Introduction to ergodic optimization

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Summary

I. Introduction

- **Additive ergodic optimization**
- Hyperbolic dynamical system and SFT
- Minimizing measures and Gibbs measures
- Mañé conjecture for SFT
- Frenkel-Kontorova model
- Linear switched systems

Introduction : **Additive ergodic optimization**

Definition

- We consider a (discrete time) topological dynamical system

$$(X, f) \text{ compact}, \quad f : X \to X \text{ continuous}$$

- We consider also a continuous observable

$$\phi : X \to \mathbb{R}, \text{ continuous}$$

- The Birkhoff average along a finite orbit

$$A_n[\phi](x) := \frac{1}{n} \sum_{i=0}^{n-1} \phi \circ f^i(x)$$

- The *ergodic minimizing value* of ϕ

$$\bar{\phi} := \lim_{n \to +\infty} \inf_{x \in X} \frac{1}{n} \sum_{i=0}^{n-1} \phi \circ f^i(x)$$

Introduction : **Additive ergodic optimization**

Questions

- How to compute the ergodic minimizing value ?

$$\bar{\phi} := \lim_{n \to +\infty} \inf_{x \in X} \frac{1}{n} \sum_{i=0}^{n-1} \phi \circ f^i(x)$$

Remark : $\min_X(\phi) \le \bar{\phi} \le \max_X(\phi)$

- Is there a notion of optimal trajectory ? A possible definition (forward optimality) coul be

$$\sup_{n \ge 1} \Big| \sum_{i=0}^{n-1} (\phi - \bar{\phi}) \circ f^i(x) \Big| = \sup_{n \ge 1} \Big| \sum_{i=0}^{n-1} \phi \circ f^i(x) - n\bar{\phi} \Big| < +\infty$$

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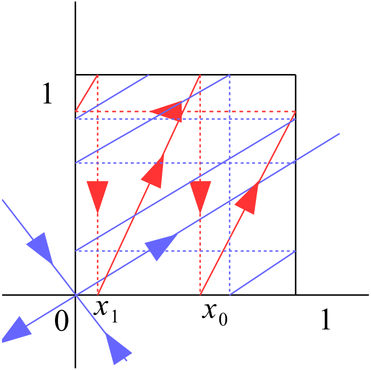
Introduction : Hyperbolic dynamical system and SFT

Example of an hyperbolic map : the Arnold map



$X = \mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ the two torus

$$f \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \mod \mathbb{Z}^2$$

$$\lambda^+ := \frac{3+\sqrt{5}}{2} > 1 > \lambda^- := \frac{3-\sqrt{5}}{2}$$

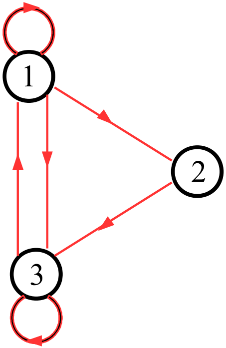
The translation by (α_1, α_2) is not hyperbolic

$$f^t \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + t\alpha_1 \\ y + t\alpha_2 \end{bmatrix} \mod \mathbb{Z}^2$$

Remark A C^1 perturbation of the Arnold map is hyperbolic. The class of hyperbolic maps is relatively large

Introduction : **Hyperbolic dynamical system and SFT**

Another example of an hyperbolic map



Directed graph $G = (V, E)$,

$$V = \{1, 2, 3\}$$
$$E = \{1 \to 1, 1 \to 2, 2 \to 2, \dots\}$$
$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

The subshift of finite type **SFT**

$$X := \{x = (x_k)_{k \in \mathbb{Z}} : x_k \in V,\ x_k \to x_{k+1}\}$$

Remark In fact the Arnold map and the SFT are very similar dynamics : they are both hyperbolic

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Introduction : Minimizing and Gibbs measures

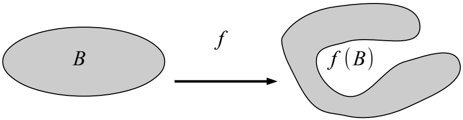
We consider a topological dynamical system (X, f) and and a continuous observable $\phi : X \to \mathbb{R}$.

Definition

- An invariant measure μ is a probability measure on X such that

$$\forall\, B \text{ Borel}, \; \mu(f^{-1}(B)) = \mu(B)$$

$$\forall\, h \in C^0(X, \mathbb{R}), \; \int h \circ f \, d\mu = \int h \, d\mu$$



Remark An hyperbolic system has many invariant measures. For instance the Arnold map preserves the normalized Lebesgue measure on $\mathbb{T}^2$

$$A := \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \det(A) = 1 \quad \int h \circ f \, \mathrm{Jac} \, d\mathrm{Leb} = \int h \, d\mathrm{Leb}$$

(change of variable)

Introduction : Minimizing and Gibbs measures

Recall The ergodic minimizing value

$$\bar{\phi} := \lim_{n\to+\infty} \inf_{x\in X} \frac{1}{n}\sum_{i=0}^{n-1} \phi\circ f^i(x)$$

Proposition We will see soon

$$\bar{\phi} = \min\Big\{ \int \phi\, d\mu : \mu \text{ is an invariant mesure } \Big\}$$

Definition

- A minimizing measure is an invariant measure satisfying

$$\int \phi\, d\mu = \bar{\phi}$$

- The Mather set is the compact invariant set

$$\mathrm{Mather}(\phi) := \bigcup \Big\{ \mathrm{supp}(\mu) : \mu \text{ is a minimizing measure } \Big\}$$

Introduction : Minimizing and Gibbs measures

Definition A Gibbs measure at temperature β^{-1} for the observable $\phi : X \to \mathbb{R}$ is an invariant measure that gives a specific mass to cylinders of size n.

- A cylinder of size n is

$$B_n(x,\epsilon) := \big\{ y \in X : d(f^k(x), f^k(y)) < \epsilon, \ \forall\, k \in [\![0, n-1]\!] \big\}$$

- the Gibbs measure at inverse temperature β

$$\mu_\beta[B_n(x,\epsilon)] \asymp \frac{1}{Z(n,\beta)} \exp\Big(-\beta \sum_{k=0}^{n-1} \phi \circ f^k(x) \Big)$$

- $Z(n,\beta) := \exp(-n\beta\bar{\phi}_\beta)$ is a normalizing factor

$$-\beta\bar{\phi}_\beta := \lim_{n \to +\infty} \inf_{E_n:\ \text{covering}} \frac{1}{n} \log\Big(\sum_{x \in E_n} \exp\big(-\beta \sum_{k=0}^{n-1} \phi \circ f^k(x) \big) \Big)$$

Remark μ_β gives a larger mass to configurations with low energy

Introduction : Minimizing and Gibbs measures

Question What is the relationship between minimizing measures and Gibbs measures ?

Theorem We will see that, by freezing an hyperbolic system, $\beta \to +\infty$, the Gibbs measure μ_β associated to a short range observable tends to a "selected" minimizing measure with maximal entropy among all minimizing measures.

Observation Some minimizing measures corresponds to "ground states" (limits up to a subsequence of Gibbs measures), ; other minimizing measures have no relationship with Gibbs measures. For non short range observables, the phenomenon of "chaotic convergence" described by van Enter takes place.

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Introduction : **Mañé conjecture for SFT**

Recall The Mather set

$$\text{Mather} := \bigcup \Big\{ \text{supp}(\mu) : \mu \text{ is a minimizing measure } \Big\}$$

Question What is the structure of the Mather set ? Is it small and reduced to a periodic orbit ? Is it a set with large complexity (or entropy) ? Could it be the whole set X ?

Mañé Conjecture For any hyperbolic dynamical system, the Mather set is reduced to a periodic orbit for generic smooth observable.

Contreras Theorem For every subshift of finite type, for every Hölder observable $\phi : X \to \mathbb{R}$, for every perturbation $\epsilon > 0$, there exists a periodic orbit $\mathcal{O}_\epsilon$ such that

$$\psi := \phi + \epsilon d(\cdot, \mathcal{O}_\epsilon)$$

has a unique minimizing measure, which is the measure supported by $\mathcal{O}$

$$\delta_{\mathcal{O}} = \frac{1}{\text{card}(\mathcal{O}_\epsilon)} \sum_{p \in \mathcal{O}_\epsilon} \delta_p$$

Introduction : **Mañé conjecture for SFT**

Obvious example Every compact invariant set $\Lambda \subset X$ can play the role of a Mather set

$$\phi(x) := d(x,\Lambda) \quad \bar\phi = 0, \quad \mu \text{ is minimizing } \Leftrightarrow \text{ supp}(\mu) \subset \Lambda$$

Another example Assume the Mather set satisfies the "subordination principle" and contains a periodic orbit $\mathcal{O}$ then

$$\psi := \phi + \epsilon d(x,\mathcal{O})$$

has a unique minimizing measure supported in $\mathcal{O}$

Proof

1. $\int \psi\, d\mu \geq \int \phi\, d\mu \quad \Rightarrow \quad \bar\psi \geq \bar\phi$
2. The Mather set satisfies the subordination principle : every measure supported in the Mather set is minimizing
3. $\delta_{\mathcal{O}}$ is minimizing : $\bar\psi \leq \int \psi\, d\delta_{\mathcal{O}} = \int \phi\, d\mu_{\mathcal{O}} = \bar\phi$
4. if μ is ψ-minimizing $\int \psi\, d\mu = \bar\psi = \bar\phi \leq \int \phi\, d\mu$

$$\epsilon \int d(\cdot,\mathcal{O})\, d\mu = \int (\psi - \phi)\, d\mu \leq 0 \ \Rightarrow \ \text{supp}(\mu) \subset \mathcal{O}$$

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Introduction : **Frenkel-Kontorova model**

Simplification The manifold is the d-torus $M = \mathbb{T}^d$, the tangent space is $TM = \mathbb{T}^d \times \mathbb{R}^d$, $\forall\,(x,v) \in TM$, x = position, v = velocity

Definition

(1) A Tonelli Lagrangian is a function $L(x,v) : TM \to \mathbb{R}$ which is C^2, periodic in x, and uniformly strictly convex in v

$$\exists\,\alpha > 0,\ \forall x \in M,\ \mathrm{Hess}(L)(x,v) := \frac{\partial^2 L}{\partial v^2}(x,v) > \alpha$$

