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Entropy and the Hausdorff Dimension for Infinite-Dimensional Dynamical Systems

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We define a sequence of uniform Lyapunov exponents in the setting of Banach spaces and prove that the Hausdorff dimension of global attractors is bounded from above by the Lyapunov dimension of the tangent map. This result generalizes the papers by Douady and Oesterlé (1980) and Ledrappier (1981) in finite dimension and Constantin *et al.* (1985) for Hilbert spaces.

KEY WORDS: Dynamical systems; Hausdorff dimension; Lyapunov exponents; entropy.

1. INTRODUCTION

Some partial differential equations with a dissipative term possess a compact global attractor K invariant with respect to the semigroup of solutions $\{\phi^t\}_{t\geqslant 0}$, $\phi^t(K)\subseteq K$ (cf. Constantin *et al.* (1985)). The fact that the tangent semigroup $\{T_x^t\}_{t\geqslant 0}$ is composed of compact operators (or at least asymptotically compact) enables us to work in a finite-dimensional setting. For these equations, the surrounding space is a Hilbert space and the definition of local Lyapunov exponents is obtained by computing the asymptotic growth of the norm of the exterior product of the tangent semigroup $\|\Lambda^p T_x^t\|$. Although the notion of p-dimensional volume does not exist in Banach spaces, one can still construct such a family of exponents (Mañé, 1983; Thieullen, 1987).

The beginning of this paper gives a geometric definition of these local exponents $\{\lambda_i(x)\}_{i\geqslant 1}$ as critical values of the α-entropy $h(T, \alpha, x)$ of the tangent semigroup. This α-entropy generalizes the usual notion of entropy and is computed by counting the number of balls $R(T_x^n, e^{-n\alpha})$ with exponentally decreasing radius which cover the image of the unit ball under the tangent semigroup:

$$h(T, \alpha, x) = \lim_{n \to +\infty} \frac{1}{n} \log R(T_x^n, e^{-n\alpha}) = \sum_{i \geq 1} d_i(x)[\lambda_i(x) + \alpha]^+$$

where the limit exists for any regular point $x \in \Lambda$, that is, on a set of points possessing good statistical properties: a set invariant under the semigroup, $\phi^t(\Lambda) = \Lambda$, with full measure for any invariant finite measure m $(m \circ \phi^{-1} = m)$. Particularly, points satisfy

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\phi^i(x)} = m_x \quad \text{(weakly)}$$

where $\{\delta_x\}_{x \in \Lambda}$ is the Dirac measure at x and m_x is an ergodic measure (the only invariant sets have measure 0 or 1).

We define after uniform Lyapunov exponents using the same formula:

$$h^u(T, \alpha) = \lim_{n \to +\infty} \frac{1}{n} \sup_{x \in K} \log R(T_x^n, e^{-n\alpha}) = \sum_{i \geq 1} d_i(T)[\lambda_i^u(T) + \alpha]^+$$

which enables us to bound from above the fractal dimension of K when $\phi(K) = K$:

$$\dim_F(K) \leqslant n + \frac{\lambda_1^u(T) + \cdots + \lambda_n^u(T)}{|\lambda_{n+1}^u(T)|}$$

Finally, we give a more precise upper bound for the Hausdorff dimension of K when $\phi(K) = K$:

$$\dim_H(K) \leqslant \sup\{\dim_L(T, m) \colon m \text{ ergodic invariant}\}$$

where $\dim_L$ is the Lyapunov dimension of m, the smallest "dimension d" from which the tangent semigroup contracts d-dimensional volumes.

These latter two results generalize the same inequality obtained by Ledrappier (1981) and by Constantin *et al.* (1985).

2. RESULTS

2.1. General Setting

Let $(E, \|\bullet\|)$ be a Banach space, $\mathscr{A}$ a nonempty compact set in E, $(\phi \colon \mathscr{A} \to \mathscr{A})$ a continuous map defined on $\mathscr{A}$ and preserving $\mathscr{A}(\phi(\mathscr{A}) \subseteq \mathscr{A})$, and $(T \colon \mathscr{A} \to L(E); x \mapsto T_x)$ a quasidifferential of ϕ: that is, (i) T_x is a continuous linear operator for each $x \in \mathscr{A}$ and continuous with respect to x, and (ii) there exists a decreasing function $(C \colon R^+ \to R^+)$ such that for all $\varepsilon > 0$, x in $\mathscr{A}$, p in the ball of radius ε centered at x $B(x, \varepsilon)$, $\|\phi(p) - \phi(x) - T_x \cdot (p - x)\| \leqslant C(\varepsilon) \|p - x\|$ and $\lim_{\varepsilon \to 0} C(\varepsilon) = 0$.

In those circumstances, we will say that the dynamical bundle $\mathscr{F} = (E, \mathscr{A}, \phi, T)$ has class C^1. If, moreover, ϕ is a homeomorphism and T_x is injective for each x in $\mathscr{A}$, $\mathscr{F}$ will be called an invertible dynamical bundle. If T is a quasidifferential of ϕ, then T^n is a quasidifferential of ϕ^n, where $T^n_x \triangleq T_{\phi^{n-1}(x)} \circ T_{\phi^{n-2}(x)} \circ \cdots \circ T_{\phi(x)} \circ T_x$ and $C_{n+1}(\varepsilon) \triangleq \tau^n C(\varepsilon) + (\tau^n + C(\varepsilon)) C_n(\varepsilon(\tau + C(\varepsilon)))$, $\tau \triangleq \sup_{x \in \mathscr{A}} \|T_x\|$.

Following Kuratowski [cf. Sadovskii (1972) for a better survey of measure of noncompactness], we define the index of compactness of any bounded subset A in E as being the smallest nonnegative real number α such that for any $r' > \alpha$, A can be covered by a finite number of balls of radius r' (not necessarily centered on $\mathscr{A}$). We define also the index of compactness of any map $S: E \to E$ as being the number

$$\|S\|_\alpha \triangleq \inf\{k > 0 : \alpha(S(A)) \leqslant k\alpha(A) \text{ for any bounded set } A \text{ in } E\}$$

If S is a continuous linear operator, then $\|S\|_\alpha = \alpha(S(B_E))$ where B_E is the open unit ball of E and $\|\bullet\|_\alpha$ is a multiplicative norm:

$$\|S + T\|_\alpha \leqslant \|S\|_\alpha + \|T\|_\alpha, \qquad \|S \circ T\|_\alpha \leqslant \|S\|_\alpha \, \|T\|_\alpha$$

Then the existence of compact global attractors for some partial differential equations (cf. Babin and Vishik, 1983) can be proved using the following proposition.

2.1.1. Proposition. *Given a continuous semiflow $(S^t)_{t \geqslant 0}$ in a complete metric space (X, d) ($S^t: X \to X$) is a continuous map for each $t \geqslant 0$, such that (1) $\lim_{t \to +\infty} 1/t \operatorname{Log}\|S^t\|_\alpha < 0$ and (2) there exists a set B in X such that $\bigcup_{t \geqslant \tau} S^t(B)$ is bounded for some $\tau > 0$. Then $\mathscr{A} \triangleq \bigcap_{t \geqslant \tau} \overline{\bigcup_{s \geqslant \tau} S^s(B)}$ is a nonempty compact set which satisfies $S^t(\mathscr{A}) = \mathscr{A}$ for all $t \geqslant \tau$ and $\lim_{t \to +\infty} \sup\{d(S^t(x), \mathscr{A}) : x \in \bigcup_{s \geqslant \tau} S^s(B)\} = 0$. If, moreover, $\bigcup_{t \geqslant \tau} S^t(B)$ is connex, then $\mathscr{A}$ is connex too.*

2.2. Oseledec's Theorem and Regular Points

The notion which generalizes the set of fixed hyperbolic points is the one of regular points in $\mathscr{A}$.

2.2.1. Definition of Regular Points. A point x in $\mathscr{A}$ is said to be regular if there exists a nonincreasing sequence $(\lambda_i)_{i \geqslant 1}$ of real numbers (possibly equal to $-\infty$) and a nonincreasing sequence of closed subvector spaces $(F_i)_{i \geqslant 1}$ satisfying the following properties:

(i) $\lambda_\infty(x) \triangleq \lim_{n\to+\infty} 1/n \ln \|T_x^n\|_\alpha = \inf_{i\geqslant 1} \lambda_i$,

(ii) $\lambda_i = \lim_{n\to+\infty} 1/n \ln \|T_x^n \mid F_i\| = \lim_{n\to+\infty} 1/n \ln \|T_x^n \bullet v\|$ for any $v \in F_i \backslash F_{i+1}$,

(iii) $F_1 = E$; if $\lambda_i > \lambda_\infty(x)$, then $\lambda_i > \lambda_{i+1}$, $1 \leqslant \operatorname{codim}(F_i/F_{i+1}) \triangleq d_i < +\infty$, if $\lambda_i = \lambda_\infty(x)$, then $F_{i+1} = F_i$ and $d_i \triangleq 0$.

We remark that $(\lambda_i)_{i\geqslant 1}$, $(F_i)_{i\geqslant 1}$, and $(d_i)_{i\geqslant 1}$ are actually functions of regular points x: $\{\lambda_i \colon \lambda_i > \lambda_\infty(x)\} = \{l > \lambda_\infty(x) \colon \exists v \in E\ l = \limsup_{n\to+\infty} 1/n \ln \|T_x^n \bullet v\|\}$, $F_i(x) = \{v \in E \colon \limsup_{n\to+\infty} 1/n \ln \|T_x^n \bullet v\| \leqslant \lambda_i(x)\}$.

We denote by $\Lambda(\mathscr{F})$ the set of regular points, $\mathscr{M}_1(\mathscr{A}, \phi)$ the set of probability ϕ-invariant measures m defined on $\mathscr{A}$ $(m(\phi^{-1}(A)) = m(A)$ for any Borel set A in $\mathscr{A})$ and $\mathscr{M}_1^{\mathrm{e}}(\mathscr{A}, \phi)$ the set of ergodic measures m in $\mathscr{M}_1(\mathscr{A}, \phi)$, $(\phi^{-1}(A) = A \Leftrightarrow m(A) = 0$ or $1)$. The main theorem about the set of regular points is the following (Osledec, 1968; Ruelle, 1979, 1982; Mañé, 1983; Thieullen, 1987).

2.2.2. Oseledec's Theorem. *For any measure m in $\mathscr{M}_1(\mathscr{A}, \phi)$, there exists a Borel set B in $\Lambda(\mathscr{F})$ such that $\phi(B) \subset B$, $m(B) = 1$, $(\lambda_i)_{i\geqslant 1}$ $(F_i)_{i\geqslant 1}$ $(d_i)_{i\geqslant 1}$ are measurable functions on B.*

If $\mathscr{F}$ is an invertible dynamical bundle, a notion of strong regular points can be defined and a stronger Oseledec's theorem can be proved (cf. Appendix B).

Since $\mathscr{A}$ is compact, any weak limitpoint of $\{1/n \sum_{i=0}^{n-1} \delta_{\phi i(x)}\}_{n\geqslant 1}$ (x fixed in $\mathscr{A}$) is a measure in $\mathscr{M}_1(\mathscr{A}, \phi)$. Therefore the set of regular points is not empty, but could be reduced to a single fixed point. When $(E, \|\bullet\|)$ is a Hilbert space, $(\lambda_i)_{i\geqslant 1}$ and $(d_i)_{i\geqslant 1}$ can be defined in a different manner (cf. Appendix A for the notations).

2.2.3. Oseledec's Theorem in Hilbert Spaces. *For any measure m in $\mathscr{M}_1(\mathscr{A}, \phi)$, and for almost every point x in $\mathscr{A}$:*

$$\lim_{n\to+\infty} \frac{1}{n} \ln \chi_i(T_x^n) = \tilde{\lambda}_i(x)$$

(where $\{\tilde{\lambda}_i(x)\}_{i\geqslant 1}$ is the sequence of Lyapunov exponents counted as many times as their multiplicity $d_i(x)$ if $\lambda_i(x) > \lambda_\infty(x)$ and once if $\lambda_i(x) = \lambda_\infty(x)$).

2.3. α-Entropy of Operators

The main new idea in this paper is the notion of α-entropy of operators. In opposition to the α-entropy of a map (cf. Thieullen, 1987), an exact formula can be given for operators. On the one hand, this new definition will enable us to prove Oseledec's theorem in Hilbert spaces [cf. Ruelle (1982) for the original proof]; on the other hand, we will define uniform Lyapunov exponents in the general case of Banach spaces.

2.3.1. Definition of Covering Number. If A is a bounded set in a metric space (X, d) and ε a positive real number, let $r(A, \varepsilon)$ be the smallest integer $n \geqslant 1$ such that A can be covered by n balls of radius strictly less than ε (not necessarily centered on A); $r(A, \varepsilon) = +\infty$ if such an integer does not exist.

If $(E, \|\bullet\|)$ is a normed space and A, B are two bounded sets, then $r(A+B, \varepsilon+\eta) \leqslant r(A, \varepsilon)\ r(B, \eta)$. If S and T are two bounded operators, then $r(S \circ T(B_E), \varepsilon\eta) \leqslant r(S(B_E), \varepsilon)\ r(T(B_E), \eta)$. To simplify the notations, we will write $r(T, \varepsilon)$ instead of $r(T(B_E), \varepsilon)$. A related notion of entropy numbers has been studied by Carl (1981), Pajor and Tomczak-Jaegermann (1985), and Tomczak-Jaegermann (1987).

