb{T}^d} L(x,v) = +\infty$.

ii. $L(x,v)$ is said to be superlinear if $\lim_{R\to+\infty} \inf_{\|v\|\geq R} \inf_{x\in\mathbb{T}^d} \dfrac{L(x,v)}{\|v\|} = +\infty$.

We call configuration any sequence $\{x_k\}_{k\in\mathbb{Z}}$ of points in $\mathbb{R}^d$. Let $\Sigma = (\mathbb{R}^d)^{\mathbb{Z}}$ be the set of configurations. We also consider the set of configurations modulo the diagonal action of $\mathbb{Z}^d$, that is, the quotient of Σ by the equivalence relation: $\{x_k\}_{k\in\mathbb{Z}} \sim \{y_k\}_{k\in\mathbb{Z}}$ if, and only if, there exists $s \in \mathbb{Z}^d$ such that $y_k = x_k + s$ for all $k \in \mathbb{Z}$. So

$$\Sigma = (\mathbb{R}^d)^{\mathbb{Z}} \quad \text{and} \quad \Sigma/_\sim = (\mathbb{R}^d)^{\mathbb{Z}}/_\sim.$$

Let us notice that, for any fundamental domain $\mathcal{D}$ of the action of $\mathbb{Z}^d$ on $\mathbb{R}^d$, the set $(\mathbb{R}^d)^{\mathbb{Z}^*_-} \times \mathcal{D} \times (\mathbb{R}^d)^{\mathbb{Z}^*_+}$ is a fundamental domain for the diagonal action of $\mathbb{Z}^d$ on Σ. Let $\sigma : \Sigma \to \Sigma$ be the left shift given by $\sigma(\{x_k\}) = \{y_k\}$ where $y_k = x_{k+1}$. Notice that σ commutes with the diagonal action.

Definition 2.2. *We call minimizing configuration any sequence $\{x_k\}_{k\in\mathbb{Z}}$ which minimizes the local interaction energy, namely,*

$$\mathcal{L}_\tau(x_n, x_{n+1}, \ldots, x_{n+m}) := \sum_{k=n}^{n+m} \mathcal{L}_\tau(x_k, x_{k+1}) \leq \mathcal{L}_\tau(y_n, y_{n+1}, \ldots, y_{n+m}),$$

for any finite configuration $\{y_k\}_{k=n}^{n+m}$ with identical boundary conditions $x_n = y_n$ and $x_{n+m} = y_{n+m}$.

Although one of our aim is to extend as much as we can the discrete Aubry-Mather theory to just C^0 coercive Lagrangian and to describe precisely the set of minimizing configurations in this general setting, we show in this section that, under a stronger hypothesis on the Lagrangian (C^2-smoothness and twist condition), we can recover the original theory, where the set of minimizing configurations can be understood through the help of a dynamical system similar to the usual standard map. Let us first recall the notion of critical configuration.

Definition 2.3. *Let $L(x,v) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^1-Lagrangian. We call critical triple a configuration (x_{-1}, x_0, x_1) of three points in $\mathbb{R}^d$ satisfying*

$$\frac{\partial \mathcal{L}_\tau}{\partial y}(x_{-1}, x_0) + \frac{\partial \mathcal{L}_\tau}{\partial x}(x_0, x_1) = 0.$$

We call critical configuration any configuration $\{x_k\}_{k\in\mathbb{Z}}$ of points in $\mathbb{R}^d$ consisting of critical triples (x_{k-1}, x_k, x_{k+1}):

$$\frac{\partial \mathcal{L}_\tau}{\partial y}(x_{k-1}, x_k) + \frac{\partial \mathcal{L}_\tau}{\partial x}(x_k, x_{k+1}) = 0, \quad \forall\, k \in \mathbb{Z}.$$

Let $\Gamma_\tau(L) \subset \Sigma$ be the set of critical configurations. We notice that $\Gamma_\tau(L)$ is invariant by both the diagonal action of $\mathbb{Z}^d$ and the shift σ. Let $\Gamma_\tau(L)/_\sim$ be the quotient of $\Gamma_\tau(L)$ by the diagonal action of $\mathbb{Z}^d$.

The equations defining $\Gamma_\tau(L)$ may be seen as a discrete version of the Euler-Lagrange equation. These equations show that, under some stronger hypothesis of twist condition, the knowledge of (x_0, x_1) implies the existence of an unique critical configuration with such initial conditions. More precisely, prefering the use of the ferromagnetic terminology instead of the twist condition as it is done in statistical mechanics, we introduce the following notion.

Definition 2.4. *A C^1-Lagrangian $L(x,v)$ is said to be ferromagnetic if, for any sufficiently small $\tau > 0$, the two maps in (I) or equivalently in (II), where*

$$(I)\left\{\begin{array}{ccc} \mathbb{R}^d & \to & \mathbb{R}^d \\ x & \mapsto & \dfrac{\partial \mathfrak{L}_\tau}{\partial y}(x,y) \\ y & \mapsto & \dfrac{\partial \mathfrak{L}_\tau}{\partial x}(x,y) \end{array}\right. \quad \text{and} \quad (II)\left\{\begin{array}{ccc} \mathbb{R}^d & \to & \mathbb{R}^d \\ v & \mapsto & \dfrac{\partial L}{\partial y}(y-\tau v, v) \\ v & \mapsto & \tau\dfrac{\partial L}{\partial x}(x,v) - \dfrac{\partial L}{\partial v}(x,v) \end{array}\right. ,$$

are homeomorphisms for all (x,y).

Similarly a discrete version of the Euler-Lagrange flow may be introduced.

Definition 2.5. *Let $L(x,v)$ be a C^1 ferromagnetic Lagrangian. For sufficiently small $\tau > 0$, we call discrete Euler-Lagrange map (or standard map), the map*

$$\Phi_\tau = \left\{\begin{array}{ccc} \mathbb{T}^d \times \mathbb{R}^d & \to & \mathbb{T}^d \times \mathbb{R}^d \\ (x,v) & \mapsto & (y,w) \end{array}\right.$$

where $y = x + \tau v$ and w is the unique solution of one of the two equivalent equations

$$\frac{\partial \mathfrak{L}_\tau}{\partial y}(x,y) + \frac{\partial \mathfrak{L}_\tau}{\partial x}(y, y+\tau w) = 0 \quad \text{or} \quad \frac{\partial L}{\partial v}(x,v) + \tau\frac{\partial L}{\partial x}(y,w) - \frac{\partial L}{\partial v}(y,w) = 0.$$

Notice that Φ_τ is a homeomorphism on $\mathbb{T}^d \times \mathbb{R}^d$. In most part of the article, the dynamical sytem $(\mathbb{T}^d \times \mathbb{R}^d, \Phi_\tau)$ will not be used, except, for instance, in section 6, where we prove that minimizing measures are supported on a graph. The main advantage of the standard map approach is that the space of critical configurations modulo the diagonal action is conjugate to a $2d$ degrees of freedom dynamical system.

Remark 2.6. *Let $\Pi_\tau : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$ be the projection given by*

$$\Pi_\tau(x_0, x_1) = \big(x_0 \bmod \mathbb{Z}^d, (x_1 - x_0)/\tau\big).$$

We extend Π_τ to Σ by writing $\Pi_\tau(\{x_k\}_{k\in\mathbb{Z}}) = \Pi_\tau(x_0, x_1)$ and notice that the projection $\Pi_\tau : \Sigma/_\sim \to \mathbb{T}^d \times \mathbb{R}^d$ is also well defined. If $L(x,v)$ is ferromagnetic, then $(\Gamma_\tau(L)/_\sim, \sigma)$ is conjugated to $(\mathbb{T}^d \times \mathbb{R}^d, \Phi_\tau)$, that is, the following diagram commutes

$$\begin{array}{ccc} \Gamma_\tau(L)/_\sim & \xrightarrow{\Pi_\tau} & \mathbb{T}^d \times \mathbb{R}^d \\ \Big\downarrow{\scriptstyle\sigma} & & \Big\downarrow{\scriptstyle\Phi_\tau} \\ \Gamma_\tau(L)/_\sim & \xrightarrow{\Pi_\tau} & \mathbb{T}^d \times \mathbb{R}^d \end{array} .$$

A critical configuration is thus completely determined by the $2d$ data (x,v) and a general configuration plays the role of a virtual deformation as in Mechanics. In order to check that a Lagrangian satisfies the ferromagnetic condition, an easier but stronger asumption may be used instead.

Notation 2.7. *Let $L(x,v) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^2-Lagrangian. We say that L is strictly convex (with respect to v) if $\frac{\partial^2 L}{\partial v^2}$ is uniformly positive definite, that is, if there exists $\alpha > 0$ such that*

$$\langle \frac{\partial^2 L}{\partial v^2}(x,v)\cdot w, w\rangle \geq \alpha \|w\|^2, \qquad \forall\, x\in\mathbb{T}^d, \quad \forall\, v,w\in\mathbb{R}^d.$$

The following proposition shows that a strictly convex Lagrangian with bounded second derivative is ferromagnetic.

Proposition 2.8. *Let $L(x,v) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^2 strictly convex Lagrangian. Then $L(x,v)$ is superlinear. If $L(x,v)$ satisfies in addition the uniform condition $\big\|\frac{\partial^2 L}{\partial x \partial v}\big\|_{\mathbb{T}^d\times\mathbb{R}^d} \leq \beta$ for some $\beta > 0$, then $L(x,v)$ is ferromagnetic. Moreover, for any $x,y\in\mathbb{T}^d$, for any sufficiently small $\tau > 0$, the two maps*

$$v\in\mathbb{R}^d \mapsto \tau\frac{\partial L}{\partial x}(x,v) - \frac{\partial L}{\partial v}(x,v) \in \mathbb{R}^d \quad \textit{and} \quad v\in\mathbb{R}^d \mapsto \frac{\partial L}{\partial v}(y-\tau v, v)\in\mathbb{R}^d,$$

or equivalently the two maps

$$y\in\mathbb{R}^d \mapsto \frac{\partial \mathcal{L}_\tau}{\partial x}(x,y)\in\mathbb{R}^d \quad \textit{and} \quad x\in\mathbb{R}^d \mapsto \frac{\partial \mathcal{L}_\tau}{\partial y}(x,y)\in\mathbb{R}^d,$$

are C^1-diffeomorphisms. In particular, the discrete Euler-Lagrange map Φ_τ is a C^1-diffeomorphism.

Proof. Taylor's formula applied to $L(x,v)$ as a function of v yields

$$L(x,v) = L(x,0) + \frac{\partial L}{\partial v}(x,0)\cdot v + \int_0^1 (1-s)\langle \frac{\partial^2 L}{\partial v^2}(x,sv)\cdot v, v\rangle ds,$$
$$L(x,v) \geq -\|L(\cdot,0)\|_{\mathbb{T}^d} - \big\|\frac{\partial L}{\partial v}(\cdot,0)\big\|_{\mathbb{T}^d}\|v\| + \frac{\alpha}{2}\|v\|^2,$$

which implies that $L(x,v)$ is superlinear. Let $\phi(v) = \tau\frac{\partial L}{\partial x}(x,v) - \frac{\partial L}{\partial v}(x,v)$, for a fixed point $x\in\mathbb{T}^d$. We want to prove that, under the uniform upper bound $\|\frac{\partial^2 L}{\partial x\partial v}\| \leq \beta$, the map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is a C^1-diffeomorphism. The same proof would show that the map $\psi(v) = \frac{\partial L}{\partial v}(y-\tau v, v)$ is also a C^1-diffeomorphism. For every $v,w\in\mathbb{R}^d$, we have

$$\frac{\partial L}{\partial v}(x,w) - \frac{\partial L}{\partial v}(x,v) = \int_0^1 \frac{\partial^2 L}{\partial v^2}\,(x, v+s(w-v))\cdot(w-v)\,ds.$$

By taking the inner product with $w-v$, we obtain

$$\langle \frac{\partial L}{\partial v}(x,w) - \frac{\partial L}{\partial v}(x,v), w-v\rangle \geq \alpha\|w-v\|^2.$$

The Cauchy-Schwartz inequality yields then

$$\|\frac{\partial L}{\partial v}(x,w) - \frac{\partial L}{\partial v}(x,v)\| \geq \alpha\|w-v\|.$$

Moreover, $\frac{\partial L}{\partial x}$ is uniformly Lipschitz in v and therefore satisfies

$$\|\frac{\partial L}{\partial x}(x,w) - \frac{\partial L}{\partial x}(x,v)\| \leq \beta\|w-v\|.$$

Combining these two inequalities, we obtain $\|\phi(w)-\phi(v)\| \geq (\alpha-\tau\beta)\|w-v\|$. So ϕ is one-to-one as soon as $\tau < \alpha\beta^{-1}$. Similar calculations would guarantee that $D\phi$ is invertible. We just have proved that ϕ is an open and injective map. In order to show that ϕ is surjective, we remark ϕ is proper and hence closed. Indeed, if B_R denotes the closed ball of center 0 and radius $R>0$, clearly $\phi(\mathbb{R}^d - B_{R'}) \cap B_R = \emptyset$, or more geometrically $\phi^{-1}(B_R) \subset B_{R'}$, whenever $R' > (\|\phi(0)\| + R)/(\alpha - \tau\beta)$. □

Remark 2.9.

- *The ferromagnetic condition can be proved under weaker hypotheses. Assume $d=1$. Let $f:\mathbb{R}\to\mathbb{R}$ be an increasing homemorphism with $f(0)=0$ and $g:\mathbb{T}^1\to\mathbb{R}$ be a C^1-function. Then $L(x,v) = g(x) + \int_0^v f(w)\,dw$ is a superlinear ferromagnetic C^1-Lagrangian.*
- *We also notice that $\frac{\partial L}{\partial x}$ is a coboundary under the dynamics $(\mathbb{T}^d\times\mathbb{R}^d,\Phi_\tau)$:*

$$\tau\frac{\partial L}{\partial x}(y,w) = \frac{\partial L}{\partial v}(y,w) - \frac{\partial L}{\partial v}\circ\Phi_\tau^{-1}(y,w).$$

3 Minimizing holonomic probabilities

We begin by recalling briefly Mather's approach of minimizing orbits theory. The Lagrangian $L(x,v)$ is usually assumed to be C^2, periodic in x (namely, $x\in\mathbb{T}^d$), strictly convex in $v\in\mathbb{R}^d$ and, for the purposes of this article, time independent.

In Mather's approach, we are interested in finding minimizing absolutely continuous trajectories, that is, trajectories $t\in\mathbb{R}\mapsto x(t)$ such that, for any $t_0<t_1$ and any other trajectory $t\in[t_0,t_1]\mapsto y(t)$ satisfying the boundary conditions $x(t_0)=y(t_0)$ and $x(t_1)=y(t_1)$, the local action of $x(t)$ on $[t_0,t_1]$ is bounded from above by the local action of $y(t)$,

$$\int_{t_0}^{t_1} L(x(t),\dot{x}(t))dt \leq \int_{t_0}^{t_1} L(y(t),\dot{y}(t))dt.$$

Actually we are interested in finding mimimizing trajectories having a prescribed rotation vector $\omega\in\mathbb{R}^d$,

$$\lim_{t\to+\infty}\frac{x(t)}{t} = \lim_{t\to+\infty}\frac{1}{t}\int_0^t \dot{x}(t)dt = \omega.$$

Notice that a minimizing trajectory must satisfy the Euler-Lagrange equation

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial v}(x,\dot{x})\Big) = \frac{\partial L}{\partial x}(x,\dot{x})$$

and is therefore governed by the Euler-Lagrange flow $\Phi_\tau(x_0, \dot{x}_0) = (x_t, \dot{x}_t)$. Suppose in addition that $(x_t, \dot{x}_t)$ is recurrent or more precisely is regular in the sense of Birkhoff's ergodic theorem for some Φ_τ-invariant ergodic probability measure μ on $\mathbb{T}^d \times \mathbb{R}^d$, then

$$\iint_{\mathbb{T}^d\times\mathbb{R}^d} L(x,v)d\mu(x,v) \leq \iint_{\mathbb{T}^d\times\mathbb{R}^d} L(x,v)d\nu(x,v)$$

for any other invariant probability measure ν on $\mathbb{T}^d \times \mathbb{R}^d$.

It is therefore natural to look for minimizing trajectories as regular orbits of the Euler-Lagrange flow located in the support of minimizing measures. Mather's approach can thus be translated into a linear optimization problem

$$\begin{cases} \mu = \text{argmin} \iint_{\mathbb{T}^d\times\mathbb{R}^d} L(x,v)d\mu(x,v) \\ \mu \text{ is a } \Phi_\tau\text{-invariant probability measure} \\ \iint_{\mathbb{T}^d\times\mathbb{R}^d} v\,d\mu(x,v) = \omega. \end{cases}$$

Following R. Mañé [23] and D. A. Gomes [17], one can weaken this optimization problem by asking μ to be only holonomic, that is, satisfying

$$\int_{\mathbb{T}^d} \phi \circ \Phi_\tau(x,v)\, d\mu(x,v) = \int_{\mathbb{T}^d} \phi(x)\, d\mu(x,v)$$

for any bounded Borel (periodic) function $\phi : \mathbb{T}^d \to \mathbb{R}$. Notice that the holonomic condition implies, for any C^1-function $\phi : \mathbb{T}^d \to \mathbb{R}$,

$$\int_{\mathbb{T}^d} \phi(x+\tau v)d\mu(x,v) = \int_{\mathbb{T}^d} \phi(x)d\mu(x,v) + o(\tau),$$

where $o(\tau)$ is some function negligable with respect to τ.

We come back to our discrete Aubry-Mather theory and, as in the weak Mather's approach, we try to look for minimizing configurations located in the support of minimizing invariant measures or more precisely in the support of minimizing holonomic measures since the discrete Euler-Lagrange map may not exist. We denote by $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ the convex set of probability measures over the Borel sets of $\mathbb{T}^d \times \mathbb{R}^d$.

Definition 3.1. *We call holonomic probability measure (or τ-holonomic if needed), a probability measure $\mu \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ satisfying*

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} \phi(x+\tau v)d\mu(x,v) = \int_{\mathbb{T}^d\times\mathbb{R}^d} \phi(x)d\mu(x,v)$$

for any bounded Borel function $\phi : \mathbb{T}^d \to \mathbb{R}$. The set of holonomic probability measures is denoted by $\mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$.

Notice that, in the ferromagnetic case, Φ_τ-invariant probability measures are holonomic. Nevertheless, the holonomic class is larger. For example, any finite configuration $(x_0, x_1, \ldots, x_{n-1})$ gives a holonomic probability $\mu = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{(x_i,v_i)}$, where $v_i = \frac{x_{i+1}-x_i}{\tau}$ and $x_n = x_0$. Notice also that the set of holonomic probability measures is closed under the narrow topology.

We want to show that, although the notion of holonomic probability measures seems to be unrelated to a dynamical system, the set of these measures is nevertheless in one-to-one correspondence with the set of normalized invariant Markov chain of (Σ, σ).

Definition 3.2. *We call normalized invariant Markov chain on (Σ, σ) a sigma-finite Markov chain $(\nu(dx), p(x, dy))$, with initial distribution $\nu(dx)$ (a sigma-finite measure defined on the Borel sets of $\mathbb{R}^d$) and transition kernel $p(x, dy)$ (a measurable family of probability measures defined on the Borel sets of $\mathbb{R}^d$), satisfying the following properties:*

i. $\nu(dx)$ is invariant under the action of $\mathbb{Z}^d$ and has mass one on any fundamental domain,

ii. $p(x, dy)$ is invariant under the action of $\mathbb{Z}^d$ in the following sense

$$\int_{\mathbb{R}^d} \psi(y+s)p(x,dy) = \int_{\mathbb{R}^d} \psi(y)p(x+s,dy),$$

for any bounded Borel function ψ, for any $s \in \mathbb{Z}^d$,

iii. $\nu(dx)$ is Markov-stationary in the following sense

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \psi(y)p(x,dy)\nu(dx) = \int_{\mathbb{R}^d} \psi(y)\nu(dy)$$

for any bounded Borel function ψ.

The sigma-finite Markov chain $\hat{\mu}$ on Σ is given as usual as

$$\int_{\Sigma} \psi(\underline{x})d\hat{\mu}(\underline{x}) = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \psi(x_0, x_1, \ldots, x_n)\nu(dx_0)p(x_0, dx_1)\cdots p(x_{n-1}dx_n),$$

for any bounded Borel function $\psi(\underline{x}) = \psi(x_0, x_1, \ldots, x_n)$, for any $n \geq 0$. Then $\hat{\mu}$ is both invariant with respect to the $\mathbb{Z}^d$ and the shift σ action.

The announced correspondence will be explained through the notion of normalized invariant transshipment measure as it is suggested by L. Evans and D. A. Gomes in [8].

Definition 3.3. *We call normalized invariant transshipment measure π a sigma-finite measure defined on the Borel sets of $\mathbb{R}^d \times \mathbb{R}^d$ verifying the following properties:*

i. π is invariant under the diagonal action of $\mathbb{Z}^d$ and has mass one on any fundamental domain,

ii. if $pr^1 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $pr^2 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ denote the two canonical projections, then $pr^1_*(\pi) = pr^2_*(\pi)$.

We can now prove the equivalence between the set of holonomic probability measures, the set of normalized invariant transshipment measures and the set of normalized invariant Markov chains.

Proposition 3.4. *The three sets of measures, holonomic probability measures μ, normalized invariant transshipment measures π and normalized invariant Markov chains $(\nu(dx), p(x, dy))$ are in one-to-one correspondence. The correspondence is given by:*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \psi(x,y)\pi(dx,dy) := \iint_{\mathbb{R}^d\times\mathbb{R}^d} \psi(x, x+\tau v)\mu(dx,dv),$$
$$\int_{\mathbb{R}^d} \phi(x)\left(\int_{\mathbb{R}^d} \psi(x,y)p(x,dy)\right)\nu(dx) := \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x)\psi(x,y)\pi(dx,dy),$$
$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x,v)\mu(dx,dv) := \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi\left(x, \frac{y-x}{\tau}\right) p(x,dy)\nu(dx),$$

where $\mu(dx, dv)$ has been extended to $\mathbb{R}^d \times \mathbb{R}^d$ by invariance under the action of $\mathbb{Z}^d$ on the first factor.

Proof. Given a holonomic probability measure $\mu(dx, dv)$ and the corresponding measure $\pi(dx, dy)$ defined above, the property $pr^1_*(\pi) = pr^2_*(\pi)$ is easily obtained:

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(y)\pi(dx,dy) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x+\tau v)\mu(dx,dv) =$$
$$= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x)\mu(dx,dv) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x)\pi(dx,dy).$$

Given a normalized invariant transshipment measure $\pi(dx, dy)$ and the associated Markov chain $(\nu(dx), p(x, dy))$ defined in the statement, we first recognize that $\nu = pr^1_*(\pi)$ and that $\{p(x, dy)\}_{x\in\mathbb{R}^d}$ is a desintegration of $\pi(dx, dy)$ over the fibers of pr^1. So we prove the invariance of $\nu(dx)$ and $p(x, dy)$ as follows

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x)\psi(y+s)p(x,dy)\nu(dy) =$$
$$= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x)\psi(y+s)\pi(dx,dy) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x-s)\psi(y)\pi(dx,dy) =$$
$$= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x-s)\psi(y)p(x,dy)\nu(dy) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x)\psi(y)p(x+s,dy)\nu(dy).$$

Besides, the stationarity of $\nu(dx)$ can be shown as follows

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(y)p(x,dy)\nu(dx) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(y)\pi(dx,dy) =$$
$$= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x)\pi(dx,dy) = \int_{\mathbb{R}^d} \phi(x)\nu(dx).$$

Given a normalized invariant Markov chain $(\nu(dx), p(x, dy))$ and the corresponding probability $\mu(dx, dv)$ as defined by the third identity, since $pr^1_*(\mu) = \nu$, the holonomic property follows from

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x+\tau v)\mu(dx,dv) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(y)p(x,dy)\nu(dx) = \\ = \int_{\mathbb{R}^d} \phi(x)\nu(dx) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x)\mu(dx,dv).$$

□

As in the weak Mather's approach, we are interested in finding particular minimizing configurations which are located in the support of minimizing holonomic probability measures. We thus introduce a similar concept equivalent to Mañé's definition of critical value.

Definition 3.5. *Let $L(x,v) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a continuous coercive Lagrangian. We call minimizing holonomic value of L the quantity*

$$\bar{L}(\tau) := \inf_{\mu} \iint_{\mathbb{T}^d\times\mathbb{R}^d} L(x,v)\,\mu(dx,dv),$$

where the infimum is taken over the set of holonomic probability measures. A measure μ attaining the infimum is called a minimizing holonomic probability measure.