(2) The action of a C^1 path $\gamma : [a,b] \to M$ is the quantity

$$\mathcal{A}(\gamma) := \int_a^b L(\gamma(t), \gamma'(t))\,dt$$

(3) The Lagrangian flow is the flow on the tangent space

$$\Phi_L^t(x,v) : TM \to TM, \quad \gamma_{x,v}(t) = pr^1 \circ \Phi_L^t(x,v),$$

$$\frac{d}{dt}\gamma_{x,v} = pr^2 \circ \Phi_L^t(x,v)$$

where $\gamma_{x,v}$ is a a local minimizer of the action :

$$\mathcal{A}(\gamma_{x,v}) \le \mathcal{A}(\gamma), \quad \forall\,\gamma : [a,b] \to M,\ C^1 \text{ close}$$

Introduction : **Frenkel-Kontorova model**

Example $M = \mathbb{T}^d$, $TM = \mathbb{T}^d \times \mathbb{R}^d$, $U : M \to \mathbb{R}$ a C^2 periodic function, $\lambda \in \mathbb{R}^d$ a constant representing a cohomological constraint

$$L(x, v) = \frac{1}{2}\|v\|^2 - U(x) - \lambda \cdot v$$

Recall The action of a C^1 path $\gamma : [a, b] \to M$ is the quantity

$$\mathcal{A}(\gamma) := \int_a^b L(\gamma(t), \gamma'(t))\, dt, \quad \gamma(a) = x,\ \gamma(b) = y$$

Discrete Aubry-Mather A discretization in time of a Laganrgian flow. Let $\tau > 0$ be a small number

$$\mathcal{A}_\tau(x, y) := \tau L\Big(x, \frac{y - x}{\tau}\Big) - \tau U(x) - \lambda \cdot (y - x)$$

Introduction : **Frenkel-Kontorova model**

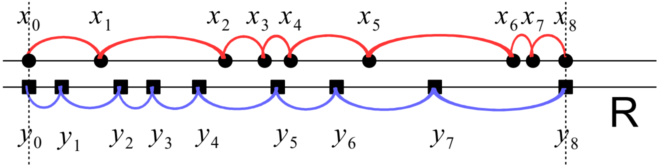
Frenkel-Kontorova model A discretization in time of the inverse pendulum : $d = 1$, $M = \mathbb{T}$, $\tilde{M} = \mathbb{R} \to M$ is the natural covering space

$$E_\tau(x, y) := \frac{1}{2\tau}|y - x|^2 + \frac{\tau K}{2\pi}\big(1 - \cos(2\pi x)\big) - \lambda(y - x)$$

E_τ is called an interaction energy

Definition A minimizing configuration $(x_k)_{k\in\mathbb{Z}}$, $x_k \in \mathbb{R}$, $\forall\, m \in \mathbb{Z}$, $\forall\, n \geq 1$

$$\sum_{k=m}^{n+n-1} E_\tau(x_k, x_{k+1}) \leq \sum_{k=m}^{m+n-1} E(y_k, y_{k+1}), \quad \forall \begin{cases} y_m = x_m \\ y_{m+n} = x_{m+n} \end{cases}$$



Introduction : Frenkel-Kontorova model

Dynamical system (Σ, σ) where Σ is the space of minimizing configurations $x = (x_k)_{k\in\mathbb{Z}}$, and $\sigma : \Sigma \to \Sigma$ is the left shift

$$\sigma(x) = y = (y_k)_{k\in\mathbb{Z}} \ \Leftrightarrow \ y_k = x_{k+1}, \ \forall\, k \in \mathbb{Z}$$

Definition The ergodic minimizing value of E, or the *effective energy*

$$\bar{E}_\tau = \lim_{n\to+\infty} \frac{1}{n} \inf_{x_0, x_1, \ldots, x_n} \sum_{k=0}^{n-1} E(x_k, x_{k+1})$$

Proposition We will see that one can define a discrete Lagrangian dynamics $\Phi_{L,\tau}(x, v) : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ such that

$$\bar{E}_\tau = \inf \Big\{ \int E(x, x + \tau v)\, d\mu(x, v) : \mu \text{ is } \Phi_{L,\tau} \text{ minimizing} \Big\}$$

Remark Although $\Phi_{L,\tau}$ is not hyperbolic, a similar theory can be applied. Numerically by discretizing the space, we get back to subshift of finite type

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Introduction : **Linear switched systems**

Question We studied in different examples the notion of ergodic minimizing value of a scalar function $\phi : X \to \mathbb{R}$. If f is multivalued what can be said ?

Definition A (discrete in time) linear switch system is a dynamical system of the form

$$v_{k+1} = A_k v_k, \ \forall\, k \geq 0$$

where $v_k \in \mathbb{R}^d$ represents the state of the system, $A_k \in \mathrm{Mat}(\mathbb{R}, d)$ is a square matrix, and v_{k+1} is the state at the next time. The action A_k can be chosen either by an external observer or by an automatic dynamical system (X, f)

Definition We consider a topological dynamical system (X, f), a continuous matrix function $A : X \to \mathrm{Mat}(\mathbb{R}, d)$, and a matrix cocycle

$$A(x, n) := A \circ f^{n-1}(x) \cdots A \circ f(x) A(x)$$

Introduction : **Linear switched systems**

Question One of the main problem in control theory is to stabilize a system, that is to find a trajectory $x \in X$ such that

$$\|A(x,n)\| = \|A \circ f^{n-1}(x) \cdots A \circ f(x) A(x)\| \leq 1$$

We are left to study the worst case, that is to compute the following characteristic of the system

Definition The *maximizing singular value* of a cocycle

$$\bar{\sigma}_1(A) := \lim_{n \to +\infty} \sup_{x \in X} \|A(x,n)\|^{1/n}$$

Actually we prefer to introduce the *maximizing Lyapunov exponent*

$$\bar{\lambda}_1 := \log(\bar{\sigma}_1(A)) = \lim_{n \to +\infty} \frac{1}{n} \sup_{x \in X} \log(\|A(x,n)\|)$$

Introduction : Linear switched systems

Definition A cocycle of order 1 over the full shift :

(1) a finite set of matrices $\mathcal{A} := \{M_1, \cdots, M_r\}$

(2) the full shift $X := \mathcal{A}^{\mathbb{N}} = \{x = (A_k)_{k \geq 0} : A_k \in \mathcal{A}, \ \forall\, k \geq 0\}$
$f : X \to X$ is the left shift

(3) the cocycle of order 1 : $A(x) = A_0$ if $x = (A_k)_{k\geq 0}$
$A(x,n) = A_{n-1} \cdots A_1 A_0$

Example A cocycle of order 1 over a set of two matrices

$$M_1 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad M_2 := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Although the spectral radius of each matrix is

$$\rho = \lim_{n\to+\infty} \|M_1^n\|^{1/n} = \lim_{n\to+\infty} \|M_2^n\|^{1/n} = 1$$

we will see that the "generalized spectral radius" of indifferent products is

$$\lim_{n\to+\infty} \sup_{A_{n-1},\ldots,A_1,A_0} \|A_{n-1}\cdots A_1 A_0\|^{1/n} > 1$$

Summary

II. Additive ergodic optimization on hyperbolic spaces

- **Basic definitions again**
- Minimal systems and Gottschalk-Hedlund
- Minimizing measures and Mather set
- An example of hyperbolic space : Subshift of finite type
- Lax-Oleinik operator and calibrated subactions
- Some extensions for Anosov systems

Additive cocycle : **Basic definitions again**

Definition We consider

(1) (X, f) a topological dynamical system, X compact, $f : X \to X$ continuous

(2) $\phi : X \to \mathbb{R}$ a continuous observable

(3) the ergodic minimizing value of ϕ

$$\bar{\phi} := \lim_{n \to +\infty} \inf_{x \in X} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f^k(x)$$

Question Can we say something for the lower bound of

$$\inf_{n \geq 1} \inf_{x \in X} \left\{ \sum_{k=0}^{n-1} \phi \circ f^k(x) - n\bar{\phi} \right\}$$

Additive cocycle : **Basic definitions again**

Definition A coboundary is a special observable of the form

$$\phi = u \circ f - u$$

for some continuous function $u : X \to \mathbb{R}$

An easy example Assume ϕ is a coboundary $\phi = u \circ f - u$ then

$$\bar{\phi} = 0 \quad \text{and} \quad \sup_{n \geq 1} \sup_{x \in X} \Big| \sum_{k=0}^{n-1} \phi \circ f^k(x) - n\bar{\phi} \Big| < +\infty$$

Proof The Birkhoff sum can be evaluated easily

$$\sum_{k=0}^{n-1} \phi \circ f^k = u \circ f^n - u$$

$$\sup_{x \in X} \Big| \sum_{k=0}^{n-1} \phi \circ f^k(x) \Big| \leq 2\|u\|_\infty$$

$$\bar{\phi} = \lim_{n \to +\infty} \inf_{x \in X} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f^k(x) = 0$$

II. Additive ergodic optimization on hyperbolic spaces

- Basic definitions again
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Additive cocycle : **Gottschalk-Hedlund theorem**

Definition A minimal system (X, f) is a topological dynamical system so that every orbit is dense

$$\forall\, x \in X,\ \overline{\{f^n(x) : n \geq 0\}} = X$$

Example The hull of the Fibonacci sequence

1. the substitution : $0 \to 1,\ 1 \to 10$

$$0 \to 1 \to 10 \to 10.1 \to 101.10 \to 10110.101 \to 10110101.10110$$

$$\omega_0 = 0,\ \omega_1 = 1,\ \omega_{n+1} = \omega_n \omega_{n-1} \quad \to \quad \omega_\infty \in \{0,1\}^{\mathbb{N}}$$

2. the hull of the bi-infinite Fibonacci sequence $\omega_{\infty\infty}$

$$\omega_{\infty\infty} := 0^\infty \mid \omega_\infty \in \Sigma := \{0,1\}^{\mathbb{Z}}$$

$$X := \bigcap_{n \geq 1} \overline{\big\{\sigma^k(\omega_{\infty\infty}) : k \geq n\big\}} \subseteq \Sigma$$

3. (X, σ) is a subshift of (Σ, σ) semi-conjugated to the rotation on the circle of rotation number

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{largest eigenvalue of} \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Additive cocycle : **Gottschalk-Hedlund theorem**

Theorem(Gottschalk-Hedlund) Let (X, f) be a minimal system and $\phi : X \to \mathbb{R}$ be a continuous function. Assume there exists a point $x_0 \in X$ such that

$$\sup_{n \geq 1} \Big| \sum_{k=0}^{n-1} \phi \circ f^k(x_0) \Big| < +\infty$$

Then there exists $u : X \to \mathbb{R}$ such that

$$\phi = u \circ f - u$$

(We say that ϕ is a coboundary)

Additive cocycle : **Gottschalk-Hedlund theorem**

Definition A function $v : X \to \mathbb{R}$ is said to be u.s.c, upper semi continuous at $x_0 \in X$ if

$$\lim_{\epsilon \to 0} \sup_{x \in B(x_0, \epsilon)} v(x) \leq v(x_0)$$

A function u is said to be l.s.c. lower semi continuous if

$$\lim_{\epsilon \to 0} \inf_{x \in B(x_0, \epsilon)} u(x) \geq u(x_0)$$

Proposition

- the supremum of a sequence of continuous functions is l.s.c.
- The infimum of a sequence of continuous functions is u.s.c.