2.3.2. Definition of α-Entropy. Given $\alpha \in R$, we define the relative α-entropy of T at x and the uniform α-entropy of T over $\mathscr{A}$ by

$$h(T, \alpha, x) \triangleq \limsup_{n \to +\infty} \frac{1}{n} \ln r(T_x^n, e^{-n\alpha})$$

$$h^u(T, \alpha) \triangleq \limsup_{n \to +\infty} \sup_{x \in \mathscr{A}} \frac{1}{n} \ln r(T_x^n, e^{-n\alpha})$$

We notice that $\{f_n(x) = r(T_x^n, e^{-n\alpha})\}_{n \geqslant 0}$ is a subadditive sequence $(f_{m+n} < f_m + f_n \circ \phi^m)$, so that the second limit is an infimum and the first limit exist m-almost everywhere for any m in $\mathscr{M}_1(\mathscr{A}, \phi)$. The following lemma is not simple and can be seen as the generalized entropic Ruelle's formula for operators:

2.3.3. Lemma. *If $\mathscr{F} = (E, \mathscr{A}, \phi, T)$ is a dynamical bundle in a Banach space, m a measure in $\mathscr{M}_1(\mathscr{A}, \phi)$, then (x) a.e. in $\mathscr{A}$ and for all real number $\alpha < -\lambda_\infty(x)$, $\lim_{n \to +\infty} 1/n \ln r(T_x^n, e^{-n\alpha}) = h(T, \alpha, x) = \sum_{i \geqslant 1} d_i(x)(\lambda_i(x) + \alpha)^+$ [where a^+ means $\max(a, 0)$].*

The next theorem is the main one in this section. It allows us to define a sequence of uniform Lyapunov exponents over $\mathscr{A}$ and, for example, will give us a sharper upper bound of the fractal dimension of $\mathscr{A}$.

2.3.4. Theorem. *Let $\mathscr{F} = (E, \mathscr{A}, \phi, T)$ be a C^1-dynamical bundle. Then there exists a nonincreasing sequence $\{\lambda_n^u(T)\}_{n \geqslant 1}$, of real numbers in $[-\infty, +\infty)$, and a sequence of integers $\{d_n^u(T)\}_{n \geqslant 1}$ such that* *(i)* $\lambda_1^u(T) = \lim_{n \to +\infty} 1/n \ln \sup_{x \in \mathscr{A}} \|T_x^n\|$;

(ii) $\lambda_\infty^u(T) \triangleq \lim_{n \to +\infty} 1/n \ln \sup_{x \in \mathscr{A}} \|T_x^n\|_\alpha = \inf_{i \geqslant 1} \lambda_i^u(T)$;

(iii) *if* $\lambda_i^u(T) > \lambda_\infty^u(T)$, *then* $d_i^u(T) \geqslant 1$, *otherwise* $d_i^u(T) = 0$;

(iv) *for any* $\alpha < -\lambda_\infty^u(T)$, $h^u(T, \alpha) = \sum_{i \geqslant 1} d_i^u(T)(\lambda_i^u(T) + \alpha)^+$, *and there exists a measure* m_α *in* $\mathscr{M}_1^e(\mathscr{A}, \phi)$ *such that* $h^u(T, \alpha) = h(T, \alpha, x)\, m_\alpha$ a.e.

This sequence depends on the tangent map as well as on the attractor itself: for example, they are increasing with respect to the ϕ-invariant sets. To prove this theorem, we need actually a great amount of ergodic theory; in particular, we need the following variational principle.

2.3.5. *Lemma.* *Let* $\{f_n\}_{n \geqslant 1}$ *be a sequence of subadditive upper semi-continuous functions defined on the compact set* $\mathscr{A}$. *Then there exists a measure m in* $\mathscr{M}_1^e(\mathscr{A}, \phi)$ *such that*

$$\lim_{n \to +\infty} \frac{1}{n} \sup_{x \in \mathscr{A}} f_n(x) = \inf_{n \geqslant 1} \frac{1}{n} \int f_n\, dm = \lim_{n \to +\infty} \frac{1}{n} f_n(x)\ m.a.e.$$

2.4. Different Notions of Uniform Lyapunov Exponents

When $(E, \|\bullet\|)$ is a Hilbert space, Constantin *et al.* (1985) defined uniform Lyapunov exponent by induction:

$$\tilde{\mu}_1^u(T) + \cdots + \tilde{\mu}_d^u(T) \triangleq \lim_{n \geqslant +\infty} \frac{1}{n} \ln \sup_{x \in \mathscr{A}} \|\Lambda^d T_x^n\|$$

$$\tilde{\mu}_d^u(T) = -\infty \qquad \text{if} \quad \lim_{n \to +\infty} \frac{1}{n} \ln \sup_{x \in \mathscr{A}} \|\Lambda^d T_x^n\| = -\infty$$

They defined the Lyapunov curve by

$$\pi^u(T, d) \triangleq \lim_{n \to +\infty} \frac{1}{n} \ln \sup_{x \in \mathscr{A}} \|\Lambda^p T_x^n\|^{1-s} \|\Lambda^{p+1} T_x^n\|^s$$

(where $d = p + s$ and $0 \leqslant s < 1$).

We define now the sequence $\{\tilde{\lambda}_i^u(T)\}_{i \geqslant 1}$, where $\lambda_i^u(T)$ is counted $d_i^u(T)$ times when $d_i^u(T) \geqslant 1$ and once otherwise, and we define another Lyapunov curve:

$$\gamma^u(T, d) \triangleq \tilde{\lambda}_1^u(T) + \cdots + \tilde{\lambda}_p^u(T) + s\tilde{\lambda}_{p+1}^u(T)$$

(where $d = p + s$ and $0 \leqslant s < 1$).

The following proposition explains the relationship between these two curves:

2.4.1. Proposition. *In the general case $[d \geqslant 0 \mapsto \gamma^u(T, d)]$ is the opposite of the Legendre transform of the curve $[\alpha \mapsto h^u(T, \alpha)]$:*

$$\gamma^u(T, d) = \inf\{h^u(T, \alpha) - \alpha d : \alpha < -\lambda_\infty(T)\}$$

If we assume, moreover, that E is a Hilbert space, then

- *for any positive d, $\pi^u(T, d) \leqslant \gamma^u(T, d)$;*
- *for any $d = d_1^u(T) + \cdots + d_r^u(T)$, $\pi^u(T, d) = \gamma^u(T, d)$.*

2.5. Uniform Hausdorff and Fractal Dimension: Entropy

The definition of these dimensions is given by Constantin *et al.* (1985). We define first what is usually called the Lyapunov dimension of the tangent map:

2.5.1. Definition. Let $\mathscr{F} = (E, \mathscr{A}, \phi, T)$ be a C^1-dynamical bundle in a Banach space E such that $\lambda_\infty^u(T) < 0$. Then there exists an integer $p \geqslant 0$ such that $\bar{\lambda}_1^u(T) + \cdots + \bar{\lambda}_p^u(T) \geqslant 0$ and $\bar{\lambda}_1^u(T) + \cdots + \bar{\lambda}_{p+1}^u(T) < 0$. We call the uniform Lyapunov dimension the real number:

$$\dim_L^u(T) \triangleq p + \frac{\bar{\lambda}_1^u(T) + \cdots + \bar{\lambda}_p^u(T)}{|\bar{\lambda}_{p+1}^u(T)|}$$

One may use two other equivalent definitions:

$$\dim_L^u(T) = \inf\left\{\frac{1}{\alpha} h^u(T, \alpha) : 0 < \alpha < -\lambda_\infty(T)\right\}$$

$$\dim_L^u(T) = \inf\{d \geqslant 0 : \gamma^u(T, d) < 0\}$$

It is not now difficult to prove the following more accurate upper bound of the fractal dimension of $\mathscr{A}$.

2.5.2. Theorem. *Let $\mathscr{F} = (E, \mathscr{A}, \phi, T)$ be a C^1-dynamical bundle in a Banach space E such that $\lambda_\infty^u(T) < 0$ and $\phi(\mathscr{A}) = \mathscr{A}$; then*

$$\dim_F(\mathscr{A}) \leqslant \dim_L^u(T)$$

Actually it is possible to prove a sharper inequality. Let $h(\phi, \alpha)$ be the metric α-entropy of the map $\phi(h(\phi, \alpha) \triangleq \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} 1/n$

$\ln r(\mathscr{A}, \varepsilon, d_n^{\phi,\alpha})$, *where* $d_n^{\phi,\alpha}(x,y) \triangleq \max_{1 \leqslant i \leqslant n} \{d(\phi^i(x), \phi^i(y))\, e^{i\alpha}\})$. *Then for any* $0 \leqslant \alpha < -\lambda_\infty^u(T)$:

$$\alpha \dim_F(\mathscr{A}) \leqslant h(\phi, \alpha) \leqslant h^u(T, \alpha)$$

$$\dim_F(\mathscr{A}) \leqslant \inf\{d \geqslant 0 : \gamma(\phi, d) < 0\} \qquad \text{where}$$

$$\gamma(\phi, d) \triangleq \inf\{h(\phi, \alpha) - \alpha d : 0 < \alpha < -\lambda_\infty^u(t)\}$$

The next theorem gives an affirmative answer to an old conjecture. For any ergodic measure m in $\mathscr{M}_1^e(\mathscr{A}, \phi)$, $\{\lambda_i(x)\}$ and $\{d_i(x)\}$ are constant almost everywhere, so that we can write $h(T, \alpha, m) = \sum_{i \geqslant 1} d_i(m)(\lambda_i(m) + \alpha)^+$ and define in the same manner:

$$\gamma(T, d, m) \triangleq \inf\{h(T, \alpha, m) - \alpha d : \alpha < -\lambda_\infty(m)\}$$
$$= \tilde{\lambda}_1(m) + \cdots + \tilde{\lambda}_p(m) + s\tilde{\lambda}_{p+1}(m)$$

$$\dim_L(T, m) \triangleq \inf\{d \geqslant 0 : \gamma(T, m, d) < 0\} = p + \frac{\tilde{\lambda}_1(m) + \cdots + \tilde{\lambda}_p(m)}{|\tilde{\lambda}_{p+1}(m)|}$$

2.5.3. Theorem. *Let* $\mathscr{F} = (E, \mathscr{A}, \phi, T)$ *a* C^1*-dynamical bundle on a Banach space* E *such that* $\lambda_\infty^u(T) < 0$ *and* $\phi(\mathscr{A}) = \mathscr{A}$. *Then*

$$\dim_H(\mathscr{A}) \leqslant \sup\{\dim_L(T, m) : m \in \mathscr{M}_1^e(\mathscr{A}, \phi)\}$$

This theorem improves an estimate of Constantin *et al.* (1985); they have defined a different notion of Lyapunov dimension in the Hilbert case, namely,

$$\dim_L^*(T) \triangleq \inf\{d \geqslant 0 : \pi^u(T, d) < 0\}$$

Under the same assumptions as in Theorem 2.5.3, we have

$$\sup\{\dim_L(T, m) : m \in \mathscr{M}_1^e(\mathscr{A}, \phi)\} \leqslant \dim_L^*(T)$$

$$\dim_L^*(T) \leqslant p + \frac{\tilde{\mu}_1^u(T) + \cdots + \tilde{\mu}_p^u(T)}{|\tilde{\mu}_{p+1}^u(T)|} \leqslant \dim_L^u(T)$$

$$\dim_L^u(T) \leqslant \max_{1 \leqslant l \leqslant p} \left\{ l + \frac{\tilde{\mu}_1(T) + \cdots + \tilde{\mu}_l(T)}{|\alpha_{p+1}^u(T)|} \right\}$$

Finally, Theorem 2.5.3 can be improved in the case of Hilbert spaces.

2.5.4. Proposition. *Let* $\mathscr{F} = (E, \mathscr{A}, \phi, T)$ *be a* C^1*-dynamical bundle on a Hilbert space* E *such that* $\lambda_\infty^u(T) < 0$, *then there exists an ergodic measure* m_0 *in* $\mathscr{M}_1^e(\mathscr{A}, \phi)$ *such that*

$$\sup\{\dim_L(T, m) : m \in \mathscr{M}_1^e(\mathscr{A}, \phi)\} = \dim_L(T, m_0)$$

3. PROOFS

3.1. General Setting

The proof of Proposition 2.1.1 is well known for compact operators (Babin and Vishik, 1983) and the generalization to uniformly asymptotically compact operators ($\lim_{t\to+\infty} 1/n \ln \|S^t\|_\alpha < 0$) does not introduce new difficulties.

3.1.1. Proof of Proposition 2.1.1. Let us call $B^* \triangleq \overline{\bigcup_{t\geq\tau} S^t(B)}$, then $S^t(B^*) \subseteq B^*$ for every $t \geq \tau$. Since $\alpha(\overline{\bigcup_{u\geq t} S^u(B)}) \leq \alpha(S^t(B^*)) \leq \|S^t\|_\alpha\, \alpha(B^*)$, $\alpha(B^*) < +\infty$ and $\|S^t\|_\alpha$ goes to $-\infty$, $\alpha(\overline{\bigcup_{u\geq t} S^u(B)})$ goes to 0 when t goes to $+\infty$. If $\{x_i\}_{i\geq 0}$ is a sequence of points in B^* and $\{t_i\}_{i\geq 0}$ is an increasing sequence of times to $+\infty$, then $\{S^{t_i}(x_i)\}_{i\geq 0}$ possesses a convergent subsequence (each set $\{S^{t_i}(x_i)\colon i\geq n\}$ can be covered by a finite number of balls of radius $\alpha_n = \alpha(\overline{\bigcup_{u\geq t_n} S^u(B)})$ and X is a complete metric space). In particular $\mathscr{A}$ is a nonempty closed set such that $\alpha(\mathscr{A}) \leq \alpha_n$ for all n: thus $\mathscr{A}$ is a nonempty compact set. Suppose $\limsup_{t\to+\infty} \sup\{d(S^t(x), \mathscr{A})\colon x \in B^*\} > \varepsilon > 0$, there exist $\{x_i\}_{i\geq 0}$ in B^* and $\{t_i\}_{i\geq 0}$ increasing such that $d(S^{t_i}(x_i), \mathscr{A}) \geq \varepsilon$, which is a contradiction since $\{S^{t_i}(x_i)\}_{i\geq 0}$ has a convergent subsequence in $\mathscr{A}$. If B^* is connexe and suppose that $\mathscr{A}$ is not, $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$ were $\mathscr{A}_i$ are nonempty disjoint compact sets, $\varepsilon = \inf\{d(x_1, x_2)\colon x_i \in \mathscr{A}_i\} > 0$, then for t large enough $S^t(B^*)$ is contained in $\mathscr{N}_\varepsilon(\mathscr{A}_1) \cup \mathscr{N}_\varepsilon(\mathscr{A}_2)$, where $\mathscr{N}_\varepsilon(\mathscr{A}_i) = \{x \in X\colon d(x, \mathscr{A}_i) < \varepsilon/2\}$, which is a contradiction since $S^t(B^*)$ is connexe and intersects each open set $\mathscr{N}_\varepsilon(\mathscr{A}_i)$.