Remark 3.6. *The three equivalent definitions given in proposition 3.4 show that any holonomic probability measure μ seen on $\mathbb{T}^d\times\mathbb{R}^d$ can be lifted to a shift-invariant probability measure $\hat{\mu}$ on $\Sigma/_\sim$ obtained from the normalized invariant Markov chain $(\nu(dx), p(x,dy))$. Conversely, the projection $\mu = (\Pi_\tau)_*(\hat{\mu})$ of any shift-invariant probability measure $\hat{\mu}$ on $\Sigma/_\sim$ is holonomic:*

$$\iint_{\mathbb{T}^d\times\mathbb{R}^d} \phi(x+\tau v)\mu(dx,dv) = \int_{\Sigma/_\sim} \phi(x_1)\hat{\mu}(d\underline{x}) = \\ = \int_{\Sigma/_\sim} \phi(x_0)\hat{\mu}(d\underline{x}) = \iint_{\mathbb{T}^d\times\mathbb{R}^d} \phi(x)\mu(dx,dv).$$

From proposition 3.4, the minimizing holonomic value of L may be computed using two different ways

$$\bar{L}(\tau) = \inf_{\pi} \iint_{\mathbb{R}^d\times\mathbb{R}^d/_\sim} L\Big(x, \frac{y-x}{\tau}\Big)\,\pi(dx,dy) = \inf_{\hat{\mu}} \int_{\Sigma/_\sim} L\Big(x_0, \frac{x_1-x_0}{\tau}\Big)\,\hat{\mu}(d\underline{x}),$$

where the infimums are taken, respectively, over the set of normalized invariant transshipment measures π and over the set of shift-invariant probability measures on $\Sigma/_\sim$.

Since $\mathbb{T}^d \times \mathbb{R}^d$ is not compact, the existence of a minimizing holonomic probability measure is not guarantee at first sight. Nevertheless, periodicity in x and coerciveness in $y - x$ implies the existence of such minimizing measures.

Proposition 3.7. *Let $L(x,v) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a continuous coercive Lagrangian. Then there exists a minimizing holonomic probability measure having a compact support.*

Before going into the proof of this result, we will make use of a special piecewise continuous map $F_\tau : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$ which enables us to replace $\mathbb{R}^d$ by a compact ball. Let $\|v\|_\infty = \max_i |v_i|$ be the maximum norm.

Definition 3.8. *Suppose $L(x,v)$ is a coercive Lagrangian, then there exists a real number $R_\tau > 1/\tau$ such that*

$$\inf_{\|v\|_\infty \geq R_\tau} \inf_{x \in \mathbb{T}^d} L(x,v) > \sup_{\|v\|_\infty \leq 1/\tau} \sup_{x \in \mathbb{T}^d} L(x,v).$$

Let $\lfloor v \rfloor \in \mathbb{Z}^d$ denotes the vector whose coordinates are the greatest integers less or equal than the respective coordinates of $v \in \mathbb{R}^d$.

Lemma 3.9. *Let $L(x,v)$ be a C^0 coercive Lagrangian and $F_\tau : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$ defined by*

$$F_\tau(x,v) = \begin{cases} \left(x, v - \frac{1}{\tau} \lfloor \tau v \rfloor\right) & \text{if } \|v\|_\infty \geq R_\tau \\ (x,v) & \text{if } \|v\|_\infty < R_\tau \end{cases}.$$

Then F_τ satisfies

i. the image $F_\tau(\mathbb{T}^d \times \mathbb{R}^d)$ is a bounded set;

ii. $L(x,v) \geq L \circ F_\tau(x,v) \quad \forall\, (x,v) \in \mathbb{T}^d \times \mathbb{R}^d$;

iii. $\mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d) \ \Rightarrow\ (F_\tau)_*\mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$.

Proof. The first item is obviously verified. The second one is just a consequence of the choice of R_τ. Finally, since $\psi(x + \tau v - \lfloor \tau v \rfloor) = \psi(x + \tau v)$ for every $\psi \in C^0(\mathbb{T}^d)$, the third item follows without difficulty. $\square$

We can now prove the existence of minimizing holonomic probability measures for C^0 coercive Lagrangians.

Proof of proposition 3.7. Consider a sequence $\{\mu_n\} \subset \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$ of holonomic probabilities satisfying $\lim_n \int L(x,v)\, d\mu_n(x,v) = \bar{L}(\tau)$. Items *ii* and *iii* of lemma 3.9 assure that the sequence $\{\nu_n = (F_\tau)_*\mu_n\}$ verifies the same properties. Furthermore, by item *i* of the same lemma, all probability measures ν_n are supported on a common compact set. Therefore, any accumulation point $\nu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$ of $\{\nu_n\}$ for the narrow topology satisfies $\int L(x,v)\, d\nu(x,v) = \bar{L}(\tau)$. $\square$

4 Lax-Oleinik operators

The Lax-Oleinik semigroup is well known in partial differential equations and in calculus of variations. It was used by A. Fathi (see [9]) for obtaining the so-called weak KAM theorem in the framework of continuous-time, autonomous, strictly convex and superlinear C^3-Lagrangians on a compact manifold.

In our context, we are interested in studying operators with similar properties to the Lax-Oleinik semigroup. We recall that $\mathcal{L}_\tau(x,y) = \tau L(x, \frac{y-x}{\tau})$.

Definition 4.1. *Given a C^0 coercive Lagrangian $L = L(x, v) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ and a constant $\tau > 0$, we call forward and backward Lax-Oleinik operators, respectively, the maps T_+ and T_- defined by*

$$\begin{aligned} T_+u(x) &= \sup_{v\in\mathbb{R}^d}[u(x+\tau v) - \tau L(x,v)] = \sup_{y\in\mathbb{R}^d}[u(y) - \mathcal{L}_\tau(x,y)], \\ T_-u(y) &= \inf_{v\in\mathbb{R}^d}[u(y-\tau v) + \tau L(y-\tau v,v)] = \inf_{x\in\mathbb{R}^d}[u(y) + \mathcal{L}_\tau(x,y)], \end{aligned}$$

for every $\mathbb{Z}^d$-periodic function $u \in C^0(\mathbb{R}^d)$ that we identify with $u \in C^0(\mathbb{T}^d)$.

Because of the choice of R_τ in definition 3.8 and the fact that the minimization of L can be made on the ball $\|v\|_\infty \le R_\tau$ as explained in lemma 3.9, $T_\pm$ are well defined and have the following more restricted definition

$$\begin{aligned} T_+u(x) &= \max_{\|v\|_\infty\le R_\tau}[u(x+\tau v) - \tau L(x,v)] = \max_{\|y-x\|_\infty\le\tau R_\tau}[u(y) - \mathcal{L}_\tau(x,y)], \\ T_-u(y) &= \min_{\|v\|_\infty\le R_\tau}[u(y-\tau v) + \tau L(y-\tau v,v)] = \min_{\|y-x\|_\infty\le\tau R_\tau}[u(y) + \mathcal{L}_\tau(x,y)]. \end{aligned}$$

Such identities are immediate consequences of the explicit construction of the application $F_\tau : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$ whose properties are described in lemma 3.9. Indeed, writing

$$\phi_+(x,v) = u(x+\tau v) - \tau L(x,v) \ \text{ and } \ \phi_-(x,v) = u(x-\tau v) + \tau L(x-\tau v, v),$$

we constate $\phi_+ \circ F_\tau \ge \phi_+$ and $\phi_- \circ F_\tau \le \phi_-$. So we have

$$\max_{v\in\mathbb{R}^d}\phi_+(x,v) = \max_{v\in\mathbb{R}^d}\phi_+ \circ F_\tau(x,v) = \max_{\|v\|_\infty\le R_\tau}\phi_+(x,v),$$

and similar equalities for ϕ_- as well.

Let $\operatorname{osc}(f, D)$ denote the oscillation of a function f on a subset D of its domain.

Lemma 4.2. *Let $L(x, v)$ be a C^0 coercive Lagrangian. Then the Lax-Oleinik operators verify the following properties.*

i. For all $u \in C^0(\mathbb{T}^d)$, for all $x, y \in \mathbb{T}^d$,

$$|T_+u(x) - T_+u(y)| \le \max_{v^*,w^*}\tau\big|L(x,v^*) - L(y,w^*)\big| \ \textit{ and}$$

$$|T_-u(x) - T_-u(y)| \le \max_{v^*,w^*}\tau\big|L(x-\tau v^*,v^*) - L(y-\tau w^*,w^*)\big|,$$

where the maxima are taken over

$$\|v^*\|_\infty, \|w^*\|_\infty \le 2R_\tau \quad \textit{and} \quad \|v^*-w^*\|_\infty \le \frac{\|x-y\|_\infty}{\tau}.$$

ii. The two operators T_+ and T_- map $C^0(\mathbb{T}^d)$ into itself.

iii. The two sets $T_+(C^0(\mathbb{T}^d))$ and $T_-(C^0(\mathbb{T}^d))$ are equicontinuous.

In particular, $\operatorname{osc}(T_+u, \mathbb{T}^d)$ *and* $\operatorname{osc}(T_-u, \mathbb{T}^d)$ *are bounded by the oscillation of* τL *on* $\mathbb{T}^d \times B_{2R_\tau}$*, where* B_{2R_τ} *denotes the closed ball of center* 0 *and radius* $2R_\tau$*.*

Proof. On the one hand, for any point x in $\mathbb{R}^d$, there exists $z^* \in \mathbb{R}^d$, such that $\|x - z^*\|_\infty \le \tau R_\tau$ and $T_+u(x) = u(z^*) - \mathfrak{L}_\tau(x, z^*)$. On the other hand, for any $y \in \mathbb{R}^d$, $T_+u(y) \ge u(z^*) - \mathfrak{L}_\tau(y, z^*)$. Combining these two estimates, we obtain

$$T_+u(y) - T_+u(x) \ge \mathfrak{L}_\tau(x, z^*) - \mathfrak{L}_\tau(y, z^*) = \tau\big[L(x, v^*) - L(y, w^*)\big]$$

where $w^* = v^* + \frac{x-y}{\tau}$ and $z^* = x + \tau v^*$. A similar estimate holds by permuting x and y which proves the first property for T_+. An analogous argument can be used to demonstrate the inequality concerning the backward Lax-Oleinik operator T_-. Since $L(x, v)$ is uniformly continuous on $\mathbb{T}^d \times B_{2R_\tau}$, the two sets $T_\pm(C^0(\mathbb{T}^d))$ are equicontinuous and the lemma is proved. □

We recall that the minimizing holonomic value $\bar{L}(\tau)$ has been introduced in definition 3.5. So the main theorem of this section can be stated as follows.

Theorem 4.3. *If* $L(x, v)$ *is a* C^0 *coercive Lagrangian, then there exist continuous periodic solutions of the Lax-Oleinik equation,* $u_+, u_- \in C^0(\mathbb{T}^d)$*, satisfying*

$$T_+u_+ = u_+ - \tau\bar{L}(\tau) \quad \textit{and} \quad T_-u_- = u_- + \tau\bar{L}(\tau).$$

Moreover $u_\pm$ *satisfies the a priori estimate:* $\|u_+\|_0,\ \|u_-\|_0 \le \operatorname{osc}(\tau L, \mathbb{T}^d \times B_{2R_\tau})$*.*

Proof. If we equip $C^0(\mathbb{T}^d)$ with the topology of the uniform convergence, it is easy to show that $T_+ : C^0(\mathbb{T}^d) \to C^0(\mathbb{T}^d)$ is 1-Lipschitz. So the Lipschitz regularity is also respected by the application $\hat{T}_+ : C^0(\mathbb{T}^d) \to C^0(\mathbb{T}^d)$ defined by

$$\hat{T}_+u = T_+u - \max(T_+u).$$

Obviously $\hat{T}_+u \le 0$ everywhere on $C^0(\mathbb{T}^d)$. Conversely, it follows from lemma 4.2 that

$$\hat{T}_+u \ge -\operatorname{osc}(\tau L, \mathbb{T}^d \times B_{2R_\tau}), \quad \forall\ u \in C^0(\mathbb{T}^d).$$

Let $\mathfrak{B} \subset C^0(\mathbb{T}^d)$ denote the closed convex hull of the closure of $\hat{T}_+(C^0(\mathbb{T}^d))$. Since the image $\hat{T}_+(C^0(\mathbb{T}^d))$ is bounded, $\mathfrak{B}$ is a compact convex set. As $\hat{T}_+(\mathfrak{B}) \subset \mathfrak{B}$, by the Schauder-Tychonoff fixed point theorem, there exists a function $u_+ \in C^0(\mathbb{T}^d)$ such that

$$T_+u_+ = u_+ + \max(T_+u_+).$$

Obviously, $\|u_+\|_0 = \|\hat{T}_+(u_+)\|_0 \le \operatorname{osc}(\tau L, \mathbb{T}^d \times B_{2R_\tau})$. It remains to show that $\max(T_+u_+) = -\tau\bar{L}(\tau)$. On the one hand

$$\tau L(x, v) + u_+(x) - u_+(x + \tau v) \ge -\max(T_+u_+)$$

everywhere on $\mathbb{T}^d \times \mathbb{R}^d$. For any holonomic probability measure $\mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$, by integrating the previous inequality, we obtain

$$\tau \int L(x, v)\, d\mu = \int \big[\tau L(x, v) + u_+(x) - u_+(x + \tau v)\big]\, d\mu \ge -\max(T_+u_+)$$

and therefore $\max(T_+u_+) \geq -\tau\bar{L}(\tau)$.

On the other hand, given $x_0 \in \mathbb{T}^d$, there exists a vector $v_0 \in \mathbb{R}^d$ such that $\|v_0\|_\infty \leq R_\tau$ and $\tau L(x_0, v_0) + u_+(x_0) - u_+(x_0 + \tau v_0) = -\max(T_+u_+)$. For every $k \geq 1$, if $x_k = x_{k-1} + \tau v_{k-1} \in \mathbb{T}^d$, we consider inductively $v_k \in \mathbb{R}^d$ such that $\|v_k\| \leq R_\tau$ and $\tau L(x_k, v_k) + u_+(x_k) - u_+(x_k + \tau v_k) = -\max(T_+u_+)$. Let $\{\mu_n\}$ be the probability measure defined by

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1}\delta_{(x_k,v_k)}.$$

Since their supports are contained in the compact set $\mathbb{T}^d \times B_{R_\tau}$, such sequence is relatively compact for the narrow topology. Let $\mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$ be some convergent subsequence limit. Note that the equality

$$\int [\tau L(x,v) + u_+(x) - u_+(x+\tau v)] \; d\mu_n(x,v) = -\max T_+u_+,$$

goes througth the limit μ. Hence, we obtain $\max T_+u_+ \leq -\tau\bar{L}(\tau)$ if we prove that μ is a holonomic probability measure. Indeed, for any function $\psi \in C^0(\mathbb{T}^d)$,

$$\begin{aligned}\left|\int [\psi(x+\tau v) - \psi(x)] \; d\mu_n(x,v)\right| &= \frac{1}{n}\left|\sum_{k=0}^{n-1}[\psi(x_k+\tau v_k) - \psi(x_k)]\right| \\ &= \frac{1}{n}\left|\psi(x_n) - \psi(x_0)\right| \leq \frac{2}{n}\|\psi\|_0.\end{aligned}$$

Letting n go to infinity, we immediately obtain that $\mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$. The existence of a function $u_- \in C^0(\mathbb{T}^d)$ is obtained in an analogous way. □

The following result is an immediate consequence of the previous proof.

Corollary 4.4. *Let $L(x,v)$ be a C^0 coercive Lagrangian. If $u \in C^0(\mathbb{T}^d)$ satisfies either $T_+u = u - c$ or $T_-u = u + c$ for some constant $c \in \mathbb{R}$, then $c = \tau\bar{L}(\tau)$.*

The previous theorem 4.3 may be seen as an important theorical tool: it gives a way to renormalize the initial Lagrangian by a coboundary

$$L_{\text{norm}}(x,v) = L(x,v) - \bar{L}(\tau) - \frac{1}{\tau}\big[u(x+\tau v) - u(x)\big] \geq 0, \quad \forall\ (x,v) \in \mathbb{T}^d \times \mathbb{R}^d.$$

The existence of a solution of the Lax-Oleinik operator also gives other characterizations of the minimizing holonomic value, either as a max-min optimal value or as an ergodic average asymptotic value.

Proposition 4.5. *Let $L(x,v)$ be a C^0 coercive Lagrangian. Then we have*

$$\tau\bar{L}(\tau) = \sup_{\psi \in C^0(\mathbb{T}^d)} \ \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \big[\tau L(x,v) + \psi(x) - \psi(x+\tau v)\big]$$

or

$$\tau\bar{L}(\tau) = \inf_{\{x_k\}\in(\mathbb{R}^d)^{\mathbb{Z}_+}} \ \liminf_{n\to\infty} \ \frac{1}{n}\sum_{k=0}^{n-1}\mathcal{L}_\tau(x_k, x_{k+1}).$$

Proof. Let $\mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$ be any minimizing holonomic probability measure. Then

$$\begin{aligned}\tau\bar{L}(\tau) &= \int \tau L(x,v)\, d\mu(x,v) \\ &= \int [\tau L(x,v) + \psi(x) - \psi(x+\tau v)]\ d\mu(x,v) \\ &\geq \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} [\tau L(x,v) + \psi(x) - \psi(x+\tau v)]\end{aligned}$$

for every $\psi \in C^0(\mathbb{T}^d)$. By taking the supremum over all $\psi \in C^0(\mathbb{T}^d)$, one obtain a lower bound of $\tau\bar{L}(\tau)$. Conversely, theorem 4.3 establishes there is $u_+ \in C^0(\mathbb{T}^d)$ such that

$$\tau\bar{L}(\tau) = \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} [\tau L(x,v) + u_+(x) - u_+(x+\tau v)]\,.$$

The first identity is proved. Consider now an arbitrary sequence $\{x_k\} \in \left(\mathbb{R}^d\right)^{\mathbb{Z}_+}$. Then for any $n > 0$

$$n\bar{L}(\tau) \leq \sum_{k=0}^{n-1} \mathcal{L}_\tau(x_k, x_{k+1}) + u_+(x_0) - u_+(x_n) \leq \sum_{k=0}^{n-1} \mathcal{L}_\tau(x_k, x_{k+1}) + 2\|u_+\|_0.$$

Dividing by n and letting n go to infinity, we obtain the above upper bound for $\tau\bar{L}(\tau)$. Conversely, choose any optimal sequence $\{x_k^*\}_{k\geq 0}$ in $\mathbb{R}^d$ such that

$$\tau\bar{L}(\tau) = \mathcal{L}_\tau(x_k^*, x_{k+1}^*) + u_+(x_k^*) - u_+(x_{k+1}^*), \quad \forall\ k \geq 0.$$

Dividing again by n and letting n go to infinity, we then obtain the above lower bound for $\tau\bar{L}(\tau)$ and the second identity is proved. □

Thanks to theorem 4.3, we know that the solutions $u_\pm$ of the Lax-Oleinik operator are continuous. If in addition L is locally α-Hölder continuous, the same estimate of part i in lemma 4.2 shows that $u_\pm$ is also α-Hölder continuous. In fact, these solutions possess a stronger regularity if L is supposed to be semiconcave.

Definition 4.6. *A function $F : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is called semiconcave if, for every $R > 0$, there exists a nondecreasing upper semicontinuous function $\theta_R : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying* $\lim_{\rho\to 0^+} \theta_R(\rho) = 0$ *and*

$$tF(\xi) + (1-t)F(\eta) - F(t\xi + (1-t)\eta) \leq t(1-t)\|\xi - \eta\|\theta_R(\|\xi - \eta\|)$$

for all $\xi = (x,v), \eta = (y,w)$ in $\mathbb{R}^d \times \mathbb{R}^d$ with $\|v\|, \|w\| \leq R$ and for any $t \in [0,1]$. We call $\{\theta_R\}_{R>0}$ a family of local modulus of semiconcavity for F. A function $G : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is called semiconvex if $-G$ is semiconcave.

Notice that in the case the function $F(x)$ depends only in $x \in \mathbb{T}^d$, semiconcavity is defined using an unique modulus θ instead of a family $\{\theta_R\}_{R>0}$. Any C^2-Lagrangian $L(x,v)$ is an example of a semiconcave function. Indeed, for every

$R > 0$, for any $\xi, \eta \in \mathbb{R}^d \times \mathbb{R}^d$ and $t \in [0,1]$, taking $\zeta_t := t\xi + (1-t)\eta$, Taylor's integral formula allows us to write

$$\begin{aligned}
tL(\xi) + (1-t)L(\eta) - L(t\xi + (1-t)\eta) &= \\
&= t(L(\xi) - L(\zeta_t)) + (1-t)(L(\eta) - L(\zeta_t)) \\
&= t\int_0^1 (1-s)D^2L(s\xi + (1-s)\zeta_t)\cdot(\xi - \zeta_t)^2\,ds + \\
&\quad + (1-t)\int_0^1 (1-s)D^2L(s\eta + (1-s)\zeta_t)\cdot(\eta - \zeta_t)^2\,ds \\
&= t(1-t)\int_0^1 (1-s)H(s)\cdot(\xi - \eta)^2\,ds,
\end{aligned}$$

where $H(s) = (1-t)\ \mathrm{Hess}(L)(s\xi + (1-s)\zeta_t) + t\ \mathrm{Hess}(L)(s\eta + (1-s)\zeta_t)$. Let

$$C_R := \frac{1}{2}\max_{x\in\mathbb{T}^d}\ \max_{\|v\|\le R} \|\ \mathrm{Hess}(L)(x,v)\|_\infty.$$

Then $\theta_R(\rho) = C_R\rho$ is a modulus of semiconcavity for L.

For more details on semiconcave functions, we refer the reader to the book of P. Cannarsa and C. Sinestrari (see [4]). Let us examine how the forward Lax-Oleinik operator T_+ deals with semiconcavity.

Proposition 4.7. *Let $L(x,v)$ be a semiconcave C^0 coercive Lagrangian. Then any solution $u \in C^0(\mathbb{T}^d)$ of the forward Lax-Oleinik equation, $T_+u = u - \tau\bar{L}(\tau)$, is semiconvex.*

Proof. Given $x, y \in \mathbb{R}^d$ and $t \in [0,1]$, set $z = tx + (1-t)y$. Then there exists an optimal $z^* \in \mathbb{R}^d$ such that

$$u(z) = u(z^*) - \mathcal{L}_\tau(z, z^*) + \tau\bar{L}(\tau)$$

with $\|z - z^*\|_\infty \le \tau R_\tau$. Moreover

$$u(x) \le u(z^*) - \mathcal{L}_\tau(x, z^*) + \tau\bar{L}(\tau) \quad \text{and} \quad u(y) \le u(z^*) - \mathcal{L}_\tau(y, z^*) + \tau\bar{L}(\tau).$$

Combining these two inequalities and the previous identity, we obtain

$$tu(x) + (1-t)u(y) - u(z) \ge -\big[t\mathcal{L}_\tau(x,z^*) + (1-t)\mathcal{L}_\tau(y,z^*) - \mathcal{L}_\tau(z,z^*)\big].$$

Let v^* and w^* be defined by $z^* = x + \tau v^*$ and $z^* = y + \tau w^*$. Then

$$z^* = z + \tau(tv^* + (1-t)w^*) \quad \text{and} \quad \|v^*\|_\infty, \|w^*\|_\infty \le R_\tau + \Big\|\frac{x-y}{\tau}\Big\|_\infty \le 2R_\tau.$$

Then

$$\begin{aligned}
tu(x) + (1-t)u(y) - u(z) &\ge \\
&\ge -\tau\big[tL(x,v^*) + (1-t)L(y,w^*) - L(z, tv^* + (1-t)w^*)\big] \ge \\
&\ge -\tau t(1-t)\frac{2}{\tau}\|x-y\|_\infty\theta_{2R_\tau}\Big(\frac{2}{\tau}\|x-y\|_\infty\Big)
\end{aligned}$$

using the fact that $\|v^* - w^*\|_\infty = \frac{1}{\tau}\|x-y\|_\infty$. We have shown that $\frac{1}{\tau}u$ is semiconvex with a modulus of convexity $\theta(\rho) = -\frac{2}{\tau}\theta_{2R_\tau}\big(\frac{2}{\tau}\rho\big)$. □

Similarly, any solution u of the backward Lax-Oleinik equation $T_-u = u + \tau\bar{L}(\tau)$ is semiconcave as soon as $L(x,v)$ is a semiconcave C^0 coercive Lagrangian.

5 Calibrated and minimizing configurations

For conciseness, for any given configuration $\{x_k\}_{k\in\mathbb{Z}}$ of points in $\mathbb{R}^d$, for any $m < n$, we call normalized interaction energy of a finite configuration the quantity

$$\bar{\mathcal{L}}_\tau(x_m, x_{m+1}, \ldots, x_n) := \sum_{k=m}^{n-1} \left[\mathcal{L}_\tau(x_k, x_{k+1}) - \tau \bar{L}(\tau)\right].$$

Let us recall from the introduction the fundamental definition of mimimizing configurations.