Proposition

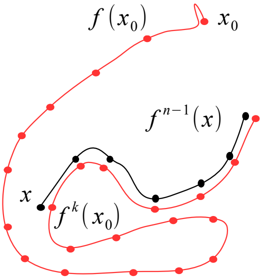
- v is u.s.c. $\Leftrightarrow \{v \geq \lambda\}$ is closed for every λ
- u is l.s.c. $\Leftrightarrow \{u \leq \lambda\}$ is closed for every λ

Additive cocycle : **Gottschalk-Hedlund theorem**

Proof of Gottschalk-Hedlund Recall we have assumed

$$R_0 := \sup_{n \geq 1} \Big| \sum_{k=0}^{n-1} \phi \circ f^k(x_0) \Big| < +\infty$$

1. We first observe that $$\sup_{x \in X} \sup_{n \geq 1} \Big| \sum_{k=0}^{n-1} \phi \circ f^k(x) \Big| \leq 2R_0$$



let $x \in X$, $n \geq 1$, $\epsilon > 0$ fixed. By minimality there exists $k \geq 0$

$$\sum_{i=0}^{n-1} |\phi \circ f^i(x) - \phi \circ f^{i+k}(x_0)| < \epsilon$$

$$\sum_{i=0}^{n-1} \phi \circ f^{i+k}(x_0) = \sum_{i=0}^{n+k-1} \phi \circ f^i(x_0) - \sum_{i=0}^{k-1} \phi \circ f^i(x_0)$$

Additive cocycle : **Gottschalk-Hedlund theorem**

Proof of Gottschalk-Hedlund

2. We define two functions

$$u := \sup_{n \geq 1} \sum_{k=0}^{n-1} \phi \circ f^k \quad v := \inf_{n \geq 1} \sum_{k=0}^{n-1} \phi \circ f^k$$

3. u is l.s.c. v is u.s.c.
4. the computation of $u \circ f$ and $v \circ f$ introduces a shift in the summation

$$u \circ f = \sup_{n \geq 1} \sum_{k=1}^{n} \phi \circ f^k \quad u \circ f + \phi = \sup_{n \geq 2} \sum_{k=0}^{n-1} \phi \circ f^k \leq u$$

$$v \circ f = \inf_{n \geq 1} \sum_{k=1}^{n} \phi \circ f^k \quad v \circ f + \phi = \inf_{n \geq 2} \sum_{k=0}^{n-1} \phi \circ f^k \geq v$$

Additive cocycle : **Gottschalk-Hedlund theorem**

Proof of Gottschalk-Hedlund

5. we just have proved : $u \circ f + \phi \leq u \qquad v \circ f + \phi \geq v$
6. define $w := v - u$, then $w \circ f \geq w$
7. w is upper semi continuous $\to$ w attains its supremum
8. let x_* be a point maximizing w
9. then $X_* := \{x \in X : w(x) = w(x_*)\}$ is invariant by f
10. X_* is closed again by u.s.c. of w
11. $X_* = X$ by minimality $w = w(x_*)$, $\forall\, x \in X$
12. $v - u = \text{const} \Rightarrow v$ and u are continuous

$$u \circ f + \phi = u \quad v \circ f + \phi = v$$

Introduction : **Gottschalk-Hedlund theorem**

Remark The assumptions in Gottschal-Hedlund implies $\bar{\phi} = 0$

$$\sup_{n\geq 1} \Big| \sum_{k=0}^{n-1} \phi \circ f^k(x_0) \Big| < +\infty \quad \Rightarrow \quad \bar{\phi} = \lim_{n\to+\infty} \inf_{x\in X} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f^k(x) = 0$$

Question Is the converse true ?

Definition An additive cocyle is nondefective from below if there exists a constant C such that

$$\forall\, x \in X,\ \forall\, n \geq 0,\ \sum_{k=0}^{n-1} \phi \circ f(x) \geq n\bar{\phi} + C$$

Proposition If (X, f) is minimal and ϕ is continuous nondefective from below then

$$\phi = u \circ f - u + \bar{\phi}$$

for some continuous $u : X \to \mathbb{R}$

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Additive cocycle : Minimizing measures and Mather set

Lemma If $(a_n)_{n\geq 0}$ is a sub additive sequence

$$a_{m+n} \leq a_m + a_n, \ \forall\, m,n \geq 0$$

then

$$\lim_{n\to+\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n}$$

Remark The following sequence $(a_n)_{n\geq 0}$ is supper additive

$$a_n := \inf_{x\in X} \sum_{k=0}^{n-1} \phi \circ f^k(x)$$

Corollary The limit in the definition of $\bar{\phi}$ exists

$$\lim_{n\to+\infty} \frac{1}{n} \inf_{x\in X} \sum_{k=0}^{n-1} \phi \circ f^k(x) = \sup_{n\geq 1} \inf_{x\in X} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f^k(x)$$

Additive cocycle : Minimizing measures and Mather set

Definition We recall that a probability measure is invariant if

$$\forall\, h \in C^0(X,\mathbb{R}),\ \int h\circ f\,d\mu = \int h\,d\mu$$

Observation Let $\mathcal{M}(X,f)$ be the set of invariant measures

$$\int \phi\,d\mu = \int \Big(\frac{1}{n}\sum_{k=0}^{n-1}\phi\circ f^k\Big)\,d\mu \geq \inf_{x\in X}\frac{1}{n}\sum_{k=0}^{n-1}\phi\circ f^k$$

$$\inf_{\mu\in\mathcal{M}(X,f)}\int\phi\,d\mu \geq \sup_{n\geq 1}\inf_{x\in X}\frac{1}{n}\sum_{k=0}^{n-1}\phi\circ f^k$$

Proposition Actually

$$\inf_{\mu\in\mathcal{M}(X,f)}\int\phi\,d\mu = \sup_{n\geq 1}\inf_{x\in X}\frac{1}{n}\sum_{k=0}^{n-1}\phi\circ f^k(x)$$

A measure realizing the infimum is called a minimizing measure

Additive cocycle : Minimizing measures and Mather set

Proof

1. for every $n \geq 1$, the infimum in $\displaystyle\inf_{x \in X} \frac{1}{n}\sum_{k=0}^{n-1} \phi \circ f^k(x)$ is realized by a point x_n
2. let μ_n be the empirical measure along the trajectory
$$\mu_n := \frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k(x_n)}$$
3. by definition $\displaystyle\int \phi \, d\mu_n = \inf_{x \in X} \frac{1}{n}\sum_{k=0}^{n-1} \phi \circ f^k(x)$
4. The space of probability measures is weak* compact, there exists a subsequence of $(\mu_n)_{n \geq 1}$ converging to some probability measure μ. We check that μ is invariant
$$\int \phi \, d\mu = \lim_{n \to +\infty} \inf_{x \in X} \frac{1}{n}\sum_{k=0}^{n-1} \phi \circ f^k(x)$$

Additive cocycle : Minimizing measures and Mather set

Definition We recall

$$\mathrm{Mather} := \bigcup \{\mathrm{supp}(\mu) : \mu \text{ is minimizing } \}$$

Proposition The Mather set is compact

$$\mathrm{Mather} = \mathrm{supp}(\mu) \quad \text{for some minimizing measure } \mu$$

Question What is the structure of the Mather set ? Is it a big set, a small set ? Can we find on the Mather set optimal trajectories x that is

$$\sup_{n\geq 1} \Big| \sum_{k=0}^{n-1} \phi \circ f^k(x) - n\bar{\phi} \Big| < +\infty$$

II. Additive ergodic optimization on hyperbolic spaces

- Basic definitions again
- Minimal systems and Gottschalk-Hedlund
- Minimizing measures and Mather set
- **An example of hyperbolic space : Subshift of finite type**
- Lax-Oleinik operator and calibrated subactions
- Some extensions for Anosov systems

Additive cocycle : **Subshift of finite type**

Definition We consider here a one-sided subshift of finite type

- $\mathcal{A} := \{1, 2, \cdots, r\}$ is a finite set of states
- M is a $r \times r$ square matrix describing the allowed transitions

  $M(i,j) \in \{0,1\} \quad M(i,j) = 1 \;\Leftrightarrow\; i \to j$ is an admissible transition

- $X = \big\{(x_n)_{n\geq 0} : \forall\, n \geq 0,\; x_n \in \mathcal{A},\; M(x_n, x_{n+1}) = 1\big\}$

  X is called a subshift of finite type SFT. The left shift $f : X \to X$

$$x = (x_0, x_1, x_2, \ldots) \quad \Rightarrow \quad y = f(x) = (x_1, x_2, x_3, \ldots)$$

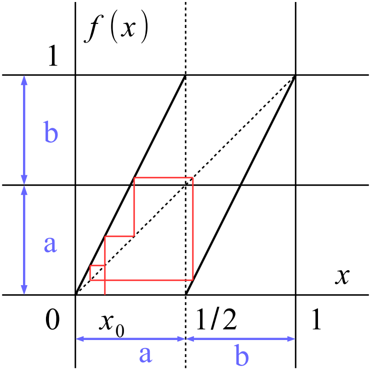
- X equipped with the product topology is compact metrizable

$$d(x,y) = e^{-n} \;\Leftrightarrow\; x_0 = y_0, \cdots, x_{n-1} = y_{n-1} \text{ and } x_n \neq y_n$$