3.2. Oseledec's Theorem and Regular Points

In Section 3.3, we will prove Lemma 2.3.3, which is the main step in the proof of Oseledec's theorem in Hilbert spaces.

3.2.1. Proof of Theorem 2.2.3. Let m be a probability measure in $\mathscr{M}_1^e(\mathscr{A}, \phi)$. Since $\{\ln \|\Lambda^p T_x^n\|\}_{n\geq 0}$ is a subadditive sequence of bounded functions, and $\|\Lambda^p T_x^n\| = \prod_{i=1}^p \chi_i(T_x^n)$ for each $p \geq 0$, m-almost everywhere $\tilde{\mu}_i(x) = \lim_{n\to+\infty} 1/n \ln \chi_i(T_x^n)$ exists (Kingman's theorem (1968)). Using Lemma 2.3.3, it is enough to prove $\lim_{n\to+\infty} 1/n \ln r(T_x^n, e^{-n\alpha}) = \sum_{i\geq 1} (\tilde{\mu}_i(x) + \alpha)^+$ m.a.e. on $\{\alpha < -\mu_\infty(x)\}$ for each real number α and $\mu_\infty(x) \triangleq \inf\{\tilde{\mu}_i(x)\colon i \geq 1\} = \lambda_\infty(x)$. To prove the second inequality, we need Kingman's and Lebesque's theorem:

$$\begin{aligned}
\int_I \mu_\infty(x)\,dm(x) &= \inf_{p \geqslant 1} \frac{1}{p} \sum_{i=1}^{p} \int_I \tilde{\mu}_i(x)\,dm(x) \\
&= \inf_{p \geqslant 1} \frac{1}{p} \int_I \lim_{n \to +\infty} \frac{1}{n} \ln \|\Lambda^p T_x^n\|\,dm(x) \\
&= \inf_{p \geqslant 1} \inf_{n \geqslant 1} \frac{1}{np} \int_I \ln \|\Lambda^p T_x^n\|\,dm(x) \\
&= \inf_{n \geqslant 1} \lim_{p \to +\infty} \frac{1}{n} \int_I \ln \chi_p(T_x^n)\,dm(x) \\
&= \lim_{n \to +\infty} \frac{1}{n} \int_I \ln \|T_x^n\|_\alpha\,dm(x) \\
&= \int_I \lambda_\infty(x)\,dm(x)
\end{aligned}$$

The last equality is true for any invariant set I, $\mu_\infty(x) = \lambda_\infty(x)$ a.e. To prove the first equality, we use Lemma A.5.3. If $\alpha < -\mu_\infty(x)$, there exists $r \geqslant 1$ such that $\tilde{\mu}_r(x) \leqslant -\alpha < \tilde{\mu}_{r+1}(x)$. Assume, moreover, that α is not one of these $\tilde{\mu}_i(x)$; then for n large enough,

$$\chi_r(T_x^n) \leqslant e^{-n\alpha}(r+1)^{-1} \leqslant \chi_{r+1}(T_x^n)$$

$$C_r^{-1} \|\Lambda^r T_x^n\|\, e^{nr\alpha} \leqslant r(T_x^n, e^{-n\alpha}) \leqslant C_r \|\Lambda^r T_x^n\|\, e^{nr\alpha}$$

3.3. α-Entropy of Operators

The main theorem of this section requires two lemmas: Lemma 2.3.3 and Lemma 2.3.5. To begin with, we prove 3.3.1, which may be omitted for a first reading. It allows us to apply Kingman's theorem to bounded functions $\{r(T_x^n, M\exp(-\alpha_n(x))\}_{n \geqslant 0}$. We then prove Lemma 3.3.3 in the invertible case and, finally, in the general case.

3.3.1. Technical Lemma. *Let be m in $\mathscr{M}_1(\mathscr{A}, \phi)$, $(\alpha\colon x \mapsto \alpha(x))$ a ϕ-invariant function such that $\alpha(x) < -\lambda_\infty(x)$ a.e. and $\varepsilon > 0$. Then there exist a bounded function $(a\colon x \mapsto a(x))$, a constant $M \geqslant 1$ and a ϕ-invariant set A such that*

(i) $m(A) > 1 - \varepsilon$;

(ii) *$r(T_x^n, M\exp - \sum_{i=0}^{n-1} a \circ \phi^i(x))$ is uniformly bounded on A for all $n \geqslant 0$;*

(iii) $\lim_{n \to +\infty} 1/n \sum_{i=0}^{n-1} a \circ \phi^i(x) = \alpha(x)$ *a.e.*

Proof of Lemma 3.3.1. Let β be a ϕ-invariant function such that $\alpha < \beta < -\lambda_\infty$ a.e., and γ a constant such that $\max(0, \sup_{x\in\mathscr{A}} \|T_x\|) < e^\gamma$. For any integer N, $R \geqslant 1$, we define the Borel set:

$$B = \{x \in \mathscr{A}: r(T_x^n, e^{-N\beta(x)}) \leqslant R\}$$

If N and R are large enough, the measure of B is close to one. Let us now define the Borel set:

$$A = \{x \in \mathscr{A}: \alpha(x) + \gamma m(B^c \mid \mathscr{T}) \leqslant \beta(x)\, m(B \mid \mathscr{T}), m(B \mid \mathscr{T}) > 0, \beta(x) \leqslant R\}$$

where $m(B \mid \mathscr{T})$ means the conditional expectation with respect to the σ-algebra of invariant sets $\mathscr{T}$. A is an invariant set and has measure at least $1-\varepsilon$ if N and R are large enough. Finally, we define the bounded function a:

$$a = \frac{\alpha + \gamma m(B^c \mid \mathscr{T})}{m(B \mid \mathscr{T})} \mathbb{1}_{B\cap A} - \gamma \mathbb{1}_{B^c \cap A}$$

The function a satisfies (iii) of the lemma, and for all x in A we have

$$r(T_x^N, e^{-Na(x)}) \leqslant R$$

$$r\left(T_x^{kN}, \exp - N \sum_{i=0}^{k-1} a \circ \phi^{iN}(x)\right) \leqslant R^k \quad (\forall k \geqslant 1)$$

$$r\left(T_x^{(k+1)N}, \exp \gamma N - N \sum_{i=0}^{k-1} a \circ \phi^{iN+j}(x)\right) \leqslant R^k \quad (\forall k \geqslant 1, \forall 0 \leqslant j < N)$$

[since $T_x^{(k+1)N} = T^{N-j}_{\phi^{iN+j}(x)} \circ T^{iN}_{\phi^j(x)} \circ T^j_x$],

$$\sum_{i=0}^{kN-1} a \circ \phi^i(x) = \sum_{j=0}^{N-1} \sum_{i=0}^{k-1} a \circ \phi^{iN+j} \leqslant N \max_{0 \leqslant j \leqslant N-1} \sum_{i=0}^{k-1} a \circ \phi^{iN+j}$$

$$r\left(T_x^{(k+1)N}, \exp \gamma N - \sum_{i=0}^{kN-1} a \circ \phi^i(x)\right) \leqslant R^k \quad (\forall k \geqslant 1)$$

We can choose $M = \exp N(\gamma + R)$.

3.3.2. Proof of Lemma 3.3.3 in the Invertible Case. We assume here that the dynamical bundle is invertible (cf. Appendix B). It is enough to prove the inequality $h(T, \alpha, x) \geqslant \sum_{i \geqslant 1} d_i(x)[\lambda_i(x) + \alpha]^+$ a.e. since the other inequality has been proved by Thieullen (1987). This proof looks like the one in Lemma A.5.2 if, in addition, we assume that

(i) $\lim_{n\to+\infty} 1/n \ln \|T^{-n}_{\phi^n(x)} \mid E_i \circ \phi^n(x)\| = \lambda_i(x)$ m.a.e.,

(ii) $\lim_{n\to+\infty} 1/n \ln \|P_r \circ \phi^n(x)\| = 0$ m.a.e. $(\forall r \geqslant 1)$, where $P_r(x)$ denotes the projection onto $\bigoplus_{i=1}^{r} E_i(x)$ parallel to $F_{r+1}(x)$. The first limit says that $T_x^n(B_E)$ is still a ball and the second says that the angle between $\bigoplus_{i=1}^{r} E_i(x)$ and $F_{r+1}(x)$ does not decrease too fast. Let r, β be such that $-\lambda_r(x) < \beta < \alpha < -\lambda_{r+1}(x)$.

Since $E = \bigoplus_{i=1}^{r} E_i \oplus F_{r+1}(x)$ we can define the projections $(\pi_1, ..., \pi_{r+1})$ onto $(E_1(x), ..., E_r(x), F_{r+1}(x))$.

$$B_E \supset \frac{1}{r} \bigoplus_{i=1}^{r} B_{E_i(x)}$$

$$T_x^n(B_E) \supset \frac{1}{r} \bigoplus_{i=1}^{r} T_x^n(B_{E_i(x)})$$

$$r(T_x^n(B_{E_i(x)}), e^{-n\beta}) \geqslant \{e^{n\beta}/(d_i(x) \, \|T_{\phi^n(x)}^{-n} \mid E_i \circ \phi^n(x)\|)\}^{d_i(x)}$$

$$s(T_x^n(B_E), 2e^{-n\alpha}) \geqslant \prod_{i=1}^{r} s(T_x^n(B_{E_i(x)}), e^{-n\beta})$$

The last inequality is true provided that

$$2e^{-n\alpha} < e^{-n\beta} / \max_{1 \leqslant i \leqslant r} \|\pi_i \circ \phi^n(x)\|$$

We complete the proof using (i) and (ii) and the inequality

$$r(T_x^n(B_E), e^{-n\alpha}) \geqslant s(T_x^n(B_E), 2e^{-n\alpha})$$

3.3.4. Proof of Lemma 3.3.3 in the General Case. Let $\mathscr{F}$ be a C^1-dynamical bundle and $\tilde{\mathscr{F}}$ its natural extension. Let m be a probability measure in $\mathscr{M}_1(\mathscr{A}, \phi)$ and $\tilde{m}$ its natural extension to $\tilde{\mathscr{F}}$. We will prove that for any real α,

$$\lim_{n \to +\infty} \frac{1}{n} \ln r(T_{\pi(x)}^n, e^{-n\alpha}) = h(\tilde{T}, \alpha, x) \text{ m̃.a.e. on } \{\alpha \circ \pi < -\lambda_\infty \circ \pi\}$$

Since $\pi(B_{\tilde{E}}) \subset B_E$, $r(T_{\pi(x)}^n, e^{-n\alpha}) \leqslant r(\tilde{T}_x^n, e^{-n\alpha})$ and so $\lim_{n \to +\infty} 1/n \ln r(T_{\pi(x)}^n, e^{-n\alpha}) \leqslant h(\tilde{T}, \alpha, x)$.

Conversely, choose α a ϕ-invariant function such that $\alpha < -\lambda_\infty$ a.e. Then there exists a ϕ-invariant measurable set A of measure at least $1 - \varepsilon$ (cf. Lemma 3.3.1) such that $r(T_x^n, M \exp -a_n(x))$ is uniformly bounded on A. If N is large enough, $e^{N\varepsilon} \geqslant M$, $b_N = a_N - N\varepsilon$, $r(T_x^n, \exp -b_N(x))$ is uniformly bounded on A. Then we can construct a finite set of vectors $V(x)$ in E such that

(i) card $V(x) = r(T_x^n, \exp -b_N(x))$,

(ii) $T_x^N(B_E) \subset V(x) + \exp(-b_N(x)) \, B_E$.

Given a vector v in B_E, there exists

$$v_1 \in V(x) \text{ such that } T_x^n \bullet v \in v_1 + \exp(-b_N(x))\, B_E, \qquad v_2 \in V \circ \phi^n(x)$$

such that

$$\begin{aligned} T_x^{2N} \bullet v \in\, & T_x^N \bullet v_1 + \exp(-b_N(x))\, v_2 \\ & + \exp(-b_N(x) - b_N \circ \phi^N(x))\, B_E, \qquad v_k \in V \circ \phi^{(k-1)N}(x) \end{aligned}$$

such that

$$\begin{aligned} T_x^{kN} \bullet v \in\, & T_x^{(k-1)N} \bullet v_1 + \exp(-b_N)\, T_x^{(k-2)N} \bullet v_2 + \cdots \\ & + \exp\left(-\sum_{i=0}^{k-1} b_N \circ \phi^{iN}(x)\right) B_E \end{aligned}$$

For any $(v_1, v_2, \ldots, v_k)$, we define a vector $w(v_1, v_2, \ldots, v_k)$ in $\tilde{E}$ in the following way:

$$w(v_1, \ldots, v_k) \triangleq (w_{kN}, w_{kN-1}, \ldots, w_N, 0, 0, \ldots)$$

where for $i \geqslant 1$ and $0 \leqslant j < N$,

$$\begin{aligned} w_{iN+j} \triangleq T^j_{\phi^{iN}(x)} \Bigg(& T_x^{(i-1)N} \bullet v_1 + \exp(-b_N)\, T_x^{(i-2)N} \bullet v_2 + \cdots \\ & + \exp\left(-\sum_{k=0}^{i-2} b_N \circ \phi^{kN}\right) v_i \Bigg) \end{aligned}$$

Then for any $w \in B_{\tilde{E}}$ and $v = \pi(w)$ we associate $(v_1, \ldots, v_k)$, and by definition of $w(v_1, \ldots, v_k)$ we have the following inequalities:

$$\begin{aligned} & \| \tilde{T}_x^{kN} \bullet w - w(v_1, \ldots, v_k) \|^2 \\ & \leqslant \sum_{N \leqslant iN+j \leqslant kN} \gamma^2_{kN-(iN+j)} \| T_{x_0}^{iN+j} \bullet v - w_{iN+j} \|^2 + \sum_{i \geqslant (k-1)N} \gamma_i^2 \|w_i\|^2 \\ & \leqslant \sum_{N \leqslant iN+j \leqslant kN} \gamma^2_{kN-(iN+j)} \exp 2\left(j\gamma - \sum_{l=0}^{i-1} b_N \circ \phi^{lN}(x_0) \right) + \gamma^2_{(k-1)N} \end{aligned}$$

where $e^\gamma = \sup_{x \in \mathscr{A}} \|T_x\|$, and $x_0 = \pi(x)$.