Definition 5.1. *Consider a bounded below C^0-Lagrangian $L(x, v)$. We say that a configuration $\{x_k\}_{k\in\mathbb{Z}}$ of points of $\mathbb{R}^d$ is a minimizing configuration if, for every pair $m < n$,*

$$\bar{\mathcal{L}}_\tau(x_m, x_{m+1}, \ldots, x_n) \le \bar{\mathcal{L}}_\tau(y_m, y_{m+1}, \ldots, y_n)$$

whenever $\{y_k\}_{k\in\mathbb{Z}}$ satisfies $y_m = x_m$ and $y_n = x_n$. A configuration $\{x_k\}_{k\in\mathbb{Z}}$ is called strongly minimizing configuration if, for any two pair $m < n$, $m' < n'$ and any configuration $\{y_k\}_{k\in\mathbb{Z}}$ satisfying $y_{m'} = x_m$ and $y_{n'} = x_n$ (mod $\mathbb{Z}^d$), we have

$$\bar{\mathcal{L}}_\tau(x_m, x_{m+1}, \ldots, x_n) \le \bar{\mathcal{L}}_\tau(y_{m'}, y_{m'+1}, \ldots, y_{n'}).$$

For a coercive Lagrangian, notice that definition 3.8 implies consecutive jumps $x_{k+1} - x_k$ are uniformly bounded for strongly minimizing configurations $\{x_k\}_{k\in\mathbb{Z}}$, namely,

$$\sup_{k\in\mathbb{Z}} \|x_{k+1} - x_k\|_\infty < \tau R_\tau.$$

The necessity of constructing minimizing configurations motivates the consideration of the following notions.

Definition 5.2. *Let $L(x, v)$ be a C^0 coercive Lagrangian. A function $u : \mathbb{R}^d \to \mathbb{R}$ is a called sub-action*[2] *with respect to L if $u(x)$ is $\mathbb{Z}^d$-periodic, continuous and satisfies*

$$\tau \bar{L}(\tau) \le \tau L(x, v) + u(x) - u(x + \tau v), \quad \forall\ (x, v) \in \mathbb{T}^d \times \mathbb{R}^d.$$

More restrictively, u is called forward calibrated sub-action if

$$\tau \bar{L}(\tau) = \inf_{v\in\mathbb{R}^d} \left[\tau L(x, v) + u(x) - u(x + \tau v)\right], \quad \forall\ x \in \mathbb{T}^d,$$

and similarly u is called backward calibrated sub-action if

$$\tau \bar{L}(\tau) = \inf_{v\in\mathbb{R}^d} \left[\tau L(x - \tau v, v) + u(x - \tau v) - u(x)\right], \quad \forall\ x \in \mathbb{T}^d.$$

Notice that C^0 periodic functions u_+, u_- are forward or backward calibrated sub-actions if, and only if, they are solutions of the forward or backward Lax-Oleinik equations given in theorem 4.3. The existence of sub-actions has been proved under the sole hypothesis of coerciveness.

[2]Perhaps one should insist on the presence of a time step $\tau > 0$ by employing an expression like *τ-sub-action*. However, in order to avoid some unnecessary accuracy and since no misunderstanding is possible, we prefer to let such dependence be implicit.

Proposition 5.3. *Let $L(x,v)$ be a C^0 coercive Lagrangian. If L is locally α-Hölder continuous, then any forward or backward calibrated sub-action $u \in C^0(\mathbb{T}^d)$ is α-Hölder continuous too. Besides, if L is semiconcave, then all forward calibrated sub-actions are semiconvex and all backward calibrated sub-actions are semiconcave.*

Proof. The Hölder case is an immediate consequence of part i of lemma 4.2. The semiconvex property is just a reinterpretation of proposition 4.7 as well as its analogue for backward calibrated sub-actions mentioned after its demonstration. □

Observe that, in terms of the associated local interaction enegy $\mathfrak{L}_\tau(x,y)$, a sub-action u satisfies $\mathfrak{L}_\tau(x,y) \geq u(y) - u(x) + \tau\bar{L}(\tau)$ everywhere on $\mathbb{R}^d \times \mathbb{R}^d$.

Definition 5.4. *Consider a $C^0(\mathbb{T}^d)$ sub-action u for a C^0 coercive Lagrangian $L(x,v)$. A configuration $\{x_k\}_{k\in\mathbb{Z}}$ in $\mathbb{R}^d$ is called u-calibrated if, for every $k \in \mathbb{Z}$, we have $\mathfrak{L}_\tau(x_k, x_{k+1}) = u(x_{k+1}) - u(x_k) + \tau\bar{L}(\tau)$.*

It is easy to see that calibrated configurations are minimizing and even strongly minimizing. We show in the following lemma that the coerciveness assumption implies the existence of calibrated configurations and therefore the existence of minimizing configurations.

Lemma 5.5. *Suppose $L(x,v)$ is a C^0 coercive Lagrangian. If $u \in C^0(\mathbb{T}^d)$ is either a forward or a backward calibrated sub-action, then there exists an u-calibrated configuration $\{x_k\}_{k\in\mathbb{Z}}$ in $\mathbb{R}^d$ passing through some point $x_0 \in [0,1)^d$ and satisfying $\|x_{k+1} - x_k\|_\infty \leq \tau R_\tau$, where $R_\tau > \frac{1}{\tau}$ has been defined in 3.8.*

Lemma 5.6. *Let $L(x,v)$ be a C^0 coercive Lagrangian. If $u \in C^0(\mathbb{T}^d)$ is an arbitrary sub-action, then any u-calibrated configuration $\{x_k\}_{k\in\mathbb{Z}}$ in $\mathbb{R}^d$ is a strongly minimizing configuration satisfying $\|x_{k+1} - x_k\|_\infty \leq \tau R_\tau$ for every $k \in \mathbb{Z}$.*

Proof of lemma 5.5. Let $u \in C^0(\mathbb{T}^d)$ be a forward calibrated sub-action. Thanks to coerciveness, u verifies $\tau\bar{L}(\tau) = \min_{\|v\|_\infty \leq R_\tau} [\tau L(x,v) + u(x) - u(x+\tau v)]$ or

$$\tau\bar{L}(\tau) = \min_{y\,:\,\|y-x\|_\infty \leq \tau R_\tau} \left[\mathfrak{L}_\tau(x,y) + u(x) - u(y)\right], \quad \forall\, x \in \mathbb{R}^d.$$

Hence, for every positive integer n, consider a configuration $\{x_k^n\}_{k\geq -n}$ in $\mathbb{R}^d$ such that $\|x_k^n - x_{k+1}^n\|_\infty \leq \tau R_\tau$ and $\tau\bar{L}(\tau) = L_\tau(x_k^n, x_{k+1}^n) + u(x_k^n) - u(x_{k+1}^n)$ for all $k \geq -n$. Since $L_\tau(x+s, y+s) = L_\tau(x,y)$ for $s \in \mathbb{Z}^d$, we may assume that $x_0^n \in [0,1)^d$ for every $n > 0$. In particular, we get that $\|x_k^n\|_\infty \leq \tau R_\tau |k| + 1$ for all $k \geq -n$. By a diagonal procedure, we extract a configuration $\{x_k\}_{k\in\mathbb{Z}}$ satisfying $\tau\bar{L}(\tau) = \mathfrak{L}_\tau(x_k, x_{k+1}) + u(x_k) - u(x_{k+1})$ for any integer k. A similar reasoning can be developed for $C^0(\mathbb{T}^d)$ backward calibrated sub-actions. □

Proof of lemma 5.6. Let $\{x_k\}_{k\in\mathbb{Z}}$ be an u-calibrated configuration. Thanks to definition 3.8, if $\|x_{k+1} - x_k\|_\infty > \tau R_\tau$, then

$$\begin{aligned} &u(x_{k+1}) - u(x_k) + \tau\bar{L}(\tau) = \mathfrak{L}_\tau(x_k, x_{k+1}) > \\ &\qquad > \mathfrak{L}_\tau(x_k, x_{k+1} - \lfloor x_{k+1} - x_k \rfloor) \geq u(x_{k+1} - \lfloor x_{k+1} - x_k \rfloor) - u(x_k) + \tau\bar{L}(\tau). \end{aligned}$$

The periodicity of u implies that the strict inequality cannot happen. Therefore, $\|x_{k+1} - x_k\|_\infty \leq \tau R_\tau$ for all $k \in \mathbb{Z}$. Moreover, for any $m < n$, for any configuration $\{y_k\}_{k\in\mathbb{Z}}$ in $\mathbb{R}^d$ satisfying $y_{m'} = x_m$ and $y_{n'} = x_n$ (mod $\mathbb{Z}^d$), since

$$\bar{\mathcal{L}}_\tau(x_m, x_{m+1}, \ldots, x_n) = u(x_n) - u(x_m) = u(y_{n'}) - u(y_{m'}) \leq \bar{\mathcal{L}}_\tau(y_{m'}, y_{m'+1}, \ldots, y_{n'}),$$

we obtain that $\{x_k\}_{k\in\mathbb{Z}}$ is a strongly minimizing configuration. □

Notice that the existence of an u-calibrated configuration of a sub-action gives an equivalent definition of the holomic minimizing value $\bar{L}(\tau)$ as defined in 3.5

$$\tau\bar{L}(\tau) = \inf_{\{x_k\}\in(\mathbb{R}^d)^\mathbb{Z}} \liminf_{n-m\to\infty} \frac{1}{n-m} \sum_{k=m}^{n-1} \mathcal{L}_\tau(x_k, x_{k+1}).$$

Furthermore, u-calibrated configurations are examples of critical configurations without assuming any ferromagnetic condition.

Lemma 5.7. *Let $L(x, v)$ be a C^1 coercive Lagrangian. Any u-calibrated configuration of some C^0 periodic sub-action u is critical.*

Proof. Let $\{x_k\}_{k\in\mathbb{Z}}$ be an u-calibrated configuration. Lemma 5.6 implies that $\{x_k\}_{k\in\mathbb{Z}}$ is minimizing and in particular satisfies

$$\mathcal{L}_\tau(x_{k-1}, x_k, x_{k+1}) \leq \mathcal{L}_\tau(x_{k-1}, x, x_{k+1}), \quad \forall\, x \in \mathbb{R}^d.$$

Therefore $\frac{\partial\mathcal{L}_\tau}{\partial y}(x_{k-1}, x_k) + \frac{\partial\mathcal{L}_\tau}{\partial x}(x_k, x_{k+1}) = 0$ for all $k \in \mathbb{Z}$ and $\{x_k\}_{k\in\mathbb{Z}} \in \Gamma_\tau(L)$. □

We have seen in remark 3.6 that any holonomic probability measure can be lifted to a shift-invariant probability measure in $\Sigma/_\sim$ and that

$$\tau\bar{L}(\tau) = \min_{\hat{\mu}\ \sigma-\text{invariant}} \int_{\Sigma/_\sim} \mathcal{L}_\tau(x_0, x_1)\, d\hat{\mu}(\underline{x}).$$

We show in the following proposition how to lift some minimizing holonomic probability measures to $(\Gamma_\tau(L), \sigma)$ or equivalently to $(\mathbb{T}^d \times \mathbb{R}^d, \Phi_\tau)$.

Proposition 5.8. *Let $L(x, v)$ be a C^1 ferromagnetic coercive Lagrangian, then the minimizing holonomic value of L is given by*

$$\bar{L}(\tau) = \min\Big\{ \int L(x, v)\, d\mu(x, v) : \mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d),\ \mu\ \Phi_\tau\text{-invariant}\Big\}.$$

Proof. We already remarked that any Φ_τ-invariant probability is holonomic, then

$$\bar{L}(\tau) \leq \min\Big\{ \int L(x, v)\, d\mu(x, v) : \mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d),\ \mu\ \Phi_\tau\text{-invariant}\Big\}.$$

If $\{x_k\}_{k\in\mathbb{Z}}$ is an u-calibrated configuration for some C^0 periodic sub-action u, then $\{x_k\}_{k\in\mathbb{Z}} \in \Gamma_\tau(L)$ by lemma 5.7. Therefore, thanks to the conjugation between $(\Gamma_\tau(L), \sigma)$ and $(\mathbb{T}^d \times \mathbb{R}^d, \Phi_\tau)$, if $v_k := \frac{x_{k+1}-x_k}{\tau}$, then $(x_k, v_k) = \Phi_\tau^k(x_0, v_0)$ for all

$k \in \mathbb{Z}$ and v_k is uniformly bounded by lemma 5.6. Let $\mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$ be a weak limit of some convergent subsequence

$$\mu_{n_l} = \frac{1}{n_l} \sum_{k=0}^{n_l-1} \delta_{\Phi_\tau^k(x_0,v_0)}.$$

Then μ is Φ_τ-invariant and we have

$$\begin{aligned} \int L(x,v)\, d\mu(x,v) &= \lim_{l\to\infty} \frac{1}{n_l} \sum_{k=0}^{n_l-1} L \circ \Phi_\tau^k(x_0,v_0) \\ &= \lim_{l\to\infty} \frac{1}{n_l} \left[\frac{u(x_{n_l}) - u(x_0)}{\tau} + n_l \bar{L}(\tau) \right] = \bar{L}(\tau). \end{aligned}$$

□

We will see in the next section that, for ferromagnetic Lagrangians, all minimizing holonomic probabilities are actually Φ_τ-invariant.

6 Graph property and Mather set

In the setting of continuous-time, periodic, strictly convex, superlinear and complete C^2-Lagrangians on a compact, connected C^∞ manifold, J. N. Mather showed (see [25]) that measures invariant under the Euler-Lagrange flow which are action minimizing can be seen as Lipschitz sections of the tangent bundle. Our main goal in this section (see theorem 6.10) is to obtain a similar graph property.

Definition 6.1. *Let $L(x,v)$ be a C^0 coercive Lagrangian. We call Mather set the set*

$$\mathcal{M}_\tau(L) = \text{closure}\Big(\bigcup \big\{ \text{supp}(\mu) \ : \ \mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d) \ \textit{and } \mu \textit{ is minimizing} \big\} \Big),$$

where supp(μ) *denotes the support of the probability μ.*

Proposition 3.7 implies that minimizing holonomic probability measures do exist, which shows that the Mather set is nonempty.

Definition 6.2. *Let $L(x,v)$ be a C^0 coercive Lagrangian and u be a C^0 periodic sub-action for L. We call nil locus of u the set*

$$\mathcal{N}_\tau(L,u) = \big\{ (x,v) \in \mathbb{T}^d \times \mathbb{R}^d \ : \ \tau L(x,v) = u(x+\tau v) - u(x) + \tau \bar{L}(\tau) \big\}.$$

We observe that coerciveness guarantees all nil loci are nonempty. The following proposition shows that $\mathcal{N}_\tau(L,u)$ actually contains the support of any minimizing holonomic probability measure.

Proposition 6.3. *Let $L(x,v)$ be a C^0 coercive Lagrangian. Then, for any sub-action $u \in C^0(\mathbb{T}^d)$ with respect to L, we have $\mathcal{M}_\tau(L) \subset \mathcal{N}_\tau(L,u)$.*

Proof. Consider a minimizing holonomic probability measure $\mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$. Since one has both $\tau L(x,v) + u(x) - u(x+\tau v) - \tau \bar{L}(\tau) \geq 0$ and

$$\int \left[\tau L(x,v) + u(x) - u(x+\tau v) - \tau \bar{L}(\tau)\right] \, d\mu(x,v) = 0,$$

$\tau L(x,v) = u(x+\tau v) - u(x) + \tau \bar{L}(\tau)$ holds everywhere on the support of μ. □

As in the proof of lemma 5.6, the coerciveness assumption of L implies that any nil locus is compact. More precisely, we have

Corollary 6.4. *Let $L(x,v)$ be a C^0 coercive Lagrangian and $u \in C^0(\mathbb{T}^d)$ be a sub-action with respect to L. If $(x,v) \in \mathcal{N}_\tau(L,u)$, then $\|v\|_\infty \leq R_\tau$. In particular, the support of any minimizing holonomic probability measure is compact.*

Proof. If $\|v\|_\infty > R_\tau$, then $L(x,v) > L(x, v - \frac{1}{\tau}\lfloor \tau v \rfloor)$. Assume $(x,v) \in \mathcal{N}_\tau(L,u)$, then

$$\begin{aligned} u(x+\tau v) - u(x) &= \tau L(x,v) - \tau \bar{L}(\tau) \\ &> \tau L\big(x, v - \frac{1}{\tau}\lfloor \tau v \rfloor\big) - \tau \bar{L}(\tau) \\ &\geq u\big(x + \tau\big(v - \frac{1}{\tau}\lfloor \tau v \rfloor\big)\big) - u(x) = u(x+\tau v) - u(x). \end{aligned}$$

We obtain a contradiction, therefore $\|v\|_\infty \leq R_\tau$. □

We assume from now on in this section that L is C^1 and coercive. We prove that any sub-action is continuously differentiable on the projected Mather set $pr^1(\mathcal{M}_\tau(L))$, where $pr^1 : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d$ denotes the first canonical projection. When L is in addition ferromagnetic, we prove that $\mathcal{M}_\tau(L)$ is a graph over its projection into $\mathbb{T}^d$. Let us recall that Π_τ has been introduced in definition 2.6.

Lemma 6.5. *Let μ be a holonomic probability measure with compact support, then for any $x \in pr^1(\mathrm{supp}(\mu))$, there exists a configuration $\underline{x} := \{x_k\}_{k\in\mathbb{Z}}$ in $\mathbb{R}^d$ such that $x_0 = x$ and $\Pi_\tau \circ \sigma^k(\underline{x}) = \big(x_k, \frac{x_{k+1}-x_k}{\tau}\big) \in \mathrm{supp}(\mu)$ for all $k \in \mathbb{Z}$.*

Proof. From proposition 3.4, we naturally associate to μ a normalized invariant transshipment π in $\mathbb{R}^d \times \mathbb{R}^d$. Let $pr^{1,2} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be the two canonical projections. Since μ has compact support, the support of π has compact horizontal and vertical slices. Then $S^{1,2} := pr^{1,2}(\mathrm{supp}(\pi))$ are closed sets. We always have $S^{1,2} \subseteq \mathrm{supp}(pr^{1,2}(\pi))$. Since $S^{1,2}$ are closed, necessarily $S^{1,2} = \mathrm{supp}(pr^{1,2}_*(\pi))$. Since π is a transshipment, $pr^1_*(\pi) = pr^2_*(\pi)$ and $S^1 = S^2$. Let $x_0 \in S^1$, then there exists x_1 such that $(x_0, x_1) \in \mathrm{supp}(\pi)$. Since $x_1 \in S^2 = S^1$, there exists x_2 such that $(x_1, x_2) \in \mathrm{supp}(\pi)$, and so on. We thus obtain a forward and backward orbit $\{x_k\}_{k\in\mathbb{Z}}$ of points (x_k, x_{k+1}) in the support of π or equivalently an orbit $\{(x_k, v_k = (x_{k+1} - x_k)/\tau)\}_{k\in\mathbb{Z}}$ of points in the support of μ. □

In order to prove the differentiability of any sub-action on the projected Mather set, we introduce two intermediate notions of calibration.

Definition 6.6. *Let $u \in C^0(\mathbb{R}^d)$ be a $\mathbb{Z}^d$-periodic sub-action for L. A couple $(x_0, x_1) \in \mathbb{R}^d \times \mathbb{R}^d$ is called u-calibrated if $\mathcal{L}_\tau(x_0, x_1) = u(x_1) - u(x_0) + \tau\bar{L}(\tau)$. A triple (x_{-1}, x_0, x_1) is called u-calibrated if both (x_{-1}, x_0) and (x_0, x_1) are u-calibrated.*

Lemma 6.7. *Let $L(x, v)$ be a C^1 coercive Lagrangian and $u \in C^0(\mathbb{T}^d)$ be a sub-action. Then, for any u-calibrated couple $(x_0, x_1) \in \mathbb{R}^d \times \mathbb{R}^d$, we have*

$$\limsup_{\|h\|_\infty \to 0} \frac{1}{\|h\|_\infty} \Big[u(x_1 + h) - u(x_1) - \big\langle \frac{\partial \mathcal{L}_\tau}{\partial y}(x_0, x_1), h\big\rangle\Big] \leq 0 \quad \textit{and}$$
$$\liminf_{\|h\|_\infty \to 0} \frac{1}{\|h\|_\infty} \Big[u(x_0 + h) - u(x_0) - \big\langle - \frac{\partial \mathcal{L}_\tau}{\partial x}(x_0, x_1), h\big\rangle\Big] \geq 0.$$

Proof. Indeed, since u is a sub-action, we have on the one hand

$$u(x_1 + h) \leq u(x_0) + \mathcal{L}_\tau(x_0, x_1 + h) - \tau\bar{L}(\tau) \quad \text{and}$$
$$u(x_1) \leq u(x_0 + h) + \mathcal{L}_\tau(x_0 + h, x_1) - \tau\bar{L}(\tau), \qquad \forall\, h \in \mathbb{R}^d.$$

On the other hand, $u(x_1) = u(x_0) + \mathcal{L}_\tau(x_0, x_1) - \tau\bar{L}(\tau)$, which implies

$$u(x_1 + h) - u(x_1) \leq \big[\mathcal{L}_\tau(x_0, x_1 + h) - \mathcal{L}_\tau(x_0, x_1)\big] \quad \text{and}$$
$$u(x_0 + h) - u(x_0) \geq \big[\mathcal{L}_\tau(x_0, x_1) - \mathcal{L}_\tau(x_0 + h, x_1)\big].$$

The lemma follows from the differentiability of L. □

Although we could use the theory of subdifferentiability and superdifferentiability of L to derive the next lemma, we prefer to give a direct proof to be the most complete possible.

Lemma 6.8. *Let $L(x, v)$ be a C^1 coercive Lagrangian and $u \in C^0(\mathbb{T}^d)$ be a sub-action. Let $\mathcal{K}_\tau(L, u)$ denote the set of mid-points x_0 of all u-calibrated triples (x_{-1}, x_0, x_1).*

i. If (x_{-1}, x_0, x_1) is u-calibrated, then u is differentiable at x_0 and

$$\begin{aligned} Du(x_0) &= \frac{\partial \mathcal{L}_\tau}{\partial y}(x_{-1}, x_0) = -\frac{\partial \mathcal{L}_\tau}{\partial x}(x_0, x_1) \\ &= \frac{\partial L}{\partial v}\big(x_{-1}, \frac{x_0 - x_{-1}}{\tau}\big) = \frac{\partial L}{\partial v}\big(x_0, \frac{x_1 - x_0}{\tau}\big) - \tau\frac{\partial L}{\partial x}\big(x_0, \frac{x_1 - x_0}{\tau}\big). \end{aligned}$$

ii. The map $Du : \mathcal{K}_\tau(L, u) \to \mathbb{R}^d$ is uniformly continuous independently of u.

iii. If L is in addition ferromagnetic, then there exists at most one u-calibrated configuration passing throught any $x_0 \in \mathbb{R}^d$.

Proof. Item i. On the one hand, an u-calibrated triple is critical as in definition 2.3. Let ∇ be the common derivative

$$\nabla := \frac{\partial \mathcal{L}_\tau}{\partial y}(x_{-1}, x_0) = -\frac{\partial \mathcal{L}_\tau}{\partial x}(x_0, x_1).$$

On the other hand, lemma 6.7 implies that

$$\limsup_{\|h\|_\infty \to 0} \frac{u(x_0+h)-u(x_0)-\big\langle \nabla, h\big\rangle}{\|h\|_\infty} \leq 0 \leq \liminf_{\|h\|_\infty \to 0} \frac{u(x_0+h)-u(x_0)-\big\langle \nabla, h\big\rangle}{\|h\|_\infty},$$

which shows that $Du(x_0) = \nabla$.