- we assume M is semi irreducible

$$\forall\, i \in \mathcal{A},\; \exists j \in \mathcal{A},\; M(i,j) = 1$$
$$\forall\, j \in \mathcal{A},\; \exists i \in \mathcal{A},\; M(i,j) = 1$$

Additive cocycle : **Subshift of finite type**



The doubling period

$$f : x \mapsto 2x \bmod 1$$

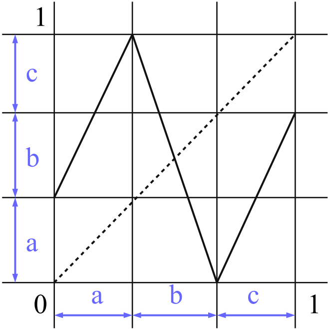
is semi conjugated (up to a countable number of points) to the full shift

$$X = \{a, b\}^{\mathbb{N}}$$

Here the hyperbolicity is related to the fact that

$$|f'(x)| > 1$$

Additive cocycle : **Subshift of finite type**



A Markov map (could be discontinuous). The states space

$$\mathcal{A} = \{a, b, c\}$$

The transition matrix

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The Markov map is semi conjugated to the SFT

$$X = \left\{ x \in \mathcal{A}^{\mathbb{N}} : M(x_k, x_{k+1}) = 1, \ \forall\, k \right\}$$

Again the hyperbolicity of the Markov map is obtained because of $|f'(x)| > 1$. Any C^2 perturbation still remaining Markov is semi conjugated to (X, f)

Additive cocycle : **Subshift of finite type**

Remark A SFT is hyperbolic in the following sense

- if $x_0 = y_0, \cdots, x_{n-1} = y_{n-1}$ and $x_n \neq y_n$ then

$$d(x,y) = e^{-n}, \quad d(f(x), f(y)) = e^{-(n-1)} = e^1 d(x,y)$$
$$\Rightarrow \ \sigma \text{ is expanding}$$

- if x and y are two configurations such that $x_0 = y_0$ and

$$\cdots \ x_{-3} \to x_{-2} \to x_{-1} \to x_0,$$

are preimages of x_0 then the new configurations

$$x' = (x_{-1}, x_0, x_1, \ldots) \quad y' = (x_{-1}, y_0, y_1, \ldots)$$
$$x'' = (x_{-2}, x_{-1}, x_0, x_1, \ldots) \quad y'' = (x_{-2}, x_{-1}, y_0, y_1, \ldots)$$

are contracted

$$d(x', y') = e^{-1} d(x,y) \quad d(x'', y'') = e^{-2} d(x,y)$$

II. Additive ergodic optimization on hyperbolic spaces

- Basic definitions again
- Minimal systems and Gottschalk-Hedlund
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- **Lax-Oleinik operator and calibrated subactions**
- Some extensions for Anosov systems

Additive cocycle : **Lax-Oleinik operator**

Recall The ergodic minimizing value of ϕ can be computed using measure

$$\bar{\phi} = \min \Big\{ \int \phi \, d\mu : \mu \text{ is an invariant measure } \Big\}$$

$$\text{Mather}(\phi) := \bigcup \Big\{ \text{supp}(\mu) : \mu \text{ is minimizing } \Big\}$$

Definition An observable is nondefective from below if

$$\forall\, x \in X,\ \forall\, n \geq 0,\ \sum_{k=0}^{n-1} \phi \circ f^k(x) \geq n\bar{\phi} + C$$

Theorem(Gottschalk-Hedlund) If (X, f) is minimal and $\phi : X \to \mathbb{R}$ is continuous then : $\quad \sup_{n \geq 1} \Big| \sum_{k=0}^{n-1} \phi \circ f^k(x_0) \Big| < +\infty \quad \Rightarrow \quad \phi = u \circ f - u$

Extension If (X, f) is minimal and ϕ is nondefective from below then

$$\phi = u \circ f - u + \bar{\phi}$$

Additive cocycle : **Lax-Oleinik operator**

Main hypothesis The observable is Lipschitz (or Hölder)

$$\forall\, x, y \in X,\ x_0 = y_0, \quad |\phi(x) - \phi(y)| \le \mathrm{Lip}(\phi) d(x,y)$$

Main result If (X, f) is a SFT, if $\phi : X \to \mathbb{R}$ is Lipschitz then there exists a Lipschitz function $u : X \to \mathbb{R}$ such that

(1) $\forall\, x \in X,\ \phi(x) \ge u \circ f(x) - u(x) + \bar{\phi}$

(2) $\forall\, x \in \text{Mather},\ \phi(x) = u \circ f(x) - u(x) + \bar{\phi}$

Definition A subaction for ϕ is a continuous function u such that

$$\forall\, x \in X,\ \phi(x) \ge u \circ f(x) - u(x) + \bar{\phi}$$

Corollary Every Lipschitz ϕ is non defective from below

$$\sum_{k=0}^{n-1} \phi \circ f^k(x) \ge u \circ f^n(x) - u(x) + n\bar{\phi} \ge n\bar{\phi} - 2\|u\|_\infty$$

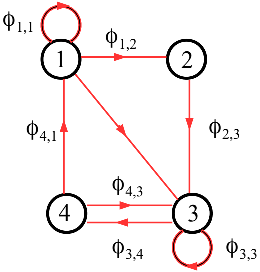
Corollary Every trajectory of the Mather set is optimal

$$x \in \text{Mather}(\phi) \quad \Rightarrow \quad \Big| \sum_{k=0}^{n-1} \big(\phi \circ f^k(x) - \bar{\phi}\big) \Big| \le 2\|u\|_\infty$$

Additive cocycle : **Lax-Oleinik operator**

Main tool The Lax-Oleinik operator is a (nonlinear) operator acting on Lipschitz function $u : X \to \mathbb{R}$ defined by

$$T[u](y) := \min\{u(x) + \phi(x) : f(x) = y\}$$



The transition matrix

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Assume ϕ is two-block : $\quad \phi(x) = \phi(x_0, x_1)$

It is enough to consider one-block function $u(x) = u(x_0)$

$$T[u](1) = \min\left\{u(1) + \phi(1,1), u(4) + \phi(4,1)\right\}$$
$$T[u](2) = u(1) + \phi(1,2)$$
$$T[u](3) = \min\left\{u(1) + \phi(1,3),\ u(2) + \phi_{2,3}, u(3) + \phi(3,3),\ u(4) + \phi(4,3)\right\}$$

Additive cocycle : **Lax-Oleinik operator**

Definition The Lax-oleinik operator $T : \mathrm{Lip}(X,\mathbb{R}) \to \mathrm{Lip}(X,\mathbb{R})$

$$T[u](y) := \min\{u(x) + \phi(x) : f(x) = y\}$$

Theorem

(1) There exists a unique "additive eigenvalue" a and an (a priori non unique) "additive eigenfunction" $u \in \mathrm{Lip}(X,\mathbb{R})$ such that

$$T[u] = u + a$$

(2) $a = \bar{\phi}$ is the unique eigenvalue

(3) Every eigenfunction u is a subaction

$$\phi(x) \geq u \circ f(x) - u(x) + \bar{\phi}$$

Definition An additive eigenfunction of the Lax-Oleinik operator is called a calibrated subaction

Additive cocycle : **Lax-Oleinik operator**

The proof uses either the Schauder theorem or a more explicit iterative scheme

Ishikawa's Algorithm(Admitted) Let $\mathbb{B}$ be a Banach space, $\mathbb{K} \subset \mathbb{B}$ be a convex compact set, and $T : \mathbb{K} \to \mathbb{K}$ be a nonexpansive map

$$\|T[u] - T[v]\| \le \|u - v\|$$

Then the sequence

$$u_0 \in \mathbb{K}, \quad u_{n+1} = \frac{u_n + T[u_n]}{2}$$

converges to a fixed point.

Notation We will apply Ishikawa's algorithm to

$$\mathbb{B} := C^0(X, \mathbb{R})/\mathbb{R} \quad \text{with} \quad u \sim v \;\Leftrightarrow\; u - v = \text{const.}$$
$$|||u||| := \inf\{\|u + c\|_\infty : c \in \mathbb{R}\}$$
$$\mathbb{K}_C := \big\{u \in \mathbb{B} : \mathrm{Lip}(u) \le C\big\} \quad \text{for some constant } C$$

Additive cocycle : **Lax-Oleinik operator**

Recall The Lax-Oleinik operator : $X \subseteq \mathcal{A}^{\mathbb{N}}, \quad \mathcal{A} = \{1, \ldots, r\}$

$$T[u](x_0, x_1, x_2, \ldots) = \min_{x_{-1} \in \mathcal{A}} \big\{ (u + \phi)(x_{-1}, x_0, x_1, \ldots) \big\}$$

Main observation Two points $x, y \in X$ starting at the same symbol $i_0 = x_0 = y_0 \in \mathcal{A}$ have a common symbolic inverse branch which contracts exponentially fast

$$x_0 = y_0 \ \Rightarrow \ \exists\, i_{-3} \to i_{-2} \to i_{-1} \to i_0$$
$$x^{(-n)} := (i_{-n}, \ldots, i_{-1}, x_0, x_1, \ldots), \quad f^n(x^{(-n)}) = x$$
$$y^{(-n)} := (i_{-n}, \ldots, i_{-1}, y_0, y_1, \ldots)$$
$$d(x^{(-n)}, y^{(-n)}) \leq \lambda^n d(x, y)$$

for some $0 < \lambda < 1 \quad (\lambda = e^{-1})$

Hyperbolicity The existence of such a contracting inverse dynamics is the main observation for the existence of u

Additive cocycle : **Lax-Oleinik operator**

Proof of the ergodic Lax-Oleinik's theorem

1. we recall the definition

$$T[u](y) = \min_{f(x)=y} \big(u(x) + \phi(x)\big)$$

2. T commutes with the constants : $\quad T[u+c] = T[u] + c$
3. T is nonexpansive :

$$\|T[u] - T[v]\|_\infty \leq \|u - v\|_\infty$$

$$\begin{array}{lcl}
y \text{ fixed} & \Rightarrow & \exists\, x \text{ optimal, } T[v](y) = v(x) + \phi(x) \\
T[u] \text{ is a min} & \Rightarrow & T[u](y) \leq u(x) + \phi(x) \\
\text{substracting} & \Rightarrow & T[u](y) - T[v](y) \leq u(x) - v(x) \leq \|u - v\| \\
\text{permuting} & \Rightarrow & |T[u](y) - T[v](y)| \leq u(x) - v(x) \leq \|u - v\|
\end{array}$$