$$\begin{aligned} & \| \tilde{T}_x^{kN} \bullet w - w(v_1, \ldots, v_k) \|^2 \\ & \qquad \leqslant M(x_0) \exp\left(-2 \sum_{l=0}^{k-1} b_N \circ \phi^{lN}(x_0) \right) + \gamma^2_{(k-1)N} \end{aligned}$$

where

$$M(x_0)=\sum_{i\geqslant 0}\gamma_i^2\exp\{2i\sup_A(a-\varepsilon)\}\sup_{0\leqslant j<N}\exp\{2j(\gamma+\sup_A(a-\varepsilon))\}$$

If k is large enough (depending on x), then (Birkhoff's theorem)

$$M(x_0)\exp\left(-2\sum_{l=0}^{k-1}b_N\circ\phi^{lN}(x_0)\right)+\gamma_{(k-1)N}^2$$
$$\leqslant \exp -k(E[B_N\circ\pi\mid\tilde{\mathscr{T}}_N]-N\varepsilon)$$

and

$$r(\tilde{T}_x^{kN},\exp -k(E[b_N\circ\pi\mid\tilde{\mathscr{T}}_N]-N\varepsilon)\leqslant\prod_{l=0}^{k-1}\operatorname{card}V\circ\phi^{lN}(x_0)\text{ a.e. on }\pi^{-1}(A)$$

where $\tilde{\mathscr{T}}_N$ is the σ-algebra of $\tilde{\phi}^N$-invariant sets. When k goes to infinity, $\tilde{m}$-almost everywhere on $\pi^{-1}(A)$, we have

$$\sum_{i\geqslant 1}\tilde{d}_i(x)\left\{\tilde{\lambda}_i(x)+\frac{1}{N}E[a_N\circ\pi\mid\tilde{\mathscr{T}}_N]-2\varepsilon\right\}^+$$
$$\leqslant\frac{1}{N}E[\ln r(T_{\pi(x)}^N,\exp(N\varepsilon-a_N\circ\pi))\mid\tilde{\mathscr{T}}_N]$$

If we integrate that inequality with respect to $\tilde{\mathscr{T}}$, we get

$$\sum_{i\geqslant 1}\tilde{d}_i(x)\left\{\tilde{\lambda}_i(x)+\frac{1}{N}E[a_N\circ\pi\mid\tilde{\mathscr{T}}]-2\varepsilon\right\}^+$$
$$\leqslant\frac{1}{N}E[\ln r(T_{\pi(x)}^N,\exp(N\varepsilon-a_N\circ\pi))\mid\tilde{\mathscr{T}}]$$

when N goes to infinity, the last inequality becomes

$$\sum_{i\geqslant 1}d_i\circ\pi(x)\{\lambda_i\circ\pi(x)+\alpha\circ\pi(x)-2\varepsilon\}^+$$
$$\leqslant\liminf_{n\to+\infty}\frac{1}{n}\ln r(T_{\pi(x)}^n,e^{-n\alpha\circ\pi(x)})$$

$\tilde{m}$-almost everywhere on $\pi^{-1}(A)$, which completes the proof.

3.3.5. Proof of Lemma 2.3.5 (cf. Ledrappier, 1981). Let $\{x_n\}_{n\geqslant 0}$ in $\mathscr{A}$ be such that $f_n(x_n)=\sup_{x\in\mathscr{A}}f_n(x)$, then $l\triangleq\inf_{n\geqslant 1}(1/n)f_n(x_n)=$

$\lim_{k \to +\infty} (1/kN) f_{kN}(x_{kN})$ for all $N \geqslant 1$. Using the subadditiveness of the sequence $\{f_n\}_{n \geqslant 1}$ we have

$$f_{kN} \leqslant f_j + \sum_{i=0}^{k-2} f_N \circ \phi^{iN+j} + f_{N-j} \circ \phi^{(k-1)N+j} \qquad \text{for all} \quad 0 \leqslant j < N$$

$$\frac{1}{kN} f_{kN} \leqslant \frac{3F}{kN} + \frac{1}{kN} \sum_{i=0}^{kN-1} \frac{1}{N} f_N \circ \phi^i$$

where

$$F \triangleq \max_{x \in \mathscr{A}} \max_{0 \leqslant i \leqslant N} |f_i(x)|$$

Let m be a weak limitpoint of $m_k \triangleq (1/kN) \sum_{i=0}^{kN-1} \delta \circ \phi^i(x_{kN})$ [where $\delta(x)$ is the Dirac measure at x]. Since $\{f_n\}_{n \geqslant 0}$ are upper semicontinuous,

$$\lim_{k \to +\infty} \frac{1}{kN} f_{kN}(x_{kN}) \leqslant \frac{1}{N} \int f_N \, dm$$

$$\lim_{n \to +\infty} \frac{1}{n} f_n(x_n) = \lim_{n \to +\infty} \frac{1}{n} \int f_n \, dm$$

Using Choquet's representation theorem, there exists a probability measure P defined on the set of ergodic ϕ-invariant probability measures $\mathscr{M}_1^e(\mathscr{A}, \phi)$ such that, for any bounded Borel function f,

$$\int_{\mathscr{A}} f \, dm = \int_{\mathscr{M}_1^e(\mathscr{A}, \phi)} \left(\int_{\mathscr{A}} f \, de \right) dP(e)$$

Then P almost everywhere on $\mathscr{M}_1^e(\mathscr{A}, \phi)$ we have

$$\lim_{n \to +\infty} \frac{1}{n} f_n(x_n) = \lim_{n \to +\infty} \frac{1}{n} \int f_n \, de$$

3.3.6. Proof of Theorem 2. For any $\alpha < -\lambda_\infty^u(T)$, we define a subadditive sequence of upper semicontinuous functions:

$$f_{n,\alpha}(x) \triangleq \ln r(T_x^n, e^{-n\alpha}) \qquad (x \in \mathscr{A})$$

For each $\alpha < -\lambda_\infty^u(T)$, there exists m_α in $\mathscr{M}_1^e(\mathscr{A}, \phi)$ such that

$$h^u(T, \alpha) = \lim_{n \to +\infty} \frac{1}{n} \int f_{n,\alpha} \, dm_\alpha = \lim_{n \to +\infty} \frac{1}{n} f_{n,\alpha}(x) \quad m_\alpha - \text{a.e.}$$

For any ergodic ϕ-invariant measure m in $\mathscr{M}_1^e(\mathscr{A}, \phi)$, $h(T, \alpha, x)$ is constant m-almost everywhere; we may write $h(T, \alpha, m)$. Thus we have proved that, for all $\alpha < -\lambda_\infty^u(T)$, there exists m_α in $\mathcal{M}_1^e(\mathcal{A}, \phi)$ such that $h^u(T, \alpha) = h(T, \alpha, m_\alpha)$, and for all m in $\mathcal{M}_1^e(\mathcal{A}, \phi)$, $h^u(T, \alpha) \geqslant h(T, \alpha, m)$. Since $(\alpha \mapsto h(T, \alpha, m))$ is a nondecreasing convex curve, $(\alpha \mapsto h^u(T, \alpha))$ is also a nondecreasing convex curve. In particular, it is a continuous curve with right and left derivative $\Delta_+^u(\alpha)$, $\Delta_-^u(\alpha)$. In the same manner, we define $\Delta_+^m(\alpha)$, $\Delta_-^m(\alpha)$ the right and left derivative of $h(T, \alpha, m)$. The main point is that $\Delta_+^m(\alpha)$ and $\Delta_-^m(\alpha)$ are integers. For any $\alpha < -\lambda_\infty^u(T)$, we have

$$\Delta_-^u(\alpha) \leqslant \Delta_-^{m_\alpha}(\alpha) \leqslant \Delta_+^{m_\alpha}(\alpha) \leqslant \Delta_+^u(\alpha)$$

If $h^u(T, \alpha)$ is differentiable at α, then its derivative is an integer. Thus there exists a nonincreasing sequence $\{\lambda_i\}_{i \geqslant 1}$ and an increasing sequence of positive integers $\{\Delta_i\}_{i \geqslant 1}$ such that

(i) $\inf_{i \geqslant 1} \lambda_i = \lambda_\infty^u(T)$;

(ii) $\lambda_i > \lambda_{i+1}$ if $\lambda_i > \lambda_\infty^u(T)$;

(iii) if $-\lambda_i < \alpha < -\lambda_{i+1}$ and $h^u(T, \alpha)$ differentiable at α, then its derivative is equal to Δ_i;

(iv) if $\alpha < -\lambda_1$ and $h^u(T, \alpha)$ differentiable at α, then its derivative is equal to zero.

The four previous properties imply $h^u(T, \alpha) = \sum_{i \geqslant 1} d_i(\lambda_i + \alpha)^+$ for all $\alpha < -\lambda_\infty^u(T)$, where $d_i \triangleq \Delta_i - \Delta_{i-1}$ $(\Delta_0 \triangleq 0)$.

Moreover, λ_1 has the property that $h^u(T, \alpha) = 0$ for $\alpha \leqslant -\lambda_1$, and $h^u(T, \alpha) > 0$ for $\alpha > -\lambda_1$. Since $\{\ln \|T_x^n\|\}_{n \geqslant 1}$ is a subadditive sequence, there exists m in $\mathcal{M}_1^e(\mathcal{A}, \phi)$ such that

$$\lim_{n \to +\infty} \frac{1}{n} \ln(\sup_{x \in \mathcal{A}} \|T_x^n\|) = \lambda_1(m) \triangleq \lim_{n \to +\infty} \frac{1}{n} \ln \|T_x^n\| \quad \text{m.a.e.}$$

For $\alpha < -\lambda_1(m)$, $\sup_{x \in \mathcal{A}} \|T_x^n\| < e^{-n\alpha}$ for large n, $r(T_x^n, e^{-n\alpha}) = 1$ and $h^u(T, \alpha) = 0$. For $\alpha > -\lambda_1(m)$, $h^u(T, \alpha) \geqslant h(T, \alpha, m) > 0$; which proves $\lambda_1 = \lambda_1(m)$.

3.4. Different Notions of Uniform Lyapunov Exponents

3.4.1. Proof of Proposition 2.4.1. For any ergodic measure m in $\mathcal{M}_1^e(\mathcal{A}, \phi)$, the opposite of the Legendre transform of $(\alpha \mapsto h(T, \alpha, m))$ is $\gamma(T, d, m)$.

$$h(T, \alpha, m) = \sum_{i \geqslant 1} (\bar{\lambda}_i(m) + \alpha)^+$$

$$\gamma(T, d, m) = \bar{\lambda}_1(m) + \cdots + \bar{\lambda}_p(m) + s\bar{\lambda}_{p+1}(m) \qquad (d = p + s, 0 \leqslant s < 1)$$

and when E is a Hilbert space (cf. Theorem 2.2.3),

$$\gamma(T, d, m) = \lim_{n \to +\infty} \frac{1}{n} \ln(\|\Lambda^p T_x^n\|^{1-s} \|\Lambda^{p+1} T_x^n\|^s) \quad \text{m.a.e.}$$

Since $\{\ln(\|\Lambda^p T_x^n\|^{1-s} \|\Lambda^{p+1} T_x^n\|^s)\}_{n \geqslant 1}$ is a subadditive sequence bounded from above; for each $d \geqslant 0$ there exists a measure m in $\mathscr{M}_1^e(\mathscr{A}, \phi)$ such that $\pi^u(T, d) = \gamma(T, d, m)$. Since $h(T, \alpha, m) \leqslant h^u(T, \alpha)$ for all $\alpha < -\lambda_\infty^u(T)$, $\gamma(T, d, m) \leqslant \gamma^u(T, d)$ for all $d \geqslant 0$. We have just proved that $\pi^u(T, d) \leqslant \gamma^u(T, d)$ for all $d \geqslant 0$.

If $d = d_1^u(T) + \cdots + d_r^u(T)$ and α has been chosen such that $-\lambda_r^u(T) < \alpha < -\lambda_{r+1}^u(T)$, there exists m in $\mathscr{M}_1^e(\mathscr{A}, \phi)$ such that $h(T, \alpha, m) = h^u(T, \alpha)$. Since $h^u(T, \alpha)$ and $h(T, \alpha, m)$ have the same derivative d at α, the value of their Legendre transform at d is the same: $\gamma^u(T, d) = \gamma(T, d, m) \leqslant \pi^u(T, d)$ and so $\pi^u(T, d) = \gamma^u(T, d)$.

3.5. Uniform Hausdorff and Fractal Dimension: Entropy

Instead of proving Theorem 2.5.2, we will prove the sharper inequality [$\mathscr{F}$ is a C^1-dynamical bundle but we do not assume $\phi(\mathscr{A}) = \mathscr{A}$]:

$$h(\phi, \alpha) \leqslant h^u(T, \alpha) \qquad \text{for all} \quad \alpha < -\lambda_\infty^u(T)$$

3.5.1. Proof of the Last Inequality. Let $\alpha < \beta < -\lambda_\infty^u(T)$, N large enough ($e^{-N\beta} < \frac{1}{4} e^{-N\alpha}$), ε small enough ($C_N(\varepsilon) < e^{-N\beta}$), then

$$\phi^N(B(x, \varepsilon)) \subset \phi^N(x) + \varepsilon T_x^n(B_E) + C_N(\varepsilon)\, \varepsilon B_E \qquad \text{(for all } x\text{)}$$

$\phi^N(B(x, \varepsilon))$ can be covered by $r(T_x^n, e^{-N\beta})$ balls of radius $2\varepsilon[C_N(\varepsilon) + e^{-N\beta}] < e^{-N\alpha}$.