Item ii. We begin by showing that there exists a positive function $C_\tau(h)$ defined for all $h \in \mathbb{R}^d$, depending only on τ and L, such that $C_\tau(h) \to 0$ when $h \to 0$ and

$$|u(x_0+h)-u(x_0)-\langle Du(x_0), h\rangle| \leq C_\tau(h)\|h\|_\infty,$$

for all $h \in \mathbb{R}^d$, all $x_0 \in \mathcal{K}_\tau(L,u)$ and all sub-action u. Let (x_{-1}, x_0, x_1) be a u-calibrated triple. On the one hand,

$$u(x_1)-u(x_0)-\mathcal{L}_\tau(x_0,x_1) = \tau\bar{L}(\tau) \geq u(x_1)-u(x_0+h)-\mathcal{L}_\tau(x_0+h,x_1),$$

and by eliminating $u(x_1)$ one obtain

$$u(x_0+h)-u(x_0)-\langle Du(x_0), h\rangle \geq -\Big[\mathcal{L}_\tau(x_0+h,x_1)-\mathcal{L}_\tau(x_0,x_1)-\langle\frac{\partial\mathcal{L}_\tau}{\partial x}(x_0,x_1).h\rangle\Big].$$

On the other hand,

$$u(x_0)-u(x_{-1})-\mathcal{L}_\tau(x_{-1},x_0) = -\tau\bar{L}(\tau) \geq u(x_0+h)-u(x_{-1})-\mathcal{L}_\tau(x_{-1},x_0+h),$$

and by eliminating $u(x_{-1})$ one obtain

$$u(x_0+h)-u(x_0)-\langle Du(x_0), h\rangle \leq \mathcal{L}_\tau(x_{-1},x_0+h)-\mathcal{L}_\tau(x_{-1},x_0)-\langle\frac{\partial\mathcal{L}_\tau}{\partial y}(x_{-1},x_0),h\rangle.$$

Notice that $\|x_0-x_{-1}\|_\infty, \|x_1-x_0\|_\infty \leq \tau R(\tau)$ whenever $(x_{-1},x_0,x_1) \in \mathcal{K}_\tau(L,u)$ and that $\mathcal{L}_\tau(x,y)$ is invariant by the diagonal $\mathbb{Z}^d$-translation. Define

$$\begin{aligned} C'_\tau(h) &:= \max_{\|x_1-x_0\|_\infty \leq \tau R(\tau)} \max_{s\in[0,1]} \Big\|\frac{\partial\mathcal{L}_\tau}{\partial x}(x_0+sh,x_1)-\frac{\partial\mathcal{L}_\tau}{\partial x}(x_0,x_1)\Big\|_\infty \\ C''_\tau(h) &:= \max_{\|x_0-x_{-1}\|_\infty \leq \tau R(\tau)} \max_{s\in[0,1]} \Big\|\frac{\partial\mathcal{L}_\tau}{\partial y}(x_{-1},x_0+sh)-\frac{\partial\mathcal{L}_\tau}{\partial y}(x_{-1},x_0)\Big\|_\infty. \end{aligned}$$

Then $C_\tau(h) = \max(C'_\tau(h), C''_\tau(h))$ is the desired function.

We now show that $Du(x_0)$ is uniformly continuous on $\mathcal{K}_\tau(L,u)$. Notice that

$$|u(x_0)-u(x_0-h)-\langle Du(x_0), h\rangle| \leq C_\tau(-h)\|h\|_\infty, \quad \forall\, h \in \mathbb{R}^d.$$

Let x_0 and x'_0 be two distinct mid-points of $\mathcal{K}_\tau(L,u)$. Then, for any h,

$$\begin{gathered} |u(x_0+h)-u(x_0)-\langle Du(x_0), h\rangle| \leq C_\tau(h)\|h\|_\infty, \\ |u(x'_0)-u(x_0+h)-\langle Du(x'_0), x'_0-x_0-h\rangle| \leq C_\tau(x_0+h-x'_0)\|x_0+h-x'_0\|_\infty, \\ |u(x_0)-u(x'_0)-\langle Du(x'_0), x_0-x'_0\rangle| \leq C_\tau(x_0-x'_0)\|x_0-x'_0\|_\infty. \end{gathered}$$

By adding the three inequalities and by taking any $\|h\|_\infty = \|x_0 - x_0'\|_\infty$, we obtain

$$|\langle Du(x_0') - Du(x_0), h\rangle| \leq C_\tau(x_0, x_0')\|h\|_\infty,$$

where

$$C_\tau(x_0, x_0') := \sup_{\|h\|_\infty = \|x_0 - x_0'\|_\infty} [C_\tau(h) + 2C_\tau(x_0 + h - x_0') + C_\tau(x_0 - x_0')].$$

Therefore $\|Du(x_0) - Du(x_0')\|_\infty \leq C_\tau(x_0, x_0')$ and $Du(x_0)$ is uniformly continuous.

Item iii. If $\{x_k\}_{k\in\mathbb{Z}}$ is an u-calibrated configuration, then $Du(x_k)$ exists for all k and the two equations $Du(x_k) = \frac{\partial \mathcal{L}_\tau}{\partial y}(x_{k-1}, x_k)$ and $Du(x_k) = -\frac{\partial \mathcal{L}_\tau}{\partial x}(x_k, x_{k+1})$ shows that x_{k-1} and x_{k+1} are known as soon as x_k is known and L is ferromagnetic. □

The following proposition is now a direct consequence of proposition 6.3 and lemmas 6.5 and 6.8.

Proposition 6.9. *Let $L(x, v)$ be a C^1 coercive Lagrangian. Then any sub-action $u \in C^0(\mathbb{T}^d)$ with respect to L is continuously differentiable on the projected Mather set $pr^1(\mathcal{M}_\tau(L))$, where $pr^1 : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d$ denotes the first canonical projection. If L is in addition $C^{1,1}$, then $Du : pr^1(\mathcal{M}_\tau(L)) \to \mathbb{R}$ is Lipschitz uniformly in u.*

Proof. Recall from lemma 6.8 that $\mathcal{K}_\tau(L, u)$ denotes the set of mid-points of u-calibrated triples. From lemmas 6.5 and proposition 6.3, we deduce that

$$pr^1(\mathcal{M}_\tau(L)) \subseteq \mathcal{K}_\tau(L, u).$$

From 6.8, we obtain that $Du : \mathbb{T}^d \to \mathbb{R}^d$ is continuous. □

Mather graph property is then an easy consequence of the previous study in the case of ferromagnetic Lagrangians.

Theorem 6.10. *Let $L(x, v)$ be a C^1 ferromagnetic coercive Lagrangian. Then there exists a continuous map*

$$V_\tau : pr^1(\mathcal{M}_\tau(L)) \to \mathbb{R}^d, \quad \|V_\tau\|_\infty \leq R_\tau,$$

such that $\mathcal{M}_\tau(L)$ is a graph over its projection, that is,

$$\mathcal{M}_\tau(L) = \mathrm{graph}(V_\tau) = \{(x, V_\tau(x)) \,|\, x \in pr^1(\mathcal{M}_\tau(L)\}.$$

Moreover, $\mathcal{M}_\tau(L)$ is compact and Φ_τ-invariant, any minimizing holonomic probability measure μ is Φ_τ-invariant and, for any sub-action $u \in C^0(\mathbb{T}^d)$, one has

$$Du(x) = \frac{\partial L}{\partial v}(x, V_\tau(x)) - \tau\frac{\partial L}{\partial x}(x, V_\tau(x)), \quad \forall\ x \in pr^1(\mathcal{M}_\tau(L)).$$

Proof. Let $u \in C^0(\mathbb{T}^d)$ be any sub-action. From lemma 6.8, we know that $Du(x)$ exists and is continuous for all $x \in \mathcal{K}_\tau(L,u)$. Since L is ferromagnetic, we define uniquely $V_\tau(x)$ by the following implicit equation

$$Du(x) = \frac{\partial L}{\partial v}(x, V_\tau(x)) - \tau \frac{\partial L}{\partial x}(x, V_\tau(x)), \quad \forall\, x \in \mathcal{K}_\tau(L,u).$$

Then V_τ becomes continuous on $\mathcal{K}_\tau(L,u)$.

Assume now that $x \in pr^1(\mathcal{M}_\tau(L))$. Consider a point $x_{-1} \in \mathbb{T}^d$ with (x_{-1}, x) u-calibrated. Take $v \in \mathbb{R}^d$ such that $(x,v) \in \mathcal{M}_\tau(L)$ and define $x_1 = x + \tau v$. Then (x_{-1}, x, x_1) is u-calibrated and, thanks to lemma 6.8, we have

$$Du(x) = \frac{\partial L}{\partial v}(x,v) - \tau \frac{\partial L}{\partial x}(x,v).$$

Necessarily $v = V_\tau(x)$, $\|V_\tau(x)\|_\infty \le R_\tau$ and $V_\tau(x)$ is independent of the choice of u. From lemma 6.5, we know there exist u-calibrated triples passing through x consisting of points of $pr^1(\mathcal{M}_\tau(L))$. From the ferromagnetic property, we deduce that this triple is unique. Thus $x_1 \in pr^1(\mathcal{M}_\tau(L)) \subseteq \mathcal{K}_\tau(L,u)$ and

$$Du(x_1) = \frac{\partial L}{\partial v}(x, V_\tau(x)) = \frac{\partial L}{\partial v}(x_1, V_\tau(x_1)) - \tau \frac{\partial L}{\partial x}(x_1, V_\tau(x_1)).$$

From the definition of Φ_τ (see definition 2.5), we obtain

$$\Phi_\tau(x,v) = \left(x + \tau v, V_\tau(x + \tau v)\right), \quad \forall\ (x,v) \in \mathcal{M}_\tau(L).$$

In particular, Φ_τ preserves the Mather set (the reverse inclusion is proved similarly)

$$\Phi_\tau(\mathcal{M}_\tau(L)) = \mathcal{M}_\tau(L).$$

Let $\mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$ be a minimizing holonomic probability measure. For any bounded Borel function $\varphi : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, from the previous identity, we have

$$\begin{aligned} \int \varphi \circ \Phi_\tau(x,v)\, d\mu(x,v) &= \int_{\mathcal{M}_\tau(L)} \varphi\left(x + \tau v, V_\tau(x + \tau v)\right)\, d\mu(x,v) \\ &= \int_{\mathcal{M}_\tau(L)} \varphi\left(x, V_\tau(x)\right)\, d\mu(x,v) \\ &= \int \varphi\left(x,v\right)\, d\mu(x,v), \end{aligned}$$

which means the Φ_τ-invariance of the measure μ. □

The last statement of theorem 6.10 is similar to a known result in the case of Lagrangian theory. For a continuous-time, periodic, strictly convex, superlinear and complete C^∞-Lagrangian on a closed Riemannian manifold, R. Mañé showed (see proposition 1.3 of [23]) that any minimizing holonomic measure is invariant under the Euler-Lagrange equations.

7 The Aubry set

In Aubry-Mather theory for continuous-time Lagrangian dynamics, there are generally two strategies for introducing the Aubry set: A. Fathi's formulation (see [11]) using the notion of *conjugate weak KAM solutions* and G. Contreras and R. Iturriaga construction (see [6]) using the notion of *static curves*. Both approaches request intrinsically a differentiable Lagrangian.

We have chosen a different approach, closer to the usual definition in ergodic optimization theory, which has the main advantage of requiring only C^0 smoothness. Aubry set will use the following notion of periodic configuration.

Definition 7.1. *We call periodic configuration of type* (q,p) *a finite configuration* $(x_0, x_1, \cdots, x_q)$ *of points of* $\mathbb{R}^d$ *such that* $x_q = x_0 + p$, $q \geq 1$ *and* $p \in \mathbb{Z}^d$. *Such a finite configuration determines uniquely a bi-infinite configuration* $\{x_k\}_{k\in\mathbb{Z}}$ *satisfying* $x_{q+k} = x_k + p$ *for all* $k \in \mathbb{Z}$.

The notion of Aubry point below is similar to the one of non-wandering point with respect a potential used in ergodic optimization (see, for instance, [7, 15, 16]). A similar projected Aubry set in [18] has also been used in the discrete Aubry-Mather problem. Recall that $\bar{\mathcal{L}}_\tau(x_0, x_1, \cdots, x_q) = \sum_{k=0}^{q-1}[\mathcal{L}_\tau(x_k, x_{k+1}) - \tau \bar{L}(\tau)]$.

Definition 7.2. *Let* $L(x,v) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ *be a* C^0 *coercive Lagrangian. A point* $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$ *is said to be an Aubry point if, for any* $\epsilon > 0$, *there exists a periodic configuration of type* (q,p), $(x_0, x_1, \cdots, x_q)$, *such that*

$$\|x - x_0\|_\infty < \epsilon, \quad \|x + \tau v - x_1\|_\infty < \epsilon \quad \textit{and} \quad |\bar{\mathcal{L}}_\tau(x_0, x_1, \cdots, x_q)| \leq \epsilon.$$

The Aubry set $\mathcal{A}_\tau(L)$ *is by definiton the set of all Aubry points.*

Notice that the Aubry set depends on L modulo any coboundary, that is, for all function $\psi \in C^0(\mathbb{T}^d)$ and any constant $c \in \mathbb{R}$, $\mathcal{A}_\tau(L) = \mathcal{A}_\tau(L - \Delta_\tau\psi - c)$, where $\Delta_\tau\psi(x,v) := \psi(x + \tau v) - \psi(x)$. Notice also that $\bar{\mathcal{L}}_\tau(x_0, x_1, \cdots, x_q) \geq 0$ for any periodic configuration of type (q,p), since $\bar{\mathcal{L}}_\tau(x_0, x_1, \cdots, x_q)$ is unchanged if, instead of L, we use $L - \frac{1}{\tau}\Delta_\tau u - \bar{L}(\tau) \geq 0$ for some sub-action u.

It is easy to see that the Aubry set is a closed subset of $\mathbb{T}^d \times \mathbb{R}^d$. The fact that it is a non empty set is proved in the following proposition.

Proposition 7.3. *Let* $L(x,v)$ *be a* C^0 *coercive Lagrangian and* $u \in C^0(\mathbb{T}^d)$ *be a sub-action with respect to* L. *Then* $\mathcal{M}_\tau(L) \subset \mathcal{A}_\tau(L) \subset \mathcal{N}_\tau(L,u)$.

Proof. We begin by proving the second inclusion. Define the associated normalized Lagrangian $E(x,v) := L(x,v) - \frac{1}{\tau}[u(x + \tau v) - u(x)] - \bar{L}(\tau)$ and the corresponding interaction energy $\mathcal{E}_\tau(x,y)$. Then $E(x,v) \geq 0$ for all $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$ and $\bar{E}(\tau) = 0$. For any $\epsilon > 0$, there exists a periodic configuration $(x_0, x_1, \cdots, x_q)$ such that

$$0 \leq \bar{\mathcal{E}}_\tau(x_0, x_1) \leq \bar{\mathcal{E}}_\tau(x_0, x_1, \cdots, x_q) = \bar{\mathcal{L}}_\tau(x_0, x_1, \cdots, x_q) \leq \epsilon.$$

Letting ϵ go to 0, we obtain $\mathcal{E}_\tau(x, x + \tau v) = 0$ or $(x,v) \in \mathcal{N}_\tau(L,u)$.

We now prove the first inclusion. The proof of this part is non trivial and requires the use of Atkinson's theorem which we recall in 7.4. Let μ be a minimizing holonomic probability measure and $(x,v) \in \mathrm{supp}(\mu)$. By proposition 3.4, the measure μ can be lifted to a normalized shift-invariant Markov chain $\hat{\mu}$ on $\Sigma/_\sim$. Remark 3.6 tells us that $\hat{\mu}$ is a minimizing shift-invariant probability in following sense

$$\tau\bar{L}(\tau) = \int_{\Sigma/\sim} \mathfrak{L}_\tau(x_0,x_1)\,d\hat{\mu}(\underline{x}) = \inf_{\hat{\nu}\ \sigma\text{-invariant}} \int_{\Sigma/\sim} \mathfrak{L}_\tau(x_0,x_1)\,d\hat{\nu}(\underline{x}).$$

Take $\epsilon > 0$. Let B_ϵ denote an open ball of radius $\eta(\epsilon) \in (0,\epsilon)$ around the point (x,v) such that the oscilation of $\mathfrak{L}_\tau(x_0,x_1)$ on $\hat{B}_\epsilon := \Pi_\tau^{-1}(B_\epsilon)$ is less than ϵ. Then $\hat{\mu}(\hat{B}_\epsilon) = \mu(B_\epsilon) > 0$ and, by the ergodic decomposition theorem (see, for instance, chapter 7 of [19]), there exists an ergodic minimizing shift-invariant probability $\hat{\nu}$ which satisfies

$$\hat{\nu}(\hat{B}_\epsilon) > 0 \quad \text{and} \quad \tau\bar{L}(\tau) = \int_{\Sigma/\sim} \mathfrak{L}_\tau(x_0,x_1)\,d\hat{\nu}(\underline{x}).$$

Atkinson's theorem implies that there exist a point $\underline{x} = \{x_k\}_{k\in\mathbb{Z}} \in \hat{B}_\epsilon$ and infinitely many positive integers q such that $\sigma^q(\underline{x}) \in \hat{B}_\epsilon$ and $|\bar{\mathfrak{L}}_\tau(x_0,x_1,\cdots,x_q)| < \epsilon$. By definition of $\hat{B}_\epsilon$, we may assume x_0 and x_1 close to x and $x+\tau v$ within $\eta(\epsilon)$. Moreover, x_q is close to x_0+p within $\eta(\epsilon)$ for some $p \in \mathbb{Z}^d$. We have obtained a periodic configuration $(x_q-p,x_1,\cdots,x_q)$ beginning close to $(x,x+\tau v)$ and satisfying $|\bar{\mathfrak{L}}_\tau(x_q-p,x_1,\cdots,x_q)| < 2\epsilon$. We have shown that $(x,v) \in \mathcal{A}_\tau(L)$. □

Atkinson's theorem is well known. We have nevertheless included a short proof.

Theorem 7.4. (Atkinson's theorem [1]) *Let $(Z,\mathfrak{C},\lambda)$ be a probability space, $T: Z \to Z$ an ergodic measure preserving map, $f: Z \to \mathbb{R}$ an integrable function, $f \in L^1(\lambda)$, and $D \in \mathfrak{C}$ a measurable set of positive measure, $\lambda(D) > 0$. Denote*

$$\Xi(f,D) := \Big\{ z \in D \ :\ \forall\ \epsilon > 0,\ \exists\ n \geq 1 \ \ \textit{with} \\ T^n(z) \in D \quad \textit{and} \quad \Big|\sum_{k=0}^{n-1} f\circ T^k(z) - n\int_Z f\,d\lambda\Big| < \epsilon \Big\}.$$

Then $\lambda(\Xi(f,D)) = \lambda(D)$.

Proof. Without loss of generality, we may assume $\int_Z f\,d\lambda = 0$. Let $\epsilon > 0$ and

$$\Xi_\epsilon(f,D) := \Big\{ z \in D \ :\ \exists\ n \geq 1 \ \text{ with } \ T^n(z) \in D \ \text{ and } \ \Big|\sum_{k=0}^{n-1} f\circ T^k(z)\Big| < \epsilon \Big\}.$$

Since $\Xi(f,D) = \cap_{k\geq 1}\Xi_{1/k}(f,D)$, it is enough to show that $\lambda(\Xi_\epsilon(f,D)) = \lambda(D)$.

Let $f(n,z) = \sum_{k=0}^{n-1} f\circ T^k(z)$ and notice the cocycle property

$$f(0,z) = 0, \quad f(m+n,z) = f(m,z) + f(n,T^m(z)), \quad \forall\ m,n \geq 0,\ \forall\ z \in Z.$$

Suppose on the contrary $C := D - \Xi_\epsilon(f,D)$ has positive measure. Let $\rho : C \to \mathbb{N}_*$ be the first return time to C, $T_C : C \to C$ the first return map to C, $T_C(z) = T^{\rho(z)}(z)$, and

$$f_C(z) = \sum_{k=0}^{\rho(z)-1} f \circ T^k(z), \quad f_C(n,z) = \sum_{k=0}^{n-1} f_C \circ T_C^k(z), \quad \forall\, z \in C,\ \forall\, n \geq 0,$$

the induced cocycle on C. By the definition of C

$$|f_C(n,z) - f_C(m,z)| = |f_C(n-m, T_C^m(z))| \geq \epsilon, \quad \forall\, n > m \geq 0.$$

Let $N \geq 1$. Then each subinterval $[-N\frac{\epsilon}{2} + i\epsilon, -N\frac{\epsilon}{2} + (i+1)\epsilon)$, $i = 0, 1, \ldots, N-1$, of the partition of $I_N := [-N\frac{\epsilon}{2}, N\frac{\epsilon}{2})$ into N intervals of length ϵ contains at most one point of the form $f_C(n,z)$. There exists therefore $k(N,z) \in \{0,1,\ldots,N\}$ such that $f_C(k(N,z),z)$ do not belong to I_N or

$$|f_C(k(N,z),z)| \geq N\frac{\epsilon}{2} \geq k(N,z)\frac{\epsilon}{2}, \quad \forall\, N \geq 1.$$

Since $\{k(N,z)\}_{N\geq 1}$ is not bounded (for $\{f_C(k(N,z),z)\}_{N\geq 1}$ is not bounded because of $|f_C(k(N,z),z)| \geq N\frac{\epsilon}{2}$), we obtain

$$\limsup_{n\to+\infty} \frac{|f_C(n,z)|}{n} \geq \frac{\epsilon}{2}$$

and by the ergodic Birkhoff's theorem,

$$\lim_{n\to+\infty} \frac{f_C(n,z)}{n} = \lim_{n\to+\infty} \frac{\sum_{k=0}^{\rho_n-1} f \circ T^k(z)}{\sum_{k=0}^{\rho_n-1} \mathbf{1}_C \circ T^k(z)} = \frac{\int_Z f\, d\lambda}{\lambda(C)}$$

where ρ_n denotes the nth return time to C, $\rho_n(z) := \sum_{k=0}^{n-1} \rho \circ T_C^k(z)$. We just have obtained the contradiction $|\int_Z f\, d\lambda| \geq \frac{\epsilon}{2}\lambda(C) > 0$. □

We want now to prove that any sub-action is continuously differentiable on the Aubry set. We first show that a finite configuration with bounded interaction energy has bounded jumps independently of the length of the configuration.

Lemma 7.5. *Let $L(x,v)$ be a C^0 coercive Lagrangian. Then for any $E > 0$ there exists $R_E > 0$ such that, for any $n \geq 1$ and any finite configuration $(x_0, x_1, \cdots, x_n)$ of length n with interaction energy bounded from above by E,*

$$\bar{\mathfrak{L}}_\tau(x_0, x_1, \cdots, x_n) \leq E \quad \Longrightarrow \quad \|x_k - x_{k-1}\|_\infty \leq R_E, \quad \forall\, k = 1, \cdots, n.$$

Proof. Let u be a fixed $C^0(\mathbb{T}^d)$ sub-action. By coerciveness of L, for every $E > 0$ there exists $R_E > 0$ such that $|\bar{\mathfrak{L}}_\tau(x,y)| \leq E + 4\|u\|_0 \Rightarrow \|y - x\|_\infty < R_E$. Then

$$\begin{aligned} 0 \leq \bar{\mathfrak{L}}_\tau(x_{k-1}, x_k) + u(x_{k-1}) - u(x_k) \leq \\ \leq \bar{\mathfrak{L}}_\tau(x_0, x_1, \cdots, x_n) + u(x_0) - u(x_n) \leq E + 2\|u\|_0, \end{aligned}$$

$|\bar{\mathfrak{L}}_\tau(x_{k-1}, x_k)| \leq E + 4\|u\|_0$ and $\|x_k - x_{k-1}\| < R_E$. □

We can now extend the conclusion of proposition 6.9.

Proposition 7.6. *Let $u \in C^0(\mathbb{T}^d)$ be a sub-action with respect to a C^1 coercive Lagrangian $L(x,v)$. Then u is continuously differentiable on the projected Aubry set $pr^1(\mathcal{A}_\tau(L))$. If L is in addition $C^{1,1}$ then $Du : pr^1(\mathcal{A}_\tau(L)) \to \mathbb{R}$ is Lipschitz uniformly in u.*

Proof. As in the proof of proposition 6.9, we just need to prove that

$$pr^1(\mathcal{A}_\tau(L) \subseteq \mathcal{K}_\tau(L,u).$$

We normalize again by defining

$$\bar{\mathcal{E}}_\tau(x_0,\cdots,x_n) = \bar{\mathcal{L}}_\tau(x_0,\cdots,x_n) + u(x_0) - u(x_n) \geq 0.$$

Let $(x,v) \in \mathcal{A}_\tau(L)$, $x_0 = x$ and $x_1 = x+\tau v$. Then there exist a sequence of periodic configurations $(x_0^l, x_1^l, \cdots, x_{q(l)}^l)$ and a sequence of integers $p^l \in \mathbb{Z}^d$ such that

$$x_0^l \to x_0, \quad x_1^l \to x_1, \quad x_{q(l)}^l = x_0^l + p^l \quad \text{and} \quad \bar{\mathcal{E}}_\tau(x_0^l, x_1^l, \cdots, x_{q(l)}^l) \to 0.$$

From lemma 7.5 we obtain that $\{x_{q(l)}^l - x_{q(l)-1}^l\}_l$ is uniformly bounded. One can extract a converging subsequence of $\{x_{q(l)-1}^l - p^l\}_l$ to some $x_{-1} \in \mathbb{R}^d$. Since

$$0 \leq \bar{\mathcal{E}}_\tau(x_{q(l)-1}^l - p^l, x_{q(l)}^l - p^l) \leq \bar{\mathcal{E}}_\tau(x_0^l, x_1^l, \cdots, x_{q(l)}^l),$$

$\bar{\mathcal{E}}_\tau(x_{-1}, x_0) = 0$ and (x_{-1}, x_0, x_1) is an u-calibrated triple: $x \in \mathcal{K}_\tau(L,u)$. □

We can now improve theorem 6.10 in the ferromagnetic case. As introduced in definition 2.5, recall that $(\mathbb{T}^d \times \mathbb{R}^d, \Phi_\tau)$ denotes the discrete Euler Lagrange map.