Proof of the ergodic Lax-Oleinik's theorem

4. T preserves the set : $\Big\{u : \mathrm{Lip}(u) \le C\Big\} \quad C := \dfrac{\lambda}{1-\lambda}\mathrm{Lip}(\phi)$

   choose y, y' such that $y_0 = y'_0$
   optimize $T[u](y')$: $\exists\, x',\ f(x') = y'$ such that

$$T[u](y') = u(x') + \phi(x')$$

   choose the same inverse branch : $\exists\, x,\ f(x) = y$ such that

$$d(x, x') \le \lambda d(y, y')$$

   by minimizing $T[u](y)$ and substracting

$$T[u](y) \le u(x) + \phi(x)$$
$$T[u](y) - T[u](y') \le (u + \phi)(x) - (u + \phi)(x')$$

5. we use now that ϕ is Lipschitz

$$T[u](y) - T[u](y') \le (\mathrm{Lip}(u) + \mathrm{Lip}(\phi))\lambda d(y, y')$$
$$\mathrm{Lip}(T[u]) \le \lambda \mathrm{Lip}(\phi) + \frac{\lambda^2}{1-\lambda}\mathrm{Lip}(\phi) = \frac{\lambda}{1-\lambda}\mathrm{Lip}(\phi)$$
$$\mathrm{Lip}(T[u]) \le C$$

Proof of the ergodic Lax-Oleinik's theorem

6. we introduce the quotient space $\mathbb{B} := C^0(X, \mathbb{R})/\mathbb{R}$
   T acts on $\mathbb{B}$ because T commutes with the constants
   T preserves the set

$$\mathbb{K} = \Big\{ u \in \mathbb{B} : \mathrm{Lip}(u) \leq \frac{\lambda}{1-\lambda} \mathrm{Lip}(\phi) \Big\}$$

   $\mathbb{K}$ is convex

7. By Ascoli's theorem $\mathbb{K}$ is compact

8. by Ishikawa's theorem T admits a fixed point u in $\mathbb{K}$:
   there exists $u : X \to \mathbb{R}$ Lipschitz and $a \in \mathbb{R}$ such that

$$T[u] = u + a$$

Proof of the ergodic Lax-Oleinik's theorem

9. We show that $a \leq \bar\phi$. For every $x, y \in X$

$$f(x) = y \ \Rightarrow \quad u(y) + a = T[u](y) \leq u(x) + \phi(x)$$
$$u \circ f(x) + a \leq u(x) + \phi(x)$$

we thus have proved that an additive eigenfunction is a subaction

$$u \circ f - u + a \leq \phi$$

$$\forall\, x \in X,\ u \circ f^n(x) - u(x) + na \leq \sum_{k=0}^{n-1} \phi \circ f^k(x)$$

$$a \leq \lim_{n \to +\infty} \inf_{x \in X} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f^k(x) = \bar\phi$$

Proof of the ergodic Lax-Oleinik's theorem

10. We show that $a \geq \bar\phi$. We choose arbitrarily a point $x^{(0)} \in X$. By optimality in the definition in Lax-Oleinik

$$u(y) + a = T[u](y) = \min_{f(x)=y} \{u(x) + \phi(x)\}$$

$$\exists\, x^{(-1)} \in X,\ f(x^{(-1)}) = x^{(0)}, \qquad u(x^{(0)}) + a = u(x^{(-1)}) + \phi(x^{(-1)})$$

$$\exists\, x^{(-2)} \in X,\ f(x^{(-2)}) = x^{(-1)}, \qquad u(x^{(-1)}) + a = u(x^{(-2)}) + \phi(x^{(-2)})$$

$$\exists\, x^{(-3)} \in X,\ f(x^{(-3)}) = x^{(-2)}, \qquad u(x^{(-2)}) + a = u(x^{(-3)}) + \phi(x^{(-3)})$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$\sum_{k=1}^{n} \phi(x^{(-k)}) = u(x^{(0)}) - u(x^{(-n)}) + na$$

$$\bar\phi = \lim_{n\to+\infty} \inf_{x\in X} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f^k(x) \leq \lim_{n\to+\infty} \frac{u(x^{(0)}) - u(x^{(-n)}) + na}{n} = a$$

Additive cocycle : **Lax-Oleinik operator**

Corollary Let (X, f) be a SFT, let ϕ be a Lipschitz function

(1) there exits a Lipschitz subaction $u : X \to \mathbb{R}$

$$\forall\, x \in X,\ \phi(x) \geq u \circ f(x) - u(x) + \bar{\phi}$$

(2) up to a coboundary, the ergodic minimizing value is a true minimum

$$\psi := \phi - (u \circ f - u) \ \Rightarrow\ \begin{cases} \bar{\psi} = \min_X(\psi) = \bar{\phi} \\ \forall\, x \in X,\ \psi(x) \geq \bar{\psi} \\ \forall\, x \in \text{Mather},\ \psi(x) = \bar{\psi} \end{cases}$$

Proof

1. for every invariant measure $\displaystyle\int \psi\, d\mu = \int \phi\, d\mu \quad \Rightarrow \quad \bar{\psi} = \bar{\phi}$
2. as $(\psi - \bar{\phi}) \geq 0$ and $\displaystyle\int (\psi - \bar{\phi})\, d\mu = 0$ for μ minimizing

$$\forall\, x \in \text{supp}(\mu),\ \psi = \bar{\phi}$$

Additive cocycle : **Lax-Oleinik operator**

Corollary Every trajectory in the Mather set is optimal

$$\forall\, x \in \text{Mather},\ \sup_{n\geq 1} \Big| \sum_{k=0}^{n-1} \phi\circ f^k(x) - n\bar\phi \Big| < +\infty$$

Proof

1. for every minimizing measure μ $\quad \int (\phi - \bar\phi)\, d\mu = 0$
2. there exists a subaction $\quad (\phi - \bar\phi) - (u\circ f - u) \geq 0$
3. $\int (\phi - \bar\phi) - (u\circ f - u)\, d\mu = 0$
4. $\phi - \bar\phi = u\circ f - u \quad \mu$ a.e.
5. $\phi - \bar\phi = u\circ f - u \quad$ everywhere on $\text{supp}(\mu)$
6. $\Big| \sum_{k=0}^{n-1} (\phi - \bar\phi)\circ f^k(x) \Big| = |u\circ f^n(x) - u(x)| \leq 2\|u\|_\infty \quad$ on $\text{supp}(\mu)$

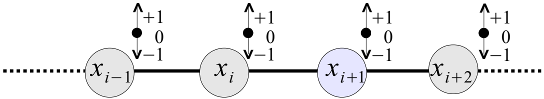
Summary

III. Zero temperature limit in thermodynamic formalism

- **Description of the BEG model**
- Gibbs measures of a directed graph
- Ground states of a directed graph
- Zero temperature limit for a SFT
- Explicit computations for the BEG model

Zero limit : Description of the BEG model

Description The Blume Emery Griffiths model (BEG model)



One considers a chain of atoms on a lattice at equilibrium at positive temperature that interact with their first neighbours.

(1) Each site of the lattice hosts a unique atom

(2) there are 3 kinds of atoms ; either He^4 with spin up or down, or an isotope He^3 with no spin. Let $\mathcal{A} = \{-1, 0, 1\}$ be the 3 kinds of atoms.

(3) a chain of atoms is an infinite sequence $x = (x_k)_{k\in\mathbb{Z}}$, $x_k \in \mathcal{A}$

(4) the interaction energy is short-range and is given by an Hamiltonian : $H : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$

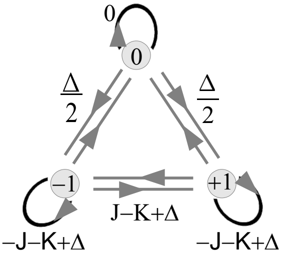
(5) the energy of a finite block of atoms

$$H(x_m, x_{m+1}, \ldots, x_{m+n}) := \sum_{k=m}^{m+n-1} H(x_k, x_{k+1})$$

Zero limit : Description of the BEG model

Hamiltonian in BGE $H : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ has the form

$$H(x,y) := -Jxy - Kx^2y^2 + \frac{\Delta}{2}(x^2+y^2)$$



(1) $x, y \in \mathcal{A} = \{-1, 0, 1\}$

(2) $J > 0 \Rightarrow$ spins tend to be aligned

(3) $K > 0 \Rightarrow$ spins tend to be neighbours

(4) $\Delta > 0 \Rightarrow$ role of a chemical potential

(5) directed graph with transition matrix

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Example of a computation

$$H(0,0) = 0, \quad H(-1,1) = J - K + \Delta, \ \cdots$$

III. Zero temperature limit in thermodynamic formalism

- Description of the BEG model
- **Gibbs measures of a directed graph**
- Ground states of a directed graph
- Zero temperature limit for a SFT
- Explicit computations for the BEG model

Zero limit : Gibbs measures of a directed graph

Formal notations

(1) $\mathcal{A} = \{1, 2, \ldots, r\}$: the possible state space of the atoms

(2) M : an $r \times r$ matrix with values in $\{0, 1\}$ called transition matrix

$$M(i,j) = 1 \ \Leftrightarrow \ \text{a transition } i \to j \text{ is allowed}$$

(3) (X, f) : the bi-infinite subshift of finite type, $f : X \to X$

$$X = \big\{x = (x_k)_{k\in\mathbb{Z}} : \forall\, k \in \mathbb{Z},\ x_k \in \mathcal{A},\ M(x_k, x_{k+1}\big\} \subseteq \mathcal{A}^{\mathbb{Z}}$$
$$f(x) = y = (y_k)_{k\in\mathbb{Z}}, \quad \forall\, k \in \mathbb{Z},\ y_k = x_{k+1}$$