We construct by induction points $y(i_0, \ldots, i_k)$ in $\phi^{kN}(\mathscr{A})$, $i_0 \in I_0, \ldots, i_k \in I_k$ such that

$$\mathscr{A} = \bigcup_{I_0} B\left(y(i_0), \frac{\varepsilon}{2}\right)$$

$$\phi^N\left[B\left(y(i_0 \cdots i_k), \frac{\varepsilon}{2} e^{-kN\alpha}\right)\right]$$

$$\subset \bigcup_{i_{k+1} \in I_{k+1}} B\left(y(i_0, \ldots, i_{k+1}), \frac{\varepsilon}{2} e^{-(k+1)N\alpha}\right)$$

$$\operatorname{card}(I_0) = r\left(\mathscr{A}, \frac{\varepsilon}{2}\right), \qquad \operatorname{card}(I_k) \leqslant \sup_{x \in \mathscr{A}} r(T_x^N, e^{-N\beta})$$

For each $(i_0 \cdots i_k)$ in $I_0 \times \cdots \times I_k$ we define a point $x(i_0 \cdots i_k)$ in $\bigcap_{l=0}^{k-1} \phi^{-lN}[B(y(i_0 \cdots i_l), (\varepsilon/2)\, e^{-lN\alpha})]$, if nonempty, which proves

$$r(\mathscr{A}, \varepsilon, d_k^{\phi^N, N\alpha}) \leqslant r\left(\mathscr{A}, \frac{\varepsilon}{2}\right) [\sup_{x \in \mathscr{A}} r(T_x^N, e^{-N\beta})]^k$$

The ideas in the proof of the next theorem are new. But the relationship between the Hausdorff dimension and the Legendre transform of the α-entropy is not well understood.

3.5.2. Proof of Theorem 2.5.3. The proof is divided into four steps.

First Step. Given nonnegative constants A, d, we will prove that $\{\inf_{0 \leqslant \alpha \leqslant A} \ln\{r(T_x^n, e^{-n\alpha})\, e^{-n\alpha d}\}\}_{n \geqslant 0}$ is a subadditive sequence. For any $0 \leqslant \alpha \leqslant A$, $0 \leqslant \beta \leqslant A$, m, $n \geqslant 0$,

$$0 \leqslant \frac{\alpha m + \beta n}{m+n} \leqslant A \qquad \text{and} \qquad r(T_x^{m+n}, e^{-m\alpha - n\beta}) \leqslant r(T_x^m, e^{-m\alpha})\, r(T_{\phi^m(x)}^n, e^{-n\beta})$$

Second Step. We define for all d, $A \geqslant 0$ the curve

$$c_A(d) = \inf_{n \geqslant 1} \sup_{x \in \mathscr{A}} \inf_{0 \leqslant \alpha \leqslant A} \frac{1}{n} \ln\{r(T_x^n, e^{-n\alpha})\, e^{-n\alpha d}\}$$

and prove

$$\dim_H(\mathscr{A}) \leqslant \inf\{d \geqslant 0 \colon C_A(d) < 0\}$$

If $d \geqslant 0$ such that $c_A(d) < 0$ and c chosen such that $c_A(d) < c < 0$, then for n large enough and for all x in $\mathscr{A}$, there exists α in $[0, A]$ such that

$$r(T_x^n, e^{-n\alpha})\, e^{-n\alpha d} \leqslant e^{nc}$$

If $\{B(x_i, \varepsilon_i)\}_{i \in I}$ is a covering of $\mathscr{A}$ with balls of radius less than ε, then each $\phi^n(B(x_i, \varepsilon_i))$ can be covered by $N_i = r(T_{x_i}^n, e^{-n\alpha_i})$ balls of radius less than $2\varepsilon_i[e^{-n\alpha_i} + C_n(\varepsilon_i)]$. Let ε_0 be small enough such that $C_n(\varepsilon_0) < e^{-nA}$ and define

$$m_d(\mathscr{A}, \varepsilon) \triangleq \inf\left\{\sum_{i \in I} r_i^d \colon \mathscr{A} = \bigcup_I B(x_i, \varepsilon_i),\, \varepsilon_i < \varepsilon\right\}$$

Then $m_d(\mathscr{A}, 4\varepsilon) \leqslant 4^d e^{nc} m_d(\mathscr{A}, \varepsilon)$ for all $\varepsilon < \varepsilon_0$. Using the same ideas, if $0 < \beta < -\lambda^u_\infty(T)$, $p \geqslant 0$ and ε_1, such that $8(e^{-p\beta} + C_p(\varepsilon_1)) \leqslant 1$, then

$$m_d(\mathscr{A}, \varepsilon) \leqslant \sup_{x \in \mathscr{A}} r(T^p_x, e^{-p\beta})\, m_d(\mathscr{A}, 4\varepsilon) \qquad \text{for all} \quad \varepsilon < \varepsilon_1$$

If n has been chosen such that $4^d e^{nc} \sup_{x \in \mathscr{A}} r(T^p_x, e^{-p\beta}) \leqslant \frac{1}{2}$,

$$m_d(\mathscr{A}, \varepsilon) \leqslant \tfrac{1}{2} m_d(\mathscr{A}, \varepsilon) \qquad \text{for} \quad \varepsilon \leqslant \min(\varepsilon_0, \varepsilon_1)$$

Third Step. Once more we will use Lemma 2.3.5. If A, $d \geqslant 0$, there exists an ergodic measure m in $\mathscr{M}^e_1(\mathscr{A}, \phi)$ such that

$$c_A(d) = \lim_{n \to +\infty} \frac{1}{n} \inf_{0 \leqslant \alpha \leqslant A} \ln\{r(T^n_x, e^{-n\alpha})\, e^{-n\alpha d}\}$$

which implies

$$c_A(d) \leqslant \inf_{0 \leqslant \alpha \leqslant A} \left\{ \lim_{n \to +\infty} \frac{1}{n} \ln\{r(T^n_x, e^{-n\alpha})\, e^{-n\alpha d}\} \right\}$$

$$\begin{aligned} c_A(d) &\leqslant \inf_{0 \leqslant \alpha \leqslant A} \left\{ \sum_{i \leqslant 1} d_i(m)(\lambda_i(m) + \alpha)^+ - \alpha d \right\} \\ &= \inf_{0 \leqslant \alpha \leqslant A} \{h(T, \alpha, m) - \alpha d\} \end{aligned}$$

Fourth Step. We will prove that, if $d > \sup\{\dim_L(T, m)$: $m \in \mathscr{M}^e_1(\mathscr{A}, \phi)\}$, then there exists $A > 0$ such that for any ergodic measure m in $\mathscr{M}^e_1(\mathscr{A}, \phi)$,

$$\inf_{0 \leqslant \alpha \leqslant A} \{h(T, \alpha, m) - \alpha d\} < 0$$

Choose δ, v, A such that

$$\begin{aligned} &d > \delta > \sup\{\dim_L(T, m) \colon m \in \mathscr{M}^e_1(\mathscr{A}, \phi)\} \\ &v > \max(\lambda^u_1(T), 0) \\ &A = \frac{\delta v}{d - \delta} \end{aligned}$$

Assume now that for any $0 < \alpha < \min(A, -\lambda_\infty(m))$,

$$h(T, \alpha, m) \geqslant \alpha d$$

In particular, $\lambda_1(m) \geqslant 0$ and $A \leqslant -\lambda_\infty(m)$. If $\alpha \in [A, -\lambda_\infty(m)]$, since $[\alpha \mapsto h(T, \alpha, m)]$ is convex,

$$h(T, \alpha, m) \geqslant \frac{Ad(\alpha + \lambda_1(m))}{A + \lambda_1(m)}$$

$$\frac{1}{\alpha} h(T, \alpha, m) \geqslant \delta \frac{A + \nu}{A + \lambda_1(m)} \geqslant \delta$$

which is a contradiction.

In the case of a Hilbert space E, we can improve Theorem 2.5.3.

3.5.3. Proof of Proposition 2.5.4. We have already shown in 3.4.1 that $\pi^u(T, d) \geqslant \gamma(T, d, m)$ for all $d \geqslant 0$ and $m \in \mathscr{M}_1^e(\mathscr{A}, \phi)$; and for all $d \geqslant 0$ there exists a measure m in $\mathscr{M}_1^e(\mathscr{A}, \phi)$ such that $\pi^u(T, d) = \gamma(T, d, m)$.

Let $d^* = \sup\{\dim_L(T, m)\colon \mathscr{M}_1^e(\mathscr{A}, \phi)\}$. Then $\pi^u(T, d) > 0$ for $0 < d < d^*$ and $\pi^u(T, d) < 0$ for $d > d^*$. We claim that $\pi(T, d^*) \geqslant 0$, which shows that there exists m_0 in $\mathscr{M}_1^e(\mathscr{A}, \phi)$ such that $\gamma(T, d^*, m_0) \geqslant 0$ or $d^* = \dim_L(T, m_0)$. To prove the claim, we choose an integer p such that $p < d^* \leqslant p+1$. If $\pi^u(T, p+1) = -\infty$, then $\lambda_{p+1}(m) = -\infty$ and so $\dim_L(T, m) \leqslant p$ for all m in $\mathscr{M}_1^e(\mathscr{A}, \phi)$. Thus $\pi^u(T, p+1) > -\infty$, the function $[d \in (p, p+1) \mapsto \pi^u(T, d)]$ is convex and so continuous. If $p < d^* < p+1$, then $\pi^u(T, d^*) = 0$; if $d^* = p+1$, then $\pi^u(T, d^*) \geqslant 0$; otherwise $\dim_L(T, m) \leqslant p + [\pi^u(T, p)]/[\pi^u(T, p) - \pi^u(T, p+1)] < p+1$ for all m in $\mathscr{M}_1^e(\mathscr{A}, \phi)$.

A. APPENDIX ON SPECTRAL ANALYSIS OF LIMIT-COMPACT OPERATORS

The notion of index of compactness has been introduced by Kuratowski. To prove Oseledec's theorem, we need to introduce this notion; even if we start with a compact dynamical bundle (each operator is compact), its natural extension is no longer compact but still remains asymptotically compact.

In this appendix, a review of Oseledec's theory for a single operator is given. In particular, we will generalize the spectral decomposition theorem and the Fredholm alternative for noncompact operators in the case of Hilbert spaces, and we will be able to give a different definition for the sequence of Lyapunov exponents. We introduce a notion of α-entropy of operators; this notion can be considered in Banach spaces as a generalization of the notion of p-dimensional volume.

A.1. Definition of Lyapunov Exponents in Banach Spaces

The main theorem about the existence of Lyapunov exponents in Banach space is the following.

A.1.1. Theorem. *Let E be a Banach space and $T: E \to E$ a continuous linear operator. We define*

$$\lambda_\infty(T) \triangleq \lim_{n \to +\infty} \frac{1}{n} \ln \|T^n\|_\alpha$$

$$F^\lambda(T) \triangleq \left\{ v \in E: \limsup_{n \to +\infty} \frac{1}{n} \ln \|T^n v\| \leqslant \lambda \right\}$$

$$E^\lambda(T) \triangleq \left\{ v \in E: \exists (w_n)_{n \geqslant 0} \text{ s.t. } w_0 = v,\ T \bullet w_{n+1} = w_n \right.$$

$$\left. \textit{and } \limsup_{n \to +\infty} \frac{1}{n} \ln \|w_n\| \leqslant -\lambda \right\}$$

Then $E^\lambda(T)$ and $F^\lambda(T)$ are vector spaces invariant with respect to T; there exists a nonincreasing sequence $\{\lambda_i\}_{i \geqslant 1}$, of numbers in $[-\infty, \infty)$ such that

(i) $\inf_{i \geqslant 1} \lambda_i = \lambda_\infty(T)$, $\lambda_1 = \lim_{n \to +\infty} (1/n) \ln \|T^n\|$;

(ii) *$F^{\lambda_i}(T)$ is a closed subvector space, and for all $v \in F^{\lambda_i}(T) \backslash F^{\lambda_{i+1}}(T)$,*

$$\lim_{n \to +\infty} \frac{1}{n} \ln \|T^n \mid F^{\lambda_i}(T)\| = \lambda_i = \lim_{n \to +\infty} \frac{1}{n} \ln \|T^n \bullet v\|$$

(iii) *if $\lambda_i > \lambda_\infty(T)$, then $\lambda_i > \lambda_{i+1}$, $1 \leqslant \dim E^{\lambda_i}(T) \triangleq d_i < +\infty$; $F^{\lambda_i}(T) = E^{\lambda_i}(T) \oplus F^{\lambda_{i+1}}(T)$, T restricted to $E^{\lambda_i}(T)$ is invertible and* $\lim_{n \to +\infty} (1/n) \ln \|T^n \mid E^{\lambda_i}(T)\| = \lambda_i = \lim_{n \to +\infty} -(1/n) \ln \|T^{-n} \mid E^{\lambda_i}(T)\|$.

The sequence $\{\lambda_i\}_{i \geqslant 1}$ is uniquely determined by T and called the sequence of Lyapunov exponents. The sequence $\{F^i(T) \triangleq F_i(T)\}_{i \geqslant 1}$ is called the sequence of Lyapunov vector spaces; $\{d_i(T)\}_{i \geqslant 1}$, the sequence of their multiplicities [$d_i(T) = 0$ by convention for $\lambda_i(T) = \lambda_\infty(T)$].

The proof of this theorem requires two lemmas, a geometric lemma and a combinatorial lemma (cf. 2.3.1 for a definition of covering numbers).