Theorem 7.7. *Let $L(x,v)$ be a C^1 ferromagnetic coercive Lagrangian. Then $\mathcal{A}_\tau(L)$ is compact, Φ_τ-invariant and equal to the graph of some continuous map $V_\tau : pr^1(\mathcal{A}_\tau(L)) \to \mathbb{R}^d$.*

Proof. The proof is similar to the proof of theorem 6.10 thanks to the fact that $pr^1(\mathcal{A}_\tau(L)) \subseteq \mathcal{K}_\tau(L,u)$ for any continuous and periodic sub-action u and to the fact that any $x \in pr^1(\mathcal{A}_\tau(L))$ is the projection of a configuration $\underline{x} = \{x_k\}_{k\in\mathbb{Z}}$ satisfying $\Pi_\tau(\sigma^k(\underline{x})) \in \mathcal{A}_\tau(L)$ for all $k \in \mathbb{Z}$. This is similar to lemma 6.5. The proof of this fact is given in the following lemma 7.8. □

Lemma 7.8. *Let $L(x,v)$ be a C^0 coercive Lagrangian. For any $(x,v) \in \mathcal{A}_\tau(L)$ there exists a configuration $\underline{x} = \{x_k\}_{k\in\mathbb{Z}}$ of points of $\mathbb{R}^d$ such that $\Pi_\tau(x_0,x_1) = (x,v)$ and $\Pi_\tau(\sigma^k(\underline{x})) \in \mathcal{A}_\tau(L)$ for all $k \in \mathbb{Z}$.*

Proof. We begin by normalizing L by assuming $L(x,v) \geq 0$ and $\bar{L} = 0$. Let $(x,v) \in \mathcal{A}_\tau(L)$, $x_0 = x$ and $x_1 = x_0 + \tau v$. Then there exists a sequence of periodic configurations $\underline{x}^l = (x_0^l, x_1^l, \cdots, x_{q(l)}^l)$, $x_{q(l)}^l = x_0^l + p^l$ for some $p^l \in \mathbb{Z}^d$ such that

$$x_0^l \to x_0, \quad x_1^l \to x_1 \quad \text{and} \quad 0 \leq \bar{\mathcal{L}}_\tau(x_0^l, x_1^l, \cdots, x_{q(l)}^l) \to 0.$$

From lemma 7.5 we know there exists $R > 0$ such that all jumps are bounded uniformly, $\|x_k^l - x_{k+1}^l\|_\infty < R$ for all $k \in \mathbb{Z}$. By a diagonal procedure of extraction, there exists a subsequence of $\{\underline{x}^l\}_l$, that we call again $\{\underline{x}^l\}_l$, such that, for all $k \geq 0$, when $l \to \infty$ one has $x_k^l \to x_k$ and $x_{q(l)-k}^l - p^l \to x_{-k}$ for some configuration $\{x_k\}_{k\in\mathbb{Z}}$. By definition of the Aubry set, each $\Pi_\tau(x_k, x_{k+1})$ belongs to $\mathcal{A}_\tau(L)$. A special care should be given in the previous argument when the length l remains bounded. □

8 Mañé potential and Peierls barrier

We introduce in this section two new definition: the Mañé potential and the Peierls barrier. We prove that these notions give an equivalent characterization of the Aubry set and that they give a different way to construct calibrated sub-actions. They will play a fundamental role in the next section to classify all calibrated sub-actions.

Definition 8.1. *Let $L(x,v) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^0 coercive Lagrangian. We call Mañé potential the function $S_\tau : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by*

$$S_\tau(x,y) = \inf_{n\geq 1} \inf_{p\in\mathbb{Z}^d} \inf_{\substack{x_0=x \\ x_n=y+p}} \bar{\mathcal{L}}_\tau(x_0,\ldots,x_n) = \inf_{n\geq 1} \inf_{p\in\mathbb{Z}^d} \inf_{\substack{x_0=x+p \\ x_n=y}} \bar{\mathcal{L}}_\tau(x_0,\ldots,x_n).$$

Notice that $S_\tau(x,y)$ is periodic in both variables x and y.

We first give obvious properties of the Mañé potential.

Remark 8.2. *For any x, y, z in $\mathbb{R}^d$, we have*

- *i.* $S_\tau(x,y) \leq \inf_{p\in\mathbb{Z}^d} \left[\mathcal{L}_\tau(x, y+p) - \tau\bar{L}(\tau)\right] \leq \mathcal{L}_\tau(x,y) - \tau\bar{L}(\tau) = \bar{\mathcal{L}}_\tau(x,y)$,
- *ii.* $u(y) - u(x) \leq S_\tau(x,y)$, *for any sub-action* $u \in C^0(\mathbb{T}^d)$,
- *iii.* $S_\tau(x,y) \leq S_\tau(x,z) + S_\tau(z,y)$,
- *iv.* $S_\tau(x,x) \geq 0$.

We just have seen that coerciveness implies the Mañé potential is a finite function. We show in the next propostion that $S_\tau(x,y)$ is continuous with respect to both x and y.

Proposition 8.3. *Let $L(x,v)$ be a C^0 coercive Lagrangian. Then*

- *i.* $S_\tau(x,y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *is continuous and periodic in x and y,*
- *ii. For every $x, y \in \mathbb{R}^d$, $S_\tau(x,\cdot)$ and $-S_\tau(\cdot,y)$ are $C^0(\mathbb{T}^d)$ sub-actions.*

Proof. We prove the first assertion. Fix $\epsilon > 0$. Take arbitrary points $(x,y), (x',y')$ in $\mathbb{R}^d \times \mathbb{R}^d$. Then there exist two configurations $(x_0,\ldots,x_m)$, $(y_0,\ldots,y_n)$ and two vectors with integer coordinates $r, s \in \mathbb{Z}^d$ such that

$$\bar{\mathcal{L}}_\tau(x_0,\ldots,x_{m-1},x_m+r) \leq S_\tau(x,y') + \epsilon/2, \quad x_0 = x, \quad x_m = y',$$
$$\bar{\mathcal{L}}_\tau(y_0+s,y_1,\ldots,y_n) \leq S_\tau(x,y) + \epsilon/2, \quad y_0 = x, \quad y_n = y.$$

Then $S_\tau(x',y')-S_\tau(x,y) = S_\tau(x',y')-S_\tau(x,y')+S_\tau(x,y')-S_\tau(x,y)$ can be bounded from above using the estimates

$$\begin{aligned} S_\tau(x',y') - S_\tau(x,y') \le& \\ &\le \bar{\mathfrak{L}}_\tau(x',x_1,\dots,x_{m-1},y'+r) - \bar{\mathfrak{L}}_\tau(x,x_1,\dots,x_{m-1},y'+r) + \epsilon/2 \\ &\le \bar{\mathfrak{L}}_\tau(x',x_1) - \bar{\mathfrak{L}}_\tau(x,x_1) + \epsilon/2, \\ S_\tau(x,y') - S_\tau(x,y) \le& \\ &\le \bar{\mathfrak{L}}_\tau(x+s,y_1,\dots,y_{n-1},y') - \bar{\mathfrak{L}}_\tau(x+s,y_1,\dots,y_{n-1},y) + \epsilon/2 \\ &\le \bar{\mathfrak{L}}_\tau(y_{n-1},y') - \bar{\mathfrak{L}}_\tau(y_{n-1},y) + \epsilon/2. \end{aligned}$$

Since $S_\tau(x,y)$ is uniformly bounded from above by periodicity and item *ii* of remark 8.2, lemma 7.5 guarantees that the points x_1 and y_{n-1} which depend on ϵ and x, y, x' and y' are uniformly bounded. So the estimation above shows S_τ is a continuous map.

The second assertion is an immediate corollary of item *iii* of remark 8.2

$$S_\tau(x,z) - S_\tau(x,y) \le S_\tau(y,z) \le \mathfrak{L}_\tau(y,z) - \tau\bar{L}, \quad \forall\ y,z \in \mathbb{R}^d,$$

or in terms of the Lagranigan L

$$S_\tau(x,y+\tau v) - S_\tau(x,y) \le \tau L(x,v) - \tau\bar{L}, \quad \forall\ (y,v) \in \mathbb{T}^d \times \mathbb{R}^d.$$

We just have proved that $S_\tau(x,\cdot)$ is a sub-action. Similarly $-S_\tau(\cdot,y)$ is a sub-action. $\square$

For a C^0 coercive Lagrangian, we clearly deduce $S_\tau(x,x) \ge 0$ from item *ii* of remark 8.2. We show in the following proposition that $S_\tau(x,x) = 0$ characterizes the Aubry set.

Proposition 8.4. *Suppose $L(x,v)$ is a C^0 coercive Lagrangian. Then $S_\tau(x,x) = 0$ if, and only if, $x \pmod{\mathbb{Z}^d} \in pr^1(\mathcal{A}_\tau(L))$.*

Proof. Let us first show that $(x,v) \in \mathcal{A}_\tau(L)$ implies $S_\tau(x,x) = 0$. One can find a sequence of periodic configurations $(x_0^l, x_1^l, \dots, x_{q(l)}^l)$, $x_{q(l)}^l = x_0^l + p^l$, such that

$$x_0^l \to x, \quad x_1^l \to x + \tau v \quad \text{and} \quad \bar{\mathfrak{L}}_\tau(x_0^l, x_1^l, \dots, x_{q(l)}^l) \to 0.$$

Since $S_\tau(x_0^l, x_0^l) = S_\tau(x_0^l, x_{q(l)}^l) \le \bar{\mathfrak{L}}_\tau(x_0^l, x_1^l, \dots, x_{q(l)}^l)$, thanks to the continuity of S_τ and item *iv* of remark 8.2, we obtain $S_\tau(x,x) = 0$.

Conversely, assume $S_\tau(x,x) = 0$. Then there exists a sequence of periodic configurations $(x_0^l, \dots, x_{q(l)}^l)$ such that $x_0^l = x = x_{q(l)}^l - p^l$ and $\bar{\mathfrak{L}}_\tau(x_0^l, x_1^l, \dots, x_{q(l)}^l) \to 0$. Thanks to lemma 7.5, $x_1^l - x_0^l$ remains uniformly bounded. So one can find a subsequence of l's such that $\{(x_1^l - x_0^l)/\tau\}_l$ converges to some $v \in \mathbb{R}^d$. By definition of the Aubry set, $(x,v) \in \mathcal{A}_\tau(L)$. $\square$

Mañé potential enable us to construct continuous sub-actions without using a Lax-Oleinik method. These sub-actions may not be calibrated. We introduce in the following definition a barrier which is similar to Mañé potential, being continuous, periodic with respect to both variables and in addition defining calibrated sub-actions (see theorem 8.10).

Definition 8.5. *Let $L(x,v) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^0 coercive Lagrangian. We call Peierls barrier the function $\mathbf{h}_\tau : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$\mathbf{h}_\tau(x,y) = \liminf_{n\to+\infty} \inf_{p\in\mathbb{Z}^d} \inf_{\substack{x_0=x \\ x_n=y+p}} \bar{\mathfrak{L}}_\tau(x_0,\ldots,x_n) = \liminf_{n\to+\infty} \inf_{p\in\mathbb{Z}^d} \inf_{\substack{x_0=x+p \\ x_n=y}} \bar{\mathfrak{L}}_\tau(x_0,\ldots,x_n).$$

Again notice that $\mathbf{h}_\tau(x,y)$ is periodic with respect to both variables x and y.

We first gather simple properties of the Peierls barrier.

Remark 8.6. *For any x,y,z in $\mathbb{R}^d$*

i. $S_\tau(x,y) \le \mathbf{h}_\tau(x,y)$,

ii. $\mathbf{h}_\tau(x,y) \le S_\tau(x,z) + \mathbf{h}_\tau(z,y)$,

iii. $\mathbf{h}_\tau(x,y) \le \mathbf{h}_\tau(x,z) + S_\tau(z,y)$.

We will prove in a moment that $\mathbf{h}_\tau(x,y)$ satisfies additional properties: $\mathbf{h}_\tau(x,y)$ takes finite values (proposition 8.7), $\mathbf{h}_\tau(x,\cdot)$ and $-\mathbf{h}_\tau(\cdot,y)$ are continuous, periodic calibrated sub-actions for all $x,y \in \mathbb{R}^d$ (theorem 8.10).

We first prove that S_τ and $\mathbf{h}_\tau$ coincide on the projected Aubry set.

Proposition 8.7. *Let $L(x,v)$ be a C^0 coercive Lagrangian. Then for any points $x,y \pmod{\mathbb{Z}^d} \in pr^1(\mathcal{A}_\tau(L))$, $S_\tau(x,\cdot) = \mathbf{h}_\tau(x,\cdot)$ and $S_\tau(\cdot,y) = \mathbf{h}_\tau(\cdot,y)$. In particular $\mathbf{h}_\tau(x,y)$ is finite for all $x,y \in \mathbb{R}^d$.*

Proof. We only prove the first identity. Let $x \pmod{\mathbb{Z}^d} \in pr^1(\mathcal{A}_\tau(L))$ and $y \in \mathbb{R}^d$. For every $\epsilon > 0$, there exists a configuration $(x, y_1, \ldots, y_{m-1}, y+s)$ in $\mathbb{R}^d$, with $m \ge 1$ and $s \in \mathbb{Z}^d$, such that

$$\bar{\mathfrak{L}}_\tau(x, y_1, \ldots, y_{m-1}, y+s) < S_\tau(x,y) + \epsilon.$$

As $S_\tau(x,x) = 0$, for every positive integer l, one can also find a finite configuration $(x, x_1, \ldots, x_{n-1}, x+r)$, with $n \ge 1$ and $r \in \mathbb{Z}^d$, such that

$$\bar{\mathfrak{L}}_\tau(x, x_1, \ldots, x_{n-1}, x+r) < \epsilon/l.$$

Notice that

$$\begin{aligned}(x, x_1, \ldots, x_{n-1}, x+r, x_1+r, \ldots, x_{n-1}+r, x+2r, x_1+2r, \ldots, x_{n-1}+2r, \ldots \\ x+(l-1)r, \ldots, x_{n-1}+(l-1)r, x+rl, y_1+lr, \ldots, y_{m-1}+lr, y+lr+s)\end{aligned}$$

is a configuration of the form $(z_0, z_1, \ldots, z_{nl+m})$ satisfying $z_0 = x$, $z_{nl+m} = y+lr+s$ and

$$\begin{aligned}\bar{\mathfrak{L}}_\tau(z_0, z_1, \ldots, z_{nl+m}) &\le l\bar{\mathfrak{L}}_\tau(x, x_1, \ldots, x_{n-1}, x) + \bar{\mathfrak{L}}_\tau(y, y_1, \ldots, y_{m-1}, y) \\ &\le S_\tau(x,y) + 2\epsilon.\end{aligned}$$

Since l can be chosen arbitrarily large, we deduce that $\mathbf{h}_\tau(x,y) \le S_\tau(x,y) + 2\epsilon$, which immediately yields $\mathbf{h}_\tau(x,y) \le S_\tau(x,y)$.

The fact that $\mathbf{h}_\tau(x,y)$ is finite comes from the inequality

$$\mathbf{h}_\tau(x,y) \le S_\tau(x,z) + \mathbf{h}_\tau(z,y) = S_\tau(x,z) + S_\tau(z,y),$$

where $z \in pr^1(\mathcal{A}_\tau(L))$ is arbitrarily chosen. $\square$

Thanks to proposition 8.3, we conclude that $\mathbf{h}_\tau(x,\cdot)$ and $-\mathbf{h}_\tau(\cdot,y)$ are continuous, $\mathbb{Z}^d$-periodic sub-actions with respect to L as soon as $x,y \in pr^1(\mathcal{A}_\tau(L))$. As a matter of fact, they are also calibrated sub-actions on the projected Aubry set (which is a first step in the proof of theorem 8.10).

Proposition 8.8. *Let $L(x,v)$ be a C^0 coercive Lagrangian.*

i. For any $x \pmod{\mathbb{Z}^d} \in pr^1(\mathcal{A}_\tau(L))$, $S_\tau(x,\cdot)$ is backward calibrated.

ii. For any $y \pmod{\mathbb{Z}^d} \in pr^1(\mathcal{A}_\tau(L))$, $-S_\tau(\cdot,y)$ is forward calibrated.

Proof. We need to show that, for every $y' \in \mathbb{R}^d$, there exists $y \in \mathbb{R}^d$ satisfying

$$\mathbf{h}_\tau(x,y') = \mathbf{h}_\tau(x,y) + \bar{\mathfrak{L}}_\tau(y,y').$$

Since $\mathbf{h}_\tau(x,\cdot)$ is a sub-action, we already know that $\mathbf{h}_\tau(x,y') \le \mathbf{h}_\tau(x,y) + \bar{\mathfrak{L}}_\tau(y,y')$. Conversely, one can find a sequence of configurations in $\mathbb{R}^d$, $(x_0^k, x_1^k, \ldots, x_{n(k)}^k)$, such that $x_0^k = x + p^k$ for some $p^k \in \mathbb{Z}^d$, $x_{n(k)}^k = y'$,

$$n(k) \to +\infty \quad \text{and} \quad \bar{\mathfrak{L}}_\tau(x_0^k, x_1^k, \ldots, x_{n(k)}^k) \to \mathbf{h}_\tau(x,y').$$

Thanks to lemma 7.5, a subsequence of $\{x_{n(k)-1}^k\}_k$ converges to some $y \in \mathbb{R}^d$. Then

$$\begin{aligned}\mathbf{h}_\tau(x, x_{n(k)-1}^k) + \bar{\mathfrak{L}}_\tau(x_{n(k)-1}^k, y') &\le S_\tau(x, x_{n(k)-1}^k) + \bar{\mathfrak{L}}_\tau(x_{n(k)-1}^k, y') \\ &\le \bar{\mathfrak{L}}_\tau(x_0^k, x_1^k, \ldots, x_{n(k)}^k).\end{aligned}$$

Letting k go to $+\infty$, we obtain $\mathbf{h}_\tau(x,y) + \bar{\mathfrak{L}}_\tau(y,y') \le \mathbf{h}_\tau(x,y')$. In an analogous way, we can prove that $-\mathbf{h}_\tau(\cdot,x) = -S_\tau(\cdot,x)$ is a forward calibrated sub-action. □

We have seen that $S_\tau(x,\cdot)$ and $-S_\tau(\cdot,y)$ are continuous, periodic sub-actions for any $x,y \in \mathbb{R}^d$. The following proposition shows that the Peierls barrier can be defined using Mañé potential. (That fact will be used in the proof of theorem 8.10.)

Proposition 8.9. *Assume $L(x,v)$ is a C^0 coercive Lagrangian. Then*

$$\mathbf{h}_\tau(x,y) = \min_{z \in pr^1(\mathcal{A}_\tau(L))} \big[S_\tau(x,z) + S_\tau(z,y)\big], \quad \forall\ x,y \in \mathbb{T}^d.$$

Proof. Propositions 8.7 tells us $\mathbf{h}_\tau(\cdot,y) = S_\tau(\cdot,y)$ and $\mathbf{h}_\tau(x,\cdot) = S_\tau(x,\cdot)$ whenever $x,y \in pr^1(\mathcal{A}_\tau(L))$. Hence, from item *ii* of remark 8.6, we immediately get

$$\mathbf{h}_\tau(x,y) \le \min_{z \in pr^1(\mathcal{A}_\tau(L))} \big[S_\tau(x,z) + S_\tau(z,y)\big].$$

So it suffices to find $z \in pr^1(\mathcal{A}_\tau(L))$ satisfying $S_\tau(x,z) + S_\tau(z,y) \le \mathbf{h}_\tau(x,y)$. Let u be a $C^0(\mathbb{T}^d)$ sub-action. By taking $L(x,v) - \frac{1}{\tau}[u(x+\tau v) - u(x)] - \bar{L}(\tau)$, we may assume $L \ge 0$ and $\bar{L}(\tau) = 0$. Let $\bar{\mathfrak{L}}_\tau(x,y) = \tau L(x, \frac{1}{\tau}(y-x))$ for $x,y \in \mathbb{R}^d$.

By definition of $\mathbf{h}_\tau(x,y)$, there exists a sequence of configurations $(x_0^k, \ldots, x_{n(k)}^k)$ in $\mathbb{R}^d$ of length $n(k)$ and a sequence $p^k \in \mathbb{Z}^d$ such that $x_0^k = x$, $x_{n(k)}^k = y + p^k$,

$$n(k) \to +\infty \quad \text{and} \quad \lim_{k\to\infty} \bar{\mathfrak{L}}_\tau(x_0^k, \ldots, x_{n(k)}^k) = \mathbf{h}_\tau(x,y).$$

Since $\mathbf{h}_\tau(x,y) < \infty$ and $\bar{\mathfrak{L}}_\tau \geq 0$, for k large enough, one can find $m(k)$ and $m'(k)$ in $\{0,\dots,n(k)-1\}$ such that

$$m'(k) - m(k) = \lfloor \sqrt{n_k} \rfloor \quad \text{and} \quad 0 \leq \bar{\mathfrak{L}}_\tau(x^k_{m(k)},\dots,x^k_{m'(k)}) < \frac{\mathbf{h}_\tau(x,y)+1}{\lfloor \sqrt{n_k} \rfloor - 1}.$$

Otherwise, we would reach a contradiction,

$$\begin{aligned} \bar{\mathfrak{L}}_\tau(x^k_0,\dots,x^k_{n_k}) &\geq \sum_{i=0}^{\lfloor \sqrt{n_k} \rfloor - 2} \bar{\mathfrak{L}}_\tau(x^k_{i\lfloor \sqrt{n_k} \rfloor},\dots,x^k_{(i+1)\lfloor \sqrt{n_k} \rfloor}) \\ &\geq (\lfloor \sqrt{n_k} \rfloor - 1)\frac{\mathbf{h}_\tau(x,y)+1}{\lfloor \sqrt{n_k} \rfloor - 1} = \mathbf{h}_\tau(x,y)+1 \end{aligned}$$

for arbitrarily large k.

Thanks to the invariance of $\mathfrak{L}_\tau$ by the diagonal action of $\mathbb{Z}^d$, $\mathfrak{L}_\tau(x+s,y+s) = \mathfrak{L}_\tau(x,y)$ for all $s \in \mathbb{Z}^d$, we may assume $x^k_{m(k)} \in [0,1)^d$. Using a diagonal procedure, lemma 7.5 allows us to find a subsequence $\{k_j\}$ of integers and a forward infinite configuration $\{z_l\}_{l\geq 0}$ of $\mathbb{R}^d$ such that

$$\lim_{j\to\infty} x^{k_j}_{m(k_j)} = z_0 \in [0,1)^d \quad \text{and} \quad \lim_{j\to\infty} x^{k_j}_{m(k_j)+l} = z_l \in \mathbb{R}^d, \quad \forall\, l \geq 1.$$

From the construction of the sequence $\{m_k\}$, it follows that $\mathfrak{L}_\tau(z_l,z_{l+1}) = 0$ for any nonnegative integer l, which clearly yields $S_\tau(z_l,z_{l+1}) = 0$. From item *iii* of remark 8.2, we get $S_\tau(z_l,z_{l'}) = 0$ whenever $l' > l \geq 0$. Therefore, if $z_\infty \in \mathbb{T}^d$ is an arbitrary accumulation point of $\{z_l \pmod{\mathbb{Z}^d}\}_{l\geq 0}$, then $S_\tau(z_\infty,z_\infty) = 0$ or $z_\infty \in pr^1(\mathcal{A}_\tau(L))$. Observe that, for any $l \geq 0$,

$$\begin{aligned} \bar{\mathfrak{L}}_\tau(x^{k_j}_0,\dots,x^{k_j}_{n(k_j)}) &= \bar{\mathfrak{L}}_\tau(x^{k_j}_0,\dots,x^{k_j}_{m(k_j)+l}) + \bar{\mathfrak{L}}_\tau(x^{k_j}_{m(k_j)+l},\dots,x^{k_j}_{n(k_j)}) \\ &\geq S_\tau(x,x^{k_j}_{m(k_j)+l}) + S_\tau(x^{k_j}_{m(k_j)+l},y). \end{aligned}$$

Passing to the limit when $j\to\infty$, we obtain $\mathbf{h}_\tau(x,y) \geq S_\tau(x,z_l)+S_\tau(z_l,y)$. Taking then a suitable subsequence of $\{z_l\}$, we get $\mathbf{h}_\tau(x,y) \geq S_\tau(x,z_\infty)+S_\tau(z_\infty,y)$. $\square$

Theorem 8.10. *Let $L(x,v)$ be a C^0 coercive Lagrangian. Then the Peierls barrier $\mathbf{h}_\tau : \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$ is continuous, $\mathbb{Z}^d\times\mathbb{Z}^d$ periodic. Moreover, $\mathbf{h}_\tau(x,\cdot) : \mathbb{R}^d \to \mathbb{R}$ is a forward calibrated sub-action and $-\mathbf{h}_\tau(\cdot,y) : \mathbb{R}^d \to \mathbb{R}$ is a backward calibrated sub-action for any $x,y \in \mathbb{R}^d$.*

Proof. Consider arbitrary points $(x,y),(x',y') \in \mathbb{T}^d\times\mathbb{T}^d$. Thanks to proposition 8.9, there exists $z_{x,y} \in pr^1(\mathcal{A}_\tau(L))$ satisfying $\mathbf{h}_\tau(x,y) = S_\tau(x,z_{x,y})+S_\tau(z_{x,y},y)$. Then

$$\mathbf{h}_\tau(x',y') - \mathbf{h}_\tau(x,y) \leq \big[S_\tau(x',z_{x,y}) - S_\tau(x,z_{x,y})\big] + \big[S_\tau(z_{x,y},y') - S_\tau(z_{x,y},y)\big].$$

Since S_τ is uniformly continuous on $\mathbb{T}^d\times\mathbb{T}^d$, the estimation above assures that $\mathbf{h}_\tau$ is a continuous map.