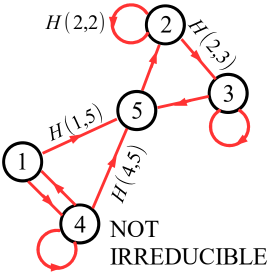
(4) $H : \mathcal{A} \times \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$: the Hamiltonian of the system describing the local energy between two successive atoms

$$H(i,j) = +\infty \ \Leftrightarrow \ M(i,j) = 0$$

(5) $\phi : X \to \mathbb{R}$: the corresponding short rang interaction on the SFT

$$\phi(x) = H(x_0, x_1)$$

Zero limit : **Gibbs measures of a directed graph**



Assumption The transition matrix (or the graph) is irreducible : for every state $i, j \in \mathcal{A}$

$$\exists\ i = i_0 \to i_1 \to i_2 \to \cdots \to i_n = j$$

Definition We introduce a weight for each transition

$$M_\beta(i,j) := \exp(-\beta H(i,j))$$

which should be proportional to the probability of the occurrence of the the transition

Remark

(1) β is supposed to be the inverse of the temperature T

(2) M_0 is the initial transition matrix corresponding to $T = +\infty$

(3) M_∞ is the frozen state corresponding to $T = 0$

Physical Ansatz The configurations prefer transitions with low energy ($\to$ which explains the sign $-\beta H$)

Zero limit : **Gibbs measures of a directed graph**

Definition A cylinder of size n is a set of configurations that have prescribed states on n consecutive sites of $\mathbb{Z}$. To simplify the notations, the cylinder starts at 0. If $i_0, i_1, \dots, i_n \in \mathcal{A}$ then

$$[i_0, i_1, \dots, i_n] := \big\{ x = (x_k)_{k\in\mathbb{Z}} \in X : x_0 = i_0,\ x_1 = i_1, \dots, x_n = i_n \big\}$$

Definition The total energy of a block is

$$H(i_0, \dots, i_n) := \sum_{k=0}^{n-1} H(i_k, i_{k+1}) = \sum_{k=0}^{n-1} \phi \circ f^k(x), \quad \forall\, x \in [i_0, \dots, i_n]$$

Definition A Gibbs measure at temperature β^{-1} is an invariant measure of the SFT (X, f) such that

$$\mu_\beta([i_0, \dots, i_n]) \asymp \exp\Big(- \beta H(i_0, \dots, i_n) + n\beta \bar{H}_\beta \Big)$$

$$\exp(-n\beta\bar{H}_\beta) \asymp \sum_{\substack{[i_0,\dots,i_n] \\ \text{admissible}}} \exp\Big(- \beta H(i_0, \dots, i_n) + n\beta \bar{H}_\beta \Big)$$

Zero limit : **Gibbs measures of a directed graph**

Theorem Let (X, f) be a SFT associated to an irreducible transition matrix and $H : \mathcal{A} \times \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ be a two-step Hamiltonian. Then there exists a unique Gibbs measure at every temperature β^{-1}

Recall $M_\beta(i,j) = \exp(-\beta H(i,j))$.

Definition A non negative matrix $M \in \mathrm{Mat}(\mathbb{R}^+, r)$ is said to be an irreducible matrix, if $\forall\, i,j \in \{1,\ldots,r\}$, there exists $i_0, i_1, \ldots, i_n$, with $i_0 = i$ and $j_0 = j$ such that

$$M(i_0,i_1)M_\beta(i_1,i_2)\cdots M(i_{n-1},i_n) > 0$$

Remember $M_\beta(i,j) = 0 \;\Leftrightarrow\; H(i,j) = +\infty \;\Leftrightarrow\; i \not\to j$

Perron Frobenius theorem If M is a non negative irreducible matrix, then the spectral radius ρ of M is strictly positive and ρ is an eigenvalue of multiplicity 1. Moreover the eigenvector associated to 1 can be chosen to have strictly positive entries

Zero limit : Gibbs measures of a directed graph

Theorem Let (X, f) be a SFT associated to an irreducible transition matrix and $H : \mathcal{A} \times \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ be a two-step Hamiltonian. Then there exists a unique Gibbs measure at every temperature β^{-1}

Proof The Perron-Frobenius theorem tells us

1. let $M_\beta(i,j) = \exp(-\beta H(i,j))$ be an irreducible $r \times r$ matrix
2. let $\rho_\beta := \exp(-\beta \bar{H}_\beta)$ be the largest eigenvalue
3. let $R_\beta(i)$ be the right eigenvector with strictly positive entries
4. let $L_\beta(i)$ be the left eigenvector with strictly positive entries
5. we normalize so that : $\sum_{i=1}^r L_\beta(i) R_\beta(i) = 1$

The Gibbs measure at temperature β^{-1} of a cylinder is

$$\mu_\beta([i_0, \ldots, i_n]) = \frac{1}{\rho_\beta^n} L_\beta(i_0) \exp\Big(- \beta H(i_0, \ldots, i_n)\Big) R_\beta(i_n)$$

We show that μ_β is a well defined probability on X and is invariant by the dynamics f

Zero limit : **Gibbs measures of a directed graph**

Recall The Gibbs measure at temperature β^{-1} is defined by

$$\mu_\beta([i_0,\ldots,i_n]) = \frac{1}{\rho_\beta^n} L_\beta(i_0)\Big[\prod_{k=0}^{n-1} M_\beta(i_k,i_{k+1})\Big] R_\beta(i_n)$$

Step 1 The measure is consistent in the Kolmogorov sense

$$\begin{aligned}\sum_{j=1}^r \mu_\beta([i_0,\ldots,i_n,j]) &= \mu_\beta([i_0,\ldots,i_n])\Big[\frac{1}{\rho_\beta}\sum_{j=1}^r M_\beta(i_n,j)\frac{R_\beta(j)}{R_\beta(i_n)}\Big] \\ &= \mu_\beta([i_0,\ldots,i_n])\end{aligned}$$

Step 2 The measure is invariant

$$\begin{aligned}\sum_{i=1}^r \mu_\beta([i,i_0,\ldots,i_n]) &= \Big[\frac{1}{\rho_\beta}\sum_{i=1}^r \frac{L_\beta(i)}{L_\beta(i_0)} M_\beta(i,i_0)\Big]\mu_\beta([i_0,\ldots,i_n]) \\ &= \mu_\beta([i_0,\ldots,i_n])\end{aligned}$$

III. Zero temperature limit in thermodynamic formalism

- Description of the BEG model
- Gibbs measures of a directed graph
- **Ground states of a directed graph**
- Zero temperature limit for a SFT
- Explicit computations for the BEG model

Zero limit : **Ground states of a directed graph**

Recall The Gibbs measure of a two steps cylinder is given by

$$\mu_\beta([i,j]) = L_\beta(i)\frac{M_\beta(i,j)}{\rho_\beta}R_\beta(j), \qquad M_\beta(i,j) = \exp(-\beta H(i,j))$$

where ρ_β is the largest eigenvalue of M_β

Definition Let $\bar{H}_\beta$ be the free energy at temperature β^{-1} defined by

$$\rho_\beta := \exp(-\beta \bar{H}_\beta)$$

Question What is the behaviour of the free energy $\bar{H}_\beta$ when the system is frozen ?

Question What is the behaviour of the Gibbs measure μ_β when the system is frozen ?

Zero limit : **Ground states of a directed graph**

Proposition The free energy converges to the ergodic minimizing value $\bar{\phi}$

$$\lim_{\beta\to+\infty} \bar{H}_\beta = \bar{H} =: \inf_\mu \sum_{i=1}^r \sum_{j=1}^r H(i,j)\mu(i,j)$$

where the infimum is realized over the set of probability measures μ on $\mathcal{A}\times\mathcal{A}$ satisfying the invariance property

$$\forall\, i \in \mathcal{A},\ \ \mu^{(1)}(i) := \sum_{k=1}^r \mu(i,k) = \sum_{k=1}^r \mu(k,i) =: \mu^{(2)}(i)$$

Theorem The Gibbs measure μ_β converges to a selected minimizing measure μ_{min}, that is a probability measure satisfying the previous invariance and

$$\sum_{i=1}^r \sum_{j=1}^r H(i,j)\mu_{min}(i,j) = \bar{H}$$

Zero limit : **Ground states of a directed graph**

Proof of $\bar{H}_\beta \to \bar{H}$

1. we recall some notations $\mathcal{A} = \{1, \ldots, r\}$

$$M_\beta(i,j) = \exp(-\beta H(i,j)), \quad \rho_\beta = exp(-\beta \bar{H}_\beta)$$

2. we choose another left eigenvector

$$\forall\, j \in \mathcal{A},\ \sum_{i=1}^{r} L_\beta(i) M_\beta(i,j) = \rho_\beta L_\beta(j), \quad \max_i L_\beta(i) = 1$$

3. we change L_β to an exponential form

$$L_\beta(i) := \exp(-\beta U_\beta(i)), \quad \min_i U_\beta(i) = 0$$

4. the eigenvalue problem becomes

$$\forall\, j \in \mathcal{A}, \quad \sum_{i=1}^{r} \exp\Big(-\beta\big(H(i,j) - \bar{H}_\beta - (U_\beta(j) - U_\beta(i))\big)\Big) = 1$$

Zero limit : **Ground states of a directed graph**

Proof of $\bar H_\beta \to \bar H$

5. we recall the new eigenvalue problem

$$\forall\, j \in \mathcal{A}, \quad \sum_{i=1}^{r} \exp\Big(-\beta\big(H(i,j) - \bar H_\beta - (U_\beta(j) - U_\beta(i))\big)\Big) = 1$$

6. first consequence

$$\begin{cases} \forall\, i \to j \in \mathcal{A}, & U_\beta(j) + \bar H_\beta \le U_\beta(i) + H(i,j) \\ \forall\, j \in \mathcal{A},\ \exists\, i \in \mathcal{A}, & \dfrac{\log(r)}{\beta} + U_\beta(j) + \bar H_\beta \ge U_\beta(i) + H(i,j) \end{cases}$$

7. second consequence, by irreducibility of the transition matrix, and the fact that there exists $i_0 \in \mathcal{A}$ such that $U_\beta(i_0) = 0$, one can find $N \ge 1$

$$0 \le \max_j U_\beta(j) \le \max_{1 \le n \le N} \max_{i=i_0 \to \cdots \to i_n = j} \big(H(i_0, \ldots, i_n) - n\bar H_\beta\big) < +\infty$$