A.1.2. Lemma. *Let E, F be two Banach spaces of dimension $d \geqslant 1$, and $[T: E \to F]$ a linear invertible operator. Then for ay $\varepsilon > 0$*

$$\max[(d\varepsilon \|T^{-1}\|)^{-d}, 1] \leqslant r(T, \varepsilon) \leqslant \{\text{ent}[d \|T\| \varepsilon^{-1}] + 1\}^d$$

A.1.3. Lemma. *Let be X a set, $[T: X \to X]$ a map, $[f: X \to R]$ a function. Let us denote*

$$S_k(f) \triangleq \sum_{i=0}^{k=1} f \circ T^i \quad \text{and} \quad A_p \triangleq \{x \in X: \exists 1 \leqslant k \leqslant p \; S_k(f) \circ T^{p-k}(x) \leqslant 0\}$$

Then for any $n \geqslant 1$ and $p \geqslant 1$,

$$S_n(f) \leqslant S_n(\mathbb{1}_{A_p^c} f) + S_p(|f|)$$

This last lemma has been proved by Silva and Thieullen (1991) in a more general setting. Actually this lemma is too strong compared to what we need in the proof of Theorem A.1.1; it shortens the proof of Oseledec's theorem for a dynamical bundle: using the notations of Thieullen (1987) we define $\tilde{X} \triangleq X \times E \backslash \{0\}$, $\{\tilde{T}: \tilde{X} \to \tilde{X}\}$ $\tilde{T}(x, v) \triangleq (\phi(x), T_x \bullet v)$, $\{\tilde{f}: X \to R\}$ $\tilde{f}(x, v) \triangleq \ln \|T_x \bullet v\|/\|v\|$, $\{\pi: \tilde{X} \to X\}$ the first projection (T_x is supposed to be one-to-one for all x). If we denote

$$B_p \triangleq \{x \in X: \exists v \in E \backslash \{0\} \; \forall k \leqslant 1 \leqslant p \; \|T_x^{p-k} \bullet v\| < \|T_x^p \bullet v\|\}$$
$$\tilde{A}_p \triangleq \{(x, v) \in \tilde{X}: \exists 1 \leqslant k \leqslant p \; S_k(\tilde{f}) \circ \tilde{T}^{p-k}(x, v) \leqslant 0\}$$

then $\pi(\tilde{A}_p^c) = B_p$ and if f is uniformly bounded by v, then

$$\frac{1}{n} S_n(\tilde{f}) \leqslant v \left[\frac{1}{n} S_n(\mathbb{1}_{B_p}) \circ \pi + \frac{p}{n}\right]$$

A.1.4. Proof of Theorem A.1.1. The proof is by induction. Let $\lambda_1 = \lambda_1(T)$, $E_1 = E^{\lambda_1(T)}(T)$, and assume that $\lambda_\infty(T) < \lambda_1$ (otherwise there is nothing to prove). The first step consists in proving that $E_1(T)$ is not reduced to $\{0\}$. Let us choose $\lambda_\infty(T) < \lambda < \lambda_1$, normed vectors $\{v_n\}_{n \geqslant 1}$ such that $\lim_{n \to +\infty} \lambda_1^n = \lambda_1$ [where $\lambda_1^n = (1/n) \ln(\|T^n \bullet v_n\|/\|v_n\|)$], $v = \ln \|T\|$, and $f(v) = \ln(\|T \bullet v\|/\|v\|)$. Using Lemma A.1.3, $(1/n) S_n(f)(v_n) = \lambda_1^n - \lambda$. $(\lambda_1 - \lambda)/(v - \lambda) \leqslant \liminf_{n \to +\infty} (1/n) S_n(\mathbb{1}_{A_p^c})$ for any p. The fact that A_p^c is not empty shows that there exist vectors u_p^k such that $\|u_p^k\| \leqslant \exp(-k\lambda) \|u_p^0\|$, $\|u_p^0\| = 1$, $T^k \bullet u_p^k = u_p^{k-1}$ for all $1 \leqslant k \leqslant p$. For fixed $k \geqslant 1$, since $\alpha(\{u_p^k: p \geqslant k\})$ is less than $\lim_{p \to +\infty} \|T^{p-k}\| e^{-p\lambda} = 0$, we can construct a normed vector u such that $T^{-n} \bullet u$ exists for all $n \geqslant 0$ and satisfies $\|T^{-n} \bullet u\| \leqslant e^{-n\lambda} \|u\|$. The second step consists in proving that $E^\lambda(T)$ has finite dimension for any $\lambda > \lambda_\infty(T)$. If $E^\lambda(T)$ contains a subspace F of dimension d and if μ has been chosen such that $\lambda_\infty(T) < \mu < \lambda$, then for large n, B_F is included in $e^{-n\mu} T^n(B_E)$; and for any $\alpha < -\lambda_\infty(T)$, $r(B_F, e^{-n(\alpha+\mu)}) \leqslant r(T^n, e^{-n\alpha})$, $d(\alpha + \lambda) \leqslant \lim_{n \to +\infty} (1/n) \ln r(T^n, e^{-n\alpha}) <$

$+\infty$. In the last step, we prove the existence of a closed subvector space F invariant under T such that $F \oplus E_1(T) = E$. If G is any closed subvector space such that $G \oplus E_1(T) = E$, π the projection onto G parallel to $E_1(T)$ and $S = (\pi \circ T \mid G)$. Then G is invariant under S and $\lambda_1(S) < \lambda_1(T)$: otherwise there would exist a finite-dimensional space $G_1(S)$ invariant under S satisyfing $\lim_{n \to +\infty} (1/n) \ln \|S^{-n} \mid G_1(S)\| = -\lambda_1(T) = \lim_{n \to +\infty} (1/n) \ln \|T^{-n} \mid E_1(T)\|$, $\tilde{G} = G_1(S) \oplus E_1(T)$ would be invariant under T, and on $\tilde{G}$ we would have

$$T^{-n} = T^{-n} \circ (I - \pi) + \sum_{k=0}^{n-1} T^{-k} \circ (I - \pi) \circ T^{-1} \circ S^{k-n+1} \circ \pi + S^{-n} \circ \pi$$

$$\|T^{-n} \mid \tilde{G}\| \leqslant K \sum_{k=0}^{n} \|T^{-k} \mid E_1(T)\| \, \|S^{k-n} \mid G_1(T)\|$$

which would show $\limsup_{n \to +\infty} (1/n) \ln \|T^{-n} \bullet v\| \leqslant -\lambda_1(T)$ for any $v \in \tilde{G}$. Thus the following series is convergent $U \triangleq \sum_{n \geqslant 0} T^{-n-1} \circ (I - \pi) \circ T \circ S^n$ and satisfies $U^2 = U$, $\mathrm{Im}(U) = E_1(T)$, $T(\ker(U)) \subset \ker(U)$; the required space is then $F = \ker(U)$.

A.2. Definition of Characteristic Exponents in Hilbert Spaces

The main Theorem A.1.1, applied to bounded symmetric operators, leads us to the notion of characteristic exponents of a general bounded operator T as Lyapunov exponents of $\sqrt{T^*T}$.

In the case of Hilbert spaces we have different definitions of index of compactness of operators.

A.2.1. Definition. If E is a Banach space, $\mathscr{L}(E)$ the space of bounded operators, $\mathscr{K}(E)$ the space of compact bounded operators, we define a new norm in $\mathscr{L}(E)/\mathscr{K}(E)$ by $\|\bar{T}\| = \inf\{\|T - K\| \colon K \in \mathscr{K}(E)\}$, satisfying $\|\bar{S} \circ \bar{T}\| \leqslant \|\bar{S}\| \, \|\bar{T}\|$.

Proof. $\|\bar{S} \circ \bar{T}\| \leqslant \|S \circ T + K \circ L - K \circ T - S \circ L\| \leqslant \|S - K\| \bullet \|T - L\|$.

A.2.2. Proposition. *If E is a Hilbert space, then $\|T\|_\alpha = \|\bar{T}\|$ for any T in $\mathscr{L}(E)$.*

Proof. $\|T\|_\alpha = \|T - K\|_\alpha \leqslant \|T - K\|$ for any $K \in \mathscr{K}(E)$, so $\|T\|_\alpha \leqslant \|\bar{T}\|$. Conversely, let $\varepsilon > \|T\|_\alpha$, then $T(B_E)$ can be covered by a finite number of ε-balls centered on $x_1, \ldots, x_r$. Let π be the orthogonal projection onto the space spanned by $\{x_i\}_{i=1}^r$, then $\|T - \pi \circ T\| \leqslant \varepsilon$.

A.2.3. Corollary. *(See Beauzamy, 1987a, b.) If T is a bounded operator of a Hilbert space E, then for any sequence $\{v_n\}_{n\geqslant 0}$ of normed vectors weakly converging to zero, $\limsup_{n\to+\infty} \|T\bullet v_n\| \leqslant \|T\|_\alpha$, and this inequality becomes an equality for at least one such sequence.*

Proof. If $K\in\mathscr{K}(E)$, then $\lim_{n\to+\infty} K(v_n)=0$, thus $\limsup_{n\to+\infty} \|T\bullet v_n\| = \limsup_{n\to+\infty} \|T\bullet v_n - K\bullet v_n\| \leqslant \|T-K\|$. To prove the second assertion we construct a sequence of orthonormal vectors $\{v_n\}_{n\geqslant 0}$ such that $\|T\bullet v_{n+1}\| \geqslant \|T\mid F_n\| - [1/(n+1)]$, where $F_n=\operatorname{span}\{v_0,..., v_n\}^\perp$ and $v_{n+1}\in F_{n+1}$. Since $B_E\subset B_{F_n}\oplus B_{F_n^\perp}$, $\alpha(T(B_{F_n})) \leqslant \alpha(T(B_E)) \leqslant \alpha(T(B_{F_n})) + \alpha(T(B_{F_n^\perp}))=\alpha(T(B_{F_n}))$, which shows $\|T\mid F_n\| \geqslant \|T\mid F_n\|_\alpha = \|T\|_\alpha$.

The only result, which can be proved for an arbitrary Banach space, is the following.

A.2.4. Proposition. *If E is a Banach space, then $\lambda_\infty(T)=\lim_{n\to+\infty} (1/n)\ln \|\bar{T}^n\|$.*

Proof. Using the main Theorem A.1.1, we construct a compact operator $K_n=\pi_n\circ T$ [π_n the projection onto $\bigoplus_{i=1}^{n-1} E_i(T)$ parallel to $F_n(T)$]. Since $\lim_{k\to+\infty} (1/k)\ln\|(T-K_n)^k\| = \lambda_n(T)$, $\lambda_\infty(T)\leqslant \lim_{k\to+\infty} (1/k)\ln\|\bar{T}^k\| \leqslant \lambda_n(T)$.

The next theorem is a simple consequence of the main one for symmetric operators.

A.2.5. Theorem. *If E is a Hilbert space and $[T\colon E\to E]$ a bounded symmetric operator, then*

(i) *if $\lambda_i(T)>\lambda_\infty(T)$, $E_i(T)=\operatorname{Ker}(T-e^{\lambda_i(T)}\mathrm{Id})\oplus\operatorname{Ker}(T+e^{\lambda_i(T)}\mathrm{Id})$ and is orthogonal to $F_{i+1}(T)$;*

(ii) $\lambda_i(T)=\ln\|T\mid F_i(T)\| = (1/n)\ln\|T^n\mid F_i(T)\|$ $(\forall i\geqslant 1, \forall n\geqslant 1)$;

(iii) $\lambda_\infty(T)=\ln\|T\|_\alpha = (1/n)\ln\|T^n\|_\alpha$ $(\forall n\geqslant 1)$;

(iv) $E=\bigoplus_{i\geqslant 1} E_i(T)\oplus\bigcap_{i\geqslant 1} F_i(T)$.

The sequence $\{\chi_i(T)=e^{\lambda_i(T)}\}_{i\geqslant 1}$ is called the sequence of characteristic exponents.

Proof. For any bounded symmetric operator T, $\|T^n\|=\|T\|^n$, which proves (ii). Since $E_i(T)$ is invariant under T and has finite dimension, T is diagonalizable; if $v\in E_i(T)$ and $w\in F_{i+1}(T)$ $[\lambda_i(T)>\lambda_\infty(T)]$, then

$$|\langle v, w\rangle| = e^{-n\lambda_i(T)}\,|\langle T^n\bullet v, w\rangle| = e^{-n\lambda_i(T)}\,|\langle v, T^n\bullet w\rangle|$$

$$|\langle v, w\rangle| \leqslant e^{-n\lambda_i(T)}\,\|T^n\mid F_{i+1}(T)\|$$

and so $\langle v, w\rangle = 0$, which proves (i). Since $\ln \|T\|_\alpha \leqslant \ln \|T \mid F_i(T)\| \leqslant \lambda_i(T)$, $\ln \|T\|_\alpha = \lambda_\infty(T)$, which proves (iii). If v is orthogonal to $\bigoplus_{i\geqslant 1} E_i(T)$, and $v \in F_i(T)$, $v = u + w$, $u \in E_i(T)$, $w \in F_{i+1}(T)$, then $0 = \langle u, v\rangle = \|u\|^2$, $v \in F_{i+1}(T)$, which proves (iv).

A.2.6. Definition. If T is any bounded operator of a Hilbert space, we generalize the notion of characteristic exponent by

$$\chi_i(T) \triangleq \chi_i(\sqrt{T^*T}), \qquad \delta_i(T) \triangleq d_i(\sqrt{T^*T}) \ (\forall i \geqslant 1)$$

$$\chi_\infty(T) \triangleq \inf_{i\geqslant 1} \chi_i(T)$$

A.2.7. Remark. For any bounded operator of a Hilbert space, $\chi_\infty(T) = \|T\|_\alpha = \|T^*\|_\alpha = \chi_\infty(T^*)$.

Proof. Following Riesz and Nagy (1968), there exist two partially isometries U and V (in particular, $\|U\| \leqslant 1$ and $\|V\| \leqslant 1$) such that $T = U\sqrt{T^*T}$ and $\sqrt{T^*T} = VT$. Thus $\|T\|_\alpha = \|\sqrt{T^*T}\|_\alpha$, $\|T^*T\|^{1/2} = \|\sqrt{T^*T}\|$ [cf. (iii) of Theorem A.2.5], which proves $\|T\|_\alpha \leqslant \|T^*\|_\alpha$.

A.3. Relationship Between Lyapunov Exponents and Spectrum

We will show that the spectrum of T inside the annulus $\exp \lambda_\infty(T) < r \leqslant \exp \lambda_1(T)$ is discrete and any point of its closure has an absolute value equal to one of the values $\exp \lambda_i(T)$.