We already know that $\mathbf{h}_\tau(x,\cdot)$ and $-\mathbf{h}_\tau(\cdot,x)$ are $\mathbb{T}^d$ periodic and continuous. Take $(y,y') \in \mathbb{R}^d \times \mathbb{R}^d$. Thanks to proposition 8.9, there exists $z \in pr^1(\mathcal{A}_\tau(L))$ such that $\mathbf{h}_\tau(x,y) = S_\tau(x,z) + S_\tau(z,y)$. Then, using the fact that $S_\tau(z,\cdot)$ is a sub-action

$$\mathbf{h}_\tau(x,y') - \mathbf{h}_\tau(x,y) \leq S_\tau(z,y') - S_\tau(z,y) \leq \bar{\mathfrak{L}}(y,y').$$

We have proved that $\mathbf{h}_\tau(x,\cdot)$ is a sub-action. Since $S_\tau(z,\cdot)$ is also backward calibrated, one can find $y'' \in \mathbb{R}^d$ such that $S_\tau(z,y) = S_\tau(z,y'') + \bar{\mathfrak{L}}(y'',y)$. Then

$$\mathbf{h}_\tau(x,y) = S_\tau(x,z) + S_\tau(z,y'') + \bar{\mathfrak{L}}(y'',y) \geq \mathbf{h}_\tau(x,y'') + \bar{\mathfrak{L}}(y'',y).$$

We have proved that $\mathbf{h}_\tau(x,\cdot)$ is calibrated. Analogously, one can show that $-\mathbf{h}_\tau(\cdot,y)$ is a calibrated sub-action too. □

9 Representation formulas for calibrated sub-actions

For a continuous-time, autonomous, strictly convex, superlinear and smooth Lagrangian on a compact Riemannian manifold, G. Contreras characterized in [5] the *weak KAM solutions* of the Hamilton-Jacobi equation in terms of their values at each *static class* and the values of the corresponding *Mañé potential*. Since the weak KAM solutions have similarities with our *calibrated sub-actions* and the set of static classes can be seen as a concept analogous to the *projection of the Aubry set*, the next theorem is comparable to the one presented by Contreras. This kind of analogy has been explored with success, for instance, in a completely abstract setting: a holonomic model of ergodic optimization for symbolic dynamics (see [15]).

Theorem 9.1. *Let $L(x,v)$ be a C^0 coercive Lagrangian. If $u_+ \in C^0(\mathbb{T}^d)$ is a forward calibrated sub-action or $u_- \in C^0(\mathbb{T}^d)$ is a backward calibrated sub-action, then, for every $x, y \in \mathbb{R}^d$,*

$$u_+(x) = \max_{y \in pr^1(\mathcal{A}_\tau(L))} [u_+(y) - S_\tau(x,y)] = \max_{y \in pr^1(\mathcal{A}_\tau(L))} [u_+(y) - \mathbf{h}_\tau(x,y)],$$
$$u_-(y) = \min_{x \in pr^1(\mathcal{A}_\tau(L))} [u_-(x) + S_\tau(x,y)] = \min_{x \in pr^1(\mathcal{A}_\tau(L))} [u_-(x) + \mathbf{h}_\tau(x,y)].$$

Proof. Thanks to proposition 8.7, we just need to prove the two first equalities. From item *ii* of remark 8.2, we verify without difficulty that

$$u_+(x) \geq \max_{y \in pr^1(\mathcal{A}_\tau(L))} [u_+(y) - S_\tau(x,y)].$$

As u_+ is a forward calibrated sub-action, one can find a forward configuration $\{x_k\}_{k\geq 0}$ of $\mathbb{R}^d$ such that $x_0 = x$ and $u_+(x_k) = u_+(x_{k+1}) - \bar{\mathfrak{L}}_\tau(x_k, x_{k+1})$ for every $k \geq 0$. From $u_+(x_l) - u_+(x_k) \leq S_\tau(x_k,x_l) \leq \bar{\mathfrak{L}}_\tau(x_k,\ldots,x_l) = u_+(x_l) - u_+(x_k)$ whenever $l > k \geq 0$, we conclude that $S_\tau(x_k,x_l) = u_+(x_l) - u_+(x_k)$. Therefore, if $y \in \mathbb{T}^d$ is an arbitrary accumulation point of $\{x_k \pmod{\mathbb{Z}^d}\}$, it follows that $S_\tau(y,y) = 0$, namely, $y \in pr^1(\mathcal{A}_\tau(L))$. Furthermore, by taking a suitable subsequence, $u_+(x) = u_+(x_k) - S_\tau(x,x_k)$ tends to $u_+(x) = u_+(y) - S_\tau(x,y)$.

Analogously, one can demonstrate the existence of a point $x \in pr^1(\mathcal{A}_\tau(L))$ achieving $u_-(y) = u_-(x) + S_\tau(x,y)$. □

Corollary 9.2. *Suppose $u, u' \in C^0(\mathbb{T}^d)$ are both either forward or backward calibrated sub-actions with respect to a C^0 coercive Lagrangian $L(x,v)$.*

i. If $u|_{pr^1(\mathcal{A}_\tau(L))} \leq u'|_{pr^1(\mathcal{A}_\tau(L))}$, then $u \leq u'$ everywhere on $\mathbb{R}^d$.

ii. If $u|_{pr^1(\mathcal{A}_\tau(L))} = u'|_{pr^1(\mathcal{A}_\tau(L))}$, then $u = u'$ everywhere on $\mathbb{R}^d$.

Theorem 9.1 admits a reciprocal.

Theorem 9.3. *Let $L(x,v)$ be a C^0 coercive Lagrangian and $\psi : pr^1(\mathcal{A}_\tau(L)) \to \mathbb{R}$ be any function.*

i. If ψ is bounded above, then

$$u(x) := \sup_{y \in pr^1(\mathcal{A}_\tau(L))} [\psi(y) - S_\tau(x,y)] = \sup_{y \in pr^1(\mathcal{A}_\tau(L))} [\psi(y) - \mathbf{h}_\tau(x,y)]$$

defines a $C^0(\mathbb{T}^d)$ forward calibrated sub-action.

ii. If $\psi : pr^1(\mathcal{A}_\tau(L)) \to \mathbb{R}$ is bounded below, then

$$u(y) := \inf_{x \in pr^1(\mathcal{A}_\tau(L))} [\psi(x) + S_\tau(x,y)] = \inf_{x \in pr^1(\mathcal{A}_\tau(L))} [\psi(x) + \mathbf{h}_\tau(x,y)]$$

defines a $C^0(\mathbb{Z}^d)$ backward calibrated sub-action.

iii. If $\psi(y) - \psi(x) \leq S_\tau(x,y)$ for all $x, y \in pr^1(\mathcal{A}_\tau(L))$, then $u|_{pr^1(\mathcal{A}_\tau)} = \psi$.

Proof. In any case, $u : \mathbb{R}^d \to \mathbb{R}$ is clearly a well defined periodic function. Since both constructions are similar, we will discuss just the second one. So let us show that $u \in C^0(\mathbb{T}^d)$. Fix $\epsilon > 0$ and consider $y, y' \in \mathbb{R}^d$. Take $x \pmod{\mathbb{Z}^d} \in pr^1(\mathcal{A}_\tau(L))$ such that $\psi(x) + S_\tau(x,y) < u(y) + \epsilon$. Thus $u(y') - u(y) \leq S_\tau(x,y') - S_\tau(x,y) + \epsilon$. Since S_τ is uniformly continuous on $\mathbb{R}^d \times \mathbb{R}^d$ and $\epsilon > 0$ is arbitrary, it is easy to deduce that

$$|u(y') - u(y)| \leq \max_{x \in pr^1(\mathcal{A}_\tau(L))} |S_\tau(x,y') - S_\tau(x,y)|,$$

which guarantees the continuity of u.

We now show that u is backward calibrated. Given $y \in \mathbb{R}^d$ and $\epsilon > 0$, choose $x \in pr^1(\mathcal{A}_\tau(L))$ satisfying $\psi(x) + S_\tau(x,y) < u(y) + \epsilon$. Thanks to proposition 8.3, $S_\tau(x,\cdot)$ is a sub-action and

$$u(y + \tau w) - u(y) - \epsilon < S_\tau(x, y + \tau w) - S_\tau(x,y) \leq \tau L(y,w) - \tau \bar{L}(\tau), \quad \forall\, w \in \mathbb{R}^d.$$

Letting ϵ go to 0, we obtain that u is a sub-action. To prove that u is a calibrated sub-action, we use the fact that the sub-actions $\{S_\tau(x,\cdot)\}_{x \in pr^1(\mathcal{A}_\tau(L))}$ are calibrated (see proposition 8.8). Let $y \in \mathbb{R}^d$. It suffices to show there exists $v \in \mathbb{R}^d$ such that $u(y) \geq u(y - \tau v) + \tau L(y - \tau v, v) - \tau \bar{L}(\tau)$. By definiton of $u(y)$, there exists a sequence of points $x_k \in pr^1(\mathcal{A}_\tau(L))$ such that $\psi(x_k) + S_\tau(x_k, y) < u(y) + \frac{1}{k}$. Moreover, there exists a sequence of $v_k \in \mathbb{R}^d$ such that

$$S_\tau(x_k, y) = S_\tau(x_k, y - \tau v_k) + \tau L(y - \tau v_k, v_k) - \tau \bar{L}(\tau).$$

Remember we can assume $\|v_k\|_\infty \leq R_\tau$ (see lemma 5.5) for some constant $R_\tau > \frac{1}{\tau}$ (see definition 3.8). Let $v \in \mathbb{R}^d$ be an accumulation point of the sequence $\{v_k\}_{k\geq 0}$. Since $u(y - \tau v_k) \leq \psi(x_k) + S_\tau(x_k, y - \tau v_k)$, we obtain

$$u(y - \tau v_k) + \tau L(y - \tau v_k, v_k) - \tau \bar{L}(\tau) < u(y) + \frac{1}{k}.$$

Taking a suitable subsequence, we get $u(y - \tau v) + \tau L(y - \tau v, v) - \tau \bar{L}(\tau) \leq u(y)$.

Suppose $\psi(y) - \psi(x) \leq S_\tau(x, y)$ for all $x, y \in pr^1(\mathcal{A}_\tau(L))$ and u is defined as in item *ii*. Let $y \pmod{\mathbb{Z}^d} \in pr^1(\mathcal{A}_\tau(L))$. On the one hand, $u(y) \leq \psi(y)$ by taking $x = y$ in the definition of u and noticing that $S_\tau(y, y) = 0$. On the other hand, for any $x \in pr^1(\mathcal{A}_\tau(L))$, $S_\tau(x, y) + \psi(x) \geq \psi(y)$ by hypothesis on ψ. By taking the infimum on x we obtain $u(y) \geq \psi(y)$. We have proved that $u|_{pr^1(\mathcal{A}_\tau(L))} = \psi$. □

Thanks to item *ii* of remark 8.2, an immediate but important consequence of theorem 9.3 is the fact that the restriction of any sub-action to the projected Aubry set behaves as a forward or backward calibrated sub-action.

Corollary 9.4. *Let $u \in C^0(\mathbb{T}^d)$ be an arbitrary sub-action for a C^0 coercive Lagrangian $L(x, v)$. Then, for every point $x \in pr^1(\mathcal{A}_\tau(L))$, we have*

$$u(x) = \max_{v\in\mathbb{R}^d}[u(x+\tau v) - \tau L(x, v) + \tau\bar{L}(\tau)] = \min_{v\in\mathbb{R}^d}[u(x-\tau v) + \tau L(x-\tau v, v) - \tau\bar{L}(\tau)].$$

Theorem 9.3 motivates the introduction of the following notion.

Definition 9.5. *Let $L(x, v)$ be a C^0 coercive Lagrangian. Suppose that u_+ is a $C^0(\mathbb{T}^d)$ forward calibrated sub-action and that u_- is a $C^0(\mathbb{T}^d)$ backward calibrated sub-action. We say that u_+ and u_- are conjugated sub-actions, and we we use the notation $u_+ \sim u_-$, if $u_+|_{pr^1(\mathcal{A}_\tau(L))} = u_-|_{pr^1(\mathcal{A}_\tau(L))}$.*

Notice that coerciveness is a sufficient condition for the existence calibrated sub-actions. Moreover, corollary 9.2 implies that, given a forward calibrated sub-action u_+, there exists at most one backward calibrated u_- conjugated to u_+ and *vice versa*. Finally, theorem 9.3 shows that such a backward calibrated sub-action do exist. More precisely, if u_- is given, the conjugated u_+ takes necessarily the form

$$u_+(x) := \max_{y\in pr^1(\mathcal{A}_\tau(L))} \big[u_-(y) - S_\tau(x, y)\big]$$

and conversely if u_+ is given, the conjugated u_- has the form

$$u_-(x) := \min_{y\in pr^1(\mathcal{A}_\tau(L))} \big[u_+(y) + S_\tau(y, x)\big].$$

A. Fathi pointed out (see [10]) that, for a continuous-time, autonomous, strictly convex, superlinear C^3-Lagrangian on a compact C^∞ manifold without boundary, the Peierls barrier admits a characterization in terms of conjugated sub-actions. We reproduce his result in the following proposition.

Proposition 9.6. *Let $L(x, v)$ be a C^0 coercive Lagrangian. Then*

$$\mathbf{h}_\tau(x, y) = \max_{u_+\sim u_-} \big[u_-(y) - u_+(x)\big], \quad \forall\ x, y \in \mathbb{R}^d.$$

Proof. For any $z \in pr^1(\mathcal{A}_\tau(L))$ and a pair of sub-actions u_+ and u_-, we have

$$u_+(z) - u_+(x) \leq S_\tau(x,z) \quad \text{and} \quad u_-(y) - u_-(z) \leq S_\tau(z,y), \quad \forall\, x,y \in \mathbb{R}^d.$$

If $u_+ \sim u_-$ are conjugated then $u_+(z) = u_-(z)$ and we obtain

$$u_-(y) - u_+(x) \leq S_\tau(x,z) + S_\tau(z,y), \quad \forall\, z \in pr^1(\mathcal{A}_\tau(L)).$$

Thanks to proposition 8.9, we get $u_-(y) - u_+(x) \leq \mathbf{h}_\tau(x,y)$, which obviously yields $\sup_{u_+\sim u_-}[u_-(y) - u_+(x)] \leq \mathbf{h}_\tau(x,y)$.

Fix $x,y \in \mathbb{R}^d$. Consider then the forward calibrated sub-action $u_+ = -\mathbf{h}_\tau(\cdot,y)$ and define a backward calibrated sub-action $u_- \in C^0(\mathbb{T}^d)$ by

$$u_-(x') := \min_{z\in pr^1(\mathcal{A}_\tau(L))}[u_+(z) + S_\tau(z,x')] = \min_{z\in pr^1(\mathcal{A}_\tau(L))}[-S_\tau(z,y) + S_\tau(z,x')].$$

By construction, u_+ and u_- are conjugated sub-actions. Furthermore, $u_-(y) = 0$. Thus $u_-(y) - u_+(x) = \mathbf{h}_\tau(x,y)$. □

Together theorems 9.1 and 9.3 provide an interesting description of the calibrated sub-actions. In order to present it, we decided to adopt a slightly different point of view.

Definition 9.7. *Let $L(x,v)$ be a C^0 coercive Lagrangian. We call positive-time Mañé-Peierls transform the application $\mathcal{F}_+$ defined on $C^0(pr^1(\mathcal{A}_\tau(L)))$ by*

$$\mathcal{F}_+(\psi)(x) = \max_{y\in pr^1(\mathcal{A}_\tau(L))}[\psi(y) - S_\tau(x,y)] = \max_{y\in pr^1(\mathcal{A}_\tau(L))}[\psi(y) - \mathbf{h}_\tau(x,y)], \quad \forall\, x \in \mathbb{R}^d.$$

In the same way, we call negative-time Mañé-Peierls transform the application $\mathcal{F}_-$ defined on $C^0(pr^1(\mathcal{A}_\tau(L)))$ by

$$\mathcal{F}_-(\psi)(y) = \min_{x\in pr^1(\mathcal{A}_\tau(L))}[\psi(x) + S_\tau(x,y)] = \min_{x\in pr^1(\mathcal{A}_\tau(L))}[\psi(x) + \mathbf{h}_\tau(x,y)], \quad \forall\, y \in \mathbb{R}^d.$$

We summarize then all the main properties of the Mañé-Peierls transforms.

Theorem 9.8. *Let $L(x,v)$ be a C^0 coercive Lagrangian. Consider arbitrary functions $\psi, \psi' \in C^0(pr^1(\mathcal{A}_\tau(L)))$. Then*

i. $\mathcal{F}_-(\psi) \leq \psi \leq \mathcal{F}_+(\psi)$ *everywhere on* $pr^1(\mathcal{A}_\tau(L))$;

ii. $\psi \leq \psi'$ *implies* $\mathcal{F}_+(\psi) \leq \mathcal{F}_+(\psi')$ *and* $\mathcal{F}_-(\psi) \leq \mathcal{F}_-(\psi')$;

iii. $\mathcal{F}_+(\psi)$ *is a continuous forward calibrated sub-action;*

iv. $\mathcal{F}_-(\psi)$ *is a continuous backward calibrated sub-action;*

v. *if* $\psi(y) - \psi(x) \leq S_\tau(x,y)$ *for all* $x,y \in pr^1(\mathcal{A}_\tau(L))$, *then* $\mathcal{F}_+$ *and* $\mathcal{F}_-$ *act as extension operators, namely*

$$\mathcal{F}_+(\psi)|_{pr^1(\mathcal{A}_\tau(L))} = \psi = \mathcal{F}_-(\psi)|_{pr^1(\mathcal{A}_\tau(L))};$$

vi. *if* $u \in C^0(\mathbb{T}^d)$ *is a forward calibrated sub-action, then*

$$\mathcal{F}_+(u|_{pr^1(\mathcal{A}_\tau(L))}) = u = \mathcal{F}_+(\mathcal{F}_-(u|_{pr^1(\mathcal{A}_\tau(L))})|_{pr^1(\mathcal{A}_\tau(L))}) \text{ everywhere on } \mathbb{R}^d$$

and $\mathcal{F}_-(u|_{\pi(\mathcal{A}_\tau(L))})$ *is the unique sub-action conjugated to* u*;*

vii. *if* $u \in C^0(\mathbb{T}^d)$ *is a backward calibrated sub-action, then*

$$\mathcal{F}_-(u|_{pr^1(\mathcal{A}_\tau(L))}) = u = \mathcal{F}_-(\mathcal{F}_+(u|_{pr^1(\mathcal{A}_\tau(L))})|_{pr^1(\mathcal{A}_\tau(L))}) \text{ everywhere on } \mathbb{R}^d$$

and $\mathcal{F}_+(u|_{pr^1(\mathcal{A}_\tau(L))})$ *is the unique sub-action conjugated to* u*;*

viii. $\mathcal{F}_+$ *is a bijective and isometric correspondence between the set of the functions* $\psi \in C^0(pr^1(\mathcal{A}_\tau(L)))$ *satisfying, for* $x, y \in pr^1(\mathcal{A}_\tau(L))$, $\psi(y) - \psi(x) \le S_\tau(x, y)$ *and the set of continuous forward calibrated sub-actions;*

ix. $\mathcal{F}_-$ *is a bijective and isometric correspondence between the set of the functions* $\psi \in C^0(\pi(\mathcal{A}_\tau(L)))$ *satisfying, for* $x, y \in pr^1(\mathcal{A}_\tau(L))$, $\psi(y) - \psi(x) \le S_\tau(x, y)$ *and the set of continuous backward calibrated sub-actions.*

Proof. Items *i* and *ii* follow immediately from the respective definitions of the Mañé-Peierls transforms. In truth, items *iii*, *iv* and *v* can be seen as theorem 9.3 rewritten. Besides, items *vi* and *vii* result from theorems 9.1 and 9.3 without difficulty.

Since items *viii* and *ix* are very similar, we will discuss just the first one. As $\mathcal{F}_+(\psi) = \psi$ everywhere on $pr^1(\mathcal{A}_\tau(L))$, $\mathcal{F}_+$ is injective. Moreover, when $u \in C^0(\mathbb{T}^d)$ is a forward calibrated sub-action, the identity $\mathcal{F}_+(u|_{\pi(\mathcal{A}_\tau(L))}) = u$ guarantees that $\mathcal{F}_+$ is surjective. In fact, this correspondence is an isometry. Indeed, fixing $x \in \mathbb{R}^d$, there exists a point $y \in pr^1(\mathcal{A}_\tau(L))$ such that $\mathcal{F}_+(\psi)(x) = \psi(y) - S_\tau(x, y)$. Hence, one has

$$\mathcal{F}_+(\psi)(x) - \mathcal{F}_+(\psi')(x) \le \psi(y) - \psi'(y) \le \|\psi - \psi'\|_0.$$

Since $x \in \mathbb{R}^d$ is arbitrary and since we can interchange the roles of ψ and ψ', we get $\|\mathcal{F}_+(\psi) - \mathcal{F}_+(\psi')\|_0 \le \|\psi - \psi'\|_0$. On the other hand, $\mathcal{F}_+(\psi)|_{\pi(\mathcal{A}_\tau(L))} = \psi$ and $\mathcal{F}_+(\psi')|_{\pi(\mathcal{A}_\tau(L))} = \psi'$ imply $\|\mathcal{F}_+(\psi) - \mathcal{F}_+(\psi')\|_0 \ge \|\psi - \psi'\|_0$. □

10 Separating sub-actions

If $u \in C^0(\mathbb{T}^d)$ is a sub-action for a C^0 coercive Lagrangian, proposition 7.3 establishes that $\mathcal{A}_\tau(L) \subset \mathcal{N}_\tau(L, u)$. So it is natural to ask if there exists a sub-action whose nill locus is the smallest possible, namely, it is equal to the Aubry set. We introduce then the following concept.

Definition 10.1. *Let* $L(x, v)$ *be a* C^0 *coercive Lagrangian. We say that a sub-action* $u \in C^0(\mathbb{T}^d)$ *is separating if* $\mathcal{N}_\tau(L, u) = \mathcal{A}_\tau(L)$.

In weak KAM theory, global critical subsolutions of the Hamilton-Jacobi equation are analogous notions to separating sub-actions. Working with continuous-time, autonomous, strictly convex and superlinear C^2-Lagrangians on a smooth manifold without boundary, A. Fathi and A. Siconolfi (see [12]) proved the existence of C^1 critical subsolutions. Keeping the hypotheses on the Lagrangians but focusing on compact manifolds, P. Bernard showed in [3] not only the existence of $C^{1,1}$ critical subsolutions but also their density in the set of C^0 subsolutions for the uniform topology.

In a surprisingly similar way, even in our discrete and topological Lagrangian context, as states theorem 10.2, separating sub-actions are quite typical. During the preparation of this paper, we become aware of a related study by M. Zavidovique [26] on separating sub-actions (or strict sub-solutions) in a general discrete setting given by cost functions defined on certain length spaces. We mention yet that the genericness of separating sub-actions was recently established also in the conceptual scheme of holonomic models of ergodic optimization for symbolic dynamics (see [16]).

Theorem 10.2. *Let $L(x,v)$ be a C^0 coercive Lagrangian. Then, in the uniform topology, the subset of the continuous separating sub-actions is generic among all continuous sub-actions.*

We will need some preliminary results.

Lemma 10.3. *Let $L(x,v)$ be a C^0 coercive Lagrangian. Then*

$$pr^1\big(\bigcap\nolimits_{u \text{ is a sub-action}} \mathcal{N}_\tau(L,u)\big) = \bigcap\nolimits_{u \text{ is a sub-action}} pr^1\big(\mathcal{N}_\tau(L,u)\big).$$

In other words, if $x \in \mathbb{R}^d$ and for any sub-action u there exists $y \in \mathbb{R}^d$ such that (x,y) is u-calibrated, then there exists $y \in \mathbb{R}^d$ such that (x,y) is u-calibrated for any sub-action u.