$\bar H_\beta$ and $U_\beta(j)$ are uniformly bounded with respect to β

Zero limit : **Ground states of a directed graph**

Proof of $\bar H_\beta \to \bar H$

8. $\bar H_\beta$ and $U_\beta(j)$ are uniformly bounded with respect to β by taking a subsequence $\beta \to +\infty$

$$\lim_{\beta\to+\infty} U_\beta(i) = U(i), \quad \lim_{\beta \emptyset +\infty} \bar H_\beta = \bar H$$

9. we recall

$$\begin{cases} \forall\, i \to j \in \mathcal{A}, & U_\beta(j) + \bar H_\beta \leq U_\beta(i) + H(i,j) \\ \forall\, j \in \mathcal{A},\ \exists\, i \in \mathcal{A}, & \dfrac{\log(r)}{\beta} + U_\beta(j) + \bar H_\beta \geq U_\beta(i) + H(i,j) \end{cases}$$

10. passing to the limit $\beta \to +\infty$

$$\begin{cases} \forall\, i \to j \in \mathcal{A}, & U(j) + \bar H \leq U(i) + H(i,j) \\ \forall\, j \in \mathcal{A},\ \exists\, i \in \mathcal{A}, & U(j) + \bar H \geq U(i) + H(i,j) \end{cases}$$

$$\forall\, j \in \mathcal{A}, \quad U(j) = \min\{U(i) + H(i,j) : i \in \mathcal{A}\}$$

Zero limit : Ground states of a directed graph

Conclusion We just have proved that $\bar{H}_\beta \to \bar{H}$ and $U_\beta \to U$

$$T[U] = U + \bar{H}$$

$$T[U](j) := \min_{i \in \mathcal{A},\ i \to j} \big(U(i) + H(i,j) \big)$$

We extend U as a function on the SFT X

$$u(x) = U(x_0), \quad x = (x_k)_{k \geq 0}$$

We extend H as a function on X

$$\phi(x) = H(x_0, x_1), \quad x = (x_k)_{k \geq 0}$$

Then

$$T[u] = u + \bar{H}$$

$$T[u](y) = \min_{x : f(x) = y} \big(u(x) + \phi(x) \big)$$

By uniqueness of the additive eigenvalue

$$\bar{H} = \bar{\phi}$$

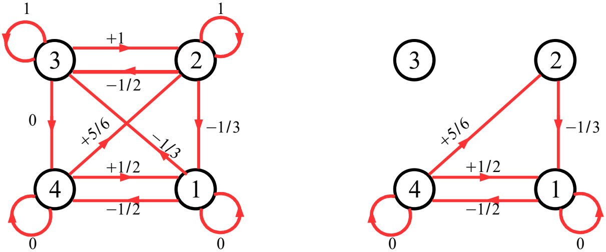
Zero limit : **Ground states of a directed graph**

Question Can we compute explicitly $\bar{H}$?

Proposition

(1) $\bar{H}$ equals the minimum of the mean energy over all simple cycles

(2) the minimizing measures are supported on the SFT made of minizing cycles



The mean energy per cycle :

order 1	$\bar{H} \in \{0, 1\}$
order 2	$\bar{H} \in \{0, \frac{1}{4}\}$
order 3	$\bar{H} \in \{0, \frac{1}{18}, \frac{1}{9}\}$

$\bar{H} = 0$

Zero limit : Ground states of a directed graph

Proof

1. We have shown the existence of a calibrated subaction U

$$\begin{cases} \forall\, i \to j \in \mathcal{A}, & U(j) + \bar{H} \le U(i) + H(i,j) \\ \forall\, i_0 \in \mathcal{A},\ \exists\, i_{-1} \in \mathcal{A}, & U(i_0) + \bar{H} = U(i_{-1}) + H(i_{-1}, i_0) \end{cases}$$

2. we construct a backward orbit that calibrates H

$$\exists\, i_{-n} \to i_{-(n-1)} \to \cdots i_{-1} \to i_0$$
$$U(i_{-k}) + \bar{H} = U(i_{-k-1}) + H(i_{-k-1} i_{-k})$$

3. because the graph is finite the backward orbit closes up

$$\exists\, p \ge 1,\ i_{-n-p} = i_{-n}$$

4. by telescoping sum U disappears

$$H(i_{-n-p}, \ldots, i_{-n-1}, i_{-n}) = p\bar{H}$$

III. Zero temperature limit in thermodynamic formalism

- Description of the BEG model
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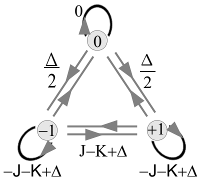
- **Explicit computations for the BEG model**

Zero limit : **Explicit computation for BEG**

The BEG model



Mean of H along simple cycles :

cycles of order 1	$0, \, (-J-K+\Delta)$
cycles of order 2	$\frac{1}{2}\Delta, \, (J-K+\Delta)$
cycles of order 3	$\frac{1}{3}(J-K+2\Delta)$

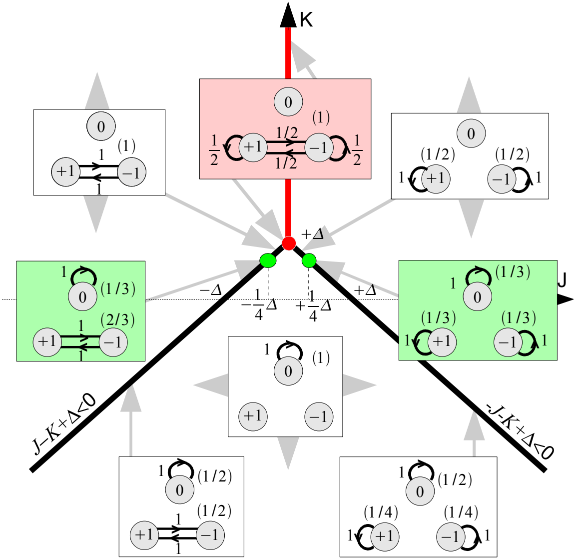
The energy matrix is

$$M_\beta = \begin{bmatrix} \exp\big(-\beta(-J-K+\Delta)\big) & \exp\big(-\beta(\frac{1}{2}\Delta)\big) & \exp\big(-\beta(J-K+\Delta)\big) \\ \exp\big(-\beta(\frac{1}{2}\Delta)\big) & 0 & \exp\big(-\beta(\frac{1}{2}\Delta)\big) \\ \exp\big(-\beta(J-K+\Delta)\big) & \exp\big(-\beta(\frac{1}{2}\Delta)\big) & \exp\big(-\beta(-J-K+\Delta)\big) \end{bmatrix}$$

We discuss the phase diagram according to the smallest term

$$\min\Big(0, \frac{\Delta}{2}, -J-K+\Delta, J-K+\Delta, \frac{1}{3}(J-K+2\Delta)\Big)$$

Zero limit : **Explicit computation for BEG**



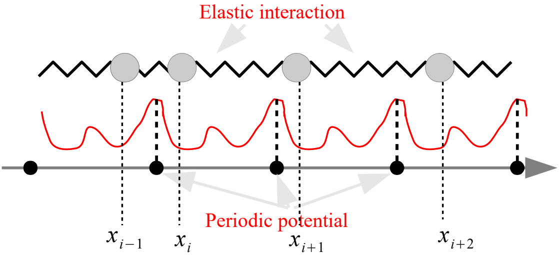
Summary

IV. Discrete Aubry-Mather and the Frenkel-Kontorova model

- **The Frenkel-Kontorova model**
- Calibrated configurations
- The algorithm

Discrete Aubry-Mather : The Frenkel-Kontorova model

The physical model The model describes the set of configuration of a chain of atoms at equilibrium in a periodic external environment



The original 1D-FK

1. $E_{\lambda,K}(x,y) = W_\lambda(x,y) + V_K(x), \quad x,y \in \mathbb{R}$
2. $W_\lambda(x,y) = \dfrac{1}{2\tau}|y-x-\lambda|^2 - \dfrac{\lambda^2}{2\tau}, \quad V_K(x) = \dfrac{K\tau}{(2\pi)^2}\Big(1-\cos(2\pi x)\Big)$
3. $E_{\lambda,K}(x,y) = E_{0,K}(x,y) - \lambda(y-x)$

Discrete Aubry-Mather : The Frenkel-Kontorova model

Question Is it possible to define a notion of configurations $\underline{x} := (x_k)_{k\in\mathbb{Z}}$, $x_k \in \mathbb{R}$, with the smallest total energy

$$E_{tot}(\underline{x}) := \sum_{k=-\infty}^{+\infty} E(x_k, x_{k+1}) \leq E_{tot}(\underline{y}), \quad \forall\ \underline{y} = (y_k)_{y\in\mathbb{Z}}$$

Definition A configuration $(x_n)_{n\in\mathbb{Z}}$ is said to be minimizing if the energy of a finite block of atoms with two fixed extremities cannot be lowered by displacing atoms inside the block :

- define $\quad E(x_m, x_{m+1}, \ldots, x_n) := \sum_{k=m}^{n-1} E(x_k, x_{k+1})$
- if $(y_m, y_{m+1}, \ldots, y_n)$ is another configuration with the two endpoints fixed, $y_m = x_m$ and $y_n = x_n$ then

$$E(x_m, x_{m+1}, \ldots, x_n) \leq E(y_m, y_{m+1}, \ldots, y_n)$$

Discrete Aubry-Mather : The Frenkel-Kontorova model

Remark The notion of minimizing configurations is NOT correct. Consider

$$E_\lambda(x,y) := E(x,y) - \lambda \cdot (y-x)$$

(λ is the distance between the atoms at rest). Then

$$(x_k)_{k\in\mathbb{Z}} \text{ is minimizing for } E_\lambda \iff (x_k)_{k\in\mathbb{Z}} \text{ is minimizing for } E_0$$

Proof

$$\sum_{k=m}^{n-1}\Big(E_0(x_k,x_{k+1}) - \lambda(x_{k+1}-x_k)\Big) = \sum_{k=m}^{n-1} E_0(x_k,x_{k+1}) - \lambda(x_n - x_m)$$

Remarks

(1) minimal geodesics have a similar definition (λ is a cohomological factor)

(2) minimizing configurations look like local minimizers of some functional energy. We need a stronger notion of global minimizers that will be called calibrated configurations

IV. Discrete Aubry-Mather and the Frenkel-Kontorova model

- The Frenkel-Kontorova model
- **Calibrated configurations**
- The algorithm

Discrete Aubry-Mather : Calibrated configurations

Definition The effective energy of a configuration is

$$\bar{E} := \lim_{n \to +\infty} \inf_{x_0, \ldots, x_n \in \mathbb{R}^d} \frac{1}{n} \sum_{k=0}^{n-1} E(x_k, x_{k+1})$$

Remark

- The limit exists by super-additivity
- By coercitivity of $E(x,y)$: $\lim_{|y-x| \to +\infty} E(x,y) = +\infty$

$$-\infty < \inf_{x,y \in \mathbb{R}} E(x,y) \le \bar{E} \le \inf_{x \in \mathbb{R}^d} E(x,x) < +\infty$$

Definition

- The Mañé potential between two positions $x, y \in \mathbb{R}$ is

$$S(x,y) := \inf_{n \ge 1} \inf_{x = x_0, \ldots, x_n = y} \sum_{k=0}^{n-1} \big(E(x_k, x_{k+1}) - \bar{E} \big)$$

- $\underline{x} = (x_k)_{k \in \mathbb{Z}}$ is said to be calibrated if

$$\forall\, m < n, \quad \sum_{k=m}^{n-1} \big(E(x_k, x_{k+1}) - \bar{E} \big) = S(x_m, x_n)$$

Discrete Aubry-Mather : Calibrated configurations

Question How can we find calibrated configurations ?