A.3.1. Proposition. *Let T be a bounded operator on a Banach space and $\sigma(T)$ the complex spectrum of T. If $l \in \sigma(T)$ and $\ln |l| \geqslant \lambda_\infty(T)$, then $\ln |l| = \lambda_i(T)$ for some $i \in N^* \cup \{\infty\}$. Conversely, for any $i \in N^* \cup \{\infty\}$, there exists $l \in \sigma(T)$ such that $\ln |l| = \lambda_i(T)$.*

Proof. If $l \in \mathbb{C}$ such that $\lambda_1(T) \geqslant \ln |l| > \lambda_\infty(T)$, we can find a decomposition of E, $E = \tilde{E} \oplus F$, $\tilde{E}$ and F are invariant under T, $\tilde{E}$ has finite dimension and $\lim_{n\to+\infty} (1/n) \ln \|T^n \mid F\| < \ln |l|$. For large n, $\|T^n \mid F\| < |l|^n$, $l^n I - T^n$ is invertible on $F_\mathbb{C}$ and so $lI - T$ is invertible on $F_\mathbb{C}$ too $[N_\mathbb{C}(lI - F) \subset N_\mathbb{C}(l^n I - T^n)$ and $R_\mathbb{C}(l^n I - T^n) \subset R_\mathbb{C}(lI - T)$ thanks to the equality $l^n I - T^n = (lI - T)(l^{n-1} I + l^{n-2} T + \cdots + T^{n-1})]$. Thus $l \in \sigma(T)$ if and only if $l \in \sigma(T \mid \tilde{E})$, and $\lambda \geqslant \ln |l|$ is a Lyapunov exponent of T if and only if λ is a Lyapunov exponent of $(T \mid \tilde{E})$. Then it is enough to prove this proposition when E has finite dimension. Since $\sigma(T)$ is compact and $\lambda_\infty(T) = \inf_{i\geqslant 1} \lambda_i(T)$, there exists $l \in \sigma(T)$ such that $\ln |l| = \lambda_\infty(T)$.

A.4. A Different Definition of Characteristic Exponents

If T is a bounded operator on a Hilbert space, we denote by $\{\tilde{\chi}_i(T)\}_{i\geqslant 1}$ the sequence of its characteristic exponents repeated as many times as their multiplicities $\{\delta_i(T)\}_{i\geqslant 1}$. The next theorem gives two different ways to compute this sequence.

A.4.1. Theorem. *If T is a bounded operator on a Hilbert space, then for any $i\geqslant 1$,*

(i) $\tilde{\chi}_i(T)=\sup\{\inf\{\|T\bullet v\|: v\in F,\ \|v\|=1\}: \dim F=i\}$,

(ii) $\|\bigwedge^i T\| = \tilde{\chi}_1(T)\cdots\tilde{\chi}_i(T)$.

This theorem is well known for compact operators. The only difficult part lies in the case $\|T\|_\alpha=\|T\|$. Furthermore, if T is a bounded operator and $R=\sqrt{T^*T}$, then $\|\bigwedge^i R\bullet v\|=\|\bigwedge^i T\bullet v\|$ for any $v\in\bigwedge^i E$ and $i\geqslant 1$, which shows that we can assume T is symmetric. To begin with we need the following lemma.

A.4.2. Lemma. *For any vectors $(e_1,..., e_p)$ in E,*

$$\|e_1\wedge\cdots\wedge e_p\|=\inf\{\|v_1\|\cdots\|v_p\|: v_1\wedge\cdots\wedge v_p=e_1\wedge\cdots\wedge e_p\}$$

Proof. We may assume that $(e_1,..., e_p)$ are linearly independent (equivalent to $e_1\wedge\cdots\wedge e_p\neq 0$). Using Gram Schmidt process, there exists a p by p uper triangular matrix $A=(a_{ij})$ with 1's on the main diagonal such that $(v_j\triangleq\sum a_{ij}e_i)$ are orthogonal and satisfy $(\|v_j\|\leqslant\|e_j\|)$. Since $v_1\wedge\cdots\wedge v_p=\det(A)\,e_1\wedge\cdots\wedge e_p$, $\|e_1\wedge\cdots\wedge e_p\|=\|v_1\|\cdots\|v_p\|$.

A.4.3. Proof of Theorem A.4.1. The proof is divided into three parts.

In the first part we prove the theorem when $\|T\|_\alpha=\|T\|=1$. Given any $\varepsilon>0$, by induction over $p\geqslant 1$, we claim that there exist $(e_1,..., e_p)$ orthonormal such that $\|T\bullet e_1\wedge\cdots\wedge T\bullet e_p\|\geqslant(1-\varepsilon)^p$ [this will prove the second assertion $1\geqslant\|\bigwedge^p T\|\geqslant(1-\varepsilon)^p$, and the first assertion, $1\geqslant\inf\{\|T\bullet v\|: v\in\operatorname{span}(e_1\cdots e_p),\ \|v\|=1\}\geqslant(1-\varepsilon)^p$]. Let us assume the claim is true for p, and let us define $G\triangleq[\operatorname{span}(e_1\cdots e_p)]^\perp$ and $H\triangleq[\operatorname{span}(T\bullet e_1,..., T\bullet e_p)]^\perp$, π the orthogonal projection onto H and $\tilde{T}\triangleq(\pi\circ T\mid G)$, which satisfies $\|\tilde{T}\|\leqslant 1$. Since $B_E\subset B_G\oplus B_{G^\perp}$, $T(B_E)\subset(\pi\circ T)(B_G)+(I-\pi)\circ T(B_G)+T(B_{G^\perp})$, $1=\alpha(T(B_E))\leqslant\alpha(\tilde{T}(B_G))\leqslant\|\tilde{T}\|\leqslant 1$. There exists a normed vector e_{p+1} in G such that $\|\tilde{T}\bullet e_{p+1}\|\geqslant 1-\varepsilon$, then $\|T\bullet e_1\wedge\cdots\wedge T\bullet e_{p+1}\|=\|T\bullet e_1\wedge\cdots\wedge T\bullet e_p\|\ \|\tilde{T}\bullet e_{p+1}\|\geqslant(1-\varepsilon)^{p+1}$.

In the second part we prove the first assertion in the general case. Let $p\geqslant 1$ be fixed. Either $\tilde{\chi}_p(T)>\|T\|_\alpha$; then there exist $(e_1\cdots e_p)$ orthonormal vectors such that $T\bullet e_i=\pm\chi_i(T)\,e_i$ and $\|T\mid G_{p-1}^\perp\|\leqslant\tilde{\chi}_p(T)$, which proves

$\inf\{\|T\bullet v\|\colon v\in G_p,\ \|v\|=1\}=\tilde{\chi}_p(T)$ [where $G_i\triangleq\mathrm{span}(e_1\cdots e_i)$ for any $i\geqslant 1$] and the inequality $\leqslant$ with F of dimension p instead of G_p, since there always exists a normed vector $v\in F$ orthogonal to G_{p-1}. Or $\tilde{\chi}_p(T)=\|T\|_\alpha$; then there exists an invariant subspace G such that $\dim G^\perp<p$, $\|T\|_\alpha=\|T\mid G\|_\alpha=\|T\mid G\|$. If F is a subspace of dimension p, $\inf\{\|T\bullet v\|\colon v\in F,\ \|v\|=1\}$, since there exists a normed vector in $F\cap G$. Using the first part, given $\varepsilon>0$ we can construct $(e_{r+1},\dots,e_p)$ orthonormal vectors in G such that $\inf\{\|T\bullet v\|\colon v\in\mathrm{span}(e_{r+1}\cdots e_p),\ \|v\|=1\}\geqslant(1-\varepsilon)^{p-r}\|T\|_\alpha$, which completes the proof.

In the last part, we prove the second assertion in the general case. We begin to prove by induction $\|\bigwedge^p T\|\leqslant\prod_{i=1}^p\tilde{\chi}_i(T)$. If $(v_1\cdots v_{p+1})$ are $p+1$ orthonormal vectors and w_{p+1} is a normed vector in $G_{p+1}\triangleq\mathrm{span}(v_1\cdots v_{p+1})$, we can construct $w_1\cdots w_p$ in G_{p+1}, orthonormal and orthogonal to w_{p+1}, then $\|T\bullet v_1\wedge\cdots\wedge T\bullet v_{p+1}\| = \|T\bullet w_1\wedge\cdots\wedge T\bullet w_{p+1}\| \leqslant \|T\bullet w_1\wedge\cdots\wedge T\bullet w_p\|\ \|T\bullet w_{p+1}\| \leqslant \|\bigwedge^p T\|\ \|T\bullet w_{p+1}\|$, which proves $\|\bigwedge^{p+1}T\|\leqslant\|\bigwedge^p T\|\ \tilde{\chi}_{p+1}(T)$. To prove the other inequality, let us assume p such that $\tilde{\chi}_p(T)=\|T\|_\alpha$ (the other case is easier). Let us define $r\geqslant 1$ such that $\tilde{\chi}_r(T)>\|T\|_\alpha$ and $\tilde{\chi}_{r+1}=\|T\|_\alpha$. If $(e_1\cdots e_r)$ is an orthonormal basis such that $T\bullet e_i=\pm\tilde{\chi}_i(T)\,e_i$, $G_r=\mathrm{span}(e_1\cdots e_r)$ and $(e_{r+1}\cdots e_p)$ any orthonormal vectors in $G_r^\perp$, then $\|\bigwedge^p T\|\geqslant\prod_{i=1}^r\tilde{\chi}_i(T)$ $\|T\bullet e_{r+1}\wedge\cdots\wedge T\bullet e_p\|$, $\|\bigwedge^p T\|\geqslant\prod_{i=1}^r\tilde{\chi}_i(T)$ $\|\bigwedge^{p-r}T\|=\prod_{i=1}^r\tilde{\chi}_i(T)$ $\|T\|_\alpha^{p-r}$.

A.4.5. Corollary. *For any bounded operator T on a Hilbert space,*

(i) $\tilde{\chi}_p(T)=\tilde{\chi}_p(T^*)$,

(ii) $\tilde{\chi}_p(T)=\inf\{\sup\{\|T\bullet v\|\colon v\in F^\perp,\ \|v\|=1\}\colon \dim F=p-1\}$.

Proof. Since $\|\bigwedge^p T\|=\|\bigwedge^p T^*\|$, by induction we have $\tilde{\chi}_p(T)=\tilde{\chi}_p(T^*)$. To prove the second assertion, we may assume T symmetric. Then for any $p\geqslant 1$, there exists a subspace F of dimension $p-1$ such that $\|T\mid F^\perp\|\leqslant\tilde{\chi}_p(T)$ (if $\tilde{\chi}_{p-1}(T)>\|T\|_\alpha$, $F\triangleq\bigoplus_{i=1}^{p-1}\mathrm{Ker}(T-\varepsilon_i\tilde{\chi}_i(T)\,Id)$; if $\tilde{\chi}_{p-1}(T)=\|T\|_\alpha$, we choose $F\supset\bigoplus_{i\geqslant 1}E_i(T)$). If G is any subspace of dimension p, and F of dimension $p-1$, $G\cap F^\perp\neq\{0\}$ and so $\|T\mid F^\perp\|\geqslant\inf\{\|T\bullet v\colon v\in G,\ \|v\|=1\}$, which proves the other inequality using Theorem A.4.1.

A.5. Relationship Between Lyapunov and Characteristic Exponents

Oseledec's (1968) theorem has been first proved in Hilbert spaces by Ruelle (1979). In this paper, Ruelle defines the sequence of Lyapunov exponents using the asymptotic limit of characteristic exponents of T^n. The following theorem shows that his definition coincides with the one given in the main Theorem A.1.1. If T is a bounded operator of a Banach space, we will write $\{\tilde{\lambda}_i(T)\}_{i\geqslant 1}$ for the sequence of Lyapunov exponents of T repeated as many times as their multiplicity $\{d_i(T)\}_{i\geqslant 1}$.

A.5.1. Theorem. *If T is a bounded operator on a Hilbert space, then*

$$\lim_{n\to+\infty}\frac{1}{n}\ln\tilde{\chi}_p(T^n)=\tilde{\lambda}_p(T)\quad(\forall p\geqslant 1)$$

The proof requires two lemmas. The main notion is the notion of α-entropy of an operator (which has been defined in 2.3.2 for a dynamical bundle): $h(T,\alpha)\triangleq\lim_{n\to+\infty}(1/n)\ln r(T^n,e^{-n\alpha})$ for $\alpha<-\lambda_\infty(T)$. The proof consists in finding an exact formula between $h(T,\alpha)$ and either the characteristic or the Lyapunov exponents.

A.5.2. Lemma. *If T is a bounded operator on a Banach space, then for any $\alpha<-\lambda_\infty(T)$,*

$$h(T,\alpha)=\sum_{i\geqslant 1}d_i(T)(\lambda_i(T)+\alpha)^+$$

Proof. Let r be such that $-\lambda_r(T)<\alpha<-\lambda_{r+1}(T)$ and $(\pi_1,\ldots,\pi_{r+1})$ the family of projections associated with the decomposition $E=E_1(T)\oplus\cdots\oplus E_r(T)\oplus F_{r+1}(T)$. Then, applying Lemma A.1.2 on each $E_i(T)$, we have

$$B_E\subset\bigoplus_{i=1}^{r}\|\pi_i\|\,B_{E_i(T)}\oplus\|\pi_{r+1}\|\,B_{F_{r+1}(T)}$$

$$T^n(B_E)\subset\bigoplus_{i=1}^{r}\|\pi_i\|\,T^n(B_{E_i(T)})$$

$$\oplus\,\|T^n\mid F_{r+1}(T)\|\,\|\pi_{r+1}\|\,B_{F_{r+1}(T)}$$

$$r(T^n,e^{-n\alpha})\leqslant\prod_{i=1}^{r}r(T^n\mid E_i(T),e^{-n\beta})$$

$$r(T^n\mid E_i(T),e^{-n\beta})\leqslant\{\mathrm{ent}[d_i(T)\,\|T^n\mid E_i(T)\|\,e^{-n\beta}]+1\}^{d_i(T)}$$

if β has been chosen in $(\alpha,-\lambda_{r+1}(T))$ and $n\geqslant 1$ such that

$$e^{-n\beta}<\left(\sum_{i=1}^{r+1}\|\pi_i\|\right)^{-1}e^{-n\alpha}$$

Conversely, if $\beta \in (-\lambda_r(T), \alpha)$ and $n \geqslant 1$ such that

$$e^{-n\alpha} < \sum_{i=1}^{r} (2r\,\|\pi_i\|)^{-1}\, e^{-n\beta}$$

$$B_E \supset \frac{1}{r}\left(\bigoplus_{i=1}^{r} B_{E_i(T)}\right)$$

$$r(T^n(B_E), e^{-n\alpha}) \geqslant \prod_{i=1}^{r} s(T^n(B_{E_i(T)}), e^{-n\beta})$$

$$s(T^n(B_{E_i(T)}), e^{-n\beta}) \geqslant \max[(2e^{n\beta} d_i(T)^{-1}\, \|T^{-n} \mid E_i(T)\|^{-1})^{d_i(T)}, 1]$$

A.5.3. ***Lemma.*** *If T is a bounded operator on a Hilbert space, then for any integer $p \geqslant 1$ and positive real $\varepsilon \in (\tilde{\chi}_{p+1}(T), \tilde{\chi}_p(T))$,*

$$C_p^{-1}\, \|\wedge^p T\|\, \varepsilon^{-p} \leqslant r(T, \varepsilon(p+1)) \leqslant C_p\, \|\wedge^p T\|\, \varepsilon^{-p}$$

where C_p is a constant which depends only on p.