Proof. The inclusion $pr^1(\cap_u \mathcal{N}_\tau(L,u)) \subset \cap_u pr^1(\mathcal{N}_\tau(L,u))$ is obvious. Consider then $x \notin pr^1(\cap_u \mathcal{N}_\tau(L,u))$. We want to show there exists a sub-action $u \in C^0(\mathbb{T}^d)$ such that $x \notin pr^1(\mathcal{N}_\tau(L,u))$. Let $u_0 \in C^0(\mathbb{T}^d)$ be a fixed sub-action. We know from corollary 6.4 that one can choose a constant $R_\tau > 0$ such that $(x,v) \notin \mathcal{N}_\tau(L,u_0)$ whenever $\|v\|_\infty > R_\tau$. By hypothesis, for any $\|v\|_\infty \le R_\tau$, there exist a sub-action $u_v \in C^0(\mathbb{T}^d)$ and a constant $\eta_v > 0$ satisfying $(x,w) \notin \mathcal{N}_\tau(L,u_v)$ whenever $\|v-w\| < \eta_v$. By extracting a finite subcover, one can find a finite collection of sub-actions $\{u_1,\ldots,u_n\} \subset C^0(\mathbb{T}^d)$, with $u_k = u_{v_k}$ for some $\|v_k\|_\infty \le R_\tau$, such that $(x,v) \notin \bigcap_{k=1}^n \mathcal{N}_\tau(L,u_k)$ for any $\|v\|_\infty \le R_\tau$.

Define thus $u := \frac{1}{n+1}\sum_{k=0}^n u_k \in C^0(\mathbb{T}^d)$. Since the set of sub-actions is convex, u turns out to be a sub-action. Besides, from $\mathcal{N}_\tau(L,u) = \bigcap_{k=0}^n \mathcal{N}_\tau(L,u_k)$, we immediately obtain $x \notin pr^1(\mathcal{N}_\tau(L,u))$. □

Lemma 10.4. *Let $L(x,v)$ be a C^0 coercive Lagrangian. If $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$, then*

$$(x,v) \in \bigcap\nolimits_{u \text{ is a sub-action}} \mathcal{N}_\tau(L,u) \Longrightarrow x + \tau v \in pr^1\big(\bigcap\nolimits_{u \text{ is a sub-action}} \mathcal{N}_\tau(L,u)\big).$$

In other words, if $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ is u-calibrated for any sub-action u, then there exists $z \in \mathbb{R}^d$ such that (y,z) is u-calibrated for any sub-action u.

Proof. Let us introduce a similar transform as in definition 9.7 by considering

$$\tilde{\mathcal{F}}_+(u)(x) := \max_{y \in \mathbb{R}^d} \big[u(y) - S_\tau(x,y)\big], \quad \forall\, x \in \mathbb{R}^d,$$

where $u \in C^0(\mathbb{T}^d)$ is any sub-action. It is easy to see that $\tilde{\mathcal{F}}_+(u) \in C^0(\mathbb{T}^d)$ is again a sub-action satisfying $\tilde{\mathcal{F}}_+(u) \le u$, with equality everywhere whenever u behaves as a forward calibrated sub-action (see corollary 9.4).

We begin by proving

$$pr^1\big(\mathcal{N}_\tau(L,u)\big) = \big\{x \in \mathbb{T}^d \,:\, \tilde{\mathcal{F}}_+(u)(x) = u(x)\big\}, \quad \forall\, u \text{ sub-action.}$$

Indeed, if (x,y) is u-calibrated, then $u(y)-u(x) \le S_\tau(x,y) \le \bar{\mathcal{L}}_\tau(x,y) = u(y)-u(x)$ which implies $\tilde{\mathcal{F}}_+(u)(x) \ge u(y) - S_\tau(x,y) = u(x)$ and therefore $\tilde{\mathcal{F}}_+(u)(x) = u(x)$. Conversely if $x \notin pr^1(\mathcal{N}_\tau(L,u))$ then, by coerciveness of L and periodicity of u, there exists $\eta > 0$ such that $\bar{\mathcal{L}}_\tau(x,y) \ge u(y) - u(x) + \eta$ for any $y \in \mathbb{R}^d$. For any finite configuration $(x_0, x_1, \ldots, x_n)$ satisfying $x_0 = x$, one has

$$\begin{aligned}\bar{\mathcal{L}}_\tau(x_0,x_1,\ldots,x_n) &= \bar{\mathcal{L}}_\tau(x_0,x_1) + \bar{\mathcal{L}}_\tau(x_1,\ldots,x_n) \\ &\ge [u(x_1) - u(x_0) + \eta] + [u(x_n) - u(x_1)] \ge u(x_n) - u(x_0) + \eta.\end{aligned}$$

By definition of $S_\tau(x,y)$, one gets $S_\tau(x,y) \ge u(y) - u(x) + \eta$ for any $y \in \mathbb{R}^d$ or equivalently $u(x) \ge \tilde{\mathcal{F}}_+(u)(x) + \eta$.

We now prove the main induction step:

$$\tilde{\mathcal{F}}_+(u)(x) = u(x) \;\text{ and }\; \bar{\mathcal{L}}_\tau(x,y) = \tilde{\mathcal{F}}_+(u)(y) - \tilde{\mathcal{F}}_+(u)(x) \implies \tilde{\mathcal{F}}_+(u)(y) = u(y).$$

Indeed, $u(y) - u(x) \le S_\tau(x,y) \le \bar{\mathcal{L}}_\tau(x,y) = \tilde{\mathcal{F}}_+(u)(y) - \tilde{\mathcal{F}}_+(u)(x)$, which implies first $u(y) \le \tilde{\mathcal{F}}_+(u)(y)$ and therefore $u(y) = \tilde{\mathcal{F}}_+(u)(y)$.

We conclude the proof. If (x,y) is u-calibrated for any sub-action u, on the one hand, $\tilde{\mathcal{F}}_+(u)(x) = u(x)$, on the other hand, since (x,y) is also $\tilde{\mathcal{F}}_+(u)$-calibrated, $\tilde{\mathcal{F}}_+(u)(y) = u(y)$. We have proved that $y \in \cap_u\{\tilde{\mathcal{F}}_+(u) = u\} = pr^1(\cap_u \mathcal{N}_\tau(L,u))$ thanks to lemma 10.3. □

The following proposition gives another equivalent definition of the Aubry set.

Proposition 10.5. *Let $L(x,v)$ be a C^0 coercive Lagrangian. Then*

$$\mathcal{A}_\tau(L) = \bigcap\nolimits_{u \text{ is a sub-action}} \mathcal{N}_\tau(L,u).$$

Proof. Let $(x,v) \in \cap_u \mathcal{N}_\tau(L,u)$. Lemma 10.4 shows there exists a configuration $\underline{x} = \{x_k\}_{k\ge0}$ such that $\Pi_\tau(\underline{x}) = (x,v)$ and (x_k, x_{k+1}) is u-calibrated for any sub-action u. Let us first show that

$$l_m := \bar{\mathcal{L}}_\tau(x_0,\ldots,x_m) + S_\tau(x_m,x_0) \to 0 \quad \text{when} \quad m \to +\infty.$$

Since $\{\bar{\mathcal{L}}_\tau(x_0,\ldots,x_m) + S_\tau(x_m,x_0)\}_{m\ge0}$ is uniformly bounded, one can choose a converging subsequence of $\{l_m\}$ and assume in addition that $\{x_m \pmod{\mathbb{Z}^d}\}$ converges to a point $x_\infty \in \mathbb{T}^d$. Define $u(x) := S_\tau(x_\infty, x)$, for all $x \in \mathbb{R}^d$. Proposition 8.3 shows that u is a sub-action. By hypothesis of calibration on $\{x_k\}$, we have

$$\bar{\mathcal{L}}_\tau(x_k,x_{k+1}) = S_\tau(x_k,x_{k+1}) = u(x_{k+1}) - u(x_k), \quad \forall\, k \ge 0.$$

More generally,

$$\bar{\mathcal{L}}_\tau(x_k,x_{k+1},\ldots,x_m) = S_\tau(x_k,x_m) = u(x_m) - u(x_k), \quad \forall\, m > k \ge 0.$$

By taking a subsequence of $\{x_m\}$, on obtains first $S_\tau(x_k, x_\infty) = u(x_\infty) - u(x_k)$, for all $k \geq 0$. By taking a subsequence of $\{k\}$, one obtains next

$$u(x_\infty) = S_\tau(x_\infty, x_\infty) = 0 \quad \text{and} \quad S_\tau(x_k, x_\infty) + S_\tau(x_\infty, x_k) = 0, \quad \forall\, k \geq 0.$$

Notice that x_∞ necessarily belongs to $pr^1(\mathcal{A}_\tau(L))$. Moreover,

$$l_m = \bar{\mathfrak{L}}_\tau(x_0, \ldots, x_m) + S_\tau(x_m, x_0) = S_\tau(x_0, x_m) + S_\tau(x_m, x_0), \quad \forall\, m \geq 0.$$

Letting $m \to +\infty$, one gets $l_m \to 0$ along a subsequence. We thus have shown that any accumulation point of $\{l_m\}$ is necessarily 0.

Let us prove now that $(x, v) \in \mathcal{A}_\tau(L)$. By definition of S_τ, there exist finite configurations $(x^\epsilon_m, x^\epsilon_{m+1}, \ldots, x^\epsilon_{n(m,\epsilon)})$ such that $x^\epsilon_m = x_m$, $x^\epsilon_{n(m,\epsilon)} = x_0 + p^\epsilon_m$ for some $p^\epsilon_m \in \mathbb{Z}^d$ and, for any m fixed,

$$\bar{\mathfrak{L}}_\tau(x^\epsilon_m, x^\epsilon_{m+1}, \ldots, x^\epsilon_{n(m,\epsilon)}) \to S_\tau(x_m, x_0) \quad \text{when} \quad \epsilon \to 0.$$

We conclude that

$$\begin{aligned} &\bar{\mathfrak{L}}_\tau(x_0, \ldots, x_{m-1}, x^\epsilon_m, x^\epsilon_{m+1}, \ldots, x^\epsilon_{n(m,\epsilon)}) = \\ &\qquad\qquad = l_m + \bar{\mathfrak{L}}_\tau(x^\epsilon_m, x^\epsilon_{m+1}, \ldots, x^\epsilon_{n(m,\epsilon)}) - S_\tau(x_m, x_0) \end{aligned}$$

tends to 0 when m is first chosen large enough and then ϵ is chosen close enough to 0. Thus $(x, v) \in \mathcal{A}_\tau(L)$. □

We now prove that separating sub-actions are generic among sub-actions.

Proof of theorem 10.2. Let $\{O_n\}_n$ be a countable family of open neighborhoods of the Aubry set such that $\cap_n O_n = \mathcal{A}_\tau(L)$. Let $\mathfrak{U}_n$ be the set of all $C^0(\mathbb{T}^d)$ sub-actions u such that $\mathcal{N}_\tau(L, u) \subset O_n$. Since the subset of $C^0(\mathbb{T}^d)$ separating sub-actions is equal to $\cap_n \mathfrak{U}_n$, the statement of the theorem will be obtained if we show that, for the uniform topology, every $\mathfrak{U}_n$ is open and dense in the set of $C^0(\mathbb{T}^d)$ sub-actions.

Suppose on the contrary that $\mathfrak{U}_n$ is not open. So there exists a sequence of $C^0(\mathbb{T}^d)$ sub-actions $\{u_k\}_{k\geq 0}$ converging to some $u \in \mathfrak{U}_n$ and a sequence of points $\{(x_k, v_k)\}_{k\geq 0}$ such that, for all $k \geq 0$, $(x_k, v_k) \in \mathcal{N}_\tau(L, u_k) - O_n$. From corollary 6.4, we know there exists a positive constant R_τ such that $\|v_k\| \leq R_\tau$ for all k. By considering a suitable subsequence, we obtain a point $(x, v) \in \mathcal{N}_\tau(L, u) - O_n$ in contradiction with $\mathcal{N}_\tau(L, u) \subset O_n$.

Let us prove now that $\mathfrak{U}_n$ is dense. We first notice that, if $t \in (0, 1)$, $u \in \mathfrak{U}_n$ and $u' \in C^0(\mathbb{T}^d)$ is any arbitrary sub-action, then

$$\mathcal{N}_\tau(L, tu + (1-t)u') = \mathcal{N}_\tau(L, u) \cap \mathcal{N}_\tau(L, u') \subset O_n$$

and therefore $tu + (1-t)u' \in \mathfrak{U}_n$. In particular, in order to prove that $\mathfrak{U}_n$ is dense, it suffices to argue that $\mathfrak{U}_n$ is nonempty.

Corollary 6.4 assures that $(x, v) \notin \mathcal{N}_\tau(L, u)$ for any $\|v\| > R_\tau$ and any sub-action u. Let $\mathbb{B}_\tau$ denote the closed ball of center $0 \in \mathbb{R}^d$ and radius R_τ. Thanks to proposition 10.5, for every point $(x,v) \in (\mathbb{T}^d \times \mathbb{B}_\tau) - O_n$, one can find a sub-action $u_{(x,v)} \in C^0(\mathbb{T}^d)$ and an open set $\mathcal{V}_{(x,v)} \subset \mathbb{T}^d \times \mathbb{R}^d$ containing (x,v) such that

$$(y,w) \notin \mathcal{N}_\tau(L, u_{(x,v)}), \quad \forall\, (y,w) \in \mathcal{V}_{(x,v)}.$$

Hence, thanks to the compactness of $(\mathbb{T}^d \times \mathbb{B}_\tau) - O_n$, there exist a finite cover by open sets $\{\mathcal{V}_1, \ldots, \mathcal{V}_m\}$ of $(\mathbb{T}^d \times \mathbb{B}_\tau) - O_n$ and a finite collection of sub-actions $\{u_1, \ldots, u_m\} \subset C^0(\mathbb{T}^d)$, where $\mathcal{V}_k = \mathcal{V}_{(x_k,v_k)}$ and $u_k = u_{(x_k,v_k)}$ for some (x_k, v_k), satisfying $\bigcap_{k=1}^m \mathcal{N}(L, u_k) \subset O_n$. Clearly $u := \frac{1}{m}\sum_{k=1}^m u_k \in C^0(\mathbb{T}^d)$ belongs to $\mathfrak{U}_n$. □

11 Aspects of rotational theory

A minimizing configuration $\{x_n\}_{n\in\mathbb{Z}}$ in $\mathbb{R}^d$ may be distributed according to a quasiperiodic pattern

$$x_n \asymp x_0 + n\tau\omega, \quad \forall\, n \in \mathbb{Z},$$

where ω is some fixed vector in $\mathbb{R}^d$ called rotation vector. In the context of monotone twist maps of the annulus, ω is a rotation number that could be interpreted as an *atomic mean distance* for one dimensional Frenkel-Kontorova models. Our purpose in this section is first to exhibit minimizing configurations with different rotation vectors and second to relate these rotation vectors to the multidimensional Mather's alpha and beta functions.

Our analysis of systems with several degrees of freedom requires a precise definition of the notion of a rotation vector.

Definition 11.1. *We call rotation vector of a configuration $\{x_k\}_{k\in\mathbb{Z}}$ of points in $\mathbb{R}^d$ the limit (when it exists)*

$$\omega(\{x_k\}) := \frac{1}{\tau} \lim_{n-m\to+\infty} \frac{x_n - x_m}{n-m}.$$

We call rotation vector of a holonomic probability measure $\mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$ with bounded support the value

$$\omega(\mu) := \int_{\mathbb{T}^d\times\mathbb{R}^d} v\, d\mu(x,v).$$

We also extend in our context the definition of the two Mather functions.

Definition 11.2. *Let $L(x,v)$ be a C^0 superlinear Lagrangian. We call Mather's alpha function the opposite of the minimizing holonomic value of the one-parameter family of Lagrangians $L_I(x,v) := L(x,v) - \langle I, v\rangle$, $I \in \mathbb{R}^d$, that is,*

$$-\alpha_L(\tau, I) := \bar{L}(\tau, I) = \min\Big\{ \int \big(L(x,v) - \langle I, v\rangle\big)\, d\mu(x,v) \,:\, \mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)\Big\}.$$

We call Mather's beta function the application

$$\beta_L(\tau,\omega) := \inf\Big\{ \int L(x,v)\, d\mu(x,v) \,:\, \mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d),$$
$$\operatorname{supp}(\mu) \textit{ is bounded and } \int v\, d\mu(x,v) = \omega\Big\}.$$

We notice that, because of the superlinearity of L, L_I is again coercive (actually superlinear) and that $\bar{L}(\tau, I)$ is indeed a minimum and not an infimum. We also point out that, in the definition of β_L, we prefer to restrict μ to have bounded support so that $\int v\, d\mu$ is well defined. We will show in a moment that the set where this infimum is taken is not empty and that the infimum is actually attained. Although α_L is a standard notation in the context of Aubry-Mather theory, we prefer to keep the notation $\bar{L}(\tau, I)$ in the rest of this section. Using standard convex analysis, we obtain the following proposition.

Proposition 11.3. *Let $L(x, v)$ be a C^0 superlinear Lagrangian. Then the two functions $I \in \mathbb{R}^d \mapsto -\bar{L}(\tau, I) \in \mathbb{R}$ and $\omega \in \mathbb{R}^d \mapsto \beta_L(\tau, \omega) \in \mathbb{R}$ are convex superlinear obtained by Legendre transform:*

$$-\bar{L}(\tau, I) = \sup_{\omega \in \mathbb{R}^d} \big[\langle I, \omega\rangle - \beta_L(\tau, \omega)\big] \quad \text{and} \quad \beta_L(\tau, \omega) = \sup_{I \in \mathbb{R}^d} \big[\langle I, \omega\rangle + \bar{L}(\tau, I)\big].$$

In particular, for every $\omega \in \mathbb{R}^d$, there exists $I \in \mathbb{R}^d$ such that

$$\beta_L(\tau, \omega) = \bar{L}(\tau, I) + \langle I, \omega\rangle.$$

We first show that Mather's beta function is well defined.

Lemma 11.4. *For every $\omega \in \mathbb{R}^d$, there exists a holonomic probability measure μ such that*

$$\int v\, d\mu(x, v) = \omega \quad \text{and} \quad \operatorname{supp}(\mu) \subset \mathbb{T}^d \times B_{\|\omega\|_\infty},$$

where $B_{\|\omega\|_\infty}$ denotes the closed ball of center 0 and radius $\|\omega\|_\infty$.

Proof. If $\omega = p/q$, with $q \in \mathbb{Z}_+^*$ and $p \in \mathbb{Z}^d$, then clearly

$$\mu_{p/q} := \frac{1}{q} \sum_{k=0}^{q-1} \delta_{\left(\frac{kp}{q}, \frac{p}{\tau q}\right)}$$

is a holonomic probability measure satisfying the statement of the lemma. For a general $\omega \in \mathbb{R}^d$, consider a sequence $\{p_n/q_n\}$, with $q_n \in \mathbb{Z}_+^*$ and $p_n \in \mathbb{Z}^d$, such that $\lim_{n\to\infty} p_n/q_n = \omega$ and $\|p_n/q_n\|_\infty \le \|\omega\|_\infty$. Let $\{\mu_{p_n/q_n}\}$ be the corresponding sequence of holonomic probabilities defined as above. Then this sequence is relatively compact for the narrow topology and any accumulation point μ_ω is holonomic, $\mu_\omega \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$, and admits ω as a rotation vector. □

We recall a standard fact in convex analysis that we prove for completeness.

Lemma 11.5. *Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be convex functions with full domain. Suppose that f is the Legendre transform of g, namely,*

$$f(I) = g^*(I) := \sup\big\{\langle I, \omega\rangle - g(\omega) : \omega \in \mathbb{R}^d\big\}, \quad \forall\, I \in \mathbb{R}^d.$$

Then f and g are superlinear and g is the Legendre transform of f. Moreover, for every I fixed (respectively ω fixed), the equation $f(I) + g(\omega) = \langle I, \omega\rangle$ admits at least one solution in ω (respectively in I).

Proof. We first show that f is superlinear. For every $R > 0$, for every $I \in \mathbb{R}^d$,

$$f(I) \geq R\|I\| - \inf\Big\{g\Big(R\frac{I}{\|I\|}\Big) \,:\, I \in (\mathbb{R}^d)^*\Big\}.$$

As g is continuous, g is bounded from bellow on the sphere of radius R and

$$\liminf_{\|I\|\to+\infty} \frac{f(I)}{\|I\|} \geq R, \quad \forall\, R > 0 \quad \Rightarrow \quad \lim_{\|I\|\to+\infty} \frac{f(I)}{\|I\|} = +\infty.$$

We show that g is the Legendre transform of f. We always have

$$g(\omega) \geq \langle I, \omega\rangle - f(\omega), \quad \forall\, I, \omega \in \mathbb{R}^d \quad \Rightarrow \quad g(\omega) \geq f^*(\omega), \quad \forall\, \omega \in \mathbb{R}^d.$$

Moreover, g admits a subdifferential I_ω at ω in the following sense

$$g(\omega') \geq g(\omega) + \langle I_\omega, \omega' - \omega\rangle, \quad \forall\, \omega' \in \mathbb{R}^d.$$

Then

$$f(I_\omega) = \sup\{\langle I_\omega, \omega'\rangle - g(\omega') : \omega' \in \mathbb{R}^d\} \leq \langle I_\omega, \omega\rangle - g(\omega)$$

and $g(\omega) \leq \langle I_\omega, \omega\rangle - f(I_\omega) \leq f^*(\omega)$. We just have proved that $g = f^*$. From the beginning of the proof, we obtain that g is superlinear. In particular, the supremum is attained in the definition of g^* and, for any fixed I, the equation $\langle I, \omega\rangle = f(I) + g(\omega)$ admits at least one solution in ω. □

Proof of proposition 11.3. We show that $-\bar{L}(x, I)$ is convex in $I \in \mathbb{R}^d$. Indeed, for any $I, J \in \mathbb{R}^d$ and $t \in [0, 1]$, if $\mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$ is a minimizing probability for $L_{tI+(1+t)J}$, then

$$\begin{aligned}
\bar{L}(\tau, tI + (1+t)J) &= \int L_{tI+(1+t)J}(x, v)\, d\mu(x, v) \\
&= t\int L_I(x, v)\, d\mu(x, v) + (1+t)\int L_J(x, v)\, d\mu(x, v) \\
&\geq t\bar{L}(\tau, I) + (1+t)\bar{L}(\tau, J).
\end{aligned}$$

We now show that $-\bar{L}(\tau, \cdot)$ is the Legendre transform of $\beta_L(\tau, \cdot)$. Thanks to corollary 6.4, we have

$$\begin{aligned}
-\bar{L}(\tau, I) = \sup\Big\{\int[\langle I, v\rangle - L(x, v)]\, d\mu(x, v) : \mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d), \\
\text{and } \operatorname{supp}(\mu) \text{ is bounded}\Big\}.
\end{aligned}$$

Therefore, one can write

$$\begin{aligned}
-\bar{L}(\tau, I) = \sup_{\omega\in\mathbb{R}^d} \sup\Big\{\langle I, \omega\rangle - \int L(x, v)\, d\mu(x, v) : \mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d), \\
\operatorname{supp}(\mu) \text{ is bounded, and } \int v\, d\mu(x, v) = \omega\Big\},
\end{aligned}$$

namely, $-\bar{L}(\tau, I) = \sup_{\omega\in\mathbb{R}^d}[\langle I, \omega\rangle - \beta_L(\omega)]$.

Proposition 11.3 follows then from lemma 11.5. □

We are now able to prove the infimum is attained in the definition of $\beta_L(\tau,\omega)$.

Proposition 11.6. *Let $L(x,v)$ be a C^0 superlinear Lagrangian. For every $\omega\in\mathbb{R}^d$, there exists a holonomic probability measure $\mu\in\mathcal{P}_\tau(\mathbb{T}^d\times\mathbb{R}^d)$ with bounded support such that*

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} v\,d\mu(x,v)=\omega \quad \textit{and} \quad \beta_L(\tau,\omega)=\int_{\mathbb{T}^d\times\mathbb{R}^d} L(x,v)\,d\mu(x,v).$$

Moreover, any holonomic probability measure, with bounded support and rotation vector ω, realizing the infimum in the definition of $\beta_L(\tau,\omega)$, minimizes a Lagrangian L_{I_ω} for some $I_\omega\in\mathbb{R}^d$.