The Lax-Oleinik operator For every periodic function $u : \mathbb{R} \to \mathbb{R}$

$$T[u](y) := \inf_{x \in \mathbb{R}} \big(u(x) + E(x, y)\big)$$

Remark

- By coercivity of E, the infimum is attained
- We have chosen an interaction energy satisfying

$$E(x+1, y+1) = E(x, y)$$

- In particular : u periodic $\Rightarrow T[u]$ periodic

Theorem There exists a Lipschitz periodic function $u : \mathbb{R} \to \mathbb{R}$ solution

$$T[u] = u + \bar{E}$$

u is called effective potential. It is not unique. The additive eigenvalue $\bar{E}$ is unique

Discrete Aubry-Mather : Calibrated configurations

Construction of calibrated configurations

1. solve $T[u](y) = u(y) + \bar{E} = \min_x (u(x) + E(x, y))$
2. choose $x_0 \in [0, 1]$ and construct a backward optimal configuration

$$u(x_{-k}) + \bar{E} = u(x_{-k-1}) + E(x_{-k-1}, x_{-k})$$

3. shift the finite configuration $(x_k + L_n)_{k=-2n}^{0}$ by an integer L_n so that $x_{-n} + L_n \in [0, 1]$
4. extract a convergent subsequence $(x_k^{\infty})_{k \in \mathbb{Z}}$ by a diagonal argument
5. the limit $(x_k^{\infty})_{k \in \mathbb{Z}}$ is calibrated

Discrete Aubry-Mather : Calibrated configurations

Theorem Recall $E_\lambda(x,y) = E_0(x,y) - \lambda(y-x)$, $\underline{x} = (x_k)_{k\in\mathbb{Z}}$

(1) $\underline{x}$ is minimizing for $E_\lambda \Leftrightarrow \underline{x}$ is minimizing for E_0

(2) A calibrated configuration for E_λ is minimizing

(3) A minimizing configuration is calibrated for some E_λ

(4) Recall

$$\bar{E}(\lambda) := \lim_{n\to+\infty} \inf_{x_0,\ldots,x_n\in\mathbb{R}^d} \frac{1}{n}\sum_{k=0}^{n-1} E_\lambda(x_k, x_{k+1})$$

(5) $\lambda \mapsto \bar{E}(\lambda)$ is a C^1 function

(6) A calibrated configuration for E_λ admits a rotation number

$$\lim_{n\to\pm\infty} \frac{x_n - x_0}{n} = \omega(\lambda) := -\frac{d\bar{E}}{d\lambda}$$

(7) Emergence of the locking phenomena at rational rotation number

$$\mathrm{Leb}\Big(\mathbb{R} \setminus \bigcup_{p/q\in\mathbb{Q}} \mathrm{interior}\Big\{\lambda\in\mathbb{R} : \omega(\lambda) = \frac{p}{q}\Big\}\Big) = 0$$

IV. Discrete Aubry-Mather and the Frenkel-Kontorova model

- The Frenkel-Kontorova model
- Calibrated configurations
- **The algorithm**

Discrete Aubry-Mather : The algorithm

The 1D-FK model

$$E_{\lambda,K}(x,y) := \frac{1}{2\tau}|y-x|^2 - \lambda(y-x) + \frac{K\tau}{(2\pi)^2}\Big(1-\cos(2\pi x)\Big)$$

Ishikawa's algorithm

1. discretize the initial cell $[0,1]$, $z_i = \frac{i}{N}$, $i = 1,\ldots,N$
2. choose a number of cells around the initial cell $R \geq 1$
3. start with the zero potential $u_0 = 0$. Assume u_n is known
4. construct the optimal backward map

$$z_j \mapsto (z_{\tau(j)}, p_j) = \underset{z_i,\ p\in[\![-R,R]\!]}{\arg\min}\Big(u_n(z_i) + E_{\lambda,K}(z_i+p, z_j)\Big)$$

5. compute Lax-Oleinik

$$T[u_n](z_j) = u_n(z_{\tau(j)}) + E_{\lambda,K}(z_{\tau(j)} + p_{\tau(j)}, z_j)$$

6. use Ishikawa's algorithm

$$u_{n+1} = \frac{u_n + T[u_n]}{2} - \min\Big(\frac{u_n+T[u_n]}{2}\Big)$$

Discrete Aubry-Mather : The algorithm

Ishikawa's algorithm

7. stop the algorithm until $\max_i \left| u_{n+1}(z_i) - u_n(z_i) \right| \leq \epsilon$
8. compute the backward minimizing cycle

$$i_0 \to i_1 = \tau(i_0), p_1 \to i_2 = \tau(i_1), p_2, \to \cdots$$

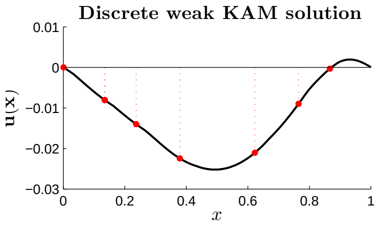
9. choose the smallest $q \geq 1$ such that $i_q = i_0$,
10. define $p = p_1 + \cdots + p_q$
11. the rotation number equals $\omega = \frac{p}{q} = -\frac{1}{\tau} \frac{\partial \bar{E}}{\partial \lambda}$
12. the Mather set is the periodic orbit

$$z_{i_0}, z_{i_1}, \ldots, z_{i_q}$$

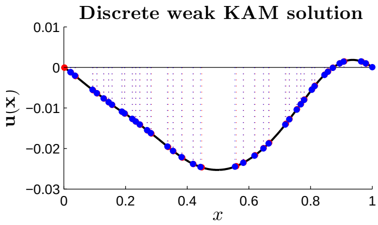
Choice of the constants

- $\tau = 1$, $N = 1000$, $R = 2$, $\epsilon = 10^-9$

Discrete Aubry-Mather : The algorithm

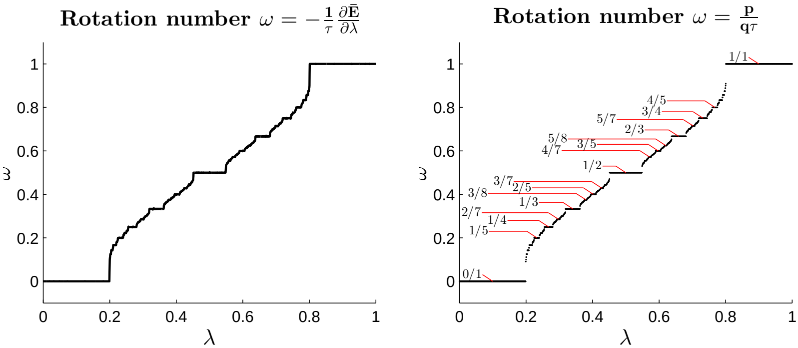


$\lambda = 0.425,\ K = 1$
$N_{Ishi} = 188$
$\bar{E}(\lambda, K) = -0.067$
The Mather = one periodic orbit (red dots) of period $q = 7$ and rotation number $\omega = 3/7$.



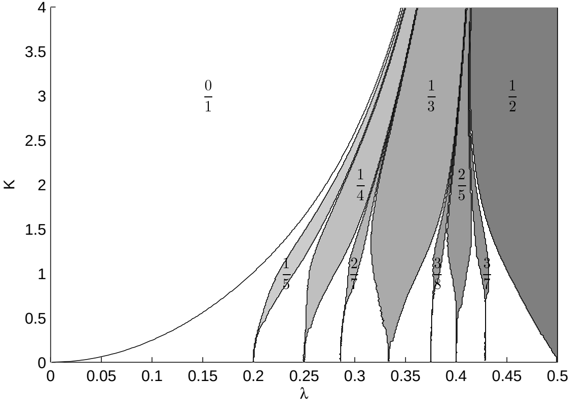
$\lambda = 0.43394,\ K = 1$
$N_{Ishi} = 1181$
$\bar{E}(\lambda, K) = -0.070614259$
Mather set = two periodic orbits of identical period $q = 39$ and rotation number $\omega = 17/39$
A grid of 2000 points shows a unique period orbit with the same period 17/39

Discrete Aubry-Mather : The algorithm



Graph of the rotation number $\omega = -\frac{1}{\tau}\frac{\partial \bar{E}}{\partial \lambda}(\lambda)$ (lefthand side), and $\omega = \frac{p(\lambda)}{\tau q(\lambda)}$ (right hand side). The coupling is $K = 1$, the grid on λ is $0 : 0.0005 : 1$. The maximum number of iteration is 198, the maximum jump is 1.286, the maximum number of cycles is 2.

Discrete Aubry-Mather : The algorithm



Phase diagram of the Frenkel-Kontorova model : $\tau = 1$, $N = 400$, $\lambda = 0 : 0.001 : 0.5$, $K = 0 : 0.01 : 4$. Each domain is parametrized by a rotation number $\omega = \frac{p}{\tau q}$

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