Proof. We may assume that T is already a symmetric operator $(r(T, \varepsilon) = r(\sqrt{T^*T}, \varepsilon))$. Let $r \geqslant 1$ be such that $\chi_r(T) = \tilde{\chi}_p(T)$ and $\chi_{r+1}(T) = \tilde{\chi}_{p+1}(T)$. Then

$$r(T, \varepsilon(p+1)) \leqslant \prod_{i=1}^{r} r(T \mid E_i(T), \varepsilon)$$

$$r(T \mid E_i(T), \varepsilon) \leqslant \{\mathrm{ent}(\delta_i(T)\, \chi_i(T)\, \varepsilon^{-1}) + 1\}^{\delta_i(T)}$$

$$r(T, \varepsilon(p+1)) \leqslant 2^p p^p\, \|\wedge^p T\|\, \varepsilon^{-p}$$

conversely,

$$r(T, \varepsilon(p+1)) \geqslant \prod_{i=1}^{r} s(T(B_{E_i(T)}), \sqrt{r}\, \varepsilon(p+1)\, 2^{-1})$$

$$r(T, \varepsilon(p+1)) \geqslant (p+1)^{-3p}\, \|\wedge^p T\|\, \varepsilon^{-p}$$

A.5.4. Proof of Theorem A.5.1. Let us define for all $i \geqslant 1$: $\tilde{\mu}_i(T) = \lim_{n \to +\infty} (1/n)\, \tilde{\chi}_i(T^n)$. We begin to prove that $\inf_p \tilde{\mu}_p(T) = \lambda_\infty(T)$: since for any $p \geqslant 1$, $(1/p) \sum_{i=1}^{p} \tilde{\mu}_i = \inf_n (1/pn) \sum_{i=1}^{p} \ln \tilde{\chi}_i^p(T^n)$, $\lambda_\infty(T) \leqslant \inf_p \tilde{\mu}_p \leqslant \inf_p \inf_n (1/pn) \sum_{i=1}^{p} \ln \tilde{\chi}_i(T^n) \leqslant \inf_n (1/n) \ln \|T^n\|_\alpha = \lambda_\infty(T)$. The proof is then complete if we prove the equality $h(T, \alpha) = \sum_{p \geqslant 1} (\tilde{\mu}_i + \alpha)^+$ for any $\alpha < -\lambda_\infty(T)$. If $\alpha \in (-\tilde{\mu}_p, -\tilde{\mu}_{p+1})$, for n large enough $(\tilde{\chi}_{p+1}(T^n) > (p+1)^{-1} e^{-n\alpha} > \tilde{\chi}_p(T^n))$,

$$C_p^{-1} \prod_{i=1}^{p} \tilde{\chi}_i(T^n)\, e^{np\alpha} \leqslant r(T^n, e^{-n\alpha}) \leqslant C_p \prod_{i=1}^{p} \tilde{\chi}_i(T^n)\, e^{np\alpha}$$

B. APPENDIX IN THE NATURAL EXTENSION

The proof of Oseledec's theorem in the Banach case assumes that the map ϕ is an homeomorphism and each operator T_x is one-to-one. But there is a natural way to get rid of these assumptions using the notion of natural extension. The original bundle then becomes a factor of the invertible one.

B.1. Extension of Regular Points

B.1.1. Definition of the Natural Extension. If $\mathscr{F}=(E, \mathscr{A}, \phi, T)$ is a C^1-dynamical bundle and $\{\gamma_n\}_{n\geqslant 0}$ a decreasing sequence satisfying $\gamma_0=1$, $0<\gamma_{m+n}\leqslant\gamma_m\gamma_n$, $\lim_{n\to+\infty}(1/n)\ln\gamma_n=-\infty$, we define its natural extension $\tilde{\mathscr{F}}=(\tilde{E}, \tilde{\mathscr{A}}, \tilde{\phi}, \tilde{T})$ by

$$\tilde{E}\triangleq\left\{v=(v_n)_{n\geqslant 0}\in E^{\mathbb{N}}\colon \sum_{n\geqslant 0}\gamma_n^2\,\|v_n\|^2<+\infty\right\}$$

$$\tilde{\mathscr{A}}\triangleq\{x=(x_n)_{n\geqslant 0}\in\mathscr{A}^{\mathbb{N}}\colon \phi(x_{n+1})=x_n \text{ for all } n\geqslant 0\}$$

$$\tilde{\phi}(x)\triangleq(\phi(x_0), x_0, x_1, \ldots)\qquad \text{for all}\quad x=(x_n)_{n\geqslant 0}$$

$$\tilde{T}_x\bullet v\triangleq(T_{x_0}\bullet v_0, v_0, v_1, \ldots)\qquad \text{for all}\quad v=(v_n)_{n\geqslant 0}$$

$$\pi\bullet v\triangleq v_0\qquad \text{(the projection onto the first coordinate)}$$

We remark that $\tilde{E}$ is a Banach space with the norm $\|v\|^2=\sum_{n\geqslant 0}\gamma_n^2\,\|v_n\|^2$ (if E is a Hilbert space, then $\tilde{E}$ is a Hilbert space, too), $\tilde{\mathscr{A}}$ is a compact subset of $\tilde{E}$, $\tilde{\phi}$ is a homeomorphism on $\tilde{\mathscr{A}}$, $\tilde{\phi}^{-1}(x)=(x_1, x_2, \ldots))$, $\tilde{T}$ is a quasidifferential of $\tilde{\phi}$, and each $\tilde{T}_x$ is one-to-one. Besides, if ϕ is $C^{1,t}$-quasidifferentiable, then $\tilde{\phi}$ is also $C^{1,t}$-quasidifferentiable.

B.1.2. Definition of Strongly Regular Points. If $\mathscr{F}$ is an invertible dynamical bundle, a point x in $\mathscr{A}$ is said to be strongly regular, if it is regular and there exists a family of finite-dimensional spaces $\{E_i\}_{i\geqslant 1}$ such that

(i) $F_i(x)=E_i\oplus F_{i+1}(x)$ for all $i\geqslant 1$;

(ii) if $\lambda_i(x)>\lambda_\infty(x)$ and $v\in E_i\backslash\{0\}$, then $T_x^{-n}\bullet v$ exists for all $n\geqslant 0$ and

$$\lim_{n\to+\infty}\frac{1}{n}\ln\|T_x^{-n}\bullet v\|=\lim_{n\to+\infty}\frac{1}{n}\ln\|T_x^{-n}\mid E_i\|=-\lambda_i(x)$$

$$\lim_{n\to+\infty}\frac{1}{n}\ln\|T_x^{n}\bullet v\|=\lim_{n\to+\infty}\frac{1}{n}\ln\|T_x^{n}\mid E_i\|=\lambda_i(x)$$

(iii) if $\lambda_i(x) > \lambda_\infty(x)$ and $v \in F_{i+1}(x) \backslash \{0\}$ such that $T_x^{-n} \bullet v$ exists for all $n \geqslant 0$, then

$$\lim_{n \to +\infty} \frac{1}{n} \ln \| T_x^{-n} \bullet v \| > -\lambda_i(x)$$

The family $\{E_i\}_{i \geqslant 1}$ is uniquely determined by the strongly regular point x. We denote $\Sigma(\mathscr{F})$ the set of strongly regular points.

B.1.3. Theorem. *If $\mathscr{F} = (E, \mathscr{A}, \phi, T)$ is a dynamical and $\tilde{\mathscr{F}} = (\tilde{E}, \tilde{\mathscr{A}}, \tilde{\phi}, \tilde{T})$ its natural extension, then $\pi(\Sigma(\tilde{\mathscr{F}})) \subseteq \Lambda(\mathscr{F})$, in particular, $F_i \circ \pi(x) = \pi \circ \tilde{F}_i(x)$, $\lambda_i \circ \pi(x) = \tilde{\lambda}_i(x)$ and $d_i \circ \pi(x) = \tilde{d}_i(x)$ for all x in $\Sigma(\mathscr{F})$.*

The proof of this theorem requires the following lemma.

B.1.4. Lemma. *If $(a_n)_{n \geqslant 0}$ and $(b_n)_{n \geqslant 0}$ are sequences of positive real numbers such that $(a_n)_{n \geqslant 0}$ is decreasing and $\lim_{n \to +\infty} (1/n) \ln b_n = -\infty$, if we denote $\sigma_n = \sum_{k=0}^{n} a_k b_{n-k}$, then*

(i) $\limsup_{n \to +\infty} (1/n) \ln a_n = \limsup_{n \to +\infty} (1/n) \ln \sigma_n$,

(ii) $\liminf_{n \to +\infty} (1/n) \ln a_n = \liminf_{n \to +\infty} (1/n) \ln \sigma_n$.

Proof. The inequality $\liminf_{n \to +\infty} (1/n) \ln \sigma_n \leqslant \liminf_{n \to +\infty} (1/n) \ln a_n$ is the main difficult one. For any n, $p \geqslant 1$, we have

$$\sigma_{n+p} = \sum_{k=0}^{p-1} a_k b_{n-k} + \sum_{k=p}^{n+p} a_k b_{n-k} \leqslant a_0 \sum_{k \geqslant n} b_k + a_p \sum_{k \geqslant 0} b_k$$

Let us suppose $\liminf_{n \to +\infty} (1/n) \ln a_n < \alpha$ and let us choose $\delta \in (0, 1)$ and $\beta < \min(0, \alpha/\delta)$.

Since $\lim (1/n) \ln b_n < \beta$, for p large enough, we have

$$\sigma_{\mathrm{ent}(\delta p) + p} \leqslant a_0 \exp(\mathrm{ent}(\delta p) \beta) + a_p \sum_{k \geqslant 0} b_k$$

and thus for infinitely many p's,

$$\sigma_{\mathrm{ent}(\delta p) + p} \leqslant \left[a_0 + \sum_{k \geqslant 0} b_k \right] \exp(\alpha p)$$

which shows

$$\liminf_{n \to +\infty} \frac{1}{n} \ln \sigma_n \leqslant \frac{\alpha}{1 + \delta} \qquad \text{for any} \quad \delta \in (0, 1)$$

B.1.5. Proof of Theorem B.1.3. We will prove that, for any $x_0 \in \pi(\Sigma(\tilde{\mathscr{F}}))$ and $x \in \Sigma(\tilde{\mathscr{F}})$ such that $\pi(x) = x_0$, $\{\tilde{\lambda}_i(x)\}_{i \geqslant 1}$ and $\{\pi \circ \tilde{F}_i(x)\}_{i \geqslant 1}$ are the sequences of Lyapunov exponents and Lyapunov spaces at x_0.

Since $\operatorname{Ker} \pi \subset \bigcap_{i \geqslant 1} \tilde{F}_i(x)$, $\operatorname{codim} \pi(\tilde{F}_i(x)) = \operatorname{codim} \tilde{F}_i(x)$ and $\pi[\tilde{F}_i(x) \backslash \tilde{F}_{i+1}(x)] = \pi[\tilde{F}_i(x)] \backslash \pi[\tilde{F}_{i+1}(x)]$.

Since $\tilde{T}_x \bullet v = (T^n_{x_0} \bullet v_0, T^{n-1}_{x_0} \bullet v_0, \dots, T_{x_0} \bullet v_0, v_0, v_1, \dots)$.

$$\| T^n_{x_0} \|_\alpha \leqslant \| \tilde{T}^n_x \|_\alpha \leqslant \sum_{k=0}^{n} \sqrt{\gamma_k} \, \| T^{n-k}_{x_0} \|_\alpha$$

For any closed subspace $\tilde{F}$ of $\tilde{E}$ containing $\operatorname{Ker} \pi$, using the fact that π is open, $\pi(\tilde{F}) \triangleq F$ is a closed subvector space and π is also open considered as a map from $\tilde{F}$ onto $\pi(\tilde{F})$. In particular, $\pi(B_{\tilde{F}})$ contains a ball of $\pi(\tilde{F}): rB_F$, for some $r \geqslant 0$ and

$$r \, \| T^n_{x_0} \mid F \| \leqslant \| \tilde{T}^n_x \mid F \| \leqslant \left[\sum_{k=0}^{n} \gamma_k \, \| T^{n-k}_{x_0} \mid F \|^2 \right]^{1/2}$$

For any vector v in $\tilde{E}$,

$$\| T^n_{x_0} \bullet v_0 \| \leqslant \| \tilde{T}^n_x \bullet v \| \leqslant \left[\sum_{k=0}^{n} \gamma_k \, \| T^{n-k}_{x_0} \bullet v_0 \|^2 \right]^{1/2} \frac{\|v\|}{\|v_0\|}$$

Using the previous lemma for any strongly regular point x,

$$\lim_{n \to +\infty} \frac{1}{n} \ln \| T^n_{x_0} \|_\alpha = \lim_{n \to +\infty} \frac{1}{n} \ln \| \tilde{T}^n_x \|_\alpha \qquad (a_n = \| T^n_{x_0} \|_\alpha \, \tau^{-n})$$

$$\lim_{n \to +\infty} \frac{1}{n} \ln \| T^n_{x_0} \mid F \| = \lim_{n \to +\infty} \frac{1}{n} \ln \| \tilde{T}^n_x \mid F \| \qquad (a_n = \| T^n_{x_0} \mid F \| \, \tau^{-n})$$

$$\lim_{n \to +\infty} \frac{1}{n} \ln \| T^n_{x_0} \bullet v_0 \| = \lim_{n \to +\infty} \frac{1}{n} \ln \| \tilde{T}^n_x \bullet v \| \qquad (a_n = \| T^n_{x_0} \bullet v_0 \| \, \tau^{-n})$$

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