Proof. We follow Mather's idea which says that the superlinearity of L implies that, given a constant $C\in\mathbb{R}$, the set of Borel measures

$$\Big\{\|v\|\,\mu(dx,dv)\,:\,\mu\in\mathcal{P}_\tau(\mathbb{T}^d\times\mathbb{R}^d),\text{ and }\int L(x,v)\,d\mu(x,v)\le C\Big\}$$

is tight. Let $\chi_R(x,v)$ be a test function taking its values in $[0,1]$ and satisfying $\chi(x,v)=1$ for all $\|v\|\le R-1$ and $\chi_R(x,v)=0$ for all $\|v\|\ge R$. Let $\{\mu_n\}_{n\ge0}$ be a sequence of Borel probability measures for which $\{\int L(x,v)\,d\mu_n(x,v)\}$ is uniformly bounded. So notice that, for every $\epsilon>0$ and R sufficiently large, we have the inequality $\|v\|(1-\chi_R)\le\epsilon(L(x,v)-\inf L)$, which clearly yields

$$\lim_{R\to+\infty}\limsup_{n\to+\infty}\int\|v\|(1-\chi_R)\,d\mu_n(x,v)=0.$$

Suppose in addition that μ_n is holonomic,

$$\lim_{n\to+\infty}\int L(x,v)\,d\mu_n(x,v)=\beta_L(\tau,\omega)\quad\text{and}\quad\int v\,d\mu_n(x,v)=\omega.$$

We first extract a subsequence, that we again call $\{\mu_n\}_{n\ge0}$, converging to a Borel measure μ in the sense that

$$\int f\,d\mu=\lim_{n\to+\infty}\int f\,d\mu_n,\quad\forall\ f\in C^0_{\text{compact}}(\mathbb{T}^d\times\mathbb{R}^d),$$
$$\int f\,d\mu\le\liminf_{n\to+\infty}\int f\,d\mu_n,\quad\forall\ f\in C^0_{\text{bounded}}(\mathbb{T}^d\times\mathbb{R}^d),\ f\ge0.$$

The tighness property actually implies

$$\int f\,d\mu=\lim_{n\to+\infty}\int f\,d\mu_n,\quad\int fv\,d\mu=\lim_{n\to+\infty}\int fv\,d\mu_n,\quad\forall\ f\in C^0_{\text{bounded}}(\mathbb{T}^d\times\mathbb{R}^d).$$

In particular, μ is a holonomic probability measure, it possesses a rotation vector ω and $\int L(x,v)\,d\mu(x,v)\le\beta_L(\tau,\omega)$. However, as $\beta_L(\tau,\cdot)$ is the Legendre transform of $-\bar L(\tau,\cdot)$, there exists $I_\omega\in\mathbb{R}^d$ such that $\beta_L(\tau,\omega)=\bar L(\tau,I_\omega)+\langle I_\omega,\omega\rangle$. We obtain

$$\int\big(L(x,v)-\langle I_\omega,v\rangle\big)\,d\mu(x,v)=\int L(x,v)\,d\mu(x,v)-\langle I_\omega,\omega\rangle\le\bar L(\tau,I_\omega),$$

which implies that μ is minimizing L_{I_ω} and therefore has bounded support as it is shown in corollary 6.4. $\square$

Our next objective is to show the equivalence between the set of subdifferentials of $-\bar{L}(\tau,\cdot)$ at the point I and the set of rotation vectors of minimizing probability measures for the Lagrangian L_I.

Definition 11.7. *Let $L(x,v)$ be a C^0 superlinear Lagrangian. Denote*

$$\Omega(\tau,I) := \Big\{ \int v\, d\mu(x,v) \,:\, \mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d) \text{ is minimizing for } L_I \Big\}, \quad \forall\, I \in \mathbb{R}^d.$$

Denote the set of subdifferentials of $-\bar{L}(\tau,\cdot)$ at I by

$$-\partial\bar{L}(\tau,I) := \Big\{ \omega \in \mathbb{R}^d \,:\, -\bar{L}(\tau,J) \geq -\bar{L}(\tau,I) + \langle \omega, J-I\rangle, \quad \forall\, J \in \mathbb{R}^d \Big\}.$$

We notice that $\Omega(\tau,I)$ is a compact convex subset of $\mathbb{R}^d$ and that proposition 11.6 implies

$$\bigcup_{I\in\mathbb{R}^d} \Omega(\tau,I) = \mathbb{R}^d.$$

Proposition 11.8. *Let $L(x,v)$ be a C^0 superlinear Lagrangian. For every $I \in \mathbb{R}^d$,*

$$\Omega(\tau,I) = \{\omega \,:\, \beta_L(\tau,\omega) = \bar{L}(\tau,I) + \langle I,\omega\rangle\} = -\partial\bar{L}(\tau,I).$$

In particular, $\Omega(\tau,I)$ is reduced to a point if, and only if, $\bar{L}(\tau,\cdot)$ is differentiable at I. In such case, for any minimizing probability measure μ for L_I,

$$\int v\, d\mu(x,v) = -\frac{\partial\bar{L}}{\partial I}(\tau,I).$$

Proof. If $\omega \in \Omega(\tau,I)$, there exists $\mu \in \mathcal{P}_\tau(\mathbb{T}^d\times\mathbb{R}^d)$ such that

$$\int v\, d\mu(x,v) = \omega \quad \text{and} \quad \int \big[L(x,v) - \langle I,v\rangle\big]\, d\mu(x,v) = \bar{L}(\tau,I).$$

Legendre transform implies the *a priori* estimate $-\bar{L}(\tau,I') + \beta_L(\tau,\omega') \geq \langle I',\omega'\rangle$ for any $I',\omega' \in \mathbb{R}^d$. As μ has bounded support, we obtain $\beta_L(\tau,\omega) \geq \bar{L}(\tau,I) + \langle I,\omega\rangle = \int L(x,v)\, d\mu(x,v) \geq \beta_L(\tau,\omega)$.

Suppose now ω satisfies $\beta_L(\tau,\omega) = \bar{L}(\tau,I) + \langle I,\omega\rangle$, then the previous *a priori* estimate implies

$$-\bar{L}(\tau,I') + \bar{L}(\tau,I) \geq \langle I'-I,\omega\rangle, \quad \forall\, I' \in \mathbb{R}^d.$$

We have shown that ω is a subdifferential of $-\bar{L}(\tau,\cdot)$ at I.

Finally, suppose $\omega \in -\partial\bar{L}(\tau,I)$. Since $\mathbb{R}^d = \cup_I \Omega(\tau,I)$, there exists J such that $\omega = \int v d\mu(x,v)$ for some $\mu \in \mathcal{P}_\tau(\mathbb{T}^d\times\mathbb{R}^d)$ minimizing for L_J. So

$$\int L(x,v)\, d\mu(x,v) = \beta_L(\tau,\omega) = \bar{L}(\tau,J) + \langle J,\omega\rangle.$$

By definition of subdifferentiability,

$$\beta_L(\tau,\omega) - \langle J,\omega\rangle = \bar{L}(\tau,J) \leq \bar{L}(\tau,I) - \langle \omega, J-I\rangle,$$

We then obtain $\beta_L(\tau,\omega) \leq \bar{L}(\tau,I) + \langle I,\omega\rangle$, that is, $\beta_L(\tau,\omega) = \bar{L}(\tau,I) + \langle I,\omega\rangle$, which guarantees μ minimizes L_I and therefore $\omega \in \Omega(\tau,I)$. □

Conversely the set $\partial\beta_L(\tau,\omega)$ of subdifferentials of $\beta_L(\tau,\cdot)$ at ω admits a dual description.

Proposition 11.9. *Let $L(x,v)$ be a C^0 superlinear Lagrangian. Then*

i. $\partial\beta_L(\tau,\omega)=\{I\in\mathbb{R}^d\ :\ \omega\in\Omega(\tau,I)\}$*;*

ii. $\beta_L(\tau,\cdot)$ *is affine of slope I on* $\mathrm{int}(\Omega(\tau,I))$*;*

iii. $\mathrm{int}(\Omega(\tau,I))\cap\mathrm{int}(\Omega(\tau,J))=\emptyset$ *as soon as $I\neq J$.*

Proof. If $\omega\in\Omega(\tau,I)$, then there exists $\mu\in\mathcal{P}_\tau(\mathbb{T}^d\times\mathbb{R}^d)$ such that $\int v\,d\mu(x,v)=\omega$,

$$\int\big[L(x,v)-\langle I,v\rangle\big]\,d\mu(x,v)=\bar L(\tau,I)\quad\text{and}\quad\beta_L(\tau,\omega)=\int L(x,v)\,d\mu(x,v).$$

The previous *a priori* estimate implies $\beta_L(\tau,\omega')\geq\bar L(\tau,I)+\langle I,\omega'\rangle$ for any ω'. We thus obtain $\beta_L(\tau,\omega')-\beta_L(\tau,\omega)\geq\langle I,\omega'-\omega\rangle$, namely, I is a subdifferential of $\beta_L(\tau,\cdot)$ at ω.

If I is a subdifferential of $\beta_L(\tau,\cdot)$ at ω, for every $\mu'\in\mathcal{P}_\tau(\mathbb{T}^d\times\mathbb{R}^d)$, with bounded support, let $\omega'=\int v\,d\mu'(x,v)$, then

$$\int\big[L(x,v)-\langle I,v\rangle\big]\,d\mu'(x,v)\geq\beta_L(\tau,\omega')-\langle I,\omega'\rangle\geq\beta_L(\tau,\omega)-\langle I,\omega\rangle.$$

By taking the infimum on μ', we obtain $\bar L(\tau,I)=\beta_L(\tau,\omega)-\langle I,\omega\rangle$. Moreover, there exists $\mu\in\mathcal{P}_\tau(\mathbb{T}^d\times\mathbb{R}^d)$, with bounded support, such that $\omega=\int v\,d\mu(x,v)$ and $\beta_L(\tau,\omega)=\int L(x,v)\,d\mu(x,v)$. Therefore, μ is minimizing for L_I and $\omega\in\Omega(\tau,I)$.

If $\omega,\omega'\in\mathrm{int}(\Omega(\tau,I))$, then I is a subdifferential of $\beta_L(\tau,\cdot)$ at both ω and ω'. We thus obtain $\beta_L(\tau,\omega')-\beta_L(\tau,\omega)=\langle I,\omega'-\omega\rangle$ and $\beta_L(\tau,\cdot)$ is affine on $\mathrm{int}(\Omega(\tau,I))$ with slope I. In particular, $\mathrm{int}(\Omega(\tau,I))\cap\mathrm{int}(\Omega(\tau,J))=\emptyset$ if $I\neq J$. □

We are now in a position to construct infinitely many minimizing configurations with different rotation vectors. By standard convexity argument, the following directional differentials exist for all $I,h\in\mathbb{R}^d$

$$\partial_h^+\bar L(\tau,I):=\lim_{\rho\to0^+}\frac{\bar L(\tau,I+\rho h)-\bar L(\tau,I)}{\rho},$$
$$\partial_h^-\bar L(\tau,I):=\lim_{\rho\to0^+}\frac{\bar L(\tau,I)-\bar L(\tau,I-\rho h)}{\rho}.$$

The following theorem improves, in the C^0 superlinear case, a result due to D. A. Gomes (see theorem 6.2 of [17]).

Theorem 11.10. *Suppose $L(x,v)$ is a C^0 superlinear Lagrangian. Let $u_I\in C^0(\mathbb{T}^d)$ be an arbitrary sub-action for L_I. Then, given $h\in\mathbb{R}^d$, any u_I-calibrated configuration $\{x_k\}_{k\in\mathbb{Z}}$ of points of $\mathbb{R}^d$ satisfies*

$$-\tau\partial_h^-\bar L(\tau,I)\leq\liminf_{n-m\to\infty}\big\langle h,\frac{x_n-x_m}{n-m}\big\rangle\leq\limsup_{n-m\to\infty}\big\langle h,\frac{x_n-x_m}{n-m}\big\rangle\leq-\tau\partial_h^+\bar L(\tau,I).$$

In particular, if $\bar{L}(\tau, \cdot)$ is differentiable at $I \in \mathbb{R}^d$, then any u_I-calibrated configuration $\{x_k\}_{k\in\mathbb{Z}}$ has a rotation vector given by

$$\omega(\{x_k\}) = -\frac{\partial \bar{L}}{\partial I}(\tau, I) = \int v \, d\mu(x, v), \quad \forall \, \mu \in \mathcal{M}_\tau(L_I).$$

There exist minimizing configurations for L with rotation vector of arbitrarily large norm.

We recall that, according to lemma 5.6, u_I-calibrated configurations are examples of minimizing configurations of L (or of any L_J since the whole family $\{L_I\}_{I\in\mathbb{R}^d}$ shares the same minimizing configurations).

Proof of theorem 11.10. Without loss of generality, we can suppose that $\bar{L}(\tau) = 0$. So by definition, notice that, whenever $m < n$,

$$u_I(x_n) = u_I(x_m) + \bar{\mathcal{L}}_\tau(x_m, \ldots, x_n) - \langle I, x_n - x_m \rangle - (n-m)\tau \bar{L}(\tau, I).$$

Take $\rho > 0$ and $h \in \mathbb{R}^d$. Set $I_h := I - \rho h$. If $u_{I_h} \in C^0(\mathbb{T}^d)$ is an arbitrary sub-action for L_{I_h}, then we obviously have

$$u_{I_h}(x_n) \leq u_{I_h}(x_m) + \bar{\mathcal{L}}_\tau(x_m, \ldots, x_n) - \langle I_h, x_n - x_m \rangle - (n-m)\tau \bar{L}(\tau, I_h).$$

Therefore, it is not difficult to obtain the following inequality

$$\tau \frac{\bar{L}(\tau, I - \rho h) - \bar{L}(\tau, I)}{\rho} - \frac{2}{\rho} \frac{\|u_I - u_{I_h}\|_0}{n - m} \leq \big\langle h, \frac{x_n - x_m}{n - m} \big\rangle,$$

from which we immediately deduce

$$-\tau \partial_h^- \bar{L}(\tau, I) = \tau \lim_{\rho \to 0^+} \frac{\bar{L}(\tau, I - \rho h) - \bar{L}(\tau, I)}{\rho} \leq \liminf_{n-m\to\infty} \big\langle h, \frac{x_n - x_m}{n - m} \big\rangle.$$

Replacing h by $-h$, one thus get

$$\limsup_{n-m\to\infty} \big\langle h, \frac{x_n - x_m}{n - m} \big\rangle \leq \tau \lim_{\rho \to 0^+} \frac{\bar{L}(\tau, I) - \bar{L}(\tau, I + \rho h)}{\rho} = -\tau \partial_h^+ \bar{L}(\tau, I).$$

Finally, if $\bar{L}(\tau, \cdot)$ is differentiable at $I \in \mathbb{R}^d$, the previous inequalities become

$$\big\langle \lim_{n-m\to\infty} \frac{x_n - x_m}{n - m} + \tau \frac{\partial \bar{L}}{\partial I}(\tau, I), h \big\rangle = 0, \quad \forall \, h \in \mathbb{R}^d.$$

We just have proved $\omega_I = \omega(\{x_k\}) = -\frac{\partial \bar{L}}{\partial I}(\tau, I)$ exists. Notice yet that ω_I satisfies the relation $\langle I, \omega_I \rangle = \beta_L(\tau, \omega_I) - \bar{L}(\tau, I)$. So if $\|I\| \to +\infty$ among the set of points of differentiability of $-\bar{L}(\tau, \cdot)$, the superlinearity of $-\bar{L}(\tau, \cdot)$ implies $\|\omega_I\| \to +\infty$. $\square$

One shall have in mind that, even in the context of C^∞ superlinear Lagrangians, a calibrated configuration may not have a well defined rotation vector. Let us present an example.

Example 11.11. *Assume $d = 1$ and $\tau = 1$. Let $\ell : \mathbb{R}\times\mathbb{R} \to [0,1]$ be a C^∞ function such that*

$$\ell^{-1}(1) = \{(0,0),(0,1/2),(1/2,0)\} \ \ \textit{and} \ \ \ell^{-1}(0) \supset \mathbb{R}^2 - (-1/4,3/4)^2.$$

Define then $\mathcal{L}_1 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}_+$ by

$$\mathcal{L}_1(x,y) = 1 - \sum_{s\in\mathbb{Z}} \ell(x+s,y+s), \quad \forall\, x,y \in \mathbb{R}.$$

Clearly, $\mathcal{L}_1$ is a C^∞ function invariant by the diagonal action of $\mathbb{Z}$,

$$\mathcal{L}_1 \geq 0, \quad \mathcal{L}_1^{-1}(0) = S := \bigcup_{s\in\mathbb{Z}}\{(s,s),(s,s+1/2),(s+1/2,s)\}$$

and $\mathcal{L}_1 > 0$ everywhere on $\mathbb{R}^2 - S$. If $\mathcal{L}_2(x,y) = |x-y|^2|x-y+1|^2|x-y-1|^2$, let us consider a nonnegative local interaction energy map given by

$$\mathcal{L}(x,y) = \mathcal{L}_1(x,y)\mathcal{L}_1(x-1,y)\mathcal{L}_1(x,y-1) + \mathcal{L}_2(x,y), \quad \forall\, x,y \in \mathbb{R}.$$

Notice that $\mathcal{L}$ is C^∞, superlinear, invariant by the diagonal action of $\mathbb{Z}$,

$$\mathcal{L} \geq 0 \quad \textit{and} \quad \mathcal{L}^{-1}(0) = \bigcup_{s\in\mathbb{Z}}\{(s,s),(s,s+1),(s+1,s)\}.$$

However, $\mathcal{L}$ does not satisfy a twist condition: there are points $(x_0,y_0) \in \mathbb{R}^2$ such that $\frac{\partial^2\mathcal{L}}{\partial x\partial y}(x_0,y_0) = 0$. Indeed, since $\frac{\partial\mathcal{L}}{\partial y}(0,0) = 0 = \frac{\partial\mathcal{L}}{\partial y}(1,0)$, Rolle's theorem states that $\frac{\partial^2\mathcal{L}}{\partial x\partial y}(x_0,0) = 0$ for some $x_0 \in (0,1)$.

Let then $L : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ denote the corresponding C^∞ superlinear Lagrangian. We will exhibit a configuration $\{x_k\}$ u-calibrated for any sub-action $u \in C^0(\mathbb{T})$ but without a well defined rotation vector. To that end, notice we have

$$(x,v) \in \mathcal{A}_1(L) \ \Leftrightarrow\ \mathcal{L}(x,x+v) = 0 \ \Leftrightarrow\ x = 0 \ (\textit{mod}\ \mathbb{Z}) \ \textit{and}\ v \in \{-1,0,1\}.$$

So consider any sequence of positive integers $\{r_i\}_{i\geq 1}$ such that $\frac{1}{n}\sum_{i=1}^n r_i$ has at least two distinct accumulation points: $1/\omega_1$ and $1/\omega_2$. We define a configuration $\{x_k\}$ by

$$x_0 = 0 \ \ \textit{and} \ \ x_k = n \ \ \textit{if} \ \ \sum_{i=1}^{n-1} r_i < |k| \leq \sum_{i=1}^{n} r_i.$$

Notice that $(x_k\ (\textit{mod}\ \mathbb{Z}), x_{k-1} - x_k) \in \{(0,-1),(0,0),(0,1)\} = \mathcal{A}_1(L)$. Therefore, proposition 7.3 guarantees $\{x_k\}$ is u-calibrated for any sub-action $u \in C^0(\mathbb{T})$. Nevertheless, the fact that

$$\frac{n}{\sum_{i=1}^n r_i} \leq \frac{x_k}{k} < \frac{n}{\sum_{i=1}^{n-1} r_i} \quad \textit{whenever} \quad \sum_{i=1}^{n-1} r_i < k \leq \sum_{i=1}^{n} r_i$$

and the choice of the sequence $\{r_i\}$ imply that, when $k \to +\infty$, x_k/k has ω_1 and ω_2 as accumulation points, which shows the configuration $\{x_k\}$ does not admit a rotation vector.

From now on $L(x,v)$ is supposed to be a C^1 superlinear ferromagnetic Lagrangian. Notice that the set of critical configurations $\Gamma_\tau(L_I)$ introduced in section 2 for $L_I(x,v) = L(x,v) - \langle I, v\rangle$ actually does not depend on $I \in \mathbb{R}^d$. Hence, from lemma 5.7, if $u_I \in C^0(\mathbb{T}^d)$ is an arbitrary sub-action for L_I, we have in particular that

$$\{x_k\}_{k\in\mathbb{Z}} \text{ is } u_I\text{-calibrated} \quad \Rightarrow \quad \{x_k\}_{k\in\mathbb{Z}} \in \Gamma_\tau(L).$$

Ferromagnetism is another property naturally inherited by the family $\{L_I\}_{I\in\mathbb{R}^d}$. Using definition 2.5, we observe that, when L is ferromagnetic, the discrete-time Lagrangian dynamics $(\mathbb{T}^d \times \mathbb{R}^d, \Phi_\tau)$ is independent of I too.

According to theorem 7.7, Aubry sets are nonempty compact Φ_τ-invariant sets. Hence, as a consequence of theorem 11.10, the next result gives a sufficient condition for the existence of disjoint invariant sets with respect to the discrete-time Lagrangian dynamics.

Proposition 11.12. *Let $L(x,v)$ be a C^1 ferromagnetic superlinear Lagrangian. Suppose $I, J \in \mathbb{R}^d$ are points of differentiability of $\bar L(\tau,\cdot)$ satisfying $\frac{\partial \bar L}{\partial I}(\tau, I) \neq \frac{\partial \bar L}{\partial I}(\tau, J)$. Then $\mathcal{A}_\tau(L_I) \cap \mathcal{A}_\tau(L_J) = \emptyset$.*

Proof. Suppose on the contrary $(x_0, v_0) \in \mathcal{A}_\tau(L_I) \cap \mathcal{A}_\tau(L_J)$. By the invariance of Aubry sets, we have

$$(x_k, v_k) := \Phi_\tau^k(x_0, v_0) \in \mathcal{A}_\tau(L_I) \cap \mathcal{A}_\tau(L_J), \quad \forall\, k \in \mathbb{Z}.$$

Define then $y_0 = x_0 \in [0,1)^d$ and recursively

$$y_{k+1} = y_k + \tau v_k \in \mathbb{R}^d \ \text{ for } k \geq 0 \quad \text{and} \quad y_{k-1} = y_k - \tau v_{k-1} \in \mathbb{R}^d \ \text{ for } k \leq 0.$$

Let $u_I \in C^0(\mathbb{T}^d)$ be a sub-action for L_I and let $u_J \in C^0(\mathbb{T}^d)$ be a sub-action for L_J. By proposition 7.3, the configuration $\{y_k\}_{k\in\mathbb{Z}}$ is simultaneously u_I-calibrated and u_J-calibrated. Hence, theorem 11.10 forces

$$-\frac{\partial \bar L}{\partial I}(\tau, I) = \omega(\{y_k\}) = -\frac{\partial \bar L}{\partial I}(\tau, J),$$

which is a contradiction. Thus $\mathcal{A}_\tau(L_I)$ and $\mathcal{A}_\tau(L_J)$ are necessarily disjoint. □

We have seen in theorem 11.10 that, if I is a point of differentiability of $\bar L(\tau,\cdot)$, then $\omega = -\frac{\partial \bar L}{\partial I}(\tau, I)$ is a rotation vector of some configuration. We are now interested in vectors $\omega \in \Omega(\tau, I)$ when $\Omega(\tau, I)$ is not any more reduced to a point, that is, when $\bar L(\tau,\cdot)$ is not any more differentiable at I. The extremal points of $\Omega(\tau, I)$ play an interesting role in the study of rotational properties for ferromagnetic Lagrangians.

Proposition 11.13. *Let $L(x,v)$ be a C^1 ferromagnetic superlinear Lagrangian. Given $I \in \mathbb{R}^d$, if ω is an extremal point of $\Omega(\tau, I)$, then there exists a configuration $\{x_k\}_{k\in\mathbb{Z}}$ of points in $\mathbb{R}^d$ which is u_I-calibrated with respect to any sub-action u_I for L_I and has a rotation vector given by*

$$\omega(\{x_k\}) = \omega.$$

Proof. Let $\omega \in \Omega(\tau, I)$ be an extremal point of $\Omega(\tau, I)$. By hypothesis, there exists a holonomic probability measure μ such that

$$\int v \, d\mu(x,v) = \omega \quad \text{and} \quad \int [L(x,v) - \langle I, v\rangle] \, d\mu(x,v) = \bar{L}(\tau, I).$$

Theorem 6.10 guarantees that μ is Φ_τ-invariant. Furthermore, thanks to the extremal conditions on ω and on $\bar{L}(\tau, I)$, we may assume that μ is Φ_τ-ergodic.

By the ergodicity of μ, for almost all $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$, if x_0 is a representant of x and $x_n = x_0 + \tau \sum_{k=0}^{n-1} pr^2 \circ \Phi_\tau(x,v)$, then

$$\frac{1}{\tau} \lim_{n-m \to +\infty} \frac{x_n - x_m}{n - m} = \lim_{n-m \to +\infty} \sum_{k=m}^{n-1} pr^2 \circ \Phi_\tau(x,v) = \int v \, d\mu(x,v) = \omega.$$

Moreover, $\{x_k\}_{k\in\mathbb{Z}}$ is u_I-calibrated for any sub-action u_I of L_I since

$$\Big(x_k \ (\bmod \mathbb{Z}^d), \frac{x_{k+1} - x_k}{\tau} \Big) = \Phi_\tau^k(x,v) \in \mathcal{M}_\tau(L_I) \subset \mathcal{N}_\tau(L_I, u_I).$$

□

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